

# Home Work #2

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## 1 Question 1

System:

$$\dot{x}(t) = -0.1x(t) + u(t)$$

Subjected to  $0 \leq u(t) \leq M$

### 1.1 part a

$$J = \int_0^{100} -x(t) dt$$

Hamiltonian matrix:

$$\begin{aligned}\mathcal{H} &= g(\vec{x}(t), u(t), t) + \vec{p}(t)^T a(\vec{x}(t), u(t), t) \\ \mathcal{H} &= -x(t) - 0.1p(t)x(t) + p(t)u(t)\end{aligned}\tag{1}$$

Euler-Lagrange equation:

$$\dot{\vec{x}} = \frac{\partial \mathcal{H}}{\partial \vec{p}} = a(\vec{x}(t), u(t), t)\tag{2}$$

$$\dot{\vec{p}} = -\frac{\partial \mathcal{H}}{\partial \vec{x}}\tag{3}$$

Now we use above equation for solve problem.

$$-\frac{\partial \mathcal{H}}{\partial x} = 1 + 0.1p$$

There is two differential equation and two unknowns.

$$\dot{x} = -x(t) - 0.1px + pu\tag{4}$$

$$\dot{p} = 1 + 0.1p\tag{5}$$

Equation 5 solved in MATLAB(Q1\_a.m) and code attached to file.

$$p(t) = C_1 \exp(t/10) - 10\tag{6}$$

Final  $x(t)$  is free so:

$$h_{\vec{x}} - \vec{p} = \vec{0} \Big|_{*, t_f} \rightarrow p(t_f) = 0$$

Use new boundry condition( $p(t_f) = 0$ ) in equation 6 to find  $p$  function( $p(t)$ ).

$$p(100) = C_1 \exp(100/10) - 10 = 0 \rightarrow C_1 = 10 \exp(-10)$$

$$p(t) = 10 \exp(0.1(t - 100)) - 10 \quad (7)$$

We know that  $u(t)$  has limit so for optimization we have another condition to select  $u(t)$  for every time.

$$u(t) = \begin{cases} \frac{\partial \mathcal{H}}{\partial u} < 0 & u(t) = M \\ \frac{\partial \mathcal{H}}{\partial u} = 0 & \mathcal{H} \text{ is not a function of } u(t) \\ \frac{\partial \mathcal{H}}{\partial u} > 0 & u(t) = 0 \end{cases} \quad (8)$$

From equation 1 we calculate  $\frac{\partial \mathcal{H}}{\partial u}$ .

$$\frac{\partial \mathcal{H}}{\partial u} = p(t)$$

From equation 7 we know that at  $t_0 \rightarrow t_f$   $p(t)$  is less than zero ( $p(t) < 0$ ), so  $u(t)$  for every time is  $M$ .

## 1.2 part b

$$J = \int_0^{100} -x(t) dt$$

Subjected to:

$$\int_0^{100} u(t) dt = K (\text{a known constant})$$

$z(t)$  is new state:

$$z(t) = \int_0^t u(t) dt \rightarrow \frac{dz}{dt} = u(t)$$

New differential constraints:

$$\begin{aligned} \frac{dz}{dt} - u(t) &= 0 \\ g_a(x, \dot{x}, \dot{z}, t, \lambda) &= g(x, \dot{x}, t) + \lambda(t)f(x, \dot{x}, \dot{z}, t) \\ g_a &= -x(t) + \lambda(\dot{z}(t) - u(t)) \end{aligned}$$

Hamiltonian matrix:

$$\mathcal{H} = g_a(\vec{x}(t), u(t), z(t), \lambda, t) + \vec{p}(t)^T a(\vec{x}(t), u(t), t)$$

We assume  $p_2 = \lambda$  and use Hamiltonian assumptions:

$$\mathcal{H} = -x(t) - 0.1p_1(t)x(t) + p_1(t)u(t) + p_2(t)u(t) \quad (9)$$

$$\begin{aligned} \dot{\vec{p}} &= -\frac{\partial \mathcal{H}}{\partial \vec{x}} = \begin{bmatrix} -\frac{\partial \mathcal{H}}{\partial x} \\ -\frac{\partial \mathcal{H}}{\partial z} \end{bmatrix} = \begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} \\ \begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} &= \begin{bmatrix} 0.1p_1 + 1 \\ 0 \end{bmatrix} \end{aligned}$$

Above equation solved in previous part.

$$\begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} = \begin{bmatrix} 10 \exp(0.1(t - 100)) - 10 \\ C_1 \end{bmatrix} \quad (10)$$

In this part we have the same condition that described in previous in equation 8. From equation 9 we calculate  $\frac{\partial \mathcal{H}}{\partial u}$ .

$$\frac{\partial \mathcal{H}}{\partial u} = p_1 - p_2 = (10 \exp(0.1(t - 100)) - 10) - C_1$$

From equation 10 we know that  $C_1$  is constant.

There is four scenario for this problem.

1. For all the time( $t_0 \rightarrow t_f$ )  $\frac{\partial \mathcal{H}}{\partial u} > 0$  so  $u(t) = 0$ , this scenario maybe possible if  $K = 0$ .
2. For all the time( $t_0 \rightarrow t_f$ )  $\frac{\partial \mathcal{H}}{\partial u} < 0$  so  $u(t) = M$ , this scenario maybe possible if  $K = M \times t_f$ .
3. For time( $t_0 \rightarrow t$ )  $\frac{\partial \mathcal{H}}{\partial u} < 0$  and for time( $t \rightarrow t_f$ )  $\frac{\partial \mathcal{H}}{\partial u} > 0$  so for time( $t_0 \rightarrow t$ ),  $u(t) = M$  and for time( $t \rightarrow t_f$ ),  $u(t) = 0$  and this scenario maybe possible for  $0 \leq K \leq M \times t_f$ .
4. For time( $t_0 \rightarrow t$ )  $\frac{\partial \mathcal{H}}{\partial u} > 0$  and for time( $t \rightarrow t_f$ )  $\frac{\partial \mathcal{H}}{\partial u} < 0$  so for time( $t_0 \rightarrow t$ ),  $u(t) = 0$  and for time( $t \rightarrow t_f$ ),  $u(t) = M$  and this scenario is not possible because  $p_1$  is growing by the time and  $p_2$  is constant all the time.

### 1.3 part c

$$J = -x(100)$$

Subjected to:

$$\int_0^{100} u(t)dt = K(\text{a known constant})$$

$z(t)$  is new state:

$$z(t) = \int_0^t u(t)dt \rightarrow \frac{dz}{dt} = u(t)$$

New differential constraints:

$$\begin{aligned} \frac{dz}{dt} - u(t) &= 0 \\ g_a(x, \dot{x}, \dot{z}, t, \lambda) &= g(x, \dot{x}, t) + \lambda(t)f(x, \dot{x}, \dot{z}, t) \\ g_a &= -x(t) + \lambda(\dot{z}(t) - u(t)) \end{aligned}$$

Hamiltonian matrix:

$$\mathcal{H} = g_a(\vec{x}(t), u(t), z(t), \lambda, t) + \vec{p}(t)^T a(\vec{x}(t), u(t), t)$$

We assume  $p_2 = \lambda$  and use Hamiltonian assumptions:

$$\mathcal{H} = -0.1p_1(t)x(t) + p_1(t)u(t) + p_2(t)u(t) \quad (11)$$

$$\begin{aligned} \dot{\vec{p}} &= -\frac{\partial \mathcal{H}}{\partial \vec{x}} = \begin{bmatrix} -\frac{\partial \mathcal{H}}{\partial x} \\ -\frac{\partial \mathcal{H}}{\partial z} \end{bmatrix} = \begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} \\ \begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} &= \begin{bmatrix} 0.1p_1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} &= \begin{bmatrix} C_1 \exp(t/10) \\ C_2 \end{bmatrix} \end{aligned} \quad (12)$$

Final  $x(t)$  is free so:

$$h_{\vec{x}} - \vec{p} = \vec{0} \Big|_{*,t_f} \rightarrow p_1(t_f) = -1 \quad (13)$$

From equation 12 and 13 we can find  $p_1(t)$  function.

$$p_1(100) = C_1 \exp(100/10) = -1 \rightarrow C_1 = -\exp(-10)$$

$$p_1(t) = -\exp(0.1(t - 100))$$

This problem is like section 1.2 and have the same scenarios.

## 2 Question 2

### 2.1 part a

Dynamic of system:

$$\ddot{x}(t) = \frac{T}{M} \cos(\beta(t)), \quad \ddot{y}(t) = \frac{T}{M} \sin(\beta(t))$$

Now we define new states:

$$x_1(t) = x(t), \quad x_2(t) = \dot{x}_1(t), \quad x_3(t) = y(t), \quad x_4(t) = \dot{x}_3(t), \quad u(t) = \beta(t)$$

New dynamic of system:

$$a(\vec{x}(t), u, t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ \frac{T}{M} \cos(u(t)) \\ x_4(t) \\ \frac{T}{M} \sin(u(t)) \end{bmatrix}$$

### 2.2 part b

For minimum time:

$$J = t_f - t_0 = \int_{t_0}^{t_f} dt$$

Hamiltonian matrix:

$$\mathcal{H} = g(\vec{x}(t), u(t), t) + \vec{p}(t)^T a(\vec{x}(t), u(t), t)$$

$$\mathcal{H} = 1 + \begin{bmatrix} p_1(t) & p_2(t) & p_3(t) & p_4(t) \end{bmatrix} \begin{bmatrix} x_2(t) \\ \frac{T}{M} \cos(u(t)) \\ x_4(t) \\ \frac{T}{M} \sin(u(t)) \end{bmatrix}$$

$$\mathcal{H} = 1 + p_1(t)x_2(t) + \frac{T}{M}p_2(t)\cos(u(t)) + p_3(t)x_4(t) + \frac{T}{M}p_4(t)\sin(u(t)) \quad (14)$$

Euler–Lagrange equation:

$$\dot{\vec{x}} = \frac{\partial \mathcal{H}}{\partial \vec{p}} = a(\vec{x}(t), u(t), t) \quad (15)$$

$$\dot{\vec{p}} = -\frac{\partial \mathcal{H}}{\partial \vec{x}} \quad (16)$$

$$\vec{0} = \frac{\partial \mathcal{H}}{\partial \vec{u}} \quad (17)$$

Now we use equation 14 to solve Euler-Lagrange equation.

$$\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \dot{p}_3 \\ \dot{p}_4 \end{bmatrix} = \begin{bmatrix} -\frac{\partial \mathcal{H}}{\partial x_1} \\ -\frac{\partial \mathcal{H}}{\partial x_2} \\ -\frac{\partial \mathcal{H}}{\partial x_3} \\ -\frac{\partial \mathcal{H}}{\partial x_4} \end{bmatrix} = \begin{bmatrix} 0 \\ -p_1 \\ 0 \\ -p_3 \end{bmatrix} \quad (18)$$

Answer of equation 18 is:

$$\begin{bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \\ p_4(t) \end{bmatrix} = \begin{bmatrix} C_1 \\ -C_1 t + C_2 \\ C_3 \\ -C_3 t + C_4 \end{bmatrix} \quad (19)$$

From Euler-Lagrange equation:

$$\vec{0} = \frac{\partial \mathcal{H}}{\partial \vec{u}} \rightarrow p_4 \cos(u) = p_2 \sin(u) \rightarrow u(t) = \tan^{-1} \left( \frac{p_4(t)}{p_2(t)} \right) \quad (20)$$

Boundary conditions:

$$t_0 = 0, \quad t_f = \text{free}$$

$$\vec{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{x}(t_f) = \begin{bmatrix} x_1(t_f) \\ x_2(t_f) \\ x_3(t_f) \\ x_4(t_f) \end{bmatrix} = \begin{bmatrix} \text{Free} \\ V \\ D \\ 0 \end{bmatrix} \quad (D, V \text{ Are known constant})$$

Final time is free so:

$$(\mathcal{H} + h_t)|_{*,t_f} = 0 \rightarrow \mathcal{H}|_{*,t_f} = 0 \quad (21)$$

Final  $x_1$  is free so:

$$\left( \frac{\partial h}{\partial x_1} - p_1 \right) \Big|_{*,t_f} = 0 \rightarrow p_1(t_f) = 0 \quad (22)$$

## 2.3 part c

There is  $11(2n+3)$  equation and 11 unknowns.

Unknowns:

$$\vec{x}(t)|_{4 \times 1}, \quad \vec{p}(t)|_{4 \times 1}, \quad t_f, \quad x_1(t_f), \quad u(t)$$

Now use previous equation to find unknowns. From equation 22 and 19 we can find out:

$$p_1(t) = C_1 = 0, \quad p_2(t) = 0t + C_2 = C_2 \quad (23)$$

Now we use equation 23, 19, 20 and 14 to have new equation and put parameters in equation 21.

$$\mathcal{H}|_{*,t_f} = 1 + 0 \times x_2(t_f) + \frac{T}{M} C_2 \cos \left( \tan^{-1} \left( \frac{-C_3 t + C_4}{C_2} \right) \right) + p_3(t_f) \times 0 + \frac{T}{M} p_4(t) \sin \left( \tan^{-1} \left( \frac{-C_3 t + C_4}{C_2} \right) \right)$$

$$\mathcal{H}|_{*,t_f} = 1 + \frac{T}{M} C_2 \cos \left( \tan^{-1} \left( \frac{-C_3 t + C_4}{C_2} \right) \right) + \frac{T}{M} p_4(t) \sin \left( \tan^{-1} \left( \frac{-C_3 t + C_4}{C_2} \right) \right) = 0 \quad (24)$$

This far i could go to solve problem analytically. :)

## 2.4 part d

For maximize horizontal range we must change cost function.

$$J = -x_1(t_f)$$

Hamiltonian matrix:

$$\begin{aligned} \mathcal{H} &= g(\vec{x}(t), u(t), t) + \vec{p}(t)^T a(\vec{x}(t), u(t), t) \\ \mathcal{H} &= \begin{bmatrix} p_1(t) & p_2(t) & p_3(t) & p_4(t) \end{bmatrix} \begin{bmatrix} x_2(t) \\ \frac{T}{M} \cos(u(t)) \\ x_4(t) \\ \frac{T}{M} \sin(u(t)) \end{bmatrix} \\ \mathcal{H} &= p_1(t)x_2(t) + \frac{T}{M}p_2(t) \cos(u(t)) + p_3(t)x_4(t) + \frac{T}{M}p_4(t) \sin(u(t)) \end{aligned} \quad (25)$$

Euler–Lagrange equation:

$$\dot{\vec{x}} = \frac{\partial \mathcal{H}}{\partial \vec{p}} = a(\vec{x}(t), u(t), t) \quad (26)$$

$$\dot{\vec{p}} = -\frac{\partial \mathcal{H}}{\partial \vec{x}} \quad (27)$$

$$\vec{0} = \frac{\partial \mathcal{H}}{\partial \vec{u}} \quad (28)$$

In this problem like previous we must satisfy Euler–Equation equation. Now we use equation 25 to solve Euler–Lagrange equation.

$$\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \dot{p}_3 \\ \dot{p}_4 \end{bmatrix} = \begin{bmatrix} -\frac{\partial \mathcal{H}}{\partial x_1} \\ -\frac{\partial \mathcal{H}}{\partial x_2} \\ -\frac{\partial \mathcal{H}}{\partial x_3} \\ -\frac{\partial \mathcal{H}}{\partial x_4} \end{bmatrix} = \begin{bmatrix} 0 \\ -p_1 \\ 0 \\ -p_3 \end{bmatrix} \quad (29)$$

Answer of equation 18 is:

$$\begin{bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \\ p_4(t) \end{bmatrix} = \begin{bmatrix} C_1 \\ -C_1 t + C_2 \\ C_3 \\ -C_3 t + C_4 \end{bmatrix} \quad (30)$$

Boundary conditions:

$$\begin{aligned} t_0 &= 0, \quad t_f = t_1 \\ \vec{x}(0) &= \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{x}(t_f) = \begin{bmatrix} x_1(t_f) \\ x_2(t_f) \\ x_3(t_f) \\ x_4(t_f) \end{bmatrix} = \begin{bmatrix} \text{Free} \\ \text{Free} \\ D \\ \text{Free} \end{bmatrix} \quad (D \text{ is known constant}) \end{aligned}$$

Final  $x_1$ ,  $x_2$  and  $x_4$  are free so:

$$\left( \frac{\partial h}{\partial \vec{x}} - \vec{p} \right) \Big|_{*, t_f} = \vec{0} \rightarrow \left[ \begin{array}{c} \frac{\partial h}{\partial \vec{x}_1} \\ \frac{\partial h}{\partial \vec{x}_2} \\ \frac{\partial h}{\partial \vec{x}_4} \end{array} \right] \Big|_{*, t_f} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} p_1(t_f) \\ p_2(t_f) \\ p_4(t_f) \end{bmatrix} \quad (31)$$

Now we use equation 31 data to get 30 constants.

$$p_1(t_f) = C_1 = -1, \quad p_2(t_f) = -t_f + C_2 \Rightarrow C_2 = t_f, \quad p_4(t_f) = -C_3 t_f + {}^\circ C_4 = 0$$

Now we use other equation that described previous part to solve problem. This far i could go to solve problem analytically. :)

### 3 Question 3

$$\ddot{x}(t) = -x(t) - 0.1\dot{x}(t) + u(t), \quad x(0) = \dot{x}(0) = 1$$

Assume:

$$x_1(t) = x(t), \quad x_2(t) = \dot{x}(t) \rightarrow a(\vec{x}, u, t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -x_1(t) - 0.1x_2(t) + u(t) \end{bmatrix}$$

$$\dot{\vec{x}} = A(t)\vec{x}(t) + B(t)\vec{u}(t) \quad (32)$$

$$\dot{\vec{x}} = \begin{bmatrix} 0 & 1 \\ -1 & -0.1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad (33)$$

$$J = \frac{1}{2}x^T(t_f)Hx(t_f) + \frac{1}{2} \int_0^{t_f} (\alpha(x^2 + \dot{x}^2) + \beta u^2) dt$$

It's LQR problem.

$$J = \frac{1}{2}x^T(t_f)Hx(t_f) + \frac{1}{2} \int_0^{t_f} \left( \dot{\vec{x}}^T(t)Q(t)\dot{\vec{x}}(t) + \dot{\vec{u}}^T(t)R(t)\dot{\vec{u}}(t) \right) dt \quad (34)$$

$$Q(t) = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}, \quad R(t) = \beta$$

#### 3.1 part a

$\alpha = \beta = 1$ ,  $t_f \rightarrow \infty$  and  $H = 0$

Riccati equation:

$$\dot{K} + KA + A^TK - KBR^{-1}B^TK + Q = 0 \quad (35)$$

$$u(t) = -R^{-1}B^TK(t)x(t); \quad (36)$$

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (37)$$

Because of  $t \rightarrow \infty$ ,  $\dot{K} = 0$  so Differential riccati equation change to Algebraic riccati equation.

$$KA + A^TK - KBR^{-1}B^TK + Q = 0 \quad (38)$$

$$\begin{aligned}
& \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -0.1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & -0.1 \end{bmatrix} \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} + \\
& \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\
& \begin{bmatrix} 1 - K_{21} - K_{12}K_{21} - K_{12} & K_{11} - K_{12}/10 - K_{22} - K_{12}K_{22} \\ K_{11} - K_{21}/10 - K_{22} - K_{21}K_{22} & K_{22}^2 - K_{22}/5 + K_{12} + K_{21} + 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

We know that  $K_{12} = K_{21}$  Equation 3.1 solved in MATLAB(Q3\_a.m). There is two answer in real numbers.

$$K_1 = \begin{bmatrix} -2.0175 & 0.4142 \\ 0.4142 & -1.4556 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 1.8175 & 0.4142 \\ 0.4142 & 1.2558 \end{bmatrix}$$

Now we use equation 36 and 37 in MATLAB ode45(Q3\_aODE.m) to simulate system and find out which K is appropriate for system.

Figure 1: System simulation with  $K_1$  matrix

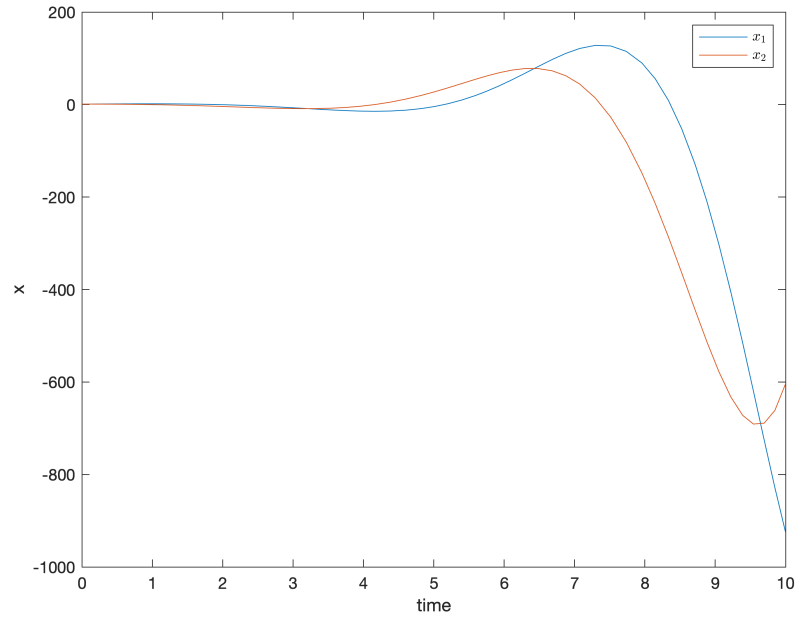
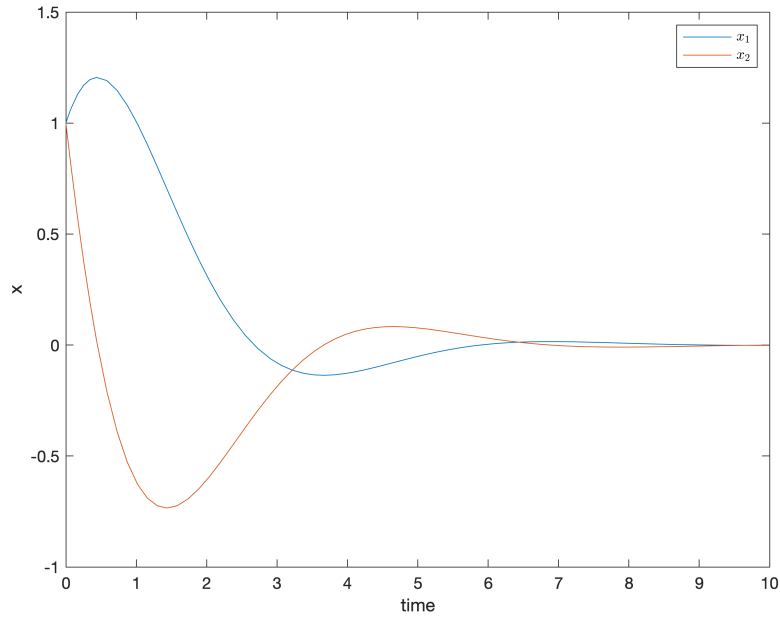




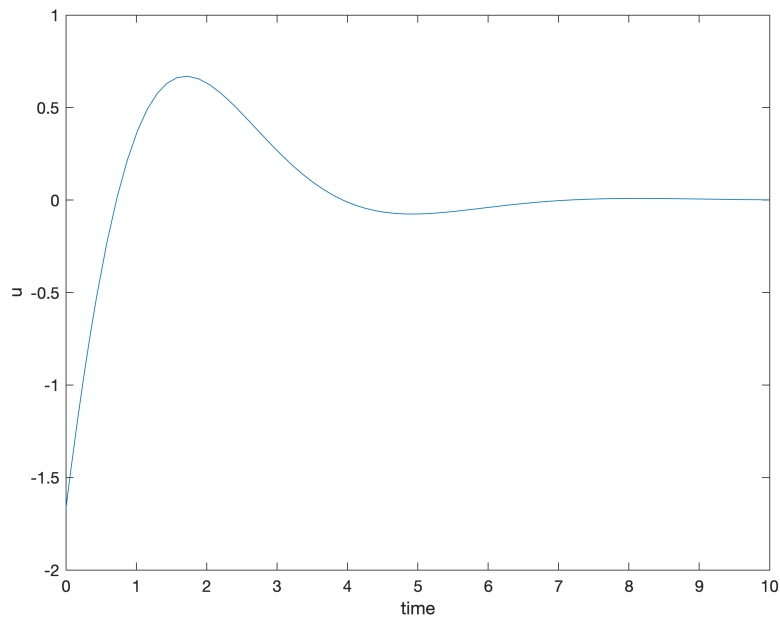
Figure 2: System simulation with  $K_2$  matrix

From ode simulation we can find out that  $K_2$  is appropriate for system. Figure of system state had plot in figure 2. Other way is to calculate eigenvalue of  $K$  matrix, if all eigenvalue are negative the system is stable so  $K$  matrix is true.

$$eig(K_1) = \begin{bmatrix} -1.236 \\ -2.237 \end{bmatrix}$$

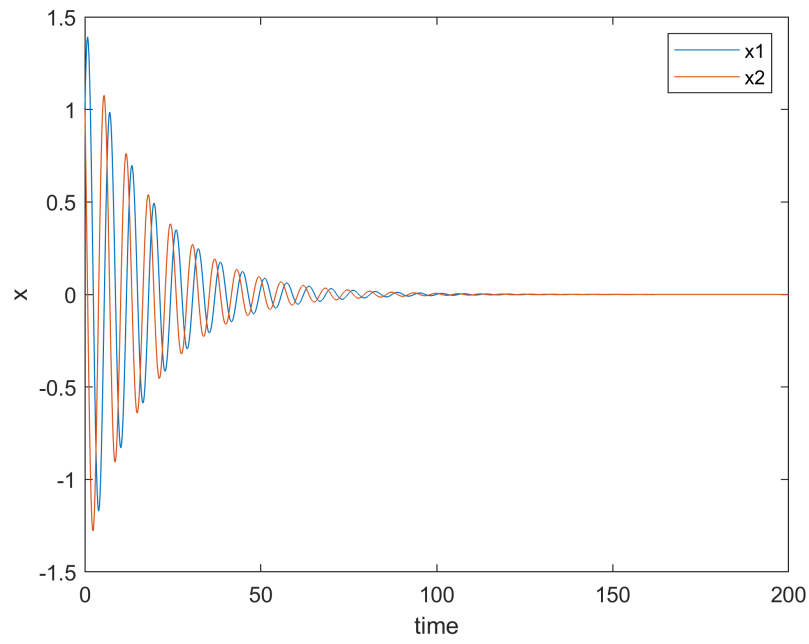
$$eig(K_2) = \begin{bmatrix} 2.0371 \\ 1.0362 \end{bmatrix}$$

Because all eigenvalue of  $K_1$  are negative so the answer is  $K_1$ .

Figure 3: System control effort  $u(t)$ 

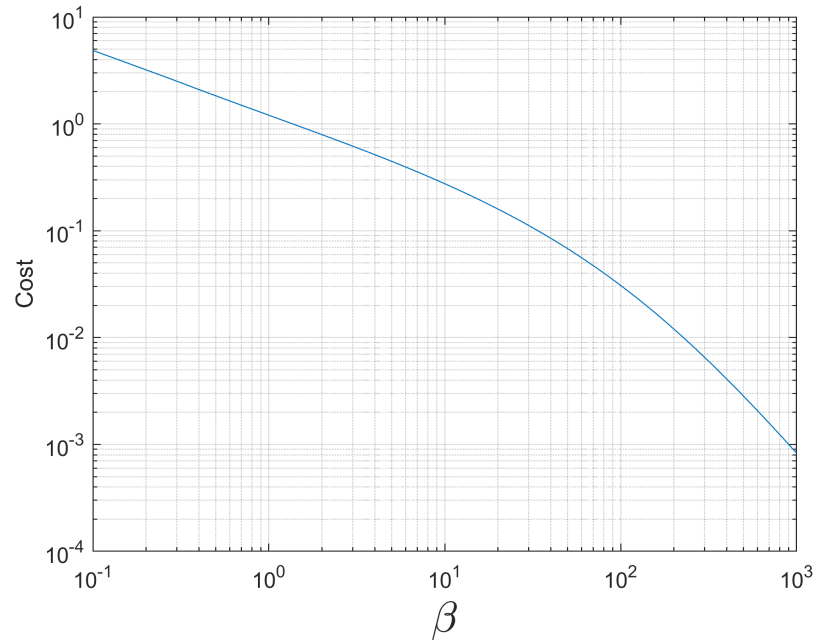
### 3.2 part b

Simulation and Code has attached to Homework(Q3\_b.m). For this  $\beta$  we can find out that for 100 second all  $J$  costs are zero and very near to zero so we neglect time after that.

Figure 4: System simulation for  $\beta = 1000$ 

**3.2.1 I** $u(t)$  Cost:

$$J_u = \int_0^\infty u(t)^2 d(t)$$

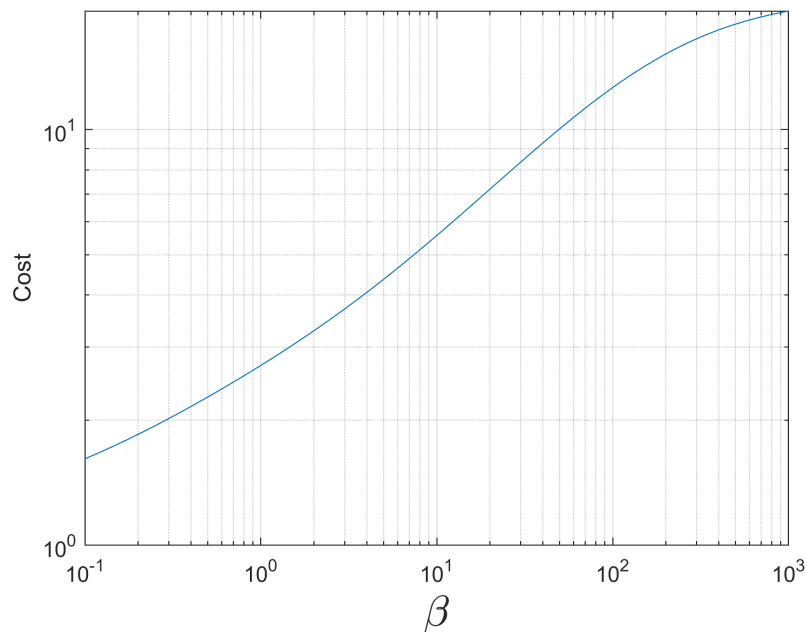
Figure 5:  $u(t)$  cost in different  $\beta$ 

### 3.2.2 II

$x_1^2 + \dot{x}_2^2$  Cost:

$$J_u = \int_0^\infty (x^2 + \dot{x}^2) d(t)$$

Figure 6:  $x^2 + \dot{x}^2$  cost in different  $\beta$



## 3.3 part c

### 3.3.1 I

$\alpha = \beta = 1$  and  $H = 0$ . Now we simulate our system for  $t_f = 3, 5, 10, 15$  sec and plot  $K(t)$  matrix,  $u(t)$  and states of system.

- $t_f = 3$  sec

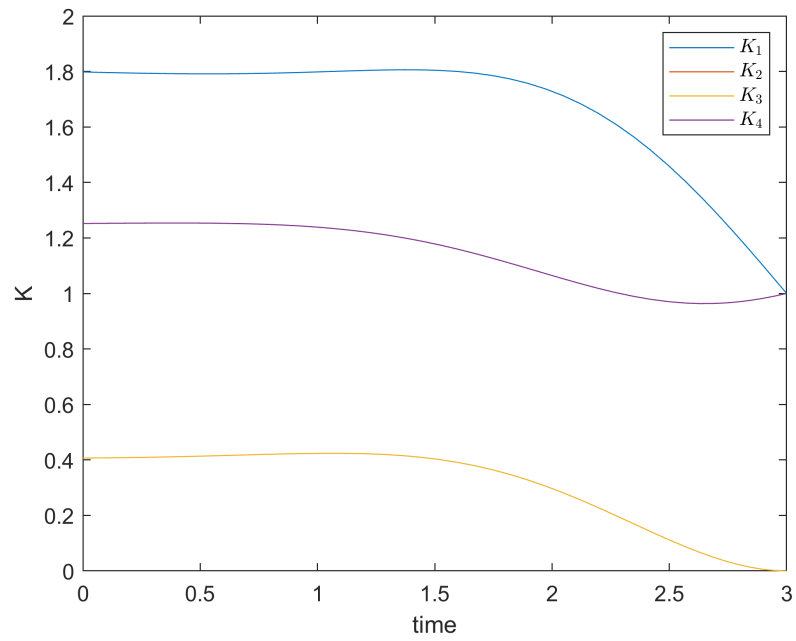
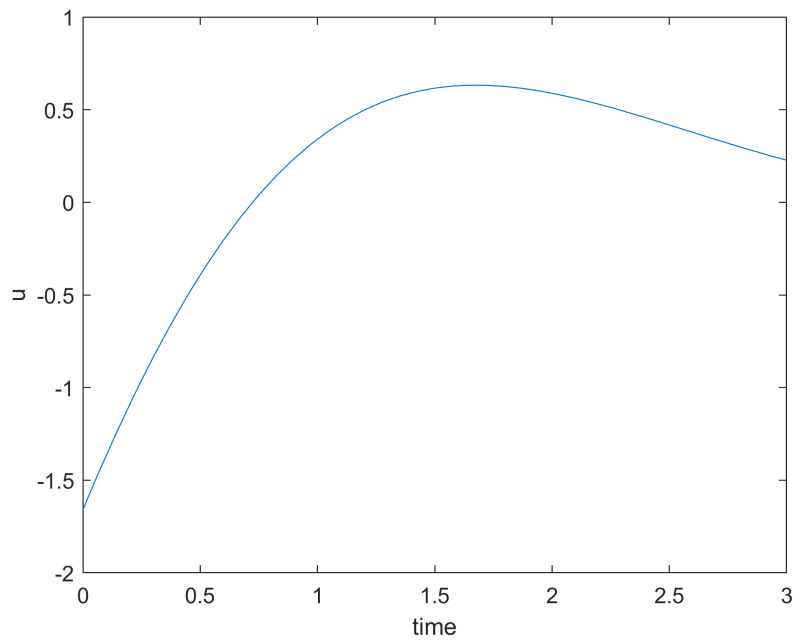
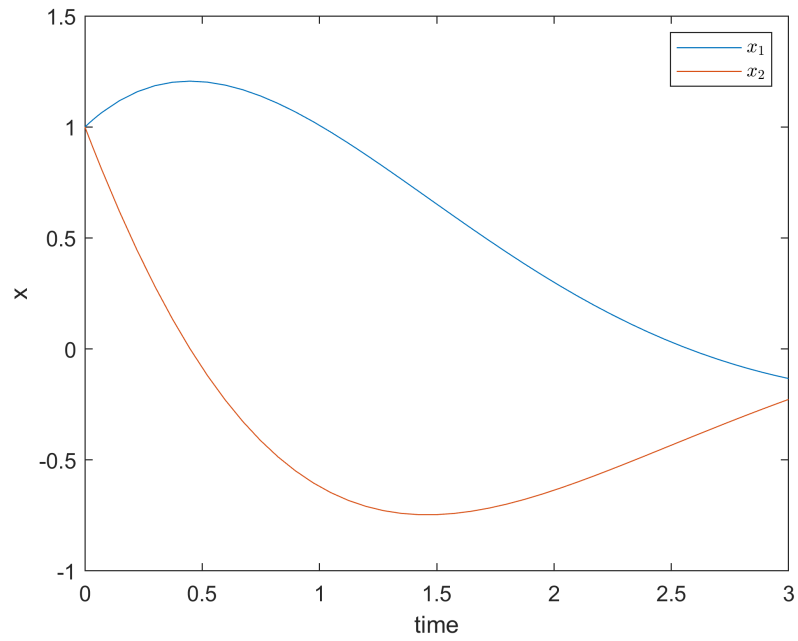
Figure 7:  $K(t)$  in  $t_f = 3$  secFigure 8:  $u(t)$  in  $t_f = 3$  sec

Figure 9: System States  $\vec{x}(t)$  in  $t_f = 3$  sec

- $t_f = 5$  sec

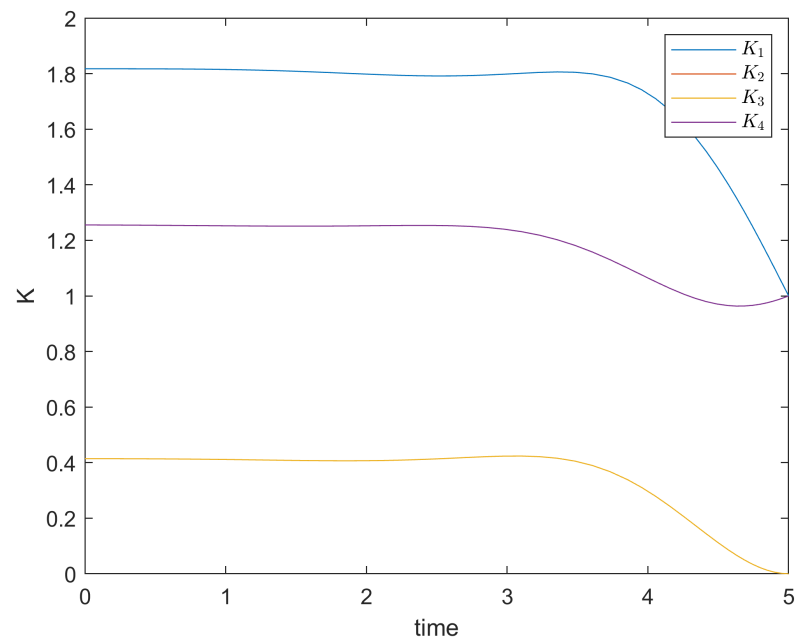
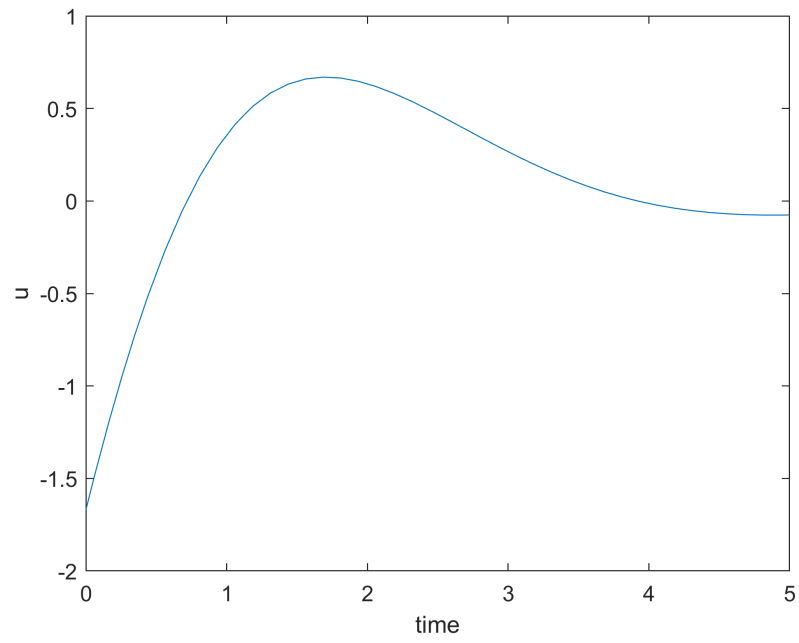
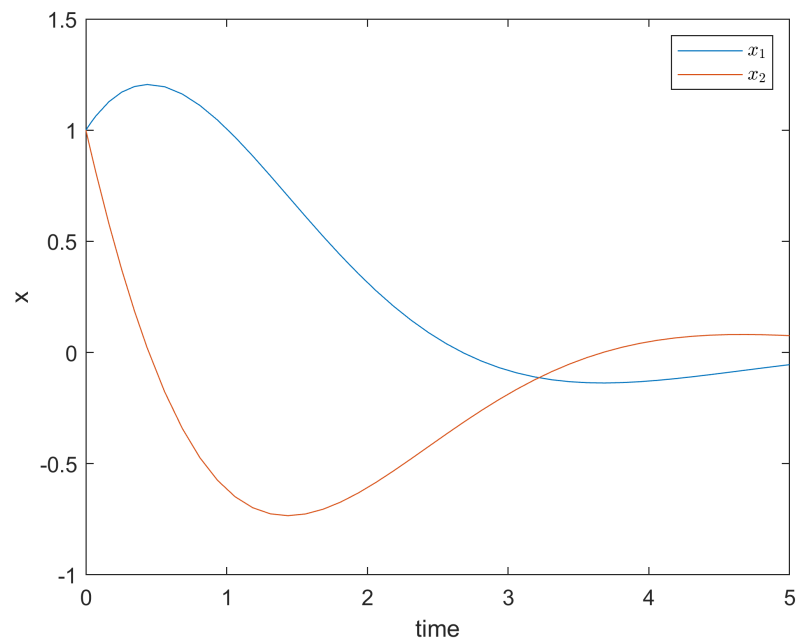
Figure 10:  $K(t)$  in  $t_f = 5$  sec

Figure 11:  $u(t)$  in  $t_f = 5$  secFigure 12: System States  $\vec{x}(t)$  in  $t_f = 5$  sec

- $t_f = 10$  sec



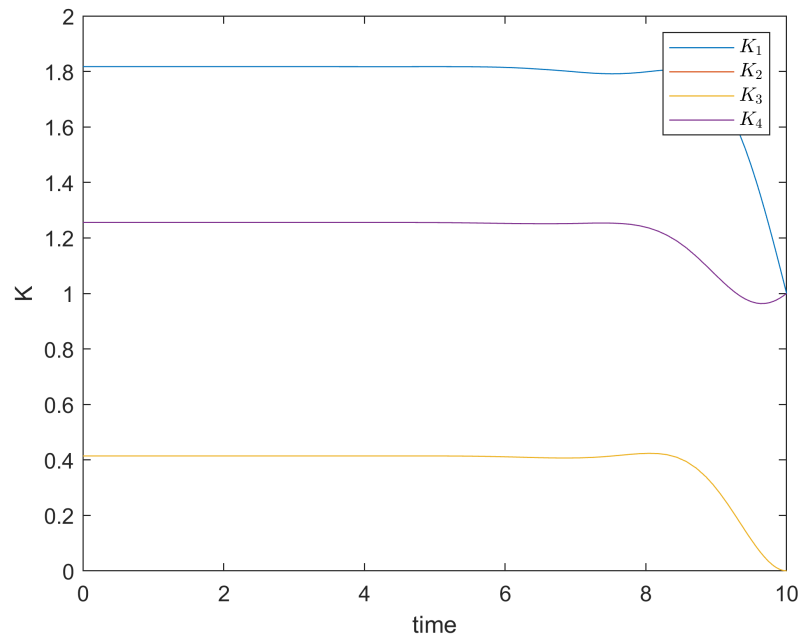
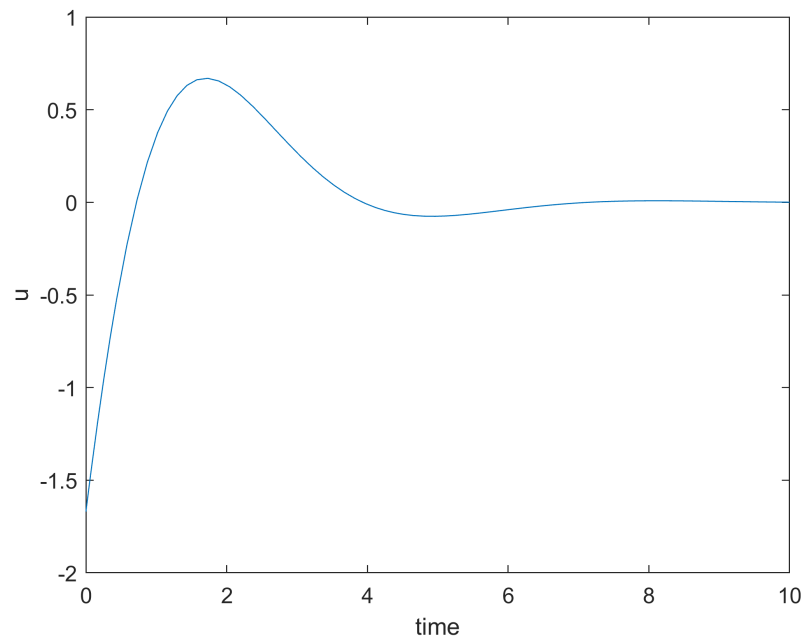
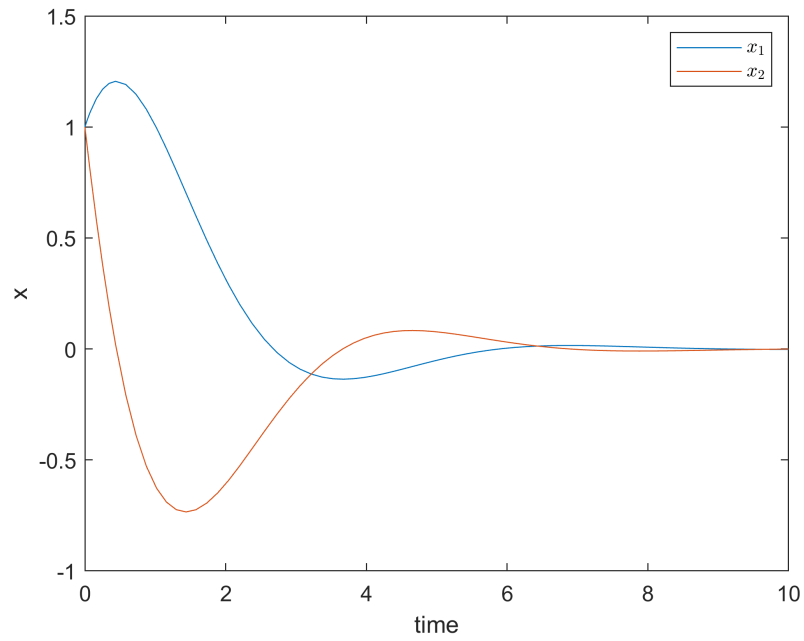
Figure 13:  $K(t)$  in  $t_f = 10$  secFigure 14:  $u(t)$  in  $t_f = 10$  sec

Figure 15: System States  $\vec{x}(t)$  in  $t_f = 10$  sec

- $t_f = 15$  sec

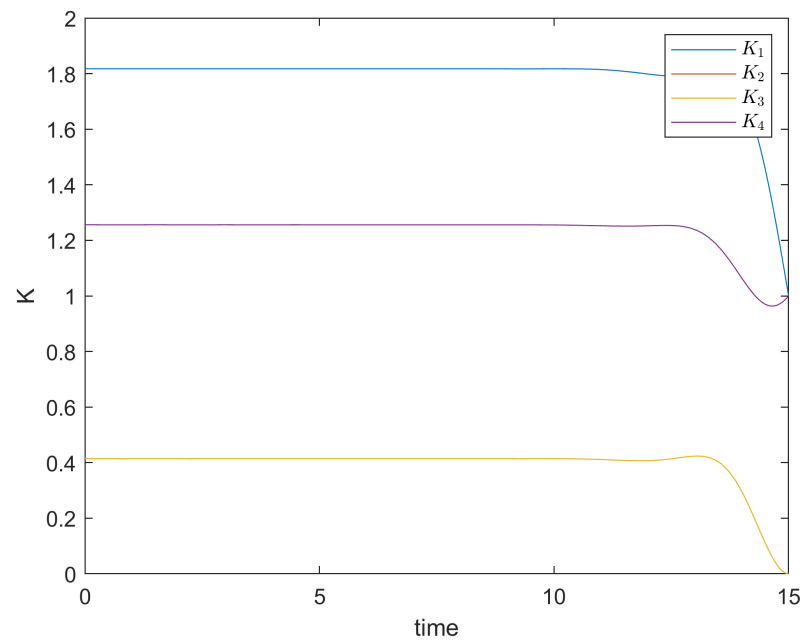
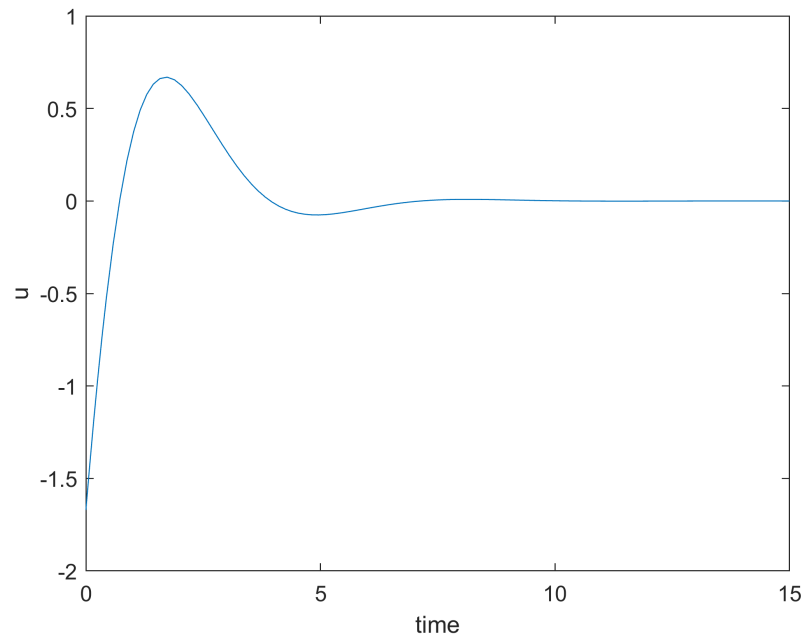
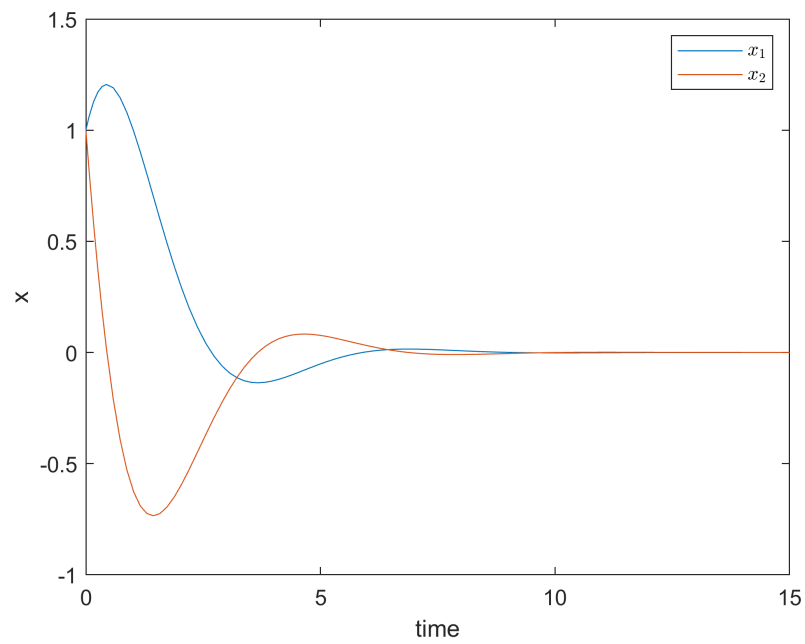
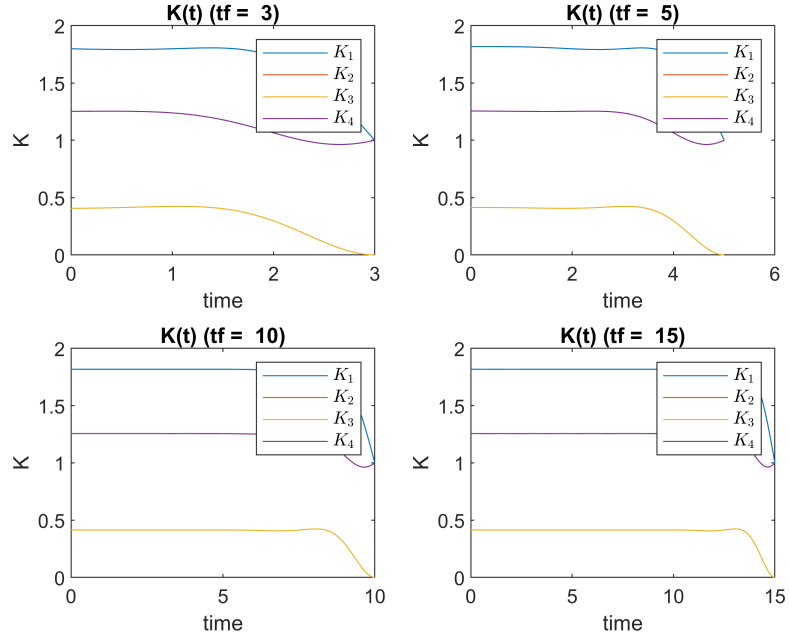
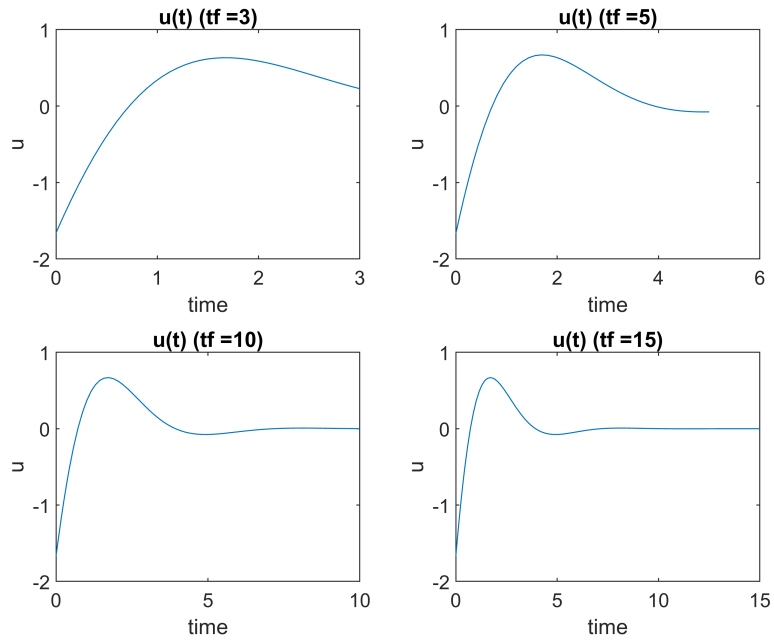
Figure 16:  $K(t)$  in  $t_f = 15$  sec

Figure 17:  $u(t)$  in  $t_f = 15$  secFigure 18: System States  $\vec{x}(t)$  in  $t_f = 15$  sec

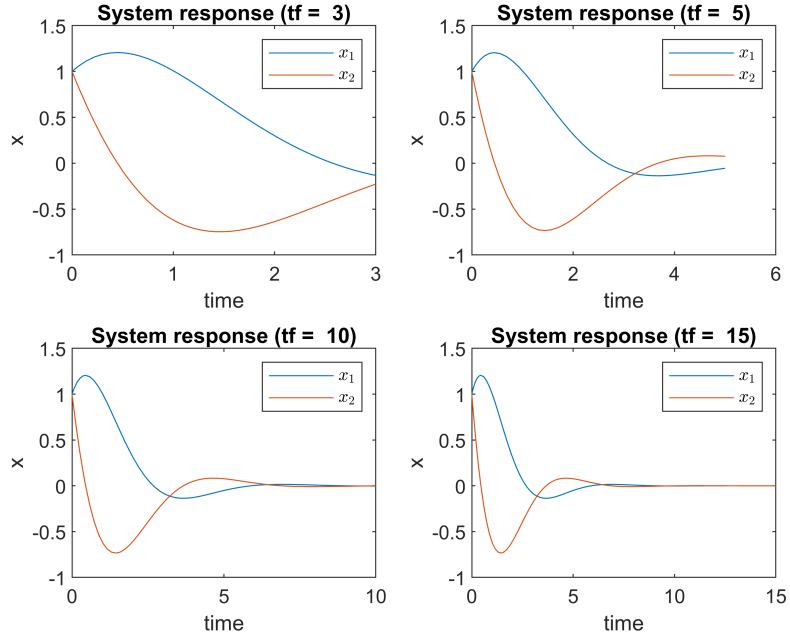
- $K(t)$  for all simulated  $t_f$

Figure 19:  $K(t)$  for all simulated  $t_f$ 

- $u(t)$  for all simulated  $t_f$

Figure 20:  $u(t)$  for all simulated  $t_f$ 

- System States  $\vec{x}(t)$  for all simulated  $t_f$

Figure 21: System States  $\vec{x}(t)$  for all simulated  $t_f$ 

For different times and constant  $\alpha, \beta, H$  the trajectory is very similar but in longer time system is more neat to target and of course use much more energy.

### 3.3.2 II

$\alpha = \beta = 1$  and  $t_f = 0$ . Now we simulate our system for  $H = 1I, 10I, 100I$  and plot  $K(t)$  matrix,  $u(t)$  and states of system.

- $H = 1I$

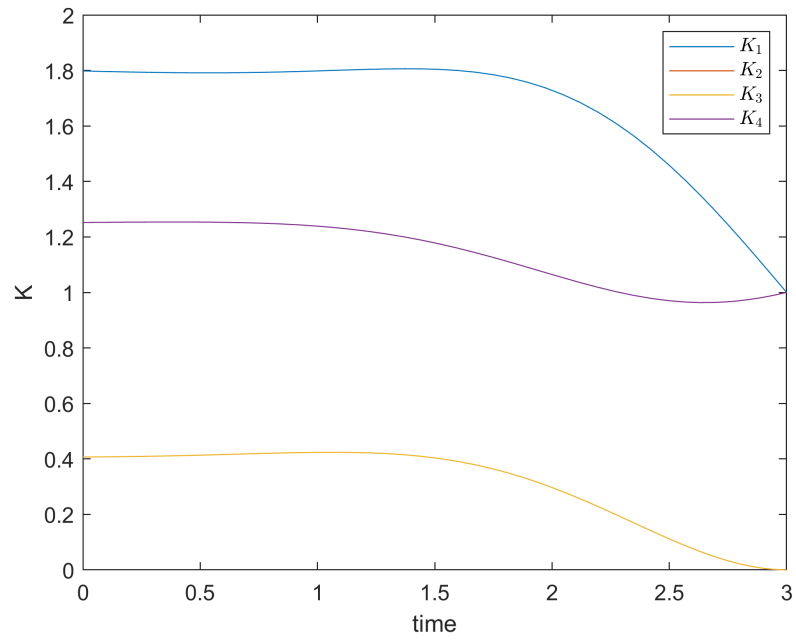
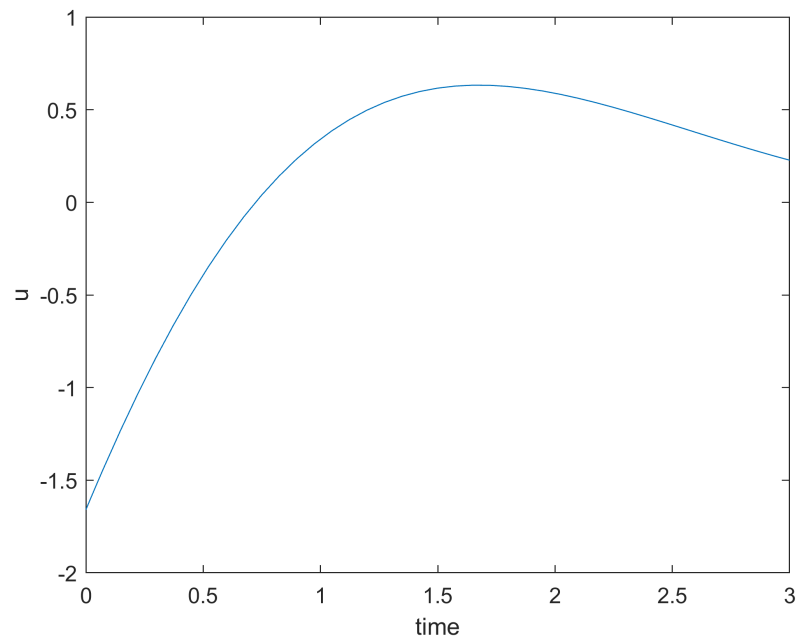
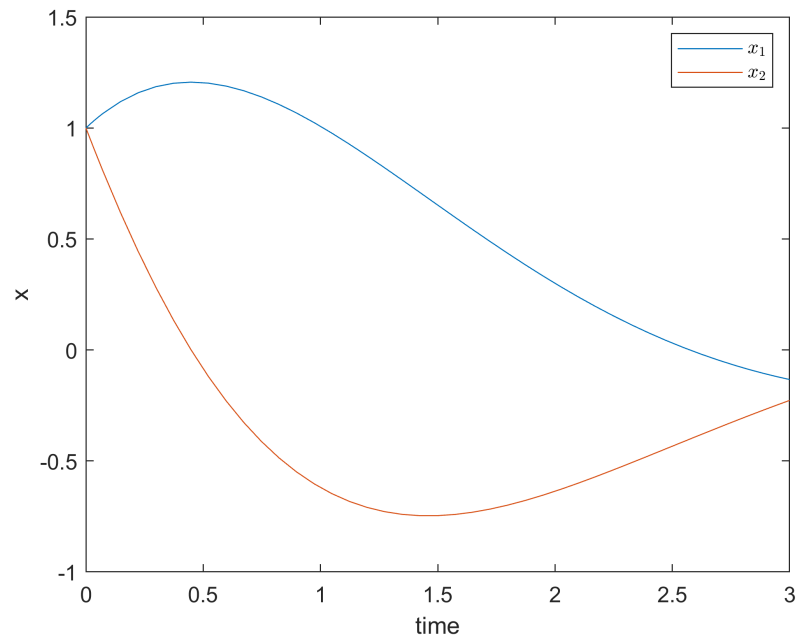
Figure 22:  $K(t)$  in  $H = 1I$ Figure 23:  $u(t)$  in  $H = 1I$ 

Figure 24: System States  $\vec{x}(t)$  in  $H = 1I$ 

- $H = 10I$

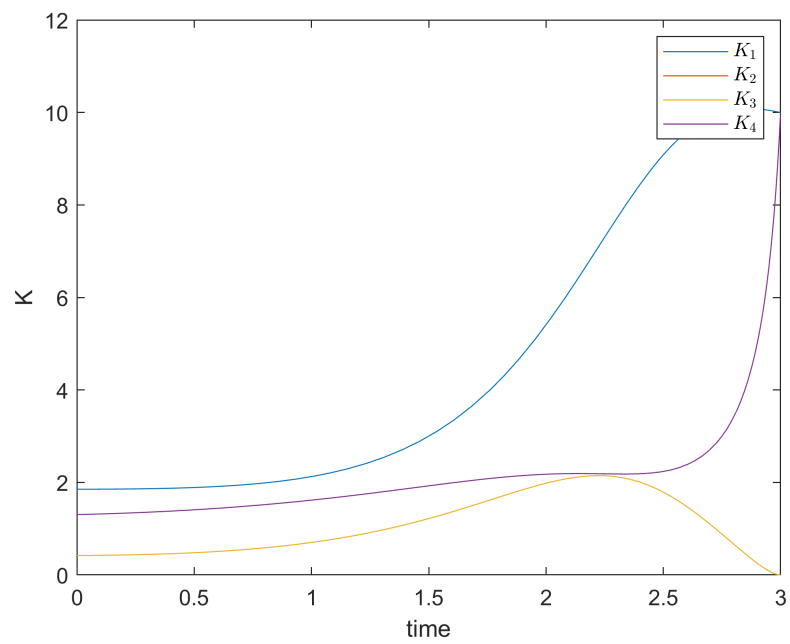
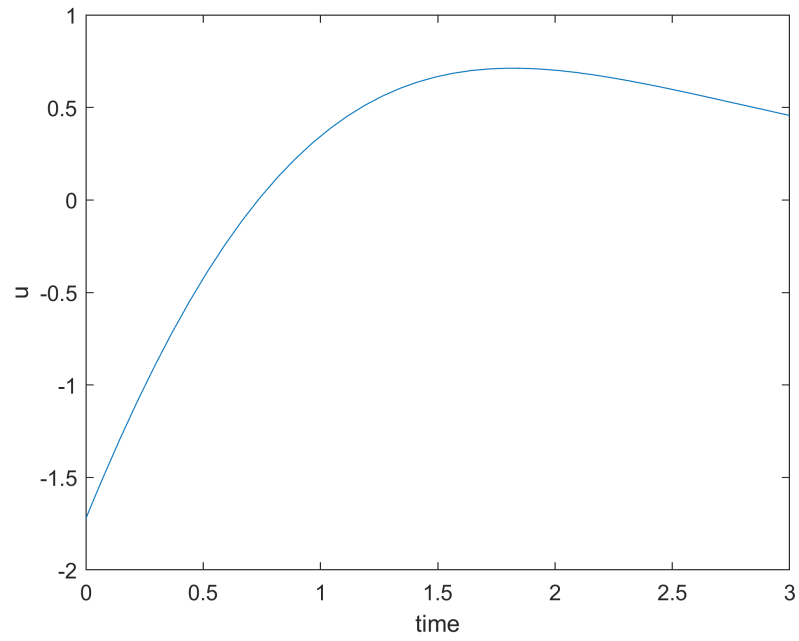
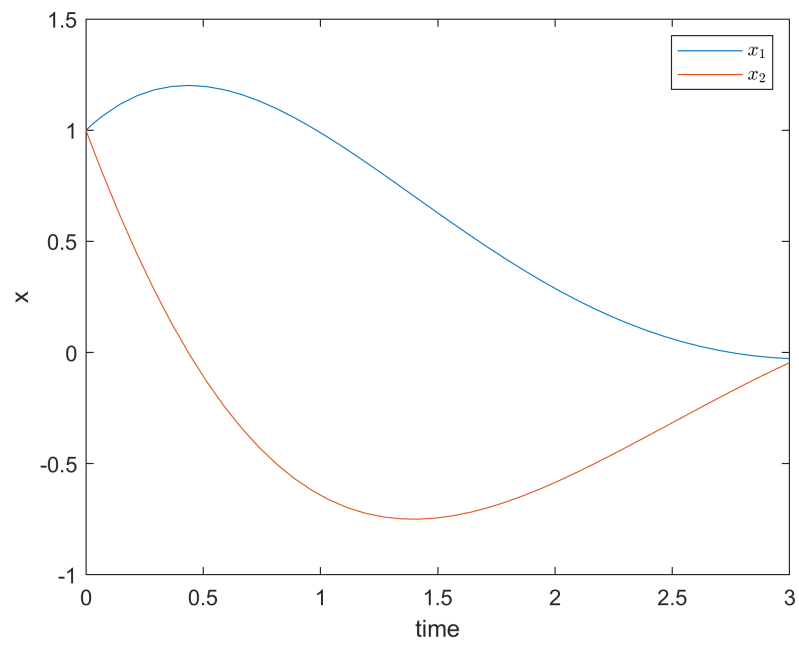
Figure 25:  $K(t)$  in  $H = 10I$ 

Figure 26:  $u(t)$  in  $H = 10I$ Figure 27: System States  $\vec{x}(t)$  in  $H = 10I$ 

- $H = 100I$



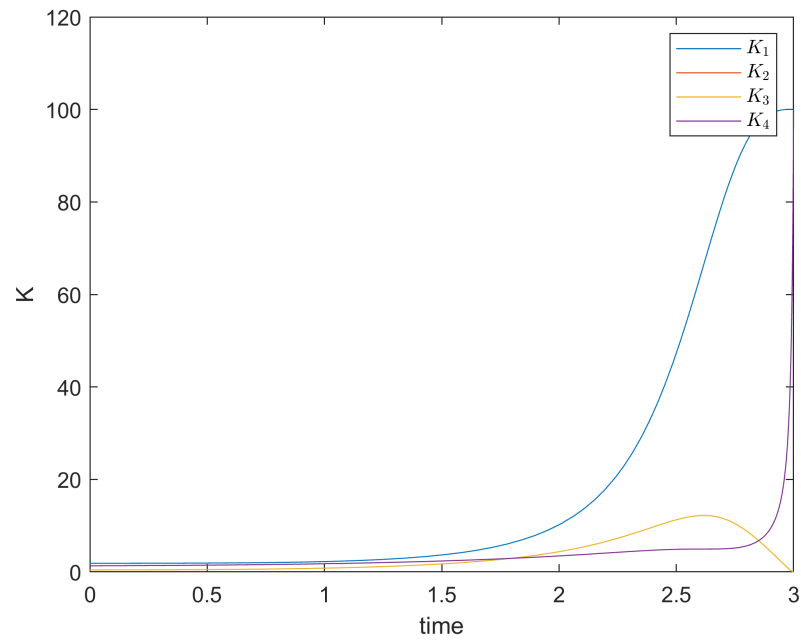
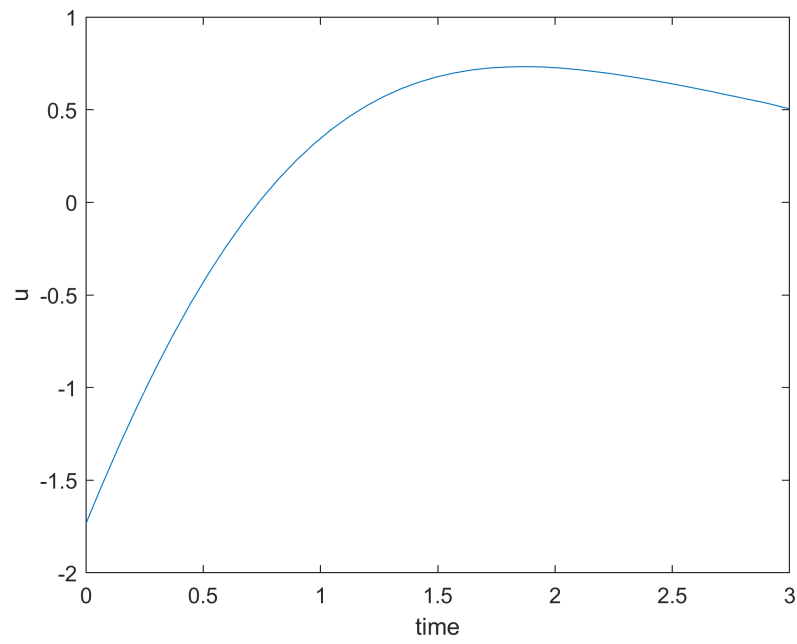
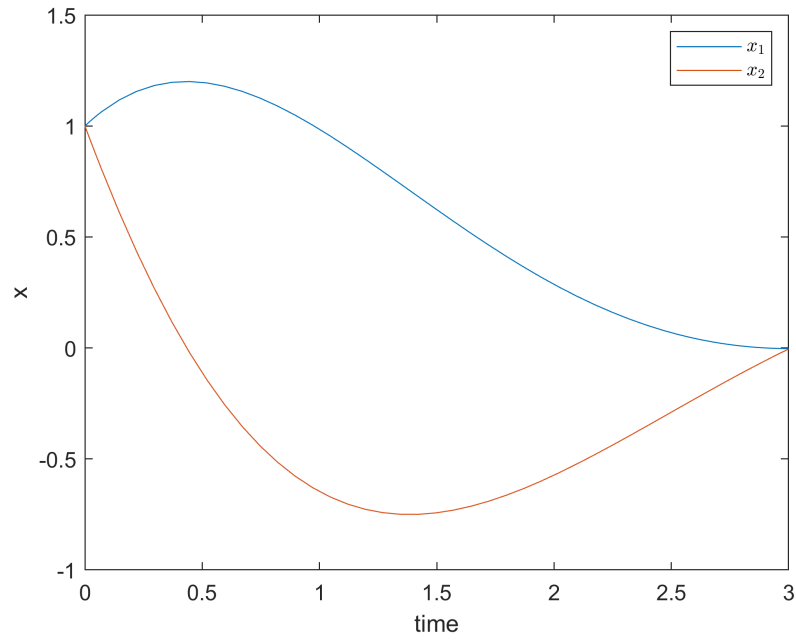
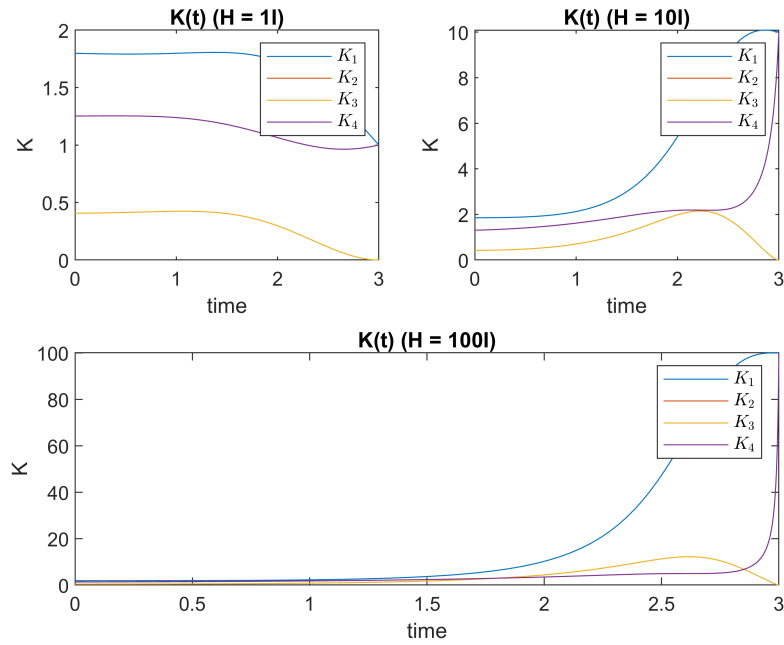
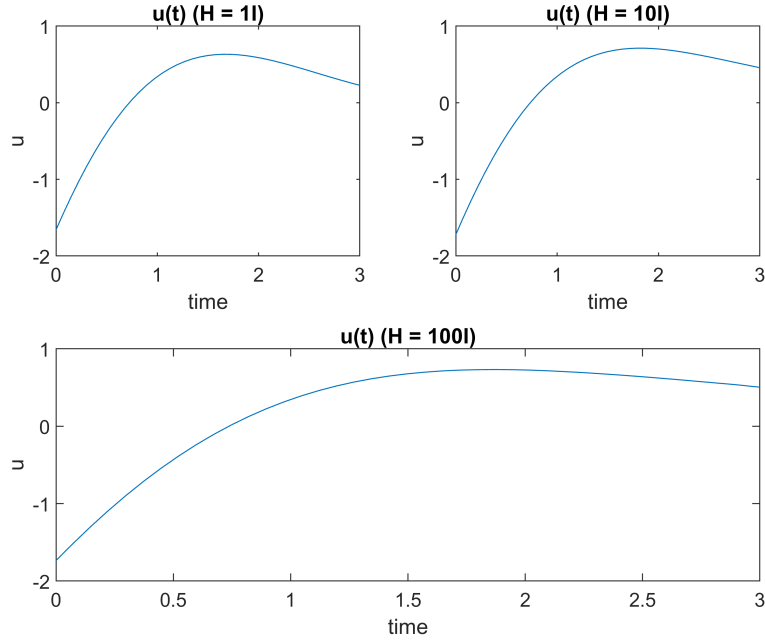
Figure 28:  $K(t)$  in  $H = 100I$ Figure 29:  $u(t)$  in  $H = 100I$ 

Figure 30: System States  $\vec{x}(t)$  in  $H = 100I$ 

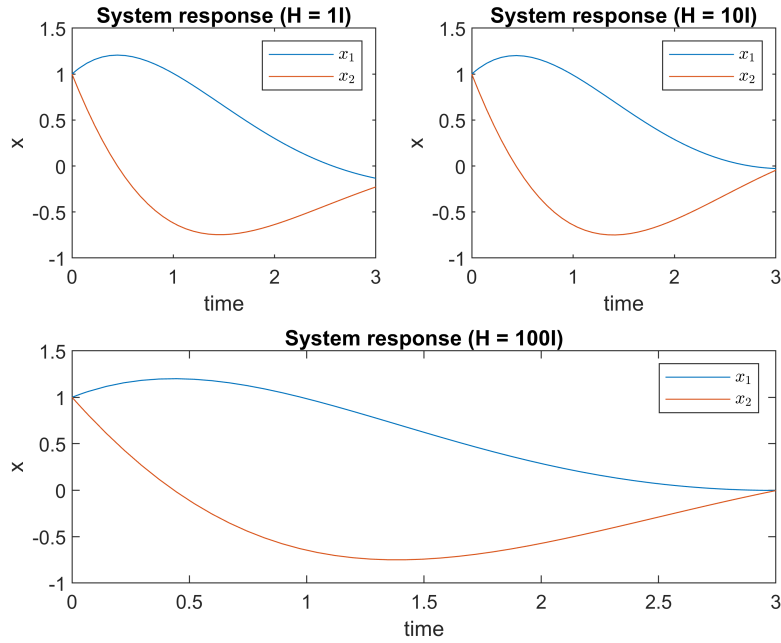
- $K(t)$  for all simulated  $H$

Figure 31:  $K(t)$  for all simulated  $H$ 

- $u(t)$  for all simulated  $H$

Figure 32:  $u(t)$  for all simulated  $H$ 

- System States  $\vec{x}(t)$  for all simulated  $H$

Figure 33: System States  $\vec{x}(t)$  for all simulated  $H$ 

We increase final position cost with  $H$  matrix so when  $H$  increase the cost of the final position increase so

controller use much more power to go to the target and because of that in  $H = 100I$  system is more near to target than  $H = 10I$  and  $H = 1I$  is near and like that for  $H = 10I$ .

### 3.4 part d

$\beta = 1$ ,  $H = 1$  and  $t_f = 10$  sec. Now we simulate our system for  $\alpha = 1, 5, 10$  and plot  $K(t)$  matrix,  $u(t)$  and states of system.

- $\alpha = 1$

Figure 34:  $K(t)$  in  $\alpha = 1$

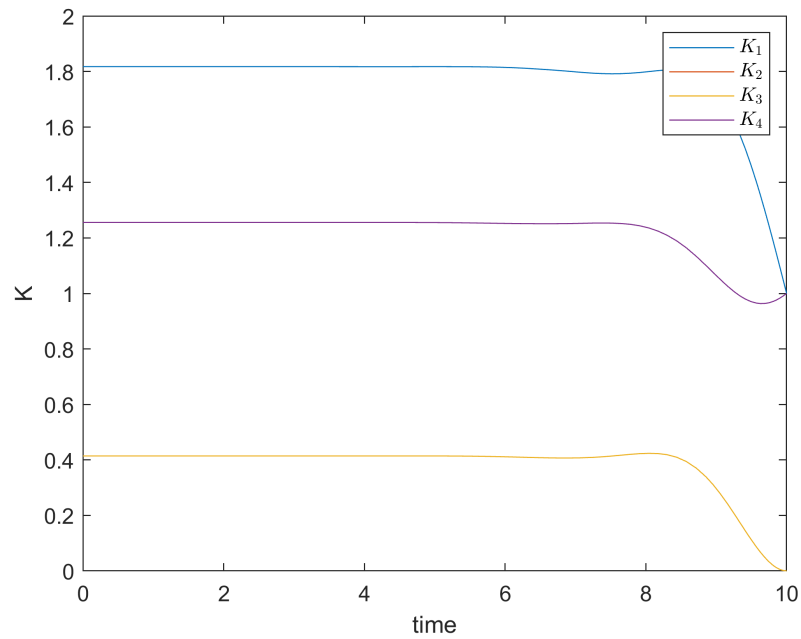
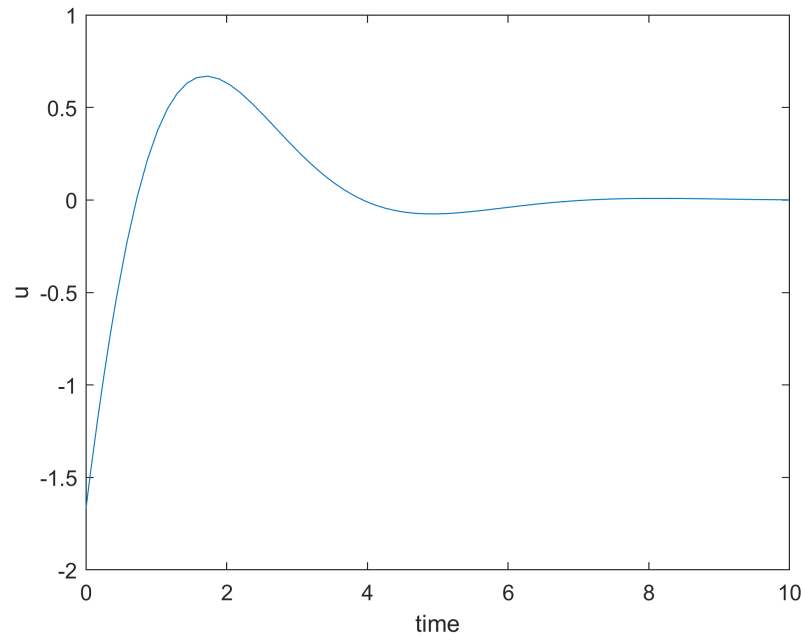
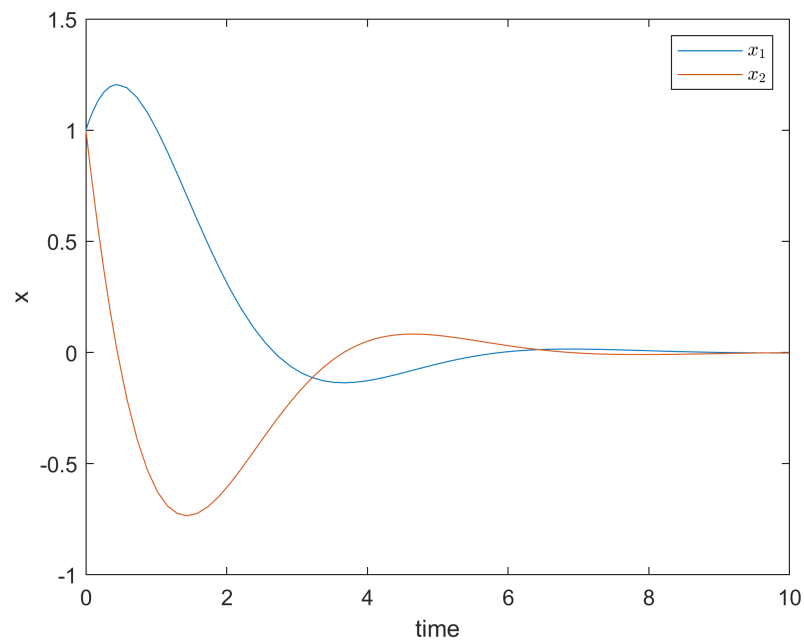


Figure 35:  $u(t)$  in  $\alpha = 1$ Figure 36: System States  $\vec{x}(t)$  in  $\alpha = 1$ 

- $\alpha = 5$

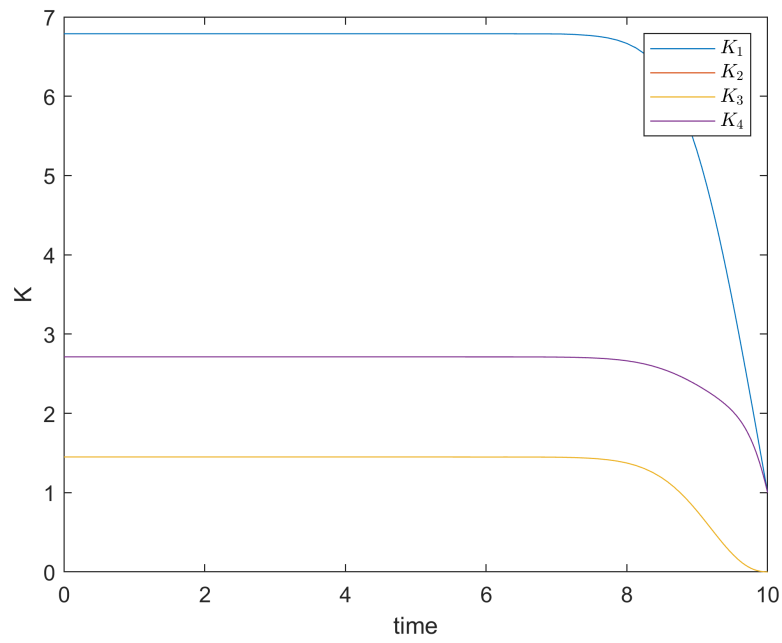
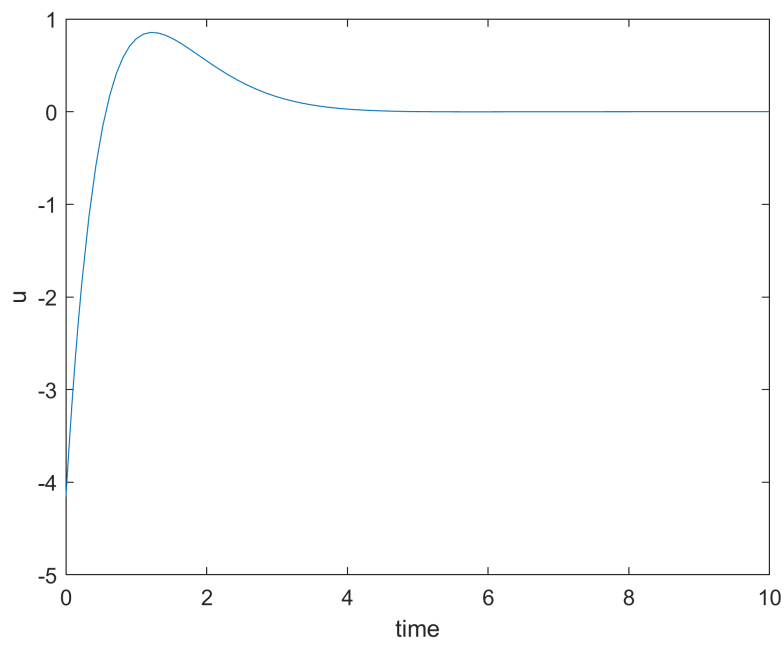
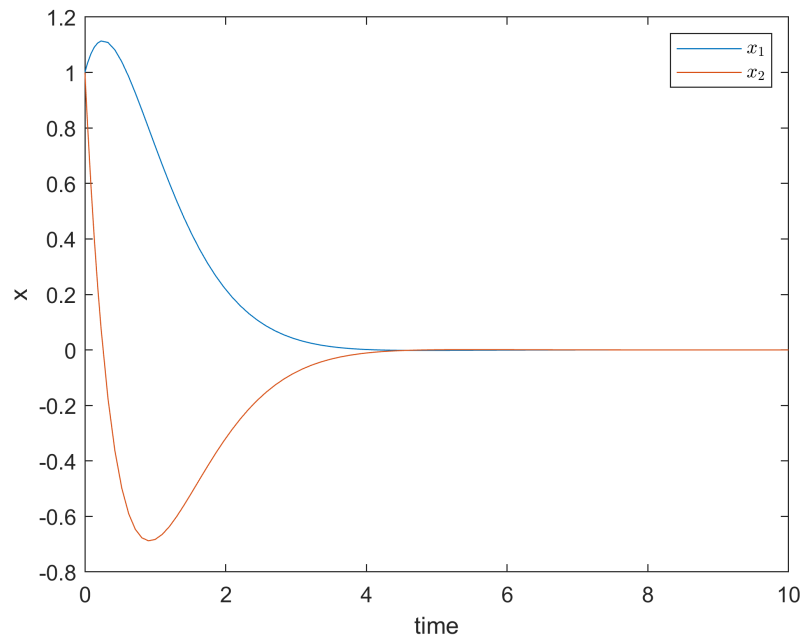
Figure 37:  $K(t)$  in  $\alpha = 5$ Figure 38:  $u(t)$  in  $\alpha = 5$ 

Figure 39: System States  $\vec{x}(t)$  in  $\alpha = 5$ 

- $\alpha = 10$

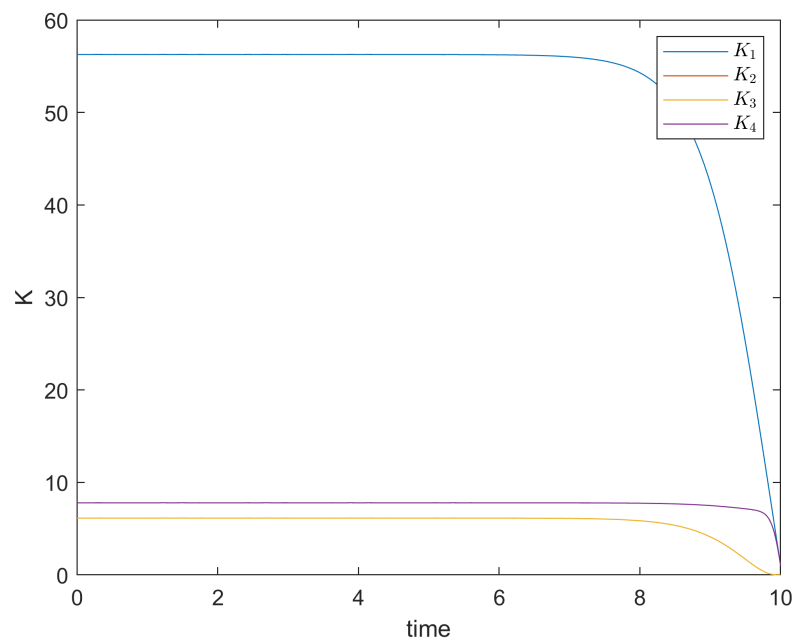
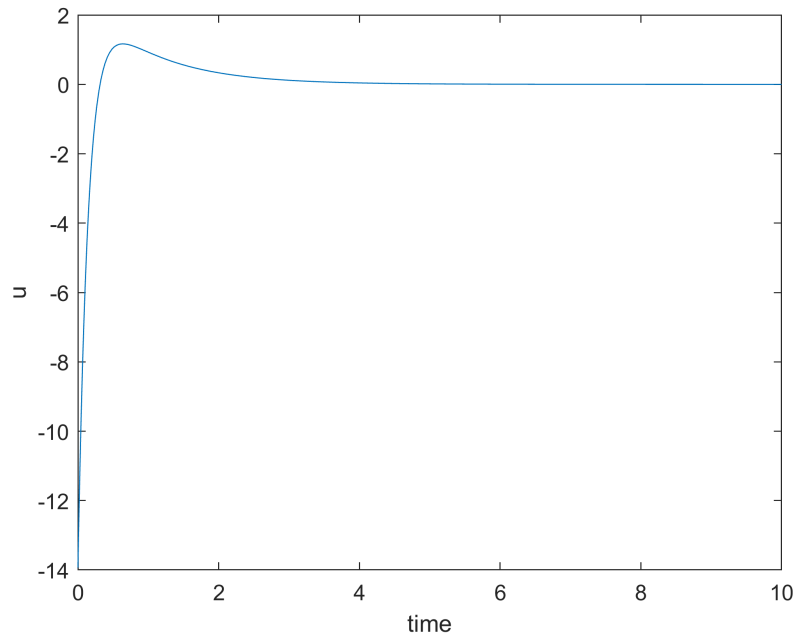
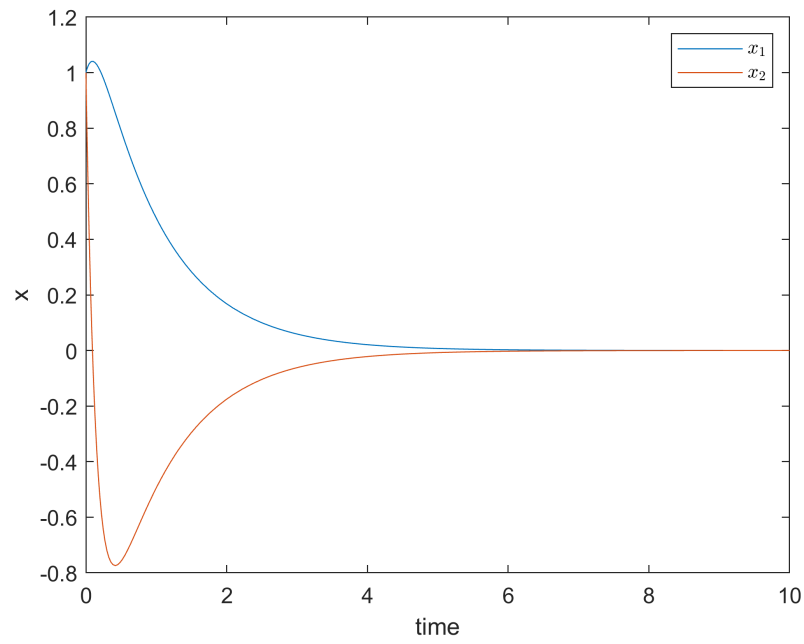
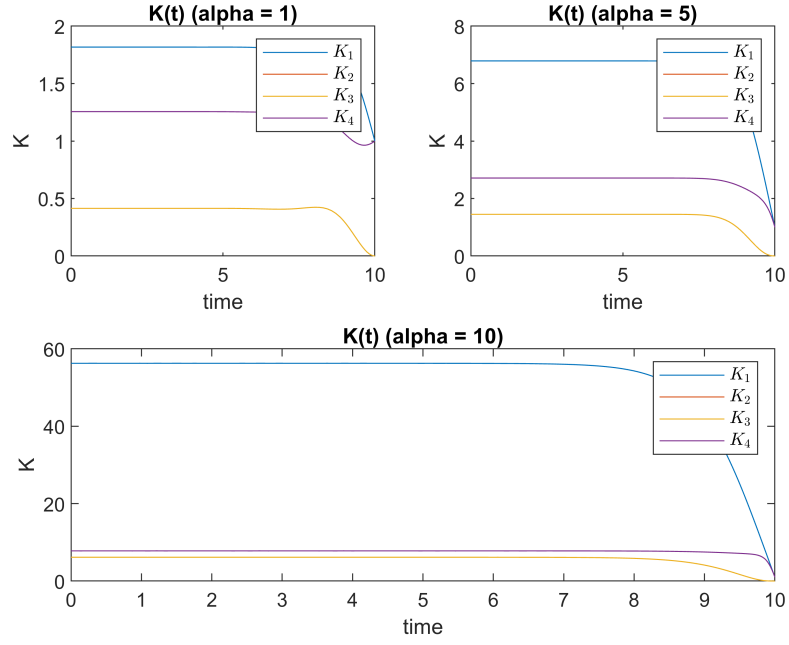
Figure 40:  $K(t)$  in  $\alpha = 10$ 

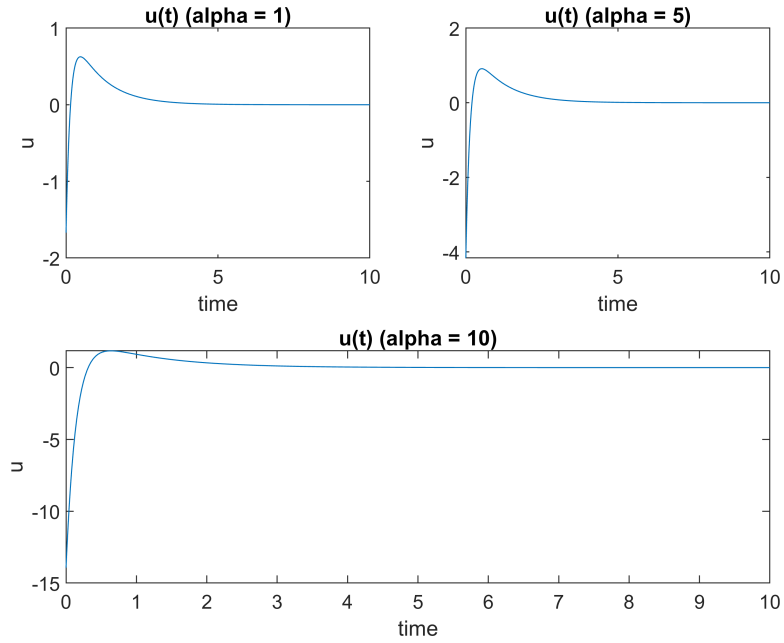
Figure 41:  $u(t)$  in  $\alpha = 10$ Figure 42: System States  $\vec{x}(t)$  in  $\alpha = 10$ 

- $K(t)$  for all simulated  $\alpha$

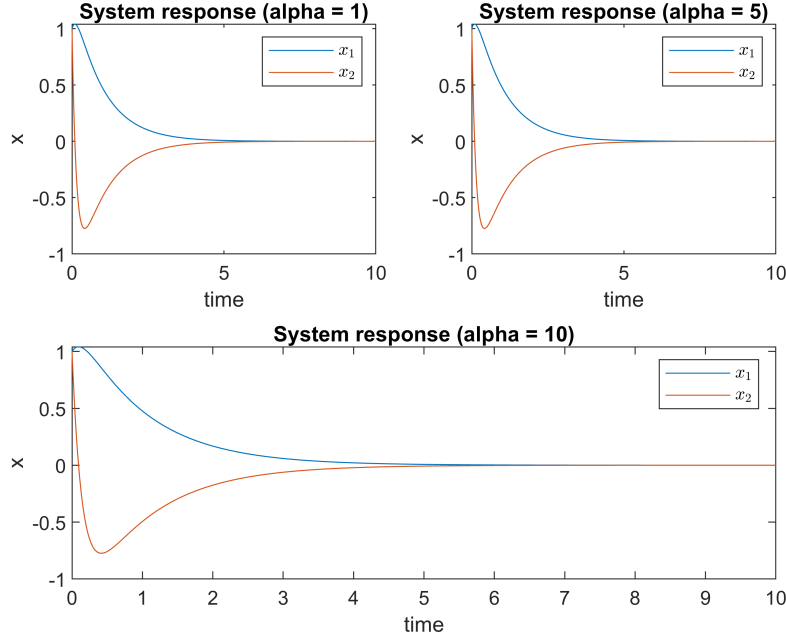


Figure 43:  $K(t)$  for all simulated  $\alpha$ 

- $u(t)$  for all simulated  $\alpha$

Figure 44:  $u(t)$  for all simulated  $\alpha$ 

- System States  $\vec{x}(t)$  for all simulated  $\alpha$

Figure 45: System States  $\vec{x}(t)$  for all simulated  $\alpha$ 

When we increase  $\alpha$  we increase position and velocity( $x_1, x_2$ ) cost then controller use more power to go to target faster but it also use more control effort.

## 4 Question 4

System:

$$a(\vec{x}(t), u(t), t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -x_1(t) + u(t) \\ -x_2(t) + u(t) \end{bmatrix} \quad (39)$$

$$\ddot{\vec{x}}(t) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \dot{\vec{x}}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \quad (40)$$

For minimum time:

$$J = t_f - t_0 = \int_{t_0}^{t_f} dt$$

Hamiltonian matrix:

$$\mathcal{H} = g(\vec{x}(t), u(t), t) + \vec{p}(t)^T a(\vec{x}(t), u(t), t)$$

$$\mathcal{H} = 1 + [p_1(t) \quad p_2(t)] \begin{bmatrix} -x_1(t) + u(t) \\ -x_2(t) + u(t) \end{bmatrix}$$

$$\mathcal{H} = 1 - p_1(t)x_1(t) + p_1(t)u(t) - p_2(t)x_2(t) + p_2(t)u(t) \quad (41)$$

Euler-Lagrange equation:

$$\dot{\vec{x}} = \frac{\partial \mathcal{H}}{\partial \vec{p}} = a(\vec{x}(t), u(t), t) \quad (42)$$

$$\dot{\vec{p}} = -\frac{\partial \mathcal{H}}{\partial \vec{x}} \quad (43)$$

$$\vec{0} = \frac{\partial \mathcal{H}}{\partial \vec{u}} \quad (44)$$

Now we use equation 41 to solve Euler-Lagrange equation.

$$\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} -\frac{\partial \mathcal{H}}{\partial x_1} \\ -\frac{\partial \mathcal{H}}{\partial x_2} \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \quad (45)$$

Answer of equation 45 is:

$$\begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} = \begin{bmatrix} C_1 \exp(-t) \\ C_2 \exp(-t) \end{bmatrix} \quad (46)$$

From Euler-Lagrange equation:

$$\vec{0} = \frac{\partial \mathcal{H}}{\partial \vec{u}} \rightarrow p_4 \cos(u) = p_1 + p_2 = (C_1 + C_2) \exp(-t) \quad (47)$$

From equation 47 we can find out sign of  $\frac{\partial \mathcal{H}}{\partial \vec{u}}$  is the same for all time so there is no switch. So for all time  $u(t)$  is constant and it may be 1 or  $-1$  for all time. Now we simulate system with this  $u(t)$ .

$$a(\vec{x}(t), u(t), t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -x_1(t) \pm 1 \\ -x_2(t) \pm 1 \end{bmatrix} \quad (48)$$

Differential equation answers:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} C_3 \exp(-t) \pm 1 \\ C_4 \exp(-t) \pm 1 \end{bmatrix} \quad (49)$$

- $u(t) = 1$

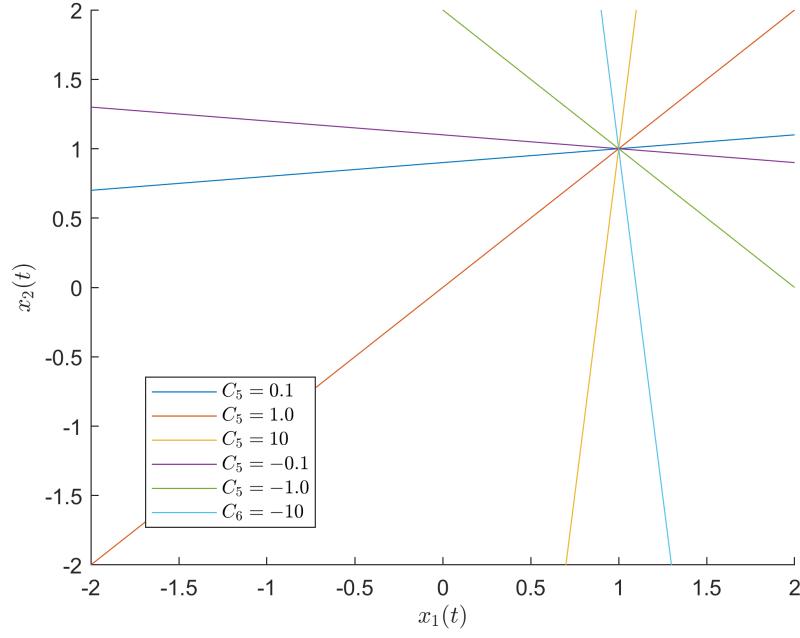
$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} C_3 \exp(-t) + 1 \\ C_4 \exp(-t) + 1 \end{bmatrix} \quad (50)$$

From equation 50:

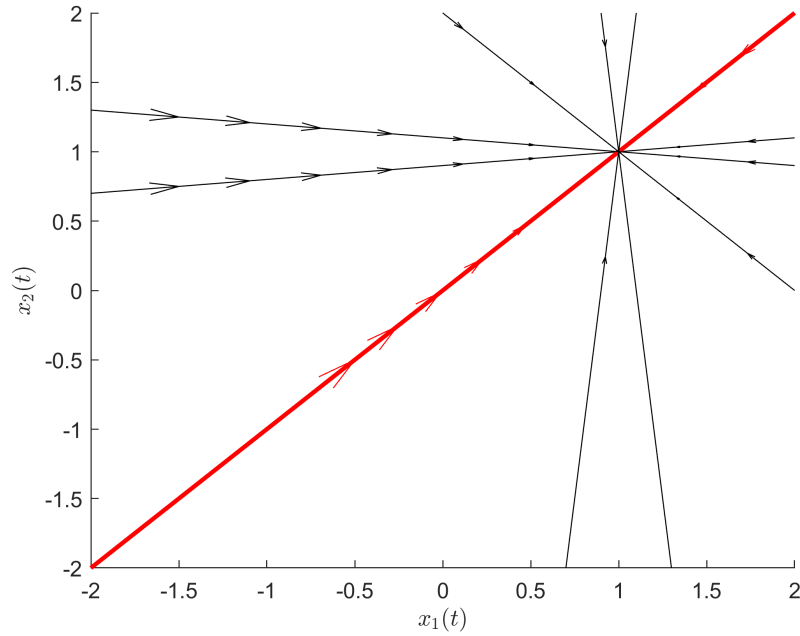
$$x_1(t) = C_3 \exp(-t) + 1 \rightarrow \exp(-t) = \frac{x_1(t) - 1}{C_3} \xrightarrow{x_2(t) = C_4 \exp(-t) + 1} x_2(t) = \frac{C_4}{C_3} x_1(t) - \frac{C_4}{C_3} + 1$$

Assume  $\frac{C_4}{C_3} = C_5$ :

$$x_2(t) = C_5 x_1(t) - C_5 + 1$$

Figure 46:  $u(t) = 1$ ,  $x_1$  and  $x_2$  for different  $C_5$ 

From figure 46 we can know that every point goes to  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$  and this is direction of function in time.  
 From figure 46 we can know that switch curve is  $x_1 = x_2$ .

Figure 47: switch curve in  $u(t) = 1$ 

In figure 47 you can see switch curve(red line).

- $u(t) = -1$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} C_3 \exp(-t) - 1 \\ C_4 \exp(-t) - 1 \end{bmatrix} \quad (51)$$

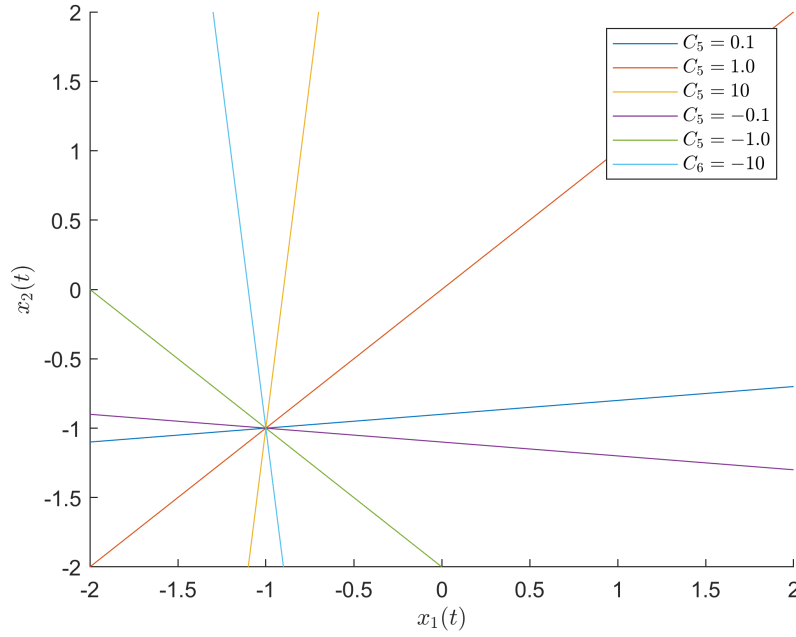
From equation 51:

$$x_1(t) = C_3 \exp(-t) - 1 \rightarrow \exp(-t) = \frac{x_1(t) + 1}{C_3} \xrightarrow{x_2(t) = C_4 \exp(-t) - 1} x_2(t) = \frac{C_4}{C_3} x_1(t) + \frac{C_4}{C_3} - 1$$

Assume  $\frac{C_4}{C_3} = C_5$ :

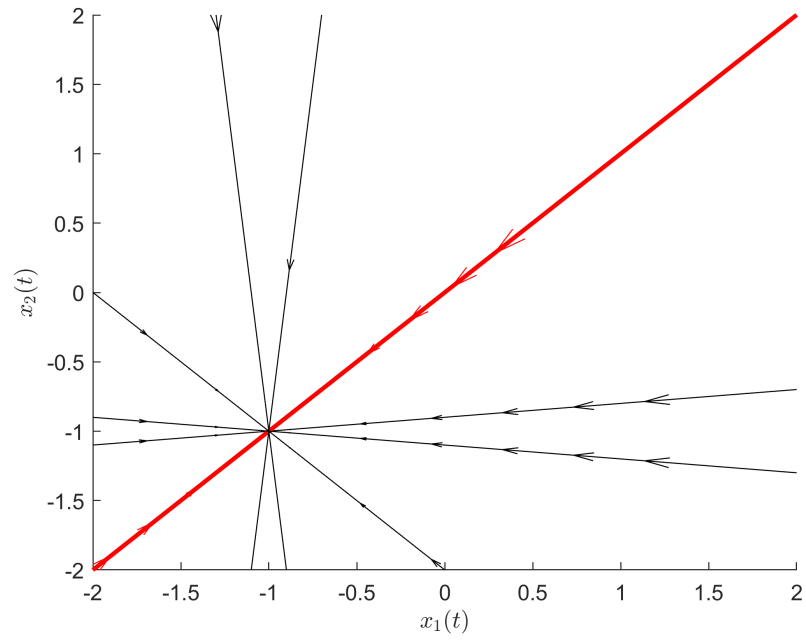
$$x_2(t) = C_5 x_1(t) + C_5 - 1$$

Figure 48:  $u(t) = -1$ ,  $x_1$  and  $x_2$  for different  $C_5$



From figure 48 we can know that every point goes to  $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$  and this is direction of function in time.  
 From figure 48 we can know that switch curve is  $x_1 = x_2$ .

Figure 49: switch curve in  $u(t) = -1$



In figure 49 you can see switch curve(red line).

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