In the Name of GOD

Guidance and Navigation I: Principles of Inertial Navigation

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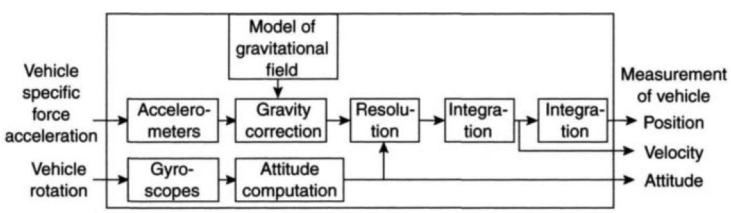
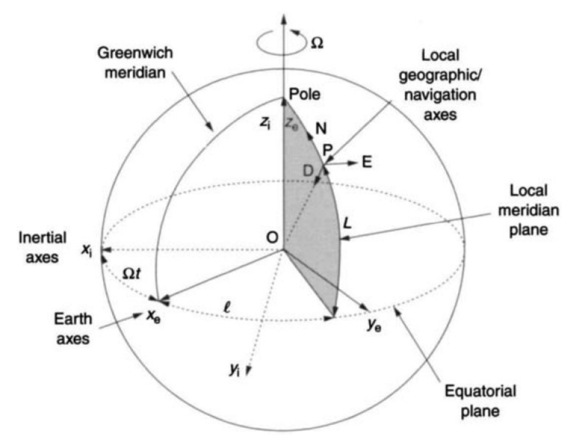
Contents

Functional components of SINS

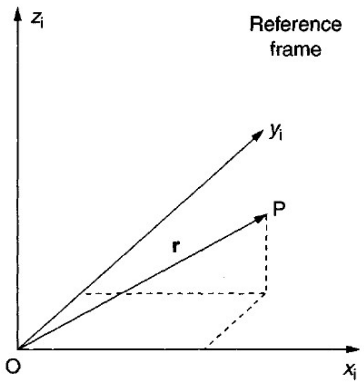
- Gyroscopes measure changes in vehicle attitude or its turn rate with respect to the inertial space.
- Accelerometers provides non-gravitational acceleration (specific force).



Reference Frames



3D Navigation with respect to inertial frame



$$\mathbf{v}_\mathrm{i} = \left.\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|_\mathrm{i}$$

$$\mathbf{a}_\mathrm{i} = \left.\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right|_\mathrm{i}$$

$$\left.\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right|_\mathrm{i} = \mathbf{f} + \mathbf{g}$$

$$\left.\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right|_\mathrm{i} = \mathbf{f}^\mathrm{i} + \mathbf{g}^\mathrm{i} = \mathbf{C}_\mathrm{b}^\mathrm{i}\mathbf{f}^\mathrm{b} + \mathbf{g}^\mathrm{i}$$

3D Navigation with respect to inertial frame

- The accelerometers usually provide a measurement of specific force in a body fixed axis set, denoted by $\mathbf{f}^\mathrm{b}$.

$$\mathbf{f}^\mathrm{i} = \mathbf{C}_\mathrm{b}^\mathrm{i}\mathbf{f}^\mathrm{b}$$

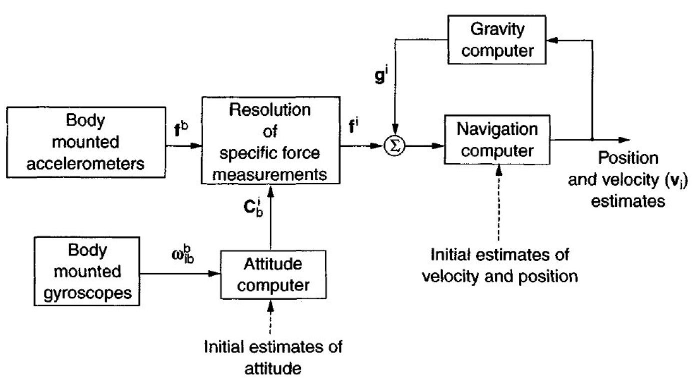
- Strapdown equation

$$\dot{\mathbf{C}}_\mathrm{b}^\mathrm{i} = \mathbf{C}_\mathrm{b}^\mathrm{i}\mathbf{\Omega}_\mathrm{ib}^\mathrm{b}$$

- where

$$\mathbf{\Omega}_\mathrm{ib}^\mathrm{b} = \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix} \qquad \boldsymbol{\omega}_\mathrm{ib}^\mathrm{b} = [p \quad q \quad r]^\mathrm{T}$$

3D Navigation with respect to inertial frame



Strapdown mechanizations: Inertial frame

Calculation of ground speed in inertial axes, $\mathbf{v}_\mathrm{e}^\mathrm{i}$

$$\left.\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|_\mathrm{i} = \left.\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|_\mathrm{e} + \boldsymbol{\omega}_\mathrm{ie} \times \mathbf{r}$$

$$\left.\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right|_\mathrm{i} = \left.\frac{\mathrm{d}\mathbf{v}_\mathrm{e}}{\mathrm{d}t}\right|_\mathrm{i} + \left.\frac{\mathrm{d}}{\mathrm{d}t}[\boldsymbol{\omega}_\mathrm{ie} \times \mathbf{r}]\right|_\mathrm{i}$$

$$\left.\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right|_\mathrm{i} = \left.\frac{\mathrm{d}\mathbf{v}_\mathrm{e}}{\mathrm{d}t}\right|_\mathrm{i} + \boldsymbol{\omega}_\mathrm{ie} \times \mathbf{v}_\mathrm{e} + \boldsymbol{\omega}_\mathrm{ie} \times [\boldsymbol{\omega}_\mathrm{ie} \times \mathbf{r}]$$

$$\left.\frac{\mathrm{d}\mathbf{v}_\mathrm{e}}{\mathrm{d}t}\right|_\mathrm{i} = \mathbf{f} - \boldsymbol{\omega}_\mathrm{ie} \times \mathbf{v}_\mathrm{e} - \boldsymbol{\omega}_\mathrm{ie} \times [\boldsymbol{\omega}_\mathrm{ie} \times \mathbf{r}] + \mathbf{g}$$

Strapdown mechanizations: Inertial frame

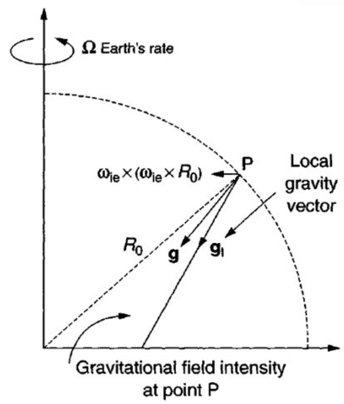
- Local Gravity Vector: the vector to which a 'plumb bob' would align itself when held above the Earth

$$\mathbf{g}_l = \mathbf{g} - \boldsymbol{\omega}_{ie} \times [\boldsymbol{\omega}_{ie} \times \mathbf{r}]$$

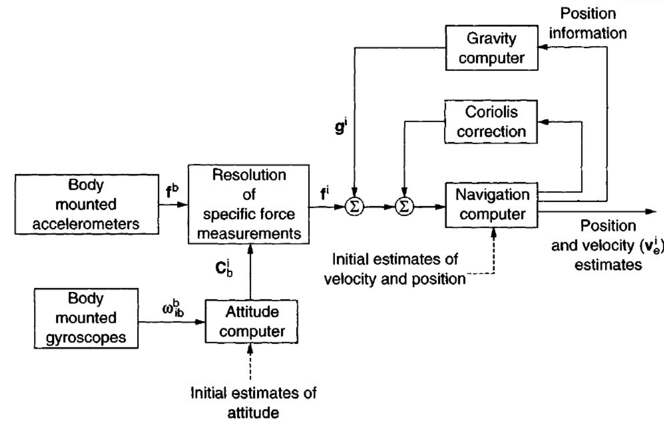
$$\left.\frac{d\mathbf{v}_e}{dt}\right|_i = \mathbf{f} - \boldsymbol{\omega}_{ie} \times \mathbf{v}_e + \mathbf{g}_l$$

$$\dot{\mathbf{v}}_e^i = \mathbf{f}^i - \boldsymbol{\omega}_{ie}^i \times \mathbf{v}_e^i + \mathbf{g}_l^i$$

$$\dot{\mathbf{v}}_e^i = \mathbf{C}_b^i \mathbf{f}^b - \boldsymbol{\omega}_{ie}^i \times \mathbf{v}_e^i + \mathbf{g}_l^i$$



Strapdown mechanizations: Inertial frame



Strapdown mechanizations: Earth frame

- In this system, ground speed is expressed in an Earth-fixed coordinate frame to give $\mathbf{v}_e^e$

$$\left.\frac{d\mathbf{v}_e}{dt}\right|_e = \left.\frac{d\mathbf{v}_e}{dt}\right|_i - \boldsymbol{\omega}_{ie} \times \mathbf{v}_e$$

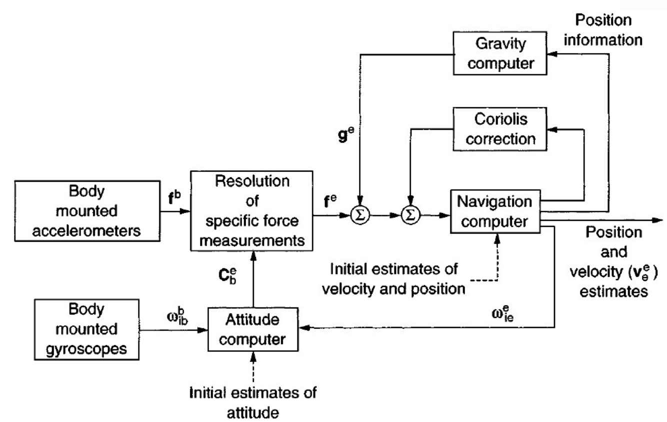
$$\left.\frac{d\mathbf{v}_e}{dt}\right|_e = \mathbf{f} - 2\boldsymbol{\omega}_{ie} \times \mathbf{v}_e + \mathbf{g}_l$$

$$\dot{\mathbf{v}}_e^e = \mathbf{C}_b^e \mathbf{f}^b - 2\boldsymbol{\omega}_{ie}^e \times \mathbf{v}_e^e + \mathbf{g}_l^e$$

$$\dot{\mathbf{C}}_b^e = \mathbf{C}_b^e \boldsymbol{\Omega}_{eb}^b$$

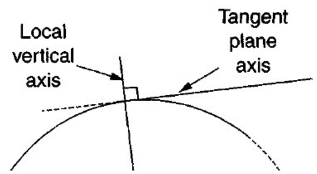
$$\boldsymbol{\omega}_{eb}^b = \boldsymbol{\omega}_{ib}^b - \mathbf{C}_e^b \boldsymbol{\omega}_{ie}^e$$

Strapdown mechanizations: Earth frame



Strapdown mechanizations: Earth frame Tactical Simplification

- Navigate over relatively short distances, with respect to a fixed point on the Earth.
- In this situation, an Earth-fixed reference frame may be defined, the axes of which is aligned with the local vertical and a plane which is tangential to the Earth's surface



Strapdown mechanizations: Earth frame Tactical Simplification

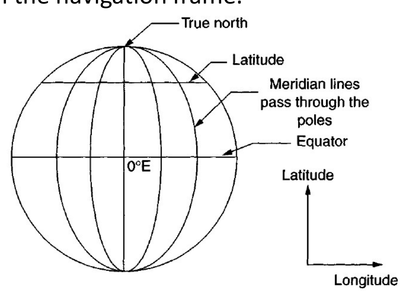
- For a short navigation period, typically less than 5 minutes,
- => the effects of Earth rotation on attitude computation can sometimes be ignored,
- => the Coriolis corrections are no longer essential

$$\dot{\mathbf{v}}_e^e = \mathbf{C}_b^e \mathbf{f}^b + \mathbf{g}_l^e$$

- In such a navigation system,
  - the permitted gyroscopic errors are in excess of the Earth rotation rate, and
  - allowable accelerometer biases are in excess of the acceleration errors introduced by ignoring the Coriolis forces.

Strapdown mechanizations: Local geographic navigation frame

- In order to navigate over large distances around the Earth, navigation information is commonly required in the local geographic or navigation axis set. Moreover, the gravity can easily be stated in the navigation frame.
- v_N =?
- v_E =?
- v_D =?
- ℓ =?
- λ =?
- h =?



Strapdown mechanizations: Local geographic navigation frame

$$\left.\frac{d\mathbf{v}_e}{dt}\right|_n = \left.\frac{d\mathbf{v}_e}{dt}\right|_i - [\boldsymbol{\omega}_{ie} + \boldsymbol{\omega}_{en}] \times \mathbf{v}_e$$

$$\left.\frac{d\mathbf{v}_e}{dt}\right|_i = \mathbf{f} - \boldsymbol{\omega}_{ie} \times \mathbf{v}_e + \mathbf{g}_l$$

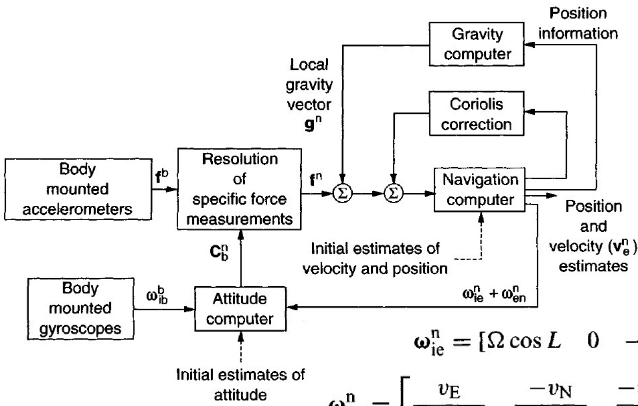
$$\Rightarrow \quad \left.\frac{d\mathbf{v}_e}{dt}\right|_n = \mathbf{f} - [2\boldsymbol{\omega}_{ie} + \boldsymbol{\omega}_{en}] \times \mathbf{v}_e + \mathbf{g}_l$$

$$\dot{\mathbf{v}}_e^n = \mathbf{C}_b^n \mathbf{f}^b - [2\boldsymbol{\omega}_{ie}^n + \boldsymbol{\omega}_{en}^n] \times \mathbf{v}_e^n + \mathbf{g}_l^n$$

$$\dot{\mathbf{C}}_b^n = \mathbf{C}_b^n \boldsymbol{\Omega}_{nb}^b$$

$$\boldsymbol{\omega}_{nb}^b = \boldsymbol{\omega}_{ib}^b - \mathbf{C}_n^b[\boldsymbol{\omega}_{ie}^n + \boldsymbol{\omega}_{en}^n]$$

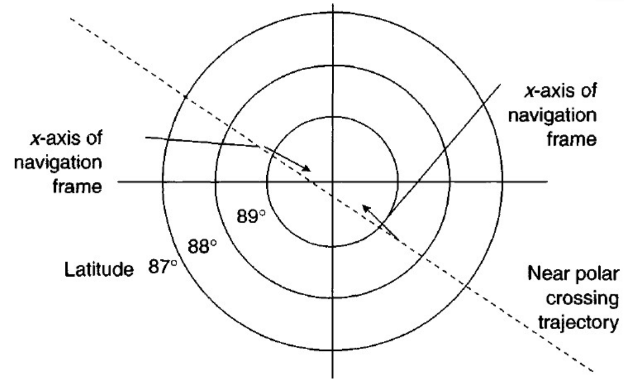
Strapdown mechanizations: Local geographic navigation frame



$$\boldsymbol{\omega}_{ie}^{n} = [\Omega \cos L \quad 0 \quad -\Omega \sin L]^{T}$$

$$\boldsymbol{\omega}_{en}^{n} = \left[\frac{v_E}{R_0 + h} \quad \frac{-v_N}{R_0 + h} \quad \frac{-v_E \tan L}{R_0 + h}\right]^{T}$$

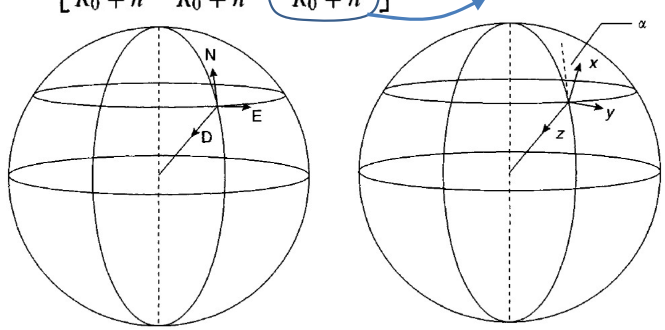
Strapdown mechanizations: Wander azimuth navigation frame



Strapdown mechanizations: Wander azimuth navigation frame

$$\boldsymbol{\omega}_{en}^{n} = \left[\frac{v_E}{R_0 + h} \quad \frac{-v_N}{R_0 + h} \quad \frac{-v_E \tan L}{R_0 + h}\right]^{T}$$

Singular at poles



Local geographic navigation frame   Wander azimuth navigation frame

Strapdown mechanizations: Wander azimuth navigation frame

- The navigation equation for a wander azimuth system is similar to a geographical navigation system:

$$\dot{\mathbf{v}}_e^w = \mathbf{C}_b^w \mathbf{f}^b - [2\mathbf{C}_e^w \boldsymbol{\omega}_e^e + \boldsymbol{\omega}_{ew}^w] \times \mathbf{v}_e^w + \mathbf{g}_l^w$$

- $\mathbf{v}_e^w$ is then used to generate the turn rate of the wander frame with respect to the Earth $\boldsymbol{\omega}_{ew}^w$

- $\mathbf{C}_e^w$ can be updated using the starpdown equation:

$$\dot{\mathbf{C}}_e^w = \mathbf{C}_e^w \boldsymbol{\Omega}_{ew}^w$$

Attitude Representation

- Various mathematical representations can be used to define the attitude of a body with respect to a coordinate reference frame.
  - Direction cosines
  - Euler angles
  - Quaternion
- Propagation of Direction cosine Matrix $\dot{\mathbf{C}}_\mathrm{b}^\mathrm{n} = \mathbf{C}_\mathrm{b}^\mathrm{n}\boldsymbol{\Omega}_\mathrm{nb}^\mathrm{b}$

$$
\begin{array}{lll}
\dot{c}_{11} = c_{12}\omega_z - c_{13}\omega_y & \dot{c}_{12} = c_{13}\omega_x - c_{11}\omega_z & \dot{c}_{13} = c_{11}\omega_y - c_{12}\omega_x \\
\dot{c}_{21} = c_{22}\omega_z - c_{23}\omega_y & \dot{c}_{22} = c_{23}\omega_x - c_{21}\omega_z & \dot{c}_{23} = c_{21}\omega_y - c_{22}\omega_x \\
\dot{c}_{31} = c_{32}\omega_z - c_{33}\omega_y & \dot{c}_{32} = c_{33}\omega_x - c_{31}\omega_z & \dot{c}_{33} = c_{31}\omega_y - c_{32}\omega_x
\end{array}
$$

Attitude Representation

- Propagation of Euler angles with time

$$
\begin{aligned}
\dot{\phi} &= (\omega_y \sin\phi + \omega_z \cos\phi)\tan\theta + \omega_x \\
\dot{\theta} &= \omega_y \cos\phi - \omega_z \sin\phi \\
\dot{\psi} &= (\omega_y \sin\phi + \omega_z \cos\phi)\sec\theta
\end{aligned}
$$

- Rotation Matrix versus Euler angles

$$
\mathbf{C}_\mathrm{b}^\mathrm{n} = \begin{bmatrix}
\cos\theta\cos\psi & \begin{array}{c} -\cos\phi\sin\psi \\ +\sin\phi\sin\theta\cos\psi \end{array} & \begin{array}{c} \sin\phi\sin\psi \\ +\cos\phi\sin\theta\cos\psi \end{array} \\
\cos\theta\sin\psi & \begin{array}{c} \cos\phi\cos\psi \\ +\sin\phi\sin\theta\sin\psi \end{array} & \begin{array}{c} -\sin\phi\cos\psi \\ +\cos\phi\sin\theta\sin\psi \end{array} \\
-\sin\theta & \sin\phi\cos\theta & \cos\phi\cos\theta
\end{bmatrix}
$$

Attitude Representation

- Quaternions (Euler parameters): A transformation from one coordinate frame to another can be performed by a single rotation about a vector $\boldsymbol{\mu}$ defined with respect to the reference frame.

$$
\mathbf{q} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} \cos(\mu/2) \\ (\mu_x/\mu)\sin(\mu/2) \\ (\mu_y/\mu)\sin(\mu/2) \\ (\mu_z/\mu)\sin(\mu/2) \end{bmatrix}
$$

$$
\mathbf{C} = \begin{bmatrix}
(a^2+b^2-c^2-d^2) & 2(bc-ad) & 2(bd+ac) \\
2(bc+ad) & (a^2-b^2+c^2-d^2) & 2(cd-ab) \\
2(bd-ac) & 2(cd+ab) & (a^2-b^2-c^2+d^2)
\end{bmatrix}
$$

Attitude Representation

- Propagation of quaternion with time

$$
\dot{\mathbf{q}} = \begin{bmatrix} \dot{a} \\ \dot{b} \\ \dot{c} \\ \dot{d} \end{bmatrix} = 0.5 \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix} \begin{bmatrix} 0 \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}
$$

- Quaternion expressed in terms of direction cosines

$$
a = \tfrac{1}{2}(1 + c_{11} + c_{22} + c_{33})^{1/2} \qquad c = \frac{1}{4a}(c_{13} - c_{31})
$$

$$
b = \frac{1}{4a}(c_{32} - c_{23}) \qquad d = \frac{1}{4a}(c_{21} - c_{12})
$$

Attitude Representation

- Quaternion expressed in terms of Euler angles

$$a = \cos\frac{\phi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2} + \sin\frac{\phi}{2}\sin\frac{\theta}{2}\sin\frac{\psi}{2}$$
$$b = \sin\frac{\phi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2} - \cos\frac{\phi}{2}\sin\frac{\theta}{2}\sin\frac{\psi}{2}$$
$$c = \cos\frac{\phi}{2}\sin\frac{\theta}{2}\cos\frac{\psi}{2} + \sin\frac{\phi}{2}\cos\frac{\theta}{2}\sin\frac{\psi}{2}$$
$$d = \cos\frac{\phi}{2}\cos\frac{\theta}{2}\sin\frac{\psi}{2} + \sin\frac{\phi}{2}\sin\frac{\theta}{2}\cos\frac{\psi}{2}$$

- *Euler angles expressed in terms of direction cosines*

$$\phi = \arctan\left[\frac{c_{32}}{c_{33}}\right] \qquad \theta = \arcsin\left[-c_{31}\right] \qquad \psi = \arctan\left[\frac{c_{21}}{c_{11}}\right]$$

Navigation equations in component form

- Mechanization in the local geographic reference frame:

$$\dot{\mathbf{v}}_e^n = \mathbf{f}^n - (2\boldsymbol{\omega}_{ie}^n + \boldsymbol{\omega}_{en}^n) \times \mathbf{v}_e^n + \mathbf{g}_l^n$$
$$\mathbf{v}_e^n = [v_N \quad v_E \quad v_D]^T$$
$$\mathbf{f}^n = [f_N \quad f_E \quad f_D]^T$$
$$\boldsymbol{\omega}_{ie}^n = [\Omega\cos L \quad 0 \quad -\Omega\sin L]^T$$
$$\boldsymbol{\omega}_{en}^n = [\dot{\ell}\cos L \quad -\dot{L} \quad -\dot{\ell}\sin L]^T$$

$$\dot{\ell} = v_E/(R_0+h)\cos L \qquad \boldsymbol{\omega}_{en}^n = \left[\frac{v_E}{R_0+h} \quad -\frac{v_N}{R_0+h} \quad -\frac{v_E\tan L}{R_0+h}\right]^T$$
$$\dot{L} = v_N/(R_0+h)$$

$$\mathbf{g}_l^n = \mathbf{g} - \boldsymbol{\omega}_{ie} \times \boldsymbol{\omega}_{ie} \times \mathbf{R} = \mathbf{g} - \frac{\Omega^2(R_0+h)}{2}\begin{pmatrix}\sin 2L \\ 0 \\ 1+\cos 2L\end{pmatrix}$$

Navigation equations in component form

$$\mathbf{g}_l = [\xi g, \quad -\eta g, \quad g]^T$$

$$\dot{v}_N = f_N - v_E(2\Omega + \dot{\ell})\sin L + v_D\dot{L} + \xi g$$
$$= f_N - 2\Omega v_E \sin L + \frac{v_N v_D - v_E^2\tan L}{R_0+h} + \xi g$$

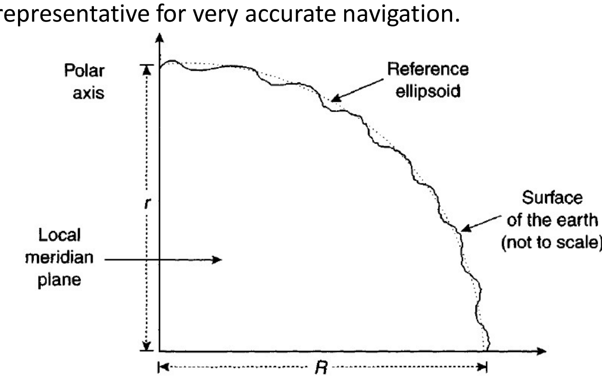
$$\dot{v}_E = f_E + v_N(2\Omega + \dot{\ell})\sin L + \dot{v}_D(2\Omega + \dot{\ell})\cos L - \eta g$$
$$= f_E + 2\Omega(v_N\sin L + v_D\cos L) + \frac{v_E}{R_0+h}(v_D + v_N\tan L) - \eta g$$

$$\dot{v}_D = f_D - v_E(2\Omega + \dot{\ell})\cos L - v_N\dot{L} + g = f_D - 2\Omega v_E\cos L - \frac{v_E^2 + v_N^2}{R_0+h} + g$$

$$\dot{L} = \frac{v_N}{R_0+h} \qquad \dot{\ell} = \frac{v_E\sec L}{R_0+h} \qquad \dot{h} = -v_D$$

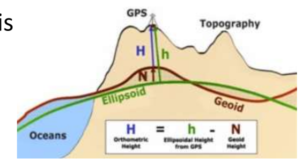
Elliptic Earth Model

- The spherical model assumed so far is not sufficiently representative for very accurate navigation.



Elliptic Earth Model

- Geoid: a surface which is perpendicular to the local gravity vector.
- Reference ellipsoid: Whilst geoid is much smoother than the physical surface of the Earth, it is too irregular to be used as a surface in which to specify spatial coordinates.
  For terrestrial navigation, a geometrical shape that approximates closely to the geoid is used;
  an ellipsoid, which in this context is a three-dimensional (3-D) shape formed by rotating an ellipse about its minor axis.



Elliptic Earth Model

the length of the semi-major axis,	R
the length of the semi-minor axis,	$r = R(1-f)$
the flattening of the ellipsoid,	$f = (R-r)/R$
the major eccentricity of the ellipsoid,	$e = [f(2-f)]^{1/2}$

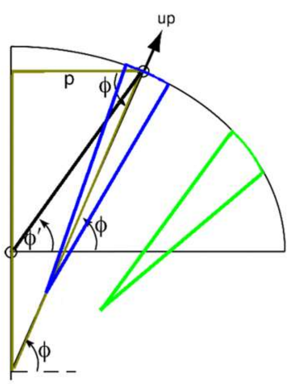
- R_N: meridian radius of curvature
- R_E: transverse radius of curvature
- $$R_N = \frac{R(1-e^2)}{(1-e^2\sin^2 L)^{3/2}}$$
  $$R_E = \frac{R}{(1-e^2\sin^2 L)^{1/2}}$$
  $$R_0 = (R_E R_N)^{1/2}$$

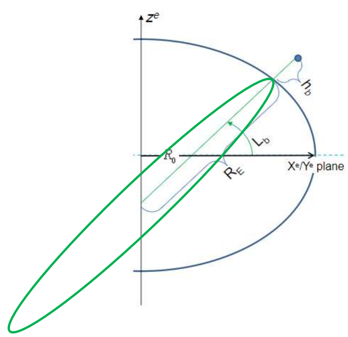
latitude and longitude rates

$$\dot{L} = \frac{v_N}{R_N + h}$$

$$\dot{\ell} = \frac{v_E \sec L}{R_E + h}$$

Elliptic Earth Model

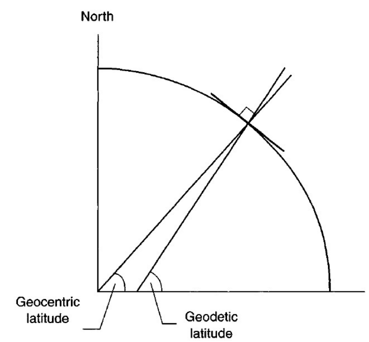




Elliptic Earth Model

- transport rates $\boldsymbol{\omega}_{en}^{n} = \left[\frac{v_E}{R_E + h} \quad \frac{-v_N}{R_N + h} \quad \frac{-v_E \tan L}{R_N + h}\right]^T$

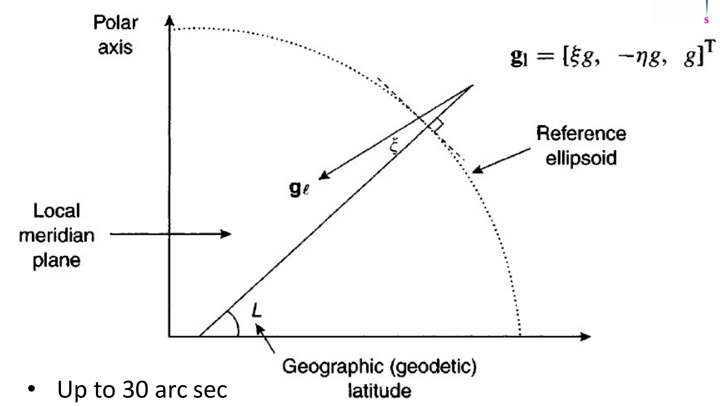
- geocentric latitude
- geodetic latitude



WGS-84

Length of the semi-major axis,	R	$= 6378137.0\,\text{m}$
Length of the semi-minor axis,	$r = R(1-f)$	$= 6356752.3142\,\text{m}$
Flattening of the ellipsoid,	$f = (R-r)/R$	$= 1/298.257223563$
Major eccentricity of the ellipsoid,	$e = [f(2-f)]^{1/2}$	$= 0.0818191908426$
Earth's rate (see Figure 3.24),	Ω	$= 7.292115 \times 10^{-5}$ rad/s (15.041067°/h)

Variation of gravitational acceleration over the Earth



- Up to 30 arc sec

Variation of gravitational acceleration over the Earth

- Steiler and Winter Model

$$g(0) = 9.780318(1 + 5.3024 \times 10^{-3} \sin^2 L - 5.9 \times 10^{-6} \sin^2 2L)\ \text{m/s}^2$$

$$\frac{\mathrm{d}g(0)}{\mathrm{d}h} = -0.0000030877(1 - 1.39 \times 10^{-3} \sin^2 L)\ \text{m/s}^2/\text{m}$$

- For many applications, precise knowledge of gravity is not required and it is sufficient to assume that the variation of gravity with altitude is as follows:

$$g(h) = \frac{g(0)}{(1 + h/R_0)^2}$$

Elliptic Gravity Model

$$\begin{cases} G_x = -\dfrac{\mu}{R^2}\left(1 + \dfrac{3}{2} J_2 \left(\dfrac{r_e}{R}\right)^2 \left(1 - 5\left(\dfrac{z}{R}\right)^2\right)\right)\dfrac{x}{R} \\ G_y = -\dfrac{\mu}{R^2}\left(1 + \dfrac{3}{2} J_2 \left(\dfrac{r_e}{R}\right)^2 \left(1 - 5\left(\dfrac{z}{R}\right)^2\right)\right)\dfrac{y}{R} \\ G_z = -\dfrac{\mu}{R^2}\left(1 + \dfrac{3}{2} J_2 \left(\dfrac{r_e}{R}\right)^2 \left(3 - 5\left(\dfrac{z}{R}\right)^2\right)\right)\dfrac{z}{R} \end{cases} \qquad R = \sqrt{x^2 + y^2 + z^2}$$

Earth's gravitational constant, μ	$3.986005 \times 10^{+14}$	m^3/s^2
Second gravitational constant, J_2	1.08263×10^{-3}	

$$\mathbf{g} = \mathbf{G} - \boldsymbol{\Omega}^{EI}\boldsymbol{\Omega}^{EI}\mathbf{r}$$

Discretization of Navigation Equations

- Incremental angles

$$\Delta\boldsymbol{\theta}_{k-1} = \int_{t_{k-2}}^{t_{k-1}} \boldsymbol{\omega}_{\mathrm{ib}}^{\mathrm{b}}\, dt \approx \frac{\boldsymbol{\omega}_{\mathrm{ib}}^{\mathrm{b}}(k-1)+\boldsymbol{\omega}_{\mathrm{ib}}^{\mathrm{b}}(k-2)}{2}(t_{k-1}-t_{k-2})$$

$$\Delta\boldsymbol{\theta}_{k} = \int_{t_{k-1}}^{t_{k}} \boldsymbol{\omega}_{\mathrm{ib}}^{\mathrm{b}}\, dt \approx \frac{\boldsymbol{\omega}_{\mathrm{ib}}^{\mathrm{b}}(k)+\boldsymbol{\omega}_{\mathrm{ib}}^{\mathrm{b}}(k-1)}{2}(t_{k}-t_{k-1})$$

- Rotation Vector

$$\boldsymbol{\varphi}_k \approx \Delta\boldsymbol{\theta}_k + \frac{1}{12}\Delta\boldsymbol{\theta}_{k-1}\times\Delta\boldsymbol{\theta}_k$$

Discretization of Navigation Equations

- Evolution of the Rotation Matrix

$$\mathbf{C}_{b(k)}^{b(k-1)} \approx \mathbf{I} + \frac{\sin(\|\boldsymbol{\varphi}_k\|)}{\|\boldsymbol{\varphi}_k\|}(\boldsymbol{\varphi}_k\times) + \frac{1-\cos(\|\boldsymbol{\varphi}_k\|)}{\|\boldsymbol{\varphi}_k\|^2}(\boldsymbol{\varphi}_k\times)(\boldsymbol{\varphi}_k\times)$$

$$\mathbf{C}_{b(k)}^{i} = \mathbf{C}_{b(k-1)}^{i}\mathbf{C}_{b(k)}^{b(k-1)}$$

- Incremental velocities

$$\Delta\mathbf{v}_{\mathbf{f},k-1}^{\mathrm{b}} = \int_{t_{k-2}}^{t_{k-1}} \mathbf{f}^{\mathrm{b}}\, dt \approx \frac{\mathbf{f}^{b}(k-1)+\mathbf{f}^{b}(k-2)}{2}(t_{k-1}-t_{k-2})$$

$$\Delta\mathbf{v}_{\mathbf{f},k}^{\mathrm{b}} = \int_{t_{k-1}}^{t_{k}} \mathbf{f}^{\mathrm{b}}\, dt \approx \frac{\mathbf{f}^{b}(k)+\mathbf{f}^{b}(k-1)}{2}(t_{k}-t_{k-1})$$

Discretization of Navigation Equations

- Evolution of velocity

$$\mathbf{v}_{\mathrm{k}}^{\mathrm{i}} = \mathbf{v}_{\mathrm{k-1}}^{\mathrm{i}} + \Delta\mathbf{v}_{\mathbf{f},k}^{i} + \Delta\mathbf{v}_{\mathbf{g},k}^{i}$$

$$\Delta\mathbf{v}_{\mathbf{f},k}^{\mathrm{i}} = \mathbf{C}_{b(k-1)}^{i}\Delta\mathbf{v}_{\mathbf{f},k}^{\mathrm{b(k-1)}}$$

$$\Delta\mathbf{v}_{\mathbf{f},k}^{\mathrm{b(k-1)}} \approx \Delta\mathbf{v}_{\mathbf{f},k}^{\mathrm{b}} + \frac{1}{2}\Delta\boldsymbol{\theta}_k\times\Delta\mathbf{v}_{\mathbf{f},k}^{\mathrm{b}} + \frac{1}{12}(\Delta\boldsymbol{\theta}_{k-1}\times\Delta\mathbf{v}_{\mathbf{f},k}^{\mathrm{b}} + \Delta\mathbf{v}_{\mathbf{f},k-1}^{\mathrm{b}}\times\Delta\boldsymbol{\theta}_k)$$

- and $\Delta\mathbf{v}_{\mathbf{g},k}^{\mathrm{i}} = \mathbf{g}_{k-\frac{1}{2}}^{i}\Delta t$

- where g(k-1/2) is calculated at: $\mathbf{r}_{k-\frac{1}{2}}^{\mathrm{i}} = \mathbf{r}_{k-1}^{\mathrm{i}} + \frac{1}{2}\mathbf{v}_{k-1}\Delta t$

Discretization of Navigation Equations

- Evolution of Position

$$\mathbf{r}_{k}^{\mathrm{i}} = \mathbf{r}_{k-1}^{\mathrm{i}} + \mathbf{v}_{k-\frac{1}{2}}^{i}\Delta t$$

$$\mathbf{v}_{k-\frac{1}{2}}^{\mathrm{i}} = \frac{1}{2}(\mathbf{v}_{k}^{\mathrm{i}} + \mathbf{v}_{k-1}^{i})$$