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A robust eigenbasis generation system for the discrete Fourier transform

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A R T I C L E I N F O

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A B S T R A C T

We have developed a systematic approach to construct an intelligible real eigenbasis for discrete Fourier transforms (DFT) by directly utilizing the eigenbases of some specific types of discrete sine and cosine transforms (DST and DCT). This methodological advancement not only enhances the comprehension of DFT spectra but also leads to a significant outcome: the identification of an explicit discrete analogue of Hermite-Gaussian functions within the context of DFT. By capitalizing on the inherent structure present in DST and DCT eigenbases, our approach facilitates a seamless transition to the domain of discrete Hermite-Gaussian functions, thereby opening up new avenues for related applications.

1. Introduction

The Fourier transform stands out as a pivotal element in both electrical engineering and mathematics. It occupies a central role in signal processing and constitutes a fundamental aspect of harmonic analysis. Substantial research in both domains has notably emphasized the spectral theory aspects, reflecting a significant level of attention [1–15].

The investigation of eigenvectors in the discrete Fourier transform (DFT) has been a subject of considerable interest, particularly within signal processing [4–6,8,16–20]. Understanding the eigenvectors of the DFT provides valuable insights into its properties, spectral analysis, and computational efficiency. In the current state-of-the-art, several key aspects have been explored. One prominent area of research involves probing the eigenbasis of the DFT matrix, potentially leading to the development of efficient algorithms for signal processing tasks such as filtering, compression, and spectral analysis. Additionally, a line of research has been focused on characterizing the structure and properties of the eigenvectors, including their orthogonality, symmetry, and relation to other mathematical constructs such as Hermite-Gaussian functions. Moreover, recent advancements have aimed to extend the understanding of eigenvectors beyond conventional DFT matrices to more specialized transforms and applications. This includes investigating eigenvectors in non-standard DFT variants, multidimensional transforms, and adapting eigenvector-based techniques for specific signal processing tasks.

In our recent work [21], a comprehensive approach to the spectrum characterization, which involves the derivation of eigenvalues and their corresponding multiplicities for non-normalized, symmetric discrete trigonometric transforms (DTT), is presented. Namely, eight types of DTT were analyzed, and new explicit analytic expressions for the eigenvalues and their multiplicities in three specific cases: DCT(1), DCT(5), and DST(8) were derived.

Both the discrete Fourier transform (DFT) and the continuous Fourier transform share the eigenvalues $\{\pm 1, \pm i\}$. As a result, in sufficiently large dimensions, there exists an infinite number of orthonormal bases containing eigenvectors or eigenfunctions. A notable example of such eigenfunctions is Hermite-Gaussian functions, which play a significant role in the continuous Fourier transform. These functions have widespread applications in quantum mechanics as solutions to the Schrödinger equation for the quantum harmonic oscillator and in signal processing for time-frequency analysis.

Direct sampling of Hermite-Gaussian functions does not provide an exact orthonormal basis for the Discrete Fourier Transform (DFT). Identifying an orthonormal basis that includes eigenvectors of the DFT and can approximately represent samples of Hermite-Gaussian functions is of significant interest. A positive answer to this question would identify specific types of orthonormal bases for the DFT, which could be referred to as discrete Hermite-Gaussian type basis.

Given the infinite number of orthonormal bases for the Discrete Fourier Transform, it is reasonable to anticipate that multiple discrete

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Table 1
Notation and symbols.

Notation	Definition
M_n	The set of all real square matrices of order n
$M_{m,n}$	The set of all real $m \times n$ matrices
$0_{m,n}$	The $m \times n$ matrix whose entries are all zeros
$\mathbf{I}_n$	The identity matrix of dimension n
$\mathbf{J}_n$	The anti-diagonal matrix of dimension n
$\mathcal{R}_n$	The set of all real square symmetric A of order n with $A^2 = \mathbf{I}_n$
$\lfloor x \rfloor$	The floor function of x
$A \otimes B$	Tensor product of two matrices A and B
$\mathrm{Eign}_A(\lambda)$	The set of eigenvectors of matrix A associated with the eigenvalue λ
$\mathrm{Tr}(A)$	The trace of matrix A
$\mathrm{rank}(A)$	The rank of matrix A
A^t	The transpose of matrix A
DFT	Discrete Fourier Transform
$\begin{pmatrix} A \\ B \end{pmatrix}$	The rectangular matrix with upper rows from A and lower rows from B
$\begin{pmatrix} A & B \end{pmatrix}$	The rectangular matrix with left columns from A and right columns from B

Hermite-Gaussian type bases exist, exhibiting a non-unique characteristic. These bases, while distinct in their construction, would still satisfy the requirements for orthonormal bases and offer different approaches to signal representation and analysis. By imposing specific properties, such as sparsity, we can identify all discrete Hermite-Gaussian type bases that meet these criteria and subsequently determine the minimal case.

In the context of Hermite-type bases, minimal refers to achieving an optimal set of properties necessary for a specific application or research goal. This concept emphasizes that minimal is relative to the desired characteristics and constraints of a particular problem. Consequently, the properties that define a minimal basis can vary widely, leading to different types of Hermite-type bases tailored to different needs.

In this study, the primary contributions comprise two key aspects:

1. *Development of a Generator System for Orthonormal Bases of DFT*: A novel generator system has been designed to yield an ensemble of comprehensible representations of orthonormal bases for the Discrete Fourier Transform. This innovative approach leverages the eigenbasis of two specific types of the discrete Sine transforms and discrete Cosine transforms to facilitate a deeper understanding of the DFT spectrum, thereby enriching the comprehension of Fourier analysis.
2. *Formulation of an Optimal Discrete Hermite-Gaussian Type Basis*: By employing the developed generator system, a concise and straightforward formula for an optimal discrete Hermite-Gaussian type basis has been derived. In this context, the notion of minimality pertains to the identification of an optimal case with respect to an appropriate uncertainty principle. The emergence of this minimal discrete Hermite-Gaussian case provides an efficient basis for Fourier analysis and demonstrates the practical utility of the proposed generator system.

1.1. Related works

The endeavor to identify an orthogonal set of eigenvectors for discrete Fourier transforms, with an emphasis on simplicity and clarity, has endured for numerous decades. An essential method for discovering orthogonal eigenvectors of the Discrete Fourier Transform involves examining specific symmetric matrices, often tridiagonal or nearly tridiagonal in structure, that share a commutative relationship with the DFT [22,23]. The eigenvectors of these matrices provide an approximate formula for the orthogonal eigenbasis of the DFT. Some of the most important approaches, particularly those using the Hermite basis that satisfies the most desirable properties, come from tridiagonal-matrix-based methods described in [6] and [7].

However, not much has been known about explicit Hermite-Gaussian-type eigenvectors of the DFT matrix, until recent work with explicit orthogonal eigenbasis of the DFT from [3]. The construction of a discrete Hermite function basis has been particularly relevant in certain signal processing applications, since it is used to form the Discrete Hermite Transform (DHT). The DHT has been explored as an alternative to the discrete cosine transform (DCT) and discrete wavelet transform (DWT) in compression techniques for digital images, audio signals, and biomedical signals [24]. Discrete Hermite-Gaussian functions have been closely linked with electrocardiogram (ECG) signals, especially the prominent QRS complexes [22,23,25,26]. They are also associated with other applications as well, including communication and radar signal processing, reconstruction of electromagnetic signals, physical optics and image and video processing [22,23,26]. We illustrate how the set of orthogonal explicit discrete analogs of Hermite-Gaussian functions can be directly generated by the proposed real eigenbasis generator. Some key papers on the generation of Hermite-Gaussian-like eigenvectors are [27–33].

The rest of paper is organized as follows. Section 2 introduces fundamental concepts related to eigenanalysis and the formation of eigenbases for discrete trigonometric transforms (DTT) and the Fourier transform. Additionally, this section outlines the underlying motivation for the study, including the investigation of the minimal Hermite-type basis of the centered discrete Fourier transform. The theoretical framework for constructing the eigenbasis of DTT is detailed in Section 3. In Section 4, Hermite-type eigenbases for the DFT matrix are elaborated upon. Section 5 provides illustrations of the Hermite-type eigenbasis. Finally, the paper wraps up with concluding remarks.

2. Basic definitions and motivation

The discrete Fourier matrix, denoted as $\mathbf{F}_n$ with an indication of its size being n, is defined as follows (Table 1):

$$\mathbf{F}_n = \frac{1}{\sqrt{n}} \left(w^{kl} \right)_{k,l=0}^{n-1}, \qquad w = \exp\left(\frac{-2\pi i}{n} \right). \tag{2.1}$$

In Theorems 2.1 and 2.2, it is shown that any eigenbasis of $\mathbf{F}$ can be obtained through straightforward modifications of the eigenbases of the discrete trigonometric transforms (DTT), as outlined in Table 2. When the size of $\mathbf{F}$ is even, it is illustrated that any eigenbasis of $\mathbf{C}_{(1)}$ leads to an eigenspace of $\mathbf{F}$ corresponding to the eigenvalues ± 1. Furthermore, it is

Table 2
The row/column scaling factors are given by $a_i = \frac{1}{2}$ for $i = 0$ and 1 otherwise; $b_i = \frac{1}{\sqrt{2}}$ for $i = 0, n-1$ and 1 otherwise.

DFT and DTTs	Math. Symbol	dim (size)	Formula (definition)
Discrete Fourier Matrix	$\mathbf{F} = \frac{1}{\sqrt{n}}\left(w^{kl}\right)_{k,l=0}^{n-1}$	n	$w = \exp\frac{-2\pi i}{n}$
Discrete Sine of Type I	$\mathbf{S}_{(1)} = \left(s_{kl}\right)_{k,l=0}^{n-1}$	n	$s_{kl} = \sqrt{\frac{2}{n+1}}\sin\frac{(k+1)(l+1)\pi}{n+1}$
Discrete Co-sine of Type I	$\mathbf{C}_{(1)} = \left(c_{kl}\right)_{k,l=0}^{n-1}$	n	$c_{kl} = \sqrt{\frac{2}{n-1}}b_k b_l\cos\frac{kl\pi}{n-1}$
Discrete Sine of Type V	$\mathbf{S}_{(5)} = \left(s_{kl}\right)_{k,l=0}^{n-1}$	n	$s_{kl} = \sqrt{\frac{2}{n+\frac{1}{2}}}\sin\frac{2(k+1)(l+1)\pi}{2n+1}$
Discrete Co-sine of Type V	$\mathbf{C}_{(5)} = \left(c_{kl}\right)_{k,l=0}^{n-1}$	n	$c_{kl} = \frac{2}{\sqrt{2n-1}}a_l\cos\frac{2kl\pi}{2n-1}$

shown that any eigenbasis of $\mathbf{S}_{(1)}$ transforms directly into an eigenspace of $\mathbf{F}$ associated with the eigenvalues $\pm i$. The process remains analogous for odd values of n, utilizing the matrices $\mathbf{C}_{(5)}$ and $\mathbf{S}_{(5)}$.

In Section 3, it is shown that the DTTs listed in Table 2 collectively belong to a specific class of orthogonal matrices. This shared characteristic facilitates a unified approach for the eigenbasis applicable to each member. The introduction of this larger class of matrices introduces a prototype for the DTTs, requiring a unified method for their eigenbases.

One notable benefit of this research is its clear exploration of the *minimal Hermite-type basis* that forms an orthonormal basis of the centered discrete Fourier transform, as outlined in [2,3]. It is a discrete analogue of the Hermite-Gaussian functions ψ_n, defined as

$$\psi_n(x) = \left(\frac{1}{2^n n!\sqrt{\pi}}\right)^{1/2}\exp(-x^2/2)\left[(-1)^n\exp(x^2)\frac{d^n}{dx^n}\exp(-x^2)\right]. \tag{2.2}$$

It is a well-established fact that Hermite-Gaussian functions constitute a complete orthonormal basis of $L^2(\mathbb{R})$, comprising eigenfunctions of the continuous Fourier transform $\mathfrak{F}$ [34,35]

$$\mathfrak{F}(g)(t) = \frac{1}{2\pi}\int_{\mathbb{R}}\exp(-ity)g(y)dy. \tag{2.3}$$

2.1. Motivation

It is a well-known that

$$\mathbf{F}_n^4 = \mathbf{I}_n \tag{2.4}$$

holds, where $\mathbf{I}_n$ represents the identity matrix of dimension n. This leads to the conclusion that $\mathbf{F}_n$ possesses precisely four eigenvalues, namely $\{\pm 1, \pm i\}$, when $n \geq 5$. Consequently, for sufficiently large matrix dimensions, an infinite array of bases, consisting of the eigenvectors exist.

This section, deeply motivated by McClellan's seminal work on eigenvalue problems [1], illustrates how the Discrete Trigonometric Transforms (DTTs) defined in Table 2 establish a generative system for producing the eigenbasis of the Discrete Fourier Transform (DFT). Notably, all four of these real symmetric DTTs are square roots of the identity matrix, a property that underpins their utility in this context.

It is worth mentioning that these DTTs, by virtue of their structure, facilitate the derivation of the eigenvectors of the DFT when employed in pairs. Through the application of a suitable converter, one can achieve a seamless transition to the eigenbasis of the DFT. This approach highlights the significant advantage of working with DTTs, as it allows us to simplify the problem. Instead of dealing with the DFT characterized by $\mathbf{F}^4 = \mathbf{I}$, we can reduce the discussion to specific real symmetric matrices satisfying $\mathbf{T}^2 = \mathbf{I}$.

This reduction not only simplifies the theoretical analysis but also enhances computational efficiency. By leveraging the intrinsic properties of these real symmetric DTTs, we can more effectively tackle problems associated with the DFT, ultimately leading to more robust solutions in various applications, including signal processing and numerical analysis.

Theorem 2.1. *Suppose $n = 2m + 2$ and consider the transforms $\mathbf{C}_{(1)}$ and $\mathbf{S}_{(1)}$ of size $m + 2$ and m, respectively.*

- *Γ maps $\mathrm{Eig}_{\mathbf{C}_{(1)}}(\pm 1)$ onto $\mathrm{Eig}_{\mathbf{F}_n}(\pm 1)$, where the converter Γ is the direct sum $A \oplus B$ and the matrices A and B are defined as follows:*

$$\begin{cases} A \text{ is the identity matrix of dimension } 1, \\ B = (X\ Y) \text{ where } X \text{ is the } (2m+1)\times m \text{ matrix } \frac{1}{\sqrt{2}}\begin{bmatrix}\mathbf{I}_m & 0 & \mathbf{J}_m\end{bmatrix}^t, \text{ and} \\ Y \text{ is the } (2m+1)\times 1 \text{ column matrix } \begin{bmatrix}0_{1\times m} & A & 0_{1\times m}\end{bmatrix}^t. \end{cases}$$

- *Λ maps $\mathrm{Eig}_{\mathbf{S}_{(1)}}(\pm 1)$ onto $\mathrm{Eig}_{\mathbf{F}_n}(\mp i)$, where the converter Λ is defined as follows:*

$$\Lambda = \begin{bmatrix}0_{m\times 1} & -\mathbf{I}_m & 0_{m\times 1} & \mathbf{J}_m\end{bmatrix}^t.$$

Proof. See appendix Section 7.2. □

Theorem 2.2. *When $n = 2m + 1$, consider the transforms $\mathbf{C}_{(5)}$ and $\mathbf{S}_{(5)}$ of sizes $m + 1$ and m respectively.*

- *Γ maps $\mathrm{Eig}_{\mathbf{C}_{(5)}}(\pm 1)$ onto $\mathrm{Eig}_{\mathbf{F}_n}(\pm 1)$, where the converter Γ is the direct sum $A \oplus B$ and the matrices A and B are defined as follows:*

$$\begin{cases} A \text{ is the identity matrix of dimension } 1, \\ B = \begin{bmatrix}\mathbf{I}_m & \mathbf{J}_m\end{bmatrix}^t. \end{cases}$$

- *Λ maps $\mathrm{Eig}_{\mathbf{S}_{(5)}}(\pm 1)$ onto $\mathrm{Eig}_{\mathbf{F}_n}(\mp i)$, where the converter Λ is defined as follows:*

$$\Lambda = \begin{bmatrix}0_{m\times 1} & -\mathbf{I}_m & \mathbf{J}_m\end{bmatrix}^t.$$

Proof. See appendix, Section 7.2. □

3. DTT-type matrices

We denote $\mathcal{R}_n$ as the set of all $n \times n$ real symmetric involutory matrices A, meaning that $A^2 = \mathbf{I}_n$. It is equivalent to write:

$$\begin{cases} A(A + \mathbf{I}_n) = (A + \mathbf{I}_n), \\ A(A - \mathbf{I}_n) = -(A - \mathbf{I}_n). \end{cases} \tag{3.1}$$

The first statement asserts that the non-zero columns of $A + \mathbf{I}_n$ are eigenvectors of A corresponding to $+1$, while the second statement confirms that the non-zero columns of $A - \mathbf{I}_n$ are eigenvectors corresponding to -1. Furthermore, the ranks of perturbed matrix A are related by

$$\mathrm{rank}(A+\mathbf{I}_n)+\mathrm{rank}(A-\mathbf{I}_n)=n. \tag{3.2}$$

As a result, the $2n$ columns derived from the matrices can jointly span $\mathbb{R}^n$, encompassing the eigenvectors of matrix A.

Definition 3.1. For a matrix A belonging to $M_n(\mathbb{R})$, we define A as having the Chebyshev property if its bottom-left submatrix, which is an $\lfloor \frac{n}{2} \rfloor \times \lfloor \frac{n}{2} \rfloor$ matrix, is nonsingular. In this context, $\lfloor x \rfloor$ denotes the floor function of x.

Remark 3.2. The definition of the Chebyshev property, as conventioned in this context, is deeply rooted in a well-known notion called the Chebyshev system [36,37]. This concept was introduced by S.N. Bernstein [38] in his work on extremal problems. The Chebyshev system is a significant tool in the approximate construction of algebraic polynomials. Some more details are given in the Section 7.1.

Example 3.3. As given in the appendix, Theorem 7.2, the symmetric discrete trigonometric transforms of size n, outlined in Table 2, are part of $\mathcal{R}_n$ and exhibit the satisfying Chebyshev property.

Theorem 3.4. *Let $n=2m+1$ and A be in $\mathcal{R}_n$ and assume that it satisfies the Chebyshev property. There are only two possibilities $-1,1$ for the trace of A. If $\mathrm{Tr}\,A=1$, then*

1st Approach *The first $(m+1)$ columns of $A+\mathbf{I}_n$ constitute a maximally linearly independent set associated with the eigenvalue $+1$. Furthermore, the first m columns of the matrix $A-\mathbf{I}_n$ comprise a maximally linearly independent set associated with the eigenvalue -1.*

2nd Approach *The last $(m+1)$ columns of the matrix $A+\mathbf{I}_n$ represent a maximally linearly independent set associated to the eigenvalue $+1$. Additionally, the last m columns of the matrix $A-\mathbf{I}_n$ form a maximally linearly independent set associated with the eigenvalue -1.*

If $\mathrm{Tr}\,A=-1$, then

1st Approach *The first $(m+1)$-columns of $A-\mathbf{I}_n$ form a maximal linearly independent set corresponded to the eigenvalue -1. Moreover, the first m-columns of $A+\mathbf{I}_n$ form a maximal linearly independent set corresponded to the eigenvalue $+1$.*

2ed Approach *The last $(m+1)$-columns of $A-\mathbf{I}_n$ form a maximal linearly independent set corresponded to the eigenvalue -1. Moreover, the last m-columns of $A+\mathbf{I}_n$ form a maximal linearly independent set corresponded to the eigenvalue $+1$.*

Proof. Our emphasis lies in the 1st approach, given that the second approach is comparably derived using the symmetry property.

The subsequent notations will aid in comprehending the progression of the proof:

- $A_{u;l,m}$ denotes the upper-left submatrix of A with size m.
- $A_{b;l,m}$ denotes the bottom-left submatrix of A with size m.
- $A_{b;r,m}$ denotes the bottom-right submatrix of A with size m.
- $\mathrm{Eigen}(A)$ denotes the set of eigenvalues of A.
- $\mathcal{E}_\lambda(A)$ denotes the eigenspace of A corresponding to the eigenvalue λ.

Furthermore, let H_+ (or H_-) denote the rectangular $n\times m$ matrix, with its columns consisting of the last m columns of $A+\mathbf{I}_n$ (or $A-\mathbf{I}_n$). By directly leveraging the orthogonality of matrix A, we obtain the following equalities:

$$H_+^t H_+ = 2\big(A_{b;r,m}+\mathbf{I}_m\big), \tag{3.3}$$

$$H_-^t H_- = 2\big(A_{b;r,m}-\mathbf{I}_m\big). \tag{3.4}$$

Firstly, we ascertain that both matrices $H_+^t H_+$ and $H_-^t H_-$ are indeed non-singular. The initial m rows of H_+ (or H_-) precisely match the submatrix $A_{b;l,m}$, a direct result of the symmetry inherent in matrix A. In conjunction with the Chebyshev property assumption, we can infer that the initial m rows of H_+ (or H_-) form a non-singular $m\times m$ matrix. This indicates that the m columns of the rectangular matrix H_+ (or H_-) constitute a linearly independent set of vectors in $\mathbb{R}^n$. Consequently, H_+ (or H_-), as a linear transformation, defines a one-to-one (injective) linear map from $\mathbb{R}^m$ into $\mathbb{R}^n$. For any vector $\mathbf{v}$ in $\mathbb{R}^m$, we have

$$\|H_+\mathbf{v}\|^2 = \langle H_+\mathbf{v}, H_+\mathbf{v}\rangle = \langle H_+^t H_+\mathbf{v}, \mathbf{v}\rangle.$$

Therefore, $H_+^t H_+\mathbf{v}=0$ implies $H_+\mathbf{v}=\mathbf{0}$. Since H_+ has a left inverse, we conclude that $\mathbf{v}=\mathbf{0}$. This implies that the square matrices $H_+^t H_+$ (and $H_-^t H_-$) are non-singular.

Combining the non-singularity of $H_+^t H_+$ (and $H_-^t H_-$) with formulas (3.3) and (3.4), we conclude that neither -1 nor 1 exist within the eigenvalue spectrum of $A_{b;r,(\frac{n-1}{2})}$.

Through the application of Lemma 3.8, we can articulate

$$\mathrm{Eign}\big(A_{u;l,(\frac{n+1}{2})}\big) = \{-\lambda : \lambda \in \mathrm{Eign}\big(A_{b;r,(\frac{n-1}{2})}\big)\} \bigcup \{x\}, \tag{3.5}$$

for some $x\in\mathbb{R}$. According to the Cauchy Interlacing theorem (Section 4 [39]), it is affirmed that x serves as an eigenvalue of matrix A. As a result, x can only take on the values of either -1 or $+1$. Moreover, the following equality holds:

$$\mathrm{Tr}\,A = \mathrm{Tr}\,A_{b;r,(\frac{n-1}{2})} + \mathrm{Tr}\,A_{u;l,(\frac{n+1}{2})} = x. \tag{3.6}$$

In the event that $\mathrm{Tr}\,A=1$, it follows that -1 is no longer an eigenvalue of $A_{u;l,(\frac{n+1}{2})}$, which is equivalent to stating that the initial $(\frac{n+1}{2})$ columns of $A+\mathbf{I}_n$ constitute a linearly independent set. Likewise, the intended result can be achieved in the case where $\mathrm{Tr}\,A=-1$. □

A brief review of the proof provided above instantly reveals the following result.

Theorem 3.5. *Let $n=2m$ and A be in $\mathcal{R}_n$ and assume that it satisfies the Chebyshev property. Then $\mathrm{Tr}\,A=0$. Furthermore,*

1st Approach *The first m columns of the matrix $A+\mathbf{I}_n$ (or $A-\mathbf{I}_n$) form a maximally linearly independent set associated with the eigenvalue $+1$ (or -1).*

2ed Approach *The last m columns of the matrix $A+\mathbf{I}_n$ (or $A-\mathbf{I}_n$) represent a maximally linearly independent set associated with the eigenvalue $+1$ (or -1).*

Corollary 3.6. *Let A be in $\mathcal{R}_n$ and assume that it satisfies the Chebyshev property.*

1. *Suppose that $n=2m$. Both eigenvalues $+1$ and -1 share the same multiplicity, and this multiplicity is equal to m.*
2. *Consider the case where $n=2m+1$. In this situation if $\mathrm{Tr}(A)=1$, the multiplicity of $+1$ is $m+1$ while the multiplicity of -1 is m.*

Consider matrix M in $M_n(\mathbb{R})$. We define a pair of principal submatrices B and C of matrix M as complementary if there exist rectangular matrices X and Y such that

$$M = \begin{pmatrix} B & X \\ Y & C \end{pmatrix}.$$

The following technical lemma provides the necessary underpinning for Theorem (3.4).

Remark 3.7. This Lemma addresses a technical point concerning eigenvalues for all matrices satisfying $M^2 = \mathbf{I}_n$, without requiring any additional assumptions.

Lemma 3.8. *Let M be in $M_n(\mathbb{R})$ with $M^2 = \mathbf{I}_n$ and suppose that B and C are a pair of complementary principal submatrices of M. Let λ be a scaler with $|\lambda| \neq 1$.*

i) *λ is an eigenvalue of matrix B if and only if $-\lambda$ is an eigenvalue of matrix C.*
ii) *The geometric multiplicity of the eigenvalue λ for matrix B is equal to the geometrical multiplicity of the eigenvalue $-\lambda$ for matrix C.*

Proof. Suppose that $B \in M_k(\mathbb{R})$ and put

$$M = \begin{pmatrix} B & X \\ Y & C \end{pmatrix}. \tag{3.7}$$

Through the utilization of the orthogonality property of matrix M, the following identities can be obtained

$$\begin{cases} XY = \mathbf{I}_k - B^2, \\ CY = -YB. \end{cases} \tag{3.8}$$

Suppose λ is an eigenvalue of B with $|\lambda| \neq 1$. The first equality confirms that the product of Y and any eigenvector of B corresponding to λ never vanishes. Utilizing this observation and applying the second identity, we deduce that $-\lambda$ is an eigenvalue of C. Furthermore, Y constitutes a one-to-one linear mapping from $\mathcal{E}_\lambda(B)$ to $\mathcal{E}_{-\lambda}(C)$. To extend this argument, relying on the equalities in (3.9), we can establish that X serves as a one-to-one linear mapping from $\mathcal{E}_{-\lambda}(C)$ to $\mathcal{E}_\lambda(B)$:

$$\begin{cases} YX = \mathbf{I}_k - C^2, \\ BX = -XC. \end{cases} \qquad \square \tag{3.9}$$

To reinforce the stated Theorem 2.2 and 2.1, we will put forth multiple specific cases concerning the eigenbases of discrete sine and cosine transforms in the forthcoming discussion.

Example 3.9. Theorems 3.4 and 3.5 present two different approaches for obtaining the eigenbases of the transforms $\mathbf{C}_{(1)}, \mathbf{S}_{(1)}, \mathbf{C}_{(5)}$, and $\mathbf{S}_{(5)}$.

Example 3.10. Let's employ $\mathbf{T}$ to denote each of the transforms $\mathbf{C}_{(1)}, \mathbf{S}_{(1)}, \mathbf{C}_{(5)}$, and $\mathbf{S}_{(5)}$. When $\mathbf{T}$ has a size of $m = 2k+1$, multiplicity of the eigenvalue $+1$ is just $k+1$. The Chebyshev property ensures that the first k columns of $(\mathbf{T}+\mathbf{I})$ form a linearly independent set.

The entries in the last column of $(\mathbf{T}+\mathbf{I})$ are all non-zero and alternate in sign. According to Theorem 4 in [1], by joining the last column of $(\mathbf{T}+\mathbf{I})$ to the union of the first k columns, a basis for $\mathrm{Eig}_{\mathbf{T}}(+1)$ is obtained.

Example 3.11. Suppose that $m = 2k$ is even. The union of even columns from the matrices $\mathbf{C}_{(1)} \pm \mathbf{I}_m$ (resp. $\mathbf{S}_{(1)} \pm \mathbf{I}_m$) constitutes a basis consisting of eigenvectors of the matrix $\mathbf{C}_{(1)}$ (resp. $\mathbf{S}_{(1)}$). To verify this point, it is directly verified that for both matrices $\mathbf{C}_{(1)}$ and $\mathbf{S}_{(1)}$, the entries (a_{kl}) exhibit column vector symmetry as follows.

$$\begin{cases} a_{n+1-j,l} = a_{j,l} & \text{for odd } l, \\ a_{n+1-j,l} = -a_{j,l} & \text{for even } l. \end{cases} \tag{3.10}$$

Through the application of this vector column symmetry condition, we deduce the following equalities:

$$\begin{cases} \mathbf{I}^t_{\text{even}}(\mathbf{C}_{(1)} \pm \mathbf{J}_m)(\mathbf{C}_{(1)} \pm \mathbf{I}_m)\mathbf{I}_{\text{even}} = \mathbf{I}_k, \\ \mathbf{I}^t_{\text{even}}(\mathbf{S}_{(1)} \pm \mathbf{J}_m)(\mathbf{S}_{(1)} \pm \mathbf{I}_m)\mathbf{I}_{\text{even}} = \mathbf{I}_k. \end{cases} \tag{3.11}$$

Table 3
The eigenvalue multiplicity of $\mathbf{F}_n$.

n	$\lambda=+1$	$\lambda=-1$	$\lambda=-i$	$\lambda=+i$
$4m$	$m+1$	m	m	$m-1$
$4m+1$	$m+1$	m	m	m
$4m+2$	$m+1$	$m+1$	m	m
$4m+3$	$m+1$	$m+1$	$m+1$	m

Here, $\mathbf{I}_{\text{even}}$ is simply the rectangular matrix whose columns consist of the even columns of $\mathbf{I}_m$. It is emphasized there that the columns of matrix $\mathbf{I}_m$ are numbered $1, 2, \ldots, m$.

Similarly, the union of odd columns from the matrices $\mathbf{C}_{(1)} \pm \mathbf{I}_m$ (resp. $\mathbf{S}_{(1)} \pm \mathbf{I}_m$) constitutes a basis consisting of eigenvectors of the matrix $\mathbf{C}_{(1)}$ (resp. $\mathbf{S}_{(1)}$).

$$\begin{cases} \mathbf{I}^t_{\text{odd}}(\mathbf{C}_{(1)} \mp \mathbf{J}_m)(\mathbf{C}_{(1)} \pm \mathbf{I}_m)\mathbf{I}_{\text{odd}} = \mathbf{I}_k, \\ \mathbf{I}^t_{\text{odd}}(\mathbf{S}_{(1)} \mp \mathbf{J}_m)(\mathbf{S}_{(1)} \pm \mathbf{I}_m)\mathbf{I}_{\text{odd}} = \mathbf{I}_k. \end{cases} \tag{3.12}$$

Here, $\mathbf{I}_{\text{odd}}$ is the rectangular matrix whose columns consist of the odd columns of $\mathbf{I}_m$.

Remark 3.12. Using the equation $\mathbf{F}_n^2 = 1 \oplus \mathbf{J}_{n-1}$, the transformations $\mathbf{P}_{+1}, \mathbf{P}_{-1}, \mathbf{P}_{+i}$, and $\mathbf{P}_{-i}$, defined as:

$$\mathbf{P}_{+1} = \frac{1}{4}\Big(\mathbf{I}_n + (1 \oplus \mathbf{J}_{n-1})\Big) + \frac{1}{4}\Big(\mathbf{I}_n + (1 \oplus \mathbf{J}_{n-1})\Big)\mathbf{F}_n.$$

$$\mathbf{P}_{-1} = \frac{1}{4}\Big(\mathbf{I}_n + (1 \oplus \mathbf{J}_{n-1})\Big) - \frac{1}{4}\Big(\mathbf{I}_n + (1 \oplus \mathbf{J}_{n-1})\Big)\mathbf{F}_n.$$

$$\mathbf{P}_{+i} = \frac{1}{4}\Big(\mathbf{I}_n - (1 \oplus \mathbf{J}_{n-1})\Big) - i\frac{1}{4}\Big(\mathbf{I}_n - (1 \oplus \mathbf{J}_{n-1})\Big)\mathbf{F}_n.$$

$$\mathbf{P}_{-i} = \frac{1}{4}\Big(\mathbf{I}_n - (1 \oplus \mathbf{J}_{n-1})\Big) + i\frac{1}{4}\Big(\mathbf{I}_n - (1 \oplus \mathbf{J}_{n-1})\Big)\mathbf{F}_n,$$

serve as projections onto eigenspaces of $\mathbf{F}$ corresponding to the eigenvalues $+1, -1, +i$, and $-i$ [9].

Assume that $n = 2m+2$. Let $\mathbf{C}_{(1)}$ and $\mathbf{S}_{(1)}$ have sizes $m+2$ and m, respectively. Through a direct computation, it can be verified that the conversion of the l-th column of $\mathbf{C}_{(1)} \pm \mathbf{I}_{m+2}$ (or $\mathbf{S}_{(1)} \pm \mathbf{I}_m$) using Γ (or Λ) coincides with the l-th column of $\mathbf{P}_{\pm 1}$ (or $\mathbf{P}_{\pm i}$).

In a similar vein, when $n = 2m+1$ and the sizes of $\mathbf{C}_{(5)}$ and $\mathbf{S}_{(5)}$ are $m+1$ and m, the conversion of the l-th column of $\mathbf{C}_{(5)} \pm \mathbf{I}_{m+1}$ (or $\mathbf{S}_{(5)} \pm \mathbf{I}_{m+1}$) using Γ (or Λ) corresponds to the l-th column of $\mathbf{P}_{\pm 1}$ (or $\mathbf{P}_{\pm i}$).

The combination of Theorems 2.2, 2.1, 3.4 and 3.5 leads directly to the following conclusion.

Corollary 3.13. *Let m_λ represent the multiplicity of the eigenvalue λ of $\mathbf{F}$. The formula for m_λ is provided in Table 3. For any eigenvalue λ of $\mathbf{F}$, let's consider the submatrix $\mathbf{P}'_\lambda$ of $\mathbf{P}_\lambda$ introduced as follows.*

1. *In the case where $\lambda = \pm 1$, $\mathbf{P}'_\lambda$ is obtained by selecting the first m_λ columns from the projection $\mathbf{P}_\lambda$.*
2. *If λ takes values of either $\pm i$, the submatrix $\mathbf{P}'_\lambda$ is composed of the columns within the projection matrix $\mathbf{P}_\lambda$ ranging from 2 to $m_\lambda + 1$.*

The column vectors within $\mathbf{P}'_\lambda$ collectively serve as an eigenbasis of $\mathbf{F}$ associated with the eigenvalue λ.

4. Exploring Hermite-type eigenbases in DFT

Here, our goal is to introduce an alternative method for deriving such discrete analogues, building upon previous discussions in [2,3]. Before delving into this, it is necessary to clarify the term *discrete analogue*.

Setting the benchmark, the following characterization Theorem 4.1 precisely describes that the Hermite-Gaussian functions stand as the unique orthonormal eigenfunctions of the continuous Fourier transform, satisfying the uncertainty principle.

Theorem 4.1 *(Uncertainty principle characterization of Hermite-Gaussian functions). Assume that the functions $f_n : \mathbb{R} \to \mathbb{R}$ satisfy the following conditions.*

- **Orthonormal eigenbasis:** *Set of functions $\{f_n\}$ forms an orthonormal basis of $L^2(\mathbb{R})$ consisting of eigenfunctions of the continuous Fourier transform with $\mathfrak{F}(f_n) = (-i)^n f_n$.*
- **Hardy's uncertainty principle:** *For every $n > 0$, the following holds:*

$$\lim_{|x|\to\infty} x^{-(n+1)} e^{\frac{x^2}{2}} f_n(x) = 0. \tag{4.1}$$

Moreover, for each positive integer n, one of the following holds true:

- *$f_n(x)$ equals $\psi_n(x)$, or*
- *$f_n(x)$ equals $-\psi_n(x)$.*

Proof. See Theorem 1 in [40]. □

This observation may pave the way for an apt counterpart in the discrete analogue of Hermite-Gaussian functions, especially when uncovering a discrete rendition of Hardy's uncertainty principle. This particular aspect is skillfully addressed in [2], and for the sake of completeness, we revisit it in the following discussion.

Definition 4.2. The width($\mathbf{v}$) of a real discrete signal $\mathbf{v}$ with a periodicity of n is defined as the minimum value of k that satisfies the following condition:

$$\begin{cases} \mathbf{v}[-k] \neq 0 \text{ or } \mathbf{v}[k] \neq 0, \\ \mathbf{v}[j] = 0, \text{ for all } |j| > k. \end{cases} \tag{4.2}$$

The following characterization theorem, previously substantiated in [2], introduces a discrete alternative to Theorem 4.1.

Theorem 4.3. *Let us consider the centered discrete Fourier matrix $\mathcal{F}$ of size n with entries $\mathcal{F}(k,l)$ defined by*

$$\mathcal{F}(k,l) = w^{kl}, \qquad w = \exp\frac{-2\pi i}{n} \tag{4.3}$$

where k and l range from $\left\lfloor -\frac{n}{2} \right\rfloor + 1$ to $\left\lfloor \frac{n}{2} \right\rfloor$. There exists a unique sequence of vectors $\{\mathbf{V}_k\}_{k=0}^{n-1}$ that adheres to the subsequent set of conditions:

- **Orthonormal eigenbasis:** *The sequence $\{\mathbf{V}_k\}$ constitutes a orthonormal basis of $\mathbb{R}^n$ that is composed of eigenvectors of $\mathcal{F}$. Specifically, for each k in $0,\dots,n-1$, the eigenvector $\mathbf{V}_k$ corresponds to the eigenvalue $(-i)^k$ of $\mathcal{F}$.*
- **Uncertainty principle:** width($\mathbf{V}_k$) $\le \left\lfloor \frac{n+k+2}{4} \right\rfloor$ *where $k = 0,\dots,n-1$.*

The orthonormal basis outlined in Theorem 4.3 is denoted as the minimal Hermite-type basis of $\mathbb{R}^n$. With the approach outlined in this study, a new proof will be presented through an innovative method. As a second approach, we illustrate that the minimal Hermite-type basis can be smoothly acquired by leveraging the eigenspaces obtained from the columns of the submatrices $\mathbf{P}'_\lambda$ outlined in Corollary 3.13. The proof is structured in two parts. We recall that λ is applied to any eigenvalues, specifically those in the set $\{\pm 1, \pm i\}$ of $\mathbf{F}$, each with the multiplicity of m_λ.

Part 1. Let us start by considering the forward shift of size n, denoted as $\mathbf{S}$:

$$\mathbf{S} = \begin{pmatrix} 0 & 1 \\ \mathbf{I}_{n-1} & 0 \end{pmatrix}. \tag{4.4}$$

Next, define $q = \left\lfloor \frac{n}{2} \right\rfloor$. With this, we can express the Fourier matrix $\mathbf{F}_n$ in terms of the forward shift $\mathbf{S}$:

$$\mathbf{F}_n = \mathbf{S}^{-q} \mathcal{F}_n \mathbf{S}^q \tag{4.5}$$

Furthermore, the operator $\mathbf{S}^q$ induces a linear isomorphism from the eigenspace $\mathrm{Eig}_{\mathbf{F}_n}(\lambda)$ to $\mathrm{Eig}_{\mathcal{F}_n}(\lambda)$.

Part 2. By employing the Gram-Schmidt algorithm on a minor modification of the columns within the submatrices $\mathbf{P}'_\lambda$, a unique orthonormal basis, consisting of eigenvectors of $\mathbf{F}$ is established. The application of the forward shift operator to transform these eigenvectors precisely generates the minimal Hermite-type basis. To address minor modification, we need to use a submatrix of $\mathbf{P}'_\lambda$ defined as follows. Consider the matrix B_λ, termed as a sparser matrix, as the submatrix with dimensions $m_\lambda \times m_\lambda$ of $\mathbf{P}'_\lambda$, as outlined below:

Suppose the size is $n = 2m+1$. The rows of matrix $\mathbf{P}'_\lambda$ range from $m+2-m_\lambda$ to $m+1$ form the matrix B_λ.

Suppose the size is $n = 2m+2$. For $\lambda = \pm 1$, the rows of $\mathbf{P}'_\lambda$ range from $m+3-m_\lambda$ to $m+2$ introduce B_λ. In the case of $\lambda = \pm i$, the rows of span from $m+2-m_\lambda$ to $m+1$ of $\mathbf{P}'_\lambda$.

As validated in the appendix, the symmetry property of $\mathbf{F}$ causes the matrix B_λ to exhibit the Chebyshev property, signifying its non-singularity. To verify the Chebyshev property of B_1 and B_{-i}, additional details are required, which are provided in Lemma (7.4).

Remark 4.4. Orthogonalizing the basis offers the significant advantage of expressing any vector as a linear combination of the basis vectors. This property simplifies the representation and analysis of vectors within the space spanned by the orthogonal basis. Applying the Gram-Schmidt algorithm not only achieves this but also yields a well-shaped, sparse form of minimal Hermit-type, as illustrated in Fig. 1.

Proof of Theorem 4.3. The nonsingularity of the sparser matrix B_λ ensures the existence of vectors $\mathbf{x}_1^{(\lambda)},\dots,\mathbf{x}_{m_\lambda}^{(\lambda)}$ in $\mathbb{R}^{m_\lambda}$ with

$$\mathbf{x}_k^{(\lambda)} = B_\lambda^{-1}\mathbf{e}_k^{(\lambda)}, \qquad k = 1,\dots,m_\lambda, \tag{4.6}$$

where $\mathbf{e}_k^{(\lambda)}$ is the standard vector in $\mathbb{R}^{m_\lambda}$ such that all its components are zero, with the exception of a single component in the k-th position, which has a value of 1. We define,

$$\mathbf{w}_k^{(\lambda)} = \mathbf{P}'_\lambda \mathbf{x}_k^{(\lambda)}, \qquad k = 1,\dots,m_\lambda. \tag{4.7}$$

The set of vectors $\{\mathbf{w}_k^{(\lambda)}\}$ form a basis of $\mathrm{Eig}_{\mathbf{F}_n}(\lambda)$. When n is odd, vectors $\mathbf{w}_k^{(\lambda)}$ take the following form:

$$\mathbf{w}_k^{(\lambda)} = \begin{bmatrix} \clubsuit \\ \mathbf{e}_k^{(\lambda)} \\ \alpha\,\mathbf{J}\left(\mathbf{e}_k^{(\lambda)}\right) \\ \clubsuit \end{bmatrix}, \quad \text{where} \quad \alpha = \begin{cases} 1 & \lambda = \pm 1, \\ -1 & \lambda = \pm i. \end{cases} \tag{4.8}$$

Here, the symbol ♣ denotes the part filled by some scalars.

We proceed with the process when n is odd. In the case of even n, the same outcome is obtained, with only a slight modification in the formula of $\mathbf{w}_k^{(\lambda)}$.

By the Gram-Schmidt process, we can transform the set $\{\mathbf{w}_k^{(\lambda)}\}$ into $\{\mathbf{z}_k^{(\lambda)}\}$, constructing an orthogonal basis for $\mathrm{Eig}_{\mathbf{F}_n}(\lambda)$ in a way that collectively forms the columns of the following matrix $\mathbf{Z}^{(\lambda)}$:

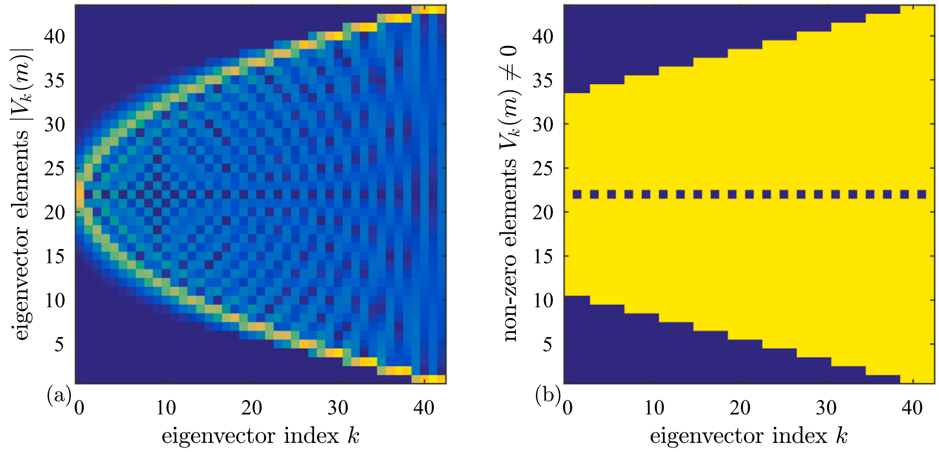

Fig. 1. Orthonormal eigenvectors $\mathbf{V}_k$ for $n = 43$: (a) absolute value, (b) non-zero elements.

$$\mathbf{Z}^{(\lambda)} = [\mathbf{z}_1^{(\lambda)}|\mathbf{z}_2^{(\lambda)}|\cdots|\mathbf{z}_{m_\lambda}^{(\lambda)}] = \begin{bmatrix} \clubsuit \\ \mathbf{U} \\ \alpha\, \mathbf{J}_{m_\lambda}\, \mathbf{U} \\ \clubsuit \end{bmatrix}, \tag{4.9}$$

where $\mathbf{U}$ is $m_\lambda \times m_\lambda$ upper-triangular matrix whose diagonal entries are all non-zero. The non-singularity of matrix $\mathbf{U}$ ensures the uniqueness of $\{\mathbf{z}_k^{(\lambda)}\}$ as the orthogonal basis for $\mathrm{Eig}_{\mathbf{F}_n}(\lambda)$ that satisfies the specified pattern in reference to (4.9).

The set of vectors $\{\mathbf{S}^q\, \mathbf{z}_k^{(\lambda)}\}_{k=1}^{m_\lambda}$ serves as an orthonormal basis for $\mathrm{Eig}_{\mathcal{F}}(\lambda)$. Moreover, each vector in this basis has a width satisfying

$$\mathrm{width}\left(\mathbf{S}^q\, \mathbf{z}_k^{(\lambda)}\right) \leq \left\lfloor \frac{n}{2} \right\rfloor - m_\lambda + k. \tag{4.10}$$

Now, it is time to introduce the desired orthonormal basis $\mathbf{V}_k$. Initially, we organize all vectors $\{\mathbf{S}^q\ \mathbf{z}_k^{(\lambda)}\}$ in four vertical partitions. The first partition contains vectors associated with $\lambda = 1$, the second with $\lambda = -1$, the third with $\lambda = -i$, and the fourth with $\lambda = i$. Moving in each row upwards to cover all $\mathbf{S}^q\, \mathbf{z}_k^{(\lambda)}$, the vectors $\mathbf{V}_k$ are defined as follows:

$$\begin{array}{cccc} \lambda = 1 & \lambda = -1 & \lambda = -i & \lambda = i \\ \mathbf{V}_0 = \mathbf{z}_1^{(1)} & \mathbf{V}_1 = \mathbf{z}_1^{(-1)} & \mathbf{V}_2 = \mathbf{z}_1^{(-i)} & \mathbf{V}_3 = \mathbf{z}_1^{(i)} \\ \mathbf{V}_4 = \mathbf{z}_2^{(1)} & \mathbf{V}_5 = \mathbf{z}_2^{(-1)} & \mathbf{V}_6 = \mathbf{z}_2^{(-i)} & \mathbf{V}_7 = \mathbf{z}_2^{(i)} \\ \vdots & \vdots & \vdots & \vdots \end{array} \tag{4.11}$$

5. Example

Fig. 1 shows the distribution of the minimal Hermite basis $\{\mathbf{V}_k\}$ for the size $n = 43$. Part (a) of Fig. 1 illustrates their arrangement based on width, with shorter vectors appearing before longer ones. Moreover, part (b) of Fig. 1 validates that each vector $\mathbf{V}_k$ contains some other zeros within its middle components, underscoring the sparsity of the minimal Hermite-type basis.

While no sample of the Hermite eigen-functions $\{\psi_k\}$ precisely forms a basis of eigen vectors for DFT, the minimal Hermite-type basis approximately addresses this issue. To elaborate further, we define

$$\mathbf{\Psi}_j(k) = (\sqrt{\frac{2\pi}{n}})^{\frac{1}{k}}\, \psi_j(\sqrt{\frac{2\pi}{n}}k). \tag{5.1}$$

It has been proven in [2] that

$$\| \mathbf{V}_k - \mathbf{\Psi}_k \| = O(n^{-1+\epsilon}), \qquad \text{as} \quad n \to \infty. \tag{5.2}$$

Fig. 2 presents a comparison between eight types of minimal Hermite-type vectors $\{\mathbf{V}_k\}$ and their corresponding Hermite eigenfunctions $\{\psi_k\}$. An observable pattern suggests that vectors $\mathbf{V_k}$ with shorter widths offer closer approximations to their corresponding Hermite-Gaussian functions. Nevertheless, this observation lacks analytical proof.

The uniqueness of the minimal Hermite-type basis highlights an important aspect. This basis stands out as the most effective sparse vectors for approximating the Hermite eigenfunctions. To accurately determine the approximation rate of the Hermite-type basis compared to that mentioned in (5.2), it is necessary to replace the uncertainty principle cited in Theorem 4.3 with another property.

6. Conclusion

This paper explores fundamental concepts related to eigenanalysis and the formation of eigenbases, with a specific emphasis on discrete trigonometric transforms (DTT) and the Fourier transform. The theoretical framework for constructing the eigenbasis of DTT has been elucidated. A notable and practically relevant outcome of this theoretical exposition is the identification of the minimal Hermite-type basis of the centered discrete Fourier transform. Unlike almost all existing methodologies, our approach facilitates the explicit form of discrete orthogonal Hermite-Gaussian functions, thereby enhancing the practical applicability and theoretical understanding of these transforms.

7. Appendix

7.1. Chebyshev property

In this section, points are presented to reinforce the Chebyshev property of DTTs in this study.

Definition 7.1. A sequence of functions $\{\phi_1, \ldots, \phi_n\}$ is called a Chebyshev system on an interval J, if any linear combination $\phi = \sum_{l=1}^n a_l \phi_l$ has at most $n-1$ distinct zeros in J. It is equivalent to say that for every strictly increasing sequence $t_1, \ldots, t_n$ in J, the $n \times n$ matrix $\Phi = \left(\phi_l(t_k)\right)$, i.e.

$$\Phi = \begin{pmatrix} \phi_1(t_1) & \cdots & \phi_n(t_1) \\ \vdots & \ddots & \vdots \\ \phi_1(t_n) & \cdots & \phi_n(t_n) \end{pmatrix} \tag{7.1}$$

is invertible.

In order to explore more details about some properties of Chebyshev systems, we refer to [36].

Theorem 7.2. *the sequence $\{\cos lx\}_{l=0}^n$ (and $\{\sin lx\}_{l=1}^n$) over the interval $[0, \pi]$ (over $(0, \pi)$) forms a Chebyshev system.*

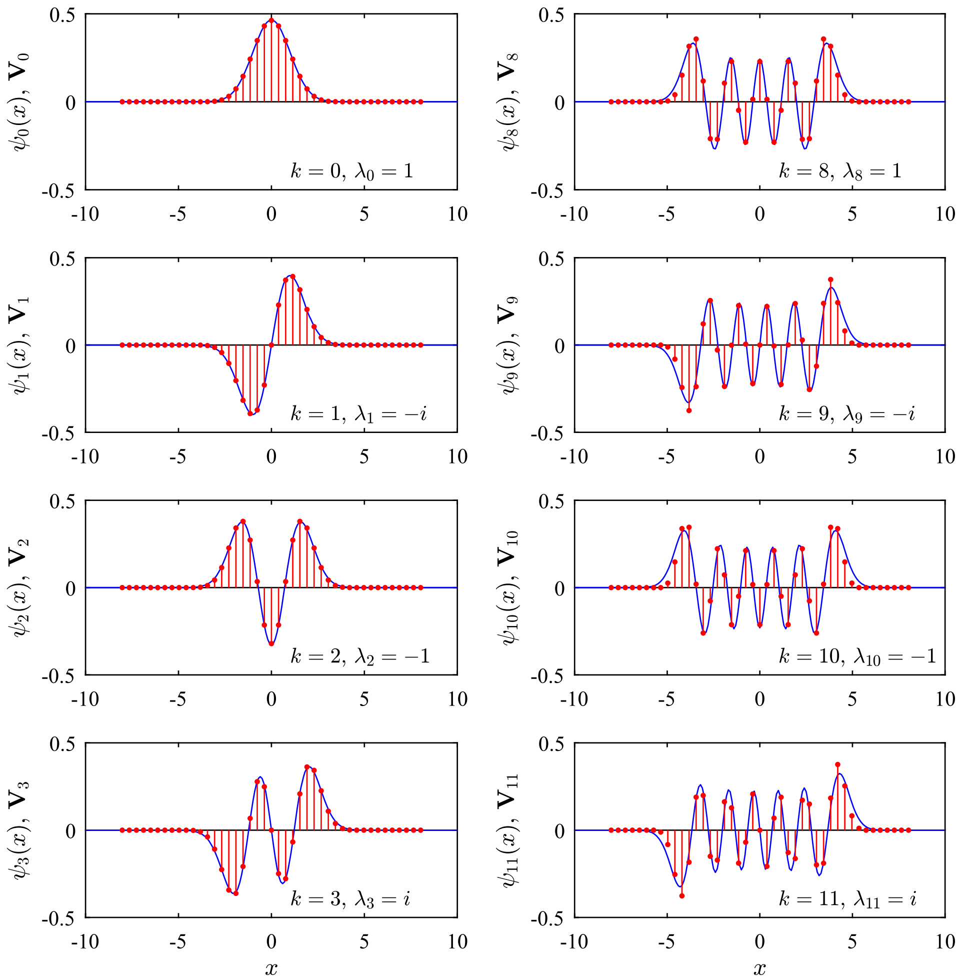

Fig. 2. Eigenvectors $\mathbf{V}_k$ for $k = 0, 1, 2, 3, 8, 9, 10, 11$ and $n = 43$ along with corresponding continuous Hermite-Gaussian functions $\psi_k(x)$.

Proof. It is proved in [1] □

Applying Theorem 7.2 enables us to derive the following Chebyshev systems.

1. Both sequences $\{\cos(2l-1)x\}_{l=1}^{n}$ and $\{\cos(2l)x\}_{l=1}^{n}$ are Chebyshev systems on the closed interval $[0, \frac{\pi}{2}]$.
2. The sequences $\{\sin(2l-1)x\}_{l=1}^{n}$ and $\{\sin(2l)x\}_{l=1}^{n}$ are Chebyshev systems on the open interval $(0, \frac{\pi}{2})$.

These two facts jointly imply the Chebyshev property of the discrete sine and cosine matrices in Table (2).

Remark 7.3. The presented trigonometric equation (7.2) assure us that all these discrete trigonometric transformations are involutory, meaning that satisfy in the equation $A^2 = \mathbf{I}$. For details about the following equation, we refer to [41].

$$\sum_{k=0}^{n-1} \cos(kt) = \frac{\cos\left(\frac{n-1}{2}t\right)\sin\left(\frac{n}{2}t\right)}{\sin\left(\frac{t}{2}\right)}. \tag{7.2}$$

7.2. *Proofs of Theorems 2.1 and 2.2*

Recalling $w = \exp -\frac{2\pi i}{m}$, we proceed to define,

$$\mathbf{G}_m = \left(w^{kl}\right)_{k,l=1}^{m} \quad \text{and} \quad \mathbf{G}_m^H = \left(w^{-kl}\right)_{k,l=1}^{m} \tag{7.3}$$

Then we have:

$$n = 2m+1: \qquad \mathbf{F}_n = \frac{1}{\sqrt{n}}\begin{bmatrix} 1 & 1_{1\times m} & 1_{1\times m} \\ 1_{m\times 1} & \mathbf{G}_m & \mathbf{G}_m^H\mathbf{J}_m \\ 1_{m\times 1} & \mathbf{J}_m\mathbf{G}_m^H & \mathbf{J}_m\mathbf{G}_m\mathbf{J}_m \end{bmatrix}, \tag{7.4}$$

$$n = 2m+2: \qquad \mathbf{F}_n = \frac{1}{\sqrt{n}}\begin{bmatrix} 1 & 1_{1\times m} & * & 1_{1\times m} \\ 1_{m\times 1} & \mathbf{G}_m & * & \mathbf{G}_m^H\mathbf{J}_m \\ * & * & * & * \\ 1_{m\times 1} & \mathbf{J}_m\mathbf{G}_m^H & * & \mathbf{J}_m\mathbf{G}_m\mathbf{J}_m \end{bmatrix}. \tag{7.5}$$

The column and row where the entries are denoted by $*$ are with following array:

$1, -1, 1, -1, \dots, (-1)^{n-1}$.

It directly leads to the following formulas.

$$\begin{cases} \left(\mathbf{I}_n + (1 \oplus \mathbf{J}_{n-1})\right)\mathbf{F}_n = \frac{2}{\sqrt{n}}\left(\cos\frac{2kl\pi}{n}\right)_{k,l=0}^{n-1} \\ \left(\mathbf{I}_n - (1 \oplus \mathbf{J}_{n-1})\right)\mathbf{F}_n = -i\frac{2}{\sqrt{n}}\left(\sin\frac{2kl\pi}{n}\right)_{k,l=0}^{n-1} \end{cases} \tag{7.6}$$

Let

$$\begin{cases} n = 2m+2 \\ \mathbf{I}'_{m+2} = 1 \oplus \frac{1}{\sqrt{2}} \mathbf{I}_m \oplus 1. \\ \mathbf{J}'_m = \frac{1}{\sqrt{2}} \mathbf{J}_m . \\ \text{dimensions of } \mathbf{C}_{(1)} \text{ and } \mathbf{S}_{(1)} \text{ are } m+2 \text{ and } m \text{ respectively.} \end{cases} \tag{7.7}$$

By utilizing the symmetry existing in each row of $\mathbf{F}_n$, as demonstrated in the equation (7.5) above, one can obtain the following matrix multiplications

$$\mathbf{F}_n \cdot \Gamma = \begin{pmatrix} \mathbf{I}'_{m+2}\,\mathbf{C}_{(1)} \\ \mathbf{J}'_m\,\mathbf{C}_{(1)} \end{pmatrix} \quad \text{and} \quad \mathbf{F}_n \cdot \Lambda = i \begin{pmatrix} 0 \\ \mathbf{S}_{(1)} \\ 0 \\ -\mathbf{J}_m\,\mathbf{S}_{(1)} \end{pmatrix}. \tag{7.8}$$

This further yields:

$$\begin{cases} \mathbf{F}_n\left(\Gamma(\mathbf{w})\right) = \pm\Gamma(\mathbf{w}) \Leftrightarrow \mathbf{C}_{(1)}(\mathbf{w}) = \pm\,\mathbf{w}, \\ \mathbf{F}_n\left(\Lambda(\mathbf{w})\right) = \mp i\Lambda(\mathbf{w}) \Leftrightarrow \mathbf{S}_{(1)}(\mathbf{w}) = \pm\,\mathbf{w}\,. \end{cases} \tag{7.9}$$

When $n = 2m+1$ the process moves forward similar to the even case. Using the symmetry property described in the equation (7.4), we have

$$\mathbf{F}_n \cdot \Gamma = \begin{pmatrix} \mathbf{C}_{(5)} \\ \mathbf{J}''_m\,\mathbf{C}_{(5)} \end{pmatrix} \quad \text{and} \quad \mathbf{F}_n \cdot \Lambda = i \begin{pmatrix} 0_{1\times m} \\ \mathbf{S}_{(5)} \\ -\mathbf{J}_m\,\mathbf{S}_{(5)} \end{pmatrix}, \tag{7.10}$$

where

$$\begin{cases} \mathbf{J}'' = \left(0 \mid \mathbf{J}_m\right) \text{ is an } m \times (m+1) \text{ matrix}\,. \\ \mathbf{C}_{(5)} \text{ and } \mathbf{S}_{(5)} \text{ are of dimensions } m+1 \text{ and } m \text{ resp.} \end{cases} \tag{7.11}$$

This further results in:

$$\begin{cases} \mathbf{F}_n\left(\Gamma(\mathbf{w})\right) = \pm\Gamma(\mathbf{w}) \Leftrightarrow \mathbf{C}_{(5)}(\mathbf{w}) = \pm\,\mathbf{w}, \\ \mathbf{F}_n\left(\Lambda(\mathbf{w})\right) = \mp i\Lambda(\mathbf{w}) \Leftrightarrow \mathbf{S}_{(5)}(\mathbf{w}) = \pm\,\mathbf{w}\,. \end{cases} \tag{7.12}$$

7.3. Non-singularity of sparser matrices B_λs

Let $\mathbf{E} = (a_{kl})$ represent the $n \times n$ matrix, where all elements are zero except for $a_{1,n} = 1/4$. By taking into account equations (7.6), we obtain the formula for the sparser matrices B_λ, which is discussed in the proof of Theorem 4.3, as follows:

$$B_{-1} = \begin{cases} \frac{-1}{2\sqrt{n}} \left(\cos \frac{2(k+m_{-1}+1)l\pi}{n}\right)_{k,l=0}^{m_{-1}-1} & n \equiv_4 0 \text{ or } n \equiv_4 1 \\ \frac{-1}{2\sqrt{n}} \left(\cos \frac{2(k+m_{-1})l\pi}{n}\right)_{k,l=0}^{m_{-1}-1} & n \equiv_4 2 \text{ or } n \equiv_4 3 \end{cases}$$

$$B_{1} = \begin{cases} \frac{1}{2\sqrt{n}} \left(\cos \frac{2(k+m_{1}-1)l\pi}{n}\right)_{k,l=0}^{m_{1}-1} + \mathbf{E} & n \equiv_4 0 \text{ or } n \equiv_4 1 \\ \frac{1}{2\sqrt{n}} \left(\cos \frac{2(k+m_{1})l\pi}{n}\right)_{k,l=0}^{m_{1}-1} & n \equiv_4 2 \text{ or } n \equiv_4 3 \end{cases}$$

$$B_{i} = \begin{cases} \frac{-1}{2\sqrt{n}} \left(\sin \frac{2(k+m_{i}+2)(l+1)\pi}{n}\right)_{k,l=0}^{m_{i}-1} & n \equiv_4 0 \text{ or } n \equiv_4 3 \\ \frac{-1}{2\sqrt{n}} \left(\sin \frac{2(k+m_{i}+1)(l+1)\pi}{n}\right)_{k,l=0}^{m_{i}-1} & n \equiv_4 1 \text{ or } n \equiv_4 2 \end{cases}$$

$$B_{-i} = \begin{cases} \frac{1}{2\sqrt{n}} \left(\sin \frac{2(k+m_{-i})(l+1)\pi}{n}\right)_{k,l=0}^{m_{-i}-1} + \mathbf{E} & n \equiv_4 0 \text{ or } n \equiv_4 3 \\ \frac{1}{2\sqrt{n}} \left(\sin \frac{2(k+m_{-i}+1)(l+1)\pi}{n}\right)_{k,l=0}^{m_{-i}-1} & n \equiv_4 1 \text{ or } n \equiv_4 2 \end{cases}$$

Regarding these sparser matrices B_λ, apart from those perturbed by the matrix $\mathbf{E}$, the remaining matrices simply exhibit the Chebyshev property. To tackle the Chebyshev property of these elements, we utilize the following lemma, which is an interesting point in and of itself.

Lemma 7.4. *Suppose that $A = (a_{kl})$ is an $n \times n$ non-singular matrix. For a given scalar x, let us define $A_x = (b_{kl})$ with,*

$$b_{kl} = \begin{cases} x, & k = 1, l = n, \\ a_{kl}, & \text{otherwise.} \end{cases} \tag{7.13}$$

There exists at most one scalar value x, that causes A_x to become singular.

Proof. Scalars s and t exist such that $\det(A) = sa_{1n} + t$. The fact that matrix A is nonsingular implies that at least one of the scalars s or t must be non-zero. When $s = 0$, then $\det A_x = t$ meaning that scalars x make A_x non-singular. While $s \neq 0$, the matrix is singular if and only if $x = -\frac{t}{s}$. □

Now, let's shift our focus to matrix B_1 when $n \equiv_4 0$, as the remaining matrices are treated in a similar manner:

$$B_1 = \frac{1}{2\sqrt{n}} \left(\cos \frac{2(k+m_\lambda-1)l\pi}{n}\right)_{k,l=0}^{m_1-1} + \mathbf{E}. \tag{7.14}$$

Furthermore, if we define

$$\tilde{B}_1 = B_1 - 2\mathbf{E} = \frac{1}{2\sqrt{n}} \left(\cos \frac{2(k+m_\lambda-1)l\pi}{n}\right)_{k,l=0}^{m_1-1} - \mathbf{E}, \tag{7.15}$$

we easily conclude that the matrix $\tilde{B}_1$ is definitely singular, as if it were nonsingular, the multiplicity of the eigenvalue -1 of $\mathbf{F}_n$ would become m_1, which is impossible. Utilizing Lemma 7.4, we can therefore conclude that B_1 is nonsingular.

CRediT authorship contribution statement

Fatemeh Zarei: Conceptualization; Formal analysis; Writing – original draft.
Ali Bagheri Bardi: Conceptualization; Formal analysis; Methodology; Writing.
Taher Yazdanpanah: Formal analysis; Methodology; Writing – review & editing.
Miloš Daković Formal analysis; Software; Validation; Writing – original draft.
Miloš Brajović: Methodology; Validation; Writing – review & editing.
Ljubiša Stanković: Formal analysis; Methodology; Writing – review & editing; Supervision.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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