

# Home Work #2

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## 1 Question 1

System:

$$\dot{x}(t) = -0.1x(t) + u(t)$$

Subjected to  $0 \leq u(t) \leq M$

### 1.1 part a

$$J = \int_0^{100} -x(t) dt$$

Hamiltonian matrix:

$$\begin{aligned}\mathcal{H} &= g(\vec{x}(t), u(t), t) + \vec{p}(t)^T a(\vec{x}(t), u(t), t) \\ \mathcal{H} &= -x(t) - 0.1p(t)x(t) + p(t)u(t)\end{aligned}\tag{1}$$

Euler-Lagrange equation:

$$\dot{\vec{x}} = \frac{\partial \mathcal{H}}{\partial \vec{p}} = a(\vec{x}(t), u(t), t)\tag{2}$$

$$\dot{\vec{p}} = -\frac{\partial \mathcal{H}}{\partial \vec{x}}\tag{3}$$

Now we use above equation for solve problem.

$$-\frac{\partial \mathcal{H}}{\partial x} = 1 + 0.1p$$

There is two differential equation and two unknowns.

$$\dot{x} = -x(t) - 0.1px + pu\tag{4}$$

$$\dot{p} = 1 + 0.1p\tag{5}$$

Equation 5 solved in MATLAB(Q1\_a.m) and code attached to file.

$$p(t) = C_1 \exp(t/10) - 10\tag{6}$$

Final  $x(t)$  is free so:

$$h_{\vec{x}} - \vec{p} = \vec{0} \Big|_{*, t_f} \rightarrow p(t_f) = 0$$

Use new boundry condition( $p(t_f) = 0$ ) in equation 6 to find  $p$  function( $p(t)$ ).

$$p(100) = C_1 \exp(100/10) - 10 = 0 \rightarrow C_1 = 10 \exp(-10)$$

$$p(t) = 10 \exp(0.1(t - 100)) - 10 \quad (7)$$

We know that  $u(t)$  has limit so for optimization we have another condition to select  $u(t)$  for every time.

$$u(t) = \begin{cases} \frac{\partial \mathcal{H}}{\partial u} < 0 & u(t) = M \\ \frac{\partial \mathcal{H}}{\partial u} = 0 & \mathcal{H} \text{ is not a function of } u(t) \\ \frac{\partial \mathcal{H}}{\partial u} > 0 & u(t) = 0 \end{cases} \quad (8)$$

From equation 1 we calculate  $\frac{\partial \mathcal{H}}{\partial u}$ .

$$\frac{\partial \mathcal{H}}{\partial u} = p(t)$$

From equation 7 we know that at  $t_0 \rightarrow t_f$   $p(t)$  is less than zero ( $p(t) < 0$ ), so  $u(t)$  for every time is  $M$ .

## 1.2 part b

$$J = \int_0^{100} -x(t) dt$$

Subjected to:

$$\int_0^{100} u(t) dt = K (\text{a known constant})$$

$z(t)$  is new state:

$$z(t) = \int_0^t u(t) dt \rightarrow \frac{dz}{dt} = u(t)$$

New differential constraints:

$$\begin{aligned} \frac{dz}{dt} - u(t) &= 0 \\ g_a(x, \dot{x}, \dot{z}, t, \lambda) &= g(x, \dot{x}, t) + \lambda(t)f(x, \dot{x}, \dot{z}, t) \\ g_a &= -x(t) + \lambda(\dot{z}(t) - u(t)) \end{aligned}$$

Hamiltonian matrix:

$$\mathcal{H} = g_a(\vec{x}(t), u(t), z(t), \lambda, t) + \vec{p}(t)^T a(\vec{x}(t), u(t), t)$$

We assume  $p_2 = \lambda$

$$\mathcal{H} = -x(t) - 0.1p_1(t)x(t) + p_1(t)u(t) + p_2(t)(\dot{z}(t) - u(t)) \quad (9)$$

$$\begin{aligned} \dot{\vec{p}} &= -\frac{\partial \mathcal{H}}{\partial \vec{x}} = \begin{bmatrix} -\frac{\partial \mathcal{H}}{\partial x} \\ -\frac{\partial \mathcal{H}}{\partial z} \end{bmatrix} = \begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} \\ \begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} &= \begin{bmatrix} 0.1p_1 + 1 \\ 0 \end{bmatrix} \end{aligned}$$

Above equation solved in previous part.

$$\begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} = \begin{bmatrix} 10 \exp(0.1(t - 100)) - 10 \\ C_1 \end{bmatrix} \quad (10)$$

In this part we have the same condition that described in previous in equation 8. From equation 9 we calculate  $\frac{\partial \mathcal{H}}{\partial u}$ .

$$\frac{\partial \mathcal{H}}{\partial u} = p_1 - p_2 = (10 \exp(0.1(t - 100)) - 10) - C_1$$

From equation 10 we know that  $C_1$  is constant.

There is four scenario for this problem.

1. For all the time( $t_0 \rightarrow t_f$ )  $\frac{\partial \mathcal{H}}{\partial u} > 0$  so  $u(t) = 0$ , this scenario maybe possible if  $K = 0$ .
2. For all the time( $t_0 \rightarrow t_f$ )  $\frac{\partial \mathcal{H}}{\partial u} < 0$  so  $u(t) = M$ , this scenario maybe possible if  $K = M \times t_f$ .
3. For time( $t_0 \rightarrow t$ )  $\frac{\partial \mathcal{H}}{\partial u} < 0$  and for time( $t \rightarrow t_f$ )  $\frac{\partial \mathcal{H}}{\partial u} > 0$  so for time( $t_0 \rightarrow t$ ),  $u(t) = M$  and for time( $t \rightarrow t_f$ ),  $u(t) = 0$  and this scenario maybe possible for  $0 \leq K \leq M \times t_f$ .
4. For time( $t_0 \rightarrow t$ )  $\frac{\partial \mathcal{H}}{\partial u} > 0$  and for time( $t \rightarrow t_f$ )  $\frac{\partial \mathcal{H}}{\partial u} < 0$  so for time( $t_0 \rightarrow t$ ),  $u(t) = 0$  and for time( $t \rightarrow t_f$ ),  $u(t) = M$  and this scenario is not possible because  $p_1$  is growing by the time and  $p_2$  is constant all the time.

### 1.3 part c

$$J = -x(100)$$

Subjected to:

$$\int_0^{100} u(t) dt = K (\text{a known constant})$$

## 2 Question 3

$$\ddot{x}(t) = -x(t) - 0.1\dot{x}(t) + u(t), \quad x(0) = \dot{x}(0) = 1$$

Assume:

$$x_1(t) = x(t), \quad x_2(t) = \dot{x}(t) \rightarrow a(\vec{x}, u, t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -x_1(t) - 0.1x_2(t) + u(t) \end{bmatrix}$$

$$J = \frac{1}{2}x^T(t_f)Hx(t_f) + \frac{1}{2} \int_0^{t_f} (\alpha(x^2 + \dot{x}^2) + \beta u^2) dt$$

### 2.1 part a

$\alpha = \beta = 1$ ,  $t_f \rightarrow \infty$  and  $H = 0$ :

$$J = \frac{1}{2} \int_0^{t_f} (x^2 + \dot{x}^2 + u^2) dt$$

$$\vec{p}(t) = \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix}$$

Hamiltonian matrix:

$$\mathcal{H} = g(\vec{x}(t), u(t), t) + \vec{p}(t)^T a(\vec{x}(t), u(t), t)$$

$$\mathcal{H} = x_1^2(t) + x_2^2(t) + u^2(t) + \begin{bmatrix} p_1(t) & p_2(t) \end{bmatrix} \begin{bmatrix} x_2(t) \\ -x_1(t) - 0.1x_2(t) + u(t) \end{bmatrix}$$

$$\mathcal{H} = x_1^2(t) + x_2^2(t) + u^2(t) + p_1(t)x_2(t) - p_2(t)x_1(t) - 0.1p_2(t)x_2(t) + p_2(t)u(t)$$

Euler–Lagrange equation:

$$\dot{\vec{x}} = \frac{\partial \mathcal{H}}{\partial \vec{p}} = a(\vec{x}(t), u(t), t) \quad (11)$$

$$\dot{\vec{p}} = -\frac{\partial \mathcal{H}}{\partial \vec{x}} = \begin{bmatrix} -\frac{\partial \mathcal{H}}{\partial x_1} \\ -\frac{\partial \mathcal{H}}{\partial x_2} \end{bmatrix} \quad (12)$$

$$\vec{0} = \frac{\partial \mathcal{H}}{\partial \vec{u}} \quad (13)$$

Now we use above equation for solve problem.

$$\begin{bmatrix} -\frac{\partial \mathcal{H}}{\partial x_1} \\ -\frac{\partial \mathcal{H}}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -2x_1 + p_2 \\ -2x_2 - p_1 + 0.1p_2 \end{bmatrix} = \begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix}$$

$$\frac{\partial \mathcal{H}}{\partial \vec{u}} = 2u + p_2 = 0 \rightarrow u = -0.5p_2$$

There is four differential equation and four unknowns.

$$\dot{x}_1 = x_2 \quad (14)$$

$$\dot{x}_2 = -x_1 - 0.1x_2 - 0.5p_2 \quad (15)$$

$$\dot{p}_1 = -2x_1 + p_2 \quad (16)$$

$$\dot{p}_2 = -2x_2 - p_1 + 0.1p_2 \quad (17)$$

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