

# Home Work #3

Ali BaniAsad 96108378

July 9, 2021

## 1 Question 1

$$z = f(x, y) = y \sin(x + y) - x \sin(x - y)$$

Gradient of  $f(x, y)$ :

$$\vec{\nabla}f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}$$

$$\vec{\nabla}f = \begin{bmatrix} y \cos(x + y) - \sin(x - y) - x \cos(x - y) \\ y \cos(x + y) + \sin(x + y) + x \cos(x - y) \end{bmatrix}$$

### 1.1 part a

$$\vec{X}_0 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Tolerance is:  $10^{-7}$

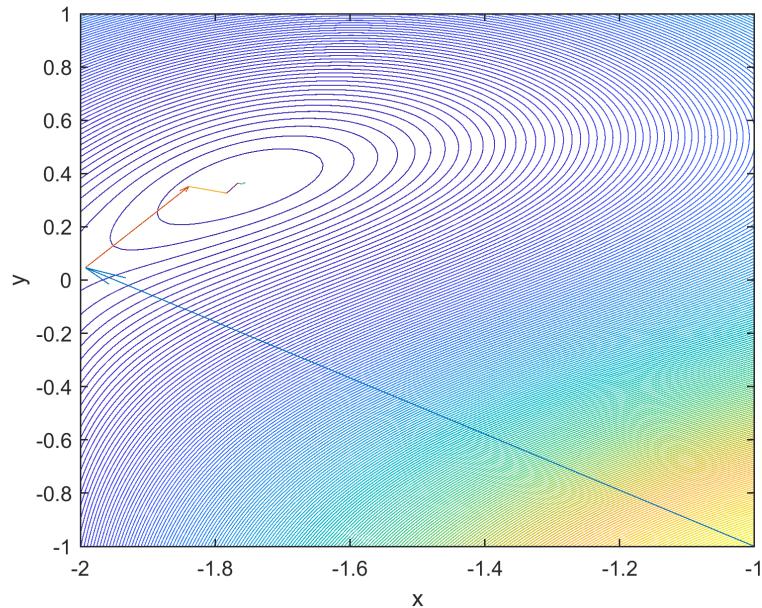
Answer is:

$$\vec{X}_{ans} = \begin{bmatrix} -1.7556 \\ 0.3655 \end{bmatrix}$$

#### 1.1.1 figures

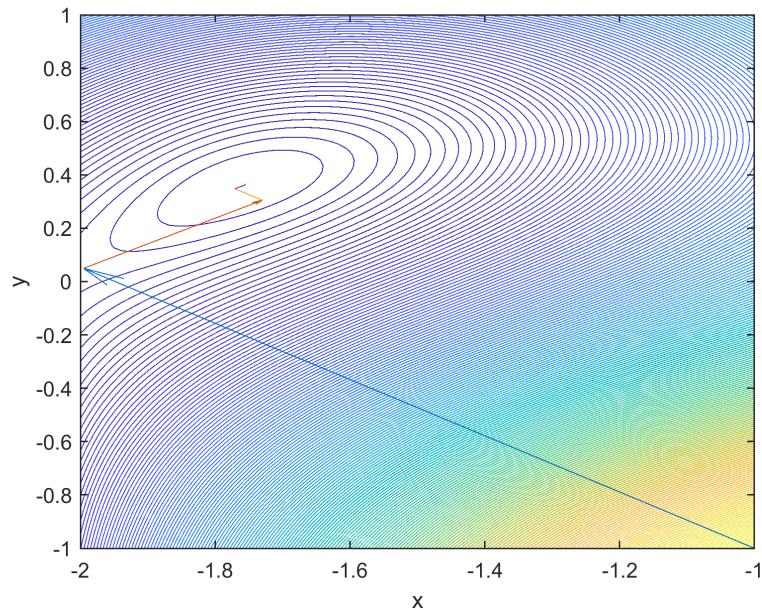
- Steepest Descent
  - Quadratic Interpolation

Figure 1: Steepest Descent and Quadratic Interpolation



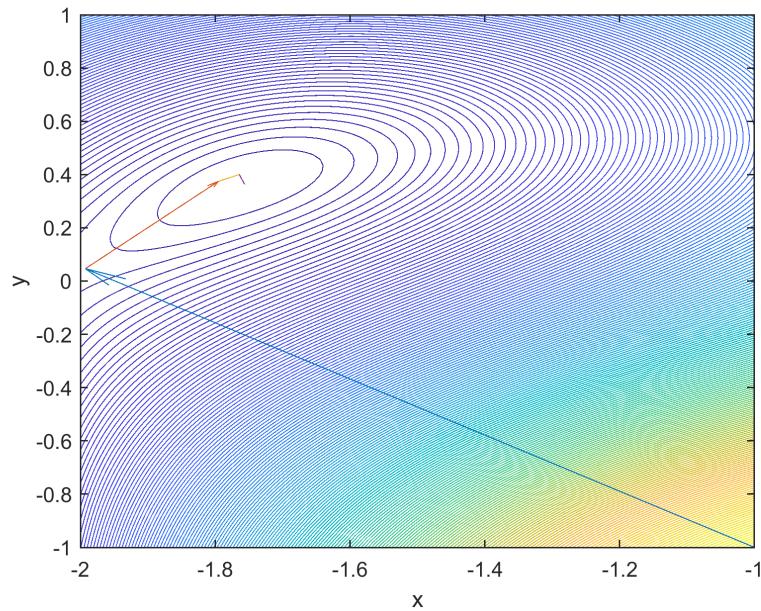
- Golden Section

Figure 2: Steepest Descent and Golden Section



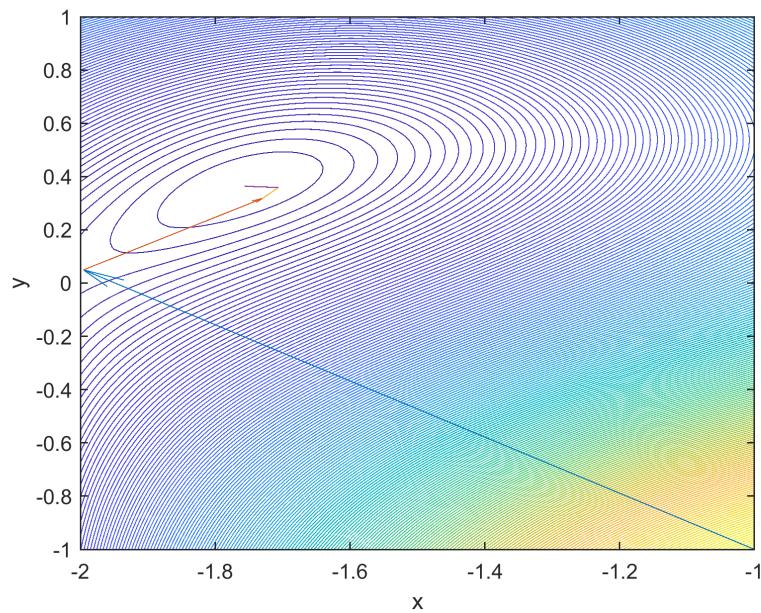
- BFGS
  - Quadratic Interpolation

Figure 3: BFGS and Quadratic Interpolation



- Golden Section

Figure 4: BFGS and Golden Section



### 1.1.2 result

- Time

Table 1: Time compare between four methods

| Steepest Descent        |                | BFGS                    |                |
|-------------------------|----------------|-------------------------|----------------|
| Quadratic Interpolation | Golden Section | Quadratic Interpolation | Golden Section |
| 0.238 sec               | 0.183 sec      | 0.164 sec               | 0.102 sec      |

- Number of Cost calculation

Table 2: Number of Cost calculation compare between four methods

| Steepest Descent        |                | BFGS                    |                |
|-------------------------|----------------|-------------------------|----------------|
| Quadratic Interpolation | Golden Section | Quadratic Interpolation | Golden Section |
| 360                     | 336            | 242                     | 213            |

- Number of Gradient calculation

Table 3: Number of Gradient calculation compare between four methods

| Steepest Descent        |                | BFGS                    |                |
|-------------------------|----------------|-------------------------|----------------|
| Quadratic Interpolation | Golden Section | Quadratic Interpolation | Golden Section |
| 19                      | 13             | 13                      | 9              |

### 1.2 part b

$$\vec{X}_0 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

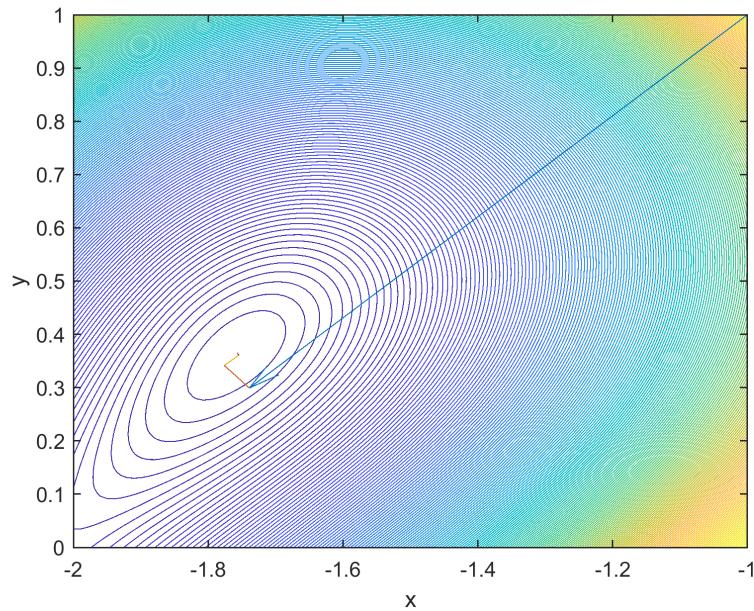
Tolerance is:  $10^{-7}$

$$\vec{X}_{ans} = \begin{bmatrix} -1.7556 \\ 0.3655 \end{bmatrix}$$

#### 1.2.1 figures

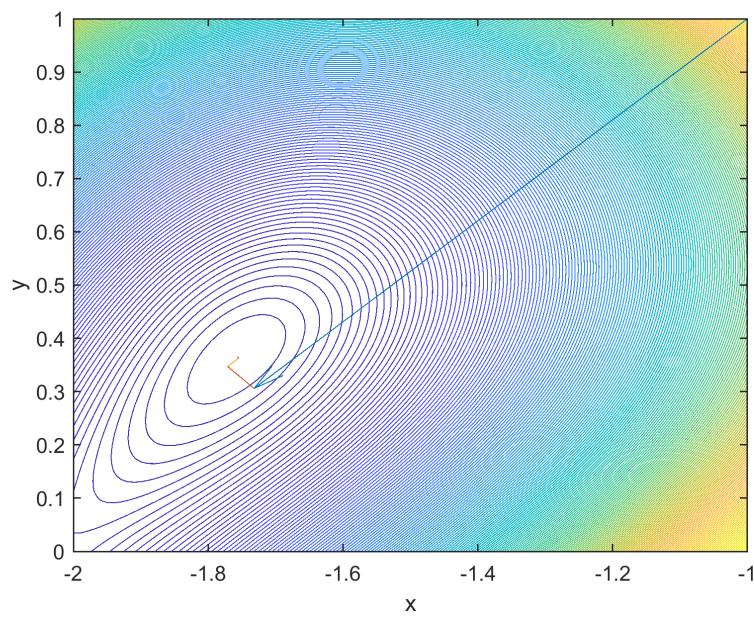
- Steepest Descent
  - Quadratic Interpolation

Figure 5: Steepest Descent and Quadratic Interpolation



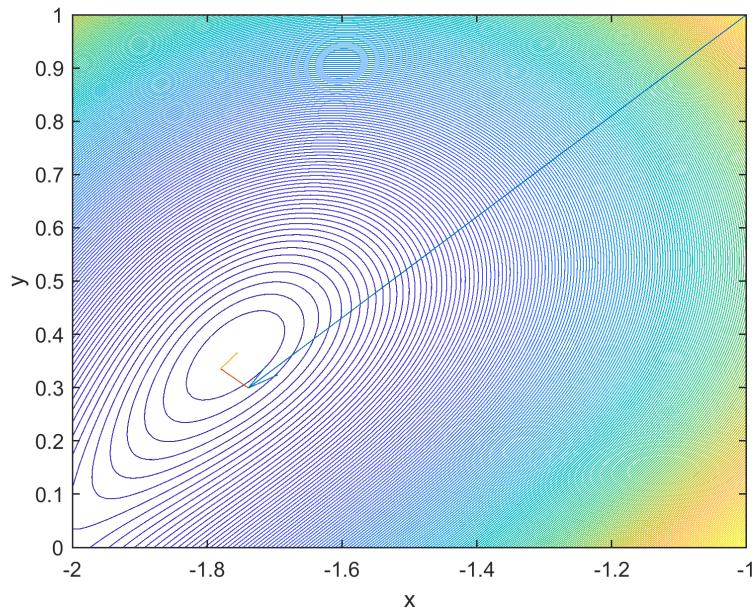
- Golden Section

Figure 6: Steepest Descent and Golden Section



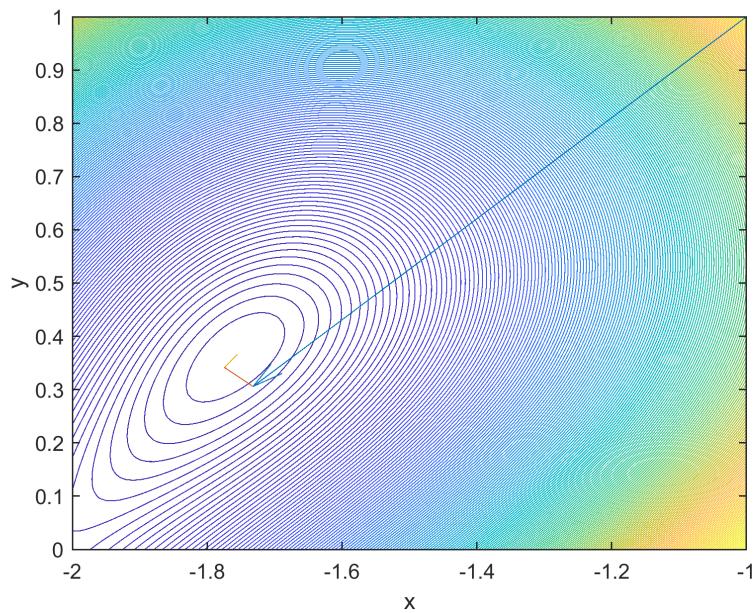
- BFGS
  - Quadratic Interpolation

Figure 7: BFGS and Quadratic Interpolation



- Golden Section

Figure 8: BFGS and Golden Section



### 1.2.2 result

- Time

Table 4: Time compare between four methods

| Steepest Descent        |                | BFGS                    |                |
|-------------------------|----------------|-------------------------|----------------|
| Quadratic Interpolation | Golden Section | Quadratic Interpolation | Golden Section |
| 0.208 sec               | 0.146 sec      | 0.106 sec               | 0.142 sec      |

- Number of Cost calculation

Table 5: Number of Cost calculation compare between four methods

| Steepest Descent        |                | BFGS                    |                |
|-------------------------|----------------|-------------------------|----------------|
| Quadratic Interpolation | Golden Section | Quadratic Interpolation | Golden Section |
| 246                     | 285            | 142                     | 142            |

- Number of Gradient calculation

Table 6: Number of Gradient calculation compare between four methods

| Steepest Descent        |                | BFGS                    |                |
|-------------------------|----------------|-------------------------|----------------|
| Quadratic Interpolation | Golden Section | Quadratic Interpolation | Golden Section |
| 14                      | 12             | 7                       | 7              |

## 2 Question 2

### 2.1 System

$$\begin{aligned} \ddot{x}(t) &= -x(t) - 0.1\dot{x}(t) + u \\ \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ 0 & -0.1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ A = \begin{bmatrix} -1 & 0 \\ 0 & -0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = 1, \quad H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \end{aligned}$$

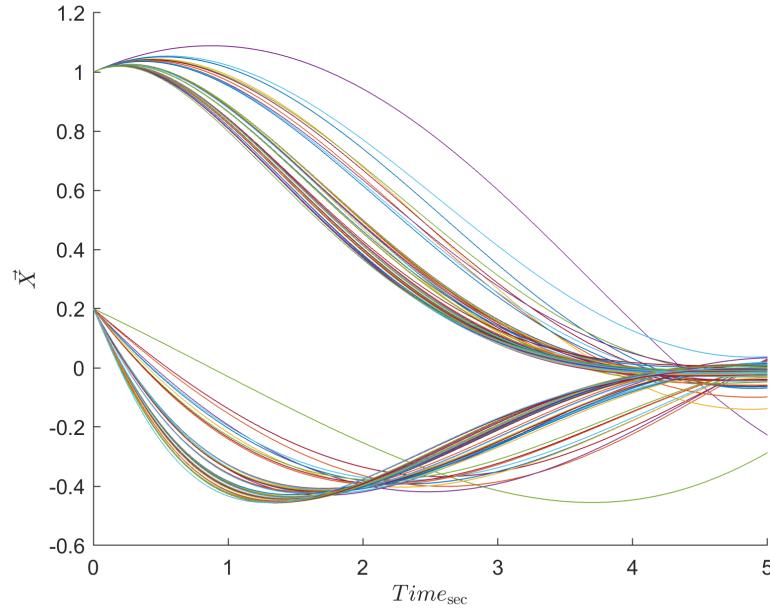
### 2.2 part a

Gradient tolerance is:  $10^{-4}$

#### 2.2.1 figures

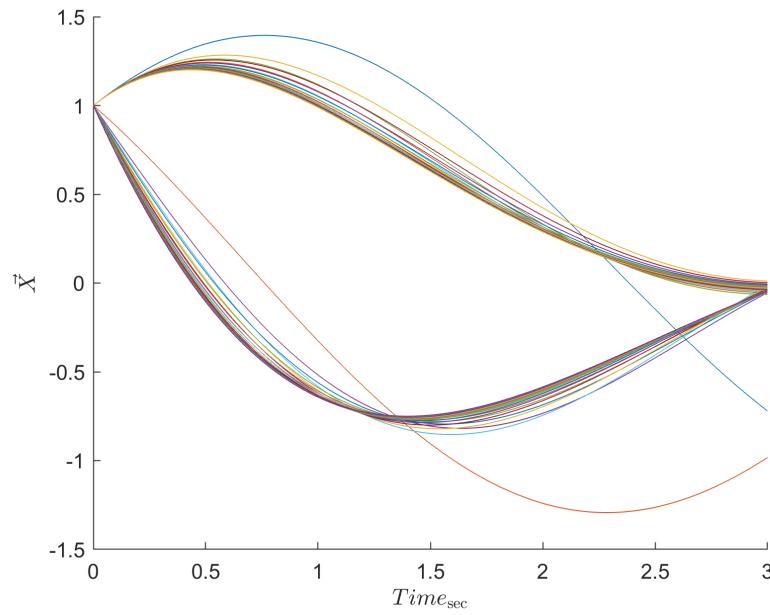
- Steepest Descent
  - Quadratic Interpolation

Figure 9: Steepest Descent and Quadratic Interpolation



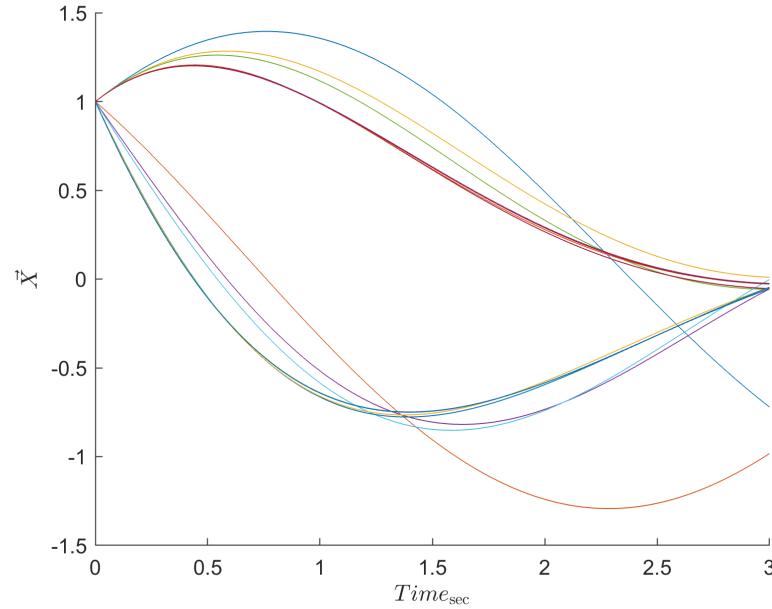
- Golden Section

Figure 10: Steepest Descent and Golden Section



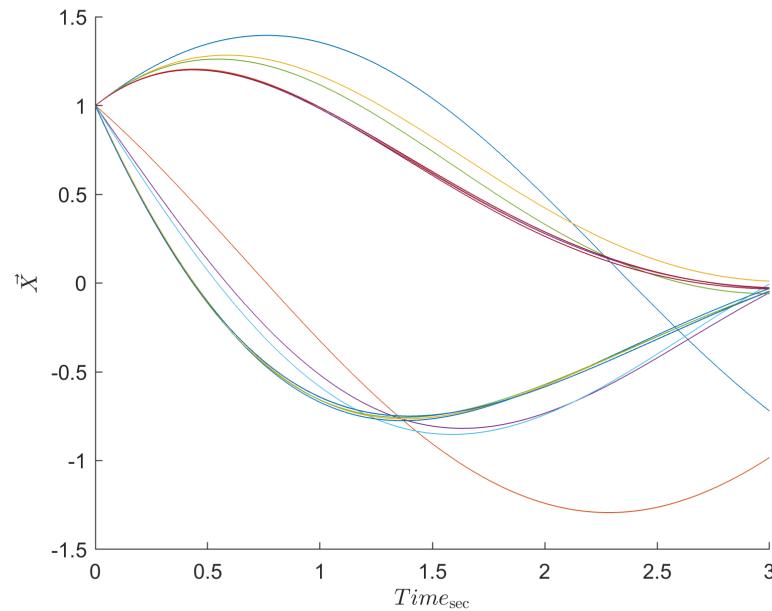
- BFGS
  - Quadratic Interpolation

Figure 11: BFGS and Quadratic Interpolation



- Golden Section

Figure 12: BFGS and Golden Section



### 2.2.2 result

- Time

Table 7: Time compare between four methods

| Steepest Descent        |                | BFGS                    |                |
|-------------------------|----------------|-------------------------|----------------|
| Quadratic Interpolation | Golden Section | Quadratic Interpolation | Golden Section |
| 17.000 sec              | 24.353 sec     | 3.905 sec               | 4.985 sec      |

- Number of Cost calculation

Table 8: Number of Cost calculation compare between four methods

| Steepest Descent        |                | BFGS                    |                |
|-------------------------|----------------|-------------------------|----------------|
| Quadratic Interpolation | Golden Section | Quadratic Interpolation | Golden Section |
| 1285                    | 1922           | 273                     | 373            |

- Number of Gradient calculation

Table 9: Number of Gradient calculation compare between four methods

| Steepest Descent        |                | BFGS                    |                |
|-------------------------|----------------|-------------------------|----------------|
| Quadratic Interpolation | Golden Section | Quadratic Interpolation | Golden Section |
| 51                      | 51             | 11                      | 11             |

### 2.2.3 Four iteration for BFGS and Quadratic interpolation

Figure 13: BFGS and Quadratic Interpolation with four iteration

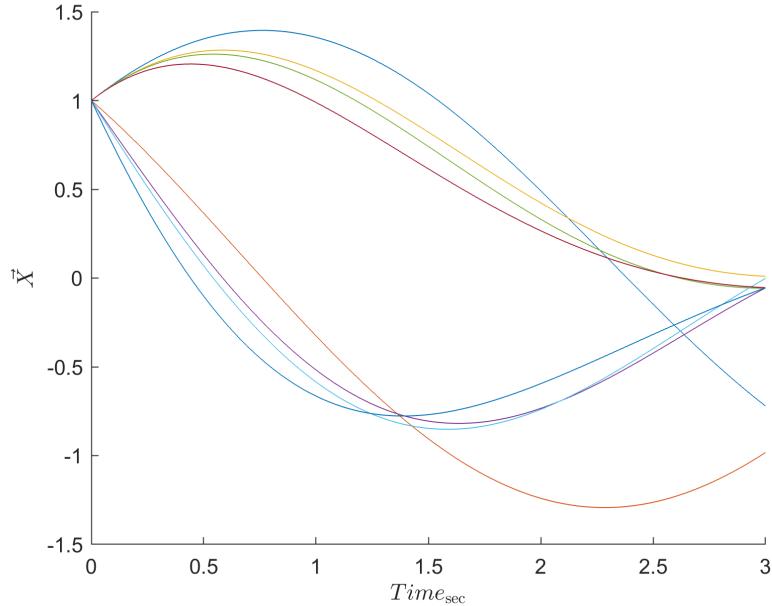


Table 10: four iteration and gradient tolerance compare

|                    | Time                 | Number of Cost calculation | Number of Gradient calculation |
|--------------------|----------------------|----------------------------|--------------------------------|
| Four iteration     | 3.905 <sub>sec</sub> | 273                        | 11                             |
| Gradient tolerance | 1.586 <sub>sec</sub> | 100                        | 4                              |

Between BFGS and Steepest Descent, BFGS is more faster but Steepest Descent is more easy to use. Quadratic interpolation can be faster when cost function use so much process but when we are so close to answer the function doesn't work well so we must increase gradient tolerance.

### 2.3 part b

Tolerance is:  $10^{-16}$  for  $\lambda S_i$  or  $10^{-4}$  for norm of gradient.

#### 2.3.1 figures

- Steepest Descent
  - Quadratic Interpolation

Figure 14: Steepest Descent and Quadratic Interpolation

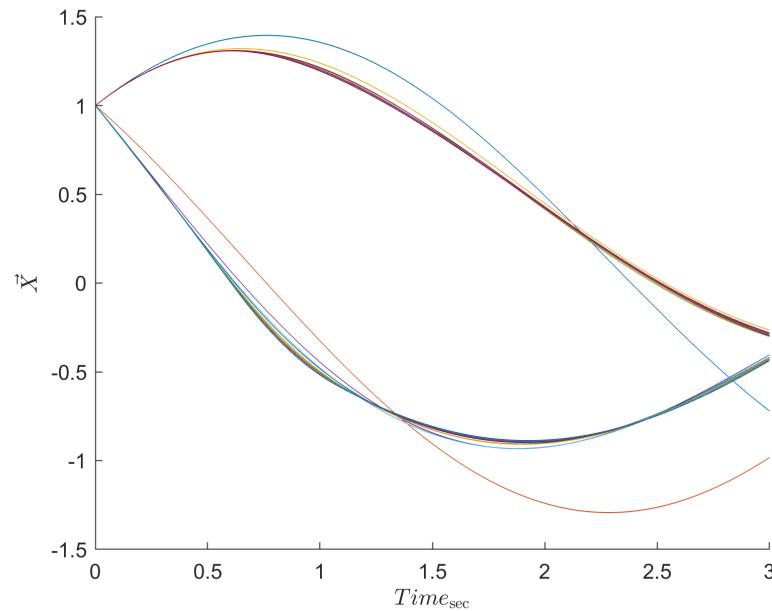
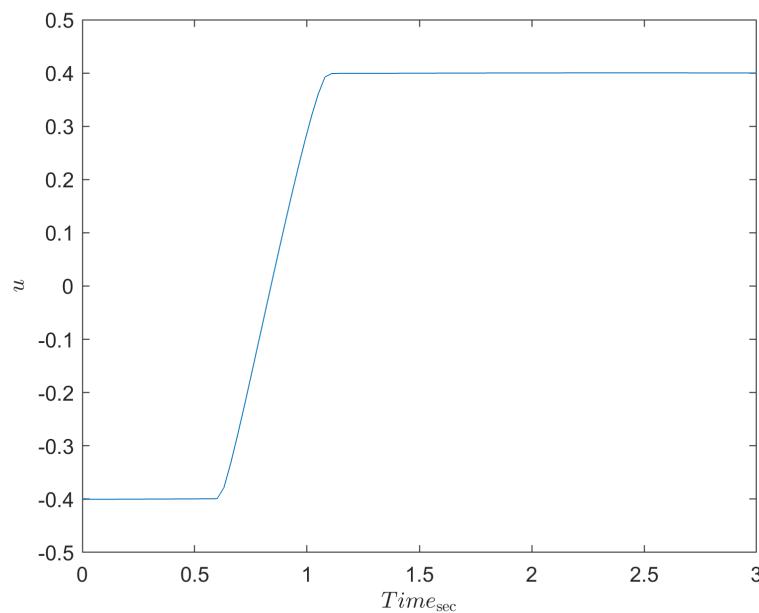


Figure 15: Steepest Descent and Quadratic Interpolation Control



– Golden Section

Figure 16: Steepest Descent and Golden Section

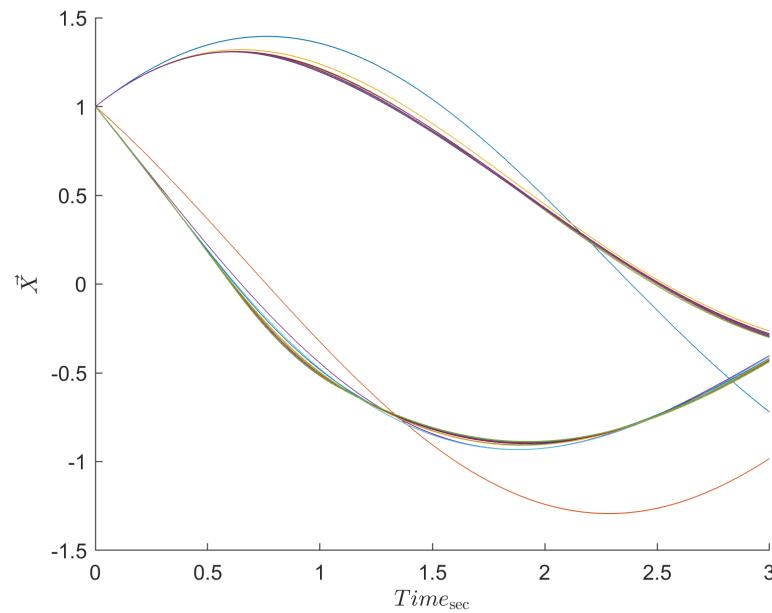
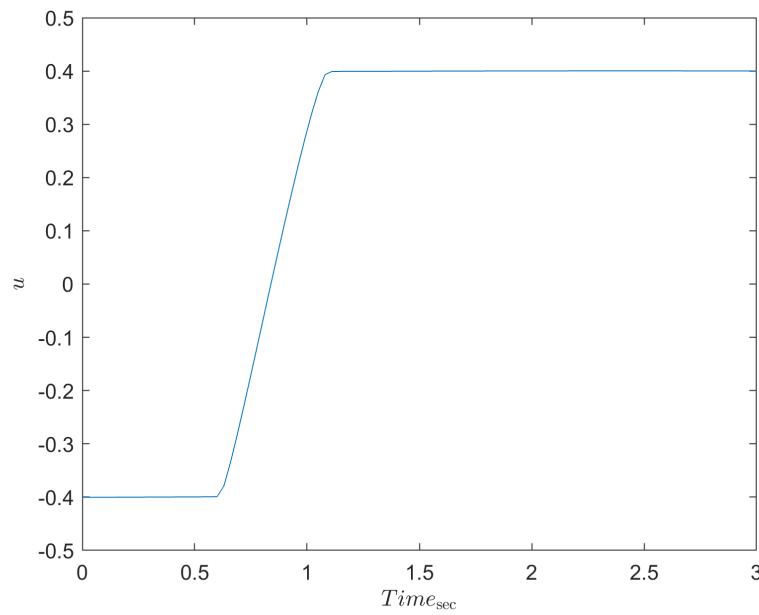


Figure 17: Steepest Descent and Golden Section Control



- BFGS
  - Quadratic Interpolation

Figure 18: BFGS and Quadratic Interpolation

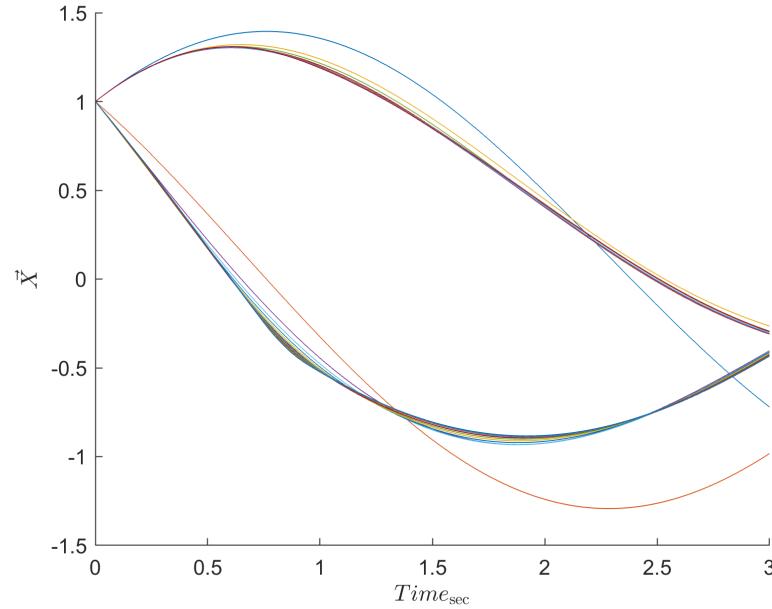
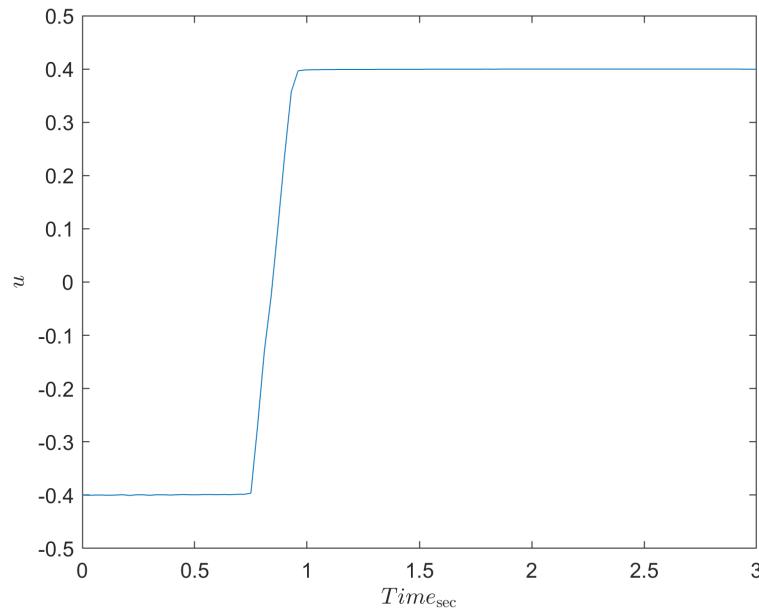


Figure 19: BFGS and Quadratic Interpolation Control



– Golden Section

Figure 20: BFGS and Golden Section

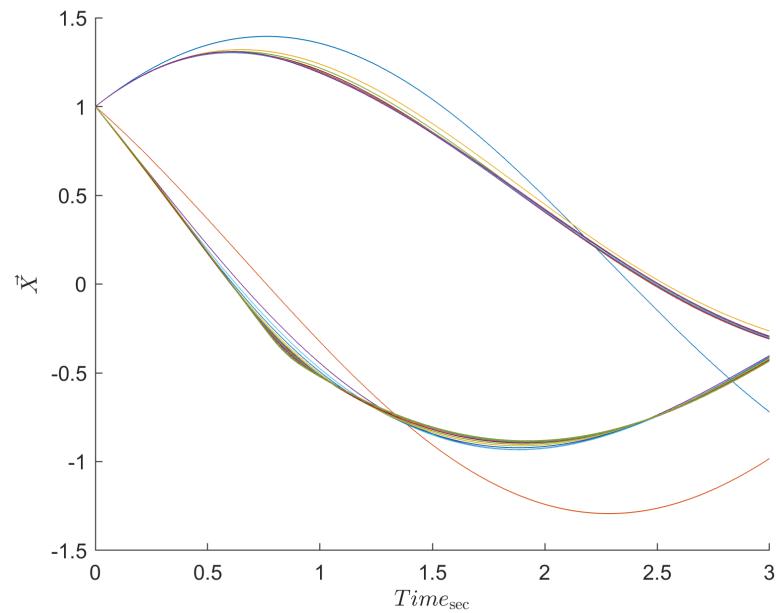
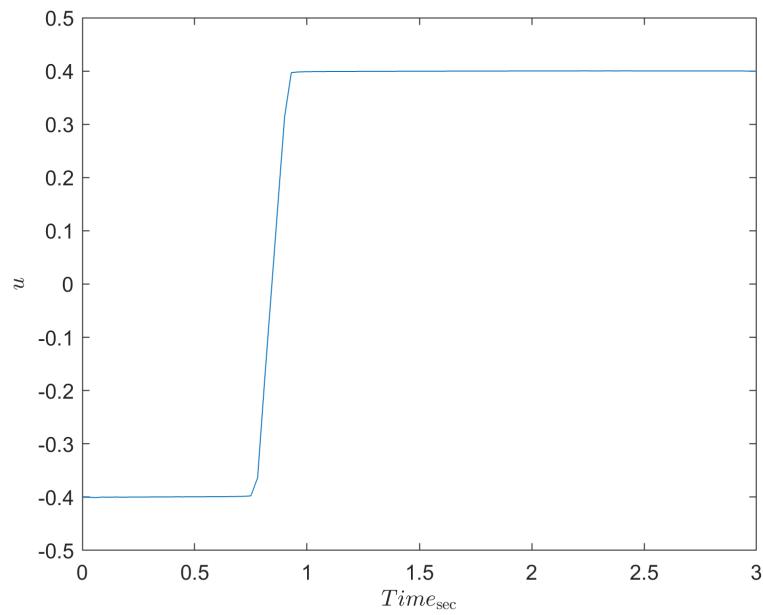


Figure 21: BFGS and Golden Section Control



### 2.3.2 result

- Time

Table 11: Time compare between four methods

| Steepest Descent        |                | BFGS                    |                |
|-------------------------|----------------|-------------------------|----------------|
| Quadratic Interpolation | Golden Section | Quadratic Interpolation | Golden Section |
| 7.595 sec               | 33.761 sec     | 86.730 sec              | 72.666 sec     |

- Number of Cost calculation

Table 12: Number of Cost calculation compare between four methods

| Steepest Descent        |                | BFGS                    |                |
|-------------------------|----------------|-------------------------|----------------|
| Quadratic Interpolation | Golden Section | Quadratic Interpolation | Golden Section |
| 442                     | 1782           | 2787                    | 2378           |

- Number of Gradient calculation

Table 13: Number of Gradient calculation compare between four methods

| Steepest Descent        |                | BFGS                    |                |
|-------------------------|----------------|-------------------------|----------------|
| Quadratic Interpolation | Golden Section | Quadratic Interpolation | Golden Section |
| 25                      | 55             | 256                     | 174            |

Cause of constrain the functions can't go to the origin but in part a function get more close to origin. In this part because of penalty function BFGS method get confused and use more iteration to get the answer, but the answer in more close to the Bang-Bang and analytical solution.

## 2.4 LQR

In LQR when control effort is grater than 0.4 we use 0.4 and when is lower than -0.4 we use -0.4.

Figure 22: LQR

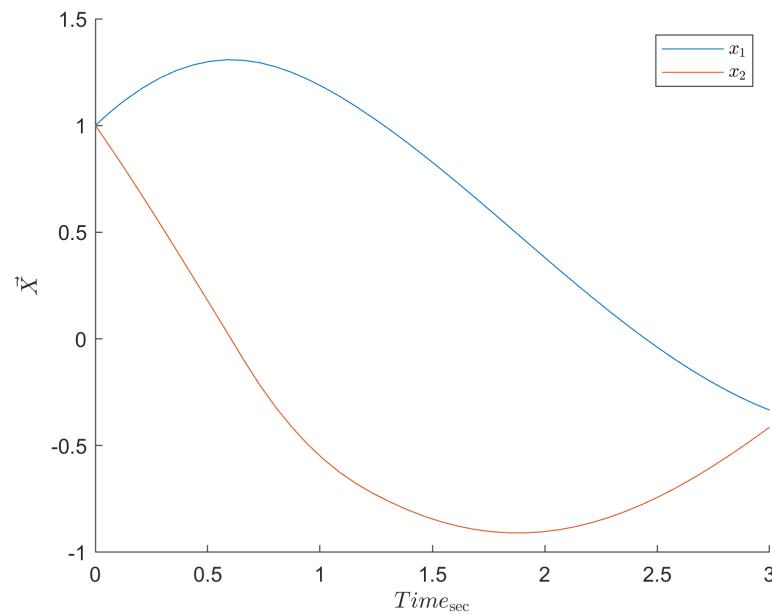
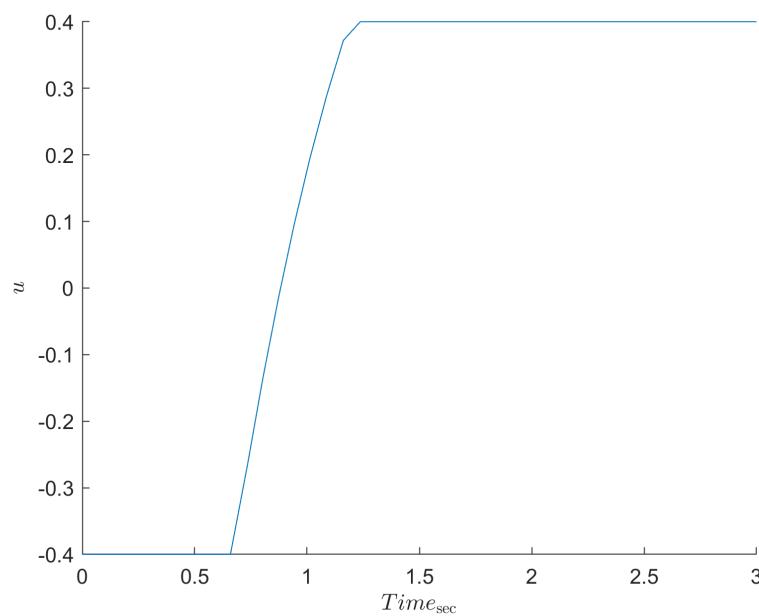


Figure 23: LQR Control



### 3 Question 3

#### 3.1 System

$$\begin{aligned}\dot{x}_1 &= -x_1 + u \\ \dot{x}_2 &= -2x_2 + 2u \\ \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u \\ A &= \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}\end{aligned}$$

#### 3.2 Direct Optimization

Time is free so we have a new unknowns. we use  $t_f$  as a new state and change some parameter that will describe here.

$$J = t_f^2, \quad 0 \leq t \leq t_f, \quad \tau = \frac{t}{t_f}, \quad 0 \leq \tau \leq 1$$

$$\vec{a}_N(\vec{x}(t), \vec{u}(t), t_f, t) = t_f \vec{a}(\vec{x}(t), \vec{u}(t), t)$$

$$g_N(\vec{x}(t), \vec{u}(t), t_f, t) = t_f g(\vec{x}(t), \vec{u}(t), t)$$

$$\mathcal{H} = g_N(\vec{x}(t), \vec{u}(t), t_f, t) + P^T \vec{a}_N(\vec{x}(t), \vec{u}(t), t_f, t)$$

$$G_1(u) = \begin{cases} -\frac{1}{g_1(u)} & g_1(u) \leq \epsilon \\ -\frac{1}{\epsilon} \left( 3 - \frac{3g_1(u)}{\epsilon} + \left( \frac{g_1(u)}{\epsilon} \right)^2 \right) & g_1(u) > \epsilon \end{cases}$$

$$G'_1(u) = \begin{cases} \frac{1}{(u-)^2} & g_1(u) \leq \epsilon \\ -\frac{1}{\epsilon} \left( -\frac{3}{\epsilon} + \frac{2u-2}{\epsilon^2} \right) & g_1(u) > \epsilon \end{cases}$$

$$G_2(x_2) = \begin{cases} -\frac{1}{g_2(u)} & g_2(u) \leq \epsilon \\ -\frac{1}{\epsilon} \left( 3 - \frac{3g_2(u)}{\epsilon} + \left( \frac{g_2(u)}{\epsilon} \right)^2 \right) & g_2(u) > \epsilon \end{cases}$$

$$G'_2(u) = \begin{cases} \frac{1}{(u+1)^2} & g_2(u) \leq \epsilon \\ -\frac{1}{\epsilon} \left( \frac{3}{\epsilon} + \frac{2u+2}{\epsilon^2} \right) & g_2(u) > \epsilon \end{cases}$$

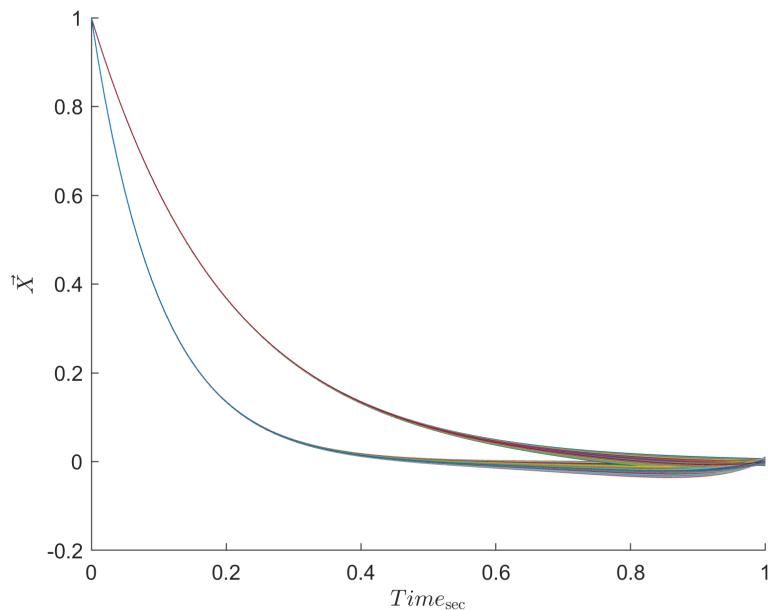
$$g_N(\vec{x}(t), \vec{u}(t), t_f, t) = t_f^2 + r_k G(u)$$

$$\frac{\partial J}{\partial t_f} = \frac{\partial h}{\partial t_f} + \int_0^1 \frac{\partial \mathcal{H}}{\partial t_f}$$

$$\frac{\partial J}{\partial \vec{X}} = \begin{bmatrix} \left. \frac{\mathcal{H}}{\partial u} \right|_{\tau_0} \\ \left. \frac{\mathcal{H}}{\partial u} \right|_{\tau_1} \\ \left. \frac{\mathcal{H}}{\partial u} \right|_{\tau_2} \\ \vdots \\ \left. \frac{\mathcal{H}}{\partial u} \right|_{\tau_f} \\ \frac{\partial h}{\partial t_f} + \int_0^1 \frac{\partial \mathcal{H}}{\partial t_f} \end{bmatrix}$$

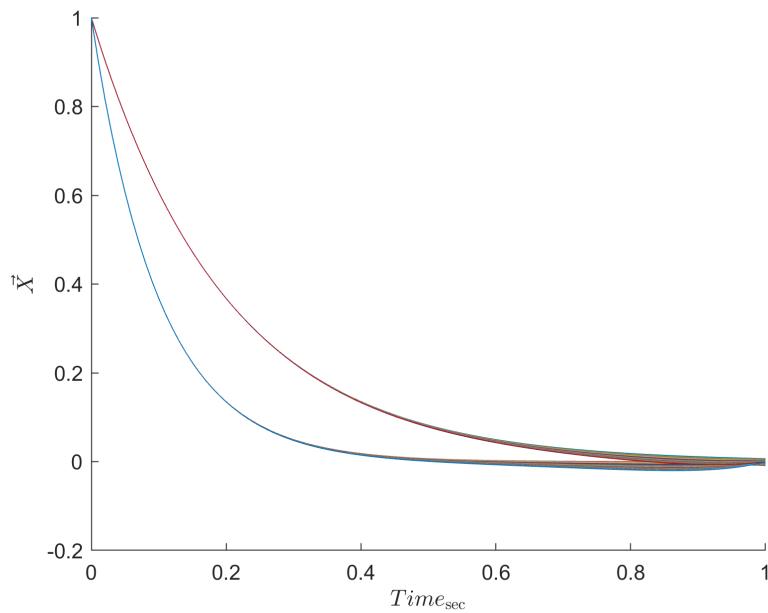
- Steepest Descent
  - Quadratic Interpolation

Figure 24: Steepest Descent and Quadratic Interpolation



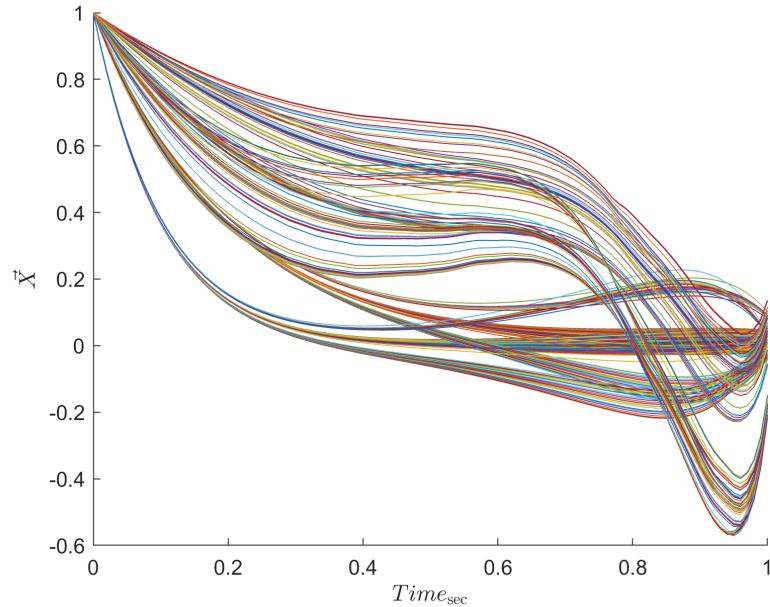
– Golden Section

Figure 25: Steepest Descent and Golden Section



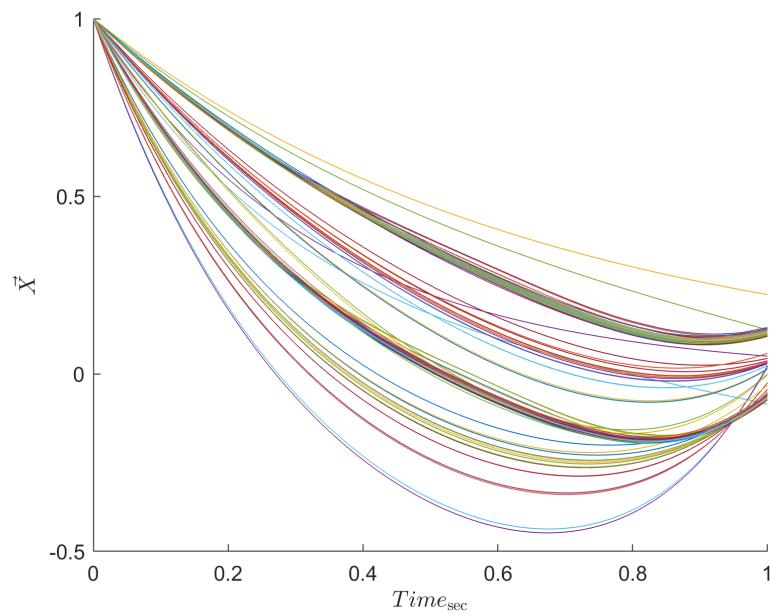
- BFGS
  - Quadratic Interpolation

Figure 26: BFGS and Quadratic Interpolation



- Golden Section

Figure 27: BFGS and Golden Section



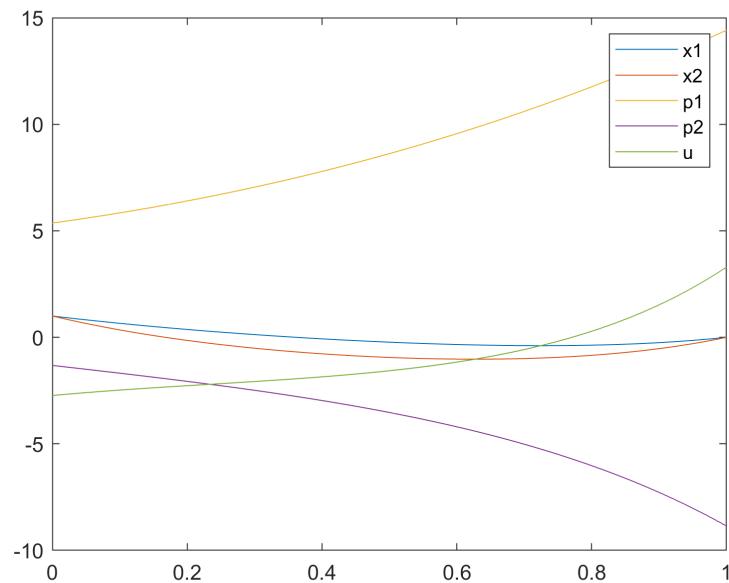
### 3.3 Shooting method

Final time is free so:

$$\vec{F} = \begin{bmatrix} x_1(t_f) - x_{1f} \\ x_2(t_f) - x_{2f} \\ (\mathcal{H} - h_t)|_{t_f} \end{bmatrix}$$

$$\vec{y}_{k+1} = \vec{y}_k - \frac{\partial \vec{F}}{\partial \vec{y}} \Big|_{\vec{y}_k} \vec{F}(\vec{y}_k)$$

Figure 28: Shooting method



## 4 Question 4

$$a = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -0.4x_1 - 0.2x_2^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$G_1(u) = \begin{cases} -\frac{1}{g_1(u)} & g_1(u) \leq \epsilon \\ -\frac{1}{\epsilon} \left( 3 - \frac{3g_1(u)}{\epsilon} + \left( \frac{g_1(u)}{\epsilon} \right)^2 \right) & g_1(u) > \epsilon \end{cases}$$

$$G'_1(u) = \begin{cases} \frac{1}{(u - 0.8)^2} & g_1(u) \leq \epsilon \\ -\frac{1}{\epsilon} \left( -\frac{3}{\epsilon} + \frac{2u - 1.6}{\epsilon^2} \right) & g_1(u) > \epsilon \end{cases}$$

$$G_2(x_2) = \begin{cases} -\frac{1}{g_2(u)} & g_2(u) \leq \epsilon \\ -\frac{1}{\epsilon} \left( 3 - \frac{3g_2(u)}{\epsilon} + \left( \frac{g_2(u)}{\epsilon} \right)^2 \right) & g_2(u) > \epsilon \end{cases}$$

$$G'_2(u) = \begin{cases} \frac{1}{(u + 0.8)^2} & g_2(u) \leq \epsilon \\ -\frac{1}{\epsilon} \left( \frac{3}{\epsilon} + \frac{2u + 1.6}{\epsilon^2} \right) & g_2(u) > \epsilon \end{cases}$$

$$\epsilon = -c(r_k)^2, \quad a = 0.5, \quad r_{k+1} = cr_k, \quad c = 0.9, \quad \min(r_k) = 0.001$$

$$\mathcal{H} = \vec{P}^T a(\vec{X}, u, t) + \frac{1}{2} (x_1^2 + x_2^2 + u^2 + r_k G(u))$$

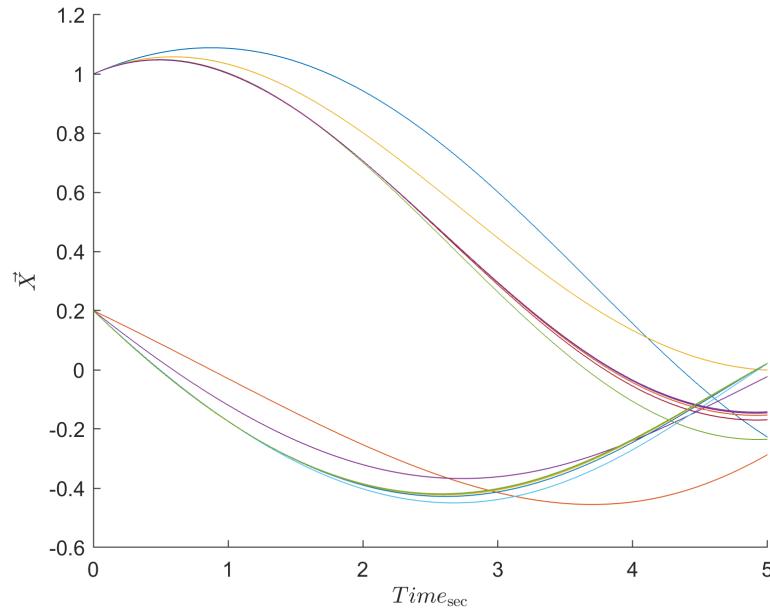
$$\dot{\vec{P}} = -\frac{\partial \mathcal{H}}{\partial \vec{X}} = \begin{bmatrix} -x_1 + 0.4p_2 \\ -x_2 - p_1 + 0.4p_2 x_2 \end{bmatrix}$$

$$\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} -x_1 + 0.4p_2 \\ x_2(0.4p_2 - 1) - p_1 \end{bmatrix}$$

### 4.1 Direct Optimization

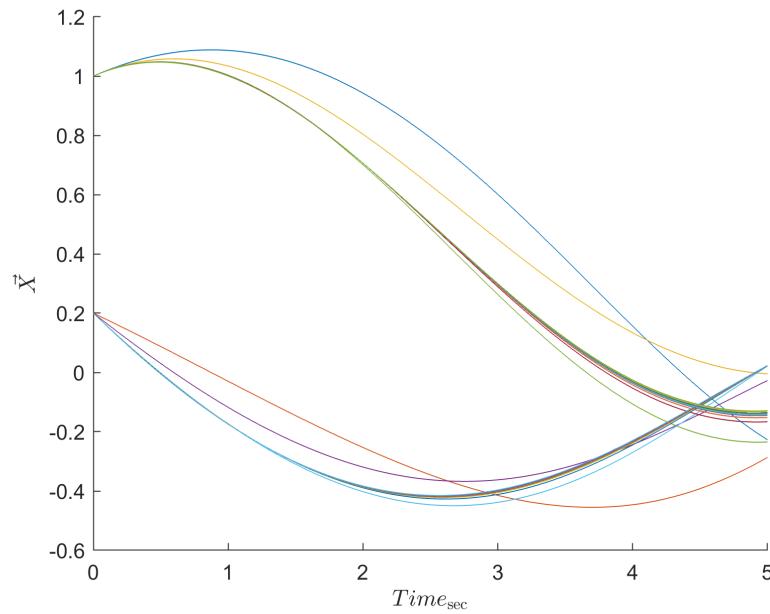
- Steepest Descent
  - Quadratic Interpolation

Figure 29: Steepest Descent and Quadratic Interpolation



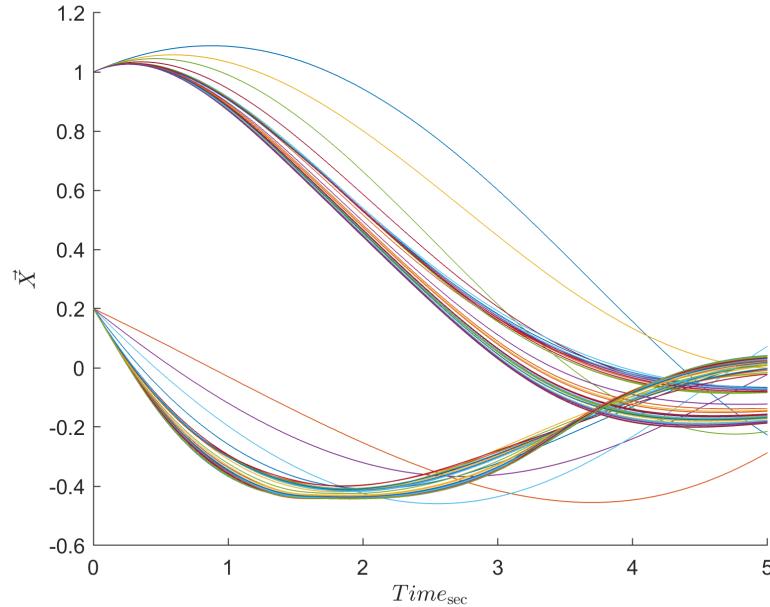
- Golden Section

Figure 30: Steepest Descent and Golden Section



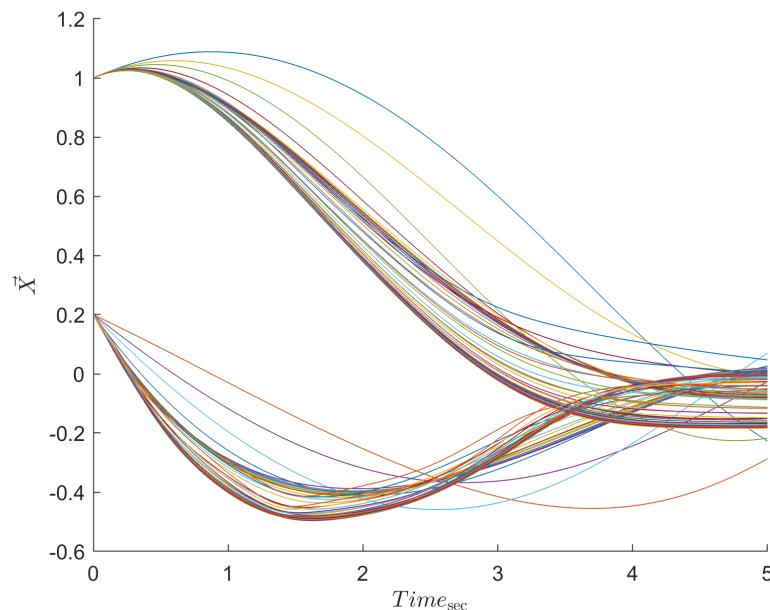
- BFGS
  - Quadratic Interpolation

Figure 31: BFGS and Quadratic Interpolation



- Golden Section

Figure 32: BFGS and Golden Section



## 4.2 Shooting method

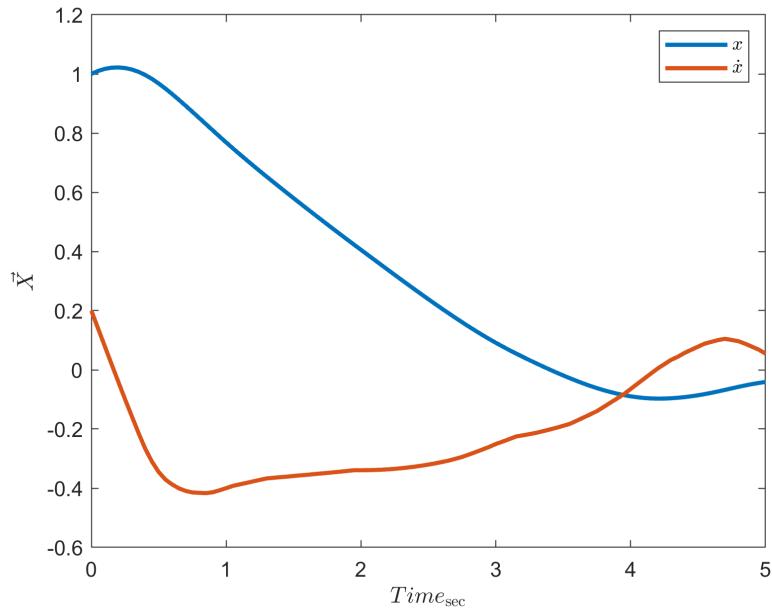
Shooting method get lost:(.

## 4.3 Dynamic programming

$$\begin{aligned} \dot{x} &= \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -0.4x_1 - 0.2x_2^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} &= \begin{bmatrix} x_2(k) \\ -0.4x_1(k) - 0.2x_2^2(k) + u(k) \end{bmatrix} \Delta t + \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \end{aligned}$$

In MATLAB Code, Control law will save in .mat file and we can use for another initial condition very fast without so much processing.

Figure 33: Dynamic Programming



## Contents

|   |           |
|---|-----------|
| <b>1 Question 1</b>   | <b>1</b>  |
| 1.1 part a . . . . .  | 1         |
| 1.1.1 figures . . . . .   | 1         |
| 1.1.2 result . . . . .  | 4         |
| 1.2 part b . . . . .  | 4         |
| 1.2.1 figures . . . . .   | 4         |
| 1.2.2 result . . . . .  | 7         |
| <b>2 Question 2</b>   | <b>7</b>  |
| 2.1 System . . . . .  | 7         |
| 2.2 part a . . . . .  | 7         |
| 2.2.1 figures . . . . .   | 7         |
| 2.2.2 result . . . . .  | 10        |
| 2.2.3 Four iteration for BFGS and Quadratic interpolation . . . . . | 11        |
| 2.3 part b . . . . .  | 11        |
| 2.3.1 figures . . . . .   | 11        |
| 2.3.2 result . . . . .  | 16        |
| 2.4 LQR . . . . .   | 16        |
| <b>3 Question 3</b>   | <b>18</b> |
| 3.1 System . . . . .  | 18        |
| 3.2 Direct Optimization . . . . .                                   | 18        |
| 3.3 Shooting method . . . . .                                       | 22        |
| <b>4 Question 4</b>   | <b>23</b> |
| 4.1 Direct Optimization . . . . .                                   | 23        |
| 4.2 Shooting method . . . . .                                       | 26        |
| 4.3 Dynamic programming . . . . .                                   | 26        |

**List of Figures**

|    |  |    |
|----|--|----|
| 1  | Steepest Descent and Quadratic Interpolation . . . . .         | 2  |
| 2  | Steepest Descent and Golden Section . . . . .                  | 2  |
| 3  | BFGS and Quadratic Interpolation . . . . .                     | 3  |
| 4  | BFGS and Golden Section . . . . .                              | 3  |
| 5  | Steepest Descent and Quadratic Interpolation . . . . .         | 5  |
| 6  | Steepest Descent and Golden Section . . . . .                  | 5  |
| 7  | BFGS and Quadratic Interpolation . . . . .                     | 6  |
| 8  | BFGS and Golden Section . . . . .                              | 6  |
| 9  | Steepest Descent and Quadratic Interpolation . . . . .         | 8  |
| 10 | Steepest Descent and Golden Section . . . . .                  | 8  |
| 11 | BFGS and Quadratic Interpolation . . . . .                     | 9  |
| 12 | BFGS and Golden Section . . . . .                              | 9  |
| 13 | BFGS and Quadratic Interpolation with four iteration . . . . . | 11 |
| 14 | Steepest Descent and Quadratic Interpolation . . . . .         | 12 |
| 15 | Steepest Descent and Quadratic Interpolation Control . . . . . | 12 |
| 16 | Steepest Descent and Golden Section . . . . .                  | 13 |
| 17 | Steepest Descent and Golden Section Control . . . . .          | 13 |
| 18 | BFGS and Quadratic Interpolation . . . . .                     | 14 |
| 19 | BFGS and Quadratic Interpolation Control . . . . .             | 14 |
| 20 | BFGS and Golden Section . . . . .                              | 15 |
| 21 | BFGS and Golden Section Control . . . . .                      | 15 |
| 22 | LQR . . . . .  | 17 |
| 23 | LQR Control . . . . .  | 17 |
| 24 | Steepest Descent and Quadratic Interpolation . . . . .         | 19 |
| 25 | Steepest Descent and Golden Section . . . . .                  | 20 |
| 26 | BFGS and Quadratic Interpolation . . . . .                     | 21 |
| 27 | BFGS and Golden Section . . . . .                              | 21 |
| 28 | Shooting method . . . . .                                      | 22 |
| 29 | Steepest Descent and Quadratic Interpolation . . . . .         | 24 |
| 30 | Steepest Descent and Golden Section . . . . .                  | 24 |
| 31 | BFGS and Quadratic Interpolation . . . . .                     | 25 |
| 32 | BFGS and Golden Section . . . . .                              | 25 |
| 33 | Dynamic Programming . . . . .                                  | 26 |

**List of Tables**

|    |   |    |
|----|---|----|
| 1  | Time compare between four methods . . . . .                           | 4  |
| 2  | Number of Cost calculation compare between four methods . . . . .     | 4  |
| 3  | Number of Gradient calculation compare between four methods . . . . . | 4  |
| 4  | Time compare between four methods . . . . .                           | 7  |
| 5  | Number of Cost calculation compare between four methods . . . . .     | 7  |
| 6  | Number of Gradient calculation compare between four methods . . . . . | 7  |
| 7  | Time compare between four methods . . . . .                           | 10 |
| 8  | Number of Cost calculation compare between four methods . . . . .     | 10 |
| 9  | Number of Gradient calculation compare between four methods . . . . . | 10 |
| 10 | four iteration and gradient tolerance compare . . . . .               | 11 |
| 11 | Time compare between four methods . . . . .                           | 16 |
| 12 | Number of Cost calculation compare between four methods . . . . .     | 16 |
| 13 | Number of Gradient calculation compare between four methods . . . . . | 16 |