Home Work #5

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1 Question 1

1.1 Part a

The development of perturbed EOMs around EPs can be found in many classical control text books, where each variable in the EOM is initially replaced by summation of two terms as follows and after the expansion, the nonlinear terms are eliminated while the steady state conditions are removed. In our case:

$$x = x_e + \delta x, y = y_e + \delta y, z = \delta z$$

Additionally, for simplicity we restrict our analysis in the xy plane where the primaries are located. The resulting perturbed EOM (about EPs) will be:

$$\begin{cases} \delta\ddot{x} - 2\delta\dot{y} - \delta x &= -A\delta x + B\delta y \\ \delta\ddot{y} + 2\delta\dot{x} - \delta y &= C\delta x - D\delta y \end{cases}, \quad \text{where:} \begin{cases} A &= (1-\mu)\left(r_{1e}^{-3} - 3A_1\right) + \mu\left(r_{2e}^{-3} - 3A_2\right) \\ B &= 3(1-\mu)B_1 + 3\mu B_2 \\ C &= 3(1-\mu)B_1 + 3\mu B_2 \\ D &= (1-\mu)\left(r_{1e}^{-3} - 3C_1\right) + \mu\left(r_{2e}^{-3} - 3A_2\right) \end{cases} \tag{1}$$

where:

$$A_1 = \frac{(x_e-\mu)^2}{r_{1e}^5}, \qquad A_2 = \frac{(x_e-\mu+1)^2}{r_{2e}^5}$$
$$B_1 = \frac{(x_e-\mu)y_e}{r_{1e}^5}, \qquad B_2 = \frac{(x_e-\mu+1)y_e^2}{r_{2e}^5}$$
$$C_1 = \frac{y_e^2}{r_{1e}^5}, \qquad C_2 = \frac{y_e^2}{r_{2e}^5}$$

As expressed above, the constants A, B, C and D can be determined (based on the EPs coordinates) and the three body system under consideration.

In the canonical system, the L4 coordinates are given below. Putting this information in the perturbed EOMs yields:

$$L4 = \begin{cases} x_e = \mu - 0.5 \\ y_e = \dfrac{\sqrt{3}}{2} \end{cases}, \quad \begin{cases} r_{1e} = 1 \\ r_{2e} = 1 \end{cases} \tag{2}$$

$$\begin{cases} \delta\ddot{x} - 2\delta\dot{y} - \dfrac{3}{4}\delta x - \dfrac{3\sqrt{3}}{4}\left(\mu - \dfrac{1}{2}\right)\delta y = 0 \\ \delta\ddot{y} + 2\delta\dot{x} - \dfrac{3\sqrt{3}}{2}\left(\mu - \dfrac{1}{2}\right)\delta x - \dfrac{9}{4}\delta y = 0 \end{cases} \tag{3}$$

where:

$$R_1\delta\ddot{r} + R_2\delta\dot{r} + R_3\delta r = 0 \tag{4}$$

where:

$$R_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}, \quad R_3 = \begin{bmatrix} -\dfrac{3}{4} & -\dfrac{3\sqrt{3}}{2}\left(\mu - \dfrac{1}{2}\right) \\ -\dfrac{3\sqrt{3}}{2}\left(\mu - \dfrac{1}{2}\right) & -\dfrac{9}{4} \end{bmatrix} \tag{5}$$

One can assume an exponential solution to determine the system characteristic equation, out of which the eigenvalues can be determined for stability analysis. So let $\delta r = ae^{\lambda t}$ where a is a constant matrix. Substituting this solution in the perturbed EOMs for L4, will give:

$$\begin{vmatrix} \lambda^2 - \dfrac{3}{4} & 2\lambda - \dfrac{3\sqrt{3}}{2}\left(\mu - \dfrac{1}{2}\right) \\ 2\lambda - \dfrac{3\sqrt{3}}{2}\left(\mu - \dfrac{1}{2}\right) & \lambda^2 - \dfrac{9}{4} \end{vmatrix} a = 0 \tag{6}$$

The expression is an eigenvalue or eigenvector problem. After simplifying, the determinant of the above equation one would get:

$$\lambda^4 + \lambda^2 - \frac{27}{4}\mu(\mu - 1) = 0 \xrightarrow{\text{solving for}\lambda^2} \lambda^2 = \frac{-1 \pm \sqrt{1 + 27\mu(\mu - 1)}}{2}, \quad 0 \le \mu \le 1 \tag{7}$$

Neutral stability requires all λ's to be imaginary, as it is not possible to obtain four roots with negative real parts for L4. Then for stability it must be:

$$\mu < \mu_1 = 0.03852, \quad \mu > \mu_2 = 0.96148 \tag{8}$$

In jupter system, $\mu = 0.00095$ which is less than μ_1 and therefore L4 is stable in this system.

1.2 Part b

In this section used Code from github and source code is is Code/Q1 directory. Here is the jaccobi constant plot for the Sun jupter system.

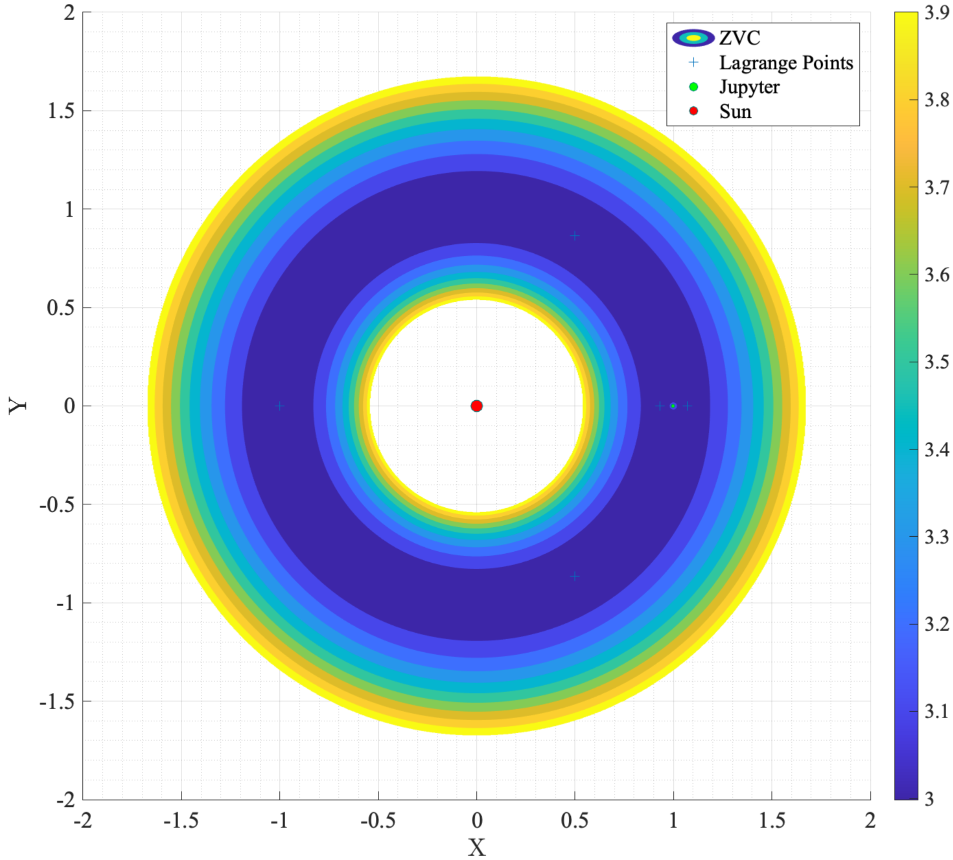

Figure 1: Jaccobi constant plot for the Sun jupter system with fiiled countour for lagrange points

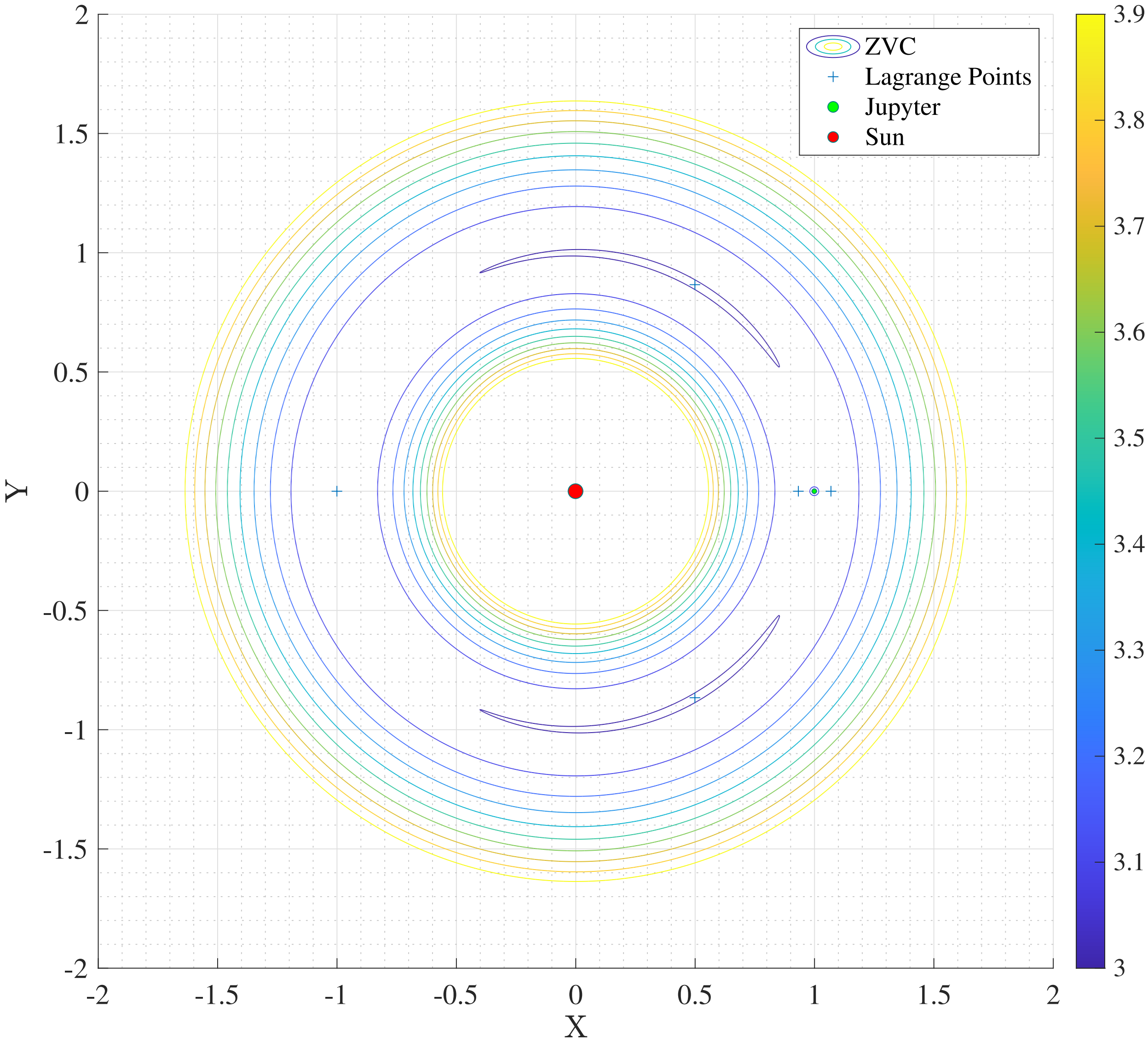

Figure 2: Jaccobi constant plot for the Sun jupter system countour for lagrange points

2 Question 2

Halo orbits are a type of periodic orbit that occur in the three-body problem, where two massive celestial bodies exert gravitational forces on a third, much smaller body. These orbits are characterized by their peculiar shape, which resembles a three-dimensional figure-eight or a halo, hence the name.

The motion of a spacecraft in a halo orbit can be described by the Hill's equations, given by:

$$\ddot{\mathbf{r}} - 2\mathbf{\Omega} \times \dot{\mathbf{r}} - \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) = \frac{G(m_1 + m_2)}{r^3}\mathbf{r} \tag{9}$$

where $\mathbf{r}$ represents the position vector of the spacecraft, $\mathbf{\Omega}$ is the angular velocity vector of the reference frame, m_1 and m_2 are the masses of the primary bodies, and G is the gravitational constant.

X	Y	Z	V_x	V_y	V_z
0.82411	0	0.05625356	0	0.166629	0

Table 1: Halo orbit parameters

Halo orbits exist in the circular restricted three-body problem, where one of the primary bodies is considered massless. The Lagrange points L_1, L_2, and L_3 serve as stable equilibrium points for halo orbits. A halo orbit around L_1 or L_2 can be defined by specifying its amplitude A, in terms of the distance between the primary bodies r_0, and the period T.

The equation for the shape of a halo orbit can be written as:

$$x = A\cos(\theta + \phi_0), \quad y = A\sin(\theta + \phi_0), \quad z = 0 \tag{10}$$

where x, y, and z represent the coordinates of the spacecraft in the rotating frame, θ is the true anomaly, and ϕ_0 is the initial phase angle.

Halo orbits have been of great interest in space exploration, as they provide stable trajectories for spacecraft to observe and study various celestial bodies.

2.1 Howell Method for Halo Orbits

Halo orbits are periodic orbits that occur in the three-body problem, characterized by their peculiar shape resembling a three-dimensional figure-eight or a halo. The Howell formula and the iterative ϕ-matrix method are two commonly used techniques to calculate halo orbits.

The Howell formula is an analytical expression that provides an approximation for the amplitude of a halo orbit. It is given by:

$$A = \left(\frac{m_2}{3\Omega^2}\right)^{1/3} \left(\frac{r_0^2}{4}\right)^{1/3} \tag{11}$$

where m_2 is the mass of the secondary body, Ω is the angular velocity of the reference frame, and r_0 is the distance between the primary bodies.

However, the Howell formula only provides an estimate of the amplitude, and it does not give the full trajectory of the halo orbit. To obtain a more accurate representation, the iterative ϕ-matrix method is often employed.

The iterative ϕ-matrix method involves solving a set of coupled linear equations iteratively to obtain the ϕ-matrix, which describes the linearized motion around the nominal halo orbit. The ϕ-matrix is updated in each iteration until convergence is achieved.

The equation for updating the ϕ-matrix is given by:

$$\phi_{k+1} = \left.\frac{\partial R(\tau)}{\partial y}\right|_{y=y_k} \phi_k \tag{12}$$

where ϕ_k represents the ϕ-matrix at the k-th iteration, $R(\tau)$ is the state transition matrix, and y_k is the state vector at the k-th iteration.

By iterating the above equation until convergence, the ϕ-matrix is obtained, which can then be used to calculate the full trajectory of the halo orbit.

The Howell formula and the iterative ϕ-matrix method are powerful tools for calculating halo orbits, providing approximations and accurate representations, respectively. These methods have been widely used in space mission design and celestial mechanics research.

2.2 Part a

Used Attached code for specific initial conditions and got the following results:

2.3 Part b

Used above attached code and got the following results:

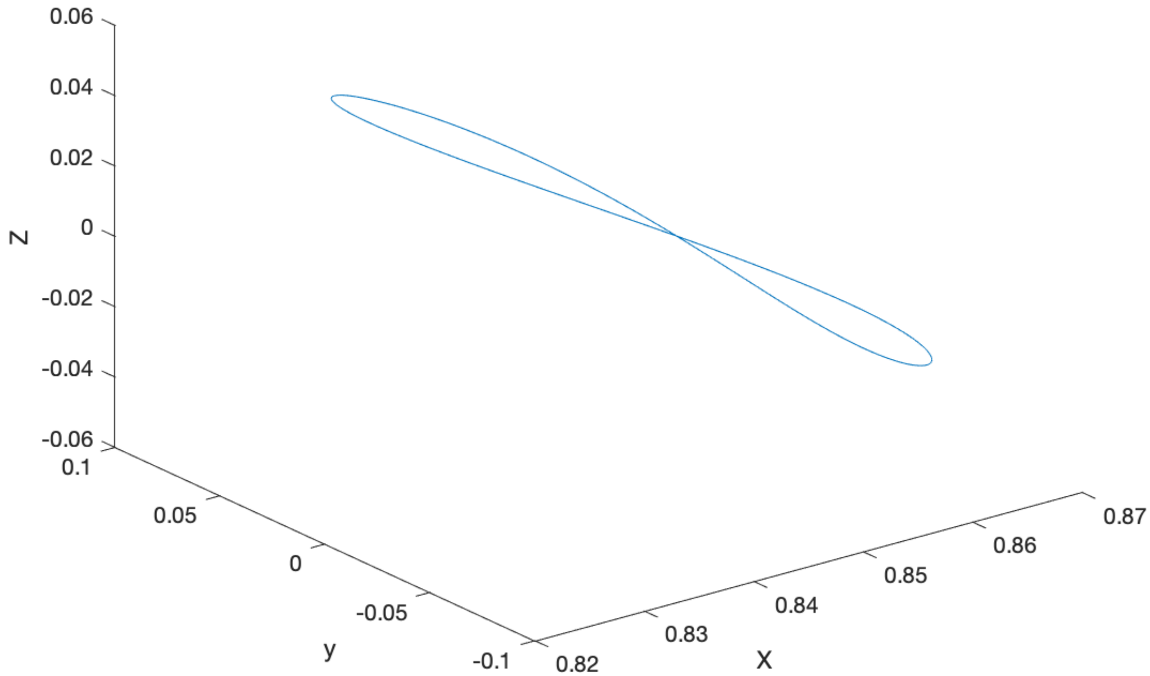

Figure 3: Halo orbit in 3D

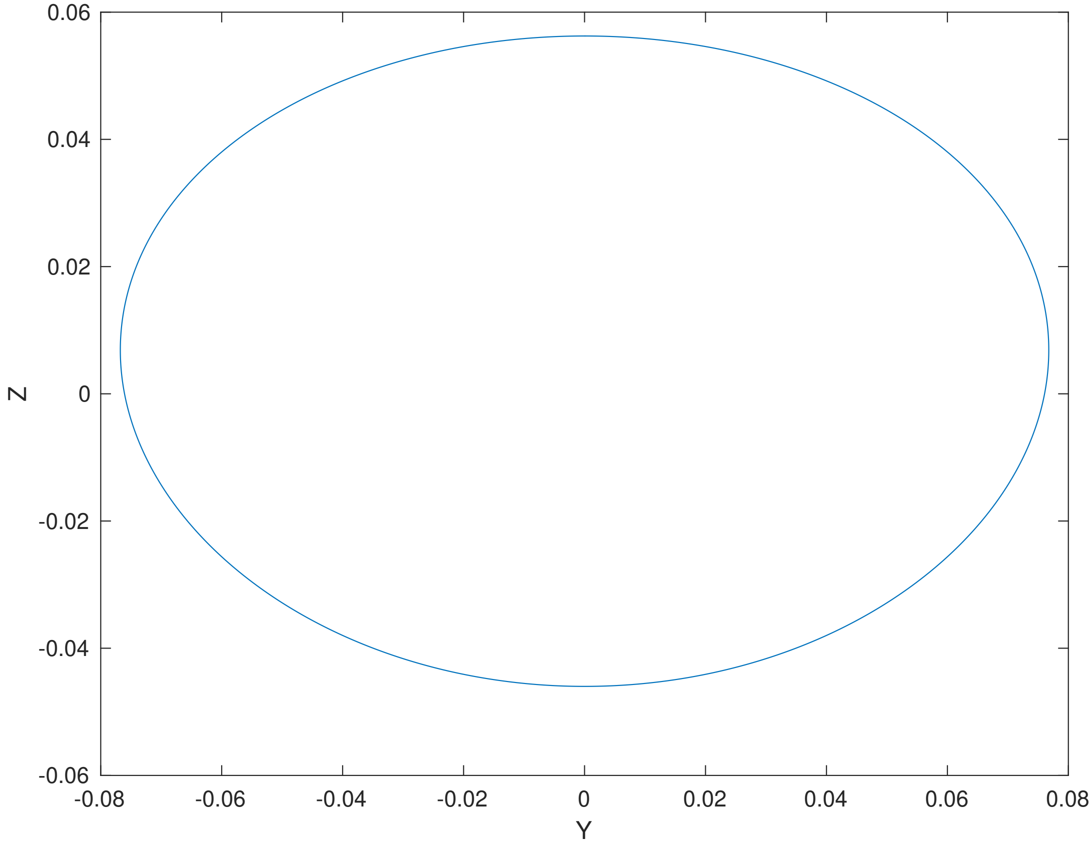

Figure 4: Halo orbit in YZ plane

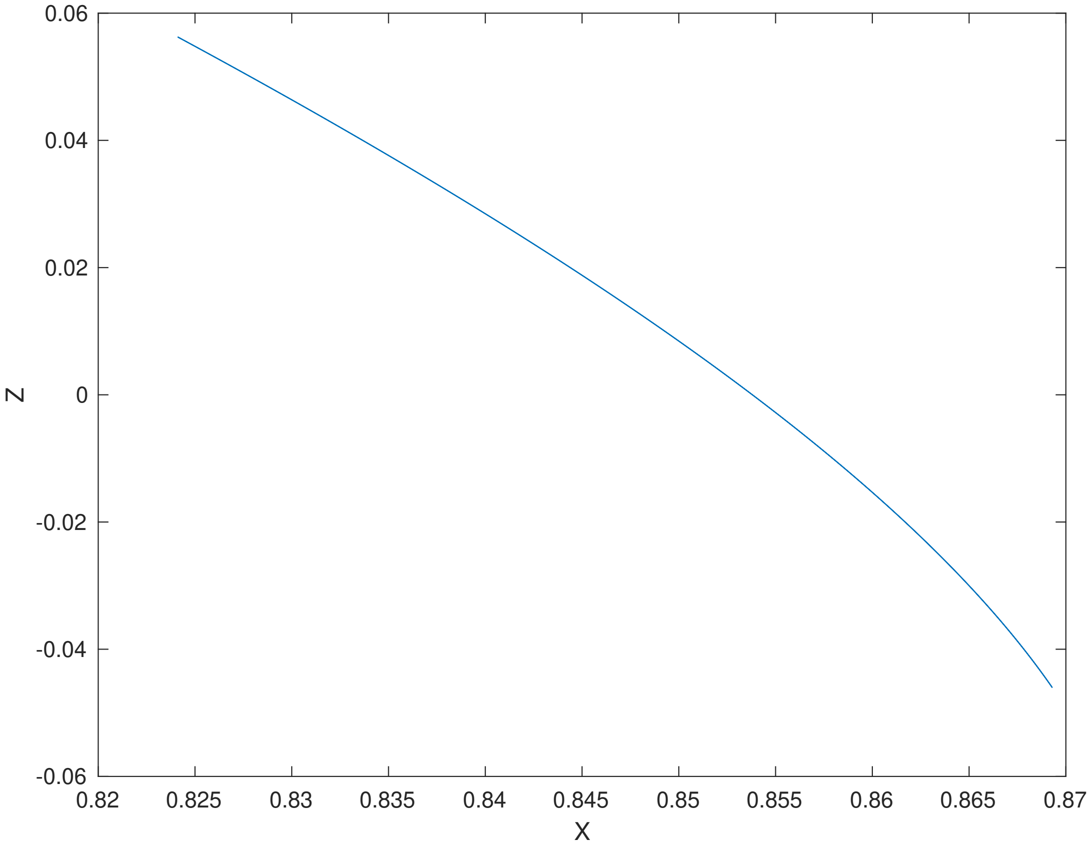

Figure 5: Halo orbit in XZ plane

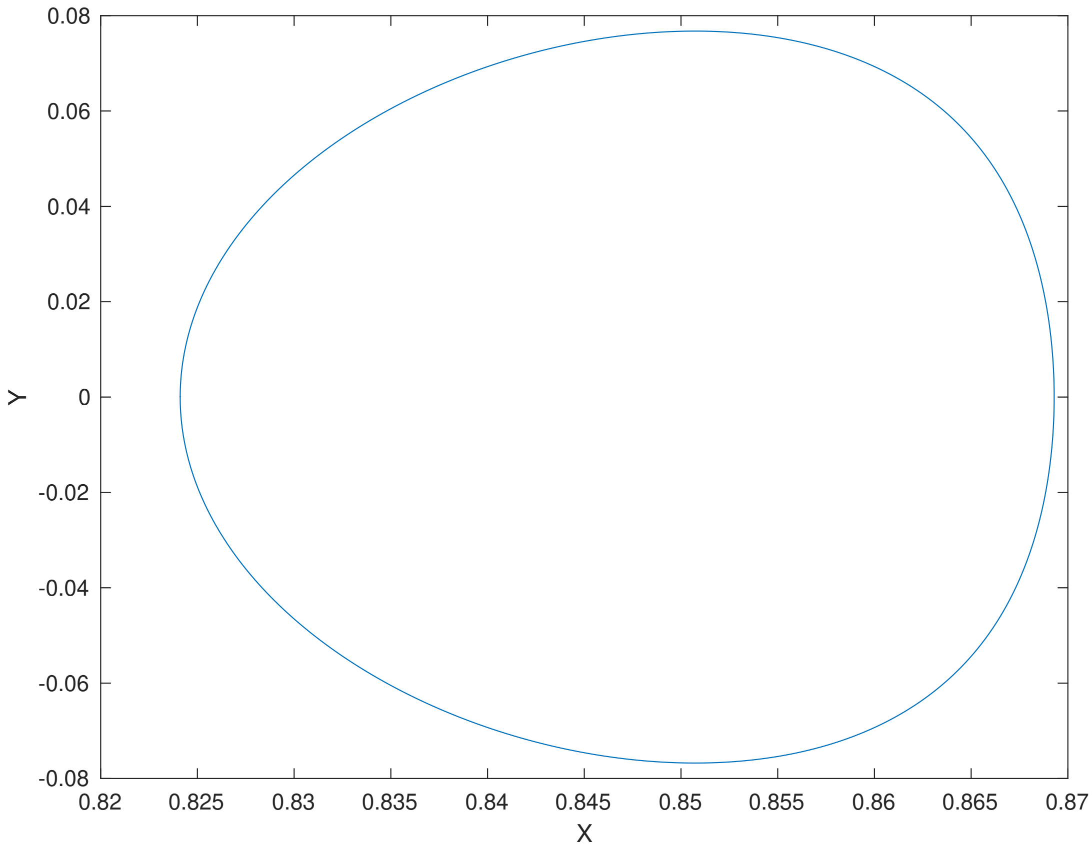

Figure 6: Halo orbit in XY plane

2.4 Part c

In this section for adding perturbation we simulate halo orbit two times with about one percent of period time difference. Then two result are divided on we calculate the effect of perturbation on the orbit. The result is shown in the following figure:

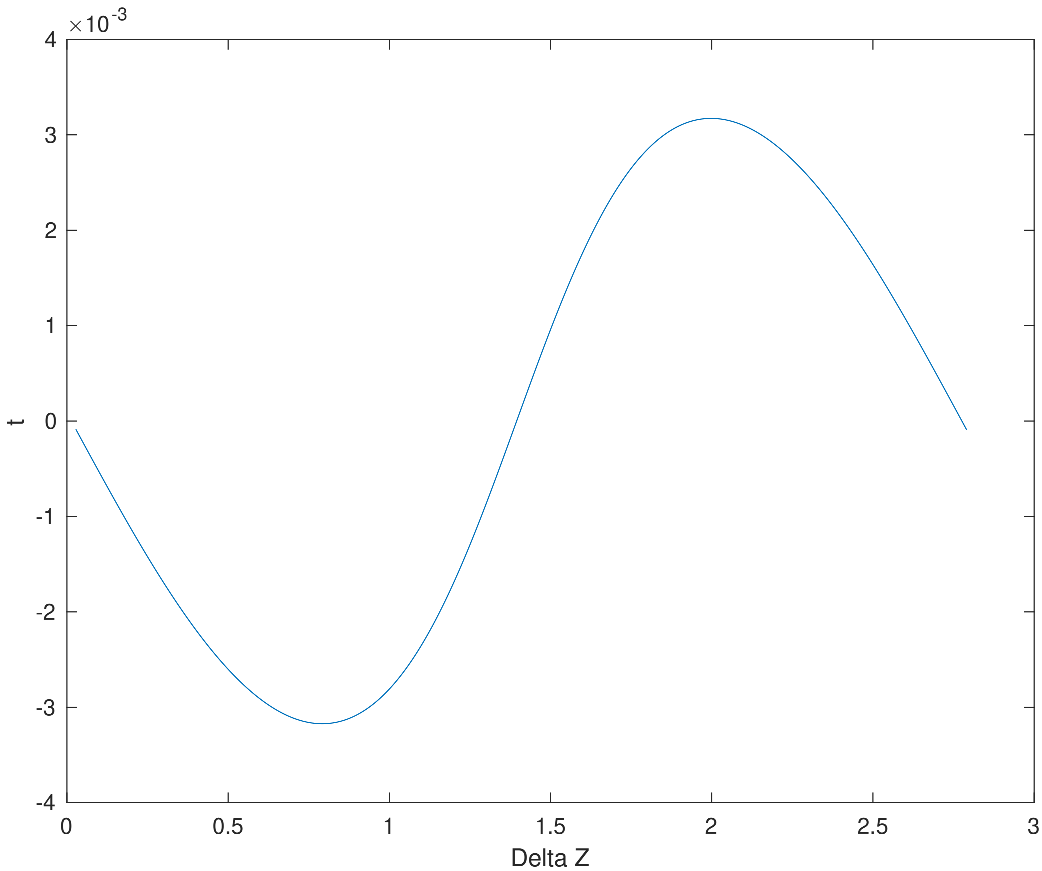

Figure 7: Z difference over time

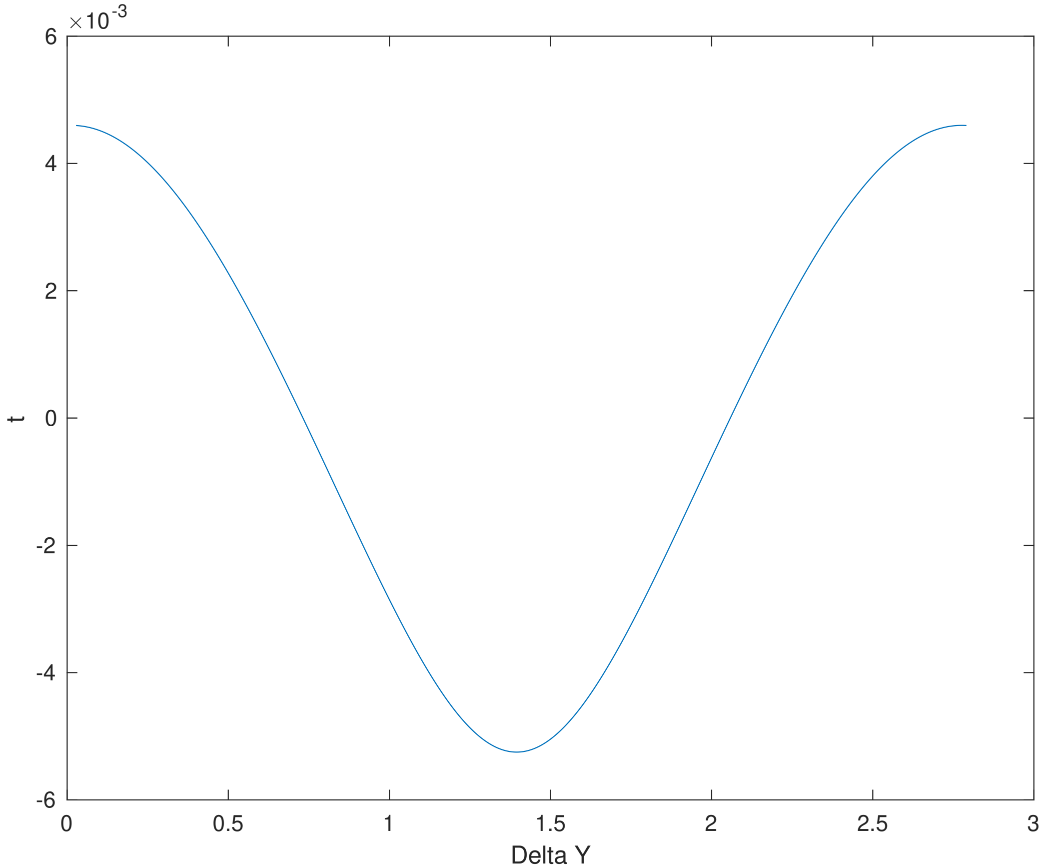

Figure 8: Y difference over time

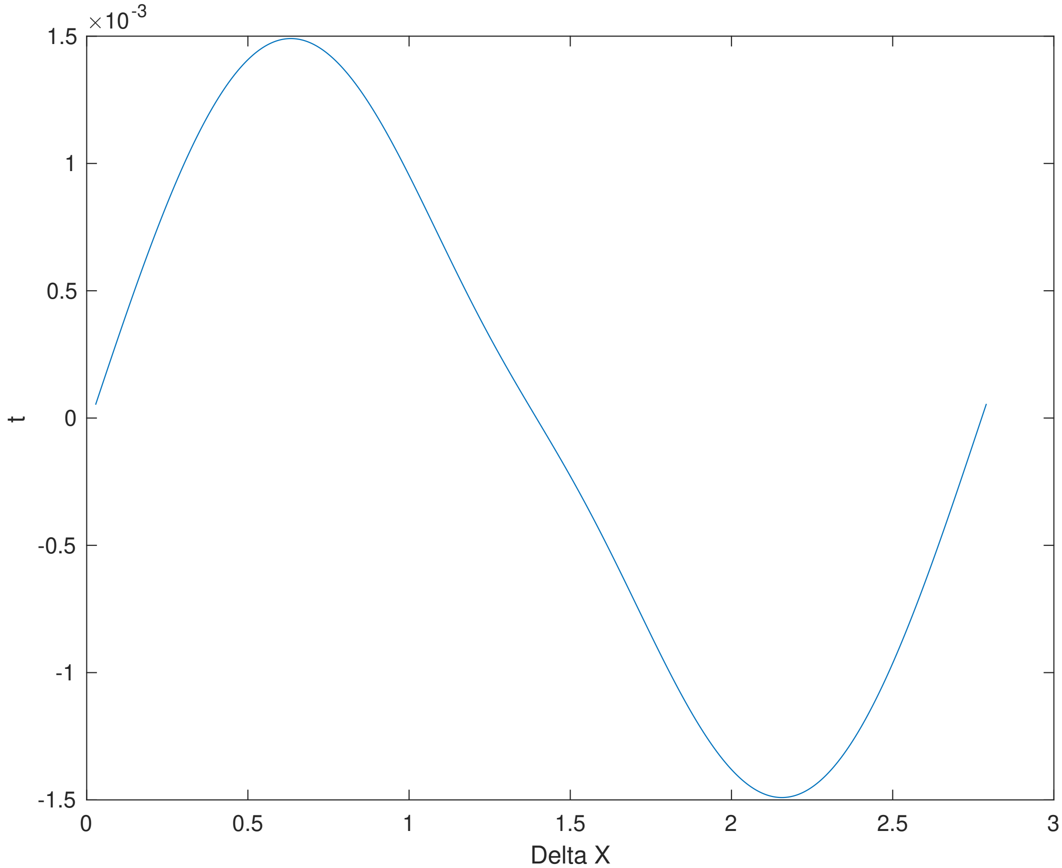

Figure 9: X difference over time

From above result we can findout the orbit have marginal stability and have oscillating behavior.

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