In the Name of GOD

Guidance and Navigation I: Ballistic/Strategic Guidance Laws

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Ballistic Guidance Laws

- Ballistic Trajectory
- Required Velocity and Flight time
- Lambert Problem
- Velocity to be Gained
- Lambert Guidance
- Cut-off Insensitive Guidance Laws
- Explicit Versus Implicit Guidance
- Q and Q* Guidance
- Preset Guidance

Ballistic Considerations

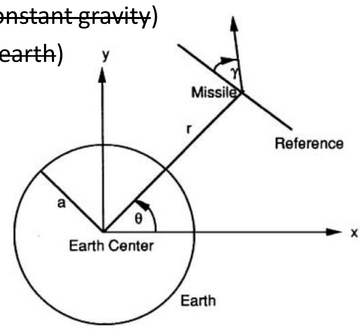
R>100 nm, h>100 kf, V>5000 ft/sec ⇨

- Elliptic or at least Spherical Earth (~~Flat Earth~~)
- Height dependent gravity (~~Constant gravity~~)
- Rotating Earth (~~non-rotating earth~~)

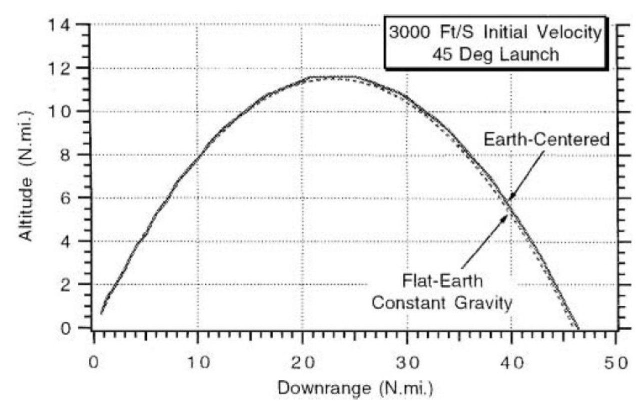
- Newton's 2nd law of motion

$$\ddot{r} = \frac{-gm\,\boldsymbol{r}}{r^3}$$

$$gm = 1.4077 * 10^{16} ft^3/s^2$$



Gravity and Earth Model



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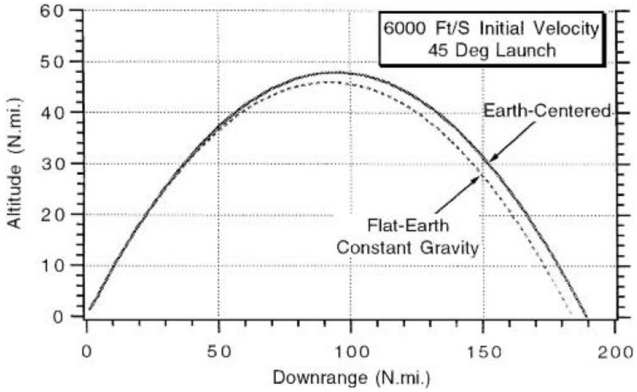
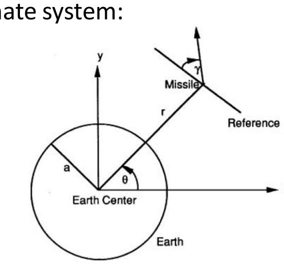


Ballistic Trajectories

- Equations of motion in polar coordinate system:

$$\frac{-gm}{r^2} = \ddot{r} - r\dot{\theta}^2$$

$$0 = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$



- It can be shown that:

$$\ddot{r} - r\dot{\theta}^2 + \frac{gm}{r^2} = 0$$

$$r^2\dot{\theta} = (a + alt_0)V_0 \cos\gamma_0$$

$$r(0) = a + alt_0$$

$$\theta(0) = 0$$

$$\dot{r}(0) = V_0 \sin\gamma_0$$

Ballistic Trajectories

- The solution:

$$\frac{r_0}{r} = \frac{1 - \cos\theta}{\lambda_0 \cos^2\gamma_0} + \frac{\cos(\theta + \gamma_0)}{\cos\gamma_0}$$

- where:

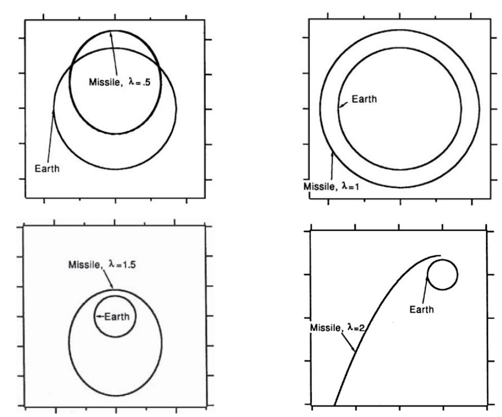
$$\lambda_0 = \frac{r_0 V_0^2}{gm}$$

- or

$$r = \frac{r_0 \lambda_0 \cos^2\gamma_0}{1 - \cos\theta + \lambda_0 \cos\gamma_0 \cos(\theta + \gamma_0)}$$

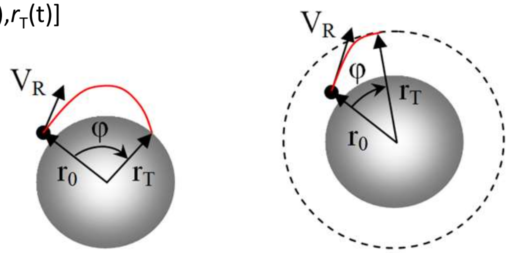
- r is a function of θ (for a given r_0, Υ_0 and λ_0)
- for Υ_0=0 ,the solution may intersect the earth, regarding the corresponding value of λ_0.

Ballistic Trajectories



Required Velocity

- Definition: The instantaneous velocity, required to reach a given position, in a ballistic trajectory.
- $V_R = f\,[r_0(t), r_T(t)]$



Required Velocity

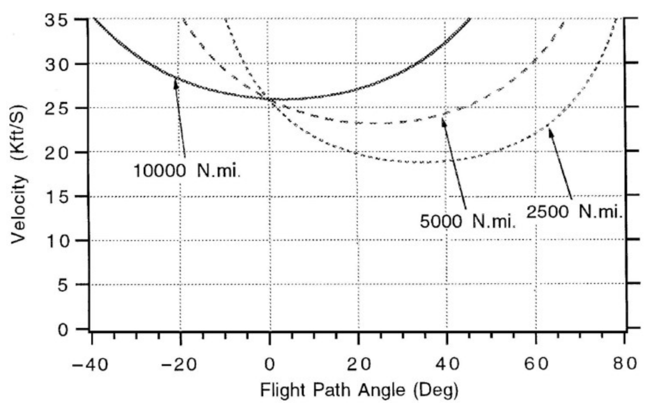
- If we desire for the missile to travel a distance *dist*,

$$\phi = dist/a \qquad r = a \text{ and } \theta = \phi \quad \Rightarrow \quad \frac{r_0}{a} = \frac{1-\cos\phi}{\lambda_0\cos^2\gamma_0} + \frac{\cos(\phi+\gamma_0)}{\cos\gamma_0}$$

$$\lambda_0 = \frac{r_0 V_0^2}{gm} \quad \Rightarrow \quad V_0 = \sqrt{\frac{gm(1-\cos\phi)}{r_0\cos\gamma_0[(r_0\cos\gamma_0/a) - \cos(\phi+\gamma_0)]}}$$

- Note: We want to launch with a certain flight-path angle, Υ_0.

Required Velocity



The feasible bounds of flight path angle

- Consider the required velocity formula and the cases that lead to escape velocity:

$$\lambda_0 = 2 = \frac{V_0^2 r_0}{gm}$$

- Therefore:

$$2 = \frac{(1-\cos\phi)}{\cos\gamma_0[(r_0\cos\gamma_0/r_F) - \cos(\phi+\gamma_0)]}$$

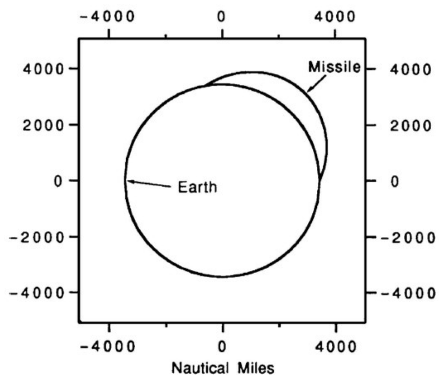
- Solving for ϒ, we get two solutions:

$$\gamma_{\min} = \tan^{-1}\left\{\left[\sin\phi - \sqrt{\frac{2r_0}{r_F}(1-\cos\phi)}\right] \Big/ (1-\cos\phi)\right\}$$

$$\gamma_{\max} = \tan^{-1}\left\{\left[\sin\phi + \sqrt{\frac{2r_0}{r_F}(1-\cos\phi)}\right] \Big/ (1-\cos\phi)\right\}$$

Sample Trajectory

- 6000 nautical miles, γ_0=23 deg



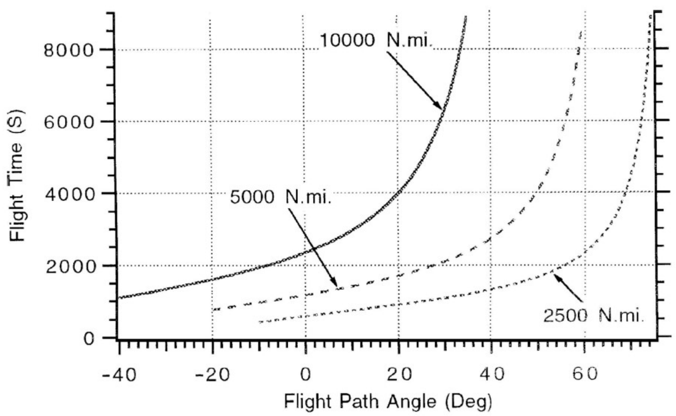
Flight Time

$$r^2\frac{d\theta}{dt} = r_0 V_0 \cos\gamma_0 \quad \Rightarrow \quad \int_0^{\phi} r^2 d\theta = \int_0^{t_F} r_0 V_0 \cos\gamma_0 \, dt$$

$$t_F = \frac{1}{r_0 V_0 \cos\gamma_0}\int_0^{\phi}\frac{r_0^2\lambda_0^2\cos^4\gamma_0}{\left[1-\cos\theta+\lambda_0\cos\gamma_0\cos(\theta+\gamma_0)\right]^2}\, d\theta$$

$$\Rightarrow \quad t_F = \frac{r_0}{V_0\cos\gamma_0}\left\{\frac{\tan\gamma_0(1-\cos\phi)+(1-\lambda_0)\sin\phi}{(2-\lambda_0)\left[\frac{1-\cos\phi}{\lambda_0\cos^2\gamma_0}+\frac{\cos(\gamma_0+\phi)}{\cos\gamma_0}\right]} + \frac{2\cos\gamma_0}{\lambda_0\left(\frac{2}{\lambda_0}-1\right)^{1.5}}\tan^{-1}\left(\frac{\sqrt{\frac{2}{\lambda_0}-1}}{\cos\gamma_0\cot\frac{\phi}{2}-\sin\gamma_0}\right)\right\}$$

Flight Time



Three dimensional formulations

$$v_R = \sqrt{\frac{\mu}{|\mathbf{r}_0|}\frac{|\mathbf{r}_F|(1-\cos\phi_0)}{|\mathbf{r}_0|\cos^2\gamma_0-|\mathbf{r}_F|\cos(\phi_0+\gamma_0)\cos\gamma_0}}$$

$$t_F = \frac{|\mathbf{r}_0|}{v_R\cos\gamma_0}\left\{\begin{array}{l}\dfrac{\tan\gamma_0(1-\cos\phi_0)+(1-\lambda)\sin\phi_0}{(2-\lambda)|\mathbf{r}_0|/|\mathbf{r}_F|} \\ +\dfrac{2\cos\gamma_0}{\lambda[(2-\lambda)-1]^{3/2}}\tan^{-1}\dfrac{[(2/\lambda)-1]^{1/2}}{\cos\gamma_0\cot(\phi_0/2)-\sin\gamma_0}\end{array}\right\}$$

Lambert's Problem

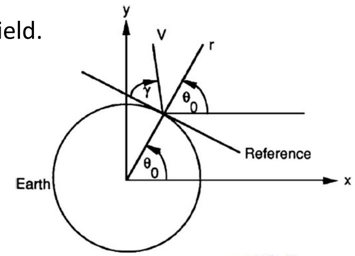
- Consider a body in the gravity field.
- initial location:

$$x(0) = x_0$$
$$y(0) = y_0$$

- Desired location after time t_F:

$$x(t_F) = x_F$$
$$y(t_F) = y_F$$

- The problem is to find the initial flight path angle, γ_0 so that the initial and final conditions are satisfied.



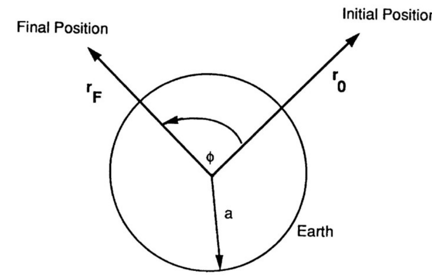
$$\ddot{x} = \frac{-gm\,x}{(x^2+y^2)^{1.5}}$$

$$\ddot{y} = \frac{-gm\,y}{(x^2+y^2)^{1.5}}$$

Solution to Lambert's Problem

- Angular distance to be traveled:

$$\phi = \cos^{-1}\frac{\boldsymbol{r}_0 \cdot \boldsymbol{r}_F}{|\boldsymbol{r}_0||\boldsymbol{r}_F|}$$



- Required velocity for a given γ:

$$V = \sqrt{\frac{gm(1-\cos\phi)}{r_0\cos\gamma\left[(r_0\cos\gamma/r_F)-\cos(\phi+\gamma)\right]}}$$

Solution to Lambert's Problem

- Flight time can be calculated as follows:

$$t_F = \frac{r_0}{V\cos\gamma}\left\{\frac{\tan\gamma(1-\cos\phi)+(1-\lambda)\sin\phi}{(2-\lambda)\left[\frac{1-\cos\phi}{\lambda\cos^2\gamma}+\frac{\cos(\gamma+\phi)}{\cos\gamma}\right]}\right.$$

$$\left.+\frac{2\cos\gamma}{\lambda(\frac{2}{\lambda}-1)^{1.5}}\tan^{-1}\left(\frac{\sqrt{\frac{2}{\lambda}-1}}{\cos\gamma\cot\frac{\phi}{2}-\sin\gamma}\right)\right\}$$

- We are going to find γ such that t_F meets the desired value.

Solution to Lambert's Problem

- Start with an initial guess for ϒ (small enough)
- Calculate ϕ, V and t_F:

$$\phi = f(\boldsymbol{r}_0, \boldsymbol{r}_F)$$
$$V = f(\boldsymbol{r}_0, \boldsymbol{r}_F, \phi, \gamma)$$
$$t_F = f(V, \phi, \gamma)$$

- If t_F is less than the desired t_F, increase ϒ and iterate, else decrease ϒ.

secant method to solve Lambert problem

- The flight path angle can be obtained using the following iterative formula:

$$\gamma_{n+1} = \gamma_n + \frac{(\gamma_n - \gamma_{n-1})(t_{\mathrm{FDES}} - t_{F_n})}{t_{F_n} - t_{F_{n-1}}}$$

secant method to solve Lambert problem

- Very accurate Lambert solutions are obtained after a few iterations:

Iteration	Flight-path angle, deg	V_{RX}, ft/s	V_{RY}, ft/s	Flight time, s
1	33.7524947	−3764.57976	18926.02426	1239.37545
2	11.2508376	−12075.71473	18072.66484	813.53185
3	21.1038504	−8103.20444	18287.99279	979.68031
4	22.3088581	−7665.88409	18332.46792	1001.50708
5	22.2256555	−7695.84063	18329.28399	999.98566
6	22.2264396	−7695.55815	18329.31392	999.99999
7	22.2264402	−7695.55795	18329.31394	1000.00000

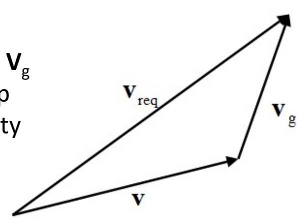
- The search is terminated when the computed flight time t_F, is sufficiently close to the desired flight time $t_{F,DES}$.

Velocity to be gained

- The difference between $\mathbf{V}_R$ and the instantaneous velocity.

$$\mathbf{V}_g = \mathbf{V}_R - \mathbf{V}$$

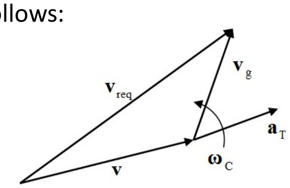
- $\mathbf{V}_g$ is a vector
- If thrust is aligned with $\mathbf{V}_g$, then $\mathbf{V}_g$ will tend to zero in a closed-loop manner and the required velocity will be obtained.
- The engine cuts-off when $\mathbf{V}_g$ becomes small enough.



Lambert Guidance (Steering)

- To achieve $\mathbf{V}_g$, it is desired to guide the total non-gravitational accelerations ($\mathbf{a}_T$), toward $\mathbf{V}_g$. For this purpose, an angular velocity command is applied as follows:

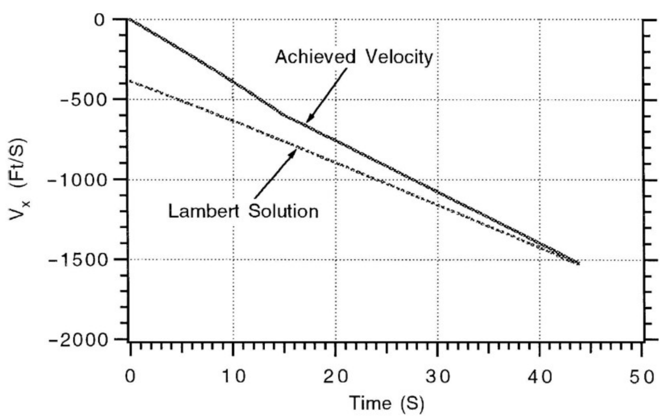
$$\boldsymbol{\omega}_C = K \frac{\mathbf{a}_T \times \mathbf{V}_g}{|\mathbf{a}_T||\mathbf{V}_g|}$$



Cross Product Steering

- K: a proportional constant (small enough)
- a_T is measured using the accelerometers.
- Here, commands are body rates instead of the body accelerations.

Lambert Guidance



Gravity Turn

Definition 1

- In ballistic flight simulation, if the thrust vector is aligned with the velocity vector, a gravity turn will be obtained.

$$\boldsymbol{\omega}_C = K \frac{\mathbf{a}_T \times \mathbf{v}}{|\mathbf{a}_T||\mathbf{v}|}$$

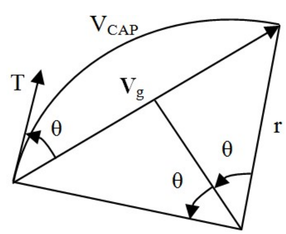
- This strategy may be used to limit the angle of attack.

Definition 2

- Make an intentional pitch rate before engine cut-off!

General Energy Management (GEM) Steering

- In the absence of a thrust termination system, all fuel must be consumed and the velocity should meet the required value at burn out time.
- A wasting technique must be employed to waste some of the excess energy (Energy Management).



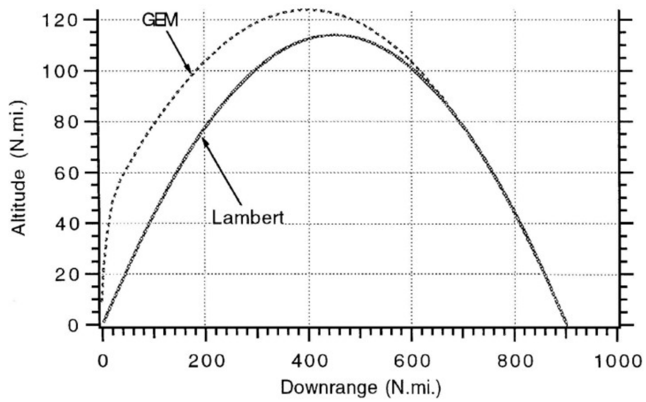
General Energy Management (GEM) Steering

$$\frac{V_g}{V_{cap}} = \frac{2r\sin\theta}{2r\theta} = \frac{\sin\theta}{\theta} \Rightarrow \frac{V_g}{V_{cap}} = \left(\theta - \frac{\theta^3}{6}\right) \Big/ \theta = 1 - \frac{\theta^2}{6}$$

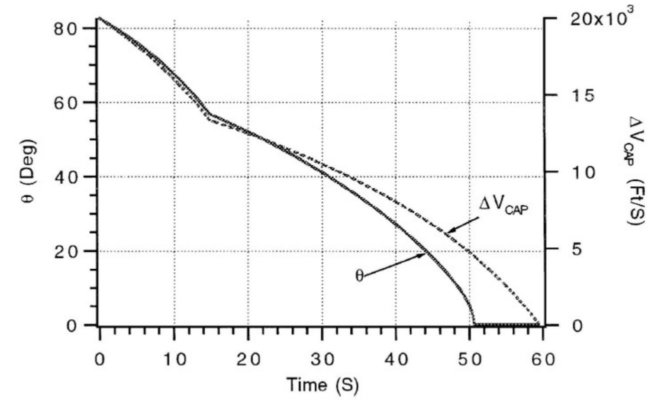
$$\Rightarrow \theta = \sqrt{6\left(1 - \frac{V_g}{V_{cap}}\right)}$$

- If the thrust vector is kept at an angle of θ over the velocity to be gained, then we can achieve the Lambert solution at the end of burn and hit the target.

General Energy Management (GEM) Steering



General Energy Management (GEM) Steering



Explicit Versus Implicit Guidance

$\mathbf{v}_g$ can be calculated explicitly or implicitly.

Explicit Guidance: **Vg** is calculated, as in Lambert guidance.
Implicit Guidance: **Vg** is obtained by numerically solving a diff. Equation.

- It can be shown that: [An Introduction to the mathematics and methods of astrodynamics]

$$\dot{\mathbf{v}}_g = -\mathbf{Q}\,\mathbf{v}_g - \mathbf{a}_T$$

- Where Q is a symmetric 3x3 matrix, defined as

$$\mathbf{Q} = \partial \mathbf{v}_{req} / \partial \mathbf{r}$$

- a_T: Total non-gravitational accelerations
- Q depends to trajectory, mass, inertia, aerodynamic and propulsion.

Q matrix

$$\mathbf{Q} = \frac{\partial \mathbf{v}_R}{\partial \mathbf{r}} = \begin{bmatrix} \frac{\partial v_{R_x}}{\partial x} & \frac{\partial v_{R_x}}{\partial y} & \frac{\partial v_{R_x}}{\partial z} \\ \frac{\partial v_{R_y}}{\partial x} & \frac{\partial v_{R_y}}{\partial y} & \frac{\partial v_{R_y}}{\partial z} \\ \frac{\partial v_{R_z}}{\partial x} & \frac{\partial v_{R_z}}{\partial y} & \frac{\partial v_{R_z}}{\partial z} \end{bmatrix} = \begin{bmatrix} q_{11}(t) & q_{12}(t) & q_{13}(t) \\ q_{21}(t) & q_{22}(t) & q_{23}(t) \\ q_{31}(t) & q_{32}(t) & q_{33}(t) \end{bmatrix} = \mathbf{Q}(t)$$

- $q_{ij}(t)$ have small variations with time.
- It can be shown that assuming flat earth and constant gravity:

$$\mathbf{Q} = \frac{-1}{t_F}\mathbf{I}$$

Q Guidance Algorithm

- A nominal Trajectory is found using a simulation based trajectory optimization.
- $\mathbf{Q}(t)$ and $\mathbf{v}_R(0)$ is calculated for the nominal trajectory **OFF-Line**

- The following differential equation is numerically solved:

$$[\dot{\mathbf{v}}_g]^I = -\mathbf{Q}[\mathbf{v}_g]^I - [\mathbf{a}_T]^I \qquad \mathbf{v}_g(0) = \mathbf{v}_R(0)$$

- Guidance commands are calculated using the Cross Product steering law:

$$\boldsymbol{\omega}_C = K \frac{\mathbf{a}_T \times \mathbf{v}_g}{|\mathbf{a}_T||\mathbf{v}_g|}$$

- $\boldsymbol{\omega}_C$ turns $\mathbf{a}_T$ such that $\mathbf{v}_g$ approaches to zero. **ON-Line**

Q* Guidance

- The differential equation of Q guidance can be reformulated to use the body accelerations instead of the inertial ones.

$$[\dot{\mathbf{v}}_g]^I = -\mathbf{Q}[\mathbf{v}_g]^I - [\mathbf{a}_T]^I$$

$$[\mathbf{v}_g]^I = [\mathbf{T}]^{IB}[\mathbf{v}_g]^B$$

$$[\dot{\mathbf{v}}_g]^I = [\dot{\mathbf{T}}]^{IB}[\mathbf{v}_g]^B + [\mathbf{T}]^{IB}[\dot{\mathbf{v}}_g]^B$$

$$[\dot{\mathbf{T}}]^{IB} = [\mathbf{T}]^{IB}[\Omega^{BI}]^B$$

$$[\dot{\mathbf{v}}_g]^I = [\mathbf{T}]^{IB}[\Omega^{BI}]^B[\mathbf{v}_g]^B + [\mathbf{T}]^{IB}[\dot{\mathbf{v}}_g]^B$$

Q* Guidance

$$[\mathbf{T}]^{IB}[\Omega^{BI}]^B[\mathbf{v}_g]^B + [\mathbf{T}]^{IB}[\dot{\mathbf{v}}_g]^B = -\mathbf{Q}[\mathbf{v}_g]^I - [\mathbf{a}_T]^I$$

- Multiplying the two sides in $[T]^{BI}$:

$$[\Omega^{BI}]^B[\mathbf{v}_g]^B + [\dot{\mathbf{v}}_g]^B = -[\mathbf{T}]^{BI}\mathbf{Q}[\mathbf{T}]^{IB}[\mathbf{v}_g]^B - [\mathbf{a}_T]^B$$

- The above DE can be rewritten as:

$$[\dot{\mathbf{v}}_g]^B = -\mathbf{Q}^*\,[\mathbf{v}_g]^B - [\mathbf{a}_T]^B$$

- Where:

$$\mathbf{Q}^* = [\mathbf{T}]^{BI}\mathbf{Q}[\mathbf{T}]^{IB} + [\Omega^{BI}]^B$$

- Flat Earth: $\mathbf{Q}^* = -\frac{1}{t_F}\mathbf{I} + [\Omega^{BE}]^B$

Preset Guidance

- A ballistic Missile or launch vehicle has a little stability immediately after launch.
- A little AOA is permitted within the atmosphere.

➔

- Closed loop guidance laws can not be implemented immediately after launch.

➔

- At first, the vehicle must be guided vertically using an attitude control system.
- Within the dense atmosphere and before Q/Q*, open-loop Pitch and Yaw Program can be executed using SAS.