Home Work #3

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1 Question 1

$z = f(x, y) = y\sin(x + y) - x\sin(x - y)$
Gradient of f(x, y):

$$\vec{\nabla} f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}$$

$$\vec{\nabla} f = \begin{bmatrix} y\cos(x + y) - \sin(x - y) - x\cos(x - y) \\ y\cos(x + y) + \sin(x + y) + x\cos(x - y) \end{bmatrix}$$

1.1 part a

$$\vec{X}_0 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

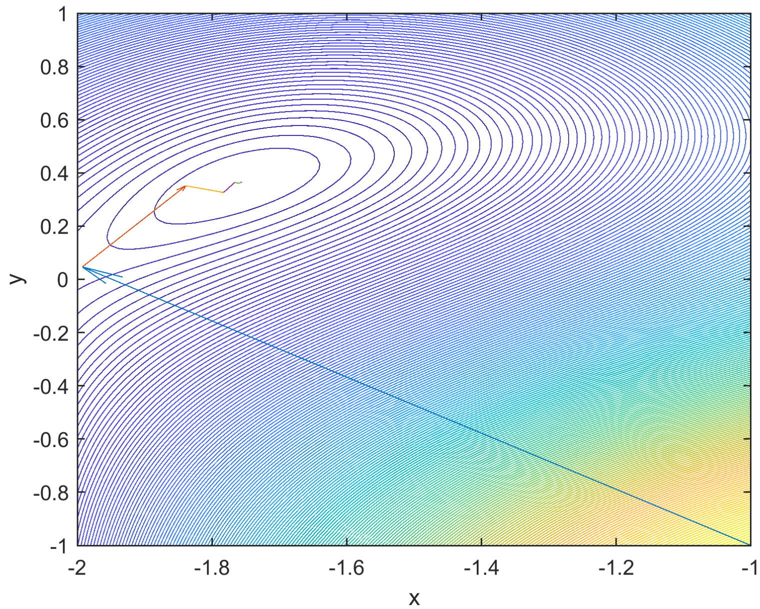
Tolerance is: 10^{-7}
Answer is:

$$\vec{X}_{ans} = \begin{bmatrix} -1.7556 \\ 0.3655 \end{bmatrix}$$

1.1.1 figures

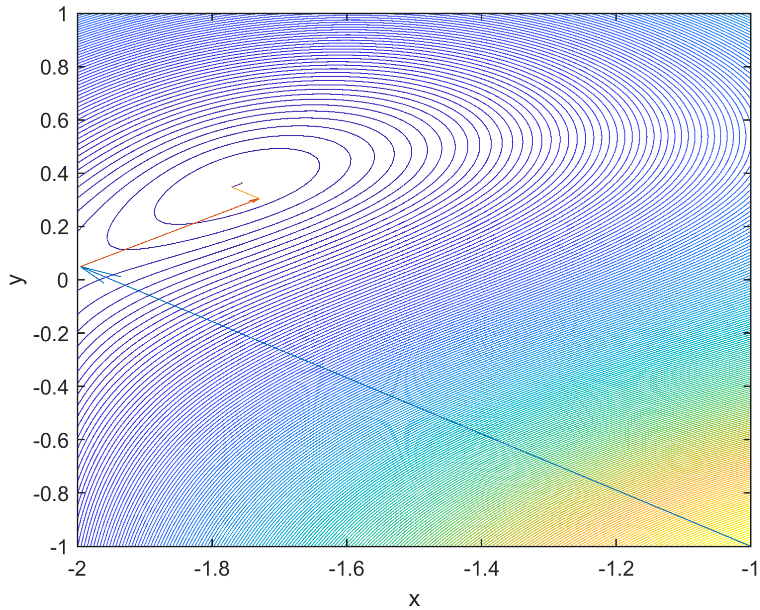
- Steepest Descent
  - Quadratic Interpolation

Figure 1: Steepest Descent and Quadratic Interpolation



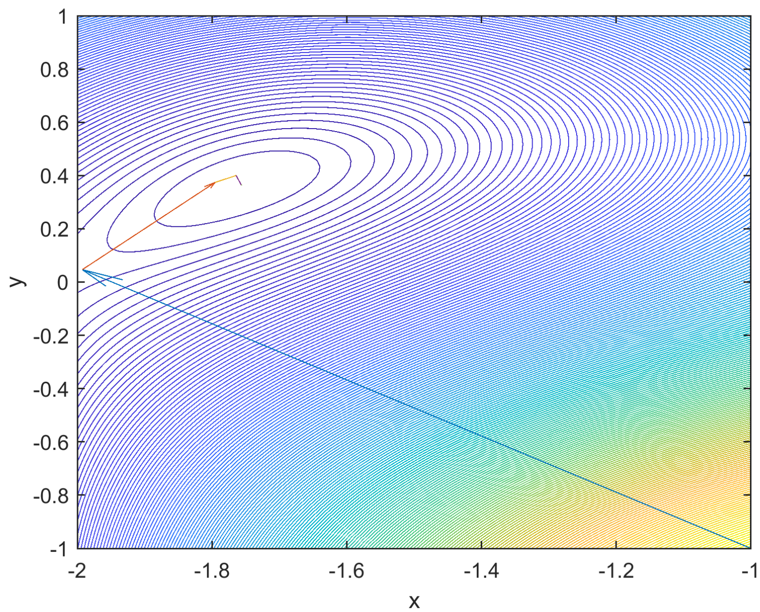
  - Golden Section

Figure 2: Steepest Descent and Golden Section



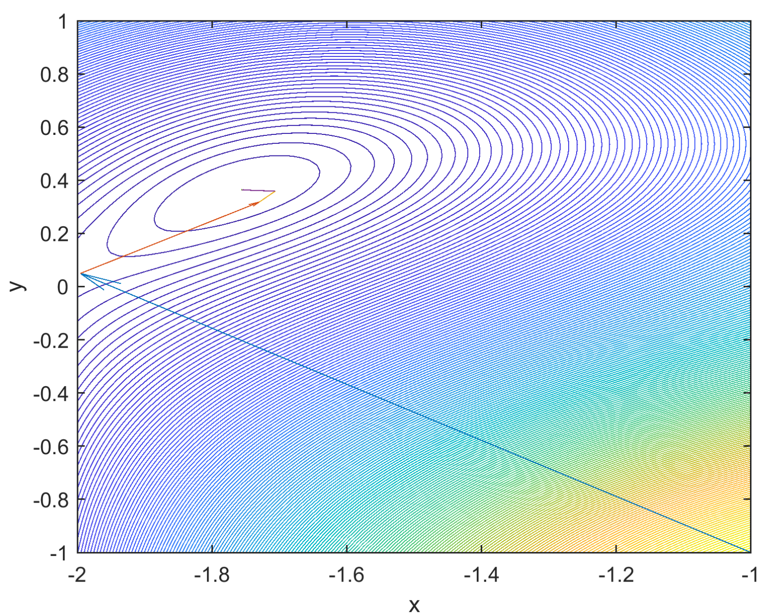
- BFGS
  - Quadratic Interpolation

Figure 3: BFGS and Quadratic Interpolation



  - Golden Section

Figure 4: BFGS and Golden Section



1.1.2 result

- Time

Table 1: Time compare between four methods

Steepest Descent		BFGS	
Quadratic Interpolation	Golden Section	Quadratic Interpolation	Golden Section
0.238 sec	0.183 sec	0.164 sec	0.102 sec

- Number of Cost calculation

Table 2: Number of Cost calculation compare between four methods

Steepest Descent		BFGS	
Quadratic Interpolation	Golden Section	Quadratic Interpolation	Golden Section
360	336	242	213

- Number of Gradient calculation

Table 3: Number of Gradient calculation compare between four methods

Steepest Descent		BFGS	
Quadratic Interpolation	Golden Section	Quadratic Interpolation	Golden Section
19	13	13	9

1.2 part b

$$\vec{X}_0 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

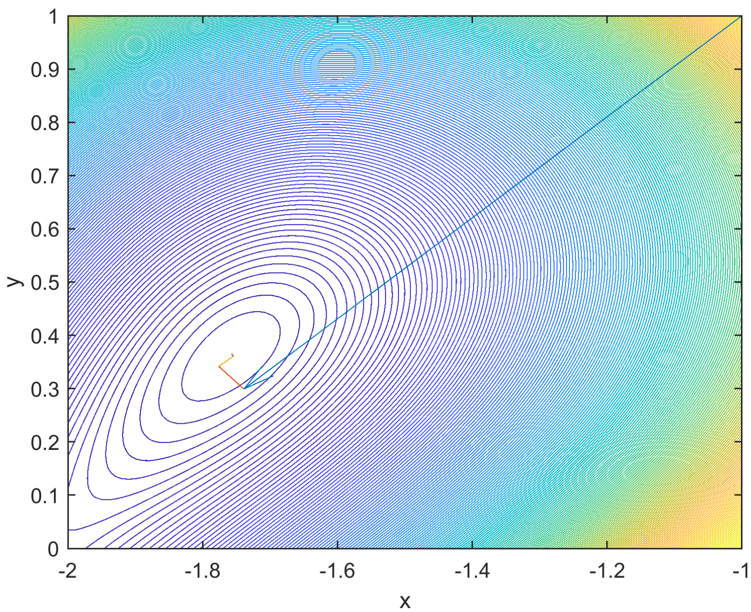
Tolerance is: 10^{-7}

$$\vec{X}_{ans} = \begin{bmatrix} -1.7556 \\ 0.3655 \end{bmatrix}$$

1.2.1 figures

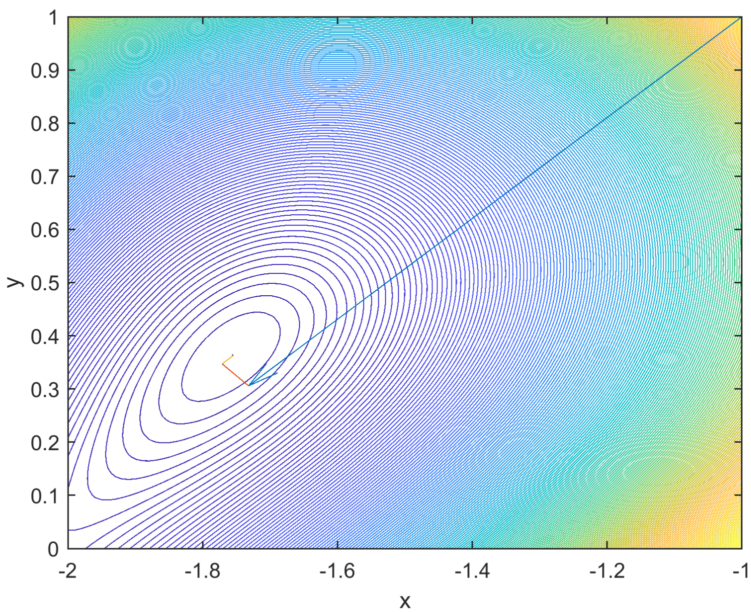
- Steepest Descent
  - Quadratic Interpolation

Figure 5: Steepest Descent and Quadratic Interpolation



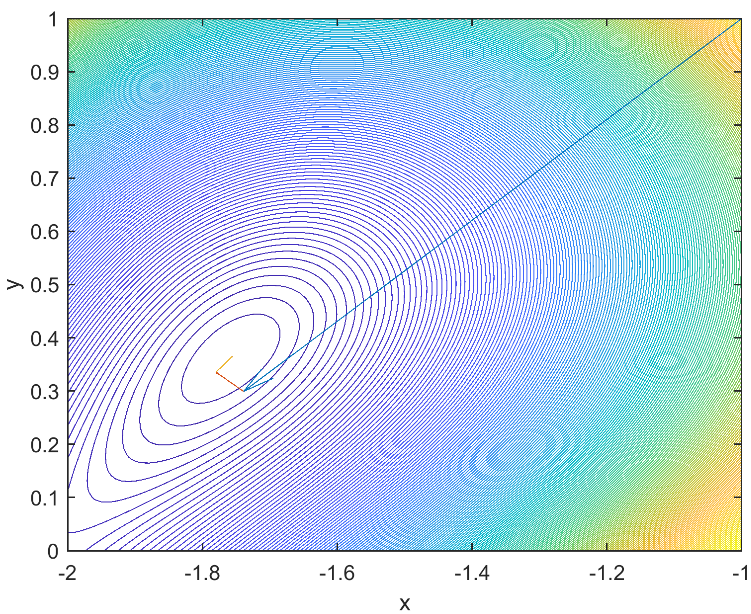
  - Golden Section

Figure 6: Steepest Descent and Golden Section



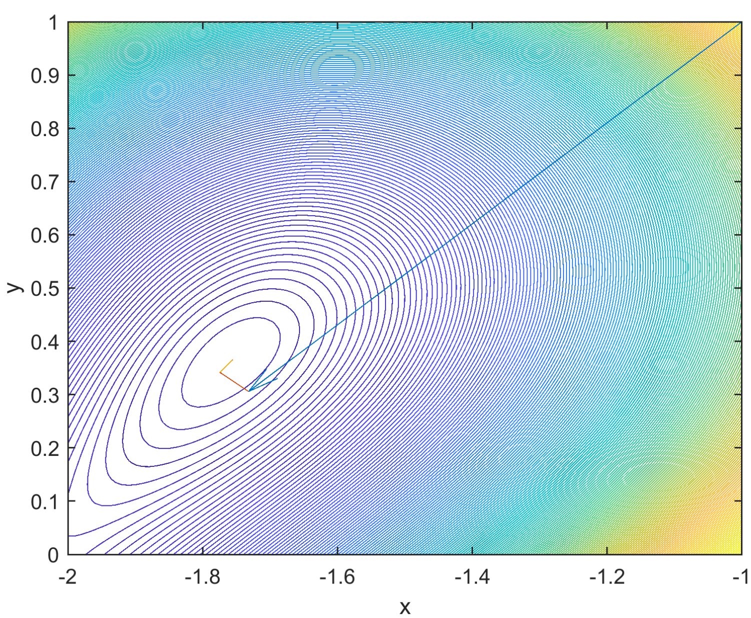
- BFGS
  - Quadratic Interpolation

Figure 7: BFGS and Quadratic Interpolation



  - Golden Section

Figure 8: BFGS and Golden Section



1.2.2 result

- Time

Table 4: Time compare between four methods

Steepest Descent		BFGS	
Quadratic Interpolation	Golden Section	Quadratic Interpolation	Golden Section
0.208 sec	0.146 sec	0.106 sec	0.142 sec

- Number of Cost calculation

Table 5: Number of Cost calculation compare between four methods

Steepest Descent		BFGS	
Quadratic Interpolation	Golden Section	Quadratic Interpolation	Golden Section
246	285	142	142

- Number of Gradient calculation

Table 6: Number of Gradient calculation compare between four methods

Steepest Descent		BFGS	
Quadratic Interpolation	Golden Section	Quadratic Interpolation	Golden Section
14	12	7	7

2 Question 2

2.1 System

$$\ddot{x}(t) = -x(t) - 0.1\dot{x}(t) + u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -0.1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad R = 1, \qquad H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

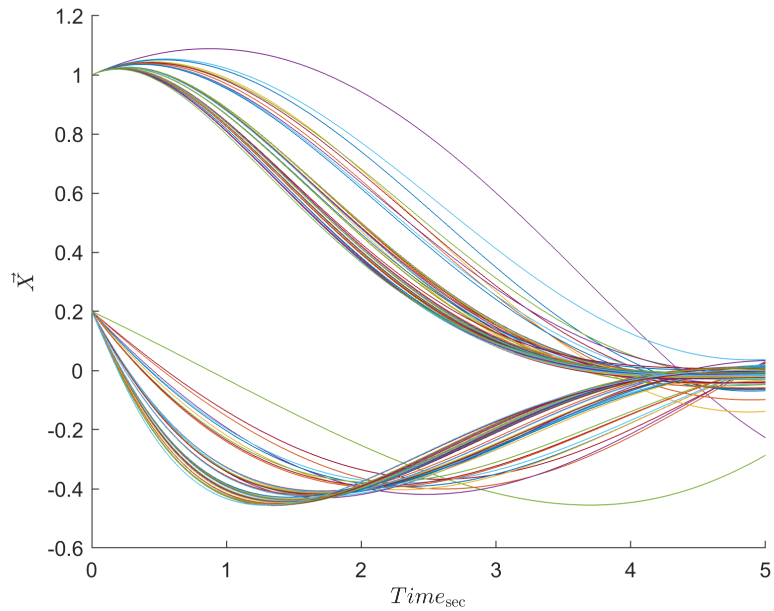
2.2 part a

Gradient tolerance is: 10^{-4}

2.2.1 figures

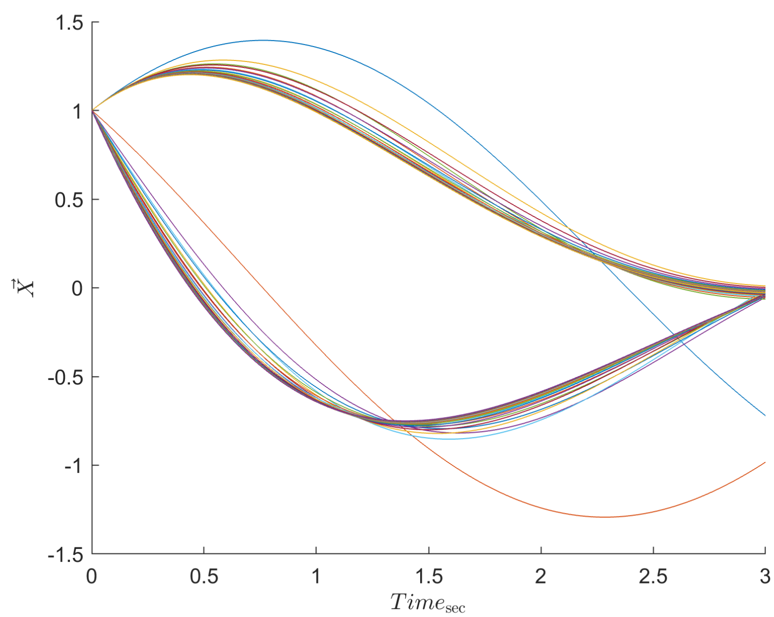
- Steepest Descent
  - Quadratic Interpolation

Figure 9: Steepest Descent and Quadratic Interpolation



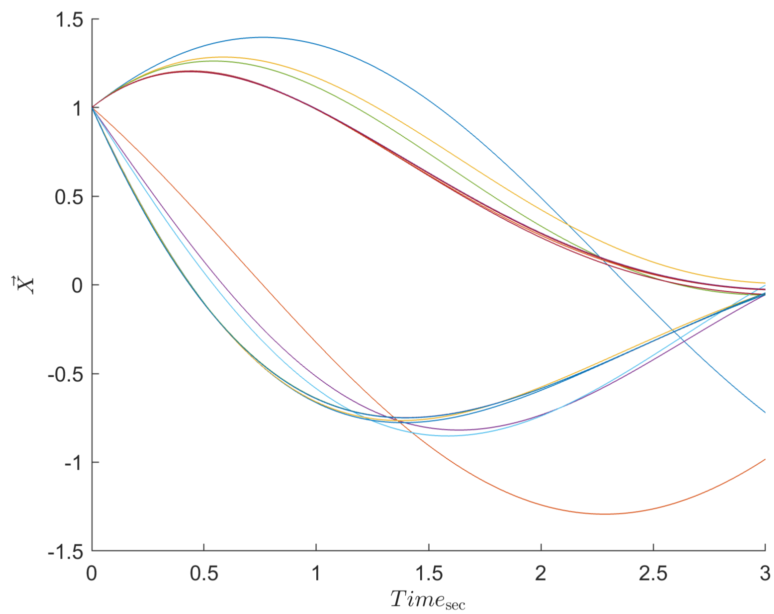
  - Golden Section

Figure 10: Steepest Descent and Golden Section



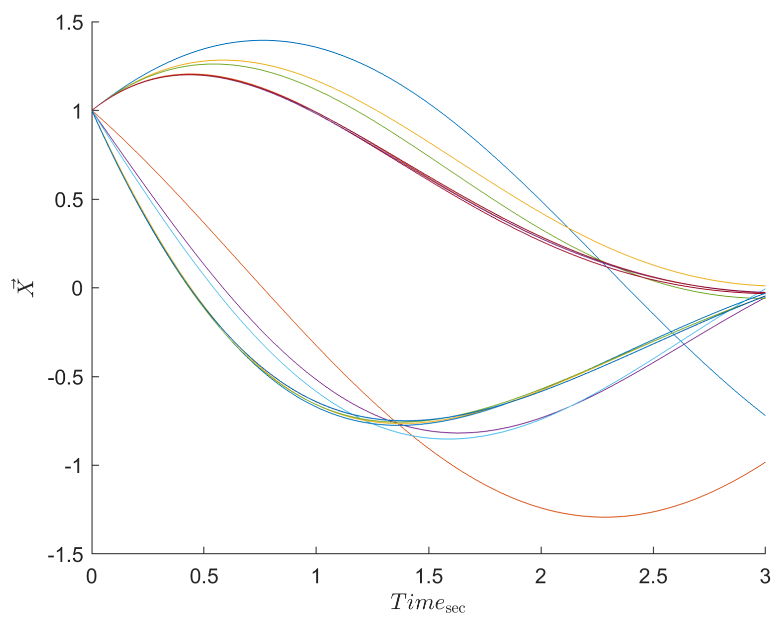
- BFGS
  - Quadratic Interpolation

Figure 11: BFGS and Quadratic Interpolation



  - Golden Section

Figure 12: BFGS and Golden Section



2.2.2 result

- Time

Table 7: Time compare between four methods

Steepest Descent		BFGS	
Quadratic Interpolation	Golden Section	Quadratic Interpolation	Golden Section
17.000 sec	24.353 sec	3.905 sec	4.985 sec

- Number of Cost calculation

Table 8: Number of Cost calculation compare between four methods

Steepest Descent		BFGS	
Quadratic Interpolation	Golden Section	Quadratic Interpolation	Golden Section
1285	1922	273	373

- Number of Gradient calculation

Table 9: Number of Gradient calculation compare between four methods

Steepest Descent		BFGS	
Quadratic Interpolation	Golden Section	Quadratic Interpolation	Golden Section
51	51	11	11

2.2.3 Four iteration for BFGS and Quadratic interpolation

Figure 13: BFGS and Quadratic Interpolation with four iteration

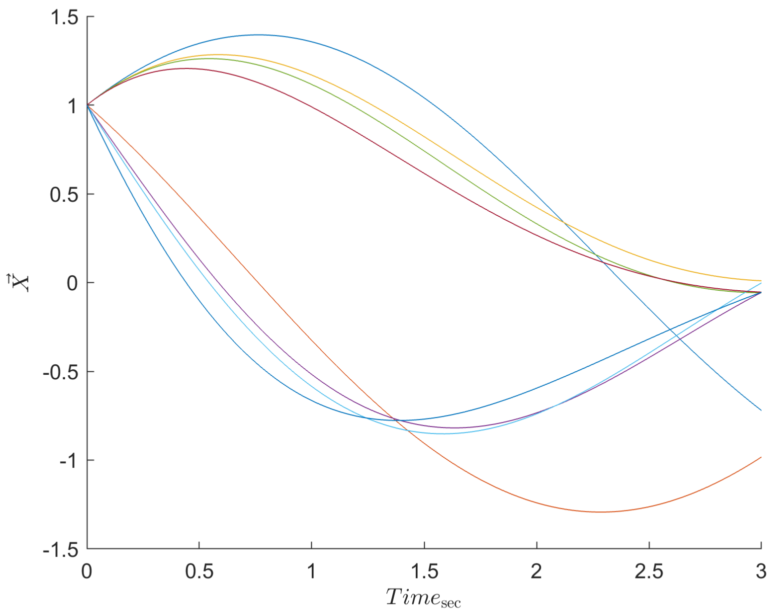


Table 10: four iteration and gradient tolerance compare

	Time	Number of Cost calculation	Number of Gradient calculation
Four iteration	3.905_{sec}	273	11
Gradient tolerance	1.586_{sec}	100	4

Between BFGS and Steepest Descent, BFGS if more faster but Steepest Descent is more easy to use. Quadratic interpolation can be faster when cost function use so much process but when we are so close to answer the function doesn't work well so we must increase gradient tolerance.

2.3 part b

Tolerance is: 10^{-16} for λS_i or 10^{-4} for norm of gradient.

2.3.1 figures

- Steepest Descent
  - Quadratic Interpolation

Figure 14: Steepest Descent and Quadratic Interpolation

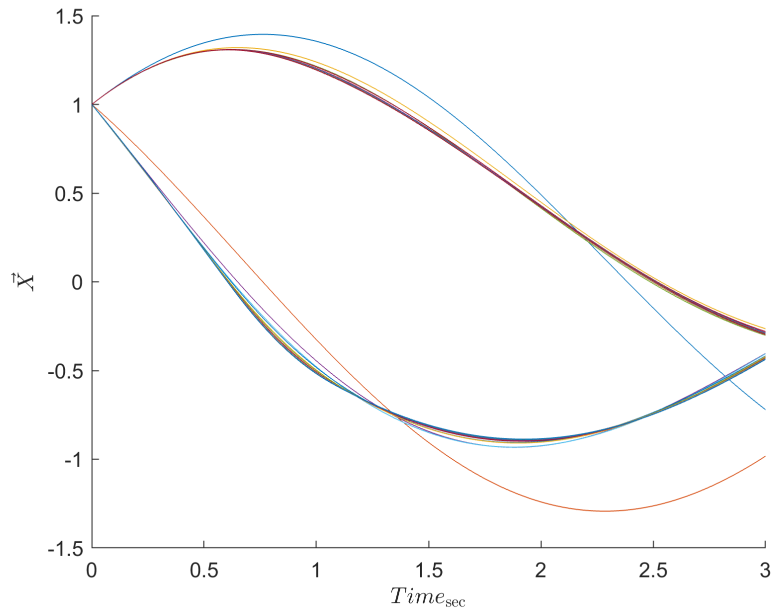
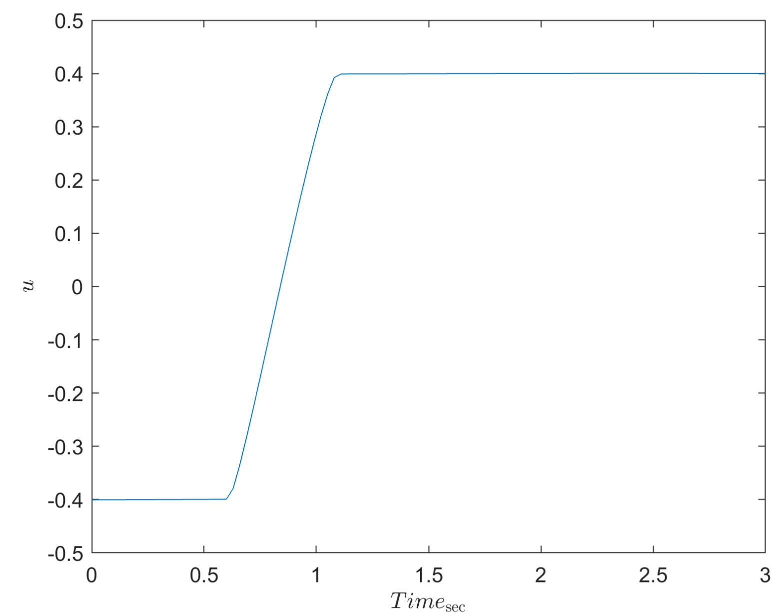


Figure 15: Steepest Descent and Quadratic Interpolation Control



– Golden Section

Figure 16: Steepest Descent and Golden Section

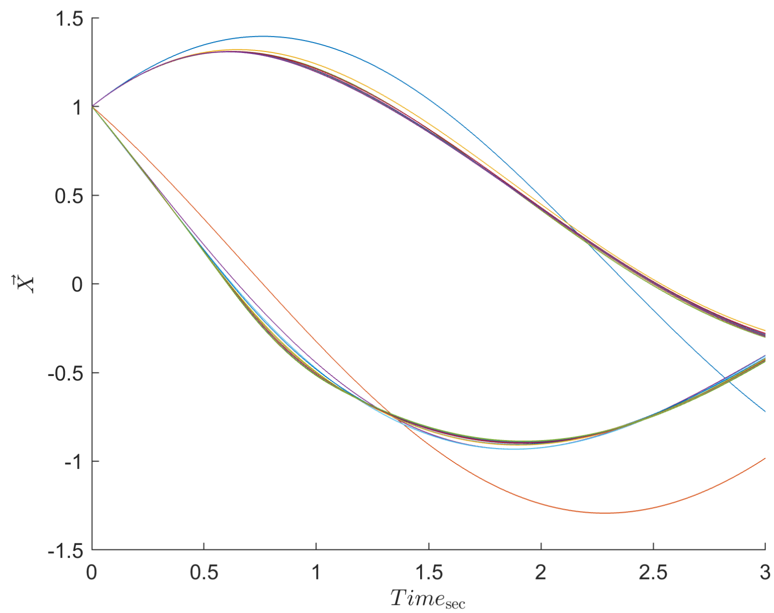
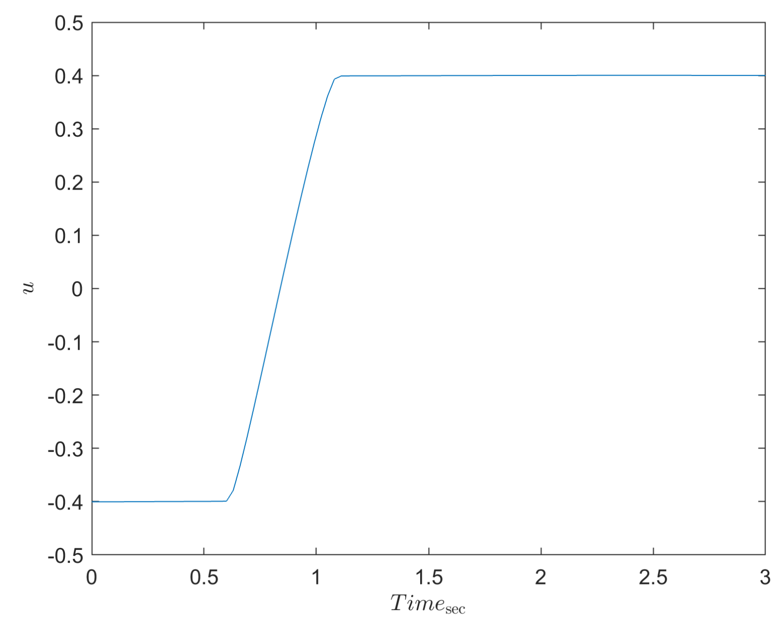


Figure 17: Steepest Descent and Golden Section Control



- BFGS
  - Quadratic Interpolation

Figure 18: BFGS and Quadratic Interpolation

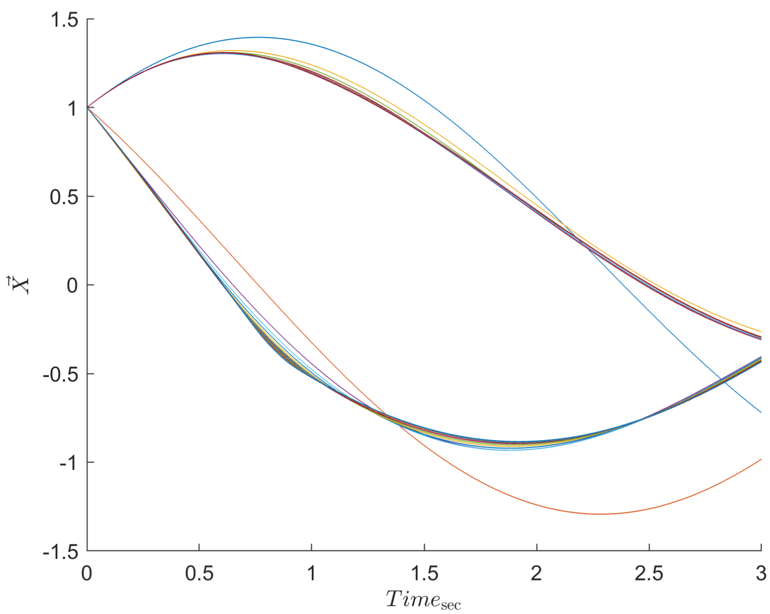
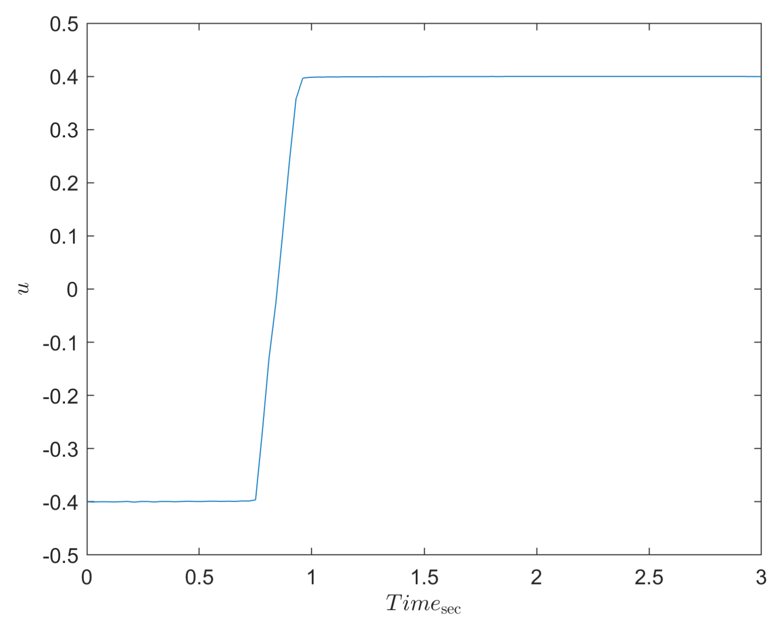


Figure 19: BFGS and Quadratic Interpolation Control



– Golden Section

Figure 20: BFGS and Golden Section

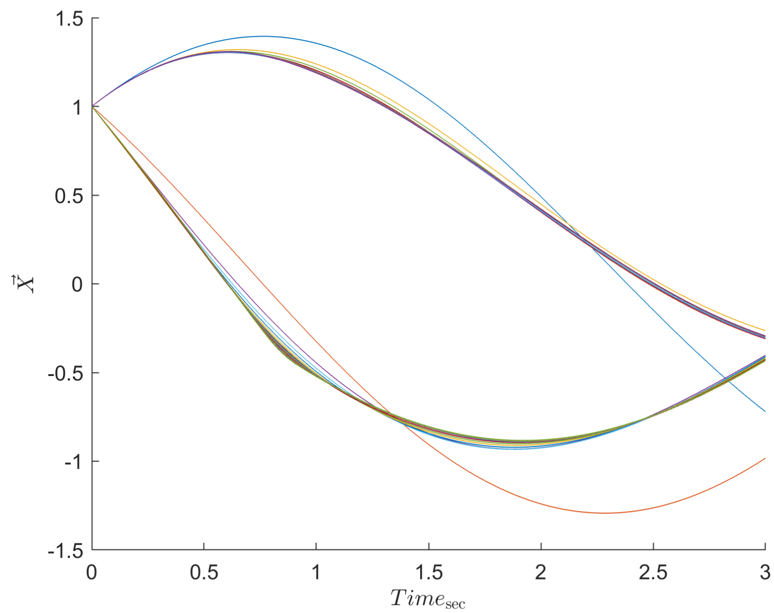
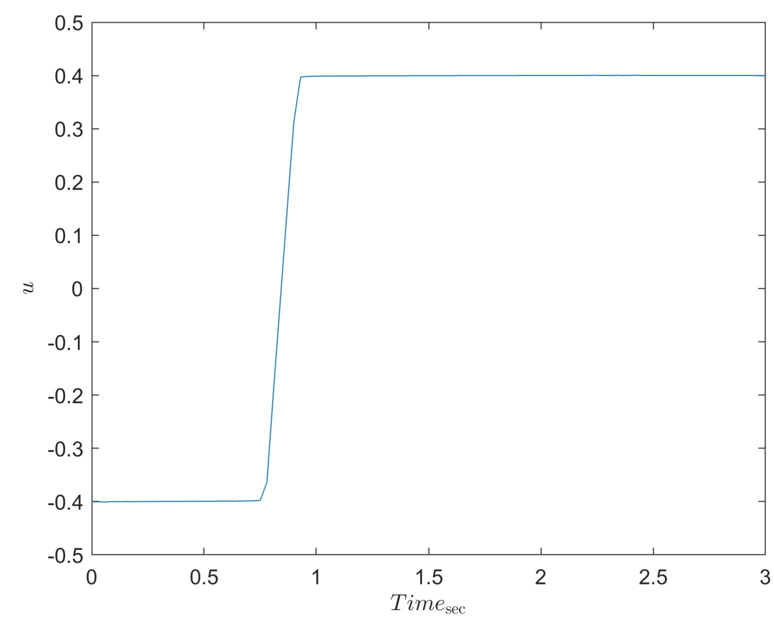


Figure 21: BFGS and Golden Section Control



2.3.2 result

- Time

Table 11: Time compare between four methods

Steepest Descent		BFGS	
Quadratic Interpolation	Golden Section	Quadratic Interpolation	Golden Section
7.595 sec	33.761 sec	86.730 sec	72.666 sec

- Number of Cost calculation

Table 12: Number of Cost calculation compare between four methods

Steepest Descent		BFGS	
Quadratic Interpolation	Golden Section	Quadratic Interpolation	Golden Section
442	1782	2787	2378

- Number of Gradient calculation

Table 13: Number of Gradient calculation compare between four methods

Steepest Descent		BFGS	
Quadratic Interpolation	Golden Section	Quadratic Interpolation	Golden Section
25	55	256	174

Cause of constrain the functions can't go to the origin but in part a function get more close to origin. In this part because of penalty function BFGS method get confused and use more iteration to get the answer, but the answer in more close to the Bang-Bang and analytical solution.

2.4 LQR

In LQR when control effort is grater than 0.4 we use 0.4 and when is lower than -0.4 we use -0.4.

Figure 22: LQR

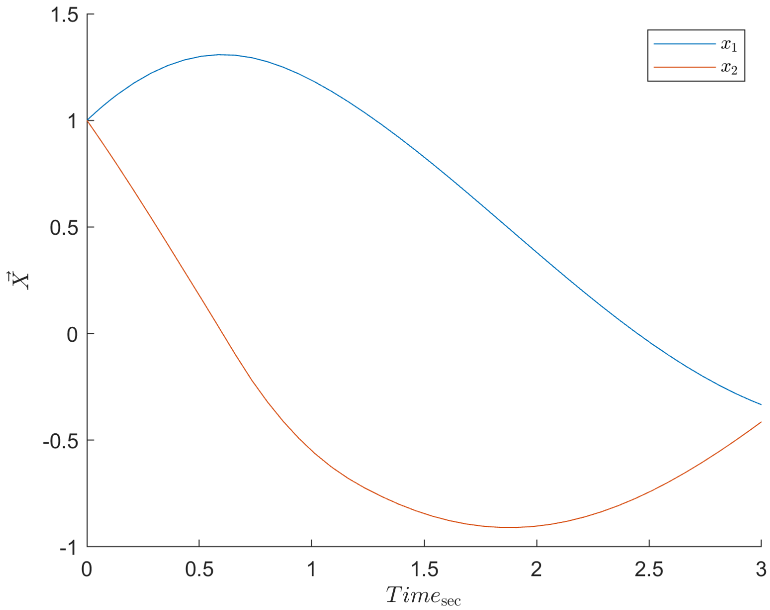
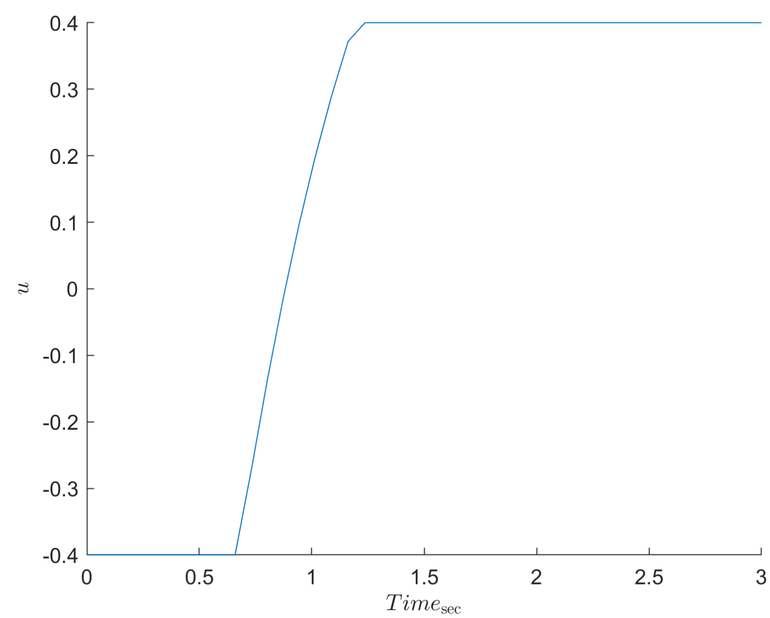


Figure 23: LQR Control



3 Question 3

3.1 System

$$\dot{x}_1 = -x_1 + u$$
$$\dot{x}_2 = -2x_2 + 2u$$
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u$$
$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

3.2 Direct Optimization

Time is free so we have a new unknowns. we use t_f as a new state and change some parameter that will describe here.

$$J = t_f^2, \qquad 0 \leq t \leq t_f, \qquad \tau = \frac{t}{t_f}, \qquad 0 \leq \tau \leq 1$$
$$\vec{a}_N(\vec{x}(t), \vec{u}(t), t_f, t) = t_f \vec{a}(\vec{x}(t), \vec{u}(t), t)$$
$$g_N(\vec{x}(t), \vec{u}(t), t_f, t) = t_f g(\vec{x}(t), \vec{u}(t), t)$$
$$\mathcal{H} = g_N(\vec{x}(t), \vec{u}(t), t_f, t) + P^T \vec{a}_N(\vec{x}(t), \vec{u}(t), t_f, t)$$

$$G_1(u) = \begin{cases} -\dfrac{1}{g_1(u)} & g_1(u) \leq \epsilon \\ -\dfrac{1}{\epsilon}\left(3 - \dfrac{3g_1(u)}{\epsilon} + \left(\dfrac{g_1(u)}{\epsilon}\right)^2\right) & g_1(u) > \epsilon \end{cases}$$
$$G_1'(u) = \begin{cases} \dfrac{1}{(u-)^2} & g_1(u) \leq \epsilon \\ -\dfrac{1}{\epsilon}\left(-\dfrac{3}{\epsilon} + \dfrac{2u-2}{\epsilon^2}\right) & g_1(u) > \epsilon \end{cases}$$

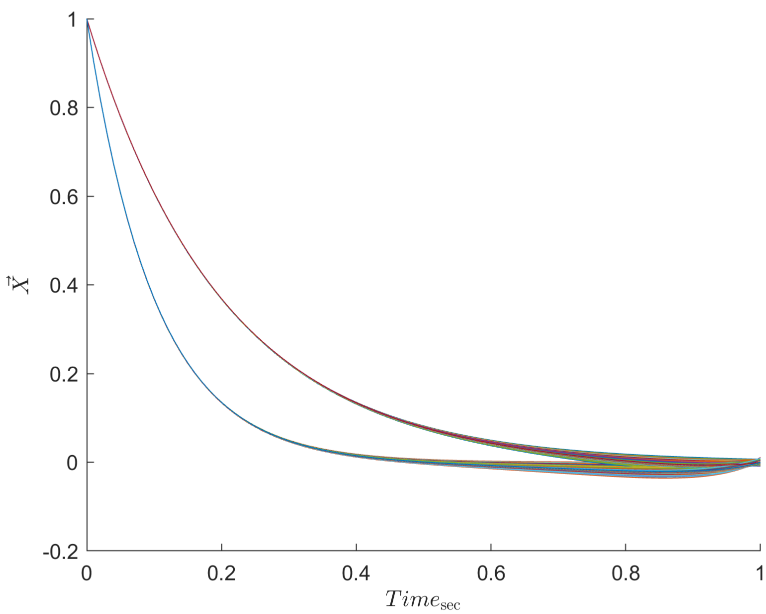
$$G_2(x_2) = \begin{cases} -\dfrac{1}{g_2(u)} & g_2(u) \leq \epsilon \\ -\dfrac{1}{\epsilon}\left(3 - \dfrac{3g_2(u)}{\epsilon} + \left(\dfrac{g_2(u)}{\epsilon}\right)^2\right) & g_2(u) > \epsilon \end{cases}$$
$$G_2'(u) = \begin{cases} \dfrac{1}{(u+1)^2} & g_2(u) \leq \epsilon \\ -\dfrac{1}{\epsilon}\left(\dfrac{3}{\epsilon} + \dfrac{2u+2}{\epsilon^2}\right) & g_2(u) > \epsilon \end{cases}$$

$$g_N(\vec{x}(t), \vec{u}(t), t_f, t) = t_f^2 + r_k G(u)$$
$$\frac{\partial J}{\partial t_f} = \frac{\partial h}{\partial t_f} + \int_0^1 \frac{\partial \mathcal{H}}{\partial t_f}$$

$$
\frac{\partial J}{\partial \vec{X}} = \begin{bmatrix} \left.\frac{\mathcal{H}}{\partial u}\right|_{\tau_0} \\ \left.\frac{\mathcal{H}}{\partial u}\right|_{\tau_1} \\ \left.\frac{\mathcal{H}}{\partial u}\right|_{\tau_2} \\ \vdots \\ \left.\frac{\mathcal{H}}{\partial u}\right|_{\tau_f} \\ \\ \frac{\partial h}{\partial t_f} + \int_0^1 \frac{\partial \mathcal{H}}{\partial t_f} \end{bmatrix}
$$

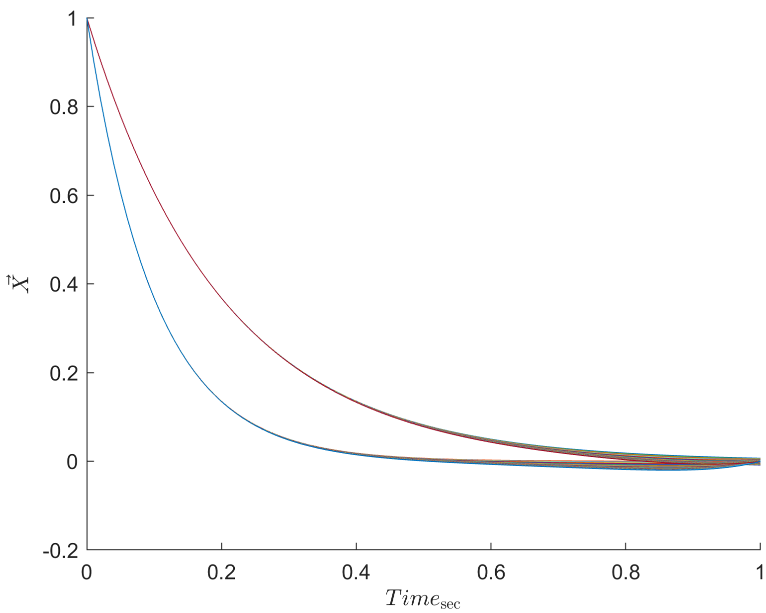
- Steepest Descent
  - Quadratic Interpolation

Figure 24: Steepest Descent and Quadratic Interpolation



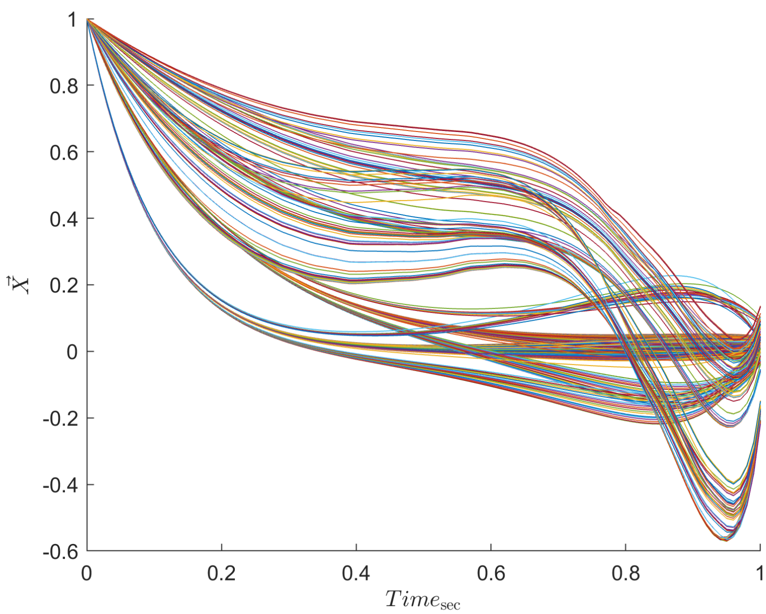
– Golden Section

Figure 25: Steepest Descent and Golden Section



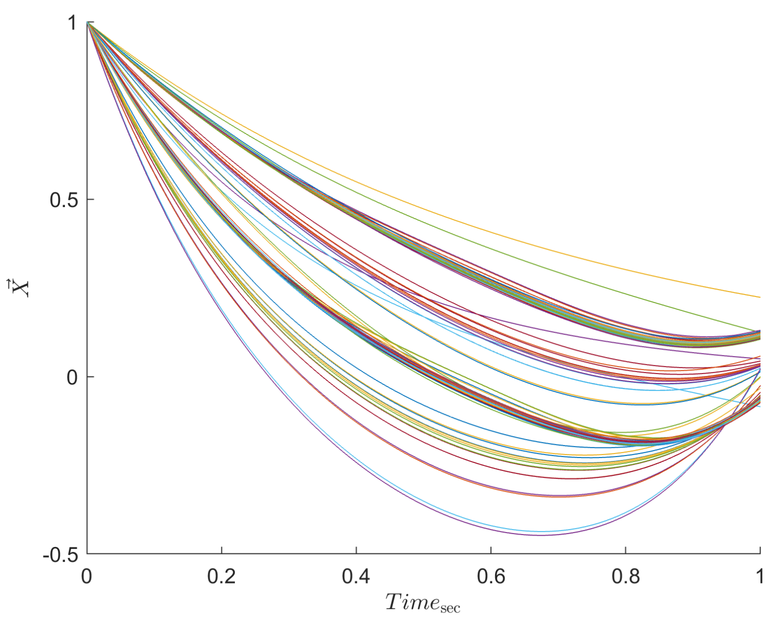
- BFGS
  - Quadratic Interpolation

Figure 26: BFGS and Quadratic Interpolation



  - Golden Section

Figure 27: BFGS and Golden Section



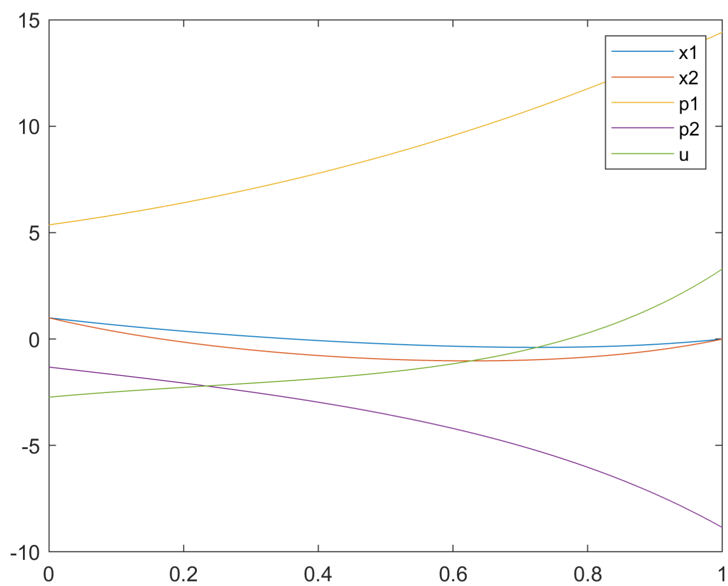
3.3 Shooting method

Final time is free so:

$$\vec{F} = \begin{bmatrix} x_1(t_f) - x_{1_f} \\ x_2(t_f) - x_{2_f} \\ (\mathcal{H} - h_t)|_{t_f} \end{bmatrix}$$

$$\vec{y}_{k+1} = \vec{y}_k - \left.\frac{\partial \vec{F}}{\partial \vec{y}}\right|_{\vec{y}_k} \vec{F}(\vec{y}_k)$$

Figure 28: Shooting method



4 Question 4

$$a = \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} x_2 \\ -0.4x_1 - 0.2x_2^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$G_1(u) = \begin{cases} -\dfrac{1}{g_1(u)} & g_1(u) \leq \epsilon \\ -\dfrac{1}{\epsilon}\left(3 - \dfrac{3g_1(u)}{\epsilon} + \left(\dfrac{g_1(u)}{\epsilon}\right)^2\right) & g_1(u) > \epsilon \end{cases}$$

$$G_1'(u) = \begin{cases} \dfrac{1}{(u-0.8)^2} & g_1(u) \leq \epsilon \\ -\dfrac{1}{\epsilon}\left(-\dfrac{3}{\epsilon} + \dfrac{2u-1.6}{\epsilon^2}\right) & g_1(u) > \epsilon \end{cases}$$

$$G_2(x_2) = \begin{cases} -\dfrac{1}{g_2(u)} & g_2(u) \leq \epsilon \\ -\dfrac{1}{\epsilon}\left(3 - \dfrac{3g_2(u)}{\epsilon} + \left(\dfrac{g_2(u)}{\epsilon}\right)^2\right) & g_2(u) > \epsilon \end{cases}$$

$$G_2'(u) = \begin{cases} \dfrac{1}{(u+0.8)^2} & g_2(u) \leq \epsilon \\ -\dfrac{1}{\epsilon}\left(\dfrac{3}{\epsilon} + \dfrac{2u+1.6}{\epsilon^2}\right) & g_2(u) > \epsilon \end{cases}$$

$$\epsilon = -c\left(r_k\right)^2, \qquad a = 0.5, \qquad r_{k+1} = cr_k, \qquad c = 0.9, \qquad \min(r_k) = 0.001$$

$$\mathcal{H} = \vec{P}^T a(\vec{X}, u, t) + \frac{1}{2}\left(x_1^2 + x_2^2 + u^2 + r_k G(u)\right)$$

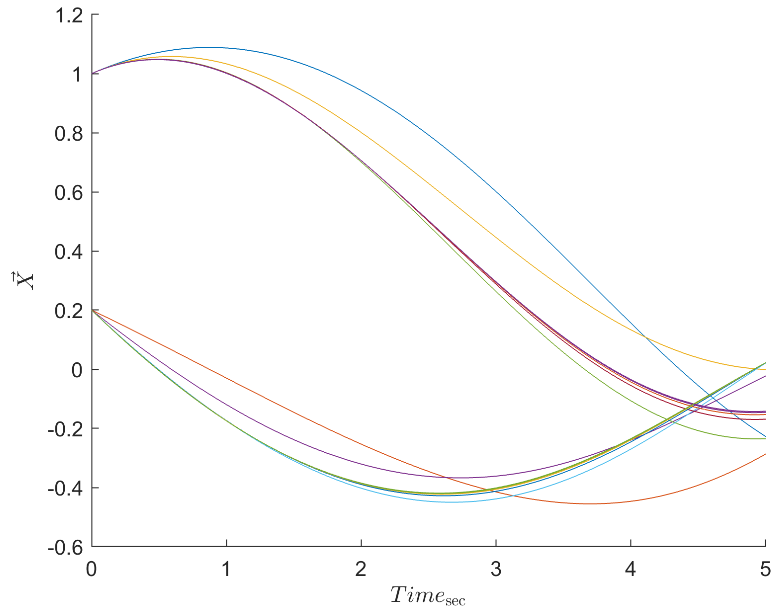
$$\dot{\vec{P}} = -\frac{\partial \mathcal{H}}{\partial \vec{X}} = \begin{bmatrix} -x_1 + 0.4p_2 \\ -x_2 - p_1 + 0.4p_2x_2 \end{bmatrix}$$

$$\begin{bmatrix} \dot{p_1} \\ \dot{p_2} \end{bmatrix} = \begin{bmatrix} -x_1 + 0.4p_2 \\ x_2(0.4p_2 - 1) - p_1 \end{bmatrix}$$

4.1 Direct Optimization

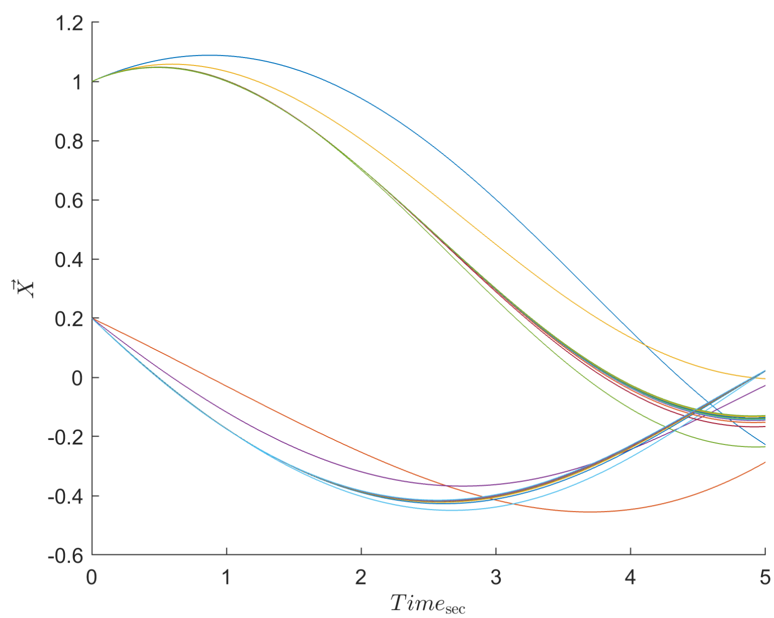
- Steepest Descent
  - Quadratic Interpolation

Figure 29: Steepest Descent and Quadratic Interpolation



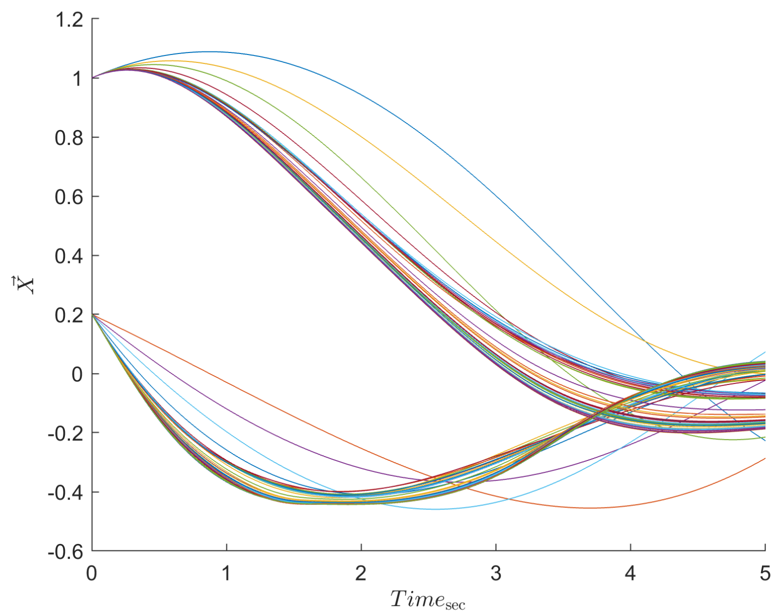
– Golden Section

Figure 30: Steepest Descent and Golden Section



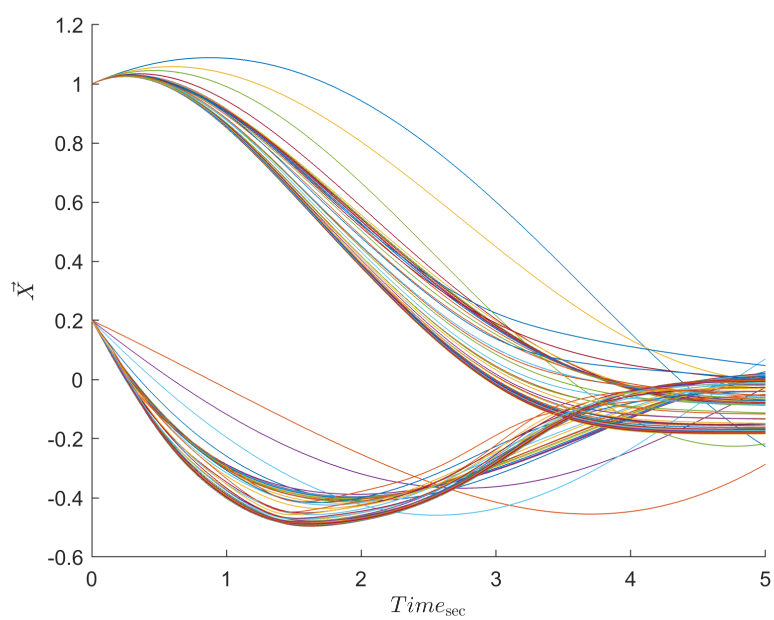
- BFGS
  - Quadratic Interpolation

Figure 31: BFGS and Quadratic Interpolation



  - Golden Section

Figure 32: BFGS and Golden Section



4.2 Shooting method

Shooting method get lost:(.

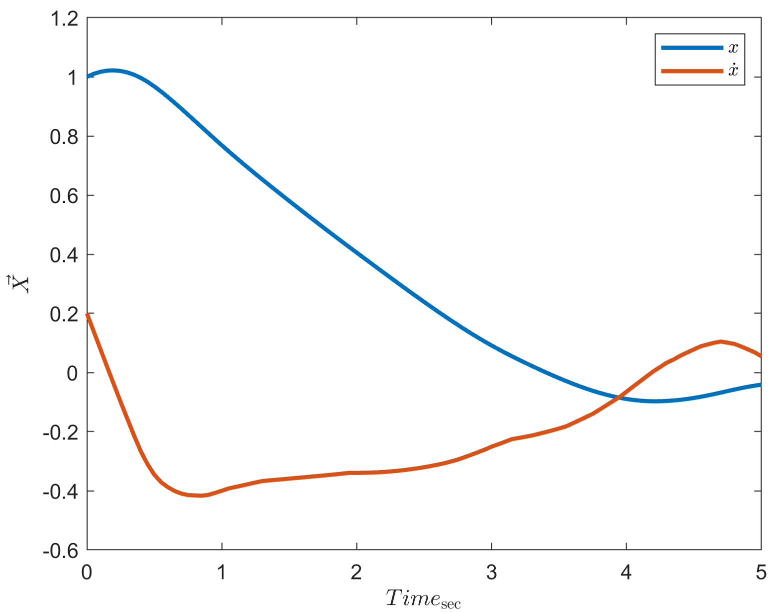
4.3 Dynamic programming

$$a = \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} x_2 \\ -0.4x_1 - 0.2x_2^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} x_2(k) \\ -0.4x_1(k) - 0.2x_2^2(k) + u(k) \end{bmatrix} \Delta t + \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

In MATLAB Code Control will save in mat file and we can use for another initial condition very fast without so much processing.

Figure 33: Dynamic Programming



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