In The Name of God

Sharif University of Technology
Department of Aerospace Engineering

45-784: Advanced Orbital Mechanics

CH#1: Review of Orbital Mechanics

Two-body Problem

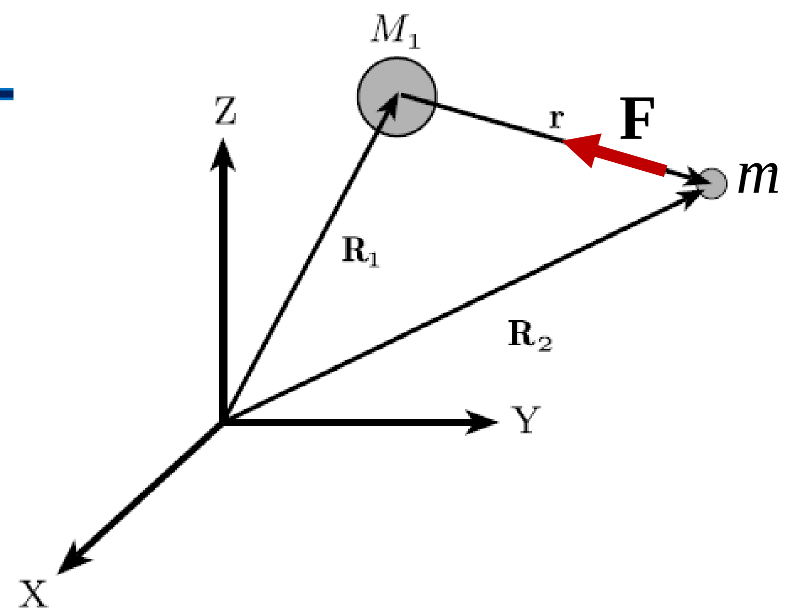
- Newton's second law: $\frac{d}{dt}\mathbf{P} = \dot{m}\dot{\mathbf{r}} + m\ddot{\mathbf{r}} = \mathbf{F}$
- Newton's universal law of gravitation: $\mathbf{F} = -G\frac{Mm}{r^2}\frac{\mathbf{r}}{r}$
- Two-body equation of motion:

$$\left.\begin{aligned} M\ddot{\mathbf{R}}_1 &= \frac{GMm\mathbf{r}}{r^3} \\ m\ddot{\mathbf{R}}_2 &= -\frac{GMm\mathbf{r}}{r^3} \end{aligned}\right\} \xrightarrow{\text{relative motion}} \mathbf{r} = \mathbf{R}_2 - \mathbf{R}_1$$

$$\Rightarrow \quad \ddot{\mathbf{r}} = -\frac{\mu\mathbf{r}}{r^3}$$

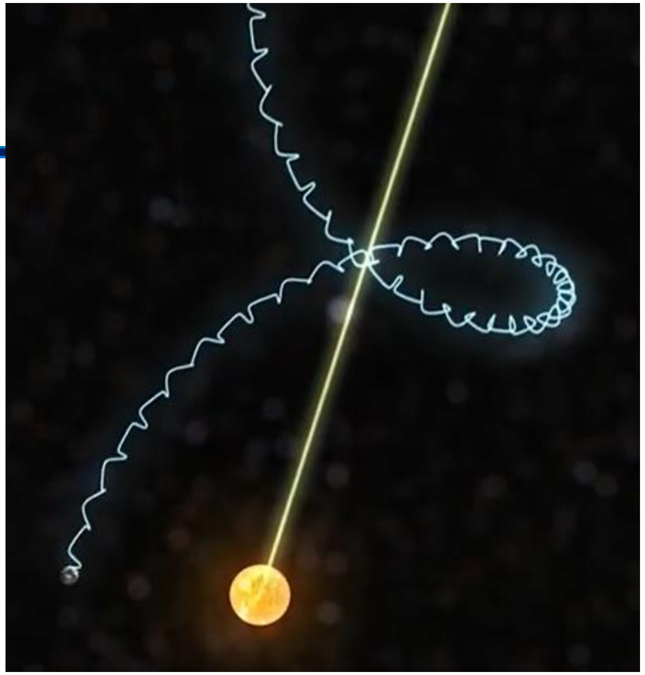
$$\mu = G(M + m) \cong GM$$

$$\mu_{EGM\,2008} = 3.986004415e+5\,km^3/s^2$$



XYZ is nonrotating, with zero acceleration; an inertial reference frame

Two-body Problem



$$\mathbf{R}_{cm} = \frac{M\mathbf{R}_1 + m\mathbf{R}_2}{M + m}$$

$$\ddot{\mathbf{R}}_{cm} = 0 \rightarrow \mathbf{R}_{cm} = \mathbf{C}_1 t + \mathbf{C}_2$$

- Integrals of motion
  - Angular momentum per unit mass

$$\mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}} = \mathbf{r} \times \mathbf{v} = \text{constant} \longrightarrow \text{motion is planar}$$

  - Energy per unit mass

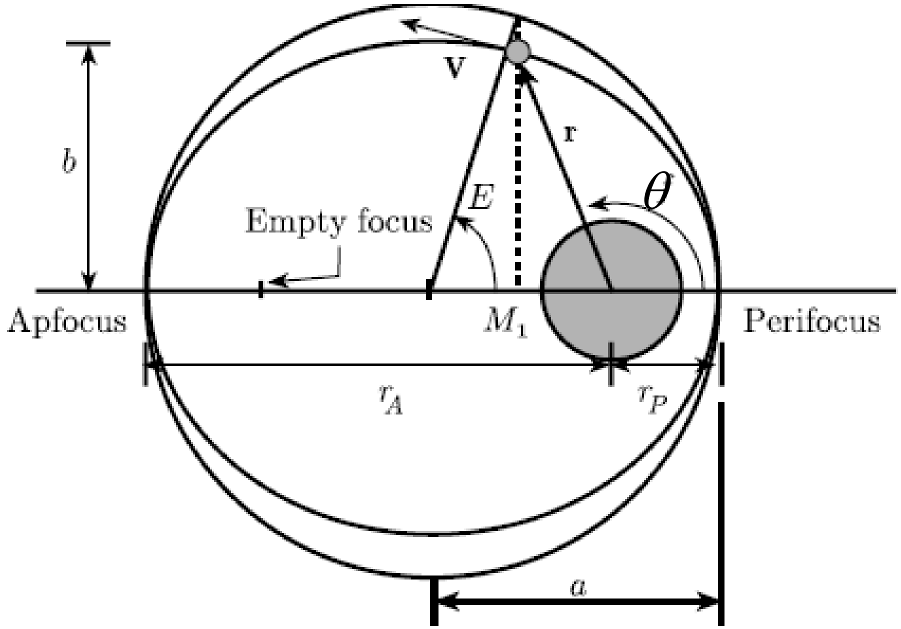
$$\varepsilon = \frac{\dot{\mathbf{r}}.\dot{\mathbf{r}}}{2} - \frac{\mu}{r} = \text{constant}$$

Two-body Problem

- Orbit equation

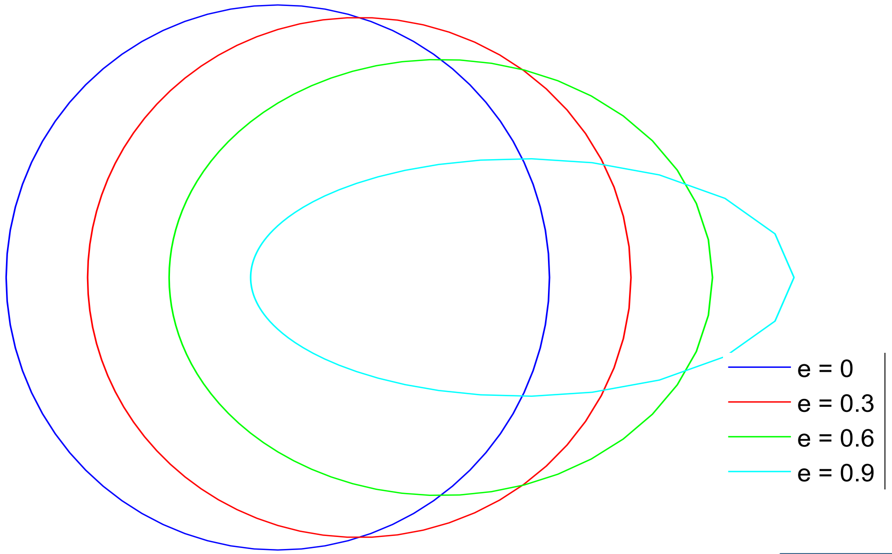
$$r = \frac{h^2/\mu}{1 + e\cos\theta} \qquad ,\theta \equiv f \equiv \upsilon$$

$$p = h^2/\mu \qquad T = 2\pi\sqrt{\frac{a^3}{\mu}}$$



- Types of orbits

Orbit Type	Eccentricity	Energy	Orbital Speed
Circle	$e = 0$	$\xi = -\frac{\mu}{2a}$	$V = \sqrt{\mu/r}$
Ellipse	$e < 1$	$\xi < 0$	$\sqrt{\mu/r} < V < \sqrt{2\mu/r}$
Parabola	$e = 1$	$\xi = 0$	$V = \sqrt{2\mu/r}$
Hyperbola	$e > 1$	$\xi > 0$	$V > \sqrt{2\mu/r}$



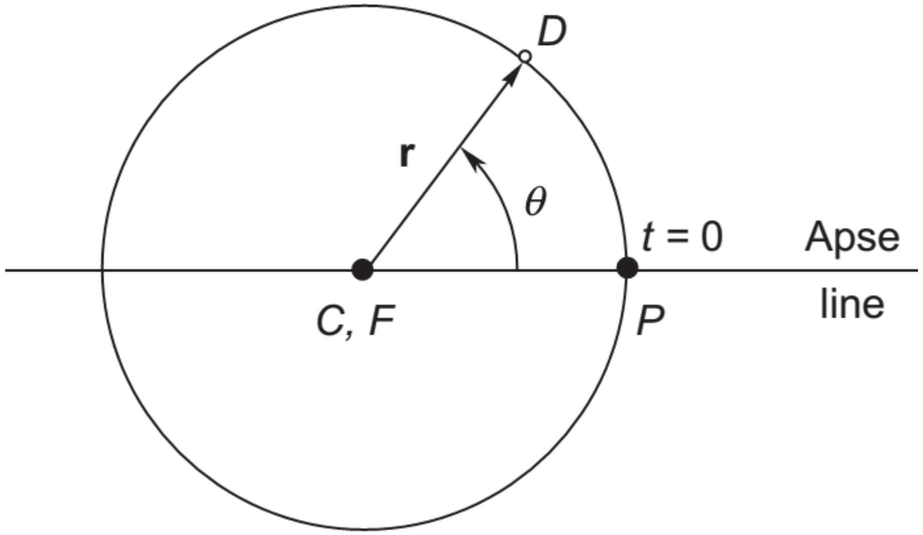
Two-body Problem

- Circular orbit (e = 0)

$$r = \frac{h^2}{\mu} \qquad v = \sqrt{\frac{\mu}{r}}$$

$$T = 2\pi\sqrt{\frac{r^3}{\mu}} \qquad r = a$$

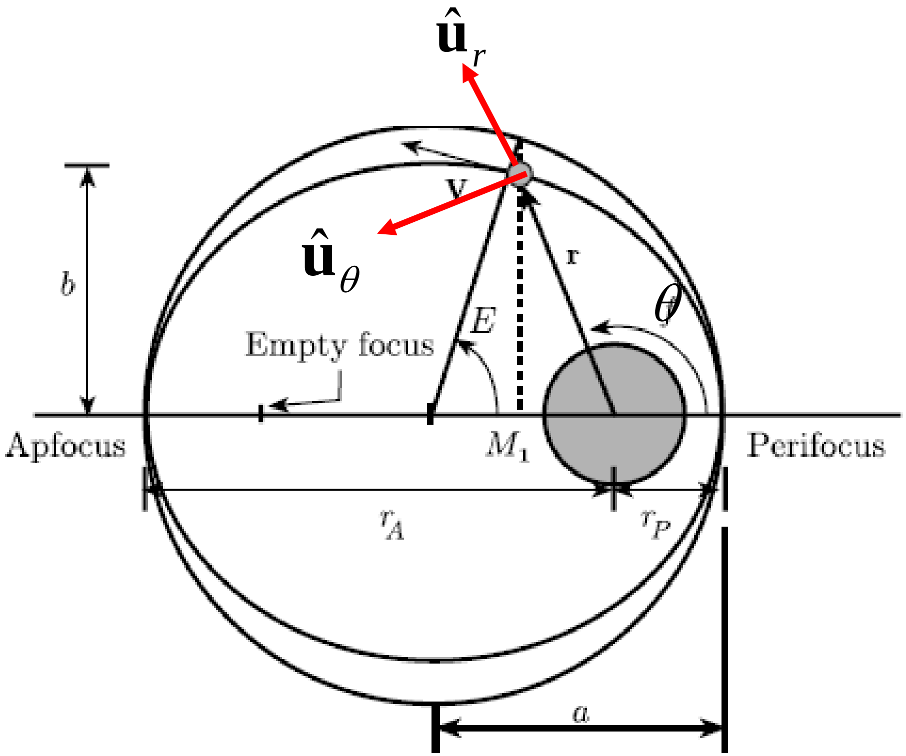
$$\dot{r} = 0$$



- Orbital position as a function of time

$$t = \frac{\theta}{2\pi}T$$

Two-body Problem



- Elliptic orbit (0 < e < 1)

$$p = a(1-e^2), \qquad b = a\sqrt{1-e^2}$$

$$r_p = a(1-e), \qquad r_a = a(1+e)$$

$$\mathbf{v} = \dot{r}\hat{\mathbf{u}}_r + r\dot{\theta}\hat{\mathbf{u}}_\theta$$

$$v_r = \dot{r} = \frac{\mu}{h} e \sin\theta$$

$$v_\theta = r\dot{\theta} = \frac{h}{r} = \frac{\mu}{h}(1 + e\cos\theta)$$

$$n = \frac{2\pi}{T} = \sqrt{\frac{\mu}{a^3}}$$

- Orbital position as a function of time

$$M_e = n(t - t_p) = E - e\sin E$$: Kepler's eqn.

$$\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2} \qquad \tan\frac{E}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}$$

Example

- ماهواره ای در یک مدار بیضوی حول زمین با کمترین ارتفاع 622 km به حداکثر سرعت 8.6038 km/s خواهد رسید. مطلوبست

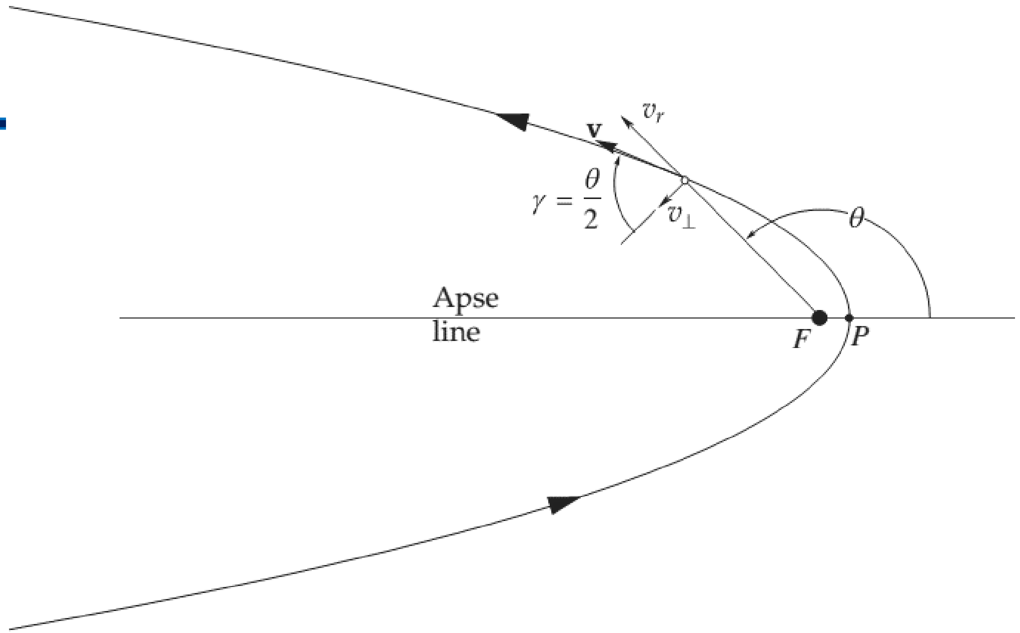
الف) تعیین دوره تناوب

ب) سرعت ماهواره در $\theta = 120^{\circ}$

Example

- ماهواره ای درنظر بگیرید که در مدار $1000\ km \times 400\ km$ حول زمین قرار دارد. مطلوبست محاسبه مدت زمانی که در ارتفاع پایین تر از 600 km قرار دارد.

Two-body Problem



- Parabolic orbit (e = 1)

$$r = \frac{h^2/\mu}{1+\cos\theta} \qquad v = \sqrt{\frac{2\mu}{r}}$$

$$\gamma = \frac{\theta}{2} \qquad \varepsilon = 0$$

- Orbital position as a function of time

$$M_p = \frac{1}{2}\tan\frac{\theta}{2} + \frac{1}{6}\tan^3\frac{\theta}{2}$$

$$M_p = \frac{\mu^2 t}{h^3}$$:Barker's eqn.

$$\tan\frac{\theta}{2} = \left(3M_p + \sqrt{(3M_p)^2+1}\right)^{\frac{1}{3}} - \left(3M_p + \sqrt{(3M_p)^2+1}\right)^{-\frac{1}{3}}$$

Example

- جسمی در یک مدار سهموی حول خورشید حرکت می‌کند. مدار این جسم مدار زمین را قطع می‌کند طوری که این دو نقطه تقاطع و خورشید تشکیل یک مثلث متساوی اضلاع می‌دهند. مدت زمانی را که این جسم در مدار زمین حرکت می‌کند را بر حسب سال بدست آورید.

Two-body Problem

- Hyperbolic orbit (e > 1)

$$r = a\frac{e^2 - 1}{1 + e\cos\theta} \rightarrow \theta_\infty = \cos^{-1}(-1/e)$$

$$r_p = a(e-1)$$

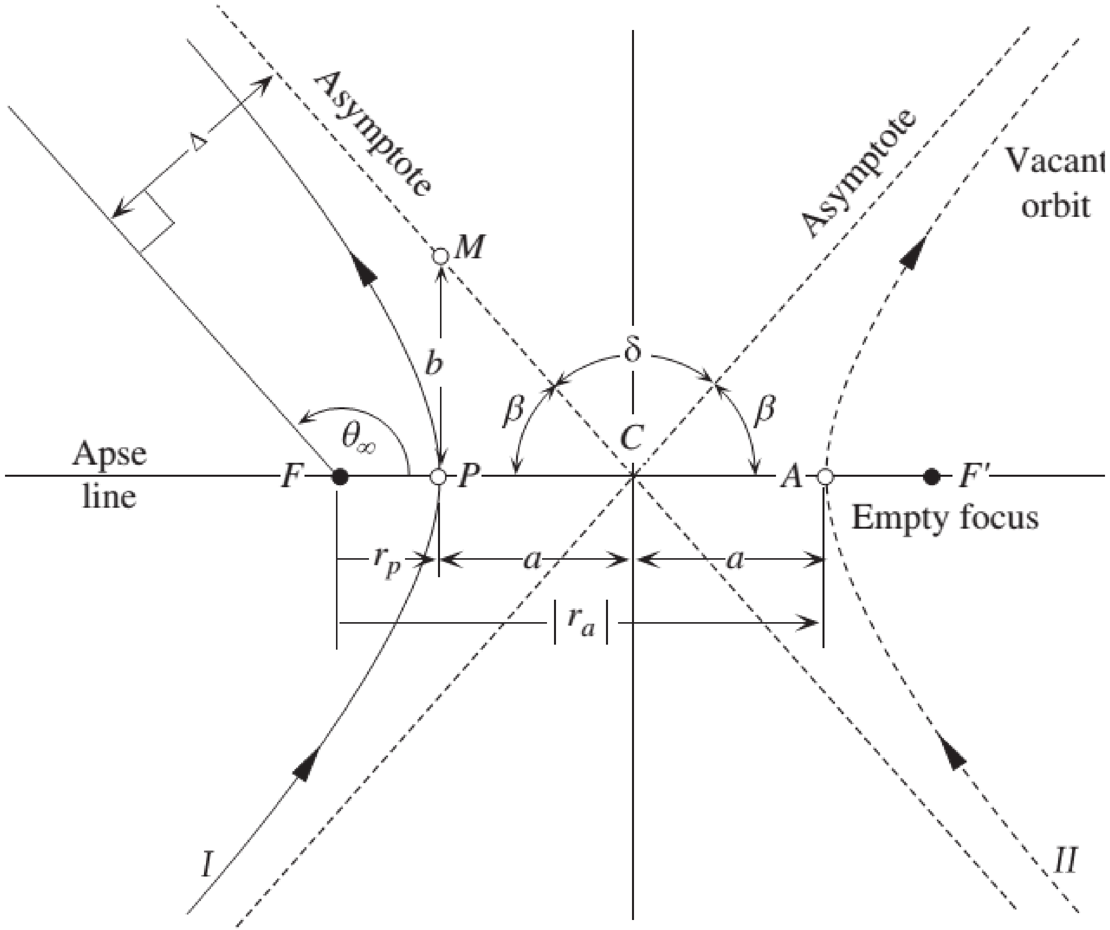
$$\beta = \cos^{-1}(1/e)$$

$$\delta = 2\sin^{-1}(1/e)$$

$$\varepsilon = \frac{\mu}{2a}$$

$$\Delta = a\sqrt{e^2 - 1}$$

$$v_\infty = \sqrt{\frac{\mu}{a}}$$



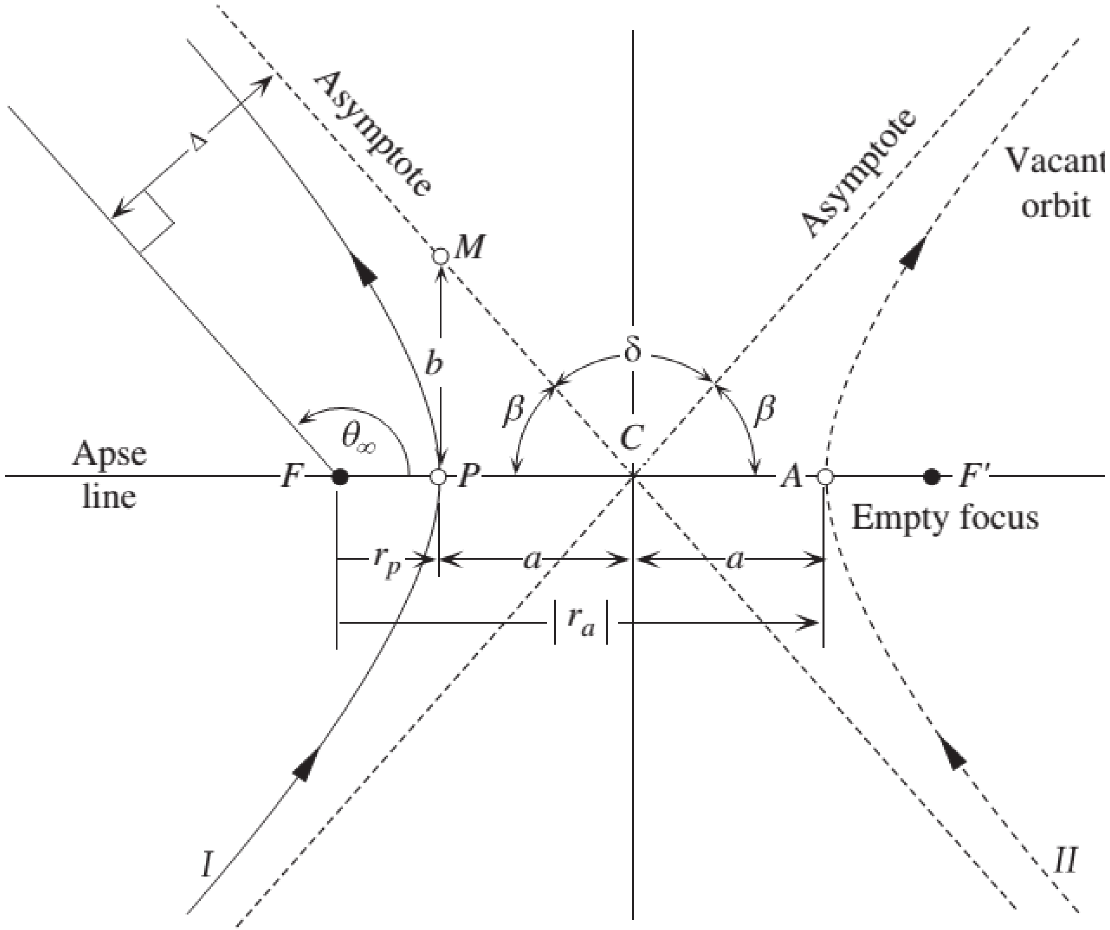
Two-body Problem

- Orbital position as a function of time

$$\sinh F = \frac{y}{b} \longrightarrow \tanh\frac{F}{2} = \sqrt{\frac{e-1}{e+1}}\tan\frac{\theta}{2}$$

$$M_{\mathrm{h}} = e\sinh F - F$$

$$M_{\mathrm{h}} = \frac{\mu^2}{h^3}(e^2-1)^{\frac{3}{2}}t$$



Examples

- شهاب‌سنگی نخستین بار در حال نزدیک‌شدن به زمین در شعاع 402000 km مرکز زمین با آنومالی حقیقی 150 deg مشاهده شده است. چنانچه سرعت شهاب‌سنگ در این لحظه برابر با 23.2 km/s باشد، مطلوبست محاسبه

الف) خروج از مرکز مسیر؛

ب) ارتفاع نزدیک‌ترین تقرب؛

پ) سرعت در نزدیک‌ترین تقرب.

Example

- یک فضاپیما در مدار هذلولوی حول زمین به شعاع حضیض 6600km و سرعت حضیض $1.2v_{esc}$ قرار دارد.

الف) چقدر طول می کشد تا فضاپیما از $\theta = -90°$ به $\theta = +90°$ برسد.

ب) فاصله فضاپیما تا مرکز زمین، $24\ hr$ پس از عبور از حضیض چقدر است؟

Universal Variables

- The overall idea is to find a unified expression relating the time and the orbit's properties.

- Angular momentum is usually employed to find the relation between time and position:

$$h = rv_\theta = r^2\dot{\theta} \rightarrow \frac{d\theta}{dt} = \frac{h}{r^2} \rightarrow dt = \frac{r^2}{h}d\theta$$

$$d\theta = \begin{cases} \dfrac{b}{r}dE & :\text{Elliptic orbit} \\ \dfrac{p}{r}d(\tan(\theta/2)) & :\text{Parabolic orbit} \\ \dfrac{b}{r}dF & :\text{Hyperbolic orbit} \end{cases} \Rightarrow dt = \frac{r^2}{h}\begin{cases} \dfrac{b}{r}dE \\ \dfrac{p}{r}d(\tan(\theta/2)) \\ \dfrac{b}{r}dF \end{cases} = \frac{r}{\sqrt{\mu}}\begin{cases} \sqrt{a}dE \\ \sqrt{p}d(\tan(\theta/2)) \\ \sqrt{-a}dF \end{cases} = \frac{r}{\sqrt{\mu}}d\chi$$

Universal Variables

$$\chi = \begin{cases} \sqrt{a}(E - E_0) \\ \sqrt{p}(\tan(\theta/2) - \tan(\theta_0/2)) \\ \sqrt{-a}(F - F_0) \end{cases} \qquad assumption: \chi_0 = 0$$

$$dt = \frac{r}{\sqrt{\mu}} d\chi \rightarrow \dot{\chi} = \frac{\sqrt{\mu}}{r}$$

$$r^2 = \mathbf{r}.\mathbf{r} \xrightarrow{d/d\chi} 2r\frac{dr}{d\chi} = 2\mathbf{r}.\frac{d\mathbf{r}}{d\chi} \rightarrow r\frac{dr}{d\chi} = \frac{r\mathbf{r}.\mathbf{v}}{\sqrt{\mu}} \Rightarrow \frac{dr}{d\chi} = \frac{\mathbf{r}.\mathbf{v}}{\sqrt{\mu}}$$

Universal Variables

$$\varepsilon = \frac{v^2}{2} - \frac{\mu}{r} = \frac{\dot{r}^2 + r^2\dot{\theta}^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a} \rightarrow \dot{r}^2 = -\frac{\mu}{a} + \frac{2\mu}{r} - r^2\dot{\theta}^2$$

$$\left(\frac{dr}{d\chi}\right)^2 = \left(\frac{\mathbf{r}.\mathbf{v}}{\sqrt{\mu}}\right)^2 = \frac{r^2\dot{r}^2}{\mu} = -\frac{r^2}{a} + 2r - p \rightarrow \frac{dr}{d\chi} = \sqrt{-\frac{r^2}{a} + 2r - p}$$

$$\rightarrow d\chi = \frac{dr}{\sqrt{-\frac{r^2}{a} + 2r - p}}$$

$a > 0$

$$\Rightarrow \chi + c_0 = -\sqrt{a}\sin^{-1}\left(\frac{2 - \frac{2r}{a}}{\sqrt{4 - \frac{4p}{a}}}\right) = \sqrt{a}\sin^{-1}\left(\frac{r/a - 1}{\sqrt{1 - p/a}}\right)$$

Universal Variables

$$\xrightarrow{p=a(1-e^2)} \chi + c_0 = \sqrt{a}\sin^{-1}\left(\frac{r/a-1}{e}\right) \to r = a\left(e\sin\left(\frac{\chi+c_0}{\sqrt{a}}\right)+1\right)$$

$$\xrightarrow{dr/dt} \dot{r} = \sqrt{a}e\cos\left(\frac{\chi+c_0}{\sqrt{a}}\right)\frac{d\chi}{dt} = \frac{\sqrt{\mu a}}{r}e\cos\left(\frac{\chi+c_0}{\sqrt{a}}\right)$$

- Assumption: $\chi(t_0) = \chi_0 = 0$

$$\begin{cases} r_0 = a\left(e\sin\left(\frac{c_0}{\sqrt{a}}\right)+1\right) \\ \dot{r}_0 = \frac{\sqrt{\mu a}}{r_0}e\cos\left(\frac{c_0}{\sqrt{a}}\right) \end{cases} \Longrightarrow \begin{cases} e\sin\left(\frac{c_0}{\sqrt{a}}\right) = \frac{r_0}{a} - 1 \\ e\cos\left(\frac{c_0}{\sqrt{a}}\right) = \frac{r_0\dot{r}_0}{\sqrt{\mu a}} = \frac{\mathbf{r}_0 . \dot{\mathbf{v}}_0}{\sqrt{\mu a}} \end{cases}$$

Universal Variables

$$\frac{d\chi}{dt} = \frac{\sqrt{\mu}}{a\left(e\sin\left(\frac{\chi + c_0}{\sqrt{a}}\right) + 1\right)} \xrightarrow{\int} \sqrt{\mu}\Delta t = \left[\chi - e\sqrt{a}\cos\left(\frac{\chi + c_0}{\sqrt{a}}\right)\right]_0^{\chi}$$

$$\sqrt{\mu}\Delta t = \left[\chi - e\sqrt{a}\left(\cos\left(\frac{\chi}{\sqrt{a}}\right)\cos\left(\frac{c_0}{\sqrt{a}}\right) - \sin\left(\frac{\chi}{\sqrt{a}}\right)\sin\left(\frac{c_0}{\sqrt{a}}\right)\right)\right]_0^{\chi}$$

$$= a + a\left[\sin\left(\frac{\chi}{\sqrt{a}}\right)\frac{\mathbf{r}_0.\mathbf{v}_0}{\sqrt{\mu a}} + \cos\left(\frac{\chi}{\sqrt{a}}\right)\left(\frac{r_0}{a} - 1\right)\right]$$

- Introducing a new variable: $\psi = \dfrac{\chi^2}{a}$

Universal Variables

$$\sqrt{\mu}\Delta t = \chi^3\left\{\frac{\sqrt{\psi}-\sin\sqrt{\psi}}{\sqrt{\psi^3}}\right\}+\frac{\mathbf{r}_0.\mathbf{v}_0}{\sqrt{\mu}}\chi^2\left\{\frac{1-\cos\sqrt{\psi}}{\psi}\right\}+\frac{r_0\chi\sin\sqrt{\psi}}{\sqrt{\psi}}$$

$$r = \chi^2\left\{\frac{1-\cos\sqrt{\psi}}{\psi}\right\}+\frac{\mathbf{r}_0.\mathbf{v}_0}{\sqrt{\mu}}\chi\left\{1-\psi\frac{\sqrt{\psi}-\sin\sqrt{\psi}}{\sqrt{\psi^3}}\right\}+r_0\left\{1-\psi\frac{1-\cos\sqrt{\psi}}{\psi}\right\}$$

- Stumpff functions:

$$C(\psi)=\begin{cases}\dfrac{1-\cos\sqrt{\psi}}{\psi} & \psi>0\\[2ex] \dfrac{1}{2} & \psi=0\\[2ex] \dfrac{\cosh\sqrt{-\psi}-1}{-\psi} & \psi<0\end{cases}$$

$$S(\psi)=\begin{cases}\dfrac{\sqrt{\psi}-\sin\sqrt{\psi}}{\sqrt{\psi^3}} & \psi>0\\[2ex] \dfrac{1}{6} & \psi=0\\[2ex] \dfrac{\sinh\sqrt{-\psi}-\sqrt{-\psi}}{\sqrt{-\psi^3}} & \psi<0\end{cases}$$

Universal Variables

$$S(\psi) = \sum_{k=0}^{\infty} (-1)^k \frac{\psi^k}{(2k+3)!} = \frac{1}{6} - \frac{\psi}{120} + \frac{\psi^2}{5040} + \ldots$$

$$C(\psi) = \sum_{k=0}^{\infty} (-1)^k \frac{\psi^k}{(2k+2)!} = \frac{1}{2} - \frac{\psi}{24} + \frac{\psi^2}{720} + \ldots$$

$$\sqrt{\mu}\Delta t = \chi^3 S(\psi) + \frac{\mathbf{r}_0.\mathbf{v}_0}{\sqrt{\mu}} \chi^2 C(\psi) + r_0 \chi \left[1 - \psi S(\psi)\right]$$

$$r = \chi^2 C(\psi) + \frac{\mathbf{r}_0.\mathbf{v}_0}{\sqrt{\mu}} \chi \left[1 - \psi S(\psi)\right] + r_0 \left[1 - \psi C(\psi)\right]$$

Universal Variables

- Kepler's problem for universal variables

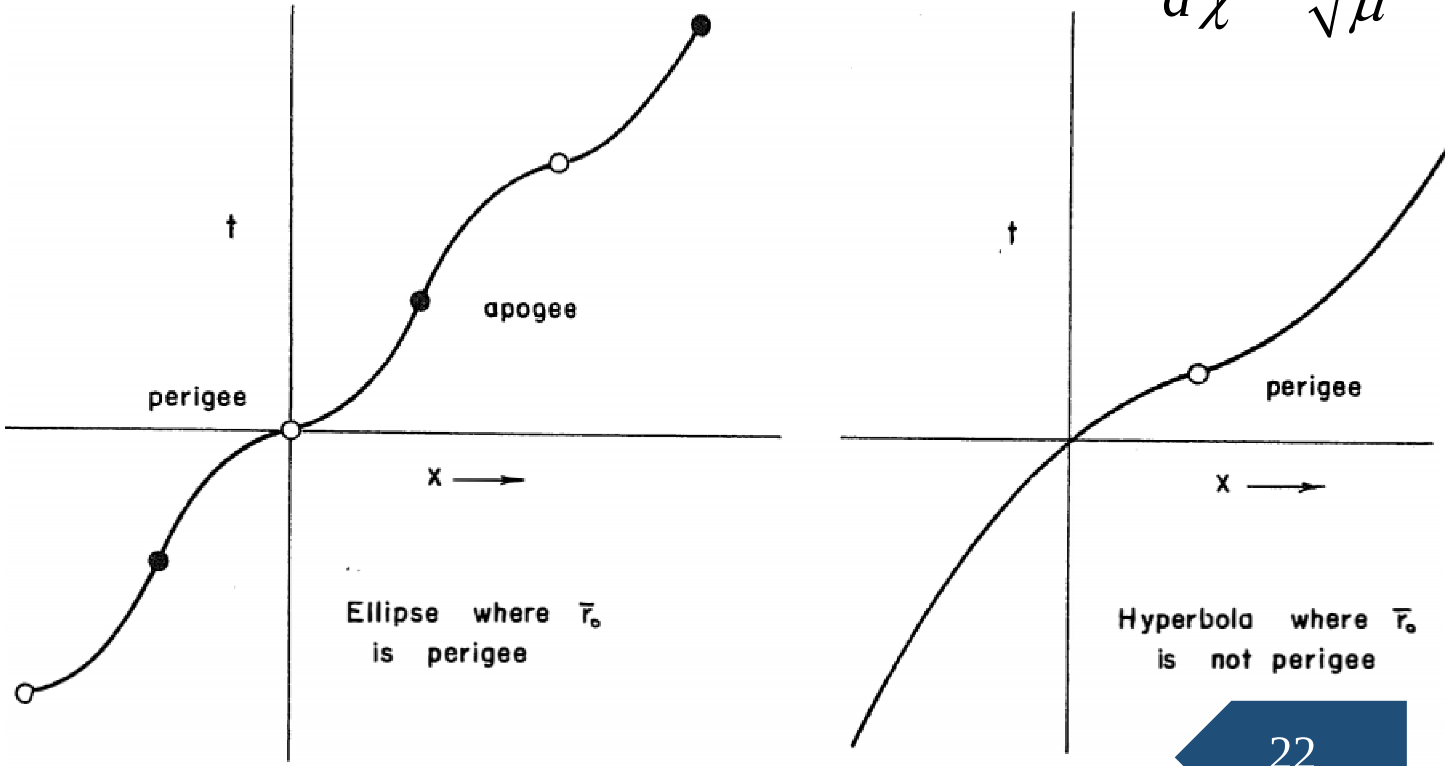
$$f(\chi)=\chi^3 S(\psi)+\frac{\mathbf{r}_0.\mathbf{v}_0}{\sqrt{\mu}}\chi^2 C(\psi)+r_0\chi[1-\psi S(\psi)]-\sqrt{\mu}\Delta t=0$$

$$\frac{dt}{d\chi}=\frac{r}{\sqrt{\mu}}$$

$$\chi_{k+1}=\chi_k-\frac{f(\chi_k)}{f'(\chi_k)}$$

$$\chi_0=\frac{\sqrt{\mu}}{|a|}\Delta t$$

$$\frac{d\psi}{d\chi}=\frac{2\chi}{a}$$



Typical t vs. χ plots

Universal Variables

- Finding $\boldsymbol{r}$ and $\mathbf{v}$

$$\mathbf{r} = f\mathbf{r_0} + g\mathbf{v_0} \qquad \textit{and} \qquad \mathbf{v} = \dot{f}\mathbf{r_0} + \dot{g}\mathbf{v_0}$$

where

$$f = 1 - \frac{\chi^2}{r_0} C(\psi)$$

$$g = t - \frac{\chi^3}{\sqrt{\mu}} S(\psi)$$

$$\dot{f} = \frac{\sqrt{\mu}}{r r_0} \chi(\psi S(\psi) - 1)$$

$$\dot{g} = 1 - \frac{\chi^2}{r} C(\psi)$$

Example

- An earth satellite moves in the x-y plane of an inertial frame with origin at the earth's center. Relative to that frame, the position and velocity of the satellite at time t_0 are

$$\mathbf{r}_0 = 7000.0\hat{\mathbf{i}} - 12,124\hat{\mathbf{j}}\ (\mathrm{km})$$
$$\mathbf{v}_0 = 2.6679\hat{\mathbf{i}} + 4.6210\hat{\mathbf{j}}\ (\mathrm{km/s})$$

Compute the position and velocity vectors of the satellite 60 min later.