Network Theorems

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Overview

Substitution Theorem

Substitution Theorem

Theorem (Sufficient Condition for Uniqueness Solution)

Suppose that N is a strictly passive LTI RLCMT network, such that all its resistors have positive resistances, all its capacitors have positive capacitances, all its inductors have positive inductances. Suppose further that every set of coupled inductors has a positive definite inductance matrix. Under these conditions, given any initial state and any set of inputs, the network N has a unique solution.

- Proof: Non-singularity of the admittance matrix $\boldsymbol{Y}_n(s)$.
- Common LTI circuits: Strictly passive LTI RLCMT networks.
- Degenerate LTI circuits: LTI circuits with unit coupling factor, dependent sources, negative resistors, ...

Substitution Theorem

Theorem (Substitution Theorem)

Consider an arbitrary network which contains a number of independent sources. Suppose that for these sources and for the given initial conditions the network has a unique solution for all its branch voltages and branch currents. Consider a particular branch, say branch k, which is not coupled to other branches of the network. Let j_k and v_k be the current and voltage waveforms of branch k. Suppose that branch k is replaced by either an independent current source with waveform j_k or an independent voltage source with waveform v_k. If the modified network has a unique solution for all its branch currents and branch voltages, then these branch currents and branch voltages are identical with those of the original network.

- Proof: Same KCL and KVL equations for the original and modified networks.
- Coupled branch: Dependent source or coupled inductive element.
- Circuits with unique solution: Strictly passive LTI RLCMT networks.
- Circuits without unique solution: Nonlinear or time-varying circuits as well as degenerate LTI circuits.

Substitution Theorem

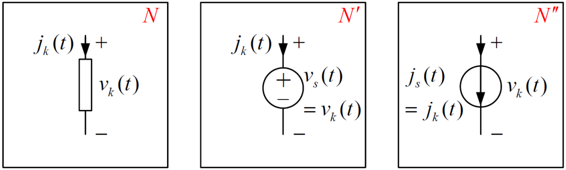


Figure: The three networks have unique solutions and branch k is not a coupled element or dependent source. The three networks have the same set of branch voltages and currents.

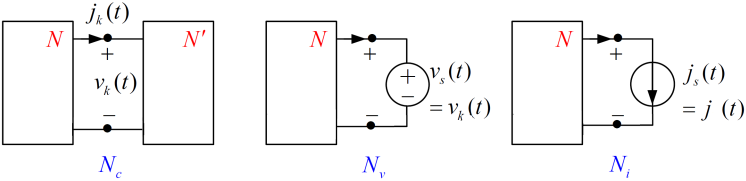
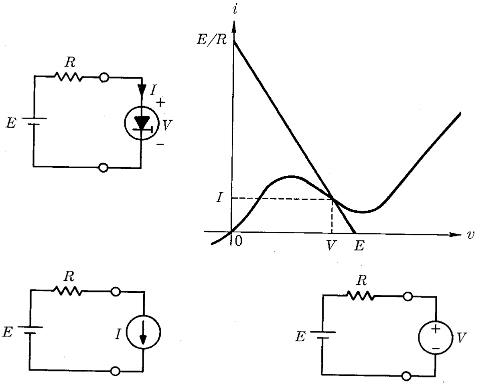


Figure: The three networks have unique solutions and sub-networks N and N' are not coupled. The sub-network N has the same solution in all three scenarios.

Substitution Theorem

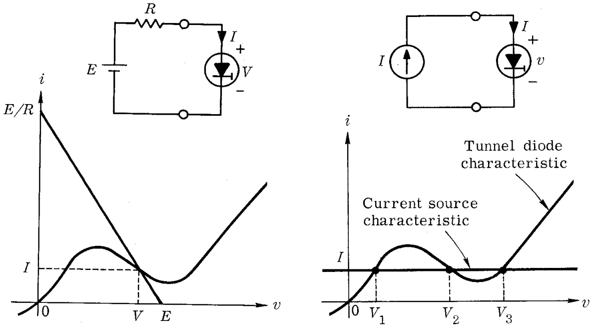
Example (Tunnel diode circuit)

The tunnel diode can be replaced by a current or voltage source according to the substitution theorem.



Example (Tunnel diode diode)

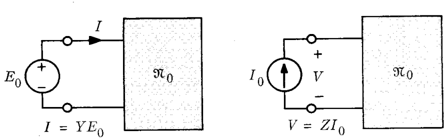
The resistor cannot be replaced by a current source due to failure of solution uniqueness condition required for the substitution theorem.



Substitution Theorem

Example (Admittance and impedance)

The admittance of a port is the inverse of the corresponding impedance of the port.

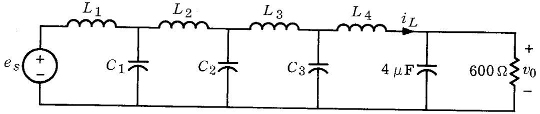


$$Y(s) = \frac{I(s)}{E_0(s)}, \quad Z(s) = \frac{V(s)}{I_0(s)}$$

$$I_0(s) = I(s) \Rightarrow V(s) = E_0(s) \Rightarrow Y(s) = \frac{I(s)}{E_0(s)} = \frac{I_0(s)}{V(s)} = \frac{1}{Z(s)}$$

Example (Ladder network)

The ladder network shown below is in the sinusoidal steady state. If $i_L(t) = 0.01\cos(377t)$ mA, then $v_0(t) = 4.45\cos(377t - 0.74)$.



$$V_0 = \frac{\frac{1}{j4\times10^{-6}\times377}}{\frac{1}{j4\times10^{-6}\times377}+600} \times 0.01 \times 600 = 4.45\underline{/-42.14^\circ} \Rightarrow v_0(t) = 4.45\cos(377t - 0.74)$$

Superposition Theorem

Superposition Theorem

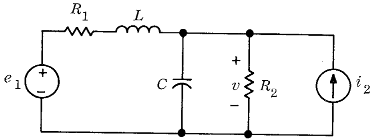
Theorem (Superposition Theorem)

Let N be a linear network; i.e., let each of its elements be either an independent source or a linear element (linear resistor, linear inductor, linear capacitor, linear transformer, or linear dependent source). The elements may be time-varying. We further assume that N has a unique zero-state response to the independent source waveforms, whatever they may be. Let the response of N be either the current in a specific branch of N, or the voltage across any specific node pair of N, or more generally any linear combination of currents and voltages. Under these conditions, the zero state response of N due to all the independent sources acting simultaneously is equal to the sum of the zero-state responses due to each independent source acting one at a time.

- Proof: Linearity of KCL, KVL, and LTI elements.
- Linear circuits: LTI or LTV circuits.
- Nonlinear networks: Superposition may not apply to nonlinear networks.
- Sinusoidal steady state: Superposition applies to sinusoidal steady state.
- Laplace analysis: $Y(s) = \sum_i H_i(s) W_i(s), \quad H_i(s) = \frac{Y(s)}{W_i(s)}\big|_{W_k(s)=0, k\neq i}$.
- Initial conditions: Can be modeled by independent sources.

Example (Transfer function)

Superposition theorem can be described in terms of transfer functions.



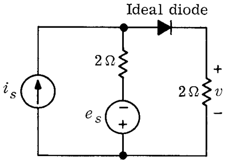
$$H_1(s) = \frac{V(s)}{E_1(s)}|_{I_2(s)=0} = \frac{R_2||\frac{1}{Cs}}{R_2||\frac{1}{Cs} + R_1 + Ls}$$

$$H_2(s) = \frac{V(s)}{I_2(s)}|_{E_1(s)=0} = R_2\frac{(R_1 + Ls)||\frac{1}{Cs}}{(R_1 + Ls)||\frac{1}{Cs} + R_2}$$

$$V(s) = H_1(s)E_1(s) + H_2(s)I_2(s)$$

Example (Nonlinear circuit)

In general, superposition does not apply to nonlinear circuits.



$$\begin{cases} i_s = 10, e_s = 0 \Rightarrow v = 10 \\ i_s = 0, e_s = 10 \Rightarrow v = 0 \\ i_s = 10, e_s = 10 \Rightarrow v = 5 \end{cases}$$

$$\begin{cases} i_s = 10, e_s = 0 \Rightarrow v = 10 \\ i_s = 0, e_s = -10 \Rightarrow v = 5 \\ i_s = 10, e_s = -10 \Rightarrow v = 15 \end{cases}$$

Thevenin-Norton Equivalent Network Theorem

Thevenin-Norton Equivalent Network Theorem

Theorem (Thevenin-Norton Equivalent Network Theorem)

Let the linear network N be connected by two of its terminals $1-1'$ *to an arbitrary load. Let N consist of independent sources and linear resistors, linear capacitors, linear inductors, linear transformers, and linear dependent sources. The elements may be time-varying. We further assume that N has a unique solution when it is terminated by the load, and when the load is replaced by an independent source. Let* N_0 *be the network obtained from N by setting all independent sources to zero and all initial conditions to zero. Let* e_{oc} *be the open-circuit voltage of N observed at terminals* $1-1'$*. Let* i_{sc} *be the short circuit current of N flowing out of* 1 *into* $1'$*. Under these conditions, whatever the load may be, the voltage waveform* $v(t)$ *across* $1-1'$ *and the current waveform* $i(t)$ *through* 1 *and* $1'$ *remain unchanged when the network N is replaced by either its Thevenin equivalent or by its Norton equivalent network.*

- Proof: Superposition theorem.
- Arbitrary load: Nonlinear time-varying load.
- Terminal interaction: Exclusive interaction with the load through the terminal.

Thevenin-Norton Equivalent Network Theorem

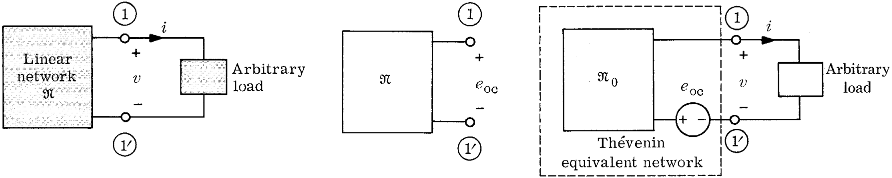


Figure: Thevenin equivalent circuit for a linear circuit.

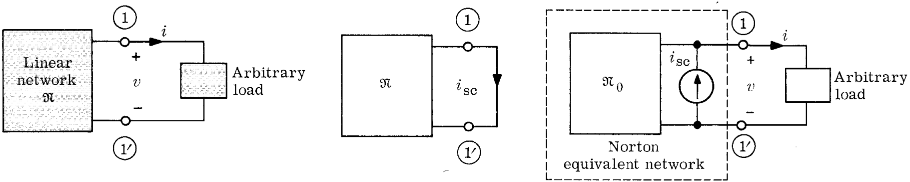


Figure: Norton equivalent circuit for a linear circuit.

Thevenin-Norton Equivalent Network Theorem

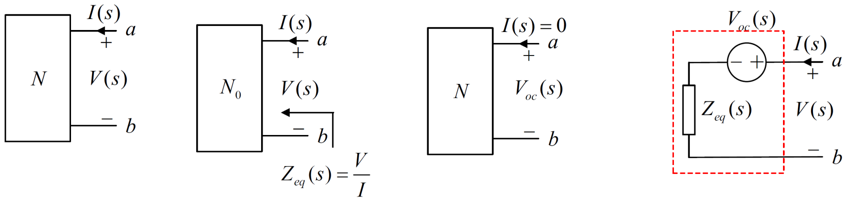


Figure: Thevenin equivalent circuit in Laplace domain for an LTI circuit. Clearly, $V_{oc} = Z_{eq} I_{sc}$.

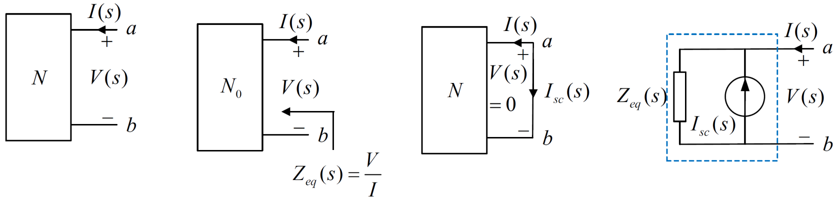
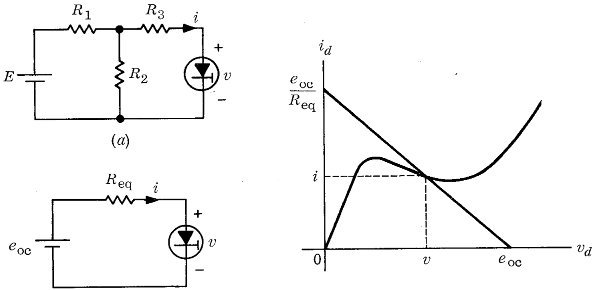


Figure: Norton equivalent circuit in Laplace domain for an LTI circuit. Clearly, $V_{oc} = Z_{eq} I_{sc}$.

Example (Nonlinear load)

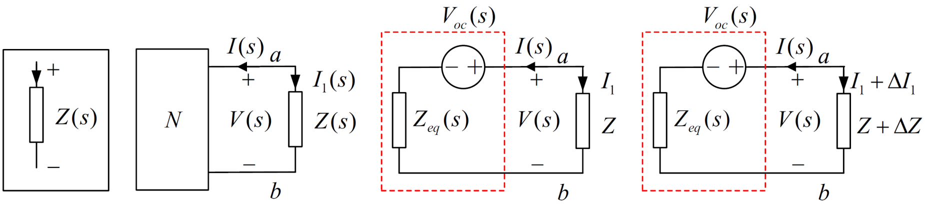
Thevenin equivalent circuit can be used to determine the working point of the nonlinear circuit below with only one nonlinear load element.



$$R_{eq} = \frac{R_1 R_2}{R_1 + R_2} + R_3, \quad e_{oc} = \frac{R_2}{R_1 + R_2} E, \quad v = e_{oc} - R_{eq} i$$

Example (Sensitivity analysis)

Thevenin equivalent circuit can facilitate sensitivity analysis.

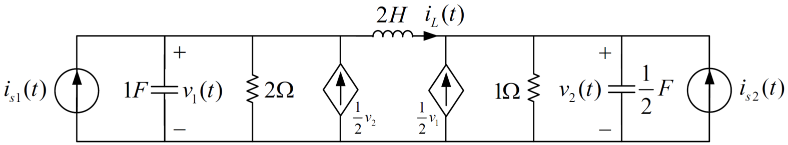


$$S_Z^{I_1} = \frac{dI_1}{dZ} = \frac{d}{dZ}\left[\frac{V_{oc}}{Z_{eq}+Z}\right] = -\frac{V_{oc}}{(Z_{eq}+Z)^2} = -\frac{I_1}{Z_{eq}+Z}$$

Thevenin-Norton Equivalent Network Theorem

Example (Laplace analysis)

Laplace analysis can be used to obtained the Thevenin or Norton equivalent circuits.



$$V_1(s) = H_1(s)I_{s_1}(s) + H_2(s)I_{s_2}(s) + \frac{F_0(s)}{A_2(s)} = Z_{eq}(s)I_{s_1} + V_{oc}(s)$$

Reciprocity Theorem

Reciprocity Theorem

Theorem (Reciprocity Theorem (first statement))

Consider a linear time-invariant network N; which consists of resistors, inductors, coupled inductors, capacitors, and transformers only. N is in the zero state and is not degenerate. Connect four wires to N thus obtaining two pairs of terminals $1-1'$ *and* $2-2'$*. Now, connect a voltage source* $e_0(t)$ *to terminals* $1-1'$ *and observe the zero state current response* $j_2(t)$ *in a short circuit connected to* $2-2'$*. Next, connect the same voltage source* $e_0(t)$ *to terminals* $2-2'$ *and observe the zero-state current response* $\hat{j}_1(t)$ *in a short circuit connected to* $1-1'$*. The reciprocity theorem asserts that whatever the topology and the element values of the network N and whatever the waveform* $e_0(t)$ *of the source,* $j_2(t) = \hat{j}_1(t)$*.*

- Proof: Tellegen's theorem.
- Reciprocal circuit: Any circuit for which reciprocity is held.
- Common reciprocal circuits: RLCMT network in zero-state without independent and dependent sources
- Nonreciprocal circuits: Gyrator, dependent sources, independent sources, ...

Reciprocity Theorem

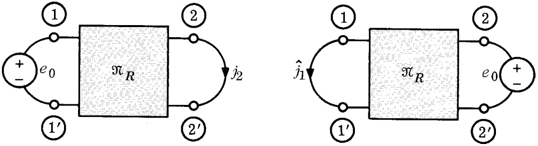


Figure: First statement of the reciprocity theorem assures $j_2(t) = \hat{j}_1(t)$.

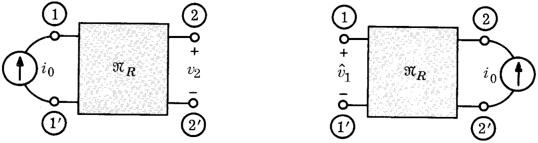


Figure: Second statement of the reciprocity theorem assures $v_2(t) = \hat{v}_1(t)$.

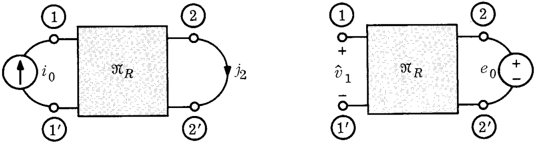


Figure: Third statement of the reciprocity theorem assures $j_2(t) \equiv \hat{v}_1(t)$ if $i_0(t) \equiv e_0(t)$.

Reciprocity Theorem

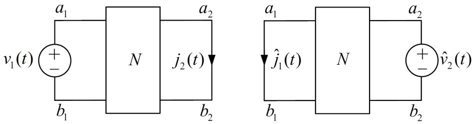


Figure: First statement of the reciprocity theorem assures $\frac{J_2(s)}{V_1(s)} = \frac{\hat{J}_1(s)}{\hat{V}_2(s)}$.

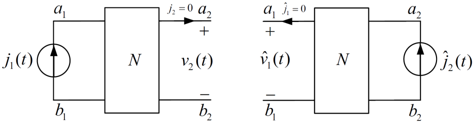


Figure: Second statement of the reciprocity theorem assures $\frac{V_2(s)}{J_1(s)} = \frac{\hat{V}_1(s)}{\hat{J}_2(s)}$.

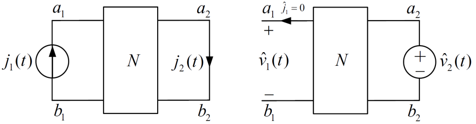


Figure: Third statement of the reciprocity theorem assures $\frac{J_2(s)}{J_1(s)} = \frac{\hat{V}_1(s)}{\hat{V}_2(s)}$.

Reciprocity Theorem

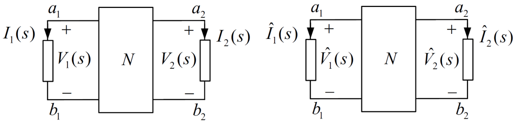


Figure: Reciprocity theorem for an RLCMT network.

$$\sum_{k=1}^{b} V_k I_k = 0, \quad \sum_{k=1}^{b} \hat{V}_k I_k = 0, \quad \sum_{k=1}^{b} V_k \hat{I}_k = 0, \quad \sum_{k=1}^{b} \hat{V}_k \hat{I}_k = 0$$

$$\sum_{k=1}^{b} V_k \hat{I}_k = \sum_{k=1}^{b} \hat{V}_k I_k \Rightarrow V_1 \hat{I}_1 + V_2 \hat{I}_2 + \sum_{k=3}^{b} V_k \hat{I}_k = \hat{V}_1 I_1 + \hat{V}_2 I_2 + \sum_{k=3}^{b} \hat{V}_k I_k$$

$$\begin{cases} \text{R,L,C: } \hat{V}_k I_k = Z_k \hat{I}_k I_k = Z_k I_k \hat{I}_k = V_k \hat{I}_k \\ \text{M: } \hat{V}_m I_m + \hat{V}_n I_n = (L_m \hat{I}_m + M_{mn} \hat{I}_n) I_m + (M_{mn} \hat{I}_m + L_n \hat{I}_n) I_n = V_m \hat{I}_m + V_n \hat{I}_n \\ \text{T: } \hat{V}_m I_m + \hat{V}_n I_n = 0 = V_m \hat{I}_m + V_n \hat{I}_n \end{cases}$$

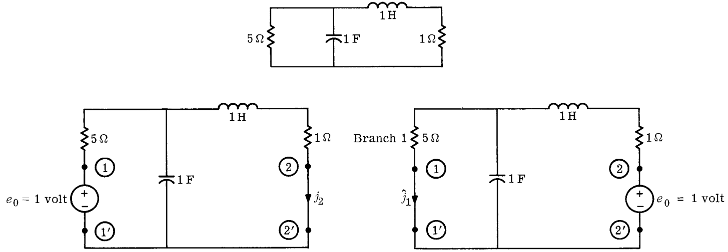
$$\Rightarrow V_1 \hat{I}_1 + V_2 \hat{I}_2 = \hat{V}_1 I_1 + \hat{V}_2 I_2$$

$$\begin{cases} \hat{V}_1 = 0, \hat{I}_1 = \hat{J}_1, \hat{V}_2, \hat{I}_2 = \hat{J}_2 \\ V_1, I_1 = J_1, V_2 = 0, I_2 = J_2 \end{cases} \Rightarrow J_2 \hat{V}_2 = V_1 \hat{J}_1 \Rightarrow \frac{J_2(s)}{V_1(s)} = \frac{\hat{J}_1(s)}{\hat{V}_2(s)}$$

Reciprocity Theorem

Example (Reciprocity theorem for an RLC network)

The RLC network below is reciprocal.

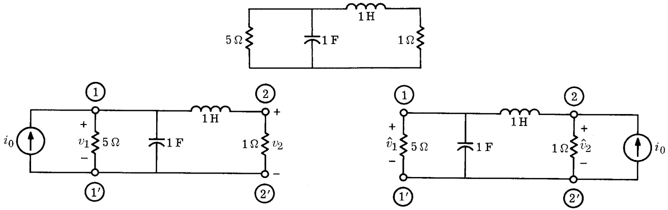


$$j_2 = \hat{j}_1$$

Reciprocity Theorem

Example (Reciprocity theorem for an RLC network (cont.))

The RLC network below is reciprocal.

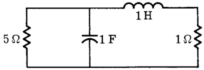


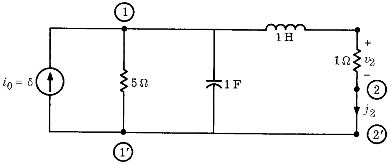
$$v_2 = \hat{v}_1$$

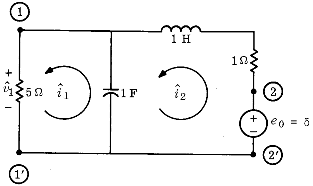
Reciprocity Theorem

Example (Reciprocity theorem for an RLC network (cont.))

The RLC network below is reciprocal.





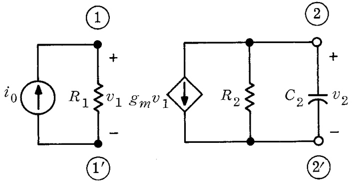


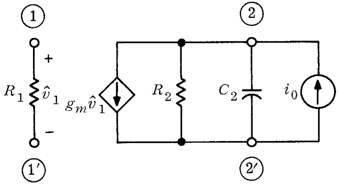
$$j_2 \equiv \hat{v}_1$$

Reciprocity Theorem

Example (Circuit with dependent source)

In general, reciprocity does not apply to the circuits with dependent sources.



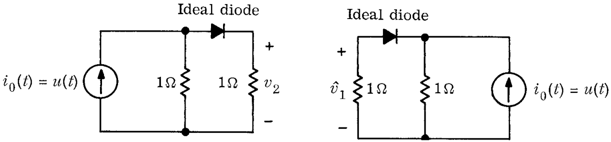


$$v_2(t) = -R_1 g_m R_2 I(1 - e^{-t/R_2 C_2}),\ t \geq 0; \quad \hat{v}_1(t) = 0,\ t \geq 0$$

Reciprocity Theorem

Example (Nonlinear circuit)

In general, reciprocity does not apply to the nonlinear circuits.

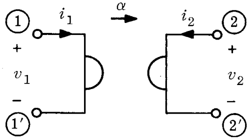


$$v_2(t) = 0.5,\ t \geq 0; \quad \hat{v}_1(t) = 0,\ t \geq 0$$

Reciprocity Theorem

Example (Gyrator)

Gyrator is a passive LTI non-reciprocal circuit.



$$\begin{cases} v_1(t) = \alpha i_2(t) \\ v_2(t) = -\alpha i_1(t) \end{cases} \Rightarrow v_1(t)i_1(t) + v_2(t)i_2(t) = 0$$

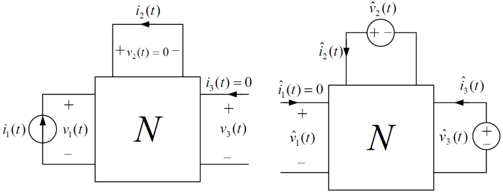
$$i_1(t) = i_0(t) \Rightarrow v_2(t) = -\alpha i_0(t)$$

$$\hat{i}_2(t) = i_0(t) \Rightarrow \hat{v}_1(t) = \alpha i_0(t)$$

Reciprocity Theorem

Example (Two-measurement experiment)

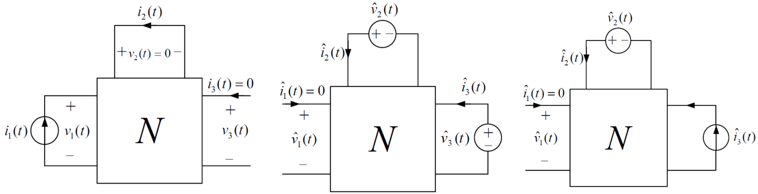
The network theorems can be used to find unknown network variables in a two-measurement experiment.



$$
\begin{cases}
v_1(t) = (-6e^{-t} + 14e^{-2t})u(t) \\
v_2(t) = 0 \\
v_3(t) = (-6e^{-t} + 12e^{-2t})u(t) \\
i_1(t) = \delta(t) \\
i_2(t) = -2e^{-2t}u(t) \\
i_3(t) = 0
\end{cases}
,\quad
\begin{cases}
\hat{v}_1(t) = ? \\
\hat{v}_2(t) = 24u(t) \\
\hat{v}_3(t) = (-12e^{-t} + 24e^{-2t})u(t) \\
\hat{i}_1(t) = 0 \\
\hat{i}_2(t) = 24e^{-2t}u(t) \\
\hat{i}_3(t) = 2\delta(t)
\end{cases}
$$

Example (Two-measurement experiment)

The network theorems can be used to find unknown network variables in a two-measurement experiment.



$$\hat{V}_1(s) = H_1(s)\hat{V}_2(s) + H_2(s)\hat{I}_3(s)$$

$$H_1(s) = \frac{\hat{V}_1(s)}{\hat{V}_2(s)}\Big|_{\hat{I}_3(s)=0} = \frac{-I_2}{I_1} = \frac{2}{s+2}$$

$$H_2(s) = \frac{\hat{V}_1(s)}{\hat{I}_3(s)}\Big|_{\hat{V}_2(s)=0} = \frac{V_3}{I_1} = \frac{-6}{s+1} + \frac{12}{s+2}$$

$$\hat{V}_1(s) = \frac{24}{s} - \frac{12}{s+1} \Rightarrow \hat{v}_2(t) = (24 - 12e^{-t})u(t)$$

The End