In The Name of God

Sharif University of Technology
Department of Aerospace Engineering

45-784: Advanced Orbital Mechanics

CH#2: Coordinate and Time Systems

The Earth

- Basic physical characteristics of the Earth:
  - the equatorial radius, $R_{\oplus} = 6378.1363\ km$ (EGM2008, GECO)

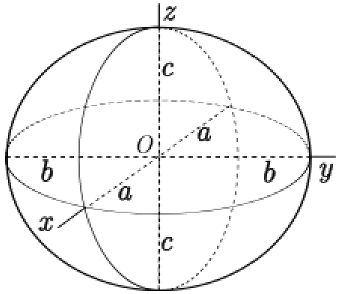
    mean semi-minor axis: $b_{\oplus} \approx 6356.751\ km$
  - The Earth's eccentricity, $e_{\oplus} = 0.081819221456$
  - The gravitational parameter, $\mu_{\oplus} = 3.986004415e+5\ km^3/s^2$
  - The rotational velocity, $\omega_{\oplus} = 7.292115e-5 \pm 1.5e-12 \quad rad/s$

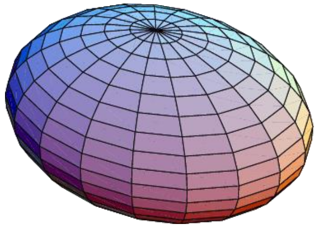
- Due to perturbative effects of the Sun, Moon, and other planets, and the Earth’s non-spherical nature

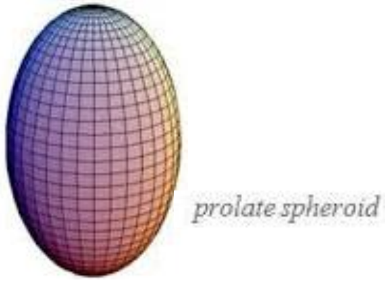
$$\omega_{\oplus} = 7.2921151467e-5 + \underbrace{7.086e-12T_{UT1} + 4.3e-15T_{UT1}^2}_{\text{secular rate of precession}} \quad rad/s$$

Shape of the Earth

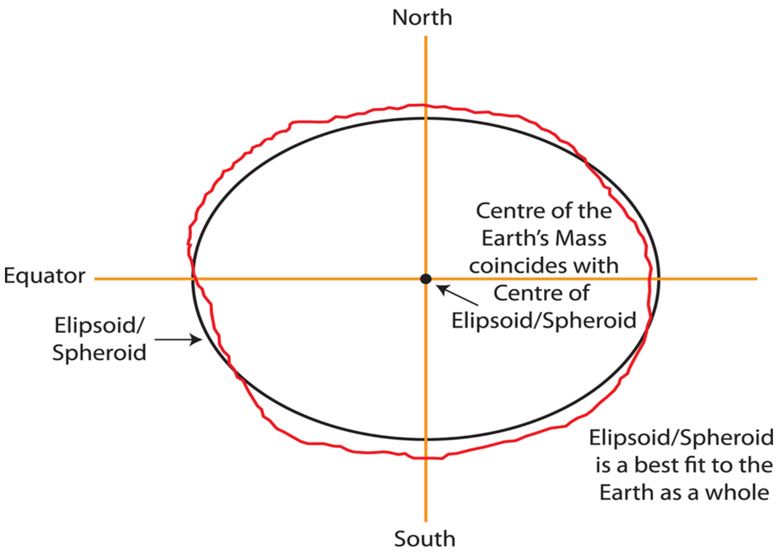
- Earth's shape
  - Spherical Earth
  - Ellipsoid (a,b,c)
    - Tri-axial ellipsoid (a, b, c)
    - Prolate ellipsoid (a, b, b)
    - Oblate ellipsoid (a, a, b)

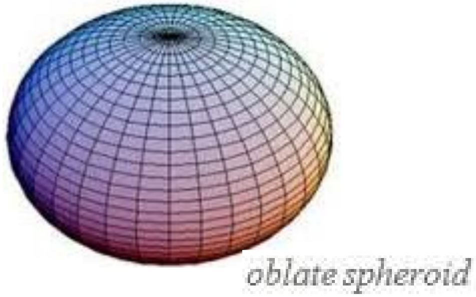




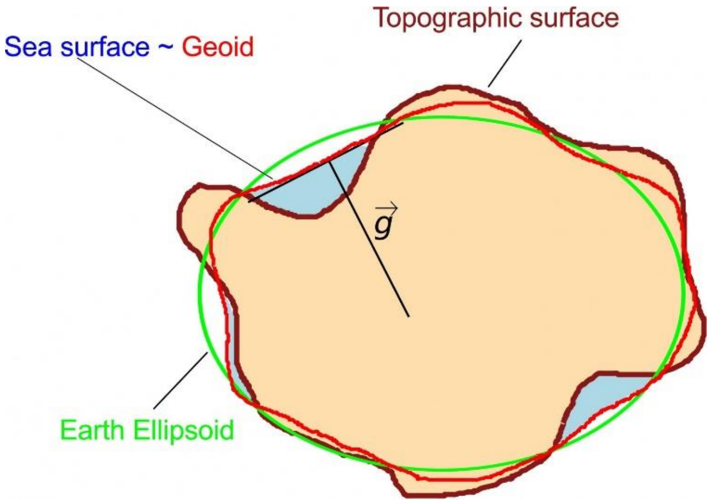


- $a_{\oplus} = 6378.1363 \ km$





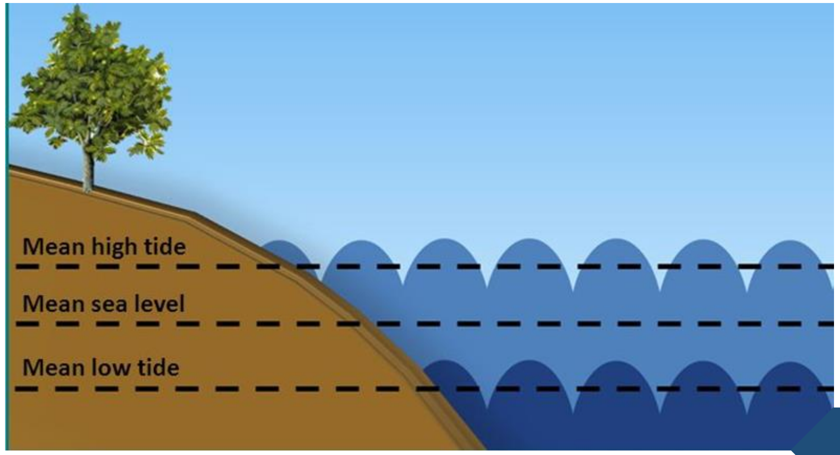
Shape of the Earth



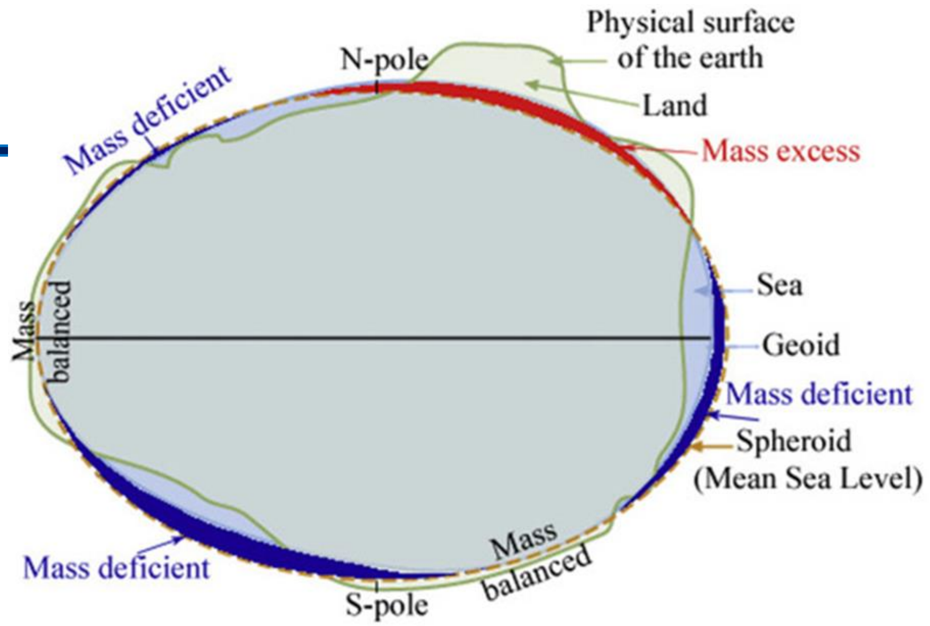
- The geoid is a geopotential surface that a plumb-bob will hang perpendicular to it at every point.

- Mean sea level (MSL) is an average level of the surface of one or more of Earth's oceans from which heights such as elevation may be measured (atmospheric pressure is measured to calibrate altitude).

- A common and relatively straightforward mean sea-level standard is the midpoint between a mean low and mean high tide at a particular location.



Mean Sea Level

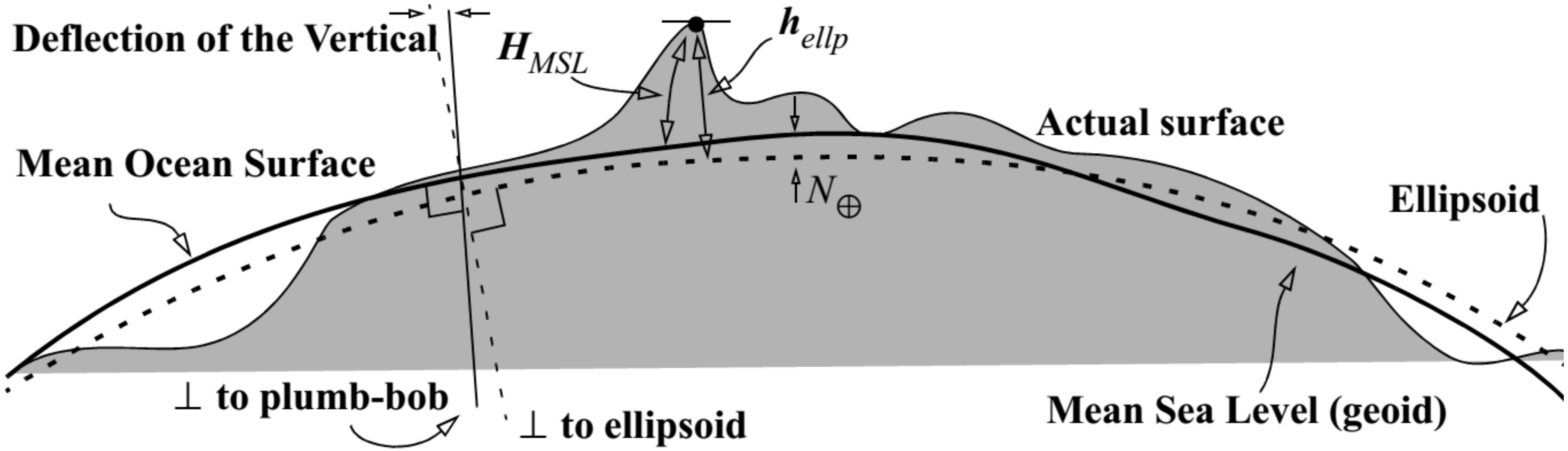


- Sea level varies quite a lot on several scales of time and space. The easiest way is selecting a location and calculating the mean sea level at that point and use it as a datum.

- In a state of rest or absence of external forces, the mean sea level would coincide with geoid surface. In reality, due to currents, air pressure variations, temperature, etc., this does not occur, not even as a long-term average. The location-dependent, but persistent in time, separation between mean sea level and the geoid is referred to as (stationary) ocean surface topography. It varies globally in a range of ± 2 m.

The Earth

- Geopotential reference surfaces



$N_\oplus \cong h_{ellp} - H_{MSL}$:geoid's undulation ([-107,+85] *m*)

$$N_\oplus = \frac{\mu}{r g_{th}} \sum_{\ell=2}^{\infty} \sum_{m=0}^{\ell} \left(\frac{R_\oplus}{r}\right)^{\ell} \bar{P}_{\ell,m}[\mathrm{SIN}(\phi_{gc})]\left\{\bar{C}^*_{\ell,m}\mathrm{COS}(m\lambda) + \bar{S}_{\ell,m}\mathrm{SIN}(m\lambda)\right\}$$

distance to the ellipsoid from the Earth's center

even zonal harmonics, modified by subtracting their geometric value

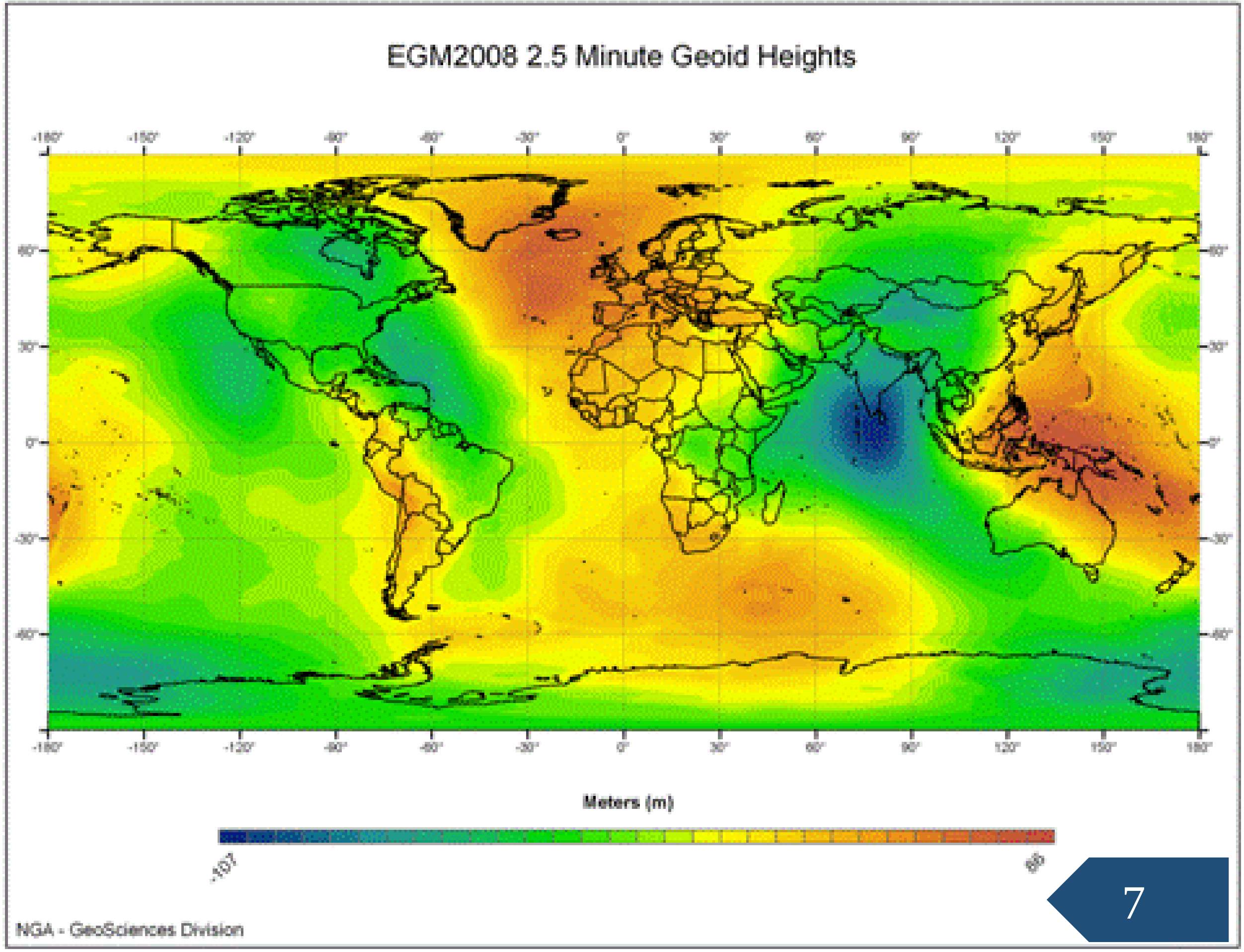
The Earth

$$g_{th} = g_{equator}\left\{\frac{1 + k_g \mathrm{SIN}^2(\phi_{gd})}{\sqrt{1 - e_\oplus^2 \mathrm{SIN}^2(\phi_{gd})}}\right\} \qquad g_{equator} = 9.780\ 325\ 335\ 9 \text{ m/s}^2$$

$$k_g = \frac{b_\oplus g_{pole}}{R_\oplus g_{equator}} - 1 = 0.001\ 931\ 852\ 652\ 41$$

$$g_{pole} = 9.832\ 184\ 937\ 8 \text{ m/s}^2$$

- $N_\oplus$ is tabulated in contour charts for grids of $10° \times 10°$, $1° \times 1°$, and so on. Locations not contained on a grid are found using interpolation techniques.



Shape of the Earth

- Oblate Earth model

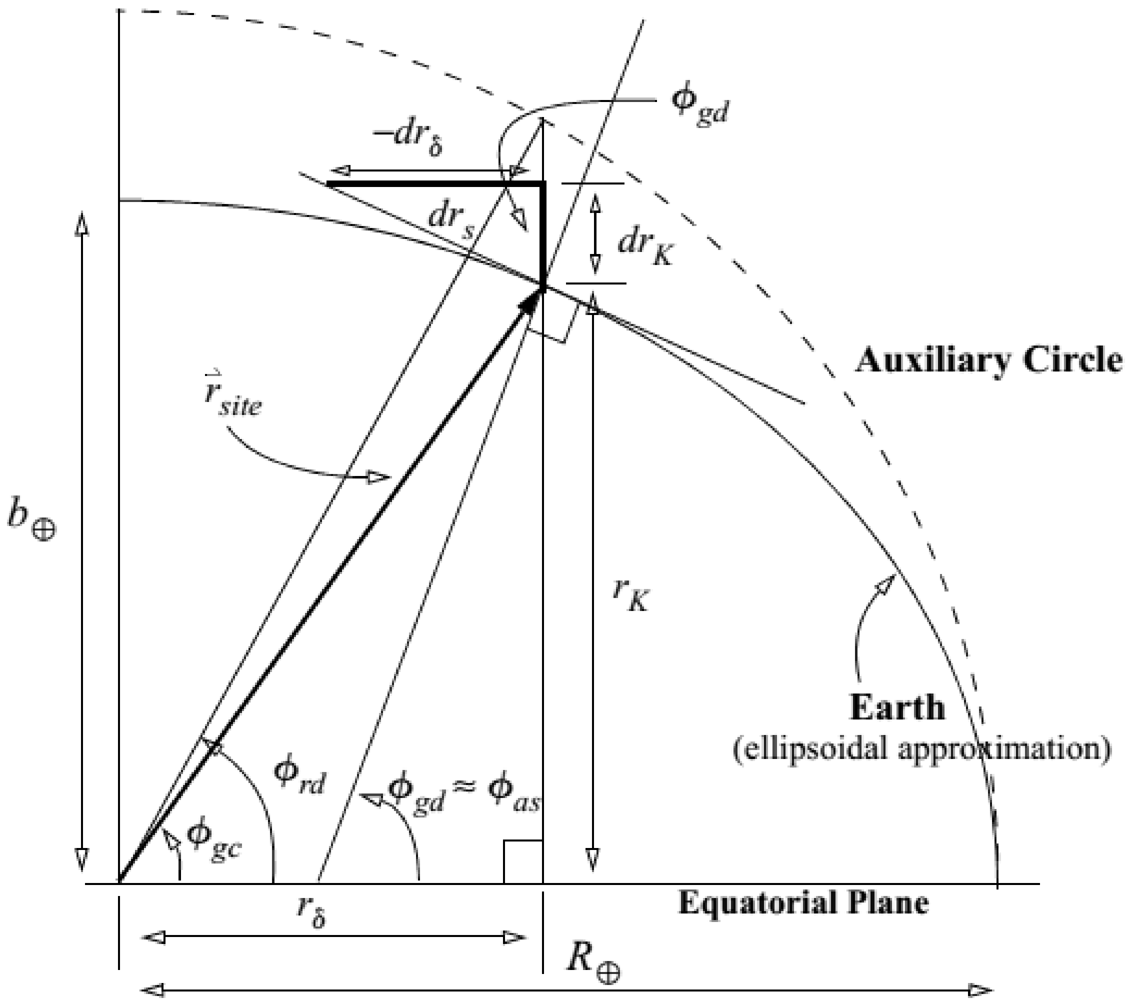
$$\tan\phi_{gc} = (1 - e_{\oplus}^{2}\frac{C_{\oplus}}{C_{\oplus} + h_{ellp}})\tan\phi_{gd}$$

where

$$C_{\oplus} = \frac{R_{\oplus}}{\sqrt{1 - e_{\oplus}^{2}\,\mathrm{SIN}^{2}(\phi_{gd})}}$$

radius of curvature in the prime vertical $(C_{\oplus}, N, R_N)$

$$\phi_{as} - \phi_{gd} = deflection\ of\ vertical$$



The Earth

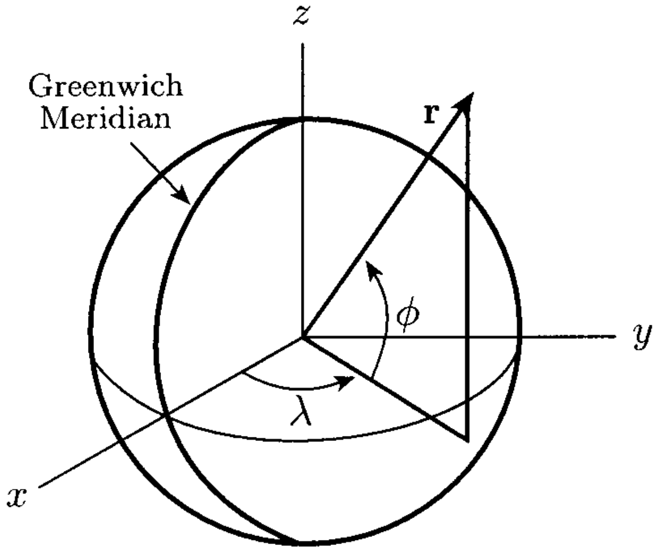
- Position vector

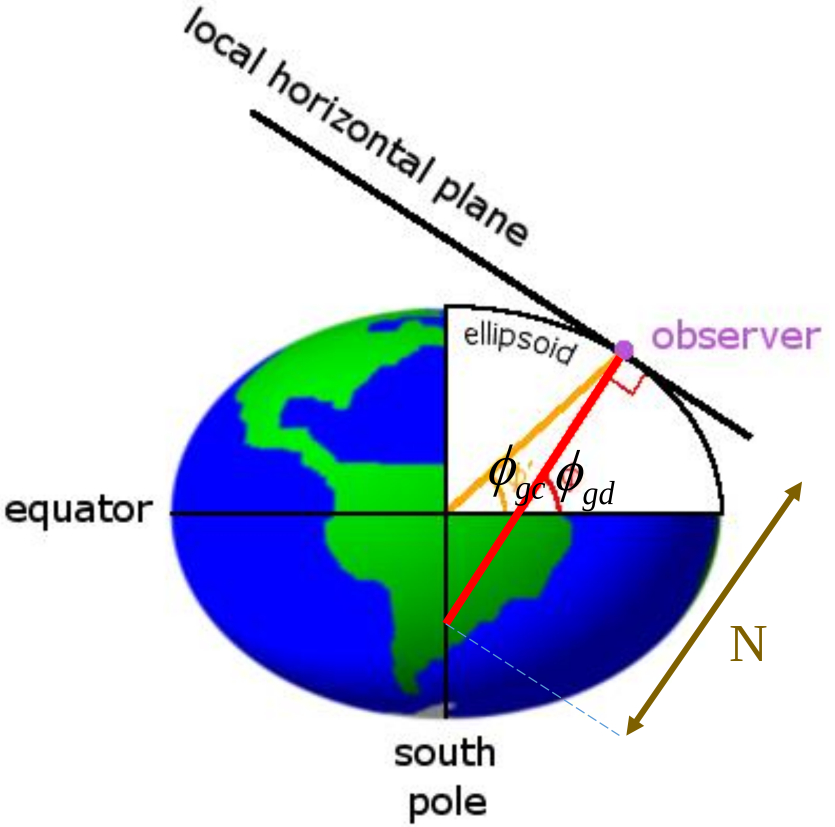
$$\mathbf{r} = r\begin{bmatrix}\cos\phi_{gc}\cos\lambda \\ \cos\phi_{gc}\sin\lambda \\ \sin\phi_{gc}\end{bmatrix}$$

$$= \begin{bmatrix}(C_\oplus + h_{ellp})\cos\phi_{gd}\cos\lambda \\ (C_\oplus + h_{ellp})\cos\phi_{gd}\sin\lambda \\ (S_\oplus + h_{ellp})\sin\phi_{gd}\end{bmatrix}$$

where

$$S_\oplus = \frac{R_\oplus(1-e_\oplus^2)}{\sqrt{1-e_\oplus^2 \, \mathrm{SIN}^2(\phi_{gd})}}$$





The Earth

- Converting $\mathbf{r} = [r_x \quad r_y \quad r_z]^T$ to longitude & latitude

$$\lambda = \tan^{-1}(r_y / r_x), \qquad r_{xy} = \sqrt{r_x^2 + r_y^2}, \qquad \phi_0 = \tan^{-1}(r_z / r_{xy})$$

$$i = 0; \qquad error = K >> \varepsilon$$

$$while\ error > \varepsilon$$

$$C_\oplus = \frac{R_\oplus}{\sqrt{1 - e_\oplus^2 \sin^2 \phi_i}}$$

$$\phi_{i+1} = \tan^{-1}\left(\frac{r_z + C_\oplus e_\oplus^2 \sin(\phi_i)}{r_{xy}}\right)$$

$$error = \left|\phi_{i+1} - \phi_i\right|$$

$$end$$

$$h_{ellp} = \frac{r_{xy}}{\cos(\phi_i)} - C_\oplus, \qquad if\ \ near\ \ poles(\sim 1°): h_{ellp} = \frac{r_z}{\sin(\phi_i)} - S_\oplus$$

Coordinate Systems (CSs)

- To define a rectangular CS: origin, fundamental plane, preferred direction

- Coordinate transformation: $[\mathbf{X}]^B = \mathbf{T}^{BA}[\mathbf{X}]^A$

    - Sequential rotations: $\mathbf{T}^{BA} = \mathbf{T}^{BC}\mathbf{T}^{CA}$

    - Unit vectors: $\mathbf{T}^{BA} = [\mathbf{a}_1^B \quad \mathbf{a}_2^B \quad \mathbf{a}_3^B] = \begin{bmatrix} \mathbf{b}_1^{A\,T} \\ \mathbf{b}_2^{A\,T} \\ \mathbf{b}_3^{A\,T} \end{bmatrix}$

- Example:

$$\mathbf{T}^{PCS-ECI} = \mathbf{R}_3(\omega)\mathbf{R}_1(i)\mathbf{R}_3(\Omega) = \begin{bmatrix} \hat{\mathbf{p}}^T \\ \hat{\mathbf{q}}^T \\ \hat{\mathbf{w}}^T \end{bmatrix}$$

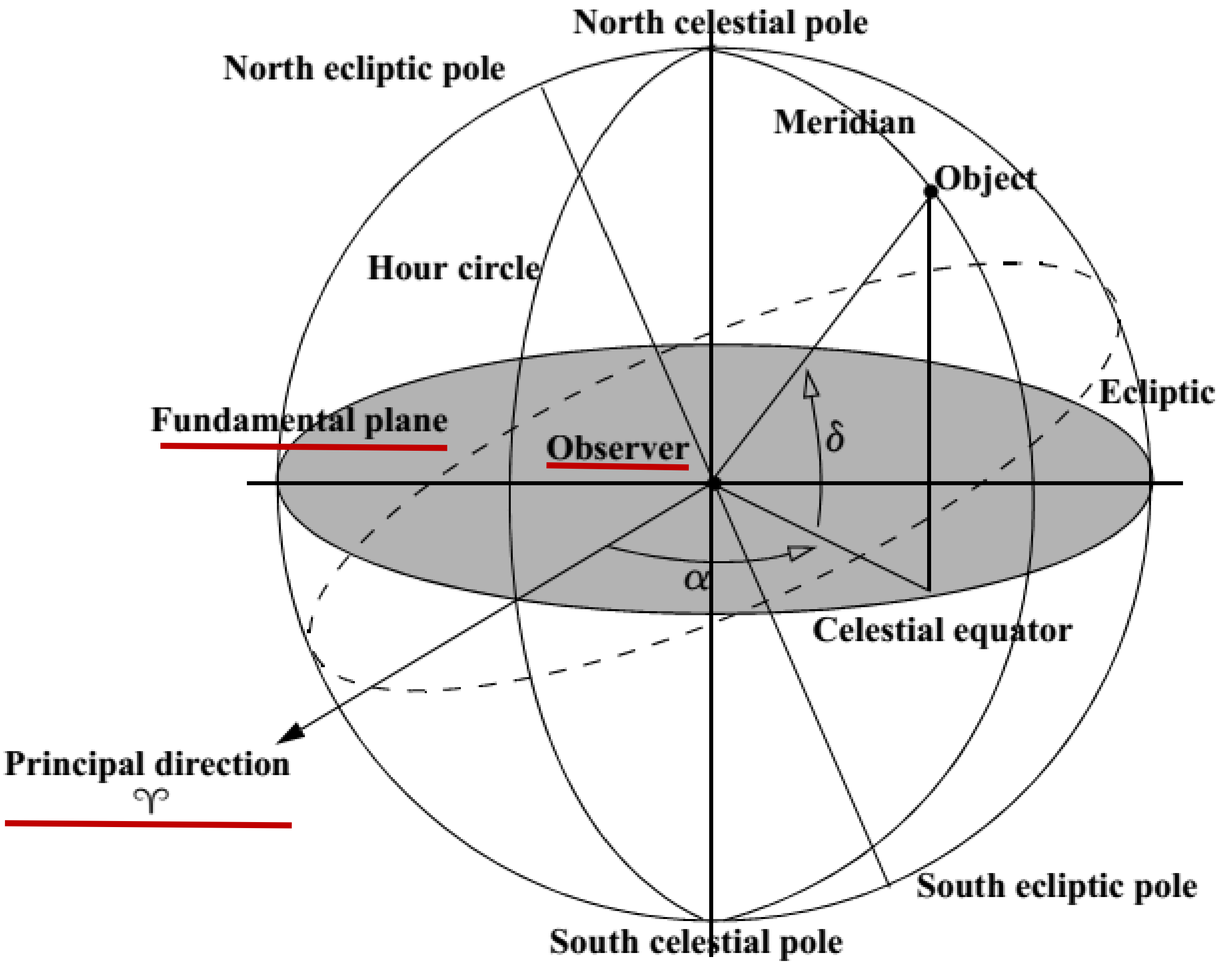
$$\hat{\mathbf{p}} = \frac{\mathbf{e}}{e}, \qquad \hat{\mathbf{w}} = \frac{\mathbf{h}}{h}, \qquad \hat{\mathbf{q}} = \hat{\mathbf{w}} \times \hat{\mathbf{p}}$$

Coordinate Systems (CSs)

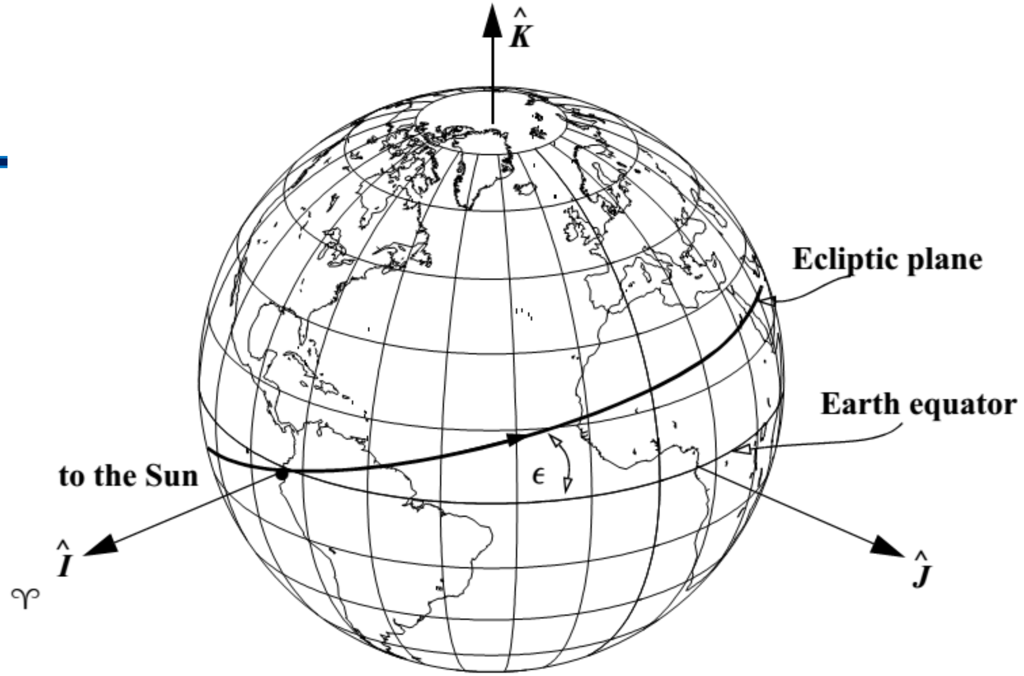
- Celestial sphere

δ : declination

α : right ascension



Coordinate Systems



- Vernal Equinox (V.E.)

  Ascending node: 21 March

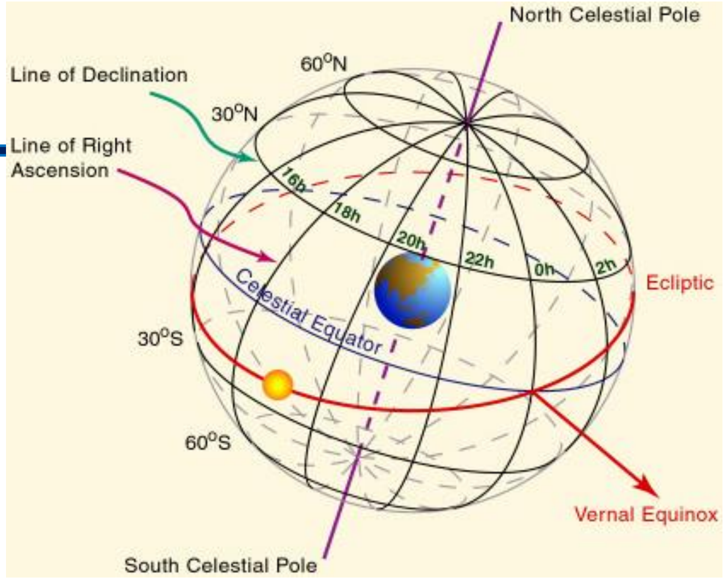
- Autumnal Equinox

  Descending Node: 23 Sep.

- Formal Definition of V.E.

it occurs when the Sun's declination is zero as it changes from negative to positive values.

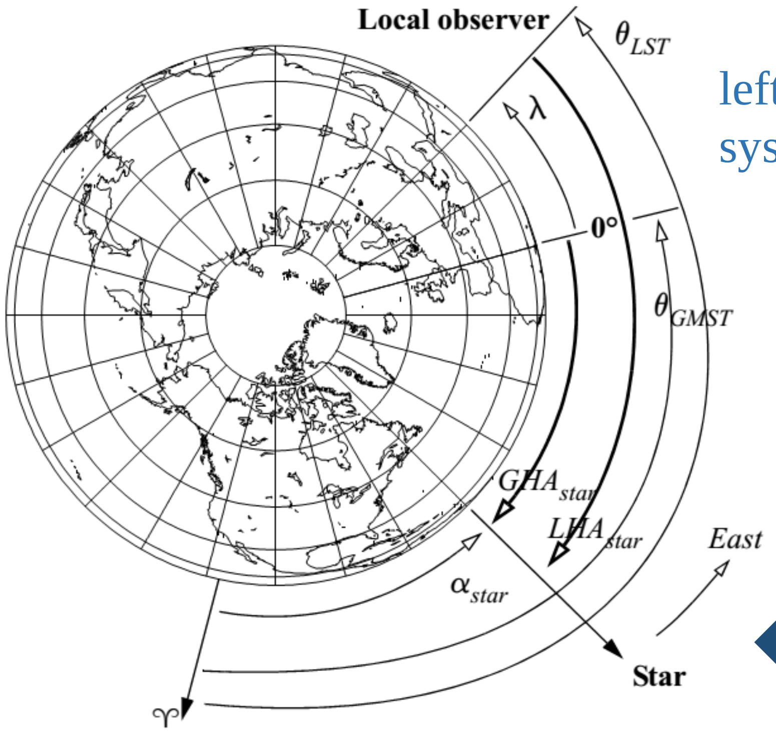
Hour Angle



- Hour angles measure the angular distance along the celestial equator of an object. The hour angle of any object is the angle from the primary hour circle to the hour circle of the object.

$$LHA = GHA + \lambda$$

- These angles can be measured in time (24 hours to a circle) or in degrees (360 degrees to a circle)- one or the other, not both.



left-handed system