Laplace Transforms

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Overview

Laplace Transform

Laplace Transform

Definition (Laplace Transform)

The unidirectional Laplace transform of $f(t)$ is defined as

$$F(s) = \mathcal{L}[f(t)] = \int_{0^-}^{\infty} f(t)e^{-st}dt, \quad s \in \text{ROC}$$

, where ROC is the region of convergence representing complex values of s for which the Laplace integral converges. $f(t)$ can be calculated from its Laplace transform as

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds, \quad \sigma \in \text{ROC}$$

Laplace Transform

Example (Laplace transform of $u(t)$)

$\mathcal{L}[u(t)] = \frac{1}{s}, \quad \Re\{s\} > 0.$

$$\int_{0^-}^{\infty} f(t)e^{-st}dt = \int_{0}^{\infty} e^{-st}dt = \frac{-e^{-st}}{s}\Big|_0^{\infty} = \frac{-e^{-(\sigma+j\omega)t}}{s}\Big|_0^{\infty} = \frac{1}{s}, \quad \Re\{s\} > 0$$

Example (Laplace transform of $e^{-at}u(t)$)

$\mathcal{L}[e^{-at}u(t)] = \frac{1}{s+a}, \quad \Re\{s\} > \Re\{-a\}.$

$$\int_{0^-}^{\infty} f(t)e^{-st}dt = \int_{0}^{\infty} e^{-at}e^{-st}dt = \frac{-e^{-(s+a)t}}{s+a}\Big|_0^{\infty} = \frac{-e^{-(s+a)t}}{s+a}\Big|_0^{\infty} = \frac{1}{s+a}, \quad \Re\{s\} > \Re\{-a\}$$

Laplace Transform Properties

Property	Time Domain	Laplace Domain
Linearity	$\alpha f_1(t) + \beta f_2(t)$	$\alpha F_1(s) + \beta F_2(s)$
Time Shift	$f(t-t_0)u(t-t_0)$	$e^{-st_0}F(s)$
Frequency Shift	$e^{s_0 t}f(t)$	$F(s-s_0)$
Scaling	$f(\alpha t), \alpha > 0$	$\frac{1}{\alpha}F(\frac{s}{\alpha})$
Conjugation	$f^*(t)$	$F^*(s^*)$
Time Differentiation	$f'(t)$	$sF(s) - f(0^-)$
Time Differentiation	$f''(t)$	$s^2F(s) - sf(0^-) - f'(0^-)$
Frequency Differentiation	$tf(t)$	$-X'(s)$
Time Integration	$\int_{0^-}^{\infty} f(\alpha)d\alpha$	$F(s)/s$
Convolution	$f_1(t) * f_2(t)$	$F_1(s)F_2(s)$
Periodicity	$f(t+T) = f(t)$	$\int_{0^-}^{T^-} f(t)e^{-st}dt/(1-e^{-sT})$
Initial Value	$f(0^+) = \lim_{s\to\infty} sF(s)$	
Final Value	$f(+\infty) = \lim_{s\to 0} sF(s)$	

Table: Properties of Laplace transform. For convolution property, $f_1(t) = 0, t < 0$ and $f_2(t) = 0, t < 0$. For initial value and final value properties, $f(t) = 0, t < 0$ and has no singular function at $t = 0$.

Laplace Transform Pairs

Time Domain	Laplace Domain
$\delta(t)$	1
$\delta'(t)$	s
$\delta^{(n)}(t)$	s^n
$u(t)$	$\frac{1}{s}$
$tu(t)$	$\frac{1}{s^2}$
$t^n u(t)$	$\frac{n!}{s^{n+1}}$
$e^{-at}u(t)$	$\frac{1}{s+a}$
$t^n e^{-at}u(t)$	$\frac{n!}{(s+a)^{n+1}}$
$\cos(\beta t)u(t)$	$\frac{s}{s^2+\beta^2}$
$\sin(\beta t)u(t)$	$\frac{\beta}{s^2+\beta^2}$

Table: Useful Laplace pairs.

Time Domain	Laplace Domain
$\cosh(\beta t)u(t)$	$\frac{s}{s^2-\beta^2}$
$\sinh(\beta t)u(t)$	$\frac{\beta}{s^2-\beta^2}$
$e^{-at}\cos(\beta t)u(t)$	$\frac{s+a}{(s+a)^2+\beta^2}$
$e^{-at}\sin(\beta t)u(t)$	$\frac{\beta}{(s+a)^2+\beta^2}$
$t\cos(\beta t)u(t)$	$\frac{s^2-\beta^2}{(s^2+\beta^2)^2}$
$t\sin(\beta t)u(t)$	$\frac{2\beta s}{(s^2+\beta^2)^2}$
$\cos(\beta t+\phi)u(t)$	$\frac{s\cos(\phi)-\beta\sin(\phi)}{s^2+\beta^2}$
$2\|K\|e^{-at}\cos(\beta t+\angle K)u(t)$	$\frac{K}{s+a-j\beta}+\frac{K^*}{s+a+j\beta}$
$2\|K\|te^{-at}\cos(\beta t+\angle K)u(t)$	$\frac{K}{(s+a-j\beta)^2}+\frac{K^*}{(s+a+j\beta)^2}$

Table: Useful Laplace pairs.

Laplace Transform Pairs

Example (Laplace Transform)

$\mathcal{L}[t\cos(\beta t)u(t)] = \frac{s^2-\beta^2}{s^2+\beta^2}$

$$e^{-at}u(t) \rightarrow \frac{1}{s+a}$$

$$e^{-j\beta t}u(t) \rightarrow \frac{1}{s+j\beta}$$

$$\cos(\beta t)u(t) = 0.5(e^{j\beta t} + e^{-j\beta t})u(t) \rightarrow 0.5(\frac{1}{s-j\beta} + \frac{1}{s+j\beta}) = \frac{s}{s^2+\beta^2}$$

$$t\cos(\beta t)u(t) \rightarrow -\frac{d}{ds}[\frac{s}{s^2+\beta^2}] = \frac{s^2-\beta^2}{(s^2+\beta^2)^2}$$

Fractional Functions

- Real-coefficient rational function:
  $F(s) = \frac{P(s)}{Q(s)} = \frac{\sum_{l=0}^{m} b_l s^l}{\sum_{k=0}^{n} a_k s^k}, \quad a_i, b_i \in \mathbb{R}, s \in \mathbb{C}$
- Transfer function zeros: $\{z_i \in \mathcal{C} | P(z_i) = 0\}$
- Transfer function poles: $\{p_i \in \mathcal{C} | Q(p_i) = 0\}$
- Zero-pole decomposition: $F(s) = \frac{P(s)}{Q(s)} = K \frac{\prod_{l=1}^{m} (s - z_l)}{\prod_{k=1}^{n} (s - p_k)}$
- Proper rational function: $F(s) = \frac{P(s)}{Q(s)} = \frac{\sum_{l=0}^{m} b_l s^l}{\sum_{k=0}^{n} a_k s^k}, \quad a_i, b_i \in \mathbb{R}, s \in \mathbb{C}, m < n$
- Proper rational-polynomial function decomposition:
  $F(s) = \frac{P(s)}{Q(s)} = \hat{P}(s) + \frac{R(s)}{Q(s)}$

Partial-fraction Expansion

Definition (Partial-fraction Expansion)

A proper fractional function is decomposed as

$$F(s) = \frac{R(s)}{Q(s)} = \frac{R(s)}{\prod_{k=1}^{r}(s-p_k)^{n_k}} = \sum_{i=1}^{r}\sum_{j=1}^{n_i}\frac{K_{i,j}}{(s-p_i)^j}$$

, where

$$K_{i,n_i-l} = \frac{1}{l!}\frac{d^l}{ds^l}\left[(s-p_i)^{n_i}F(s)\right]\Big|_{s=p_i}, \quad l = 0, 1, \cdots, n_i - 1$$

Inverse Laplace Transform

Example (Proper rational Laplace function with two simple poles)

$\mathcal{L}^{-1}[\frac{s+3}{(s+1)(s+2)}] = (2e^{-t} - e^{-2t})u(t)$

$$F(s) = \frac{s+3}{(s+1)(s+2)} = \frac{K_{11}}{s+1} + \frac{K_{21}}{s+2}$$

$$K_{11} = (s+1)F(s)\big|_{s=-1} = \frac{s+3}{s+2}\big|_{s=-1} = 2, \quad K_{21} = (s+2)F(s)\big|_{s=-2} = \frac{s+3}{s+1}\big|_{s=-2} = -1$$

$$F(s) = \frac{s+3}{(s+1)(s+2)} = \frac{2}{s+1} + \frac{-1}{s+2}$$

$$F(s) = \frac{s+3}{(s+1)(s+2)} = \frac{K_{11}}{s+1} + \frac{K_{21}}{s+2} = \frac{(K_{11}+K_{21})s + (2K_{11}+K_{21})}{(s+1)(s+2)}$$

$$\begin{cases} K_{11} + K_{21} = 1 \\ 2K_{11} + K_{21} = 3 \end{cases} \Rightarrow \begin{cases} K_{11} = 2 \\ K_{12} = -1 \end{cases}$$

$$F(s) = \frac{s+3}{(s+1)(s+2)} = \frac{2}{s+1} + \frac{-1}{s+2}$$

$$f(t) = (2e^{-t} - e^{-2t})u(t)$$

Inverse Laplace Transform

Example (Improper rational Laplace function with two simple poles)

$\mathcal{L}^{-1}[\frac{s^2+1}{s(s+1)}] = \delta(t) + (1 - 2e^{-t})u(t)$

$$F(s) = \frac{s^2+1}{s(s+1)} = 1 + \frac{-s+1}{s(s+1)} = 1 + \frac{K_{11}}{s} + \frac{K_{21}}{s+1}$$

$$K_{11} = \frac{-s+1}{s+1}\Big|_{s=0} = 1, \quad K_{21} = \frac{-s+1}{s}\Big|_{s=-1} = -2$$

$$F(s) = \frac{s^2+1}{s(s+1)} = 1 + \frac{-s+1}{s(s+1)} = 1 + \frac{1}{s} + \frac{-2}{s+1}$$

$$f(t) = \delta(t) + (1 - 2e^{-t})u(t)$$

Example (Proper rational Laplace function with simple, repeated, and complex poles)

$$\mathcal{L}^{-1}\left[\frac{4s^4+19s^3+36s^2+34s+16}{(s+2)(s+1)^2(s^2+2s+2)}\right] = \left[2e^{-2t} + 2e^{-t} + 3te^{-t} + 2\sqrt{2}e^{-t}\cos(t + \angle 45^\circ)\right]u(t)$$

$$F(s) = \frac{4s^4 + 19s^3 + 36s^2 + 34s + 16}{(s+2)(s+1)^2(s^2+2s+2)} = \frac{K_{11}}{s+2} + \frac{K_{21}}{s+1} + \frac{K_{22}}{(s+1)^2} + \frac{K_{31}}{s+1-j} + \frac{K_{41}}{s+1+j}$$

$$K_{11} = \frac{4s^4 + 19s^3 + 36s^2 + 34s + 16}{(s+1)^2(s^2+2s+2)}\Big|_{s=-2} = 2$$

$$K_{21} = \frac{d}{ds}\left[\frac{4s^4 + 19s^3 + 36s^2 + 34s + 16}{(s+2)(s^2+2s+2)}\right]\Big|_{s=-1} = 2$$

$$K_{22} = \frac{4s^4 + 19s^3 + 36s^2 + 34s + 16}{(s+2)(s^2+2s+2)}\Big|_{s=-1} = 3$$

$$K_{31} = \frac{4s^4 + 19s^3 + 36s^2 + 34s + 16}{(s+2)(s+1)^2(s+1+j)}\Big|_{s=-1+j} = 1+j$$

$$K_{41} = \frac{4s^4 + 19s^3 + 36s^2 + 34s + 16}{(s+2)(s+1)^2(s+1-j)}\Big|_{s=-1-j} = 1-j$$

$$F(s) = \frac{4s^4 + 19s^3 + 36s^2 + 34s + 16}{(s+2)(s+1)^2(s^2+2s+2)} = \frac{2}{s+2} + \frac{2}{s+1} + \frac{3}{(s+1)^2} + \frac{1+j}{s+1-j} + \frac{1-j}{s+1+j}$$

$$f(t) = \left[2e^{-2t} + 2e^{-t} + 3te^{-t} + 2\sqrt{2}e^{-t}\cos(t + \angle 45^\circ)\right]u(t)$$

Example (Proper rational Laplace function with repeated complex poles)

$$\mathcal{L}^{-1}\left[\frac{768}{(s^2+6s+25)^2}\right] = \left[6e^{-3t}\cos(4t + \angle 90^\circ) - 24te^{-3t}\cos(4t)\right]u(t)$$

$$F(s) = \frac{768}{(s^2+6s+25)^2} = \frac{K_{11}}{s+3-4j} + \frac{K_{12}}{(s+3-4j)^2} + \frac{K_{21}}{s+3+4j} + \frac{K_{22}}{(s+3+4j)^2}$$

$$K_{11} = \frac{d}{ds}\left[\frac{768}{(s+3+4j)^2}\right]\Big|_{s=-3+4j} = -3j$$

$$K_{12} = \frac{768}{(s+3+4j)^2}\Big|_{s=-3+4j} = -12$$

$$K_{21} = \frac{d}{ds}\left[\frac{768}{(s+3-4j)^2}\right]\Big|_{s=-3-4j} = +3j$$

$$K_{22} = \frac{768}{(s+3-4j)^2}\Big|_{s=-3-4j} = -12$$

$$F(s) = \frac{768}{(s^2+6s+25)^2} = \frac{-3j}{s+3-4j} + \frac{-12}{(s+3-4j)^2} + \frac{3j}{s+3+4j} + \frac{-12}{(s+3+4j)^2}$$

$$f(t) = \left[6e^{-3t}\cos(4t + \angle 90^\circ) - 24te^{-3t}\cos(4t)\right]u(t)$$

Laplace Analysis

Circuit Analysis

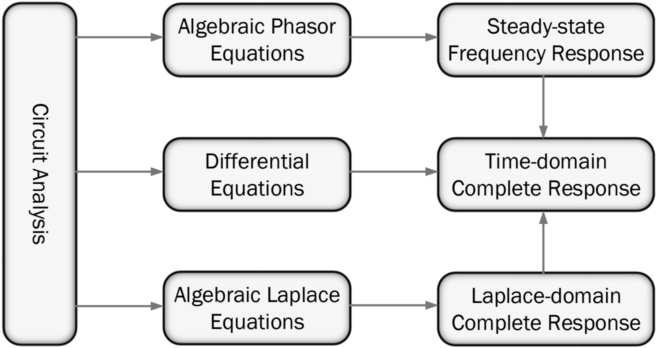


Figure: Different methods of circuit analysis.

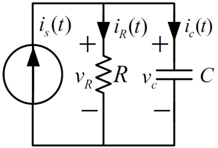
Example (First-order RC circuit)

The complete response to the impulse and step inputs for a first-order circuit can be found using Laplace transform.

$$C\frac{dv_C(t)}{dt} + \frac{v_C(t)}{R} = i_s(t), \quad v_C(0^-) = V_0$$

$$\frac{dv_C(t)}{dt} + \frac{v_C(t)}{RC} = \frac{\delta(t)}{C} \Rightarrow (sV_C(s) - V_0) + \frac{V_c(s)}{RC} = \frac{1}{C}$$

$$V_C(s) = \frac{\frac{1}{C} + V_0}{s + \frac{1}{RC}} \Rightarrow v_C(t) = [\frac{1}{C} + V_0]e^{-\frac{t}{RC}}u(t), t \geq 0$$



$$\frac{dv_C(t)}{dt} + \frac{v_C(t)}{RC} = \frac{u(t)}{C} \Rightarrow (sV_C(s) - V_0) + \frac{V_c(s)}{RC} = \frac{1}{Cs}$$

$$V_C(s) = \frac{V_0}{s + \frac{1}{RC}} + \frac{R}{s} - \frac{R}{s + \frac{1}{RC}}$$

$$v_C(t) = V_0 e^{-\frac{t}{RC}}u(t) + R(1 - e^{-\frac{t}{RC}})u(t), t \geq 0$$

Complete Response

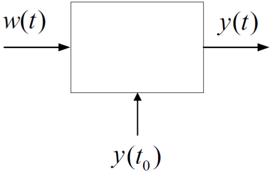


Figure: Laplace transform for constant-coefficient linear differential equations. Laplace analysis confirms that for linear systems, the complete response is the sum of zero-input and zero-state responses.

$$\sum_{k=0}^{n} a_k y^{(k)}(t) = \sum_{l=0}^{m} b_l w^{(l)}(t), \quad y(0^-), y'(0^-), \cdots, y^{(n-1)}(0^-)$$

$$\sum_{k=0}^{n} \left[a_k s^k Y(s) - \sum_{k'=1}^{k} s^{k-k'} y^{k'-1}(0^-)\right] = \sum_{l=0}^{m} b_l s^l W(s)$$

$$Y(s) \sum_{k=0}^{n} a_k s^k - F_0(s) = W(s) \sum_{l=0}^{m} b_l s^l$$

$$Y(s) = \frac{\sum_{l=0}^{m} b_l s^l}{\sum_{k=0}^{n} a_k s^k} W(s) + \frac{F_0(s)}{\sum_{k=0}^{n} a_k s^k}$$

$$Y(s) = H(s) W(s) + \frac{F_0(s)}{\sum_{k=0}^{n} a_k s^k}$$

Zero-state Response

Figure: Transfer function for zero-state response of LTI systems.

- Laplace-domain zero-state response: $Y(s) = H(s)W(s)$
- Transfer function: $H(s) = \frac{\sum_{l=0}^{m} b_l s^l}{\sum_{k=0}^{n} a_k s^k}$
- Transfer function zeros: $\{s_i \in \mathcal{C} | H(s_i) = 0\}$
- Transfer function poles: $\{s_i \in \mathcal{C} | H(s_i) = \infty\}$
- Time-domain zero-state response: $y(t) = h(t) * w(t), \quad h(t) = \mathcal{L}^{-1}[H(s)]$
- Frequency response: $H(j\omega) = H(s)|_{s=j\omega}$
- Multi-input zero-state response: $Y(s) = \sum_i H_i(s)W_i(s), \quad H_i(s) = \frac{Y(s)}{W_i(s)}|_{W_k(s)=0, k\neq i}$

Impedance and Admittance

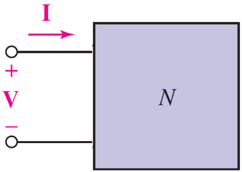


Figure: Impedance $Z(s) = \frac{V(s)}{I(s)}$ and admittance $Y(s) = \frac{I(s)}{V(s)} = \frac{1}{Z(s)}$ for a one-port network containing no independent sources and initial conditions.

Element	**Impedance** $Z(s) = \frac{V(s)}{I(s)}$	**Admittance** $Y(s) = \frac{I(s)}{V(s)}$
Resistor	R	G
Capacitor	$\frac{1}{Cs}$	Cs
Inductor	jLs	$\frac{1}{Ls}$

Table: Impedance and admittance for basic LTI one-port circuit elements. Series and parallel combinations as well as delta-why conversion can be used for impedance and admittance.

Impedance and Admittance

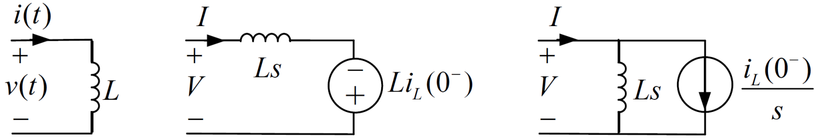


Figure: Series and parallel model of LTI inductor with initial condition in Laplace-domain.

- Time-domain element equation: $v(t) = Li'(t), \quad i_L(0^-)$
- Series Laplace-domain element equation: $V(s) = LsI(s) - Li_L(0^-)$
- Parallel Laplace-domain element equation: $I(s) = \frac{V(s)}{Ls} + \frac{i_L(0^-)}{s}$

Impedance and Admittance

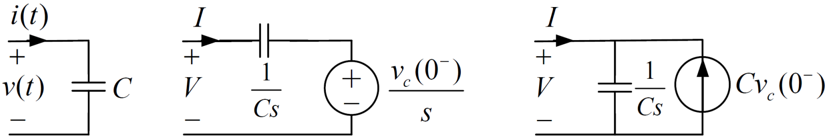
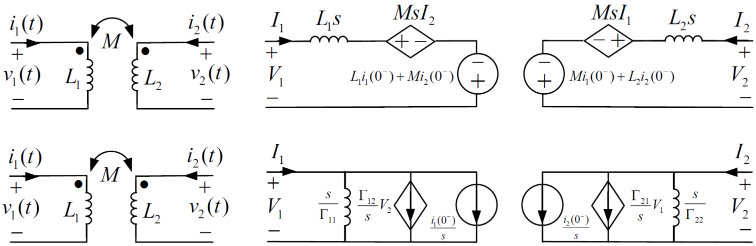


Figure: Series and parallel model of LTI capacitor with initial condition in Laplace-domain.

- Time-domain element equation: $i(t) = Lv'(t), \quad v_C(0^-)$
- Parallel Laplace-domain element equation: $I(s) = CsV(s) - Cv_C(0^-)$
- Series Laplace-domain element equation: $V(s) = \frac{I(s)}{Cs} + \frac{v_C(0^-)}{s}$

Example (Laplace model of coupled inductors)

Coupled inductors can be modeled in Laplace domain using dependent sources.



$$\begin{cases} v_1(t) = L_1 i_1'(t) + M i_2'(t) \\ v_2(t) = M i_1'(t) + L_2 i_2'(t) \end{cases} \quad , \quad i_1(0^-), i_2(0^-)$$

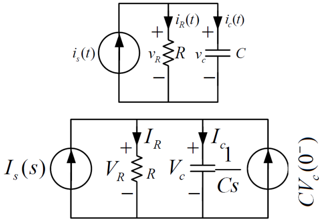
$$\begin{cases} V_1(s) = L_1 s I_1(s) + M s I_2(s) - (L_1 i_1(0^-) + M i_2(0^-)) \\ V_2(s) = M s I_1(s) + L_2 s I_2(s) - (M i_1(0^-) + L_2 i_2(0^-)) \end{cases}$$

$$\begin{cases} I_1(s) = \frac{\Gamma_{11}}{s} V_1(s) + \frac{\Gamma_{12}}{s} V_2(s) + \frac{i_1(0^-)}{s} \\ I_2(s) = \frac{\Gamma_{21}}{s} V_1(s) + \frac{\Gamma_{22}}{s} V_2(s) + \frac{i_2(0^-)}{s} \end{cases}$$

Example (First-order RC circuit)

The complete response to the impulse and step inputs for a first-order circuit can be found using Laplace transform.

$$-I_s(s) + \frac{V_C(s)}{R} + \frac{V_C(s)}{1/Cs} - CV_0 = 0 \Rightarrow V_C(s) = \frac{\frac{I_s(s)}{C} + V_0}{s + \frac{1}{RC}}$$



$$I_s(s) = 1 \Rightarrow V_C(s) = \frac{\frac{1}{C} + V_0}{s + \frac{1}{RC}}$$

$$v_C(t) = [\frac{1}{C} + V_0]e^{-\frac{t}{RC}}u(t), t \geq 0$$

$$I_s(s) = \frac{1}{s} \Rightarrow V_C(s) = \frac{\frac{1}{Cs} + V_0}{s + \frac{1}{RC}}$$

$$V_C(s) = \frac{V_0}{s + \frac{1}{RC}} + \frac{R}{s} - \frac{R}{s + \frac{1}{RC}}$$

$$v_C(t) = V_0 e^{-\frac{t}{RC}}u(t) + R(1 - e^{-\frac{t}{RC}})u(t), t \geq 0$$

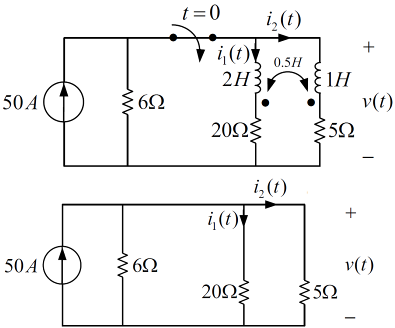
Example (Coupled circuit)

A coupled circuit can be analyzed using Laplace transform.

$$i_1(0^-) = 50\frac{6||5}{6||5+20} = 6$$

$$i_2(0^-) = 50\frac{6||20}{6||20+5} = 24$$

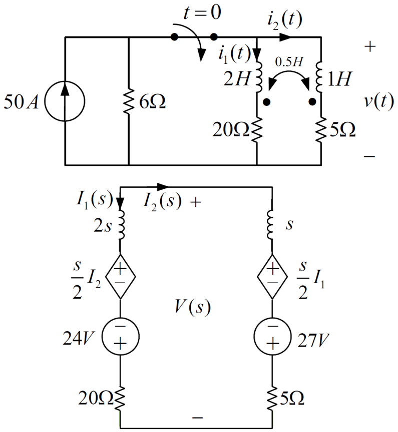
$$v(0^-) = 5i_2(0^-) = 120$$



Laplace Analysis

Example (Coupled circuit)

A coupled circuit can be analyzed using Laplace transform.



$$\begin{cases} 20I_2 + 24 - 0.5sI_2 + 2sI_2 + sI_2 + 0.5sI_1 - 27 + 5I_2 = 0 \\ I_1 = -I_2 \end{cases}$$

$$I_2(s) = \frac{1.5}{s + 12.5} \Rightarrow i_2(t) = -i_1(t) = 1.5e^{-12.5t}u(t)$$

$$V(s)2sI_1 + 0.5sI_2 - 24 + 20I_1 = (1.5s + 20)I_1 - 24$$

$$V(s) = \frac{-105}{4} - \frac{15}{8}\frac{1}{s + 12.5}$$

$$v(t) = \frac{-105}{4}\delta(t) - \frac{15}{8}e^{-12.5t}u(t)$$

The End