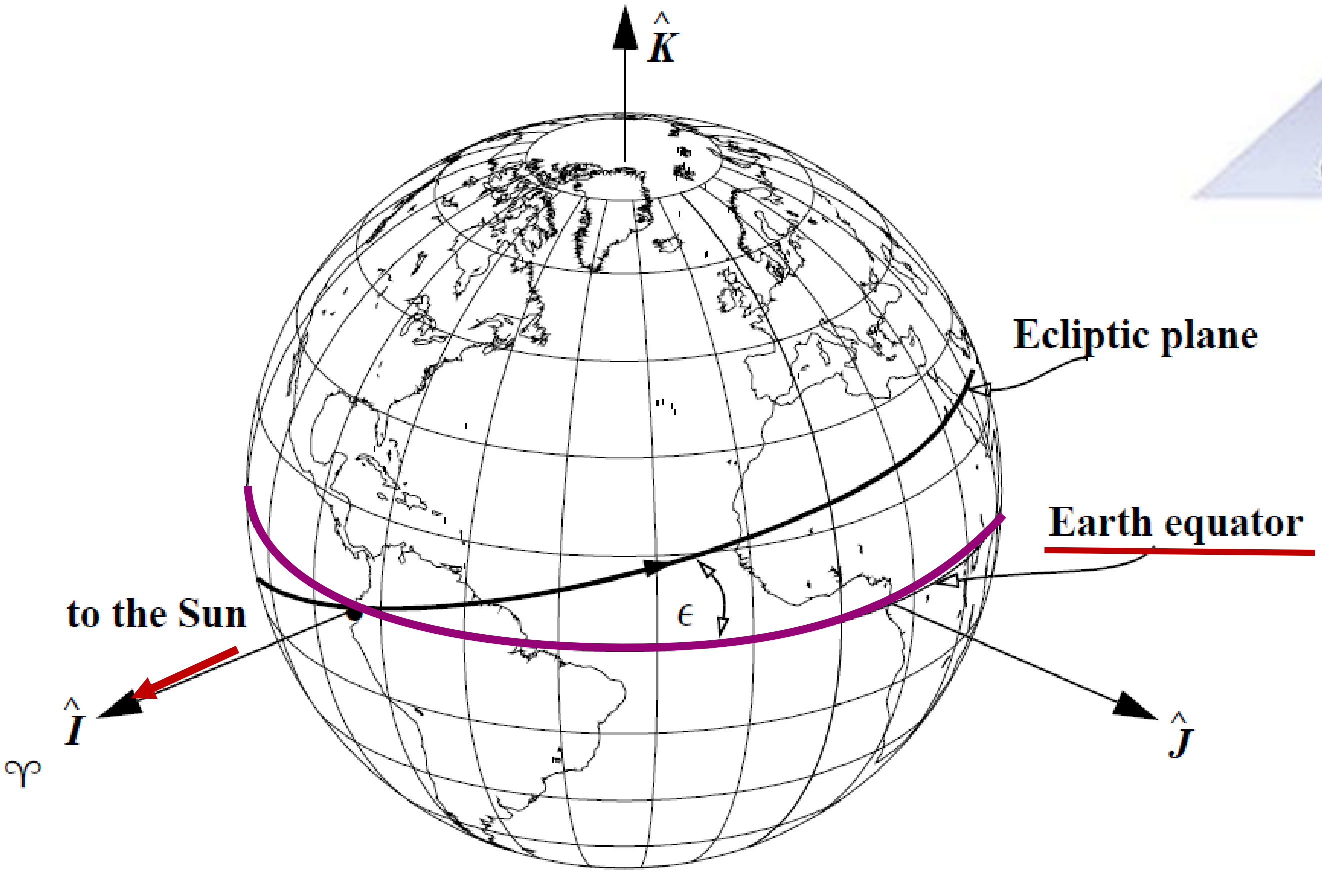
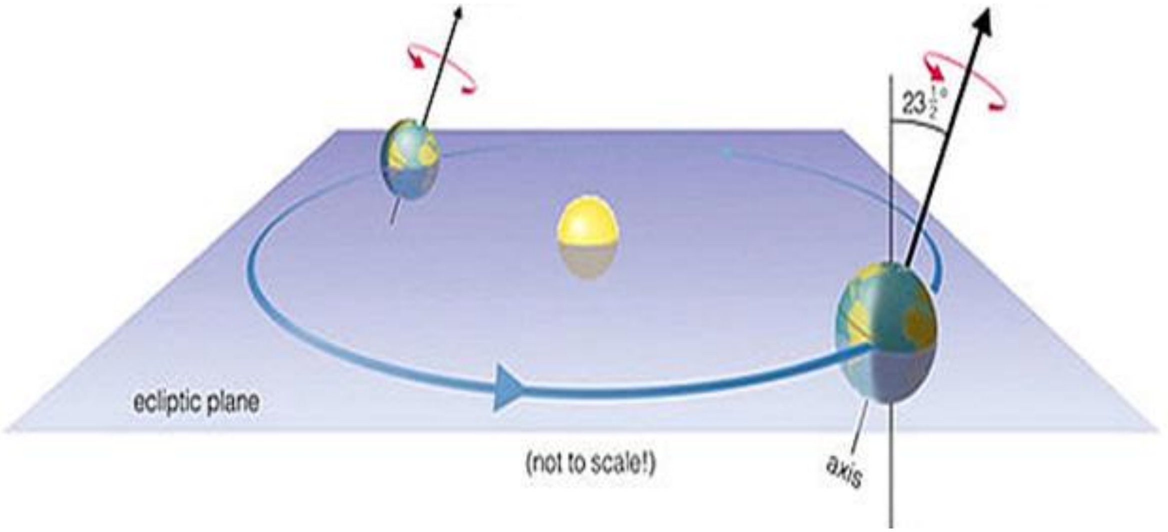
Earth centered Inertial Coordinate Systems





Orbits in Three Dimensions

- True classical element set

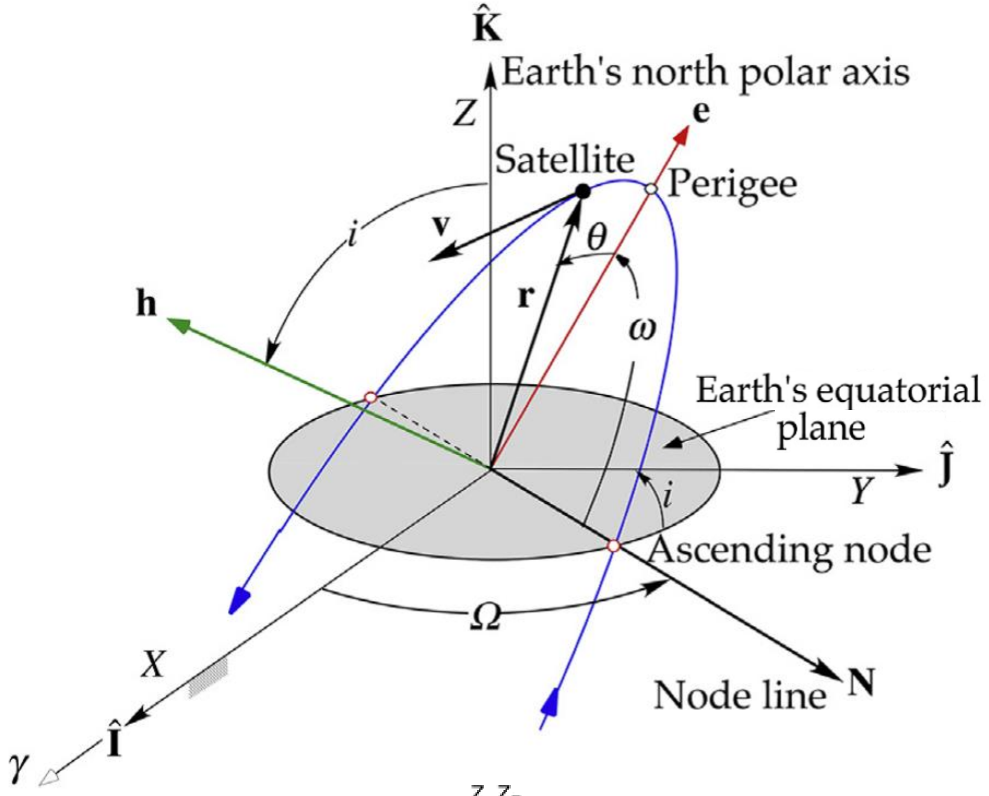
$$\mathbf{X} = [a \quad e \quad i \quad \Omega \quad \omega \quad t]^T$$

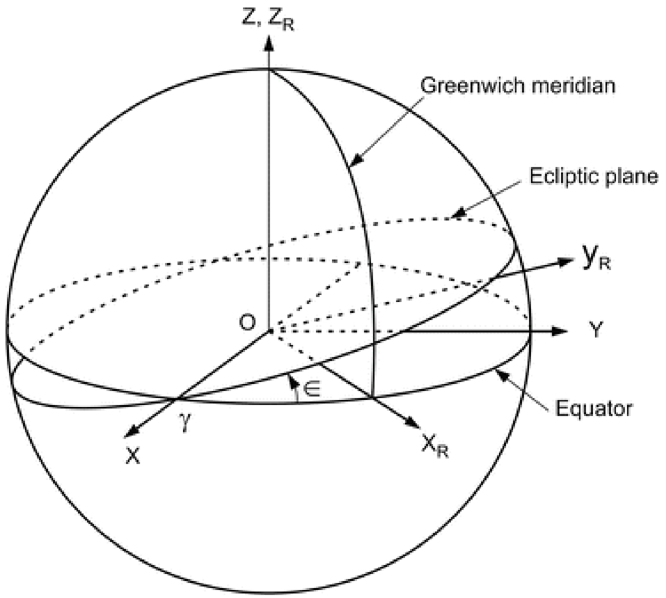
- Modified classical element set

$$\mathbf{X} = [a \quad e \quad i \quad \Omega \quad \omega \quad \theta]^T$$

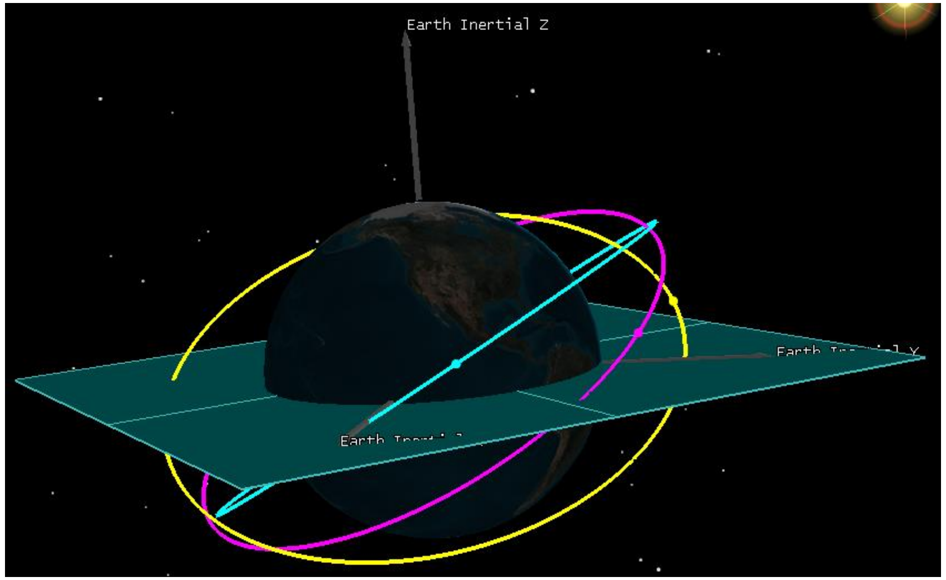
or

$$\mathbf{X} = [a \quad e \quad i \quad \Omega \quad \omega \quad M]^T$$

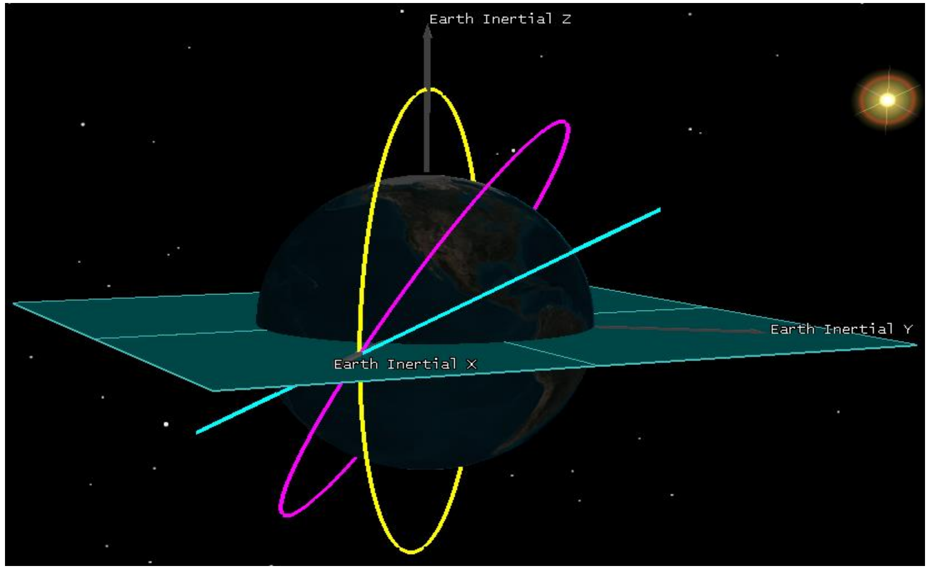




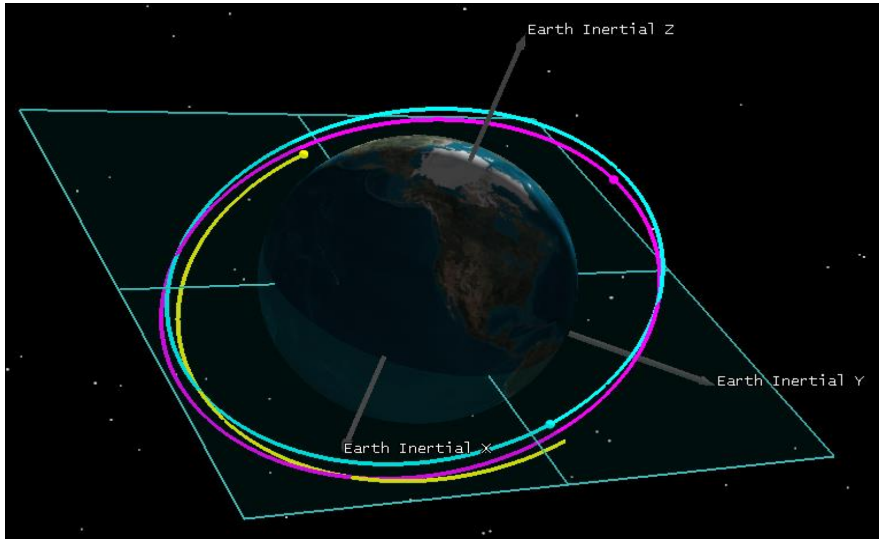
Orbits in Three Dimensions



RAAN



Inclination



Argument of perigee

Orbits in Three Dimensions

- How to calculate orbital elements in terms of Cartesian coordinates?

($\mathbf{r} = [X \quad Y \quad Z]^T, \mathbf{v} = [v_X \quad v_Y \quad v_Z]^T$ are given, orbital elements?)

$$\mathbf{X} = \boldsymbol{\alpha} = [\underbrace{a \quad e}_{\text{Shape}} \quad \underbrace{i \quad \Omega \quad \omega}_{\text{Orientation}} \quad \theta]^T$$

h

Position, Reference time

$\theta \equiv M \equiv t$

1. $r = \sqrt{\mathbf{r}.\mathbf{r}} = \sqrt{X^2 + Y^2 + Z^2}$

$$v = \sqrt{\mathbf{v}.\mathbf{v}} = \sqrt{{v_X}^2 + {v_Y}^2 + {v_Z}^2}$$

$$v_r = \mathbf{r}.\mathbf{v} / r = \left(Xv_X + Yv_Y + Zv_Z\right) / r$$

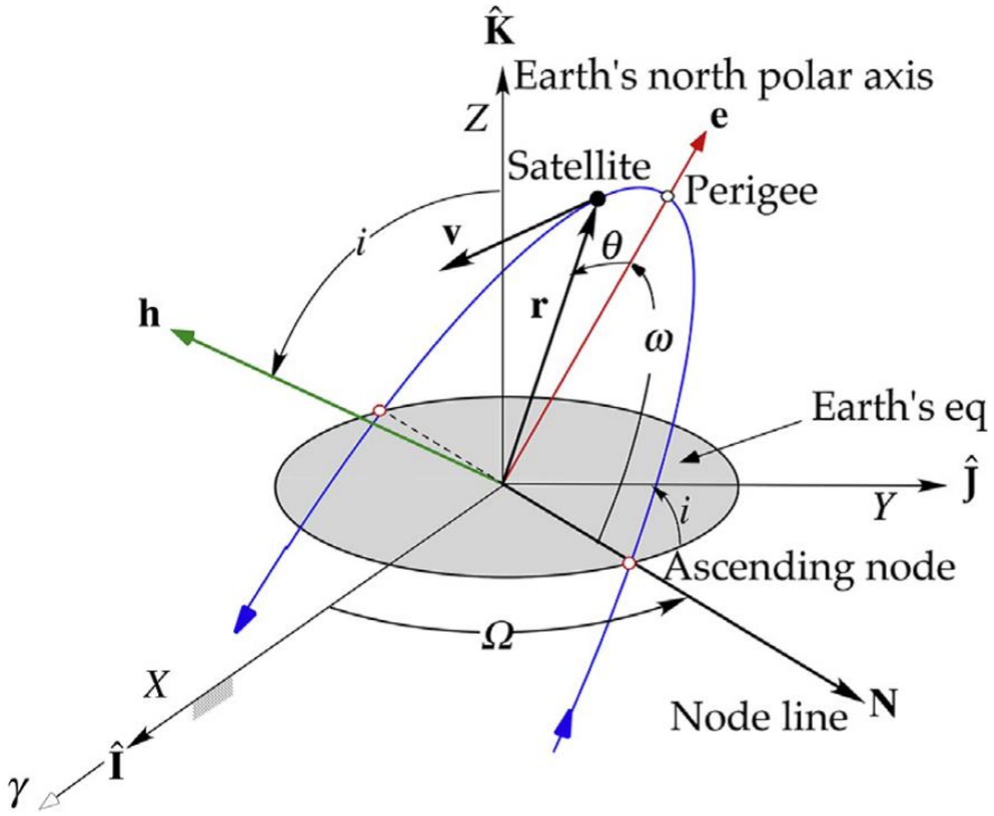
Orbits in Three Dimensions

$$2.\ \mathbf{h}=\mathbf{r}\times\mathbf{v}=\begin{vmatrix}\hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}}\\ X & Y & Z\\ v_X & v_Y & v_Z\end{vmatrix}\quad ,\ h=\sqrt{\mathbf{h}.\mathbf{h}}$$

$$\frac{v^2}{2}-\frac{\mu}{r}=-\frac{\mu}{2a}\rightarrow a=\frac{\mu}{\frac{2\mu}{r}-v^2}$$

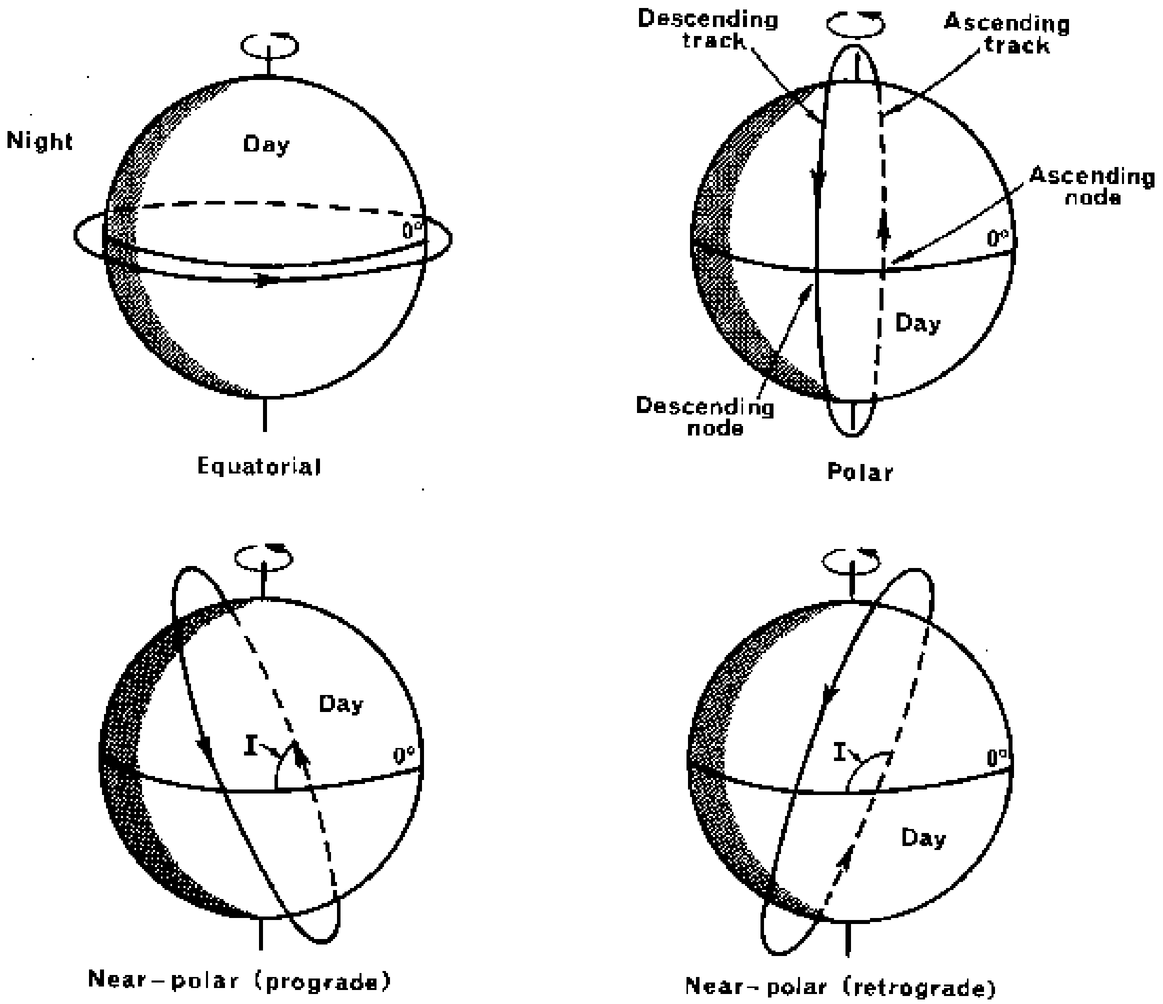
$$3.\ \mathbf{e}=\frac{1}{\mu}\left[\left(v^2-\frac{\mu}{r}\right)\mathbf{r}-rv_r\mathbf{v}\right]\quad\rightarrow e=\sqrt{\mathbf{e}.\mathbf{e}}$$

$$4.\ i=\cos^{-1}\left(\frac{h_Z}{h}\right)$$



Orbits in Three Dimensions

- Equatorial Orbits: $i \approx 0°$
  - GEO
- Prograde/Direct Orbits: $0 < i < 90°$
- Polar Orbits: $i \approx 90°$
- Retrograde Orbits: $90° < i < 180°$

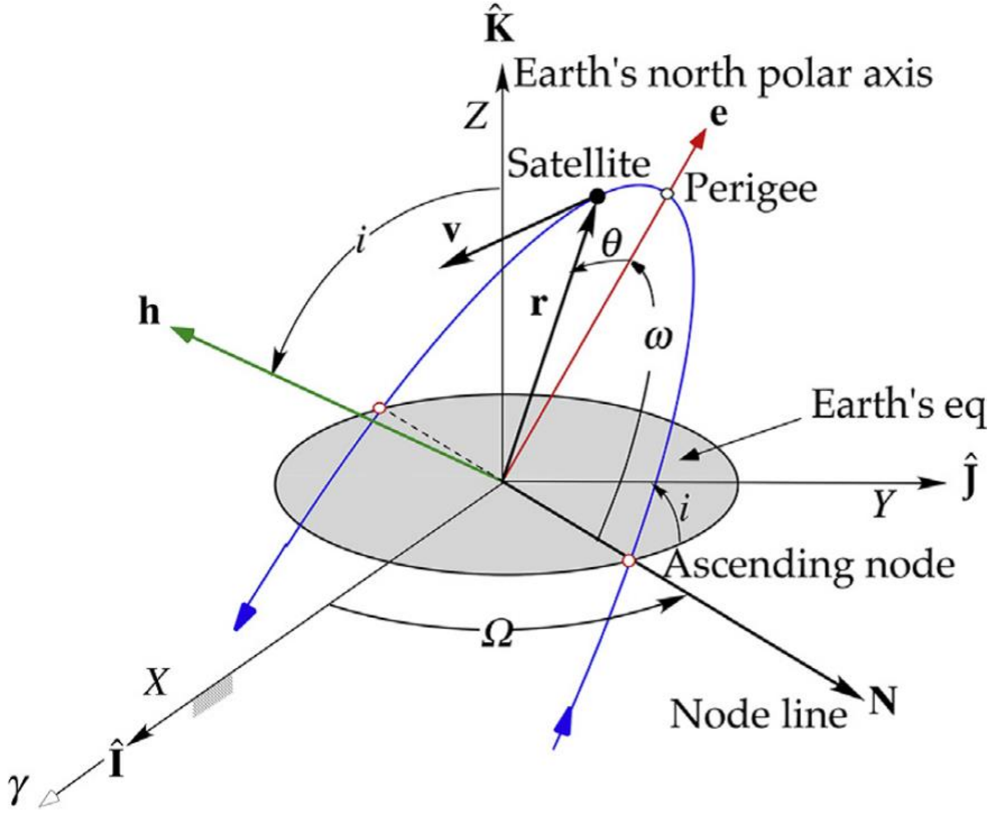


Orbits in Three Dimensions

$$5.\ \ \mathbf{N} = \hat{\mathbf{K}} \times \mathbf{h} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ 0 & 0 & 1 \\ h_X & h_Y & h_Z \end{vmatrix}$$

$$\mathbf{N}.\hat{\mathbf{I}} = N\cos\Omega \qquad \textit{where} \qquad N = \sqrt{\mathbf{N}.\mathbf{N}}$$

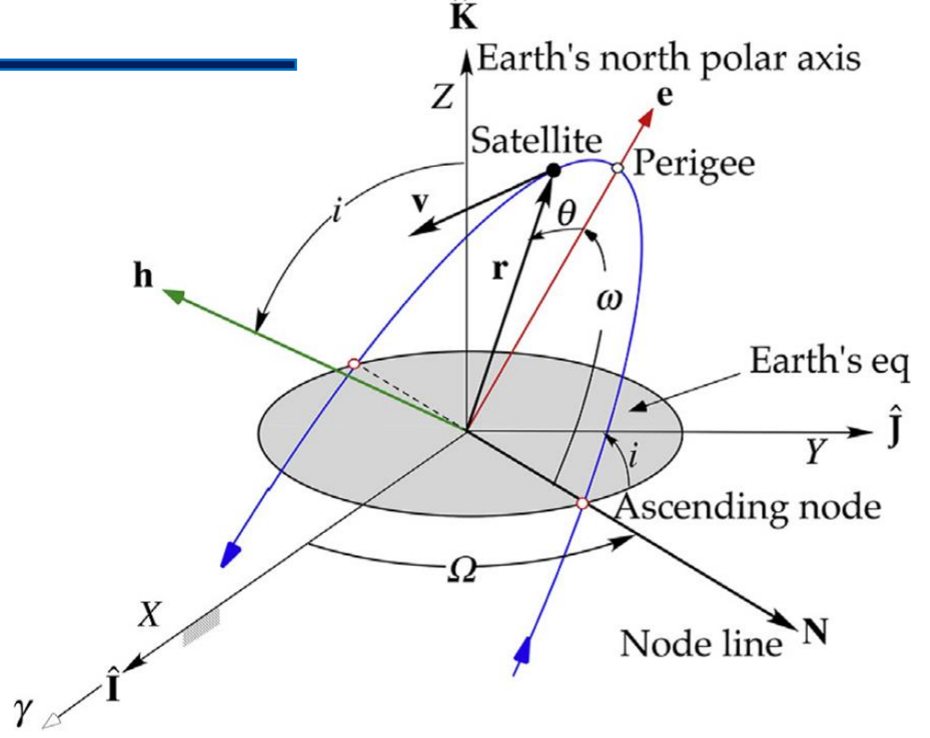
$$\Omega = \begin{cases} cos^{-1}\left(\dfrac{N_X}{N}\right) & (N_Y \geq 0) \\ 360° - cos^{-1}\left(\dfrac{N_X}{N}\right) & (N_Y < 0) \end{cases}$$



Orbits in Three Dimensions

$$6.\ \ \mathbf{N.e} = Ne\cos\omega \quad \rightarrow \quad \omega = \begin{cases} \cos^{-1}\left(\dfrac{\mathbf{N.e}}{Ne}\right) & e_Z \geq 0 \\ 360° - \cos^{-1}\left(\dfrac{\mathbf{N.e}}{Ne}\right) & e_Z < 0 \end{cases}$$

$$7.\ \ \mathbf{r.e} = re\cos\theta \quad \rightarrow \quad \theta = \begin{cases} \cos^{-1}\left(\dfrac{\mathbf{e}}{e}.\dfrac{\mathbf{r}}{r}\right) & v_r \geq 0 \\ 360° - \cos^{-1}\left(\dfrac{\mathbf{e}}{e}.\dfrac{\mathbf{r}}{r}\right) & v_r < 0 \end{cases}$$



Special Cases

- Equatorial orbits ($i = 0$)

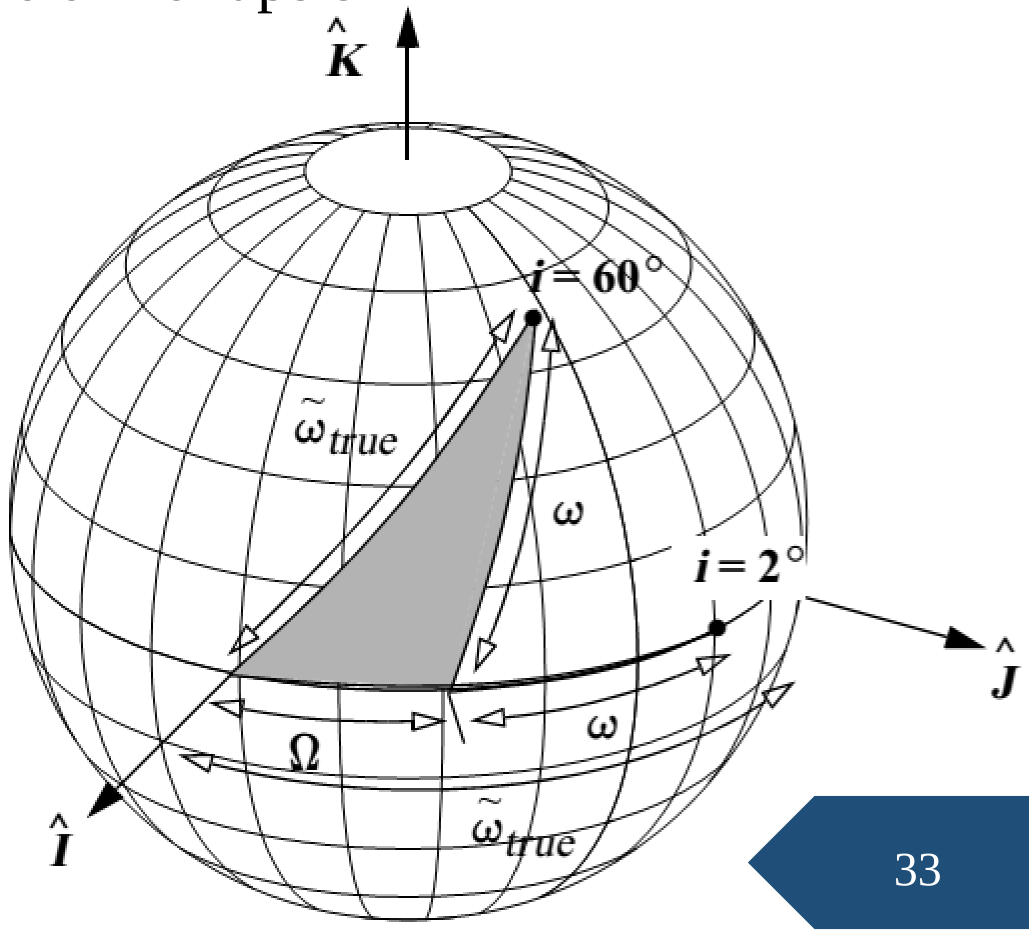
Ω is undefined in this case → True Longitude of Periapsis

Non-circular Equatorial

$$\cos(\tilde{\omega}_{true}) = \frac{\hat{I} \cdot \vec{e}}{|\hat{I}||\vec{e}|}$$

$$\text{IF } (e_J < 0) \text{ THEN } \tilde{\omega}_{true} = 360° - \tilde{\omega}_{true}$$

$$if \;\; i \le 5° \Rightarrow \tilde{\omega} = \Omega + \omega$$



Special Cases

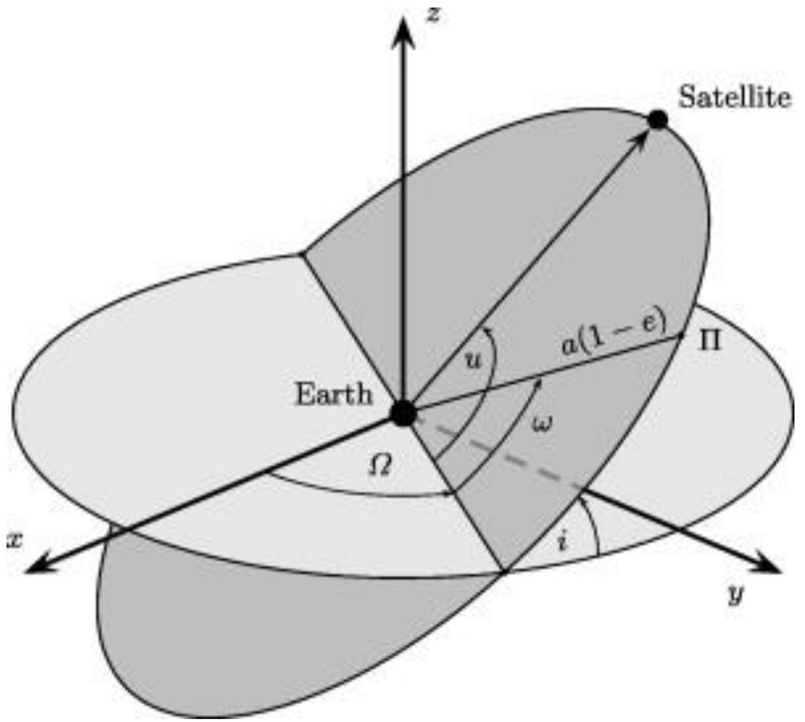
- Circular orbits ($e = 0$)

ω is undefined → Argument of Latitude ($u = \omega + \theta$)

$$\cos(u) = \frac{\vec{n} \cdot \vec{r}}{|\vec{n}||\vec{r}|}$$

$$\text{IF } (r_K < 0) \text{ THEN } u = 360° - u$$

mean argument of latitude: $u_M = \omega + M$



Special Cases

- Circular and Equatorial Orbits $(e = 0, i = 0)$

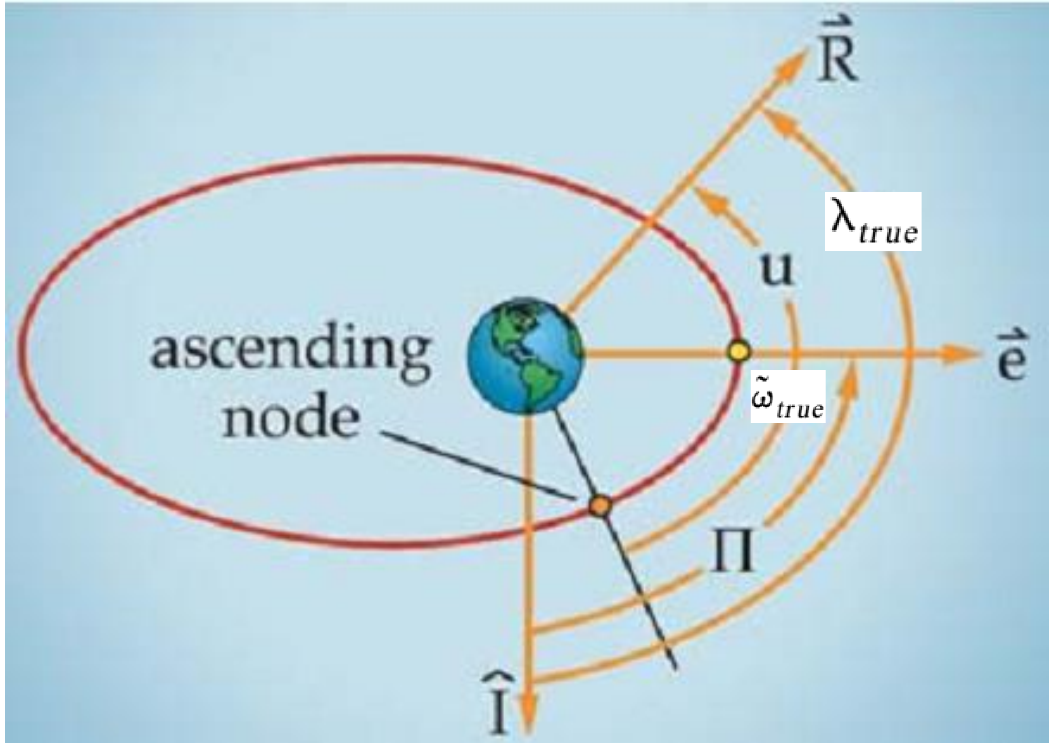
neither ω nor Ω are defined ⟶ True Longitude

$$\cos(\lambda_{true}) = \frac{\hat{I} \cdot \vec{r}}{|\hat{I}||\vec{r}|}$$

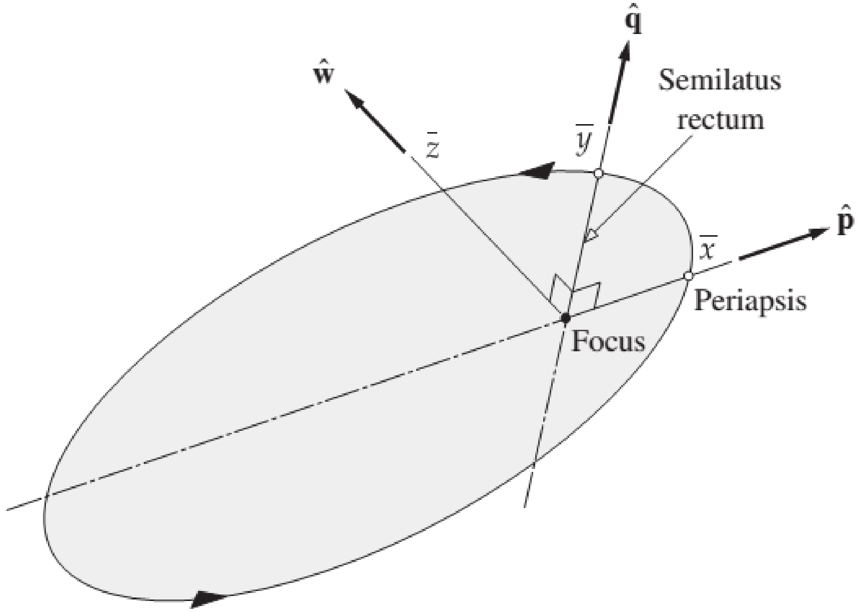
$$\text{IF } (r_J < 0) \text{ THEN } \lambda_{true} = 360° - \lambda_{true}$$

$$\lambda_{true} \approx \Omega + \omega + \theta$$

mean longitude: $\lambda_M = \Omega + \omega + M = \tilde{\omega} + M$



Orbits in Three Dimensions



- Position and Velocity Transformation

$$[\mathbf{r}]^{PCS} = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta}\begin{bmatrix}\cos\theta\\ \sin\theta\\ 0\end{bmatrix}$$

$$[\mathbf{v}]^{PCS} = \frac{\mu}{h}\begin{bmatrix}-\sin\theta\\ e+\cos\theta\\ 0\end{bmatrix}$$

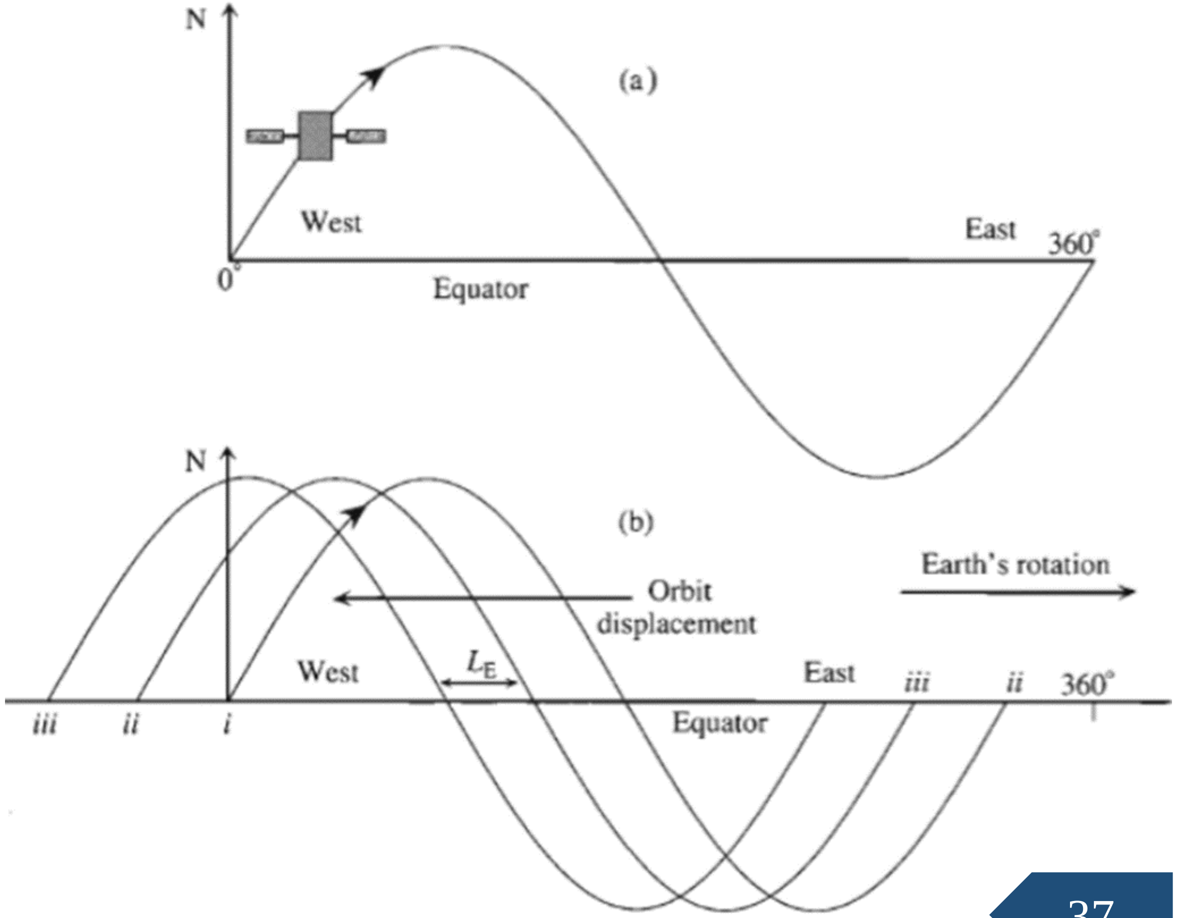
$$[\mathbf{r}]^{ECI} = \mathbf{T}^{ECI-PCS}[\mathbf{r}]^{PCS}$$
$$= (\mathbf{T}^{PCS-ECI})^T[\mathbf{r}]^{PCS}$$
$$[\mathbf{r}]^{PCS} = \mathbf{T}^{PCS-ECI}[\mathbf{r}]^{ECI}$$

$$[\mathbf{v}]^{ECI} = \mathbf{T}^{ECI-PCS}[\mathbf{v}]^{PCS}$$
$$= (\mathbf{T}^{PCS-ECI})^T[\mathbf{v}]^{PCS}$$
$$[\mathbf{v}]^{PCS} = \mathbf{T}^{PCS-ECI}[\mathbf{v}]^{ECI}$$

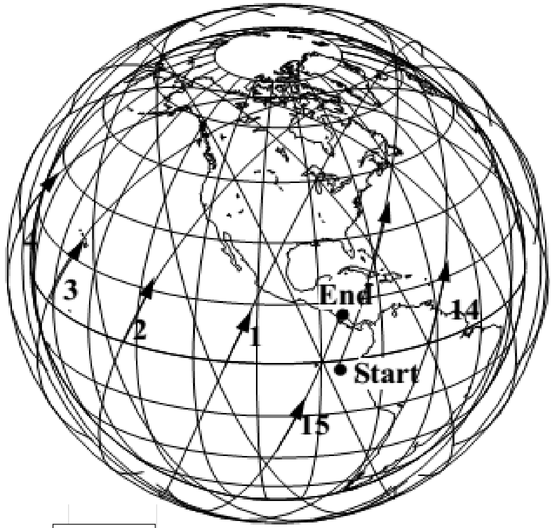
Ground Track

- Ground track (GT) or ground trace is the projection of the satellite's orbit onto the surface of the Earth.

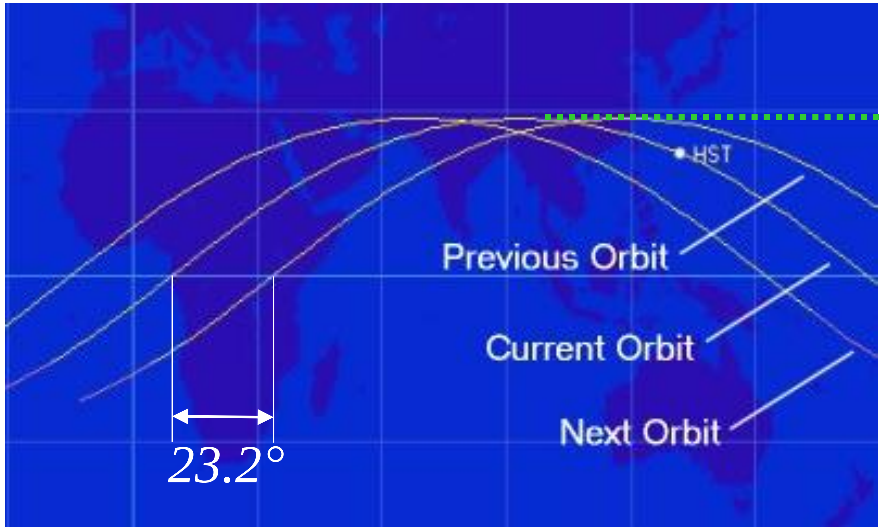
- GT is a powerful tool for determining orbit position and location relative to a ground site.



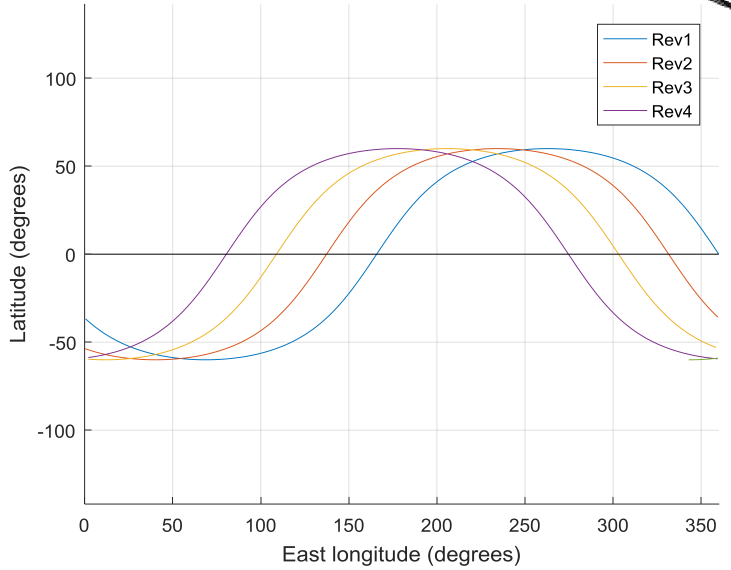
Ground Track



- Revolution vs. pass
- Exploring orbital elements on ground tracks



$i \approx 30°$



$$\begin{cases} T = \dfrac{23.2°}{15.04\ °/hr} = 1.54\ \ hr\ \ (LEO) \\ \\ T = \dfrac{23.2°}{(15.04 - \dot{\Omega})\ °/hr} \end{cases}$$

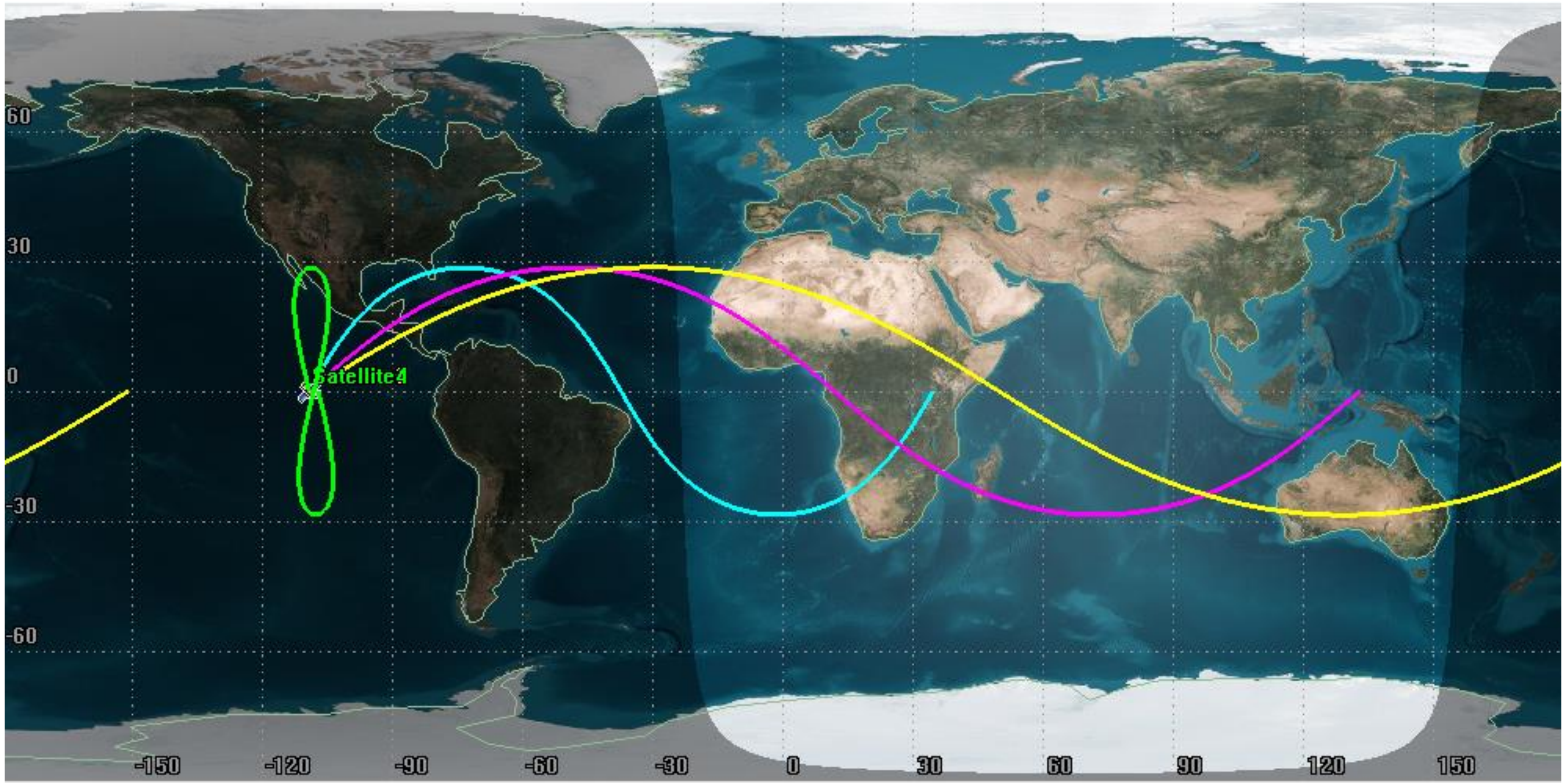
$$T = 2\pi\sqrt{\frac{a^3}{\mu}} \rightarrow a = 6770.3\ km$$

- Retrograde Orbit

  (i=120 deg)

Ground Track

- Effect of the semi-major axis



a = 10000 *km*

a = 20000 *km*

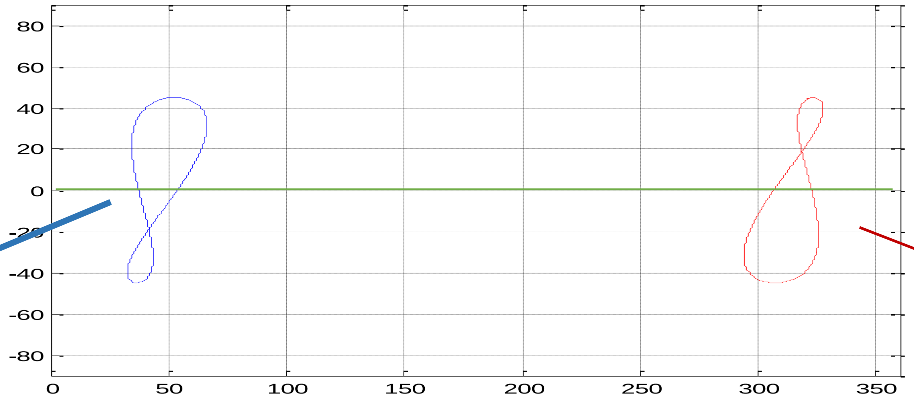
a = 30000 *km*

a = 42000 *km*

Ground Track

- Geosynchronous orbits: Figure eight (8)
  - GEO: a single point on the Earth's equator

- Argument of perigee
  - Zero
  - Non-zero



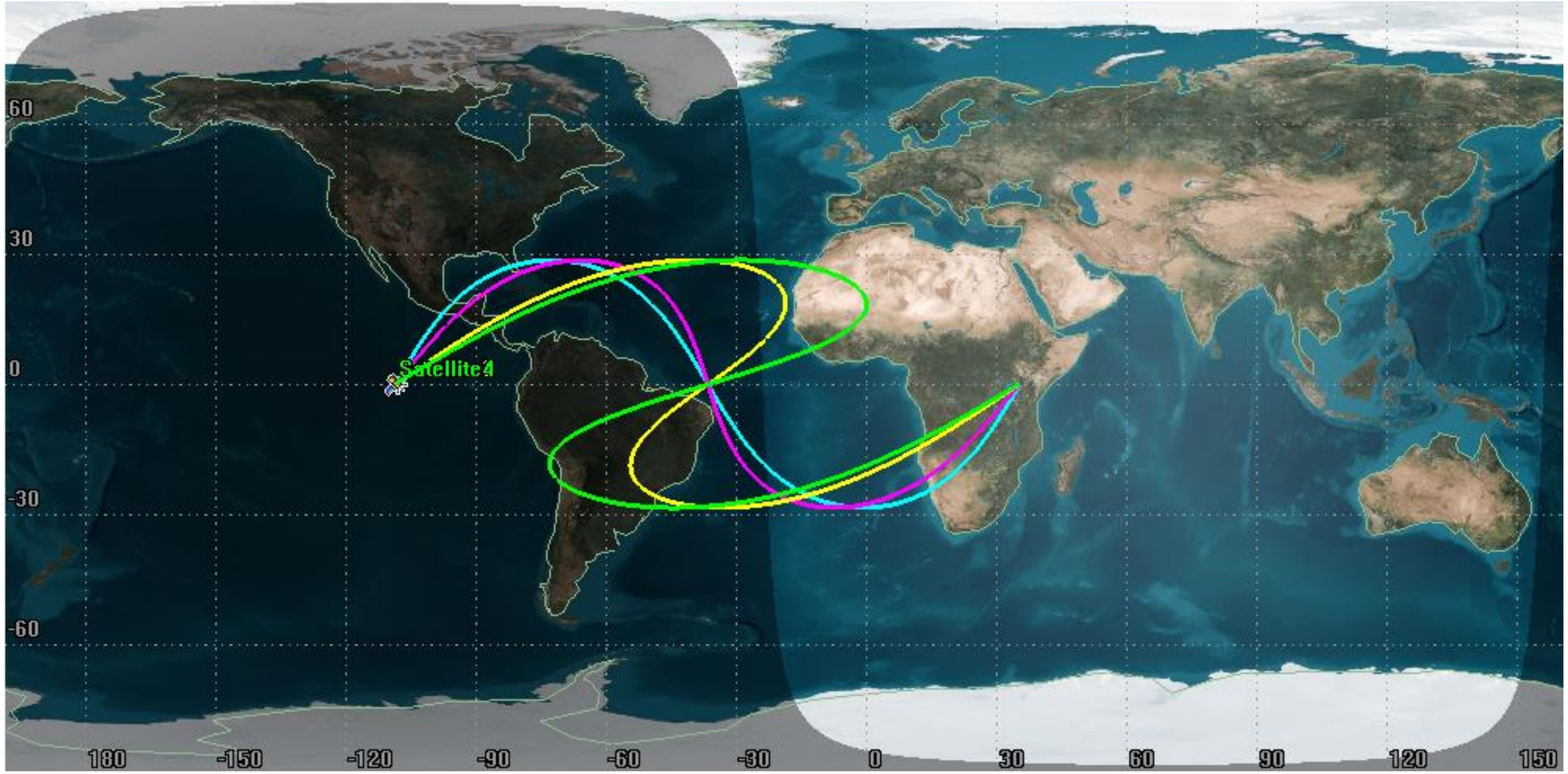
$\omega > 0$

$\omega < 0$

$$T = \frac{360° - \delta\lambda}{15.04\ °/hr}$$

Ground Track

- Effect of the eccentricity



e = 0

e = 0.1

e = 0.5

e = 0.7

Earth-Centered/Earth-Fixed Frame

- Earth-Centered/Earth-Fixed Frame (ECEF)

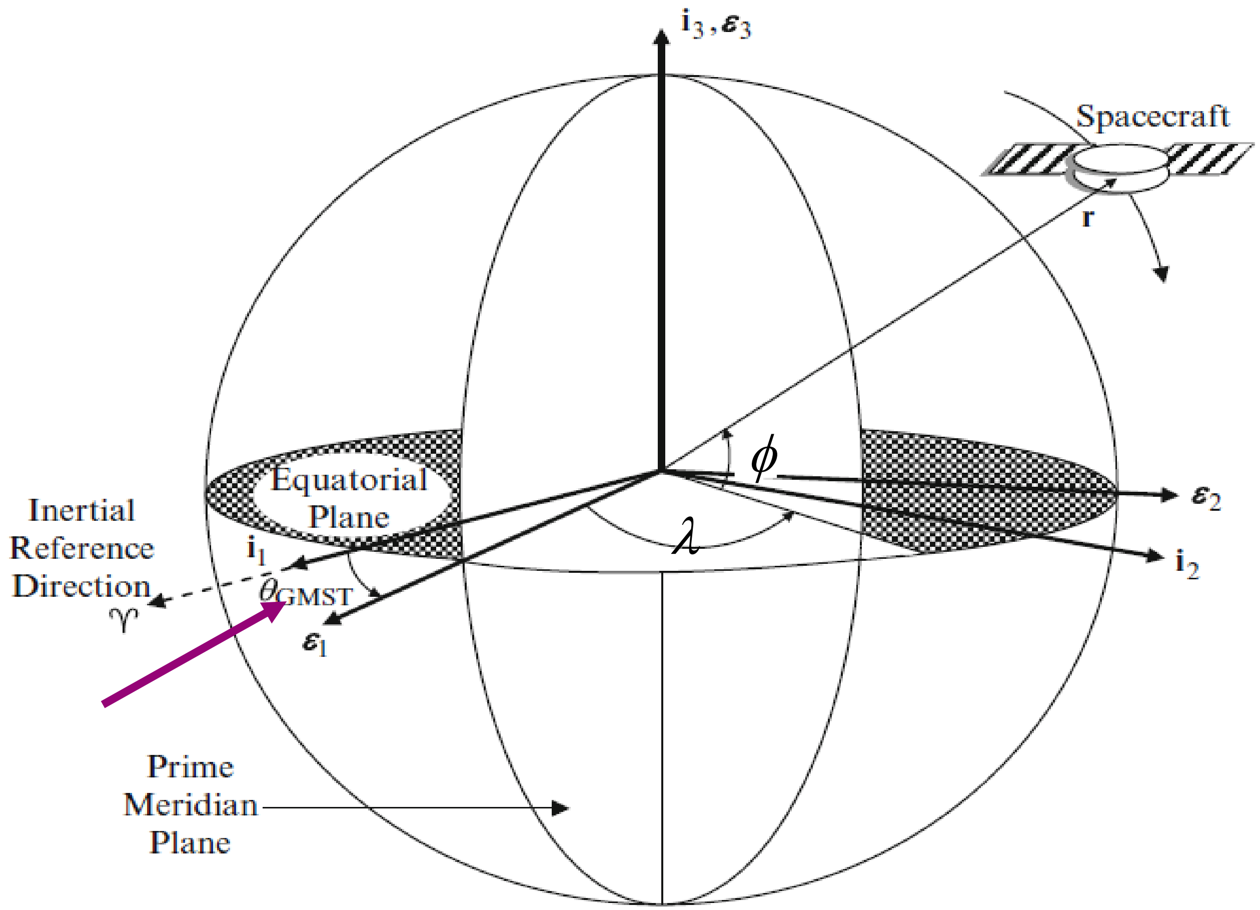
  Origin: c.m. of the Earth,

  Fundamental plane: Equatorial plane,

  preferred direction: Greenwich meridian

  Coordinates:

  - Longitude (λ),
  - Latitude (ϕ)   ?!



Ground Track

- How to plot?

$$[\mathbf{R}_3(\theta_{GMST})] = \begin{bmatrix} \cos\theta_{GMST} & \sin\theta_{GMST} & 0 \\ -\sin\theta_{GMST} & \cos\theta_{GMST} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\theta_{GMST} \approx \omega_E (t - t_0)$$

$$[\mathbf{r}]^{ECEF} = [\mathbf{R}_3(\theta_{GMST})][\mathbf{r}]^{ECI}$$