In the Name of GOD

Guidance and Navigation I: Initial Alignment of INS + Error Analysis

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Alignment of INS

- The process by which the orientation of an INS is determined with respect to the reference axis system.
- Accurate alignment is crucial, if precision navigation is to be achieved over long periods of time without any form of aiding.
- In many applications, alignment must be performed within a very short period of time.
- Fundamental Types of Alignment
  - self-alignment, using gyro-compassing techniques
  - the alignment of a slave system with respect to a master reference

Coarse Alignment on a fixed IMU

- the accelerometers measure three orthogonal components of the specific force needed to overcome gravity
- the gyroscopes measure the components of the Earth's turn rate.

$$[\tilde{\mathbf{f}}]^{B} = [-\mathbf{g}_L]^{B} = \begin{bmatrix} \tilde{f}_x \\ \tilde{f}_y \\ \tilde{f}_z \end{bmatrix} \qquad [\tilde{\boldsymbol{\omega}}^{BI}]^{B} = [\boldsymbol{\omega}^{EI}]^{B} = \begin{bmatrix} \tilde{\omega}_x \\ \tilde{\omega}_y \\ \tilde{\omega}_z \end{bmatrix}$$

- Moreover,

$$[-\mathbf{g}_L]^{B} = [T]^{BN}[-\mathbf{g}_L]^{N} \qquad [\boldsymbol{\omega}^{EI}]^{B} = [T]^{BN}[\boldsymbol{\omega}^{EI}]^{N}$$

Coarse Alignment on a fixed IMU

- Therefore

$$\Big[[-\mathbf{g}_L]^{B} \quad [\boldsymbol{\omega}^{EI}]^{B} \quad [-\mathbf{g}_L]^{B} \times [\boldsymbol{\omega}^{EI}]^{B}\Big] = [T]^{BN}\Big[[-\mathbf{g}_L]^{N} \quad [\boldsymbol{\omega}^{EI}]^{N} \quad [-\mathbf{g}_L]^{N} \times [\boldsymbol{\omega}^{EI}]^{N}\Big]$$

- Define:

$$\mathbf{M} = \Big[[-\mathbf{g}_L]^{N} \quad [\boldsymbol{\omega}^{EI}]^{N} \quad [-\mathbf{g}_L]^{N} \times [\boldsymbol{\omega}^{EI}]^{N}\Big]$$

- Therefore:

$$[T]^{BN} = \Big[[-\mathbf{g}_L]^{B} \quad [\boldsymbol{\omega}^{EI}]^{B} \quad [-\mathbf{g}_L]^{B} \times [\boldsymbol{\omega}^{EI}]^{B}\Big]\mathbf{M}^{-1}$$

Coarse Alignment on a fixed IMU

- Analytically:

$$[-\mathbf{g}_L]^N = \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} \qquad [\boldsymbol{\omega}^{EI}]^N = \begin{bmatrix} \omega^{EI}\cos\lambda \\ 0 \\ -\omega^{EI}\sin\lambda \end{bmatrix}$$

- Therefore:

$$\mathbf{M}^{-1} = \left[[-\mathbf{g}_L]^N \quad [\boldsymbol{\omega}^{EI}]^N \quad [-\mathbf{g}_L]^N \times [\boldsymbol{\omega}^{EI}]^N \right]^{-1}$$

$$= \begin{bmatrix} \dfrac{-\tan\lambda}{g} & 0 & \dfrac{-1}{g} \\ \dfrac{1}{\omega^{EI}\cos\lambda} & 0 & 0 \\ 0 & \dfrac{-1}{g\omega^{EI}\cos\lambda} & 0 \end{bmatrix}$$

Coarse Alignment on a fixed IMU

- Finally:

$$[T]^{BN} \approx \begin{bmatrix} \dfrac{\sec\lambda}{\omega^{EI}}\tilde{\omega}_x - \dfrac{\tan\lambda}{g}\tilde{f}_x & \dfrac{\sec\lambda}{g\omega^{EI}}(\tilde{f}_z\tilde{\omega}_y - \tilde{f}_y\tilde{\omega}_z) & \dfrac{-1}{g}\tilde{f}_x \\ \dfrac{\sec\lambda}{\omega^{EI}}\tilde{\omega}_y - \dfrac{\tan\lambda}{g}\tilde{f}_y & \dfrac{\sec\lambda}{g\omega^{EI}}(\tilde{f}_x\tilde{\omega}_z - \tilde{f}_z\tilde{\omega}_x) & \dfrac{-1}{g}\tilde{f}_y \\ \dfrac{\sec\lambda}{\omega^{EI}}\tilde{\omega}_z - \dfrac{\tan\lambda}{g}\tilde{f}_z & \dfrac{\sec\lambda}{g\omega^{EI}}(\tilde{f}_y\tilde{\omega}_x - \tilde{f}_x\tilde{\omega}_y) & \dfrac{-1}{g}\tilde{f}_z \end{bmatrix}$$

- Euler angles can be obtained from $[T]^{BN}$.
- The time average of sensor outputs is used.

$$\mathbf{C}_b^n = \begin{bmatrix} \cos\theta\cos\psi & -\cos\phi\sin\psi + \sin\phi\sin\theta\cos\psi & \sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi \\ \cos\theta\sin\psi & \cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi & -\sin\phi\cos\psi + \cos\phi\sin\theta\sin\psi \\ -\sin\theta & \sin\phi\cos\theta & \cos\phi\cos\theta \end{bmatrix}$$

Alignment Error

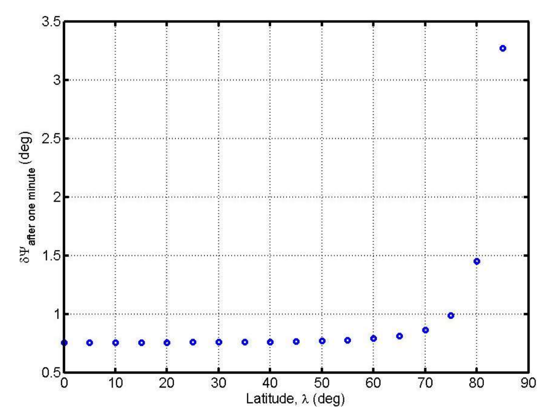
- It can be shown that:

$$\begin{cases} \varepsilon\phi = \dfrac{1}{g}\delta f_e \\ \varepsilon\theta = -\dfrac{1}{g}\delta f_n \\ \varepsilon\psi = \dfrac{\sec\lambda}{\omega^{EI}}\delta\omega_e - \dfrac{\tan\lambda}{g}\delta f_e \end{cases}$$

- For example:

$$|\varepsilon\psi| \le \frac{|\sec\lambda|}{\omega^{EI}}|\delta\omega_e| + \frac{|\tan\lambda|}{g}|\delta f_e|$$

Effect of Latitude on Alignment Error



Propagation of Errors in a 2D Strapdown INS

Navigation error equations

$$\delta\dot{\theta} = \delta\omega_{yb}$$

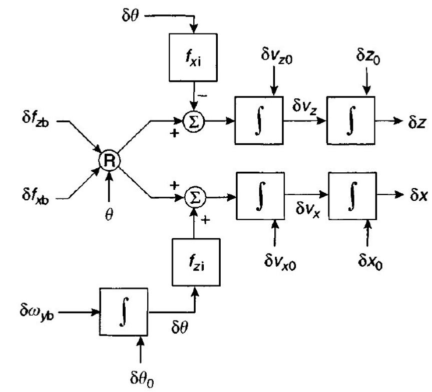
$$\delta f_{xi} = (-f_{xb}\sin\theta + f_{zb}\cos\theta)\delta\theta + \delta f_{xb}\cos\theta + \delta f_{zb}\sin\theta$$
$$= f_{zi}\delta\theta + \delta f_{xb}\cos\theta + \delta f_{zb}\sin\theta$$

$$\delta f_{zi} = -(f_{xb}\cos\theta + f_{zb}\sin\theta)\delta\theta - \delta f_{xb}\sin\theta + \delta f_{zb}\cos\theta$$
$$= -f_{xi}\delta\theta - \delta f_{xb}\sin\theta + \delta f_{zb}\cos\theta$$

$$\delta\dot{v}_{xi} = \delta f_{xi} \qquad \delta\dot{v}_{zi} = \delta f_{zi} \qquad \delta\dot{x}_i = \delta v_{xi} \qquad \delta\dot{z}_i = \delta v_{zi}$$

Error Block Diagram in Inertial Frame

- Error Sources
  - Initial Position Error
  - Initial Velocity Error
  - Initial Attitude Error
  - Accelerometers' Bias
  - Gyros' Bias
- More Sources
  - Imperfections in the Gravitational Field
  - Computational Error



Propagation of Errors in Inertial Frame

Error source		Position error: x-axis	Position error: z-axis
Initial position errors	δx_0	δx_0	–
	δz_0	–	δz_0
Initial velocity errors	δv_{x0}	$\delta v_0 t$	–
	δv_{z0}	–	$\delta v_{z0} t$
Initial attitude error	$\delta\theta_0$	$\delta\theta_0 f_{zi}\frac{t^2}{2}$	$-\delta\theta_0 f_{xi}\frac{t^2}{2}$
Accelerometer biases	δf_{xb}	$\delta f_{xb}\cos\theta\frac{t^2}{2}$	$-\delta f_{xb}\sin\theta\frac{t^2}{2}$
	δf_{zb}	$\delta f_{zb}\sin\theta\frac{t^2}{2}$	$\delta f_{zb}\cos\theta\frac{t^2}{2}$
Gyroscope bias	$\delta\omega_{yb}$	$\delta\omega_{yb} f_{zi}\frac{t^3}{6}$	$-\delta\omega_{yb} f_{xi}\frac{t^3}{6}$

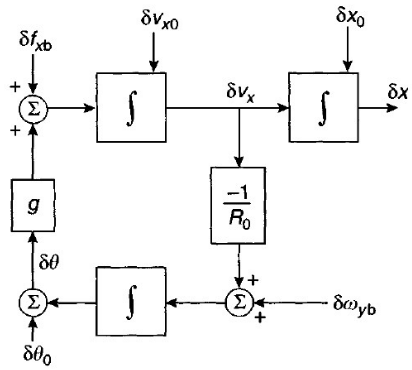
2D Navigation in the Local Geographic Frame

- x and z are coincident with the local horizontal and local vertical, respectively.
- The propagation of error can be stated as follows:

Horizontal channel equations	Vertical channel equations
$\delta\dot{\theta} = \delta\omega_{yb} - \delta v_x/R_0$ $\delta\dot{v}_x = g\delta\theta + \delta f_{xb}$ $\delta\dot{x} = \delta v_x$	$\delta\dot{v}_z = \delta f_{zb}$ $\delta\dot{z} = \delta v_z$

2D Navigation in the Local Geographic Frame

- Schuler Loop
- The horizontal error dynamic is oscillatory.
- Schuler Pendulum



Stability of the Schuler Loop

- Characteristic Equation

$$1+\frac{g}{s^2 R_0}=0 \qquad s^2+\frac{g}{R_0}=0 \qquad s^2+\omega_s^2=0$$

- A simple harmonic motion

$$\omega_s = \sqrt{g/R_0} = 0.00124\,\text{rad/s}$$

$$T_s = \frac{2\pi}{\omega_s} = 2\pi\sqrt{\frac{R_0}{g}} = 84.4\,\text{min}$$

Single Axis Error Propagation

Error source	Position error
Initial position error (δx_0)	δx_0
Initial velocity error (δv_0)	$\delta v_0\left(\frac{\sin\omega_s t}{\omega_s}\right)$
Initial attitude error ($\delta\theta_0$)	$\delta\theta_0 R_0(1-\cos\omega_s t)$
Fixed acceleration bias (δf_{xb})	$\delta f_{xb}\left(\frac{1-\cos\omega_s t}{\omega_s^2}\right)$
Fixed angular rate bias ($\delta\omega_{yb}$)	$\delta\omega_{yb} R_0\left(t-\frac{\sin\omega_s t}{\omega_s}\right)$

Single Axis Error Propagation

- Over long periods of time, several Schuler periods or more, the errors in the single navigation system are bounded as a result of Schuler tuning.
- The above conclusion is true for all sources of error with except for the bias of gyroscope which gives rise to a linearly increasing position error, in addition to an oscillatory component.
- Performance of gyroscope is critical in the achievement of long term system accuracy.
- Performance of the INS can be deducted solely from the knowledge of gyroscopic measurement accuracy.