Sharif University of Technology
Institute of Aerospace

Homework #1

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Course: *Optimal Control I* – Professor: *Dr. Assadian*
Due date: *March 28th, 2025*

Problem 1

(a) $z = f(x, y) = y\sin(x+y) - x\sin(x-y)$
Gradient of f(x, y):

$$\vec{\nabla} f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}$$

$$\vec{\nabla} f = \begin{bmatrix} y\cos(x+y) - \sin(x-y) - x\cos(x-y) \\ y\cos(x+y) + \sin(x+y) + x\cos(x-y) \end{bmatrix}$$

Two nonlinear equations with two unknowns. We use MATLAB to solve this equations. MATLAB file is attached. Answers are provided in table 1 .

Table 1: Answers

x	y
-3.41877	-1.82764
-2.88904	1.84693
-2.02875	0.00000
-1.84693	-2.88904
-1.82764	3.41877
-1.75560	0.36547
-0.36547	-1.7556
0.00000	-2.02875
0.00000	0.00000
0.00000	2.02875
0.36547	1.7556
1.75560	-0.36547
1.82764	-3.41877
1.84693	2.88904
2.02875	0.00000
2.88904	-1.84693
3.41877	1.82764

Hessian matrix:

$$H = \frac{\partial^2 f}{\partial \vec{X}^2} = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial xy} \\ \frac{\partial^2 f}{\partial yx} & \frac{\partial f}{\partial y^2} \end{bmatrix}$$

$$-y\sin(x+y) - 2\cos(x-y) + x\cos(x-y) \qquad \cos(x+y) - y\sin(x+y) + cos(x-y) - x\sin(x-y)$$

$$cos(x+y) - y\sin(x+y) + \cos(x-y) - x\sin(x-y) \qquad x\sin(x-y) + 2\cos(x+y) - y\sin(x+y)$$

Hessian matrix and eigenvalues have calculated in MATLAB and attached.

Table 2: Answers With Conditions

x	y	Point Condition
-3.41877	-1.82764	Minimum
-2.88904	1.84693	Saddle Point
-2.02875	0.00000	Saddle Point
-1.84693	-2.88904	Saddle Point
-1.82764	3.41877	Maximum
-1.75560	0.36547	Minimum
-0.36547	-1.7556	Maximum
0.00000	-2.02875	Saddle Point
0.00000	0.00000	Saddle Point
0.00000	2.02875	Saddle Point
0.36547	1.7556	Maximum
1.75560	-0.36547	Saddle Point
1.82764	-3.41877	Maximum
1.84693	2.88904	Saddle Point
2.02875	0.00000	Saddle Point
2.88904	-1.84693	Saddle Point
3.41877	1.82764	Minimum

Answers and conditions are provided in table 2

Figure 1: 3D figure of function

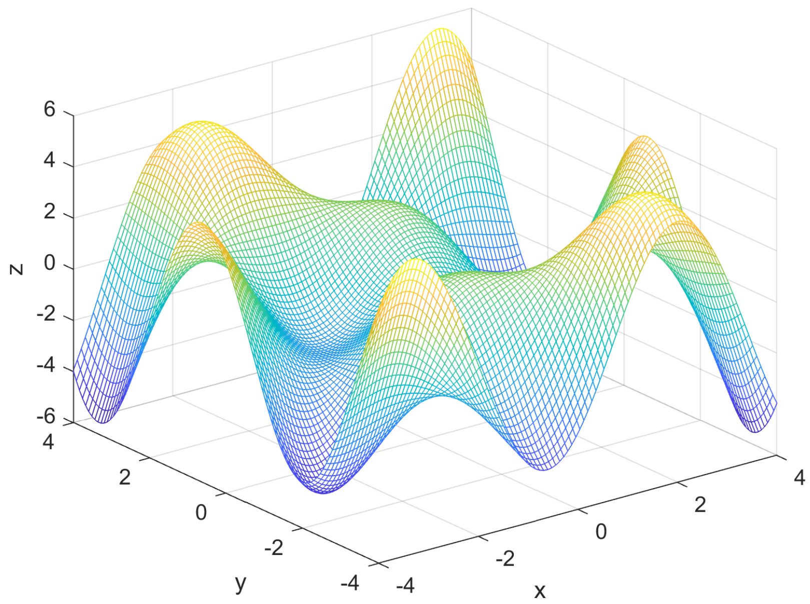


Figure 2: 3D figure of function with Points

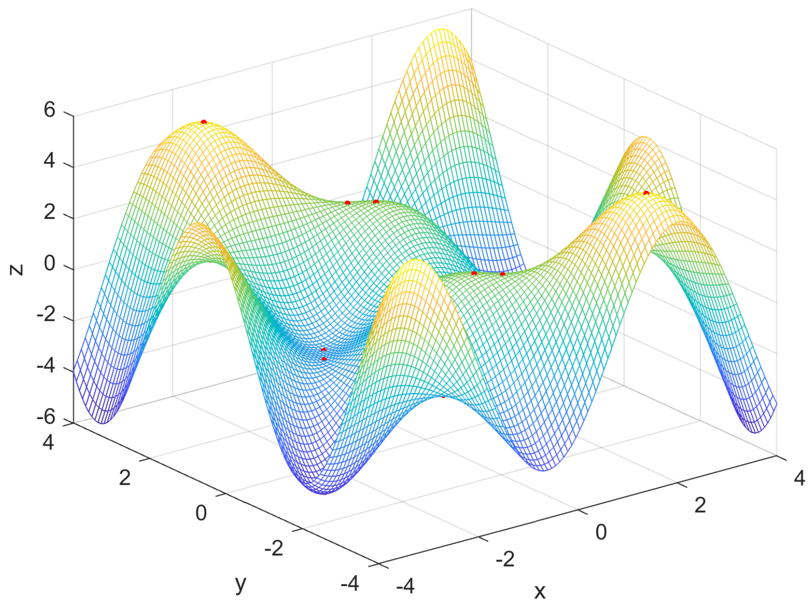


Figure 3: Contour figure of function

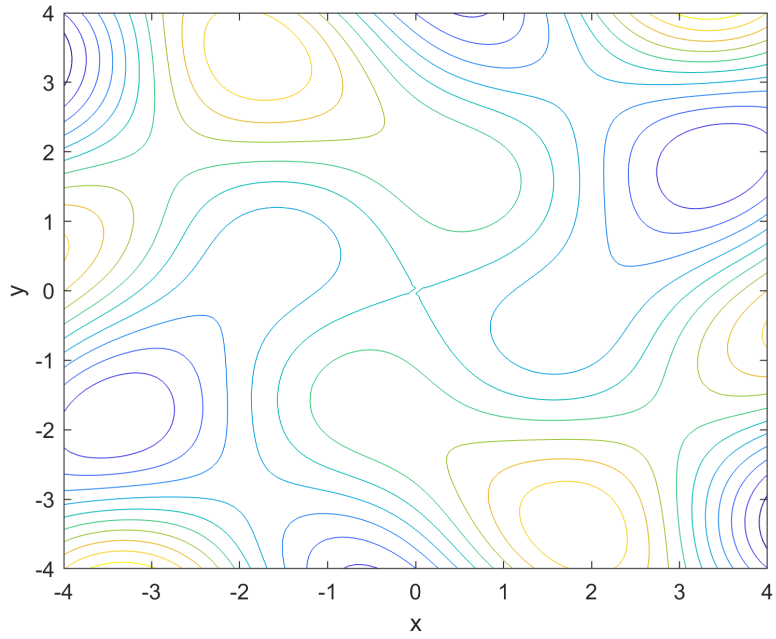
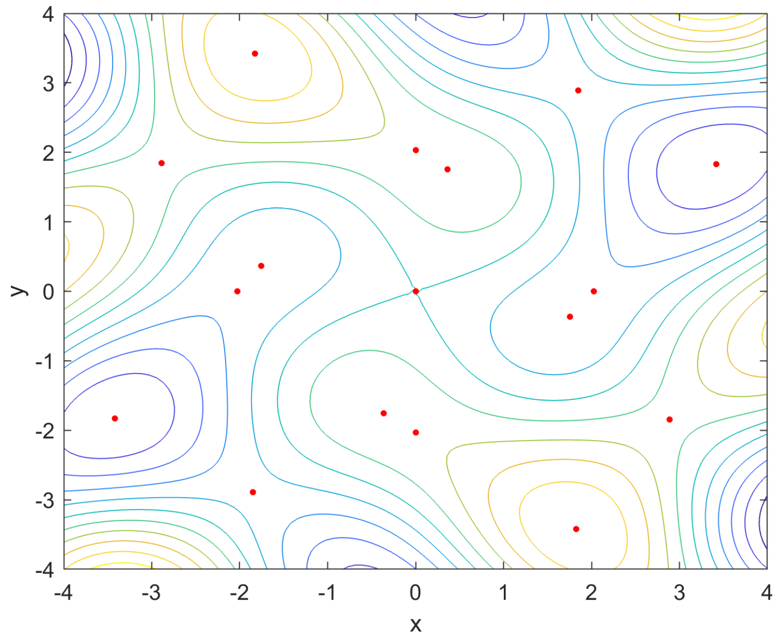


Figure 4: Contour figure of function with Points



(b) $z = f(x, y) = x^3 - 3xy^2$
Gradient of f(x, y):

$$\vec{\nabla} f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}$$

$$\vec{\nabla} f = \begin{bmatrix} 3x^2 - 3y^2 \\ -6xy \end{bmatrix}$$

Two linear equations with two unknowns.

$$3x^2 - 3y^2 = 0$$

$$-6xy = 0$$

Answers is $x = 0$ and $y = 0$.

Hessian matrix:

$$H = \frac{\partial^2 f}{\partial \vec{X}^2} = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial xy} \\ \frac{\partial^2 f}{\partial yx} & \frac{\partial f}{\partial y^2} \end{bmatrix}$$

$$H = \begin{bmatrix} 6x & -6y \\ -6y & -6x \end{bmatrix}$$

In $x = 0$ and $y = 0$ Hessian matrix in :

$$H = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

so this point is saddle point.

Figure 5: 3D figure of function

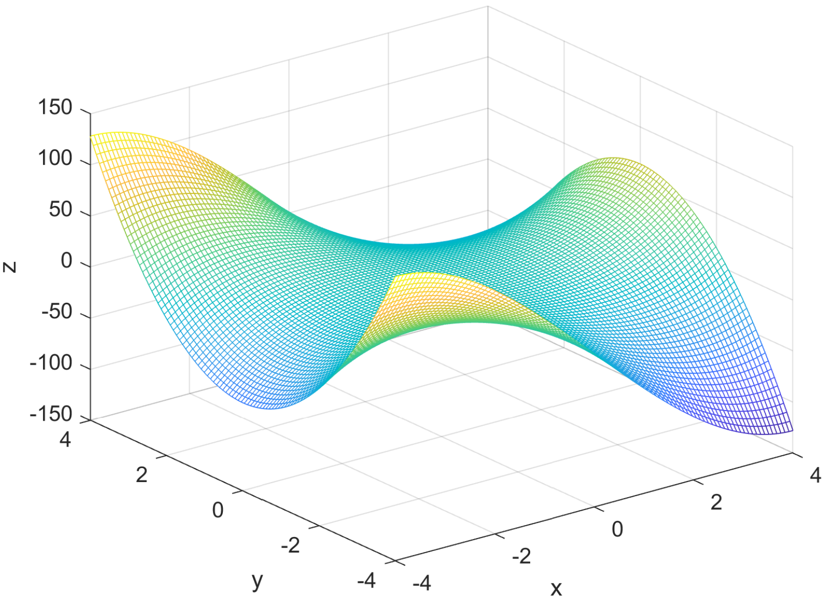


Figure 6: 3D figure of function with Points

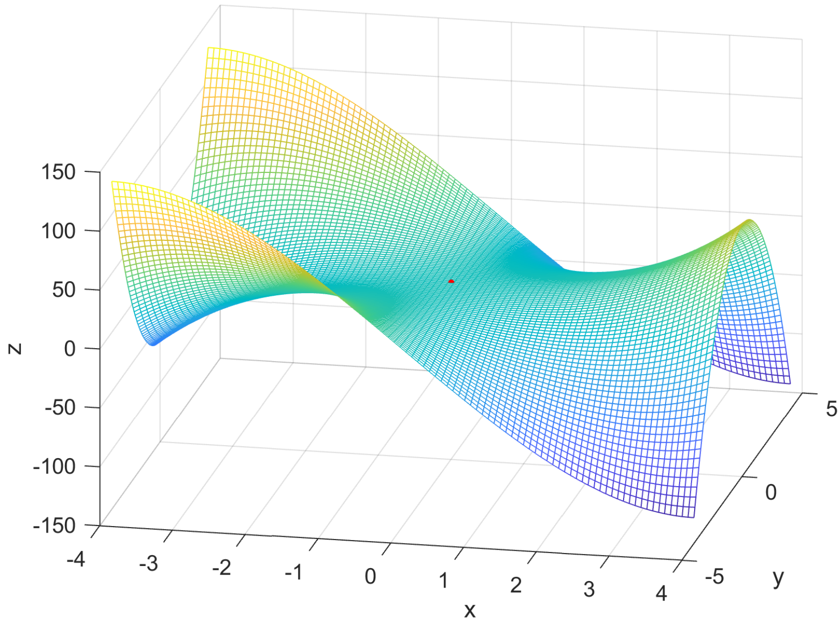


Figure 7: Contour figure of function

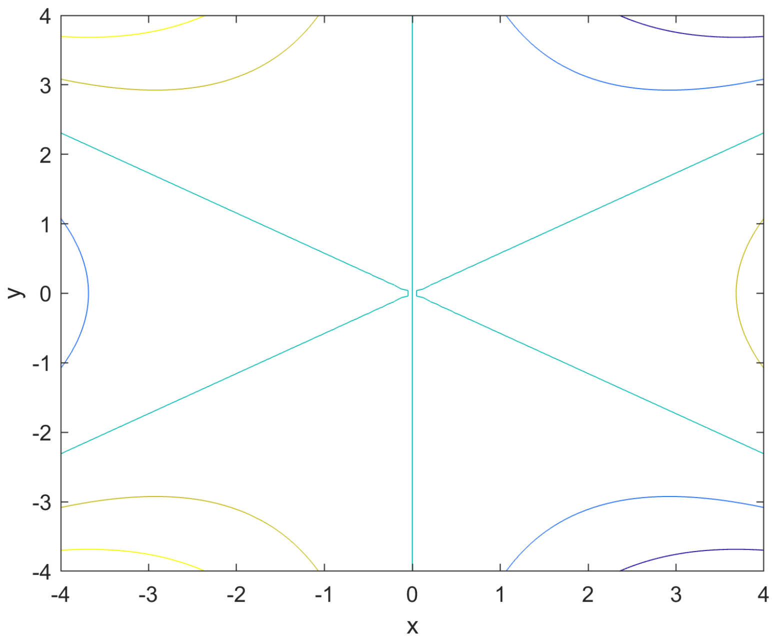
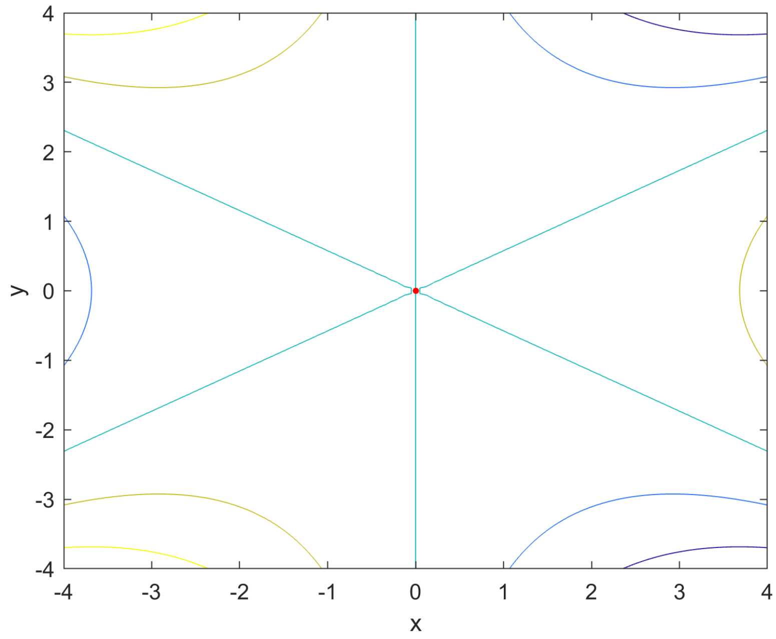


Figure 8: Contour figure of function with Points



(c) $z = f(x_1, x_2, x_3) = x_1^2 + x_1x_2 - 4x_2^2 - x_3^2 + 3x_2x_3$
Gradient of $f(x_1, x_2, x_3)$:

$$\vec{\nabla} f = \frac{\partial f}{\partial \vec{X}} = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_3} \end{bmatrix}$$

$$\vec{\nabla} f = \begin{bmatrix} 2x_1 + x_2 \\ x_1 - 8x_2 + 3x_3 \\ 3x_2 - 2x_3 \end{bmatrix} = \vec{0}$$

Three linear equations with Three unknowns.

$$2x_1 + x_2 = 0$$

$$x_1 - 8x_2 + 3x_3 = 0$$

$$3x_2 - 2x_3 = 0$$

Answers is $x_1 = x_2 = x_3 = 0$ Hessian matrix:

$$H = \frac{\partial^2 f}{\partial \vec{X}^2} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 x_2} & \frac{\partial^2 f}{\partial x_1 x_3} \\ \frac{\partial^2 f}{\partial x_2 x_1} & \frac{\partial f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 x_3} \\ \frac{\partial^2 f}{\partial x_3 x_1} & \frac{\partial f}{\partial x_3 x_2} & \frac{\partial^2 f}{\partial x_3^2} \end{bmatrix}$$

$$H = \begin{bmatrix} 2 & 1 & 0 \\ 1 & -8 & 3 \\ 0 & 3 & -2 \end{bmatrix}$$

All Hessian eigenvalues are:

$$eig(H) = \begin{bmatrix} -9.3182 \\ -0.8077 \\ 2.1259 \end{bmatrix}$$

So $(0, 0, 0)$ is a saddle point.

Problem 2

$$\min f(x,y,z) = x^2 + y^2 + z^2$$

subject to :

$$z = \sin(x) + \cos(y)$$

(a) Direct Substitution:

$f(x,y,z) = x^2 + y^2 + z^2 \xrightarrow{z=\sin(x)+\cos(y)} f(\vec{X}) = f(x,y) = x^2 + y^2 + (\sin(x)+\cos(y))^2$
$f(x,y) = x^2 + y^2 + \sin(x)^2 + 2\sin(x)\cos(y) + \cos(y)^2$ Gradient of f(x, y):

$$\vec{\nabla} f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}$$

$$\vec{\nabla} f = \begin{bmatrix} 2x + 2\cos(x)\cos(y) + 2\cos(x)\sin(x) \\ 2y - 2\cos(y)\sin(y) - 2\sin(x)\sin(y) \end{bmatrix}$$

$$2x + 2\cos(x)\cos(y) + 2\cos(x)\sin(x) = 0$$

$$2y - 2\cos(y)\sin(y) - 2\sin(x)\sin(y) = 0$$

Above equation solved in MATLAB and code (Q2_a.m) has attached to homework.
$x = -0.47872, \quad y = 0.0 \rightarrow z = 0.5393$

Table 3: Answers

x	y	z
-0.47872	0.000	0.5393

(b) Lagrange multipliers:
$\min \mathcal{L}(\vec{X},\vec{\lambda}) = f(\vec{X}) + \vec{\lambda}^T \vec{g}$
necessary condition:

$$\vec{\nabla}\mathcal{L} = \begin{bmatrix} \vec{\nabla}_{\vec{X}}\mathcal{L} \\ \vec{\nabla}_{\vec{\lambda}}\mathcal{L} \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathcal{L}}{\partial \vec{X}} \\ \frac{\partial \mathcal{L}}{\partial \vec{\lambda}} \end{bmatrix} = \vec{0}$$

$f(\vec{X}) = x^2 + y^2 + z^2, \quad g(\vec{X}) = \sin(x) + \cos(y) - z = 0$

$$\min \mathcal{L}(\vec{X},\vec{\lambda}) = x^2 + y^2 + z^2 + \lambda(\sin(x) + \cos(y) - z)$$

$$\vec{\nabla}\mathcal{L} = \begin{bmatrix} \frac{\partial \mathcal{L}}{\partial x} \\ \frac{\partial \mathcal{L}}{\partial y} \\ \frac{\partial \mathcal{L}}{\partial z} \\ \frac{\partial \mathcal{L}}{\partial \lambda} \end{bmatrix} = \begin{bmatrix} 2x + \lambda\cos(x) \\ 2y - \lambda\sin(y) \\ 2z - \lambda \\ \sin(x) + \cos(y) - z \end{bmatrix}$$

Above equation solved in MATLAB and code (Q2_b.m) has attached to homework.

Table 4: Answers

x	y	z	λ
-0.47872	0.000	0.5393	1.078708

Figure 9: Figure with Sphere Answer

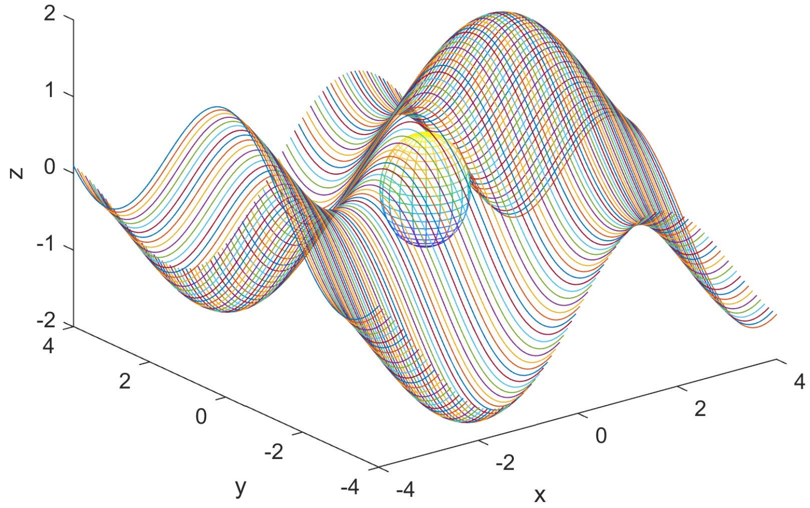
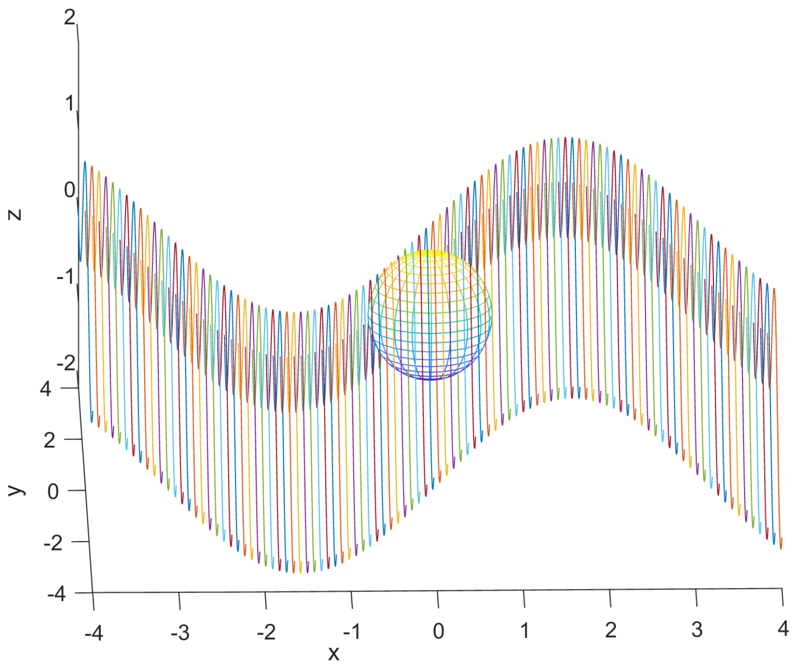


Figure 10: Figure with Sphere Answer Another view



Problem 3

$$\min f(x_1, x_2, y_1, y_2) = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

subject to :

$$y_1 = x_1^2, \quad y_2 = x_2 - 1$$

(a) Direct Substitution:

$$f(x_1, x_2, y_1, y_2) = (x_1 - x_2)^2 + (y_1 - y_2)^2 \xrightarrow{y_1 = x_1^2, \quad y_2 = x_2 - 1} f(x_1, x_2) = (x_1 - x_2)^2 + (x_1^2 - (x_2 - 1)^2)^2$$

$$f(x_1, x_2) = x_1^4 - 2x_1^2 x_2^2 + 4x_1^2 x_2 - x_1^2 - 2x_1 x_2 + x_2^4 - 4x_2^3 + 7x_2^2 - 4x_2 + 1$$

Gradient of $f(x_1, x_2)$:

$$\vec{\nabla} f = \begin{bmatrix} \dfrac{\partial f}{\partial x_1} \\ \dfrac{\partial f}{\partial x_2} \end{bmatrix}$$

$$\vec{\nabla} f = \begin{bmatrix} 4x_1^3 - 4x_1 x_2^2 + 8x_1 x_2 - 2x_1 - 2x_2 \\ -4x_1^2 x_2 + 4x_1^2 - 2x_1 + 4x_2^3 - 12x_2^2 + 14x_2 - 4 \end{bmatrix}$$

Two nonlinear equations with two unknowns. We use MATLAB to solve this equations. MATLAB file (Q3_a.m) is attached.

$$4x_1^3 - 4x_1 x_2^2 + 8x_1 x_2 - 2x_1 - 2x_2 = 0$$
$$-4x_1^2 x_2 + 4x_1^2 - 2x_1 + 4x_2^3 - 12x_2^2 + 14x_2 - 4 = 0$$

$x_1 = \dfrac{1}{2} \to y_1 = \dfrac{1}{4}, \quad x_2 = \dfrac{7}{8} \to y_2 = -\dfrac{1}{8}$

Table 5: Answers

x_1	y_1	x_2	y_2
0.5	0.25	0.875	-0.125

(b) Lagrange multipliers:

$\min \mathcal{L}(\vec{X}, \vec{\lambda}) = f(\vec{X}) + \vec{\lambda}^T \vec{g}$ necessary condition:

$$\vec{\nabla}\mathcal{L} = \begin{bmatrix} \vec{\nabla}_{\vec{X}}\mathcal{L} \\ \vec{\nabla}_{\vec{\lambda}}\mathcal{L} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \mathcal{L}}{\partial \vec{X}} \\ \dfrac{\partial \mathcal{L}}{\partial \vec{\lambda}} \end{bmatrix} = \vec{0}$$

$$f(\vec{X}) = (x_1 - x_2)^2 + (y_1 - y_2)^2, \quad g(\vec{X}) = \begin{bmatrix} y_1 - x_1^2 & y_2 - x_2 + 1 \end{bmatrix} = \vec{0}$$

$$\min \mathcal{L}(\vec{X}, \vec{\lambda}) = (x_1 - x_2)^2 + (y_1 - y_2)^2 + \lambda_1(y_1 - x_1^2) + \lambda_2(y_2 - x_2 + 1)$$

$$\vec{\nabla}\mathcal{L} = \begin{bmatrix} \dfrac{\partial \mathcal{L}}{\partial x_1} \\ \dfrac{\partial \mathcal{L}}{\partial y_1} \\ \dfrac{\partial \mathcal{L}}{\partial x_2} \\ \dfrac{\partial \mathcal{L}}{\partial y_2} \\ \dfrac{\partial \mathcal{L}}{\partial \lambda_1} \\ \dfrac{\partial \mathcal{L}}{\partial \lambda_2} \end{bmatrix} = \begin{bmatrix} 2x_1 - 2x_2 - 2\lambda_1 x_1 \\ 2y_1 - 2y_2 + \lambda_1 \\ 2x_2 - 2x_1 - \lambda_2 \\ 2y_2 - y_1 + \lambda_2 \\ y_1 - x_1^2 \\ y_2 - x_2 + 1 \end{bmatrix}$$

Six nonlinear equations with six unknowns. We use MATLAB to solve this equations. MATLAB file (Q3_b.m) is attached.

Table 6: Answers

x_1	y_1	x_2	y_2	λ_1	λ_2
0.5	0.25	0.875	-0.125	-0.75	0.75

(c) Calculus of Variation:

Euler–Lagrange equation:

$$g_x - \frac{d}{dt} g_{\dot{x}} = 0$$

In problem g function is that we must minimize is:

$$g(\dot{x}, x, t) = g(\dot{x}) = \sqrt{1 + \dot{x}^2} =$$

$$\to \frac{\partial g_{\dot{x}}}{\partial \dot{x}} \frac{\dot{x}}{dt} = 0 \to g_{\dot{x}\dot{x}} \ddot{x} = 0$$

The solution is line:

$$\to x(t) = c_1 t + c_2$$

Boundary condition:

Initial time: $\theta(t) = t^2$

$$(g + (\dot{\theta} - \dot{x}) g_{\dot{x}})|_{t=t_0} = 0 \to \sqrt{1 + \dot{x}^2} + (2t - \dot{x}) \frac{\dot{x}}{\sqrt{1 + \dot{x}^2}}|_{t=t_0} = 0 \to t_0 = -\frac{1}{2\dot{x}}|_{t=t_0}$$

Final time: $\theta(t) = t - 1$

$$(g + (\dot{\theta} - \dot{x}) g_{\dot{x}})|_{t=t_f} = 0 \to \sqrt{1 + \dot{x}^2} + (1 - \dot{x}) \frac{\dot{x}}{\sqrt{1 + \dot{x}^2}}|_{t=t_f} = 0 \to \dot{x}_f = -1$$

Because of x function $\dot{x}$ is constant in $t_0 \to t_f$,

$$\to c_1 = -1 \to t_0 = \frac{1}{2\dot{x}}|_{t=t_0} = -\frac{1}{2c_1} = \frac{1}{2} \to x_0 = \frac{1}{4}$$

$$x(t) = c_1 t + c_2 \to x(t_0) = c_1 t_0 + c_2 \to c_2 = 0.75$$

Final time:

$$x(t_f) = \theta(t_f) \to -t_f + 0.75 = t_f - 1 \to t_f = 0.875 \xrightarrow{x(t) = -t + 0.75} x_f = -0.125$$

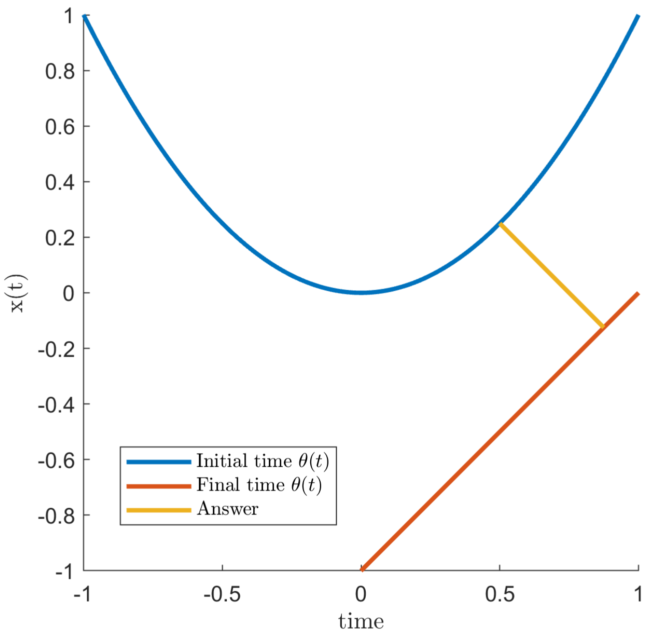
Answer:

$$x(t) = -t + 0.75 t_0 \to t_f$$

Table 7: Answers

t_0	x_0	t_f	x_f
0.5	0.25	0.875	-0.125

Figure 11: Answer function



(d) From last part we know that answer is a line but in this question we must minimize two cost function so there is two line and seven unknowns.

Equations:

$$x_1(t) = c_1 t + c_2 \quad 0 \le t \le t_1$$

$$x_2(t) = c_3 t + c_4 \quad t_1 \le t \le t_f$$

Boundary condition:

Initial time: $\theta_0(t) = t^2$

$$(g + (\dot{\theta} - \dot{x})g_{\dot{x}})|_{t=t_0} = 0 \to \sqrt{1+\dot{x}^2} + (2t - \dot{x})\frac{\dot{x}}{\sqrt{1+\dot{x}^2}}|_{t=t_0} = 0 \to t_0 = -\frac{1}{2\dot{x}}|_{t=t_0}$$

$$\to t_0 = -\frac{1}{2c_1} \to t_0 + \frac{1}{2c_1} = 0$$

$$\theta_0(t_0) = x_1(t_0) = t_0^2 = c_1 t_0 + c_2 \to t_0^2 - c_1 t_0 - c_2 = 0$$

Final time: $\theta_f(t) = t - 1$

$$(g + (\dot{\theta} - \dot{x})g_{\dot{x}})|_{t=t_f} = 0 \to \sqrt{1+\dot{x}^2} + (1 - \dot{x})\frac{\dot{x}}{\sqrt{1+\dot{x}^2}}|_{t=t_f} = 0 \to \dot{x}_f = -1$$

$$\to c_3 = -1$$

$$\theta_f(t_f) = x_2(t_f) = t_f - 1 = -t_f + c_4 \to 2t_f - c_4 - 1 = 0$$

Corner conditions: $\theta_c(t) = 4 + 2(t-4)^2$

$$(g + (\dot{\theta} - \dot{x})g_{\dot{x}})|_{t=t_1^-} = (g + (\dot{\theta} - \dot{x})g_{\dot{x}})|_{t=t_1^+}$$

$$\to -4t_1 + 16 = 4t_1 c_1 - 16c_1 \to 4t_1(c_1+1) - 16c_1 - 16 = 0$$

In $t = t_1$:

$$\theta_0(t_1) = \theta_f(t_1) \to c_1 t_1 + c_2 = -t_1 + c_4$$

$$c_1 t_1 + c_2 + t_1 - c_4 = 0$$
$$\theta_0(t_1) = \theta_c(t_1) = \theta_f(t_1)$$
$$\rightarrow c_1 t_1 + c_2 = 4 + 2(t_1 - 4)^2 \rightarrow 2(t_1 - 4)^2 + 4 - c_1 t_1 - c_2 = 0$$

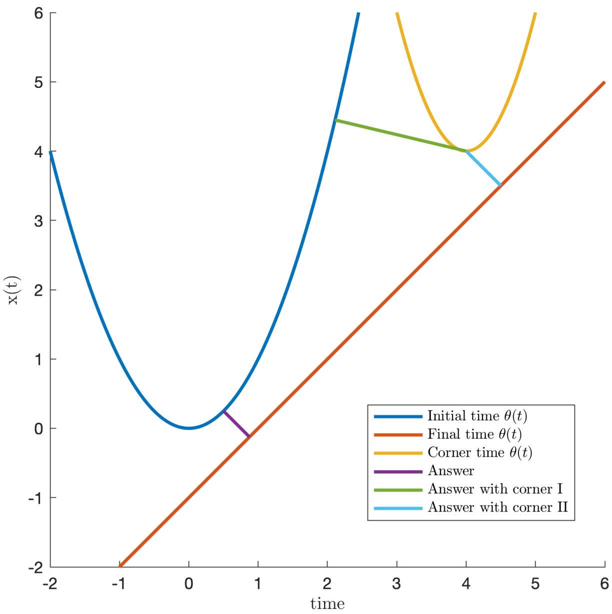
The above equations solved if MATLAB (Q3_d.m).

Table 8: Answers

t_0	x_0	t_1	x_1	t_f	x_f
2.11	4.448	4	4	4.5	3.5

$x_1(t) = -0.237t + 4.9483, \quad x_2(t) = -t + 8$

Figure 12: Answer function with corner



Problem 4

Potential function:

$$dP = xg_0 dm = xg_0 \rho_s ds = xg_0 \rho_s \sqrt{1+\dot{x}^2}dt \rightarrow dP = g_0 \rho_s x \sqrt{1+\dot{x}^2}dt \rightarrow P = \int_{t_0}^{t_f} g_0 \rho_s x \sqrt{1+\dot{x}^2}dt$$

$$P = g_0 \rho_s \int_{t_0}^{t_f} x\sqrt{1+\dot{x}^2}dt$$

$$\min J(\dot{x}, x) = \int_{t_0}^{t_f} x\sqrt{1+\dot{x}^2}dt$$

length of rope is L so problem subject to:

$$z(t) = \int_{t_0}^{t} \sqrt{1+\dot{x}^2}dt = L$$

$z(t)$ is new state:

$$z(t) = \int_{t_0}^{t_f} \sqrt{1+\dot{x}^2}dt \to \frac{dz}{dt} = \sqrt{1+\dot{x}^2}$$

New differential constraints:

$$\frac{dz}{dt} - \sqrt{1+\dot{x}^2} = 0$$

$$g_a(x,\dot{x},\dot{z},t,\lambda) = g_a(x,\dot{x},t) + \lambda(t)f(x,\dot{x},\dot{z},t)$$

$$g_a = x\sqrt{1+\dot{x}^2} + \lambda(\dot{z} - \sqrt{1+\dot{x}^2})$$

Euler–Lagrange equation:

$$\frac{\partial g_a}{\partial x} - \frac{d}{dt}\frac{\partial g_a}{\partial \dot{x}} = 0$$

$$\frac{\partial g_a}{\partial \lambda} - \frac{d}{dt}\frac{\partial g_a}{\partial \dot{\lambda}} = 0 \to \dot{z} - \sqrt{1+\dot{x}^2} = 0$$

$$\frac{\partial g_a}{\partial z} - \frac{d}{dt}\frac{\partial g_a}{\partial \dot{z}} = 0 \to \lambda = C$$

λ is constant.

$$\frac{\partial g_a}{\partial x} = \sqrt{1+\dot{x}^2}$$

$$g_{a_{\dot{x}}} = \frac{\partial g_a}{\partial \dot{x}} = \frac{x\dot{x}}{\sqrt{1+\dot{x}^2}} - \frac{\lambda\dot{x}}{\sqrt{1+\dot{x}^2}}$$

$$\frac{d}{dt}\frac{\partial g_a}{\partial \dot{x}} = \frac{\partial g_{a_{\dot{x}}}}{\partial t} + \frac{\partial g_{a_{\dot{x}}}}{\partial x}\dot{x} + \frac{\partial g_{a_{\dot{x}}}}{\partial \dot{x}}\ddot{x} + \frac{\partial g_{a_{\dot{x}}}}{\partial \ddot{x}}\dddot{x}$$

g is a function of x and $\dot{x}$ so:

$$g - \dot{x}g_{\dot{x}} = a$$

a is constant.

$$\frac{x-\lambda}{\sqrt{1+\dot{x}^2}} = a \to dt = \frac{adx}{\sqrt{(x-\lambda)^2 - a^2}}$$

$$x(t) = \lambda + a\cosh(\frac{t-b}{a})$$

Boundary conditions:

$$t_0 = 0, \quad t_f = t_f, \quad x(t_0) = 0, \quad x(t_f) = 0, \quad z(t_0) = 0, \quad z(t_f) = L$$

$$\int_{t_0}^{t_f} \sqrt{1+\dot{x}^2} = \int_0^{t_f} \cosh(\frac{t-b}{a}) = L$$

$$a\sinh(\frac{t_f-b}{a}) = L \tag{1}$$

$$\lambda + a\cosh(\frac{-b}{a}) = 0 \tag{2}$$

$$\lambda + a\cosh(\frac{t_f-b}{a}) = 0 \tag{3}$$

Three nonlinear equations with Three unknowns.

Problem 5

(a) $g = \sin(\dot{x}^2)$

g is only a function of $\dot{x}$ so the answer is a line.

Answer: $c_1 t + c_2$

Final time is fix and final location is free so the boundary condition is:

$$g_{\dot{x}}\big|_{t_f} = 0 \rightarrow 2\dot{x}_{t_f}\cos(\dot{x}_{t_f}^2) = 0 \xrightarrow{x(t)=c_1t+c_2} 2c_1\cos(c_1^2) = 0$$

$$\rightarrow c_1 = \pm\sqrt{\frac{\pi}{2}}, \quad c_1 = 0, \quad c_1 = \pm\sqrt{\frac{3\pi}{2}}$$

$$J(x) = \int_0^2 \sin(\dot{x}^2) \xrightarrow{c_1=\pm\sqrt{\frac{\pi}{2}}} J(x) = 2 \tag{4}$$

$$J(x) = \int_0^2 \sin(\dot{x}^2) \xrightarrow{c_1=0} J(x) = 0 \tag{5}$$

$$J(x) = \int_0^2 \sin(\dot{x}^2) \xrightarrow{c_1=\pm\sqrt{\frac{3\pi}{2}}} J(x) = -2 \tag{6}$$

The point $c_1 = \pm\sqrt{\frac{\pi}{2}}$ is maximum, $c_1 = 0$ is saddle and $c_1 = \pm\sqrt{\frac{3\pi}{2}}$ is minimum.

In t_f $x(t_0) = 1$ so c_2 is one.

Maximum $J(x)$

$$x(t) = \pm\sqrt{\frac{\pi}{2}}t + 1$$

Saddle $J(x)$

$$x(t) = 1$$

Maximum $J(x)$

$$x(t) = \pm\sqrt{\frac{3\pi}{2}}t + 1$$

(b) $g = t\dot{x}^2(t) + \ln(\dot{x}^2(t))$

g is a function of $\dot{x}$ and t so:

$$\frac{dg_{\dot{x}}}{dt} = 0 \rightarrow g_{\dot{x}} = C$$

C is constant.

$$g_{\dot{x}} = 2t\dot{x} + \frac{2}{\dot{x}} \rightarrow \dot{x}^2 + \dot{x}(\frac{2-C}{2t}) = 0$$

Above equation solved with MATLAB(Q5_d.m).

Answer:

There is two answer for above differential equation:

- first

$$x(t) = C_1$$

- second

$$C_2 + \ln(t)(C/2 - 1)$$

Boundary conditions:

$$x(0) = 1, \quad x(2) = 2$$

With those boundary conditions there is no acceptable answer.

(c) $g = \dot{x}_1^2 + \dot{x}_2^2 - 2x_1x_2$ Euler–Lagrange equation:

$$\frac{\partial g_{\vec{x}}}{\partial \vec{x}} - \frac{d}{dt}\frac{\partial g_{\vec{x}}}{\partial \dot{\vec{x}}} = 0$$

$$\begin{bmatrix} \dfrac{\partial g}{\partial x_1} \\ \dfrac{\partial g}{\partial x_2} \end{bmatrix} - \frac{d}{dt}\begin{bmatrix} \dfrac{\partial g}{\partial \dot{x_1}} \\ \dfrac{\partial g}{\partial \dot{x_2}} \end{bmatrix} = \begin{bmatrix} -2x_2 + 2\ddot{x}_1 \\ -2x_1 + 2\ddot{x}_2 \end{bmatrix} = 0$$

We can find out from above equation that $x_2 = \ddot{\ddot{x}}_2$ and this equation solve in MATLAB(Q5_c.m).

$$x_1(t) = C_3\cos(t) - C_2\exp(t) - C_4\sin(t) - C_1\exp(-t)$$

$$x_2(t) = C_3\cos(t) + C_2\exp(t) - C_4\sin(t) + C_1\exp(-t)$$

On of the boundry conditions are free so we must make another equation.

$$g_{\dot{x}_2}|_{t_f} = 0 \to \dot{x}_2(t_f) = 0$$

Boundary conditions:

$$x_1(0) = x_2(0) = 1, \quad x_1(\pi) = 4, \quad \dot{x}_2(\pi) = 0$$

$C_3 - C_2 - C_1 = 1, \quad C_3 + C_2 + C_1 = 1, \quad -C_3 - C_2\exp(\pi) - C_1\exp(-\pi) = 4$

$C_3 = 1, \quad C_1 = -C_2 = \dfrac{5}{\exp(\pi) - \exp(-\pi)}$

$-C_1\exp(-\pi) + C_2\exp(\pi) + C_4 = 0 \to C_4 = \dfrac{10(\exp(\pi) + \exp(-\pi))}{\exp(\pi) - \exp(-\pi)}$

$$x_1(t) = \cos(t) - \frac{5}{\exp(-\pi) - \exp(\pi)}\exp(t) - \frac{10(\exp(\pi) + \exp(-\pi))}{\exp(\pi) - \exp(-\pi)}\sin(t) - \frac{5}{\exp(\pi) - \exp(-\pi)}\exp(-t)$$

$$x_2(t) = \cos(t) + \frac{5}{\exp(-\pi) - \exp(\pi)}\exp(t) - \frac{10(\exp(\pi) + \exp(-\pi))}{\exp(\pi) - \exp(-\pi)}\sin(t) + \frac{5}{\exp(\pi) - \exp(-\pi)}\exp(-t)$$

(d) $g(x, \dot{x}, \ddot{x}) = x^2 + 2\dot{x}^2 + \ddot{x}$

Euler–Lagrange equation:

$$\frac{\partial g}{\partial x} - \frac{d}{dt}\frac{\partial g}{\partial \dot{x}} + \frac{d^2}{dt^2}\frac{\partial g}{\partial \ddot{x}} = 0$$

$$\frac{\partial g}{\partial x} = 2x, \quad \frac{\partial g}{\partial \dot{x}} = 4\dot{x}, \quad \frac{\partial g}{\partial \ddot{x}} = 2\ddot{x}$$

In below equation we use this syntax:

$$\frac{\partial g}{\partial \dot{x}} = g_{\dot{x}}, \quad \frac{\partial g}{\partial \ddot{x}} = g_{\ddot{x}}$$

$$\frac{dg_{\dot{x}}}{dt} = \frac{\partial g_{\dot{x}}}{\partial t} + \frac{\partial g_{\dot{x}}}{\partial x}\dot{x} + \frac{\partial g_{\dot{x}}}{\partial \dot{x}}\ddot{x} + \frac{\partial g_{\dot{x}}}{\partial \ddot{x}}\dddot{x} = 4\ddot{x}$$

$$\frac{dg_{\ddot{x}}}{dt} = \frac{\partial g_{\ddot{x}}}{\partial t} + \frac{\partial g_{\ddot{x}}}{\partial x}\dot{x} + \frac{\partial g_{\ddot{x}}}{\partial \dot{x}}\ddot{x} + \frac{\partial g_{\ddot{x}}}{\partial \ddot{x}}\dddot{x} = 2\dddot{x}$$

$$\frac{d^2}{dt^2}\frac{\partial g}{\partial \ddot{x}} = \frac{\partial(\frac{dg_{\ddot{x}}}{dt})}{\partial t} + \frac{\partial(\frac{dg_{\ddot{x}}}{dt})}{\partial x}\dot{x} + \frac{\partial(\frac{dg_{\ddot{x}}}{dt})}{\partial \dot{x}}\ddot{x} + \frac{\partial(\frac{dg_{\ddot{x}}}{dt})}{\partial \ddot{x}}\dddot{x} + \frac{\partial(\frac{dg_{\ddot{x}}}{dt})}{\partial \dddot{x}}\ddddot{x} = 2\ddddot{x}$$

Euler–Lagrange equation:

$$2\ddddot{x} - 4\ddot{x} + 2x = 0 \to \ddddot{x} - 2\ddot{x} + x = 0$$

Above differential equation solve with MATLAB(Q5_5d.m). Answer:

$$x(t) = C_3 \exp(t) + C_1 \exp(-t) + C_2 t \exp(-t) + C_4 t \exp(t)$$
$$\dot{x}(t) = (C_3 + C_4 + C_4 t) \exp(t) + (C_2 - C_2 - C_2 t) \exp(-t)$$

Boundary conditions:

$$x(0) = 1, \quad \dot{x}(0) = 2, \quad x(\infty) = \dot{x}(\infty) = 0$$

$C_1 + C_3 = 1, \quad C_4 = C_3 = 0, \quad C_2 - C_1 = 2$

$$x(t) = \exp(-t) + 3t \exp(-t)$$

(e) minimize:

$$\int_0^1 (x^2(t) + \dot{x}^2(t))dt$$

Subjected to:

$$\int_0^1 x(t)dt = 1$$

$z(t)$ is new state:

$$z(t) = \int_0^t x(t)dt \to \frac{dz}{dt} = x(t)$$

New g function:

$$g_a(x, \dot{x}, \dot{z}, \lambda, t) = g_a(x, \dot{x}, t) + \lambda(t) f(x, \dot{z}, t)$$

Euler–Lagrange equation:

$$\frac{\partial g_a}{\partial x} - \frac{d}{dt}\frac{\partial g_a}{\partial \dot{x}} = 0$$
$$\frac{\partial g_a}{\partial \lambda} - \frac{d}{dt}\frac{\partial g_a}{\partial \dot{\lambda}} = 0 \to \dot{z} - x(t) = 0$$
$$\frac{\partial g_a}{\partial z} - \frac{d}{dt}\frac{\partial g_a}{\partial \dot{z}} = 0 \to \lambda = C$$

λ is constant.

$$\frac{\partial g_a}{\partial x} = 2x - \lambda$$
$$g_{a_{\dot{x}}} = 2\dot{x}$$
$$\frac{d}{dt}\frac{\partial g_a}{\partial \dot{x}} = \frac{\partial g_{a_{\dot{x}}}}{\partial t} + \frac{\partial g_{a_{\dot{x}}}}{\partial x}\dot{x} + \frac{\partial g_{a_{\dot{x}}}}{\partial \dot{x}}\ddot{x} + \frac{\partial g_{a_{\dot{x}}}}{\partial \ddot{x}}\dddot{x} = 2\ddot{x}$$

Euler–Lagrange equation:

$$\ddot{x} - 2x + \lambda = 0$$

Above differential equation solve with MATLAB(Q5_5e.m). Answer:

$$x(t) = C_1 \exp(\sqrt{2}t) + C_2 \exp(-\sqrt{2}t) + \lambda/2$$

Boundary conditions:

$$\int_0^1 x(t)dt = 1 \to \left. \frac{\sqrt{2}}{2} C_1 \exp(\sqrt{2}t) - \frac{\sqrt{2}}{2} C_2 \exp(-\sqrt{2}t) + \lambda t/2 \right|_0^1$$
$$x(0) = C_1 + C_2 + \lambda/2 = 0$$
$$x(1) = C_1 \exp(\sqrt{2}) + C_2 \exp(-\sqrt{2}) + \lambda/2 = 1$$

There is There nonlinear equation and there Unknowns. Above differential equation solve with MATLAB(Q5_5e.m).

Answers:

$$C_1 = \frac{1}{\exp(\sqrt{2}) - \exp(-\sqrt{2})}, \quad C_2 = \frac{-1}{\exp(\sqrt{2}) - \exp(-\sqrt{2})}, \quad \lambda = 0$$
$$x(t) = \frac{\exp(\sqrt{2}t)}{\exp(\sqrt{2}) - \exp(-\sqrt{2})} + \frac{-\exp(-\sqrt{2}t)}{\exp(\sqrt{2}) - \exp(-\sqrt{2})}$$

(f) $g = \dfrac{\sqrt{1+\dot{x}^2}}{x}$

Euler–Lagrange equation:

$$\frac{\partial g}{\partial x} - \frac{d}{dt}\frac{\partial g}{\partial \dot{x}} = 0$$

g is a function of x and $\dot{x}$ so:

$$g - \dot{x} g_{\dot{x}} = C$$

C is constant.

$$g_{\dot{x}} = \frac{\dot{x}}{x\sqrt{1+\dot{x}^2}}$$
$$\frac{1}{x\sqrt{1+\dot{x}^2}} = C$$

Above equation solved in MATLAB(Q5_f.m) with boundary condition $x(0) = 0$.

$$x(t) = \pm\frac{\sqrt{-c\,t\,(c\,t+2)}}{c}$$

Final time: $(t_f - 9)^2 + x^2(t_f) = 9$

We can change θ_t to:

$$\theta(t) = \pm\sqrt{9-(t_f-9)^2}$$

$$(g + (\dot{\theta} - \dot{x})g_{\dot{x}})\Big|_{t=t_f} = 0 \rightarrow (9-t_f)^2(\dot{x}_{tf}^2 - 1) = 9$$

$$\dot{x} = \pm\frac{ct-1}{\sqrt{-c\,t\,(c\,t-2)}}$$

$$(t_f-9)^2 + \frac{-t_f\,(c\,t_f+2)}{c} = 9$$
$$(9-t_f)^2\left(\frac{(ct_f-1)^2}{-c\,t_f\,(c\,t_f-2)}\right) = 9$$

There is two equation and two unknowns.