Fourier Series and Transform Problem Set

Undergraduate Level

Instructions

Attempt all problems. Use graphing tools where necessary. Justify all conceptual answers with appropriate reasoning and mathematical evidence.

Fourier Series Problems

Easy Problems (25%)

E1. Compute the Fourier series for the periodic extension of $f(x) = x$ on $[-\pi, \pi]$.

E2. Show that the Fourier coefficients of an even function contain only cosine terms.

E3. Find the Fourier series of the square wave defined as:

$$f(x) = \begin{cases} 1, & 0 < x < \pi \\ -1, & -\pi < x < 0 \end{cases}$$

E4. Plot $f(x) = |\sin x|$ and sketch its Fourier approximation using the first 3 nonzero terms.

E5. Determine whether $f(x) = x^2$ on $[-\pi, \pi]$ is even, odd, or neither. Then find its Fourier series.

Medium Problems (30%)

M1. Derive the Fourier series of the triangle wave defined as $f(x) = |x|$ on $[-\pi, \pi]$.

M2. Show that the Fourier coefficients of an odd function contain only sine terms.

M3. Prove that the Fourier series of a function with jump discontinuities converges to the average of the left- and right-hand limits at the point of discontinuity.

M4. Let $f(x) = x(\pi - x)$ on $[0, \pi]$, extended as an even function. Find its Fourier cosine series.

M5. Find the Fourier sine series of $f(x) = x$ on $[0, L]$.

M6. Analyze the convergence of the Fourier series for a function that is continuous but not differentiable at a point.

Difficult Problems (45%)

D1. Prove that Fourier coefficients decay faster for smoother functions. *Hint: Use integration by parts.*

D2. Derive the Fourier series for $f(x) = x^2$ on $[-\pi, \pi]$ and use Parseval's identity to evaluate $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

D3. Analyze the convergence behavior of the Fourier series of a periodic sawtooth function.

D4. Consider a periodic function with a discontinuity: derive its Fourier series and illustrate how the Gibbs phenomenon manifests.

D5. Show how Fourier series can approximate discontinuous functions, noting the overshoot at jump points.

D6. Construct a piecewise-defined function that is neither even nor odd. Compute its full Fourier series.

D7. Using Parseval's theorem, prove that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ via the Fourier series of $f(x) = x$ on $[-\pi, \pi]$.

D8. For a periodic function $f(x)$ with period $2L$, show how the interval length affects the frequency of the sine and cosine terms.

D9. Derive the complex form of the Fourier series and apply it to $f(x) = x$ on $[-\pi, \pi]$.

D10. Let $f(x) = x \sin x$. Find the Fourier series of the even extension of f defined on $[0, \pi]$.

Conceptual Problems

VH1. Why do Fourier coefficients decay as $n \to \infty$? Relate this to function smoothness and integration by parts.

VH2. Explain why cosine and sine naturally arise in Fourier series due to orthogonality principles.

VH3. Let f be periodic with bounded variation. Prove convergence in the L^2-norm and discuss the uniqueness of Fourier coefficients.

VH4. Compare the decay rates of Fourier coefficients for $f(x) = |x|$ and $f(x) = x^2$. Provide visual evidence.

VH5. Let $f(x) \in C^\infty$. Show that the Fourier coefficients decay faster than any polynomial rate.

VH6. Examine the implications of Fourier series in solving the heat equation.

Sampling Theory Problems

Easy

S-E1. Define the Nyquist rate and explain its importance in signal sampling.

S-E2. A signal has a maximum frequency component of 2 kHz. What is the minimum sampling rate required to avoid aliasing?

Medium

S-M1. Given a band-limited signal $f(t)$ with a maximum frequency of 1.5 kHz, what happens if you sample it at 2 kHz? Justify your answer.

S-M2. Explain the aliasing phenomenon and demonstrate it using the formula $f_a = |f_s - f|$, where f_s is the sampling rate.

S-M3. Sketch or simulate the aliasing that occurs when sampling a 3 kHz signal at 4 kHz. Indicate the apparent (aliased) frequency.

Hard

S-H1. Derive the sampling theorem (Shannon-Nyquist) and state the assumptions required for perfect reconstruction.

S-H2. A continuous-time signal is given by $f(t) = \cos(2\pi \cdot 1000t) + \cos(2\pi \cdot 3000t)$. Analyze the outcome if it's sampled at 4000 Hz.

S-H3. Explain and derive the reconstruction formula using sinc interpolation:

$$f(t) = \sum_{n=-\infty}^{\infty} f(nT) \cdot \operatorname{sinc}\left(\frac{t - nT}{T}\right)$$

Fourier Transform: Easy and Medium Problems

Easy

FT-E1. State the definition of the continuous-time Fourier transform (FT) of a function $f(t)$.

FT-E2. Compute the Fourier transform of the unit impulse $\delta(t)$.

FT-E3. Find the Fourier transform of the rectangular pulse function:

$$f(t) = \begin{cases} 1, & |t| < \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

FT-E4. State the Fourier transform of a time-shifted function $f(t - t_0)$ in terms of the transform of $f(t)$.

FT-E5. What is the physical meaning of the magnitude and phase of the Fourier transform in signal analysis?

FT-E6. Sketch the function $\text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$. Comment on its symmetry and decay.

FT-E7. Show that $\text{sinc}(0) = 1$ using L'Hôpital's Rule.

Medium

FT-M1. Let $f(t) = e^{-a|t|}$ for $a > 0$. Compute its Fourier transform.

FT-M2. Use the duality property of the Fourier transform to derive the transform of the sinc function:

$$\text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$

FT-M3. Prove the time-scaling property:

$$f(at)\mathcal{F}\frac{1}{|a|}F\left(\frac{\omega}{a}\right)$$

FT-M4. Suppose $f(t) = \cos(2\pi f_0 t)$. Find and sketch its Fourier transform.

FT-M5. Let $f(t)$ be a real, even function. Show that its Fourier transform is real and even.

FT-M6. Compute the Fourier transform of $f(t) = \text{rect}(t/T)$, where:

$$\text{rect}(t/T) = \begin{cases} 1, & |t| < T/2 \\ 0, & \text{otherwise} \end{cases}$$

and show that it results in a scaled sinc function.

FT-M7. Consider $f(t) = \text{sinc}(t)$. Use Fourier transform duality to find its transform. What does this imply about bandwidth?