Home Work #2

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1 Question 1

System:

$$\dot{x}(t) = -0.1x(t) + u(t)$$

Subjected to $0 \leq u(t) \leq M$

1.1 part a

$$J = \int_0^{100} -x(t)dt$$

Hamiltonian matrix:

$$\mathcal{H} = g(\vec{x}(t), u(t), t) + \vec{p}(t)^T a(\vec{x}(t), u(t), t)$$
$$\mathcal{H} = -x(t) - 0.1p(t)x(t) + p(t)u(t) \tag{1}$$

Euler–Lagrange equation:

$$\dot{\vec{x}} = \frac{\partial \mathcal{H}}{\partial \vec{p}} = a(\vec{x}(t), u(t), t) \tag{2}$$
$$\dot{\vec{p}} = -\frac{\partial \mathcal{H}}{\partial \vec{x}} \tag{3}$$

Now we use above equation for solve problem.

$$-\frac{\partial \mathcal{H}}{\partial x} = 1 + 0.1p$$

There is two differential equation and two unknowns.

$$\dot{x} = -x(t) - 0.1px + pu \tag{4}$$
$$\dot{p} = 1 + 0.1p \tag{5}$$

Equation 5 solved in MATLAB(Q1_a.m) and code attached to file.

$$p(t) = C_1 * \exp(t/10) - 10 \tag{6}$$

Final $x(t)$ is free so:

$$h_{\vec{x}} - \vec{p} = \vec{0}\Big|_{*,t_f} \rightarrow p(t_f) = 0$$

Use new boundry condition($p(t_f) = 0$) in equation 6 to find p function($p(t)$).

$$p(100) = C_1 * \exp(100/10) - 10 = 0 \rightarrow C_1 = 10\exp(-10)$$

$$p(t) = 10\exp\left(0.1(t-100)\right) - 10 \tag{7}$$

We know that $u(t)$ has limit so for optimization we have another condition to select $u(t)$ for every time.

$$u(t) = \begin{cases} \dfrac{\partial \mathcal{H}}{\partial u} & < 0 \quad u(t) = M \\ \dfrac{\partial \mathcal{H}}{\partial u} & = 0 \quad \mathcal{H} \text{ is not a function of } u(t) \\ \dfrac{\partial \mathcal{H}}{\partial u} & > 0 \quad u(t) = 0 \end{cases} \tag{8}$$

From equation 1 we calculate $\dfrac{\partial \mathcal{H}}{\partial u}$.

$$\frac{\partial \mathcal{H}}{\partial u} = p(t)$$

From equation 7 we know that at $t_0 \to t_f$ $p(t)$ is less than zero($p(t) < 0$), so $u(t)$ for every time is M.

1.2 part b

$$J = \int_0^{100} -x(t)dt$$

Subjected to:

$$\int_0^{100} u(t) = K(\text{a known constant})$$

2 Question 3

$$\ddot{x}(t) = -x(t) - 0.1\dot{x}(t) + u(t), \qquad x(0) = \dot{x}(0) = 1$$

Assume:

$$x_1(t) = x(t), \quad x_2(t) = \dot{x}(t) \to a(\vec{x}, u, t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -x_1(t) - 0.1x_2(t) + u(t) \end{bmatrix}$$

$$J = \frac{1}{2}x^T(t_f)Hx(t_f) + \frac{1}{2}\int_0^{t_f} \left(\alpha(x^2 + \dot{x}^2) + \beta u^2)\right) dt$$

2.1 part a

$\alpha = \beta = 1$, $t_f \to \infty$ and $H = 0$:

$$J = \frac{1}{2}\int_0^{t_f} (x^2 + \dot{x}^2 + u^2)dt$$

$$\vec{p}(t) = \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix}$$

Hamiltonian matrix:

$$\mathcal{H} = g(\vec{x}(t), u(t), t) + \vec{p}(t)^T a(\vec{x}(t), u(t), t)$$

$$\mathcal{H} = x_1^2(t) + x_2^2(t) + u^2(t) + \begin{bmatrix} p_1(t) & p_2(t) \end{bmatrix} \begin{bmatrix} x_2(t) \\ -x_1(t) - 0.1x_2(t) + u(t) \end{bmatrix}$$

$$\mathcal{H} = x_1^2(t) + x_2^2(t) + u^2(t) + p_1(t)x_2(t) - p_2(t)x_1(t) - 0.1p_2(t)x_2(t) + p_2(t)u(t)$$

Euler–Lagrange equation:

$$\dot{\vec{x}} = \frac{\partial \mathcal{H}}{\partial \vec{p}} = a(\vec{x}(t), u(t), t) \tag{9}$$

$$\dot{\vec{p}} = -\frac{\partial \mathcal{H}}{\partial \vec{x}} = \begin{bmatrix} -\dfrac{\partial \mathcal{H}}{\partial x_1} \\ -\dfrac{\partial \mathcal{H}}{\partial x_2} \end{bmatrix} \tag{10}$$

$$\vec{0} = \frac{\partial \mathcal{H}}{\partial \vec{u}} \tag{11}$$

Now we use above equation for solve problem.

$$\begin{bmatrix} -\dfrac{\partial \mathcal{H}}{\partial x_1} \\ -\dfrac{\partial \mathcal{H}}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -2x_1 + p_2 \\ -2x_2 - p_1 + 0.1p_2 \end{bmatrix} = \begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix}$$

$$\frac{\partial \mathcal{H}}{\partial \vec{u}} = 2u + p_2 = 0 \rightarrow u = -0.5p_2$$

There is four differential equation and four unknowns.

$$\dot{x}_1 = x_2 \tag{12}$$

$$\dot{x}_2 = -x_1 - 0.1x_2 - 0.5p_2 \tag{13}$$

$$\dot{p}_1 = -2x_1 + p_2 \tag{14}$$

$$\dot{p}_2 = -2x_2 - p_1 + 0.1p_2 \tag{15}$$