

# Home Work #2

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## 1 Question 1

System:

$$\dot{x}(t) = -0.1x(t) + u(t)$$

Subjected to  $0 \leq u(t) \leq M$

### 1.1 part a

$$J = \int_0^{100} -x(t) dt$$

Hamiltonian matrix:

$$\begin{aligned}\mathcal{H} &= g(\vec{x}(t), u(t), t) + \vec{p}(t)^T a(\vec{x}(t), u(t), t) \\ \mathcal{H} &= -x(t) - 0.1p(t)x(t) + p(t)u(t)\end{aligned}\tag{1}$$

Euler-Lagrange equation:

$$\dot{\vec{x}} = \frac{\partial \mathcal{H}}{\partial \vec{p}} = a(\vec{x}(t), u(t), t)\tag{2}$$

$$\dot{\vec{p}} = -\frac{\partial \mathcal{H}}{\partial \vec{x}}\tag{3}$$

Now we use above equation for solve problem.

$$-\frac{\partial \mathcal{H}}{\partial x} = 1 + 0.1p$$

There is two differential equation and two unknowns.

$$\dot{x} = -x(t) - 0.1px + pu\tag{4}$$

$$\dot{p} = 1 + 0.1p\tag{5}$$

Equation 5 solved in MATLAB(Q1\_a.m) and code attached to file.

$$p(t) = C_1 \exp(t/10) - 10\tag{6}$$

Final  $x(t)$  is free so:

$$h_{\vec{x}} - \vec{p} = \vec{0} \Big|_{*, t_f} \rightarrow p(t_f) = 0$$

Use new boundry condition( $p(t_f) = 0$ ) in equation 6 to find  $p$  function( $p(t)$ ).

$$p(100) = C_1 \exp(100/10) - 10 = 0 \rightarrow C_1 = 10 \exp(-10)$$

$$p(t) = 10 \exp(0.1(t - 100)) - 10 \quad (7)$$

We know that  $u(t)$  has limit so for optimization we have another condition to select  $u(t)$  for every time.

$$u(t) = \begin{cases} \frac{\partial \mathcal{H}}{\partial u} < 0 & u(t) = M \\ \frac{\partial \mathcal{H}}{\partial u} = 0 & \mathcal{H} \text{ is not a function of } u(t) \\ \frac{\partial \mathcal{H}}{\partial u} > 0 & u(t) = 0 \end{cases} \quad (8)$$

From equation 1 we calculate  $\frac{\partial \mathcal{H}}{\partial u}$ .

$$\frac{\partial \mathcal{H}}{\partial u} = p(t)$$

From equation 7 we know that at  $t_0 \rightarrow t_f$   $p(t)$  is less than zero ( $p(t) < 0$ ), so  $u(t)$  for every time is  $M$ .

## 1.2 part b

$$J = \int_0^{100} -x(t) dt$$

Subjected to:

$$\int_0^{100} u(t) dt = K (\text{a known constant})$$

$z(t)$  is new state:

$$z(t) = \int_0^t u(t) dt \rightarrow \frac{dz}{dt} = u(t)$$

New differential constraints:

$$\begin{aligned} \frac{dz}{dt} - u(t) &= 0 \\ g_a(x, \dot{x}, \dot{z}, t, \lambda) &= g(x, \dot{x}, t) + \lambda(t)f(x, \dot{x}, \dot{z}, t) \\ g_a &= -x(t) + \lambda(\dot{z}(t) - u(t)) \end{aligned}$$

Hamiltonian matrix:

$$\mathcal{H} = g_a(\vec{x}(t), u(t), z(t), \lambda, t) + \vec{p}(t)^T a(\vec{x}(t), u(t), t)$$

We assume  $p_2 = \lambda$

$$\mathcal{H} = -x(t) - 0.1p_1(t)x(t) + p_1(t)u(t) + p_2(t)(\dot{z}(t) - u(t)) \quad (9)$$

$$\begin{aligned} \dot{\vec{p}} &= -\frac{\partial \mathcal{H}}{\partial \vec{x}} = \begin{bmatrix} -\frac{\partial \mathcal{H}}{\partial x} \\ -\frac{\partial \mathcal{H}}{\partial z} \end{bmatrix} = \begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} \\ \begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} &= \begin{bmatrix} 0.1p_1 + 1 \\ 0 \end{bmatrix} \end{aligned}$$

Above equation solved in previous part.

$$\begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} = \begin{bmatrix} 10 \exp(0.1(t - 100)) - 10 \\ C_1 \end{bmatrix} \quad (10)$$

In this part we have the same condition that described in previous in equation 8. From equation 9 we calculate  $\frac{\partial \mathcal{H}}{\partial u}$ .

$$\frac{\partial \mathcal{H}}{\partial u} = p_1 - p_2 = (10 \exp(0.1(t - 100)) - 10) - C_1$$

From equation 10 we know that  $C_1$  is constant.

There is four scenario for this problem.

1. For all the time( $t_0 \rightarrow t_f$ )  $\frac{\partial \mathcal{H}}{\partial u} > 0$  so  $u(t) = 0$ , this scenario maybe possible if  $K = 0$ .
2. For all the time( $t_0 \rightarrow t_f$ )  $\frac{\partial \mathcal{H}}{\partial u} < 0$  so  $u(t) = M$ , this scenario maybe possible if  $K = M \times t_f$ .
3. For time( $t_0 \rightarrow t$ )  $\frac{\partial \mathcal{H}}{\partial u} < 0$  and for time( $t \rightarrow t_f$ )  $\frac{\partial \mathcal{H}}{\partial u} > 0$  so for time( $t_0 \rightarrow t$ ),  $u(t) = M$  and for time( $t \rightarrow t_f$ ),  $u(t) = 0$  and this scenario maybe possible for  $0 \leq K \leq M \times t_f$ .
4. For time( $t_0 \rightarrow t$ )  $\frac{\partial \mathcal{H}}{\partial u} > 0$  and for time( $t \rightarrow t_f$ )  $\frac{\partial \mathcal{H}}{\partial u} < 0$  so for time( $t_0 \rightarrow t$ ),  $u(t) = 0$  and for time( $t \rightarrow t_f$ ),  $u(t) = M$  and this scenario is not possible because  $p_1$  is growing by the time and  $p_2$  is constant all the time.

### 1.3 part c

$$J = -x(100)$$

Subjected to:

$$\int_0^{100} u(t)dt = K(\text{a known constant})$$

$z(t)$  is new state:

$$z(t) = \int_0^t u(t)dt \rightarrow \frac{dz}{dt} = u(t)$$

New differential constraints:

$$\begin{aligned} \frac{dz}{dt} - u(t) &= 0 \\ g_a(x, \dot{x}, \dot{z}, t, \lambda) &= g(x, \dot{x}, t) + \lambda(t)f(x, \dot{x}, \dot{z}, t) \\ g_a &= -x(t) + \lambda(\dot{z}(t) - u(t)) \end{aligned}$$

Hamiltonian matrix:

$$\mathcal{H} = g_a(\vec{x}(t), u(t), z(t), \lambda, t) + \vec{p}(t)^T a(\vec{x}(t), u(t), t)$$

We assume  $p_2 = \lambda$

$$\mathcal{H} = -0.1p_1(t)x(t) + p_1(t)u(t) + p_2(t)(\dot{z}(t) - u(t)) \quad (11)$$

$$\begin{aligned} \dot{\vec{p}} &= -\frac{\partial \mathcal{H}}{\partial \vec{x}} = \begin{bmatrix} -\frac{\partial \mathcal{H}}{\partial x} \\ -\frac{\partial \mathcal{H}}{\partial z} \end{bmatrix} = \begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} \\ \begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} &= \begin{bmatrix} 0.1p_1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} &= \begin{bmatrix} C_1 \exp(t/10) \\ C_2 \end{bmatrix} \end{aligned} \quad (12)$$

Final  $x(t)$  is free so:

$$h_{\vec{x}} - \vec{p} = \vec{0} \Big|_{*,t_f} \rightarrow p_1(t_f) = -1 \quad (13)$$

From equation 12 and 13 we can find  $p_1(t)$  function.

$$p_1(100) = C_1 \exp(100/10) = -1 \rightarrow C_1 = -\exp(-10)$$

$$p_1(t) = -\exp(0.1(t - 100))$$

This problem is like section 1.2 and have the same scenarios.

## 2 Question 2

### 2.1 part a

Dynamic of system:

$$\ddot{x}(t) = \frac{T}{M} \cos(\beta(t)), \quad \ddot{y}(t) = \frac{T}{M} \sin(\beta(t))$$

Now we define new states:

$$x_1(t) = x(t), \quad x_2(t) = \dot{x}_1(t), \quad x_3(t) = y(t), \quad x_4(t) = \dot{x}_3(t), \quad u(t) = \beta(t)$$

New dynamic of system:

$$a(\vec{x}(t), u, t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ \frac{T}{M} \cos(u(t)) \\ x_4(t) \\ \frac{T}{M} \sin(u(t)) \end{bmatrix}$$

### 2.2 part b

For minimum time:

$$J = t_f - t_0 = \int_{t_0}^{t_f} dt$$

Hamiltonian matrix:

$$\mathcal{H} = g(\vec{x}(t), u(t), t) + \vec{p}(t)^T a(\vec{x}(t), u(t), t)$$

$$\mathcal{H} = 1 + \begin{bmatrix} p_1(t) & p_2(t) & p_3(t) & p_4(t) \end{bmatrix} \begin{bmatrix} x_2(t) \\ \frac{T}{M} \cos(u(t)) \\ x_4(t) \\ \frac{T}{M} \sin(u(t)) \end{bmatrix}$$

$$\mathcal{H} = 1 + p_1(t)x_2(t) + \frac{T}{M}p_2(t)\cos(u(t)) + p_3(t)x_4(t) + \frac{T}{M}p_4(t)\sin(u(t)) \quad (14)$$

Euler–Lagrange equation:

$$\dot{\vec{x}} = \frac{\partial \mathcal{H}}{\partial \vec{p}} = a(\vec{x}(t), u(t), t) \quad (15)$$

$$\dot{\vec{p}} = -\frac{\partial \mathcal{H}}{\partial \vec{x}} \quad (16)$$

$$\vec{0} = \frac{\partial \mathcal{H}}{\partial \vec{u}} \quad (17)$$

Now we use equation 14 to solve Euler-Lagrange equation.

$$\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \dot{p}_3 \\ \dot{p}_4 \end{bmatrix} = \begin{bmatrix} -\frac{\partial \mathcal{H}}{\partial x_1} \\ -\frac{\partial \mathcal{H}}{\partial x_2} \\ -\frac{\partial \mathcal{H}}{\partial x_3} \\ -\frac{\partial \mathcal{H}}{\partial x_4} \end{bmatrix} = \begin{bmatrix} 0 \\ -p_1 \\ 0 \\ -p_3 \end{bmatrix} \quad (18)$$

Answer of equation 18 is:

$$\begin{bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \\ p_4(t) \end{bmatrix} = \begin{bmatrix} C_1 \\ -C_1 t + C_2 \\ C_3 \\ -C_3 t + C_4 \end{bmatrix} \quad (19)$$

From Euler-Lagrange equation:

$$\vec{0} = \frac{\partial \mathcal{H}}{\partial \vec{u}} \rightarrow p_4 \cos(u) = p_2 \sin(u) \rightarrow u(t) = \tan^{-1} \left( \frac{p_4(t)}{p_2(t)} \right) \quad (20)$$

Boundary conditions:

$$t_0 = 0, \quad t_f = \text{free}$$

$$\vec{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{x}(t_f) = \begin{bmatrix} x_1(t_f) \\ x_2(t_f) \\ x_3(t_f) \\ x_4(t_f) \end{bmatrix} = \begin{bmatrix} \text{Free} \\ V \\ D \\ 0 \end{bmatrix} \quad (D, v \text{ Are known constant})$$

Final time is free so:

$$(\mathcal{H} + h_t)|_{*,t_f} = 0 \rightarrow \mathcal{H}|_{*,t_f} = 0 \quad (21)$$

Final  $x_1$  is free so:

$$\left( \frac{\partial h}{\partial x_1} - p_1 \right) \Big|_{*,t_f} = 0 \rightarrow p_1(t_f) = 0 \quad (22)$$

## 2.3 part c

There is  $11(2n+3)$  equation and 11 unknowns.

Unknowns:

$$\vec{x}(t)|_{4 \times 1}, \quad \vec{p}(t)|_{4 \times 1}, \quad t_f, \quad x_1(t_f), \quad u(t)$$

Now use previous equation to find unknowns. From equation 22 and 19 we can find out:

$$p_1(t) = C_1 = 0, \quad p_2(t) = 0t + C_2 = C_2 \quad (23)$$

Now we use equation 23, 19, 20 and 14 to have new equation and put parameters in equation 21.

$$\mathcal{H}|_{*,t_f} = 1 + 0 \times x_2(t_f) + \frac{T}{M} C_2 \cos \left( \tan^{-1} \left( \frac{-C_3 t + C_4}{C_2} \right) \right) + p_3(t_f) \times 0 + \frac{T}{M} p_4(t) \sin \left( \tan^{-1} \left( \frac{-C_3 t + C_4}{C_2} \right) \right)$$

$$\mathcal{H}|_{*,t_f} = 1 + \frac{T}{M} C_2 \cos \left( \tan^{-1} \left( \frac{-C_3 t + C_4}{C_2} \right) \right) + \frac{T}{M} p_4(t) \sin \left( \tan^{-1} \left( \frac{-C_3 t + C_4}{C_2} \right) \right) = 0 \quad (24)$$

This far i could go to solve problem analytically. :)

## 2.4 part d

KObs

## 3 Question 3

$$\ddot{x}(t) = -x(t) - 0.1\dot{x}(t) + u(t), \quad x(0) = \dot{x}(0) = 1$$

Assume:

$$x_1(t) = x(t), \quad x_2(t) = \dot{x}(t) \rightarrow a(\vec{x}, u, t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -x_1(t) - 0.1x_2(t) + u(t) \end{bmatrix}$$

$$J = \frac{1}{2}x^T(t_f)Hx(t_f) + \frac{1}{2} \int_0^{t_f} (\alpha(x^2 + \dot{x}^2) + \beta u^2) dt$$

### 3.1 part a

$\alpha = \beta = 1$ ,  $t_f \rightarrow \infty$  and  $H = 0$ :

$$J = \frac{1}{2} \int_0^{t_f} (x^2 + \dot{x}^2 + u^2) dt$$

$$\vec{p}(t) = \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix}$$

Hamiltonian matrix:

$$\mathcal{H} = g(\vec{x}(t), u(t), t) + \vec{p}(t)^T a(\vec{x}(t), u(t), t)$$

$$\mathcal{H} = x_1^2(t) + x_2^2(t) + u^2(t) + \begin{bmatrix} p_1(t) & p_2(t) \end{bmatrix} \begin{bmatrix} x_2(t) \\ -x_1(t) - 0.1x_2(t) + u(t) \end{bmatrix}$$

$$\mathcal{H} = x_1^2(t) + x_2^2(t) + u^2(t) + p_1(t)x_2(t) - p_2(t)x_1(t) - 0.1p_2(t)x_2(t) + p_2(t)u(t)$$

Euler-Lagrange equation:

$$\dot{\vec{x}} = \frac{\partial \mathcal{H}}{\partial \vec{p}} = a(\vec{x}(t), u(t), t) \quad (25)$$

$$\dot{\vec{p}} = -\frac{\partial \mathcal{H}}{\partial \vec{x}} = \begin{bmatrix} -\frac{\partial \mathcal{H}}{\partial x_1} \\ -\frac{\partial \mathcal{H}}{\partial x_2} \end{bmatrix} \quad (26)$$

$$\vec{0} = \frac{\partial \mathcal{H}}{\partial \vec{u}} \quad (27)$$

Now we use above equation for solve problem.

$$\begin{bmatrix} -\frac{\partial \mathcal{H}}{\partial x_1} \\ -\frac{\partial \mathcal{H}}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -2x_1 + p_2 \\ -2x_2 - p_1 + 0.1p_2 \end{bmatrix} = \begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix}$$

$$\frac{\partial \mathcal{H}}{\partial \vec{u}} = 2u + p_2 = 0 \rightarrow u = -0.5p_2$$

There is four differential equation and four unknowns.

$$\dot{x}_1 = x_2 \tag{28}$$

$$\dot{x}_2 = -x_1 - 0.1x_2 - 0.5p_2 \tag{29}$$

$$\dot{p}_1 = -2x_1 + p_2 \tag{30}$$

$$\dot{p}_2 = -2x_2 - p_1 + 0.1p_2 \tag{31}$$

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