

Home Work #2

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1 Question 1

System:

$$\dot{x}(t) = -0.1x(t) + u(t)$$

Subjected to $0 \leq u(t) \leq M$

1.1 part a

$$J = \int_0^{100} -x(t) dt$$

Hamiltonian matrix:

$$\begin{aligned}\mathcal{H} &= g(\vec{x}(t), u(t), t) + \vec{p}(t)^T a(\vec{x}(t), u(t), t) \\ \mathcal{H} &= -x(t) - 0.1p(t)x(t) + p(t)u(t)\end{aligned}$$

Euler–Lagrange equation:

$$\dot{\vec{x}} = \frac{\partial \mathcal{H}}{\partial \vec{p}} = a(\vec{x}(t), u(t), t) \quad (1)$$

$$\dot{\vec{p}} = -\frac{\partial \mathcal{H}}{\partial \vec{x}} \quad (2)$$

Now we use above equation for solve problem.

$$-\frac{\partial \mathcal{H}}{\partial x} = 1 + 0.1p$$

There is two differential equation and two unknowns.

$$\dot{x} = -x(t) - 0.1px + pu \quad (3)$$

$$\dot{p} = 1 + 0.1p \quad (4)$$

Equation 4 solved in MATLAB(Q1.a.m) and code attached to file.

$$p(t) = C_1 * \exp(-t/10) - 10 \quad (5)$$

Final $x(t)$ is free so:

$$h_{\vec{x}} - \vec{p} = \vec{0} \Big|_{*, t_f} \rightarrow p(t_f) = 0$$

Use new boundry condition($p(t_f) = 0$) in equation 5 to find p function($p(t)$).

$$p(100) = C_1 * \exp(-100/10) - 10 = 0 \rightarrow C_1 = 10 \exp(10)$$

$$p(t) = 10 \exp(0.1(100 - t)) - 10 \quad (6)$$

2 Question 3

$$\ddot{x}(t) = -x(t) - 0.1\dot{x}(t) + u(t), \quad x(0) = \dot{x}(0) = 1$$

Assume:

$$x_1(t) = x(t), \quad x_2(t) = \dot{x}(t) \rightarrow a(\vec{x}, u, t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -x_1(t) - 0.1x_2(t) + u(t) \end{bmatrix}$$

$$J = \frac{1}{2}x^T(t_f)Hx(t_f) + \frac{1}{2} \int_0^{t_f} (\alpha(x^2 + \dot{x}^2) + \beta u^2) dt$$

2.1 part a

$\alpha = \beta = 1$, $t_f \rightarrow \infty$ and $H = 0$:

$$J = \frac{1}{2} \int_0^{t_f} (x^2 + \dot{x}^2 + u^2) dt$$

$$\vec{p}(t) = \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix}$$

Hamiltonian matrix:

$$\mathcal{H} = g(\vec{x}(t), u(t), t) + \vec{p}(t)^T a(\vec{x}(t), u(t), t)$$

$$\mathcal{H} = x_1^2(t) + x_2^2(t) + u^2(t) + \begin{bmatrix} p_1(t) & p_2(t) \end{bmatrix} \begin{bmatrix} x_2(t) \\ -x_1(t) - 0.1x_2(t) + u(t) \end{bmatrix}$$

$$\mathcal{H} = x_1^2(t) + x_2^2(t) + u^2(t) + p_1(t)x_2(t) - p_2(t)x_1(t) - 0.1p_2(t)x_2(t) + p_2(t)u(t)$$

Euler-Lagrange equation:

$$\dot{\vec{x}} = \frac{\partial \mathcal{H}}{\partial \vec{p}} = a(\vec{x}(t), u(t), t) \tag{7}$$

$$\dot{\vec{p}} = -\frac{\partial \mathcal{H}}{\partial \vec{x}} = \begin{bmatrix} -\frac{\partial \mathcal{H}}{\partial x_1} \\ -\frac{\partial \mathcal{H}}{\partial x_2} \end{bmatrix} \tag{8}$$

$$\vec{0} = \frac{\partial \mathcal{H}}{\partial \vec{u}} \tag{9}$$

Now we use above equation for solve problem.

$$\begin{bmatrix} -\frac{\partial \mathcal{H}}{\partial x_1} \\ -\frac{\partial \mathcal{H}}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -2x_1 + p_2 \\ -2x_2 - p_1 + 0.1p_2 \end{bmatrix} = \begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix}$$

$$\frac{\partial \mathcal{H}}{\partial \vec{u}} = 2u + p_2 = 0 \rightarrow u = -0.5p_2$$

There is four differential equation and four unknowns.

$$\dot{x}_1 = x_2 \tag{10}$$

$$\dot{x}_2 = -x_1 - 0.1x_2 - 0.5p_2 \tag{11}$$

$$\dot{p}_1 = -2x_1 + p_2 \tag{12}$$

$$\dot{p}_2 = -2x_2 - p_1 + 0.1p_2 \tag{13}$$