In the Name of GOD

Guidance and Navigation I: Two-point Guidance Laws

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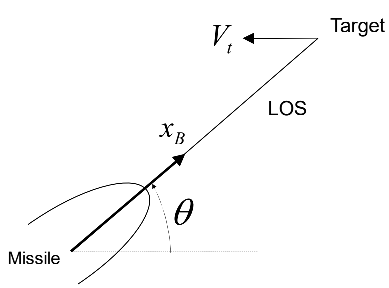
Two-point Guidance Laws

- Pursuit Guidance
- Proportional Navigation (PN) Guidance
- 3D Implementation of PN
- Analytical Solution of PN
- Simulation of PN
- Performance of PN
- Important Implementation Issues
- Linearization of PN
- Analysis of the Homing Loop Using Adjoint Theory
- Optimal Two-point Guidance
- Proportional Navigation Command Guidance
- Pulsed Guidance

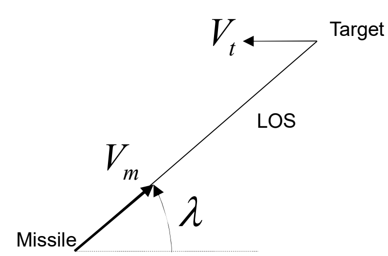
Pursuit Guidance

- Pure Pursuit
  - The longitudinal axis is always kept toward the target



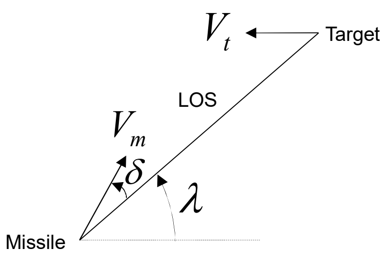
Pursuit Guidance

- Velocity Pursuit
  - The velocity vector is always kept toward the target



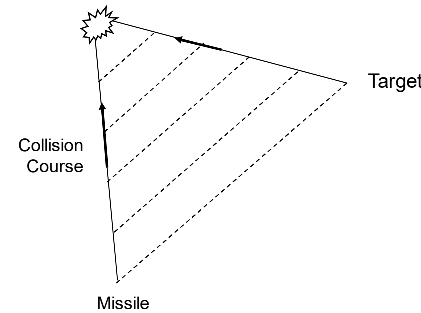
Pursuit Guidance

- Deviated Pursuit
  - The velocity vector is always kept with a lead angle WRT the target



Proportional Navigation Guidance (PN)

- Foundation
  - If the angular rate of the LOS between the interceptor and the target is kept on zero, the interceptor will intercept the target. (**provided that the interceptor is closing the target**)



Proportional Navigation Guidance (PN)

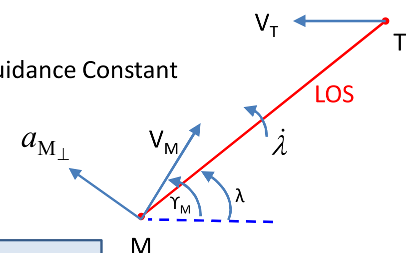
- Implementation
  - The angular LOS rate is measured by a seeker
  - The guidance commands must be applied such that they decrease the LOS rate
- History
  - The idea was born in 1942 in a technical report!
  - First Publish: Yuan, C. L., "Homing and Navigation Courses of Automatic Target-Seeking Devices," *Journal of Applied Physics, 1948*
  - First successful test: Lark missile (SAM), 1950.
  - The optimality proof (under certain assumptions): 1969

Pure PN (PPN)

- If one rotates the velocity vector proportional to the LOS rate, then the LOS rate will be decreased. That is why PN is called proportional.
- $\dot{\gamma}_M = N\dot{\lambda}$ N: Guidance Constant

  $a_{M_\perp} = V_M \dot{\gamma}_M$

  $a_{M_\perp} = N V_M \dot{\lambda}$



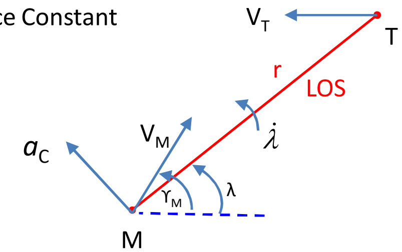
- $\boxed{a_C = N V_M \dot{\lambda} \qquad a_C = K\dot{\lambda}}$

True PN (TPN)

- It is more effective to apply the acceleration commands perpendicular to the LOS instead of the velocity vector.
- It is more effective to use the closing velocity (V_C) instead of the missile velocity (V_M). [It is optimal]
- N': Effective Guidance Constant

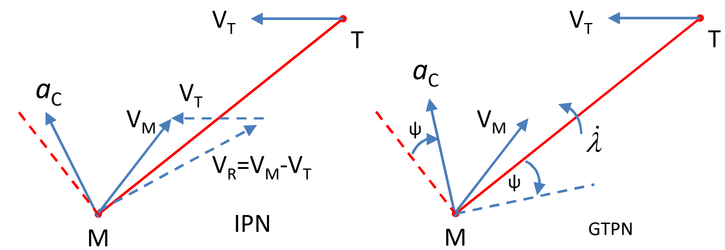
$$a_C = N' V_C \dot{\lambda}$$

$$V_C = -\dot{r}$$



Generalized TPN (GTPN) and Ideal PN (IPN)

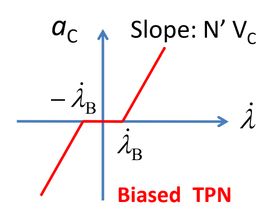
- In IPN, the acceleration commands are issued perpendicular to the relative velocity.
- In GTPN, the acceleration commands are issued with an angle, ψ, WRT the perpendicular plane to the LOS.



Biased TPN and Dead Space TPN

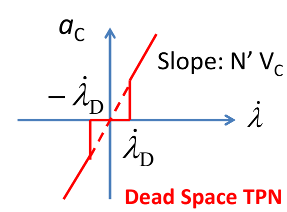
- Seeker output contains measurement noises.
- When λ_dot is small (or zero), the acceleration commands oscillates between positive and negative random values.
- There are two solutions:
  - Noise Filtering
    - Drawbacks?
  - Using Dead bands, etc.
    - Drawbacks?

Biased TPN and Dead Space TPN



$$a_C = \begin{cases} N' V_C (\dot{\lambda} - \dot{\lambda}_B) & \text{if } \dot{\lambda} > \dot{\lambda}_B \\ N' V_C (\dot{\lambda} + \dot{\lambda}_B) & \text{if } \dot{\lambda} < -\dot{\lambda}_B \\ 0 & \text{if } |\dot{\lambda}| < \dot{\lambda}_B \end{cases}$$

$$a_C = \begin{cases} N' V_C \dot{\lambda} & \text{if } |\dot{\lambda}| \geq \dot{\lambda}_D \\ 0 & \text{if } |\dot{\lambda}| < \dot{\lambda}_D \end{cases}$$



Augmented TPN (ATPN)

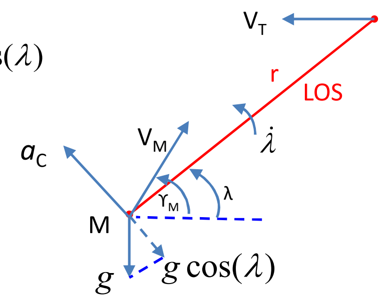
- If the target is highly maneuverable, a fraction of target maneuver can be augmented to TPN to modify the performance.
- It will be shown that:

$$a_{\mathrm{C}} = N' V_{\mathrm{C}}\, \dot{\lambda} + \frac{N'}{2} a_{\mathrm{T},\perp_{\mathrm{LOS}}}$$

Gravity acceleration compensation

- Since the accelerometers can not measure the gravity acceleration, it should be compensated.
- In TPN:

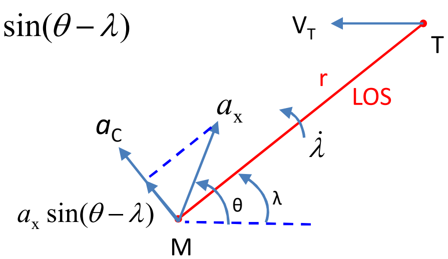
$$a_{\mathrm{C}} = N' V_{\mathrm{C}}\, \dot{\lambda} + g\cos(\lambda)$$



Axial acceleration compensation

- The component of missile axial acceleration which is perpendicular to the LOS should be compensated.

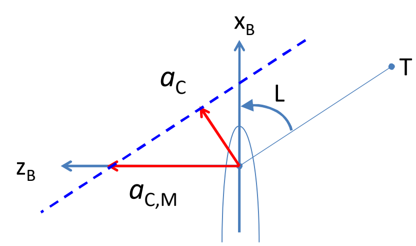
$$a_{\mathrm{C}} = N' V_{\mathrm{C}}\, \dot{\lambda} - a_{\mathrm{x}} \sin(\theta - \lambda)$$



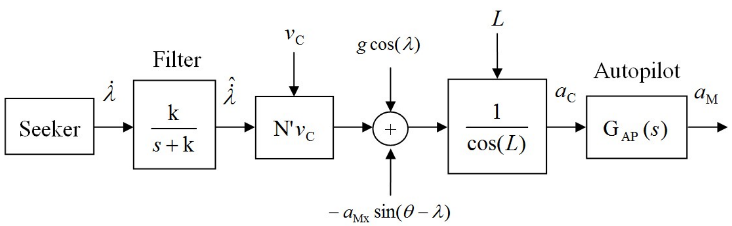
Compensation of Look Angle

- Flight control system can not execute the acceleration commands perpendicular to the LOS.
- Lateral accelerometers are perpendicular to the longitudinal axis.
- L: look angle

$$a_{\mathrm{C,M}} = \frac{N' V_{\mathrm{C}}\, \dot{\lambda}}{\cos \mathrm{L}}$$



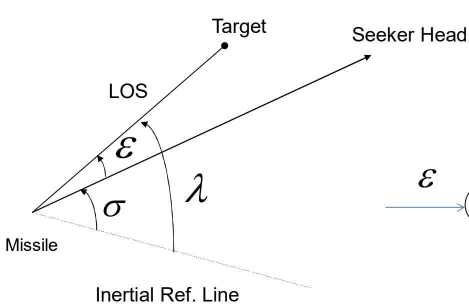
Block Diagram of TPN

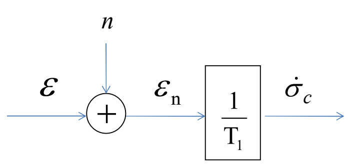


3D Formulation of PNG

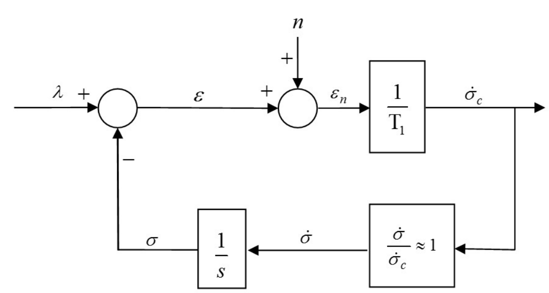
- PPN $\quad a_C = N V_M \dot{\lambda} \quad \mathbf{a}_C = N\,\boldsymbol{\omega} \times \mathbf{v}_M$
- TPN $\quad a_C = N' v_C \dot{\lambda} \quad \mathbf{a}_C = N'\boldsymbol{\omega} \times \mathbf{v}_C$
- ITPN $\quad a_C = N' v_R \dot{\lambda} \quad \mathbf{a}_C = N'\boldsymbol{\omega} \times \mathbf{v}_R$

LOS rate measurement





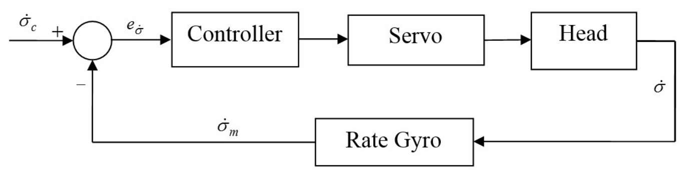
Tracking Loop and LOS rate measurement



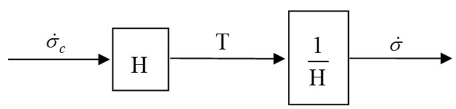
$$\frac{\dot{\sigma}_c}{\lambda} = \frac{s}{T_1 s + 1} \quad \Rightarrow \quad \frac{\dot{\sigma}_c}{\dot{\lambda}} = \frac{1}{T_1 s + 1}$$

Seeker Stabilization Loop

- Servo stabilized Seekers



- Gyro stabilized Seekers

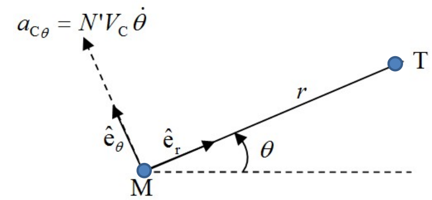


$$T = \omega \times H$$

Analytical solution of 2D TPN

- Assumptions: 1) Planar motion, 2) No target Maneuver

$$a = a_r \hat{e}_r + a_\theta \hat{e}_\theta$$



$$a_r = a_{T_r} - a_{M_r} = \ddot{r} - r\dot{\theta}^2$$

$$a_\theta = a_{T_\theta} - a_{M_\theta} = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

$$a_{M_r} = 0$$

$$a_{M_\theta} = N'V_C\dot{\theta}$$

$$a_{T_r} = a_{T_\theta} = 0$$

$$\Rightarrow \quad r\ddot{\theta} + 2\dot{r}\dot{\theta} = -N'V_C\dot{\theta}$$

$$\Rightarrow \quad r\ddot{\theta} + 2\dot{r}\dot{\theta} - N'\dot{r}\dot{\theta} = 0$$

Analytical solution of 2D TPN

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} - N'\dot{r}\dot{\theta} = 0$$

$$\frac{\ddot{\theta}}{\dot{\theta}} = (N'-2)\frac{\dot{r}}{r}$$

$$\frac{d\dot{\theta}}{\dot{\theta}} = (N'-2)\frac{dr}{r}$$

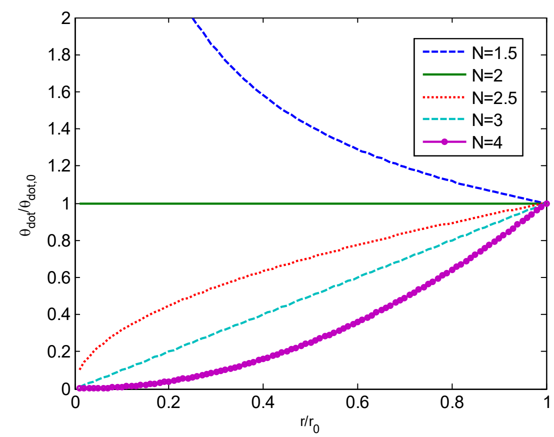
$$\mathrm{Ln}\,\dot{\theta}\Big|_{t_0}^{t} = (N'-2)\,\mathrm{Ln}\,r\Big|_{t_0}^{t}$$

$$\mathrm{Ln}\frac{\dot{\theta}}{\dot{\theta}_0} = (N'-2)\,\mathrm{Ln}\frac{r}{r_0}$$

$$\mathrm{Ln}\frac{\dot{\theta}}{\dot{\theta}_0} = \mathrm{Ln}\left(\frac{r}{r_0}\right)^{(N'-2)}$$

$$\frac{\dot{\theta}}{\dot{\theta}_0} = \left(\frac{r}{r_0}\right)^{(N'-2)}$$

Analytical solution of 2D TPN



Simulation of PNG

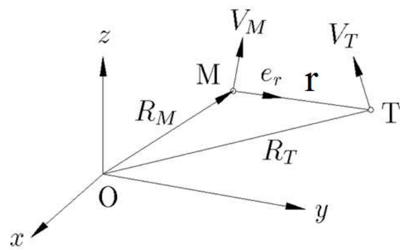
- Seeker can measure the components of LOS rate which are perpendicular to the LOS.
- To simulate the output of seeker, $\underline{\Omega}_{\perp}$ must be calculated

$$\underline{R} = \underline{R}_{Target} - \underline{R}_{Missile}$$

$$\underline{V} = \underline{V}_{Target} - \underline{V}_{Missile}$$

$$\underline{R} = r\,\hat{e}_r$$

$$\underline{V} = \dot{r}\,\hat{e}_r + \underline{\Omega}\times\underline{R}$$



Calculation of LOS rates

$$\underline{\Omega}_{\perp} = \underline{\Omega} - (\underline{\Omega}\cdot\hat{e}_r)\,\hat{e}_r$$

$$\underline{R}\times\underline{V} = \dot{r}\,(\cancelto{0}{\underline{R}\times\hat{e}_r}) + \underline{R}\times(\underline{\Omega}\times\underline{R})$$

$$\underline{R}\times\underline{V} = \|\underline{R}\|^2[\hat{e}_r\times(\underline{\Omega}\times\hat{e}_r)]$$

$$\underline{A}\times(\underline{B}\times\underline{C}) = (\underline{C}\cdot\underline{A})\underline{B} - (\underline{A}\cdot\underline{B})\underline{C}$$

$$\hat{e}_r\times(\underline{\Omega}\times\hat{e}_r) = (\hat{e}_r\cdot\hat{e}_r)\underline{\Omega} - (\hat{e}_r\cdot\underline{\Omega})\hat{e}_r = \underline{\Omega} - (\underline{\Omega}\cdot\hat{e}_r)\hat{e}_r = \underline{\Omega}_{\perp}$$

$$\underline{\Omega}_{\perp} = \frac{\underline{R}\times\underline{V}}{\|\underline{R}\|^2}$$

Simulation of Seeker Outputs

$$\underline{\Omega}_{\perp}^{L} = \frac{\underline{R}^{L}\times\underline{V}^{L}}{\|\underline{R}^{L}\|^2}$$

$$\underline{\Omega}_{\perp}^{S} = \mathbf{C}_{B}^{S}\,\mathbf{C}_{L}^{B}\,\underline{\Omega}_{\perp}^{L} = \begin{bmatrix}0 & \dot{\sigma}_{2s} & \dot{\sigma}_{3s}\end{bmatrix}$$

$$\dot{\sigma}_{2m} = G(s)\,\dot{\sigma}_{2s} \qquad \dot{\sigma}_{3m} = G(s)\,\dot{\sigma}_{3s}$$

$$\underline{\Omega}_{\perp m}^{B} = \begin{Bmatrix}\dot{\sigma}_{1B}\\ \dot{\sigma}_{2B}\\ \dot{\sigma}_{3B}\end{Bmatrix} = \mathbf{C}_{S}^{B}\begin{Bmatrix}0\\ \dot{\sigma}_{2m}\\ \dot{\sigma}_{3m}\end{Bmatrix}$$

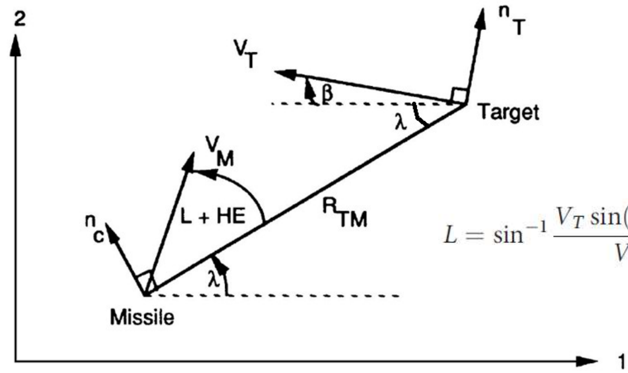
Calculation of Guidance Commands

- e.g.: 3D PPN $\quad \underline{a}_{c}^{B} = \mathrm{N}\,\underline{\Omega}_{\perp m}^{B}\times\underline{V}_{M}^{B}$
- Assuming Small angle of attack and side slip:

$$\underline{a}_{c}^{B} = \mathrm{N}\begin{bmatrix}0 & -\dot{\sigma}_{3B} & \dot{\sigma}_{2B}\\ \dot{\sigma}_{3B} & 0 & -\dot{\sigma}_{1B}\\ -\dot{\sigma}_{2B} & \dot{\sigma}_{1B} & 0\end{bmatrix}\begin{bmatrix}V_M\\ 0\\ 0\end{bmatrix} = \begin{bmatrix}0\\ \mathrm{N}V_M\dot{\sigma}_{3B}\\ -\mathrm{N}V_M\dot{\sigma}_{2B}\end{bmatrix}$$

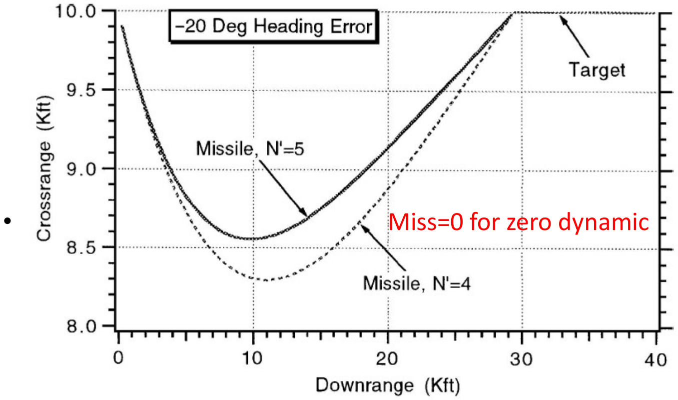
- Compensation of gravity and axial acceleration can also be made before applying the commands to AP.

Performance of TPN

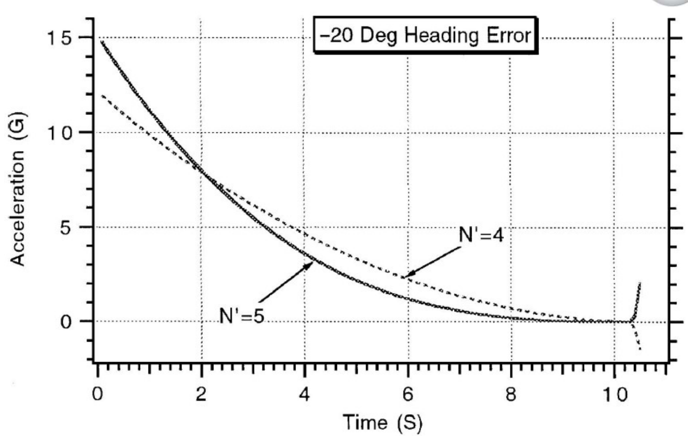


$$L = \sin^{-1} \frac{V_T \sin(\beta + \lambda)}{V_M}$$

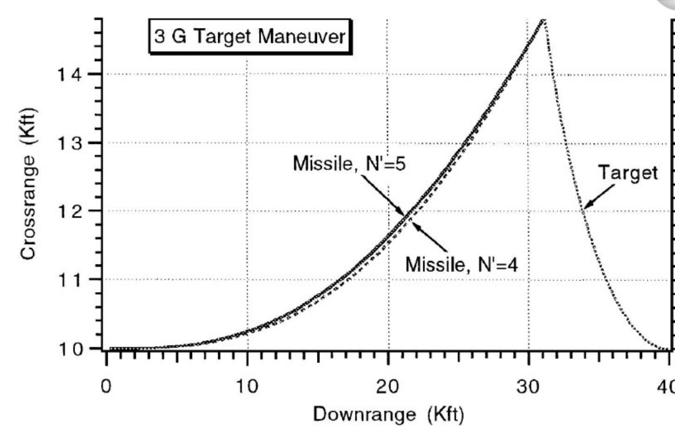
Performance of TPN



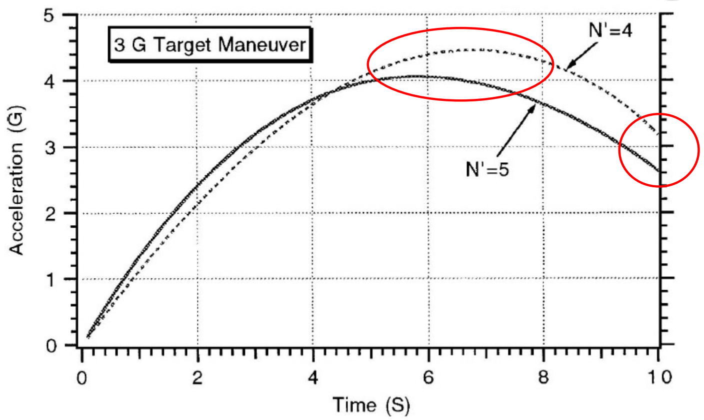
Performance of TPN



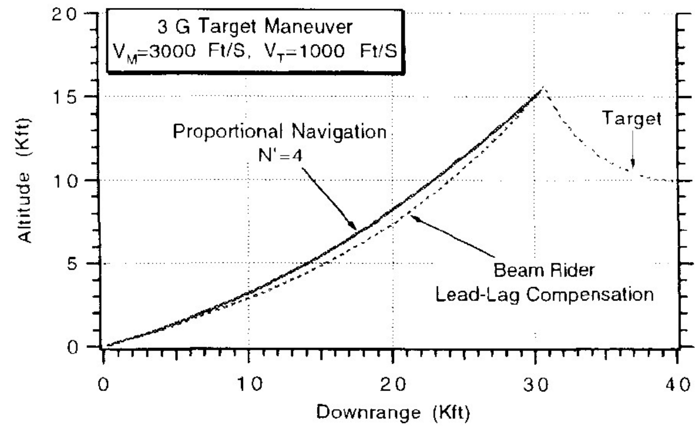
Performance of TPN



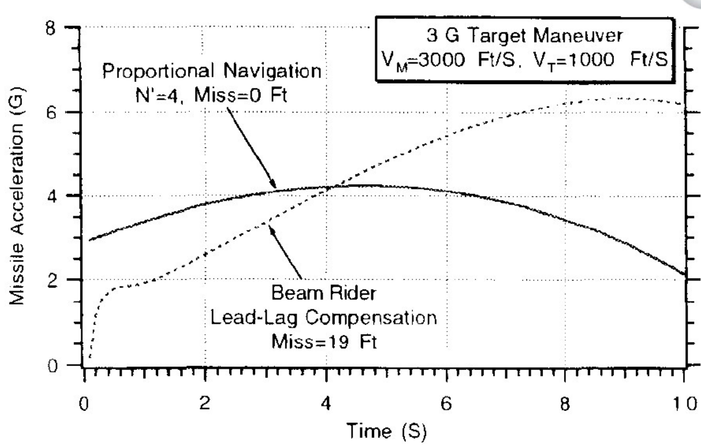
Performance of TPN



Comparison of Basic LOS and PN: Maneuvering Target



Comparison of Beam Rider and PN: Maneuvering Target



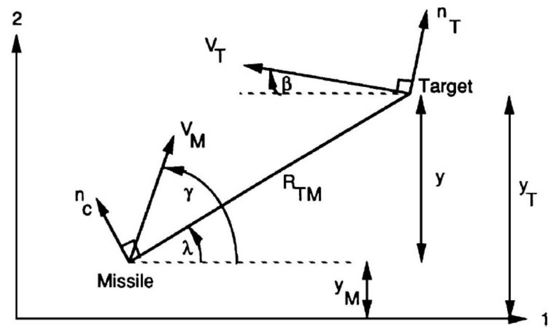
Important Implementation Issues

- The acceleration commands must be saturated.
- Rate of Commands is limited.
- A strapdown seeker has a limited field of view.
- A gimbaled seeker has a limited look angle (field of regard) and a limited gimbal rate.
- Seeker has a minimum range and a maximum range.
- Although it is not theoretically necessary, it is better to control the roll angle in a homing missile (or at least roll rate).

Linearization of PN

- TPN has a nonlinear differential equation as follows:

$$\ddot{y} = n_T \cos\beta - n_c \cos\lambda$$



Linearized differential equation of TPN

- y is defined as the relative separation between the missile and target perpendicular to a fixed reference line.
- Flight path angles are small near head-on or tail chase case.

$$\boxed{\ddot{y} = n_T - n_c}$$

- Assuming small λ,

$$\lambda = y/R_{\mathrm{TM}}$$

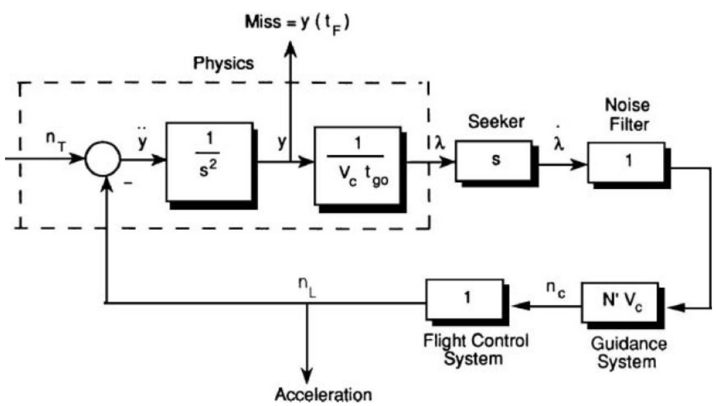
- Where

$$R_{\mathrm{TM}} = V_c(t_F - t)$$

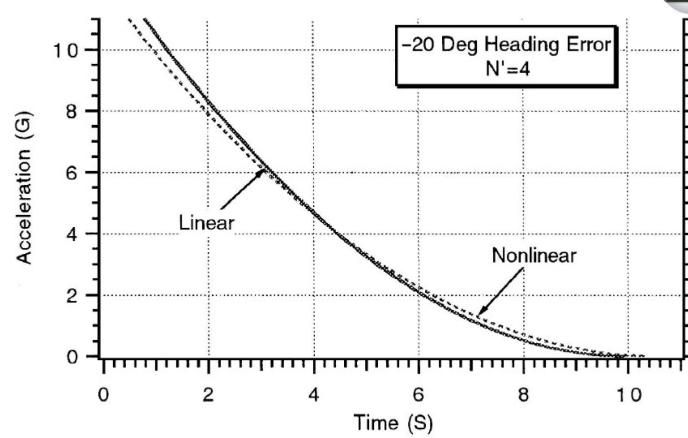
- t_F is the total flight time
- The linearized miss distance is defined as

$$\mathrm{Miss} = y(t_F)$$

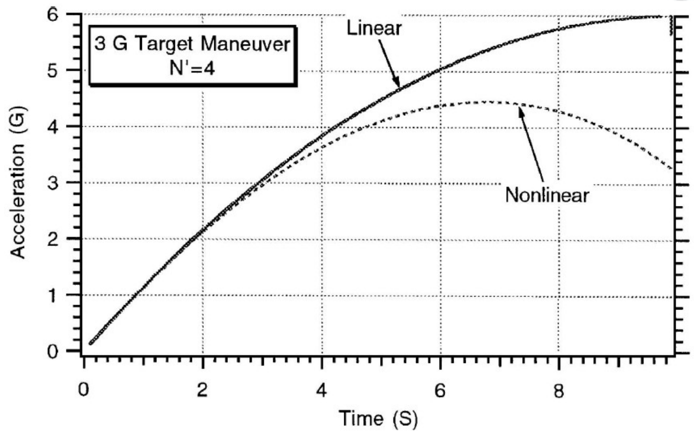
Simple proportional navigation guidance loop: a **zero-lag** guidance system



Validation of the linear TPN in the presence of Heading Error



Validation of the linear TPN in the presence of Target Maneuver



Closed-form Solution of Linear TPN against Heading Error

- Considering No target maneuver, linear differential equation of TPN:

$$\ddot{y} = n_T - n_c \Rightarrow \ddot{y} = -N' V_c \dot{\lambda}$$

- Initial Conditions:

$$y(0) = 0 \qquad \dot{y}(0) = -V_M HE$$

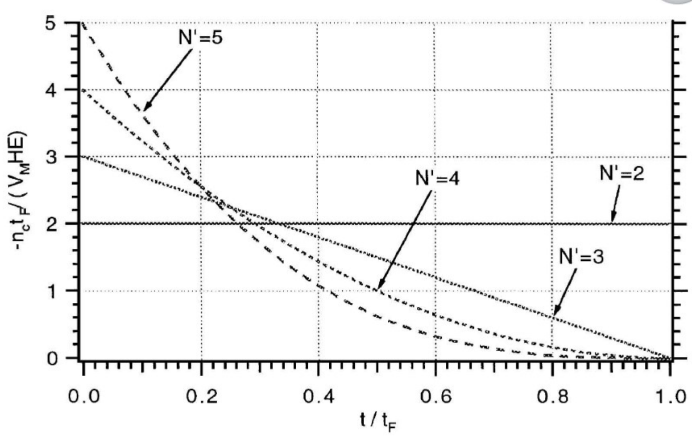
- Where:

$$\lambda = y/R_{\mathrm{TM}} \qquad R_{\mathrm{TM}} = V_c(t_F - t)$$

- Solving the above linear differential equation yields:

$$n_c = \frac{-V_M HE\, N'}{t_F}\left(1 - \frac{t}{t_F}\right)^{N'-2}$$

Normalized acceleration due to heading error



Closed-form Solution of Linear TPN against Target Maneuver

- TPN linear differential equation

$$\ddot{y} = n_T - n_c \Rightarrow \ddot{y} = -N' V_c \dot{\lambda} + n_T$$

- Initial Conditions:

$$y(0) = 0 \qquad \dot{y}(0) = 0$$

- Where:

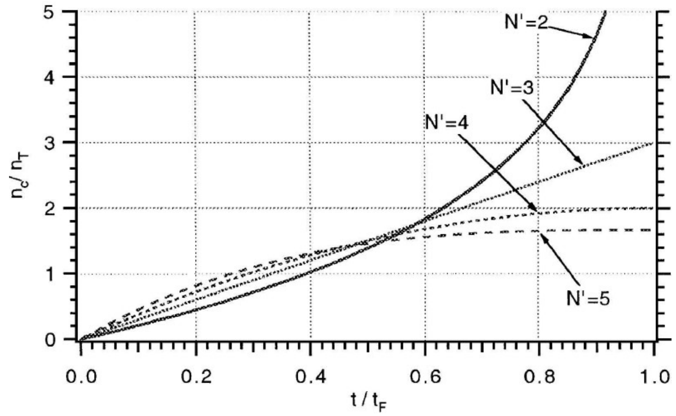
$$\lambda = y/R_{\mathrm{TM}} \qquad R_{\mathrm{TM}} = V_c(t_F - t)$$

- Solving the above linear differential equation yields:

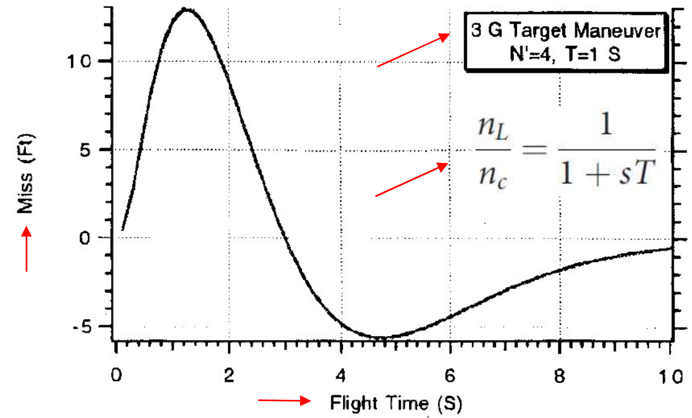
$$n_c = \frac{N'}{N'-2}\left[1 - \left(1 - \frac{t}{t_F}\right)^{N'-2}\right] n_T$$

$$\lim_{N' \to 2} n_c = -2\,\ell n\left(\frac{t_F - t}{t_F}\right) n_T$$

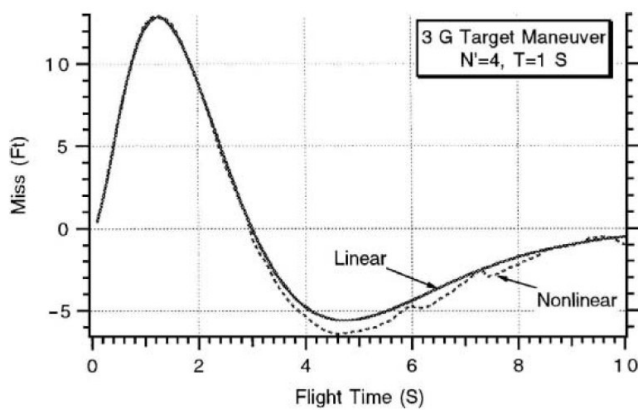
Normalized acceleration due to Target Maneuver



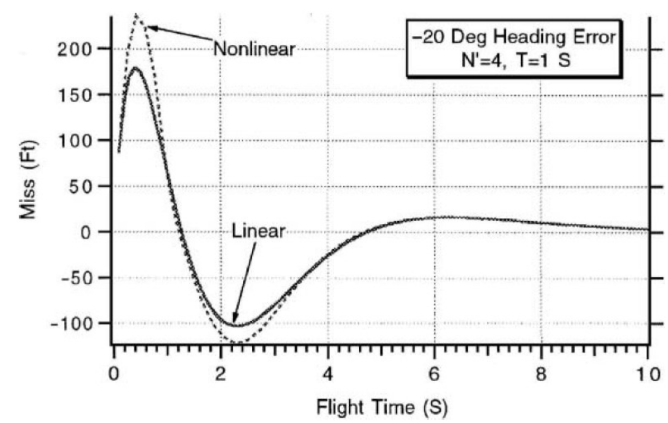
Analysis of the Homing Loop Using Adjoint Theory: Single Time Constant Guidance System



The comparison between linear and nonlinear model in predicting Miss due to Target Maneuver



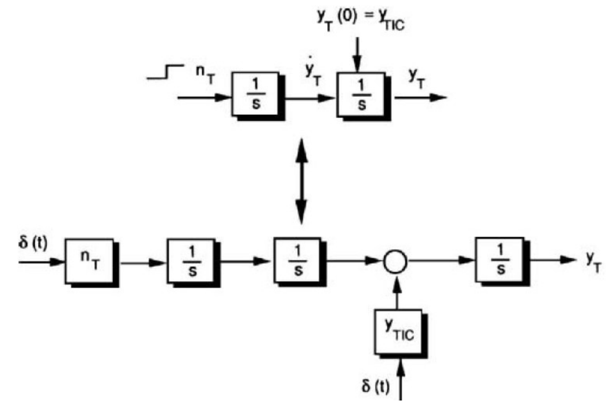
The comparison between linear and nonlinear model in predicting Miss due to Heading Error



Adjoint System

- The miss distance, obtained from many simulation trials, can be obtained in one computer run using the simulation of adjoint system instead of the original system
- Definition: For every linear deterministic system there exists an adjoint system that can be constructed from the block diagram of the original system using the following rules:
  - Convert all system inputs and initial conditions to impulses
  - Replace t by $t_F - t$ in all time-varying coefficients
  - Reverse all signal flow
  - Convert nodes to summing junctions and vice versa

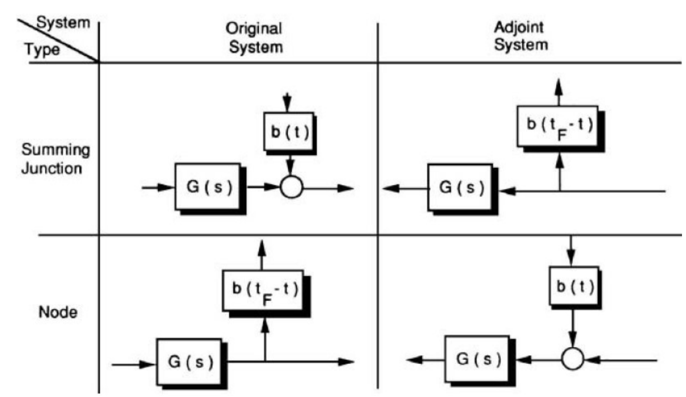
1- Replace steps and initial conditions by impulses



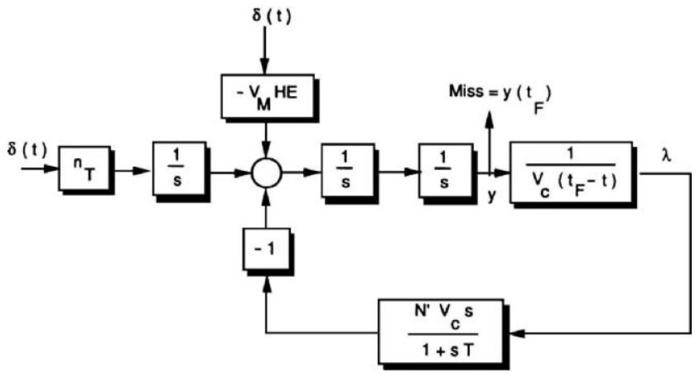
2- Replace t by $t_F - t$

System / Function	Original System	Adjoint System
Time Varying Gain	$K(t) = at+b$ $K(t) = \frac{1}{a(t_F - t) + b}$	$K(t_F - t) = a(t_F - t) + b$ $K(t_F - t) = \frac{1}{at + b}$
Table	t: 0, 1, 2, 3 K: 8, 4, 3, 9	t: 0, 1, 2, 3 K: 9, 3, 4, 8

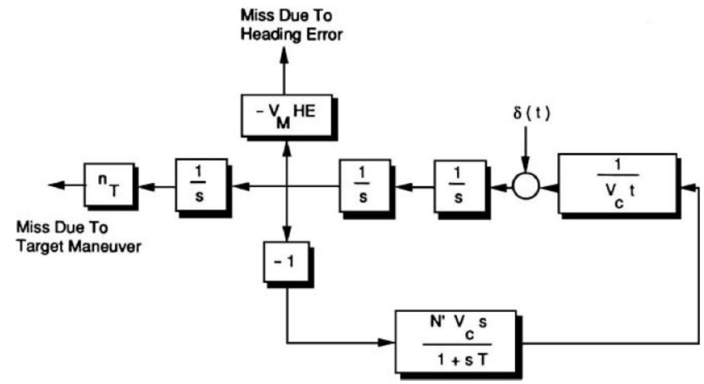
3,4- Reverse all signal flow, convert nodes to summing junctions and vice versa



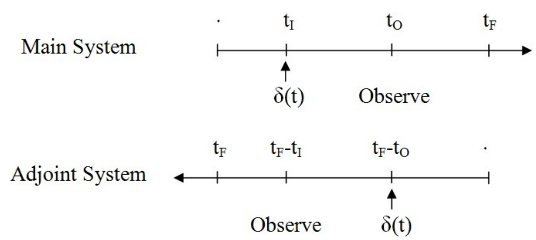
Example:
Single-lag proportional navigation homing loop



Example:
Miss distance adjoint of single-lag PN loop



Adjoint Mathematics

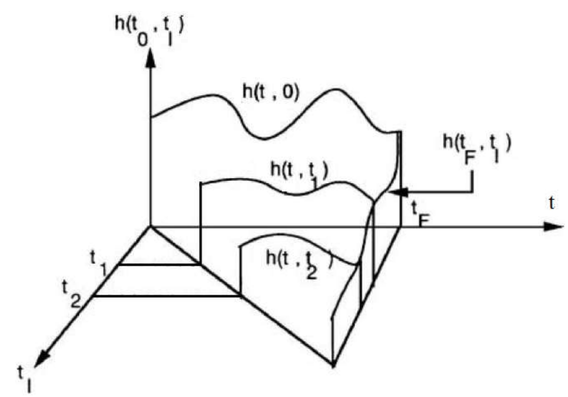


- The relation between the impulse response of the adjoint system (h^*) and the original system (h) can be stated as:

$$h^*(t_F - t_I, t_F - t_O) = h(t_O, t_I)$$

- t_I: impulse application time
- t_O: impulse observation time

Adjoint Mathematics



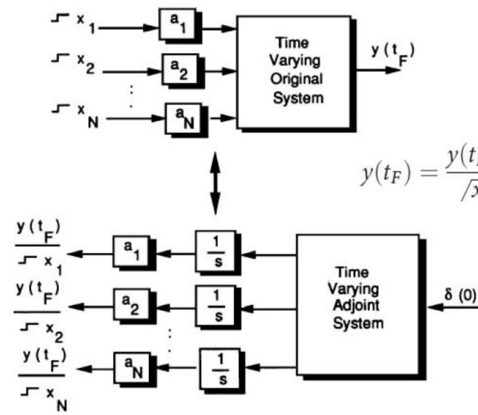
Adjoint Mathematics

- In special case when $t_O=t_F$:

$$h^*(t_F - t_I, 0) = h(t_F, t_I)$$

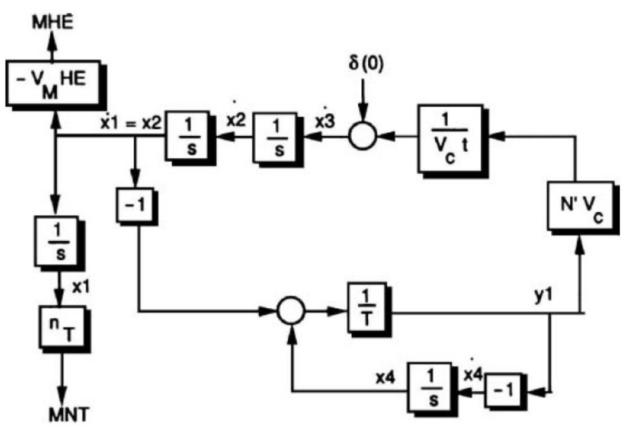
- If one is going to observe the impulse response of the original system at time t_F due to various impulse application times t_I, i.e. $h(t_F, t_I)$, instead of running the original system for different impulse times, only one adjoint response is needed to be simulated.
- If one obtain the adjoint system response to the impulse, applied in t=0, each point of the response corresponds to one flight time, considered for the original system.

Error budgeting with adjoint theory



$$y(t_F) = \frac{y(t_F)}{/x_1} + \frac{y(t_F)}{/x_2} + \cdots + \frac{y(t_F)}{/x_N}$$

Adjoint System and Sensitivity Analysis



Adjoint Closed Form Solution for 1st order FCS

$$\left.\frac{\text{Miss}}{-V_M HE}\right|_{N'=3} = t_F e^{-t_F/T}\left(1 - \frac{t_F}{2T}\right)$$

$$\left.\frac{\text{Miss}}{-V_M HE}\right|_{N'=4} = t_F e^{-t_F/T}\left(1 - \frac{t_F}{T} + \frac{t_F^2}{6T^2}\right)$$

$$\left.\frac{\text{Miss}}{-V_M HE}\right|_{N'=5} = t_F e^{-t_F/T}\left(1 - 1.5\frac{t_F}{T} + \frac{t_F^2}{2T^2} + \frac{t_F^3}{24T^3}\right)$$

Due to Heading Error

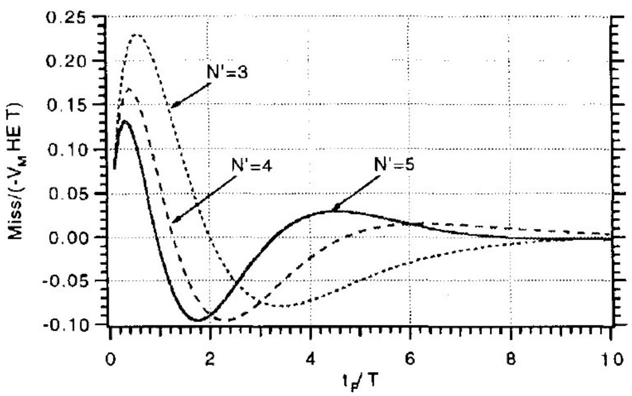
$$\left.\frac{\text{Miss}}{n_T}\right|_{N'=3} = 0.5 t_F^2 e^{-t_F/T}$$

$$\left.\frac{\text{Miss}}{n_T}\right|_{N'=4} = t_F^2 e^{-t_F/T}\left(0.5 - \frac{t_F}{6T}\right)$$

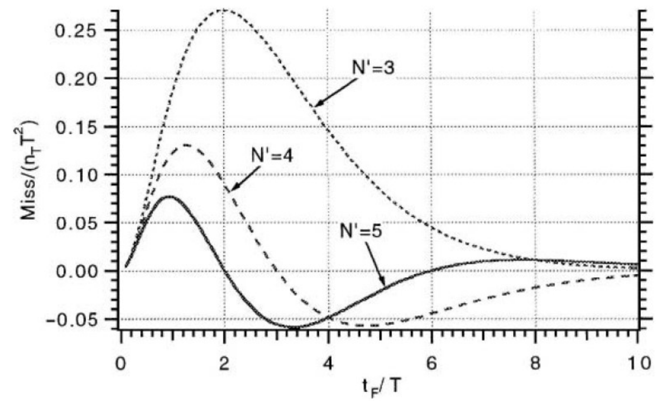
$$\left.\frac{\text{Miss}}{n_T}\right|_{N'=5} = t_F^2 e^{-t_F/T}\left(0.5 - \frac{t_F}{3T} + \frac{t_F^2}{24T^2}\right)$$

Due to Target Maneuver

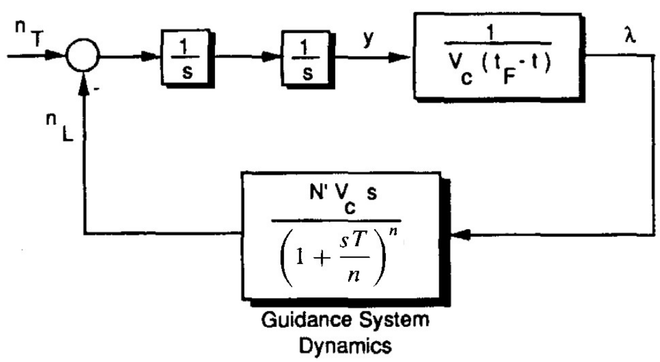
Normalized Heading Error Miss Sensitivity



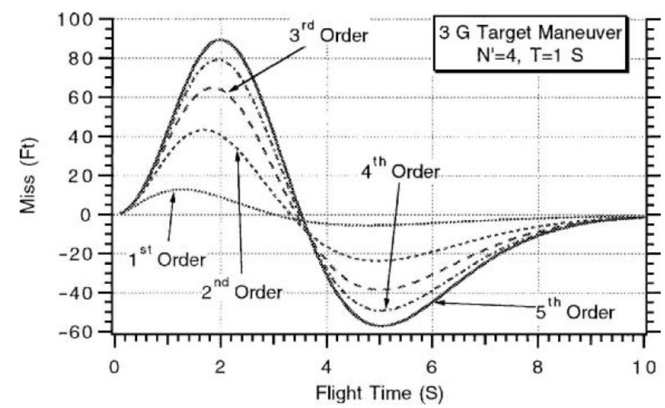
Normalized Target Maneuver Miss Sensitivity



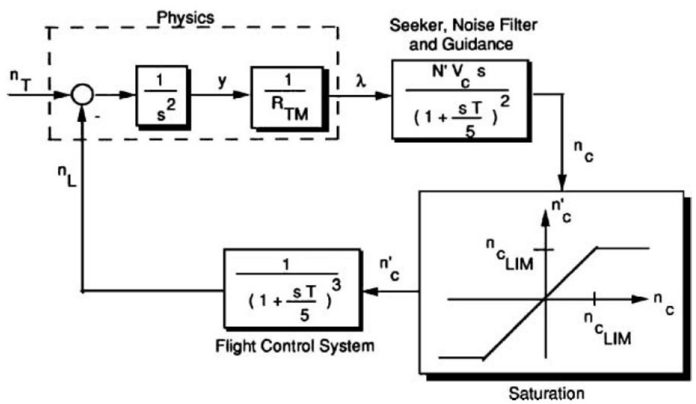
Performance of TPN: Effect of system order



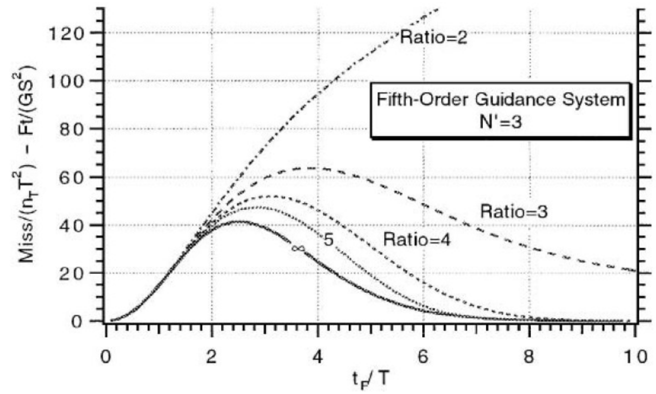
Performance of TPN: Effect of system order



Performance of TPN: Saturation Effect



Performance of TPN: Saturation Effect (N'=3)



Performance of TPN: Saturation Effect (N'=4)