In The Name of God

Sharif University of Technology
Department of Aerospace Engineering

45-784: Advanced Orbital Mechanics

CH#4: Orbit Representation Methods

Satellite State Representations

- We need six quantities to define the translational state of a satellite in space. These quantities may take on many equivalent forms:

- Cartesian (inertial) coordinates

$$\mathbf{X} = \begin{bmatrix} \mathbf{r}^T & \mathbf{v}^T \end{bmatrix}^T = \begin{bmatrix} x & y & z & \dot{x} & \dot{y} & \dot{z} \end{bmatrix}^T$$

- True classical element set

$$\mathbf{X} = \begin{bmatrix} a & e & i & \Omega & \omega & t \end{bmatrix}^T$$

Singular at e = 0 or 1, i = 0 or π

- Modified classical element set

$$\mathbf{X} = \begin{bmatrix} a & e & i & \Omega & \omega & \theta \end{bmatrix}^T$$

M

Singular at e = 0 or 1, i = 0 or π

Satellite State Representations

- Flight elements (ADBARV elements)

Spherical elements	Geographic elements	Flight elements
α	h	α
δ	λ	δ
α_v	ϕ_{gc}	β
δ_v	v	A
r	γ	r
v	A	v

Singular at poles

$$\sin(\alpha) = \frac{y}{\sqrt{x^2 + y^2}}, \quad \sin(\delta) = \frac{z}{r}, \quad r = \sqrt{x^2 + y^2 + z^2}, \quad v = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$$

Satellite State Representations

$$[\mathbf{v}]^{SEZ} = T^{SEZ-GCRF}[\mathbf{v}]^{GCRF} = T^{SEZ-GCRF}\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{bmatrix}$$

$$T^{SEZ-GCRF} = R_2(90-\delta)R_3(\alpha)$$

$$[\mathbf{v}]^{SEZ} = \begin{bmatrix}v_s\\ v_E\\ v_z\end{bmatrix} = v\begin{bmatrix}-\cos A\cos\gamma\\ \sin A\cos\gamma\\ \sin\gamma\end{bmatrix}$$

$$\sin\gamma = \frac{v_z}{v}, \qquad \sin A = \frac{v_E}{\sqrt{v_s^2+v_E^2}}$$

$$\sin\phi_{gc} = \frac{r_z}{r}, \qquad \tan\lambda = \frac{r_y}{r_x}$$

where $[\mathbf{r}]^{ITRF} = T^{ITRF-GCRF}[\mathbf{r}]^{GCRF} \rightarrow \begin{bmatrix}r_x\\ r_y\\ r_z\end{bmatrix} = T^{ITRF-GCRF}\begin{bmatrix}x\\ y\\ z\end{bmatrix}$

Satellite State Representations

- Nonsingular (e < 1) Euler parameters $\mathbf{X} = \begin{bmatrix} a & e & q_1 & q_2 & q_3 & M \end{bmatrix}^T$

$$T^{GCRF-PCS} = R_3(-\Omega)R_1(-i)R_3(-\omega)$$

$$= \begin{bmatrix} q_1^2 - q_2^2 - q_3^2 + q_4^2 & 2(q_1q_2 - q_3q_4) & 2(q_1q_3 + q_2q_4) \\ 2(q_1q_2 + q_3q_4) & -q_1^2 + q_2^2 - q_3^2 + q_4^2 & 2(q_2q_3 - q_1q_4) \\ 2(q_1q_3 - q_2q_4) & 2(q_2q_3 + q_1q_4) & -q_1^2 - q_2^2 + q_3^2 + q_4^2 \end{bmatrix}$$

$$where \ \ \mathbf{q} = \begin{bmatrix} q_1 & q_2 & q_3 & q_4 \end{bmatrix}^T, \quad \mathbf{q}^T\mathbf{q} = 1$$

$$q_4 = \frac{1}{2}\sqrt{1 + t_{11} + t_{22} + t_{33}} = \cos\frac{i}{2}\cos\frac{\Omega + \omega}{2}$$

Undefined for e ≥1.

$$q_1 = \frac{t_{32} - t_{23}}{4q_4} = \sin\frac{i}{2}\cos\frac{\Omega - \omega}{2}$$

$$q_2 = \frac{t_{13} - t_{31}}{4q_4} = \sin\frac{i}{2}\sin\frac{\Omega - \omega}{2}, \qquad q_3 = \frac{t_{21} - t_{12}}{4q_4} = \cos\frac{i}{2}\sin\frac{\Omega + \omega}{2}$$

Quaternions

- How to compute unit quaternion from a given DCM?

1. Form the symmetric matrix

$$\mathbf{K} = \frac{1}{3}\begin{bmatrix} t_{11}-t_{22}-t_{33} & t_{21}+t_{12} & t_{31}+t_{13} & t_{23}-t_{32} \\ t_{21}+t_{12} & -t_{11}+t_{22}-t_{33} & t_{32}+t_{23} & t_{31}-t_{13} \\ t_{31}+t_{13} & t_{32}+t_{23} & -t_{11}-t_{22}+t_{33} & t_{12}-t_{21} \\ t_{23}-t_{32} & t_{31}-t_{13} & t_{12}-t_{21} & t_{11}+t_{22}+t_{33} \end{bmatrix}$$

2. Solve the eigenvalue problem $\mathbf{Ke} = \lambda\mathbf{e}$ for the largest eigenvalue ($\lambda_{\max}$). The corresponding eigenvector is the quaternion.

Satellite State Representations

- Nonsingular (e < 1) quaternion element set: $\mathbf{X} = [\mathbf{q}^T \quad e_x \quad e_y]^T$

$$q_1 = p^{\frac{1}{4}} \sin\frac{i}{2} \cos\frac{\Omega - \omega - M}{2}$$

$$q_2 = p^{\frac{1}{4}} \sin\frac{i}{2} \sin\frac{\Omega - \omega - M}{2}$$

$$q_3 = p^{\frac{1}{4}} \cos\frac{i}{2} \sin\frac{\Omega + \omega + M}{2}$$

$$q_4 = p^{\frac{1}{4}} \cos\frac{i}{2} \cos\frac{\Omega + \omega + M}{2}$$

$$e_x = e \cos M$$

$$e_y = e \sin M$$

Satellite State Representations

- Equinoctial elements: $\mathbf{X} = [a \quad h \quad k \quad p \quad q \quad \lambda]^T$

$$a = a$$

$$h = e\sin(\omega + f_r\Omega)$$

$$k = e\cos(\omega + f_r\Omega)$$

$$f_r = \begin{cases} +1 & \text{: Prograde orbits} \\ -1 & \text{: Retrograde orbits} \end{cases}$$

$$p = \tan^{f_r}(i/2)\sin\Omega = \frac{\sin i \sin\Omega}{1+\cos^{fr} i}$$

$$q = \tan^{f_r}(i/2)\cos\Omega = \frac{\sin i \cos\Omega}{1+\cos^{fr} i}$$

$$\lambda = M + \omega + \Omega : \quad mean \quad longitude$$

Undefined for e ≥1.

- Modified Equinoctial elements: $\mathbf{X} = [p \;\; f \;\; g \;\; h \;\; k \;\; L]^T$

$$p = a(1-e^2) \qquad h = \tan^{f_r}(i/2)\cos(\Omega)$$

$$f = e\cos(\omega + f_r\Omega) \qquad k = \tan^{f_r}(i/2)\sin(\Omega)$$

$$g = e\sin(\omega + f_r\Omega) \qquad L = \omega + f_r\Omega + \theta$$

Nonsingular for all eccentricities and inclinations

Satellite State Representations

- Classical OE from MEOE

$$e = \sqrt{f^2 + g^2}$$

$$a = \frac{p}{1 - e^2}$$

Assuming $f_r = 1$: $\tan(i/2) = \sqrt{h^2 + k^2}$

$$\tan\Omega = \frac{k}{h}$$

$$\omega = \tan^{-1}(g/f) - \Omega$$

$$\theta = L - (\Omega + \omega)$$

Satellite State Representations

- Delaunay elements: $\mathbf{X} = [M \quad \Omega \quad \omega \quad L \quad h \quad H]^T$

$$L = \sqrt{\mu a}$$

$$H = \sqrt{\mu p} \cos i$$

Singular at e = 0 or 1, *i* = 0 or π

- Poincaré elements: $\mathbf{X} = [\Lambda \quad \xi \quad \eta \quad u \quad \upsilon \quad \lambda]^T$

$$\Lambda = \sqrt{\mu a}$$

$$\xi = e \sin(\Omega + \omega)\sqrt{2\Lambda / (1 + \sqrt{1 - e^2})}$$

$$\eta = e \cos(\Omega + \omega)\sqrt{2\Lambda / (1 + \sqrt{1 - e^2})}$$

$$u = \sin i \sin \Omega \sqrt{2\Lambda \sqrt{1 - e^2} / (1 + \cos i)}$$

$$\upsilon = \sin i \cos \Omega \sqrt{2\Lambda \sqrt{1 - e^2} / (1 + \cos i)}$$

$$\lambda = M + \Omega + \omega$$

Nonsingular for small eccentricities and inclinations

Undefined for e ≥1.

Satellite State Representations

- Alternate form of Poincaré elements: $\mathbf{X} = [L \quad g_p \quad h_p \quad G \quad H \quad \lambda]^T$

$$L = \sqrt{\mu a}$$

$$g_p = \cos(\Omega + \omega)\sqrt{2L(1 - \sqrt{1 - e^2})}$$

$$h_p = \cos(\Omega)\sqrt{2L\sqrt{1 - e^2}(1 - \cos i)}$$

$$G = \sin(\Omega + \omega)\sqrt{2L(1 - \sqrt{1 - e^2})}$$

$$H = \sin(\Omega)\sqrt{2L\sqrt{1 - e^2}(1 - \cos i)}$$

$$\lambda = M + \Omega + \omega$$

Nonsingular for small eccentricities and inclinations

Undefined for e ≥ 1.

Satellite State Representations

- Two-line Element Set (TLE) is a data format encoding a list of orbital elements of an Earth-orbiting object for a given point in time.
- A TLE set consists of two 69-character lines of data which can be used together with SGP4/SDP4 orbital model to determine the position and velocity of the associated satellite.

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ISS (ZARYA)
1 25544U 98067A   08264.51782528 -.00002182  00000-0 -11606-4 0  2927
2 25544  51.6416 247.4627 0006703 130.5360 325.0288 15.72125391563537
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- Line 0 is a twenty-four character name.

Satellite State Representations

- Line 1

1 25544U 98067A   08264.51782528 -.00002182 00000-0 -11606-4 0 2927

Field	Columns	Content	Example
1	01–01	Line number of Element Data	1
2	03–07	Satellite number	25544
3	08–08	Classification (U=Unclassified)	U
4	10–11	International Designator (Last two digits of launch year)	98
5	12–14	International Designator (Launch number of the year)	067
6	15–17	International Designator (piece of the launch)	A
7	19–20	Epoch Year (last two digits of year)	08
8	21–32	Epoch (day of the year and fractional portion of the day)	264.51782528
9	34–43	First Time Derivative of the Mean Motion divided by two $(\dot{n}/2)$	−.00002182
10	45–52	Second Time Derivative of Mean Motion divided by six (decimal point assumed) $(\ddot{n}/6)$	00000-0
11	54–61	BSTAR drag term (decimal point assumed)	-11606-4
12	63–63	The number 0 (originally this should have been "Ephemeris type")	0
13	65–68	Element set number. Incremented when a new TLE is generated for this object	292
14	69–69	Checksum (modulo 10) (Letters, blanks, periods, plus signs = 0; minus signs = 1)	7

$$B^* = \frac{1}{2}\frac{c_D A}{m}\rho_o R_\oplus \longrightarrow BC = \frac{1}{12.741\,621 B^*} \frac{\text{kg}}{\text{m}^2}$$

Satellite State Representations

- Line 2

2 25544 51.6416 247.4627 0006703 130.5360 325.0288 15.72125391563537

Field	Columns	Content	Example
1	01–01	Line number of Element Data	2
2	03–07	Satellite number	25544
3	09–16	Inclination (degrees)	51.6416
4	18–25	Right ascension of the ascending node (degrees)	247.4627
5	27–33	Eccentricity (decimal point assumed)	0006703
6	35–42	Argument of perigee (degrees)	130.5360
7	44–51	Mean Anomaly (degrees)	325.0288
8	53–63	Mean Motion (revolutions per day)	15.72125391
9	64–68	Revolution number at epoch (revolutions)	56353
10	69–69	Checksum (modulo 10)	7