

# Home Work #3

Ali BaniAsad 96108378

July 9, 2021

## 1 Question 1

$$z = f(x, y) = y \sin(x + y) - x \sin(x - y)$$

Gradient of  $f(x, y)$ :

$$\vec{\nabla}f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}$$

$$\vec{\nabla}f = \begin{bmatrix} y \cos(x + y) - \sin(x - y) - x \cos(x - y) \\ y \cos(x + y) + \sin(x + y) + x \cos(x - y) \end{bmatrix}$$

### 1.1 part a

$$\vec{X}_0 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Tolerance is:  $10^{-7}$

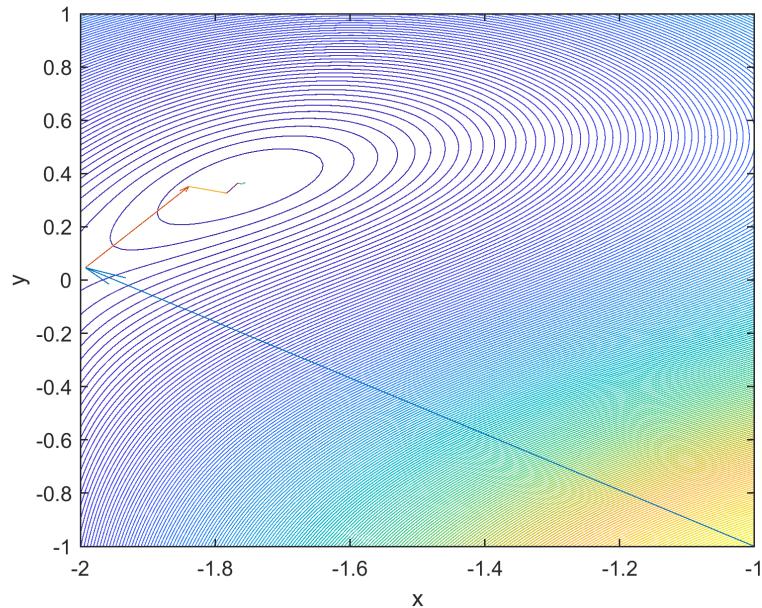
Answer is:

$$\vec{X}_{ans} = \begin{bmatrix} -1.7556 \\ 0.3655 \end{bmatrix}$$

#### 1.1.1 figures

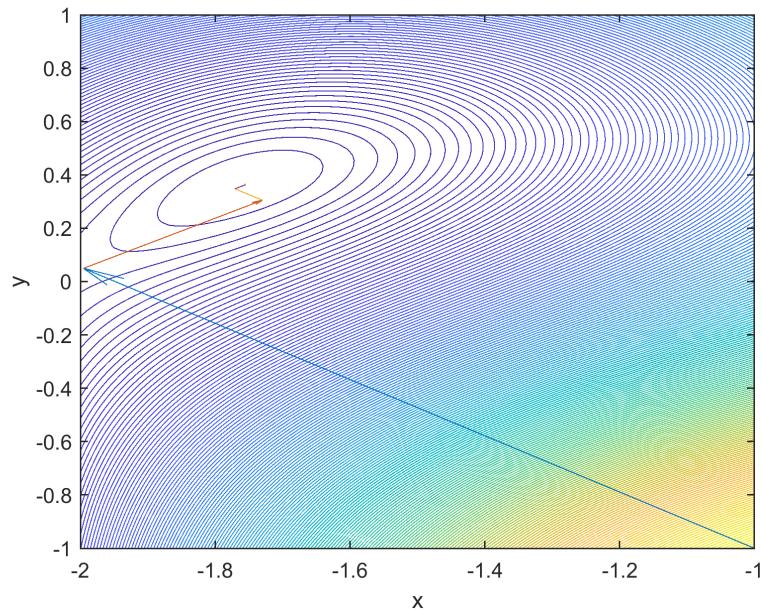
- Steepest Descent
  - Quadratic Interpolation

Figure 1: Steepest Descent and Quadratic Interpolation



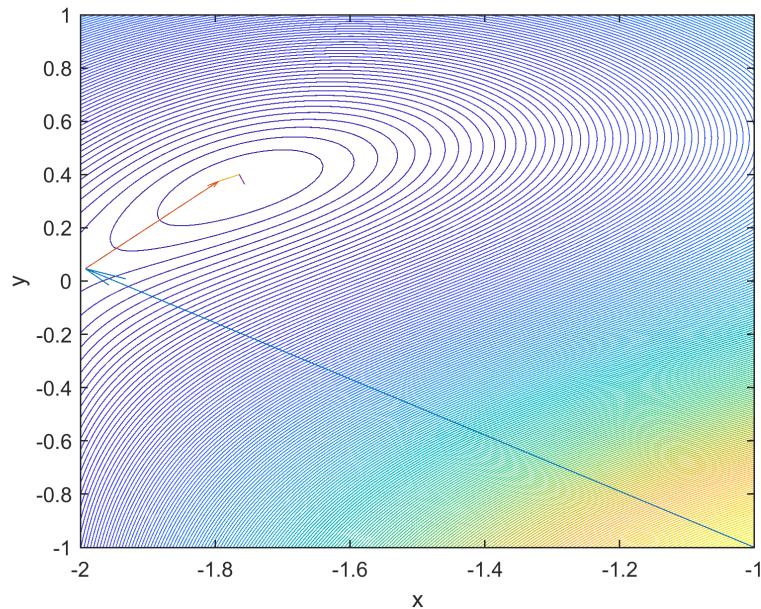
- Golden Section

Figure 2: Steepest Descent and Golden Section



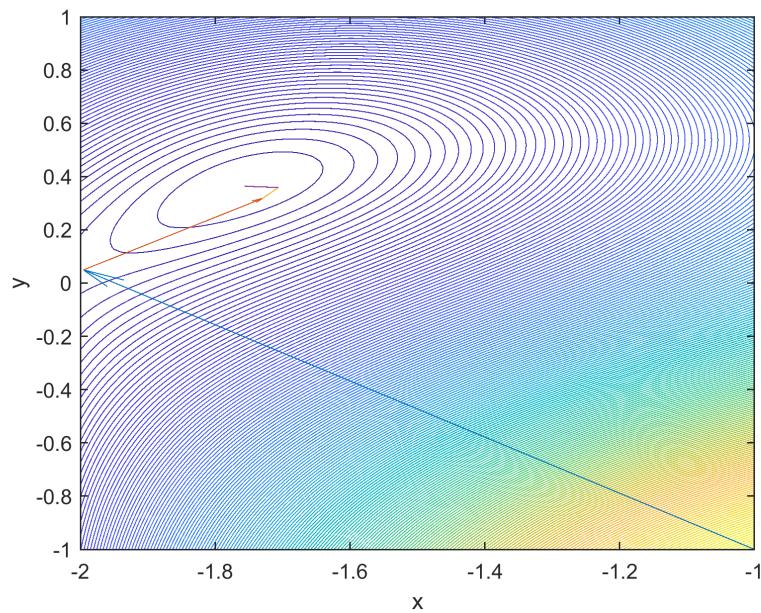
- BFGS
  - Quadratic Interpolation

Figure 3: BFGS and Quadratic Interpolation



- Golden Section

Figure 4: BFGS and Golden Section



### 1.1.2 result

- Time

Table 1: Time compare between four methods

Steepest Descent		BFGS	
Quadratic Interpolation	Golden Section	Quadratic Interpolation	Golden Section
0.238 sec	0.183 sec	0.164 sec	0.102 sec

- Number of Cost calculation

Table 2: Number of Cost calculation compare between four methods

Steepest Descent		BFGS	
Quadratic Interpolation	Golden Section	Quadratic Interpolation	Golden Section
360	336	242	213

- Number of Gradient calculation

Table 3: Number of Gradient calculation compare between four methods

Steepest Descent		BFGS	
Quadratic Interpolation	Golden Section	Quadratic Interpolation	Golden Section
19	13	13	9

### 1.2 part b

$$\vec{X}_0 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

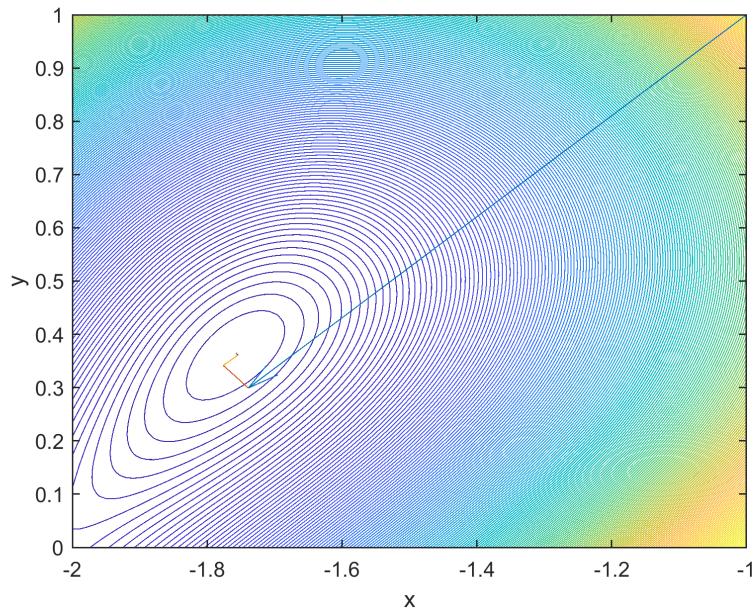
Tolerance is:  $10^{-7}$

$$\vec{X}_{ans} = \begin{bmatrix} -1.7556 \\ 0.3655 \end{bmatrix}$$

#### 1.2.1 figures

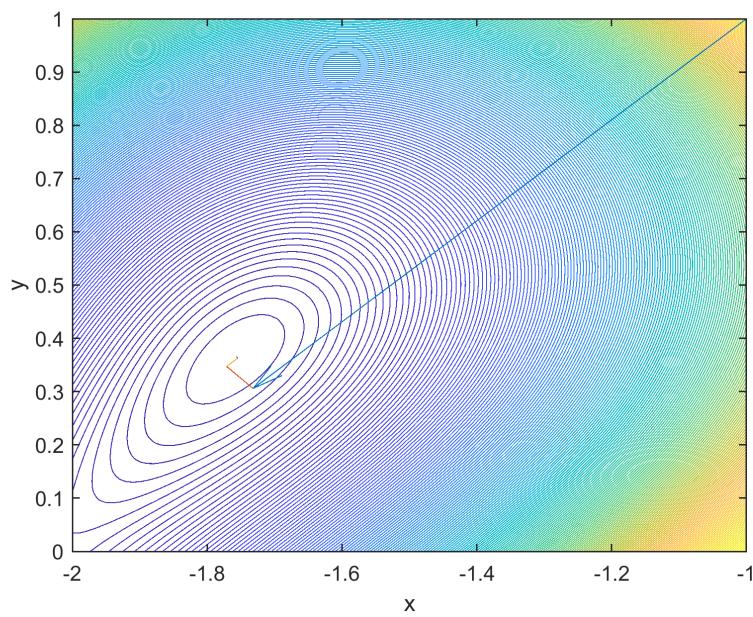
- Steepest Descent
  - Quadratic Interpolation

Figure 5: Steepest Descent and Quadratic Interpolation



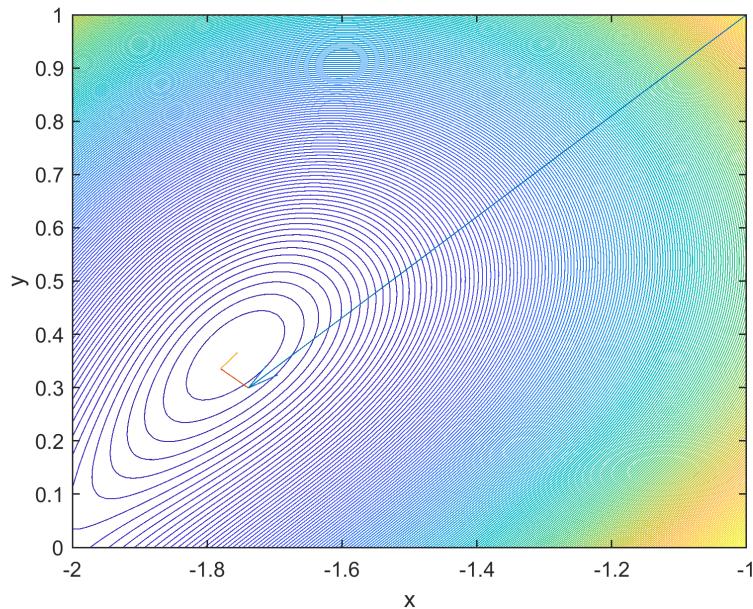
- Golden Section

Figure 6: Steepest Descent and Golden Section



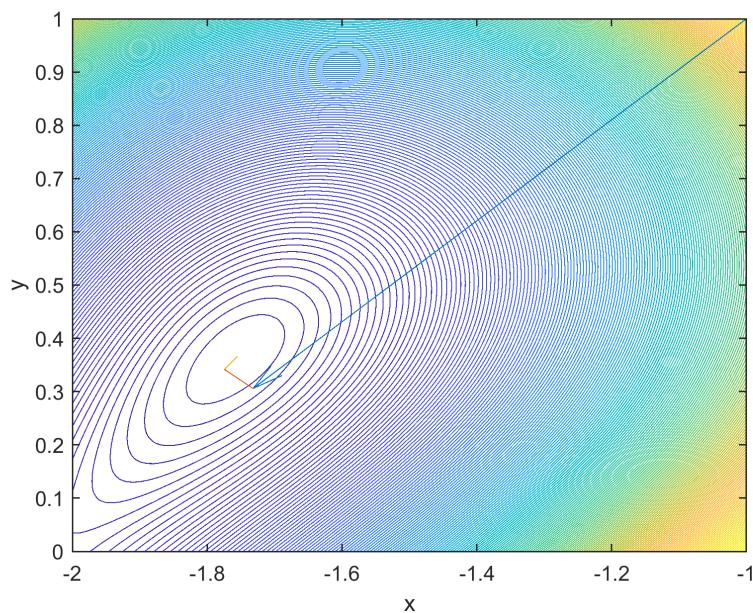
- BFGS
  - Quadratic Interpolation

Figure 7: BFGS and Quadratic Interpolation



- Golden Section

Figure 8: BFGS and Golden Section



### 1.2.2 result

- Time

Table 4: Time compare between four methods

Steepest Descent		BFGS	
Quadratic Interpolation	Golden Section	Quadratic Interpolation	Golden Section
0.208 sec	0.146 sec	0.106 sec	0.142 sec

- Number of Cost calculation

Table 5: Number of Cost calculation compare between four methods

Steepest Descent		BFGS	
Quadratic Interpolation	Golden Section	Quadratic Interpolation	Golden Section
246	285	142	142

- Number of Gradient calculation

Table 6: Number of Gradient calculation compare between four methods

Steepest Descent		BFGS	
Quadratic Interpolation	Golden Section	Quadratic Interpolation	Golden Section
14	12	7	7

## 2 Question 2

### 2.1 part a

Gradient tolerance is:  $10^{-4}$

#### 2.1.1 System

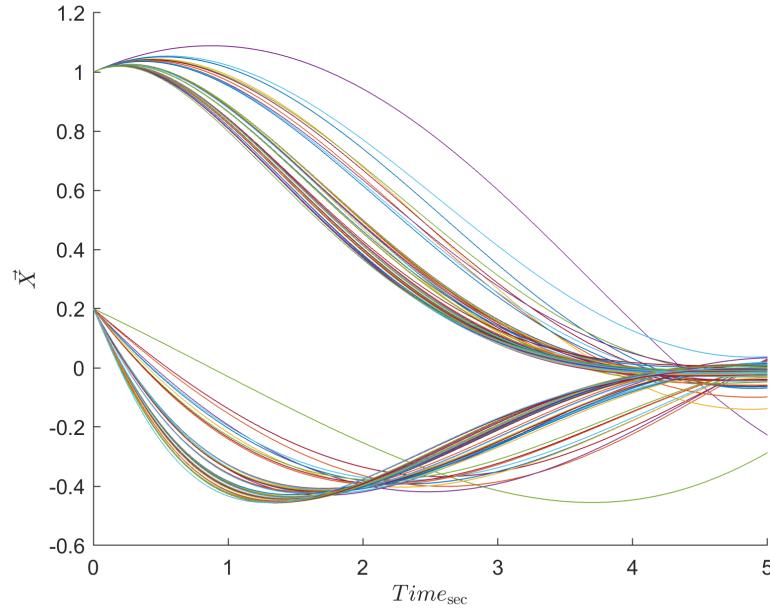
#### 2.2 System

$$\begin{aligned} \ddot{x}(t) &= -x(t) - 0.1\dot{x}(t) + u \\ \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ 0 & -0.1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ A = \begin{bmatrix} -1 & 0 \\ 0 & -0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = 1, \quad H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \end{aligned}$$

#### 2.2.1 figures

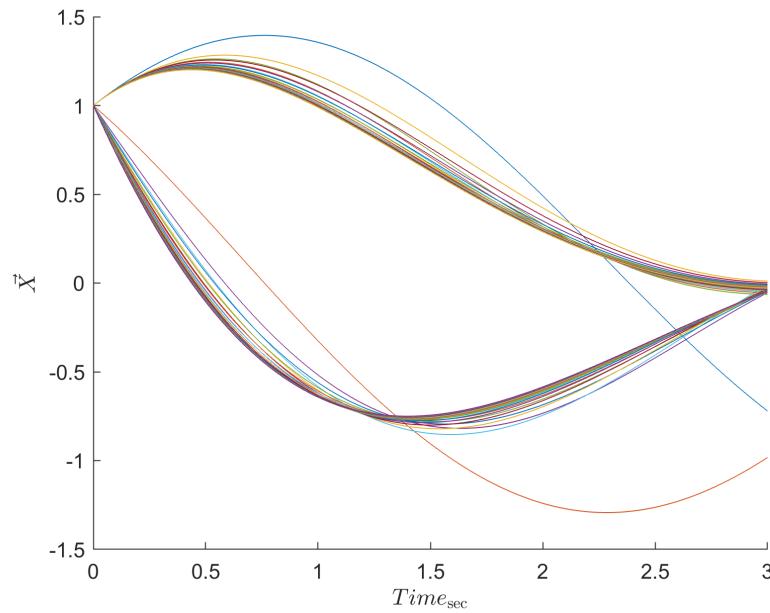
- Steepest Descent
  - Quadratic Interpolation

Figure 9: Steepest Descent and Quadratic Interpolation



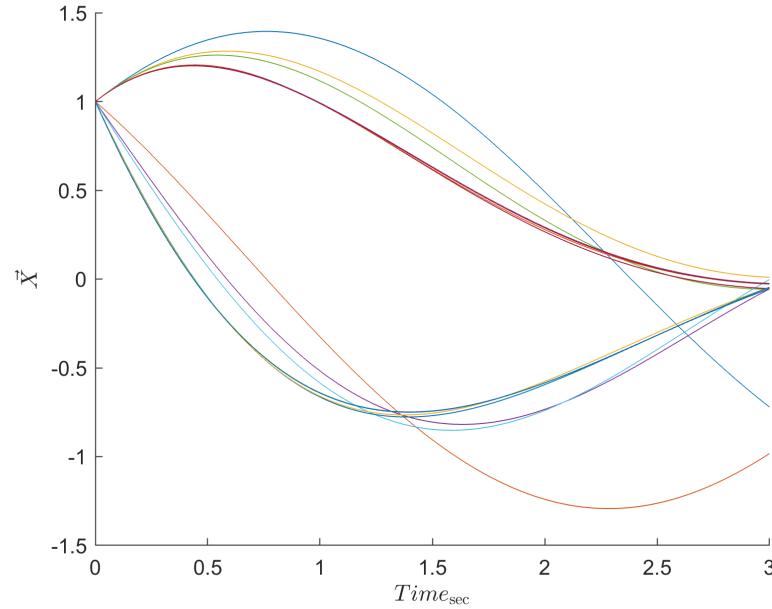
- Golden Section

Figure 10: Steepest Descent and Golden Section



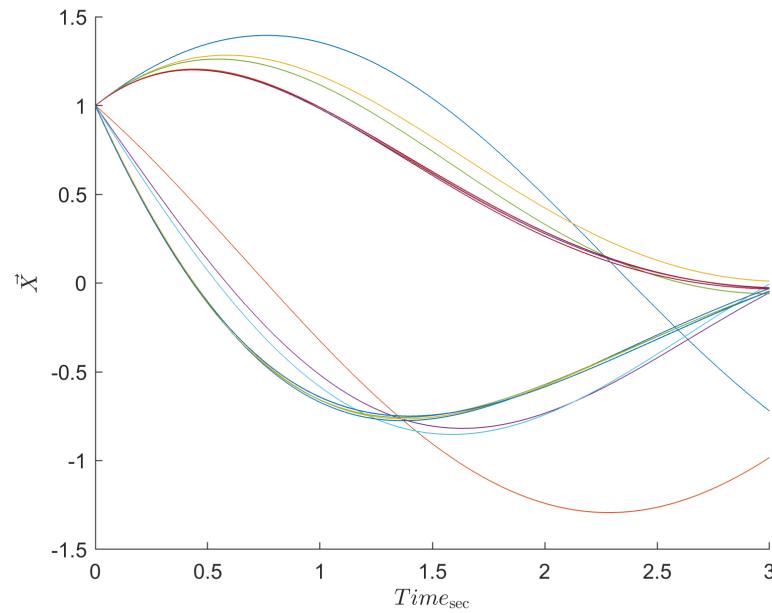
- BFGS
  - Quadratic Interpolation

Figure 11: BFGS and Quadratic Interpolation



- Golden Section

Figure 12: BFGS and Golden Section



### 2.2.2 result

- Time

Table 7: Time compare between four methods

Steepest Descent		BFGS	
Quadratic Interpolation	Golden Section	Quadratic Interpolation	Golden Section
17.000 sec	24.353 sec	3.905 sec	4.985 sec

- Number of Cost calculation

Table 8: Number of Cost calculation compare between four methods

Steepest Descent		BFGS	
Quadratic Interpolation	Golden Section	Quadratic Interpolation	Golden Section
1285	1922	273	373

- Number of Gradient calculation

Table 9: Number of Gradient calculation compare between four methods

Steepest Descent		BFGS	
Quadratic Interpolation	Golden Section	Quadratic Interpolation	Golden Section
51	51	11	11

### 2.2.3 Four iteration for BFGS and Quadratic interpolation

Figure 13: BFGS and Quadratic Interpolation with four iteration

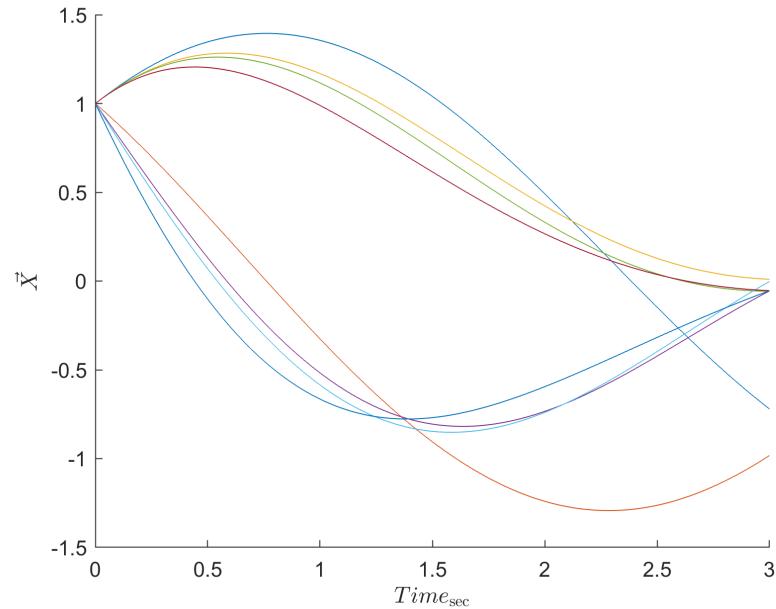


Table 10: four iteration and gradient tolerance compare

	Time	Number of Cost calculation	Number of Gradient calculation
Four iteration	3.905 <sub>sec</sub>	273	11
Gradient tolerance	1.586 <sub>sec</sub>	100	4

## 2.3 part b

Tolerance is:  $10^{-16}$  for  $\lambda S_i$  or  $10^{-4}$  for norm of gradient.

### 2.3.1 figures

- Steepest Descent
  - Quadratic Interpolation

Figure 14: Steepest Descent and Quadratic Interpolation

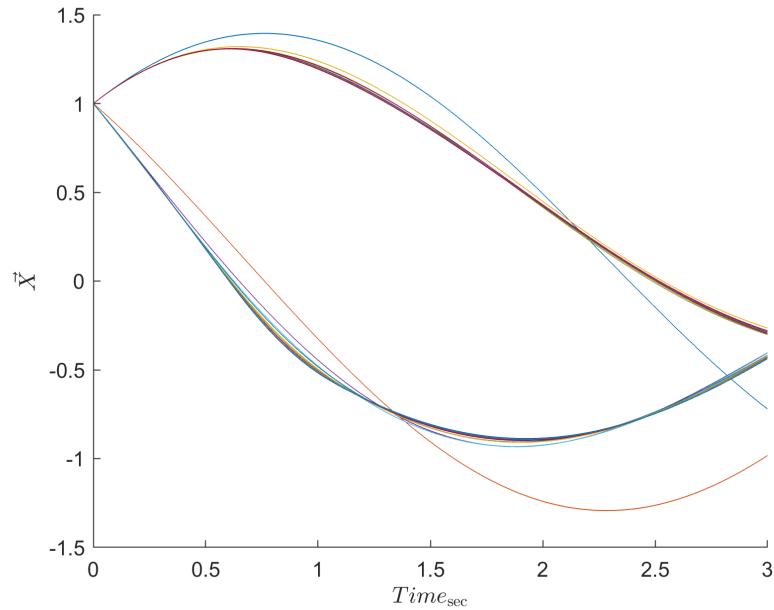
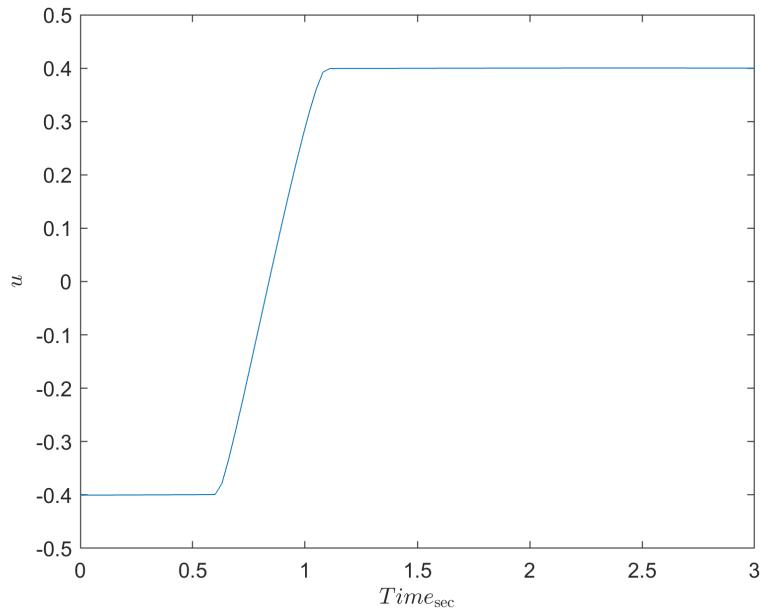


Figure 15: Steepest Descent and Quadratic Interpolation Control



- Golden Section

Figure 16: Steepest Descent and Golden Section

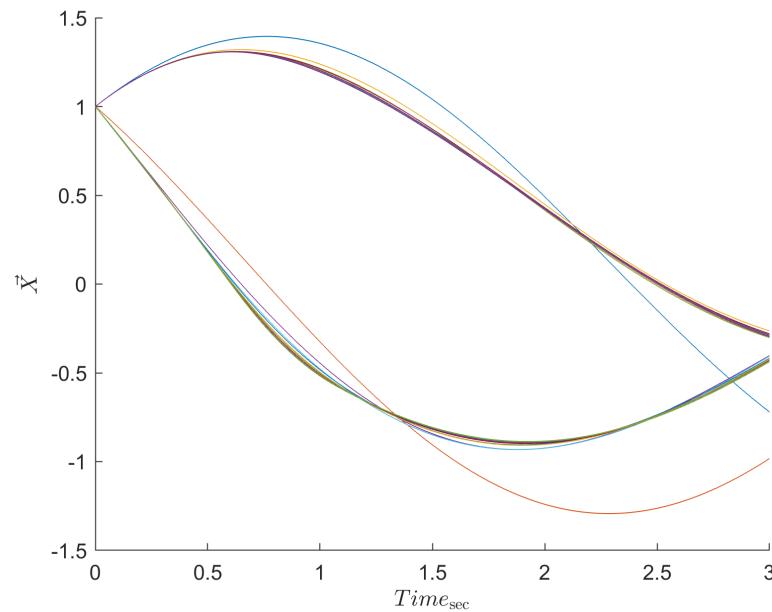
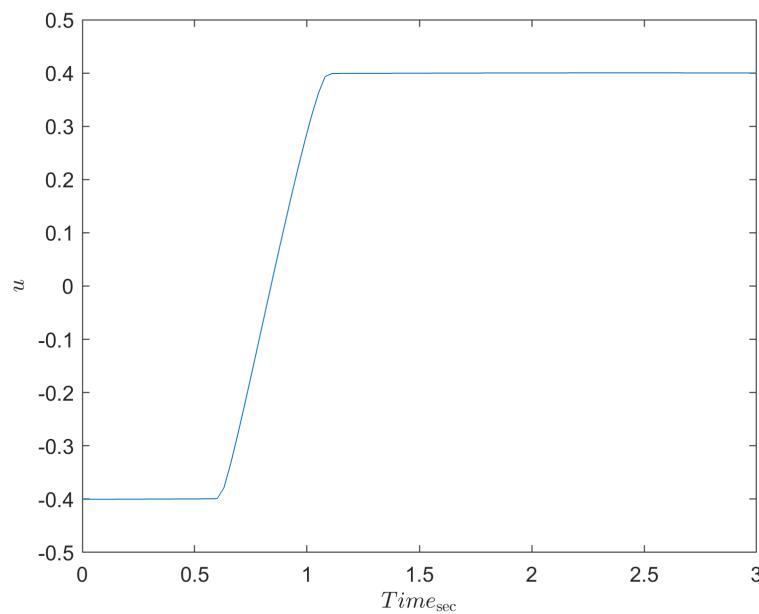


Figure 17: Steepest Descent and Golden Section Control



- BFGS
  - Quadratic Interpolation

Figure 18: BFGS and Quadratic Interpolation

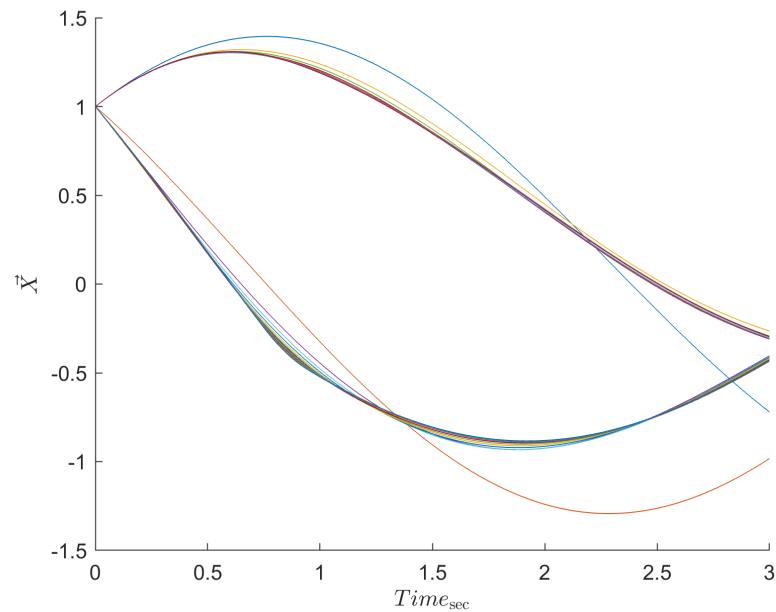
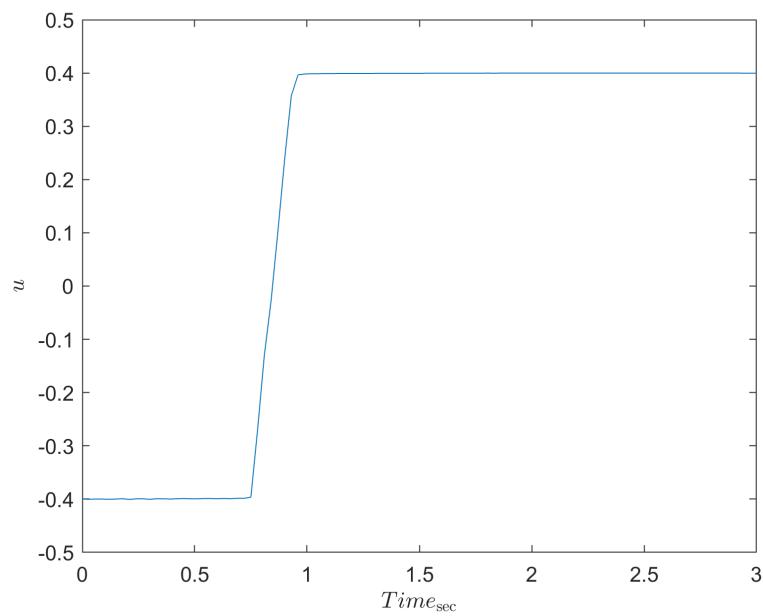


Figure 19: BFGS and Quadratic Interpolation Control



– Golden Section

Figure 20: BFGS and Golden Section

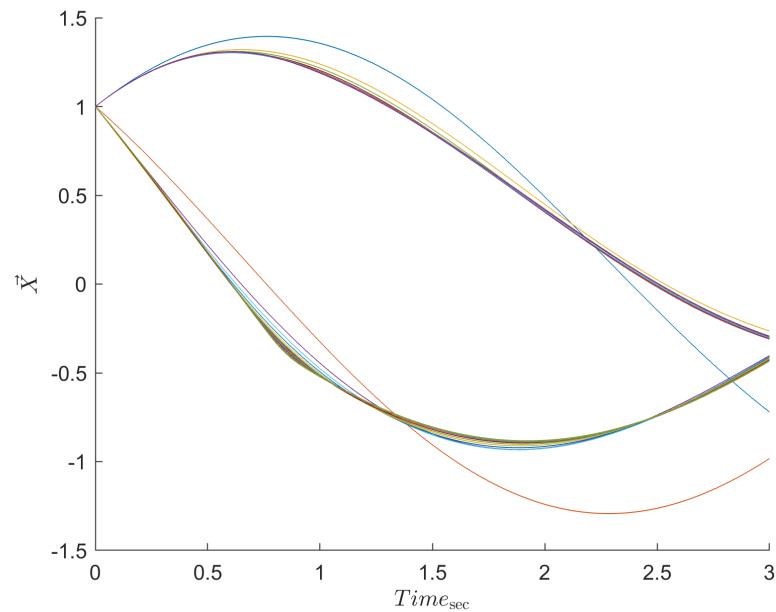
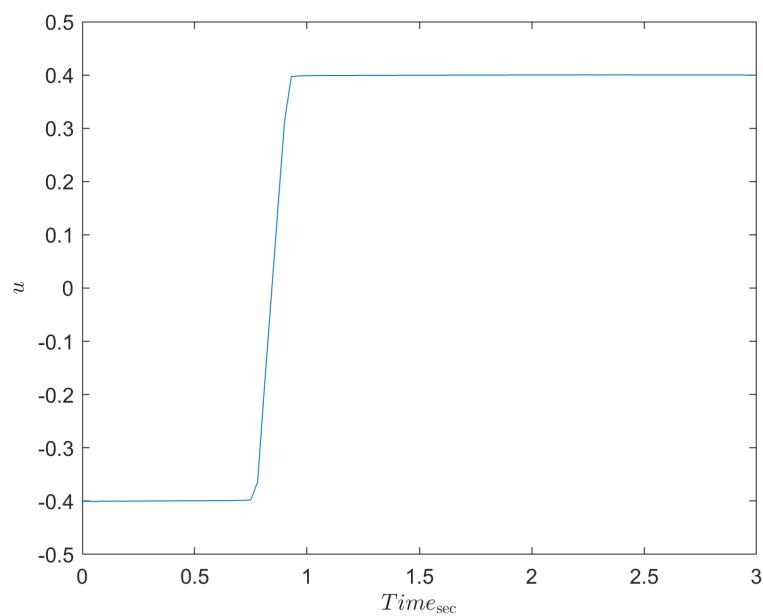


Figure 21: BFGS and Golden Section Control



### 2.3.2 result

- Time

Table 11: Time compare between four methods

Steepest Descent		BFGS	
Quadratic Interpolation	Golden Section	Quadratic Interpolation	Golden Section
7.595 sec	33.761 sec	86.730 sec	72.666 sec

- Number of Cost calculation

Table 12: Number of Cost calculation compare between four methods

Steepest Descent		BFGS	
Quadratic Interpolation	Golden Section	Quadratic Interpolation	Golden Section
442	1782	2787	2378

- Number of Gradient calculation

Table 13: Number of Gradient calculation compare between four methods

Steepest Descent		BFGS	
Quadratic Interpolation	Golden Section	Quadratic Interpolation	Golden Section
25	55	256	174

### 3 Question 3

#### 3.1 System

$$\begin{aligned} \dot{x}_1 &= -x_1 + u \\ \dot{x}_2 &= -2x_2 + 2u \\ \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u \\ A &= \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{aligned}$$

#### 3.2 Direct Optimization

Time is free so we have a new unknowns. we use  $t_f$  as a new state and change some parameter that will describe here.

$$J = t_f^2, \quad 0 \leq t \leq t_f, \quad \tau = \frac{t}{t_f}, \quad 0 \leq \tau \leq 1$$

$$\vec{a}_N(\vec{x}(t), \vec{u}(t), t_f, t) = t_f \vec{a}(\vec{x}(t), \vec{u}(t), t)$$

$$g_N(\vec{x}(t), \vec{u}(t), t_f, t) = t_f g(\vec{x}(t), \vec{u}(t), t)$$

$$\mathcal{H} = g_N(\vec{x}(t), \vec{u}(t), t_f, t) + P^T \vec{d}_N(\vec{x}(t), \vec{u}(t), t_f, t)$$

$$G_1(u) = \begin{cases} -\frac{1}{g_1(u)} & g_1(u) \leq \epsilon \\ -\frac{1}{\epsilon} \left( 3 - \frac{3g_1(u)}{\epsilon} + \left( \frac{g_1(u)}{\epsilon} \right)^2 \right) & g_1(u) > \epsilon \end{cases}$$

$$G'_1(u) = \begin{cases} \frac{1}{(u-)^2} & g_1(u) \leq \epsilon \\ -\frac{1}{\epsilon} \left( -\frac{3}{\epsilon} + \frac{2u-2}{\epsilon^2} \right) & g_1(u) > \epsilon \end{cases}$$

$$G_2(x_2) = \begin{cases} -\frac{1}{g_2(u)} & g_2(u) \leq \epsilon \\ -\frac{1}{\epsilon} \left( 3 - \frac{3g_2(u)}{\epsilon} + \left( \frac{g_2(u)}{\epsilon} \right)^2 \right) & g_2(u) > \epsilon \end{cases}$$

$$G'_2(u) = \begin{cases} \frac{1}{(u+1)^2} & g_2(u) \leq \epsilon \\ -\frac{1}{\epsilon} \left( \frac{3}{\epsilon} + \frac{2u+2}{\epsilon^2} \right) & g_2(u) > \epsilon \end{cases}$$

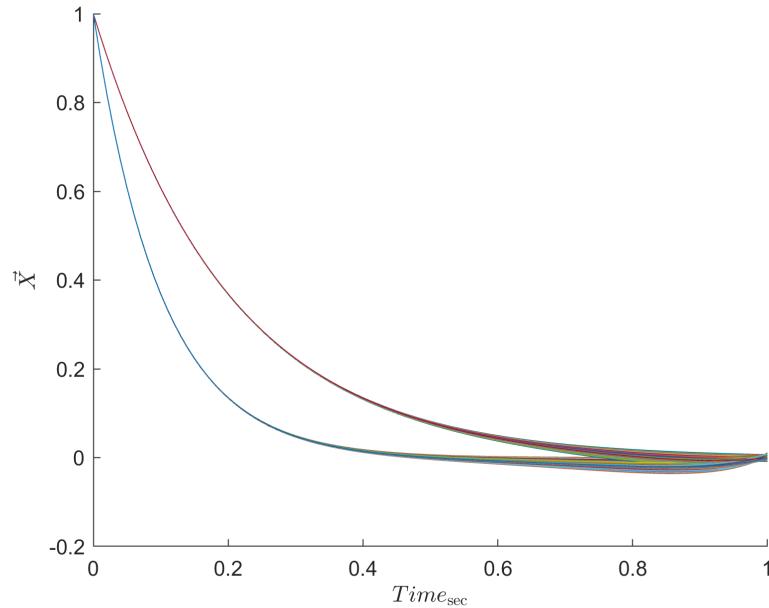
$$g_N(\vec{x}(t), \vec{u}(t), t_f, t) = t_f^2 + r_k G(u)$$

$$\frac{\partial J}{\partial t_f} = \frac{\partial h}{\partial t_f} + \int_0^1 \frac{\partial \mathcal{H}}{\partial t_f}$$

$$\frac{\partial J}{\partial \vec{X}} = \begin{bmatrix} \frac{\mathcal{H}}{\partial u} \Big|_{\tau_0} \\ \frac{\mathcal{H}}{\partial u} \Big|_{\tau_1} \\ \frac{\mathcal{H}}{\partial u} \Big|_{\tau_2} \\ \vdots \\ \frac{\mathcal{H}}{\partial u} \Big|_{\tau_f} \\ \frac{\partial h}{\partial t_f} + \int_0^1 \frac{\partial \mathcal{H}}{\partial t_f} \end{bmatrix}$$

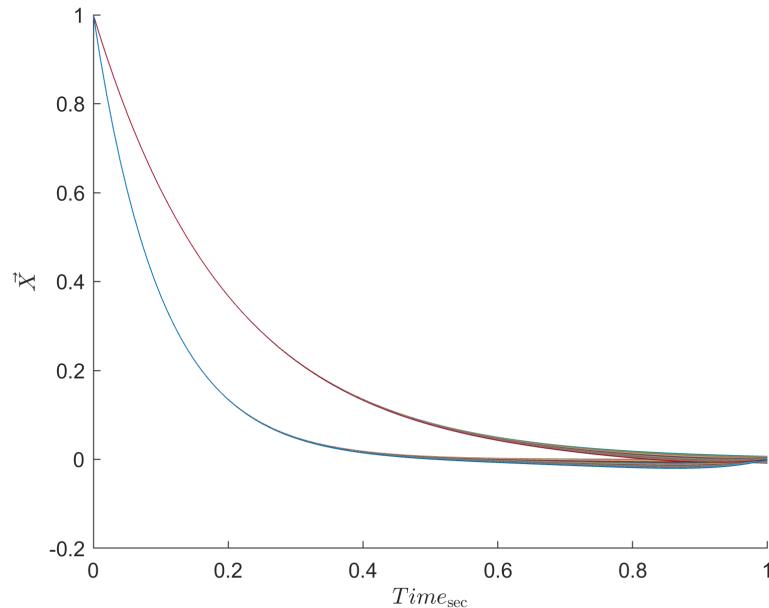
- Steepest Descent
  - Quadratic Interpolation

Figure 22: Steepest Descent and Quadratic Interpolation



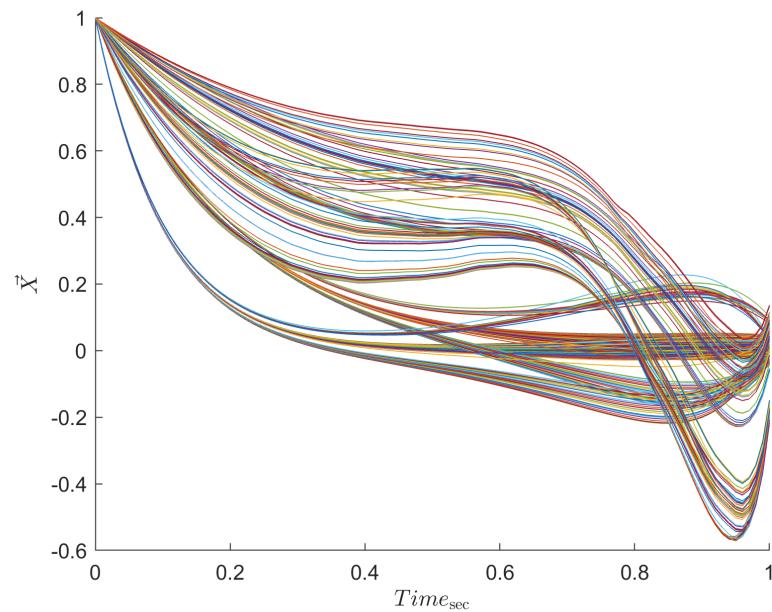
– Golden Section

Figure 23: Steepest Descent and Golden Section



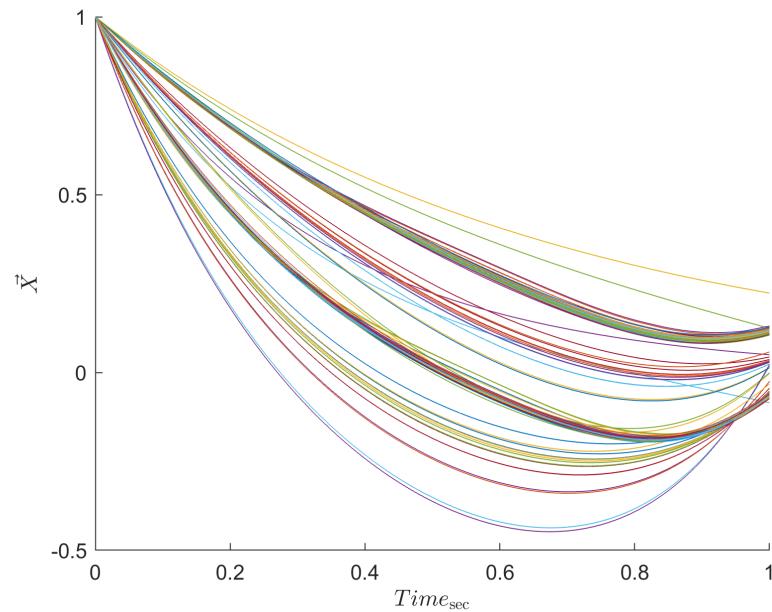
- BFGS
  - Quadratic Interpolation

Figure 24: BFGS and Quadratic Interpolation



– Golden Section

Figure 25: BFGS and Golden Section



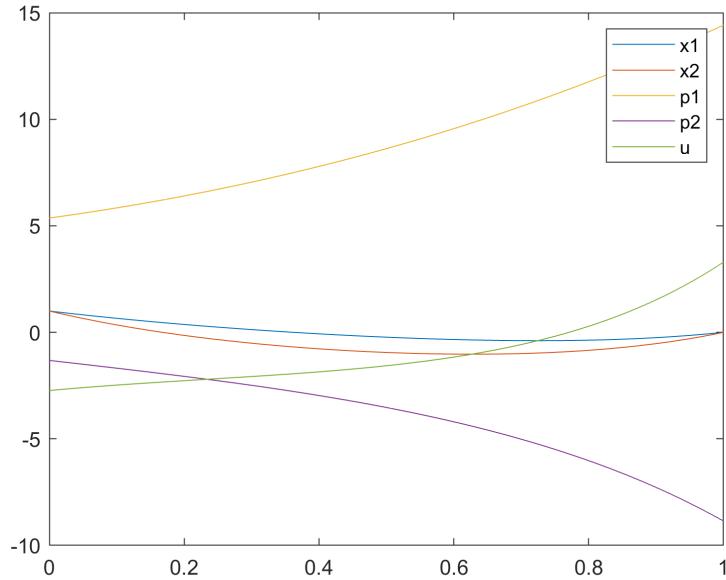
### 3.3 Shooting method

Final time is free so:

$$\vec{F} = \begin{bmatrix} x_1(t_f) - x_{1f} \\ x_2(t_f) - x_{2f} \\ (\mathcal{H} - h_t)|_{t_f} \end{bmatrix}$$

$$\vec{y}_{k+1} = \vec{y}_k - \frac{\partial \vec{F}}{\partial \vec{y}} \Big|_{\vec{y}_k} \vec{F}(\vec{y}_k)$$

Figure 26: Shooting method



## 4 Question 4

$$a = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -0.4x_1 - 0.2x_2^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$G_1(u) = \begin{cases} -\frac{1}{g_1(u)} & g_1(u) \leq \epsilon \\ -\frac{1}{\epsilon} \left( 3 - \frac{3g_1(u)}{\epsilon} + \left( \frac{g_1(u)}{\epsilon} \right)^2 \right) & g_1(u) > \epsilon \end{cases}$$

$$G'_1(u) = \begin{cases} \frac{1}{(u - 0.8)^2} & g_1(u) \leq \epsilon \\ -\frac{1}{\epsilon} \left( -\frac{3}{\epsilon} + \frac{2u - 1.6}{\epsilon^2} \right) & g_1(u) > \epsilon \end{cases}$$

$$G_2(x_2) = \begin{cases} -\frac{1}{g_2(u)} & g_2(u) \leq \epsilon \\ -\frac{1}{\epsilon} \left( 3 - \frac{3g_2(u)}{\epsilon} + \left( \frac{g_2(u)}{\epsilon} \right)^2 \right) & g_2(u) > \epsilon \end{cases}$$

$$G'_2(u) = \begin{cases} \frac{1}{(u+0.8)^2} & g_2(u) \leq \epsilon \\ -\frac{1}{\epsilon} \left( \frac{3}{\epsilon} + \frac{2u+1.6}{\epsilon^2} \right) & g_2(u) > \epsilon \end{cases}$$

$$\epsilon = -c(r_k)^2, \quad a = 0.5, \quad r_{k+1} = cr_k, \quad c = 0.9, \quad \min(r_k) = 0.001$$

$$\mathcal{H} = \vec{P}^T a(\vec{X}, u, t) + \frac{1}{2} (x_1^2 + x_2^2 + u^2 + r_k G(u))$$

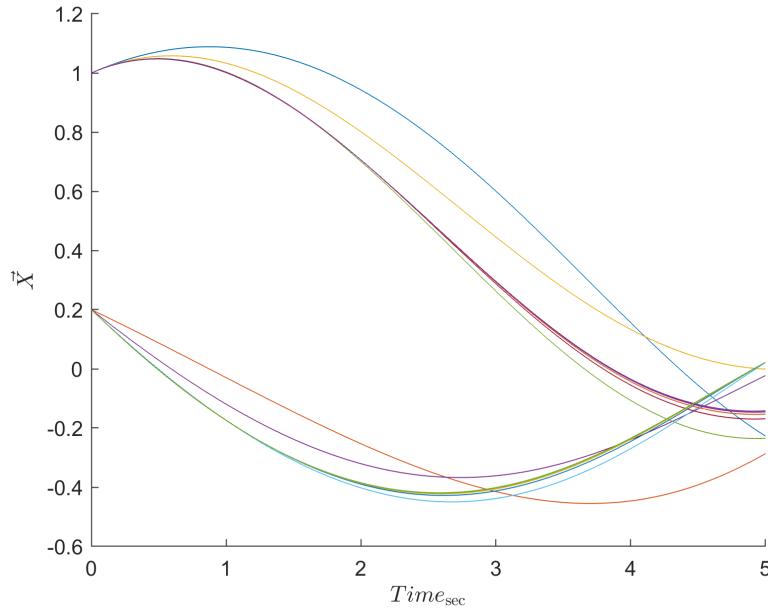
$$\dot{\vec{P}} = -\frac{\partial \mathcal{H}}{\partial \vec{X}} = \begin{bmatrix} -x_1 + 0.4p_2 \\ -x_2 - p_1 + 0.4p_2 x_2 \end{bmatrix}$$

$$\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} -x_1 + 0.4p_2 \\ x_2(0.4p_2 - 1) - p_1 \end{bmatrix}$$

#### 4.1 part a

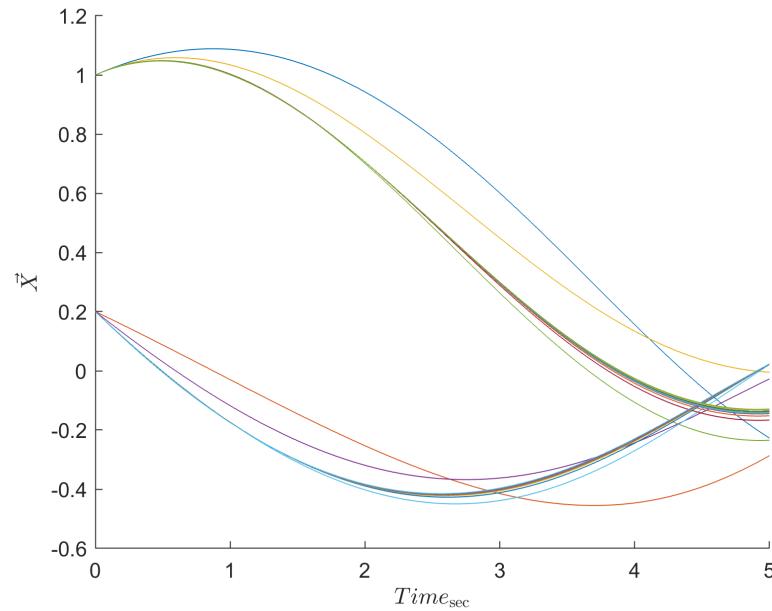
- Steepest Descent
  - Quadratic Interpolation

Figure 27: Steepest Descent and Quadratic Interpolation



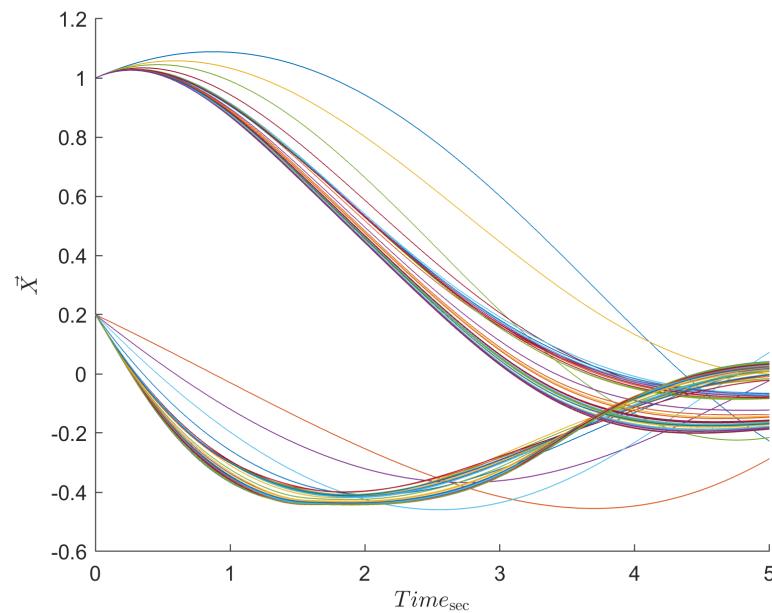
- Golden Section

Figure 28: Steepest Descent and Golden Section



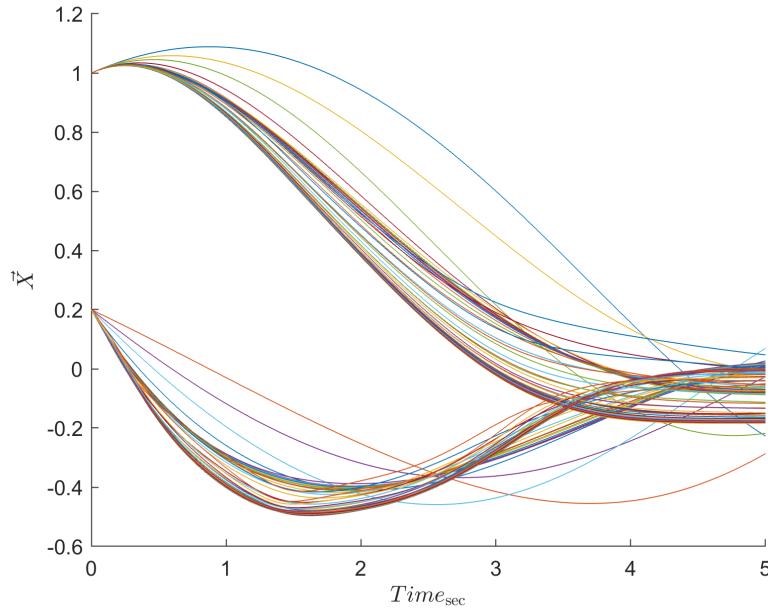
- BFGS
  - Quadratic Interpolation

Figure 29: BFGS and Quadratic Interpolation



– Golden Section

Figure 30: BFGS and Golden Section

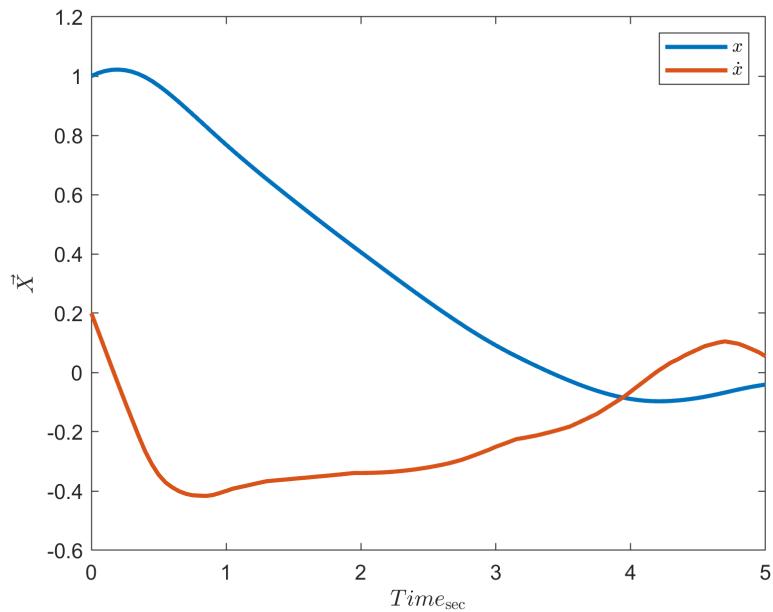


## 4.2 part c

$$\begin{aligned} a &= \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -0.4x_1 - 0.2x_2^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} &= \begin{bmatrix} x_2(k) \\ -0.4x_1(k) - 0.2x_2^2(k) + u(k) \end{bmatrix} \Delta t + \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \end{aligned}$$

In MATLAB Code Control will save in mat file and we can use for another initial condition very fast without so much processing.

Figure 31: Dynamic Programming



## Contents

<b>1 Question 1</b>	<b>1</b>
1.1 part a . . . . .	1
1.1.1 figures . . . . .	1
1.1.2 result . . . . .	4
1.2 part b . . . . .	4
1.2.1 figures . . . . .	4
1.2.2 result . . . . .	7
<b>2 Question 2</b>	<b>7</b>
2.1 part a . . . . .	7
2.1.1 System . . . . .	7
2.2 System . . . . .	7
2.2.1 figures . . . . .	7
2.2.2 result . . . . .	10
2.2.3 Four iteration for BFGS and Quadratic interpolation . . . . .	11
2.3 part b . . . . .	11
2.3.1 figures . . . . .	11
2.3.2 result . . . . .	15
<b>3 Question 3</b>	<b>16</b>
3.1 System . . . . .	16
3.2 Direct Optimization . . . . .	16
3.3 Shooting method . . . . .	20
<b>4 Question 4</b>	<b>20</b>
4.1 part a . . . . .	21
4.2 part c . . . . .	23

## List of Figures

1	Steepest Descent and Quadratic Interpolation . . . . .	2
2	Steepest Descent and Golden Section . . . . .	2
3	BFGS and Quadratic Interpolation . . . . .	3
4	BFGS and Golden Section . . . . .	3
5	Steepest Descent and Quadratic Interpolation . . . . .	5
6	Steepest Descent and Golden Section . . . . .	5
7	BFGS and Quadratic Interpolation . . . . .	6
8	BFGS and Golden Section . . . . .	6
9	Steepest Descent and Quadratic Interpolation . . . . .	8
10	Steepest Descent and Golden Section . . . . .	8
11	BFGS and Quadratic Interpolation . . . . .	9
12	BFGS and Golden Section . . . . .	9
13	BFGS and Quadratic Interpolation with four iteration . . . . .	11
14	Steepest Descent and Quadratic Interpolation . . . . .	12
15	Steepest Descent and Quadratic Interpolation Control . . . . .	12
16	Steepest Descent and Golden Section . . . . .	13
17	Steepest Descent and Golden Section Control . . . . .	13
18	BFGS and Quadratic Interpolation . . . . .	14
19	BFGS and Quadratic Interpolation Control . . . . .	14
20	BFGS and Golden Section . . . . .	15
21	BFGS and Golden Section Control . . . . .	15
22	Steepest Descent and Quadratic Interpolation . . . . .	18
23	Steepest Descent and Golden Section . . . . .	18
24	BFGS and Quadratic Interpolation . . . . .	19
25	BFGS and Golden Section . . . . .	19
26	Shooting method . . . . .	20
27	Steepest Descent and Quadratic Interpolation . . . . .	21
28	Steepest Descent and Golden Section . . . . .	22
29	BFGS and Quadratic Interpolation . . . . .	22
30	BFGS and Golden Section . . . . .	23
31	Dynamic Programming . . . . .	24

**List of Tables**

1	Time compare between four methods . . . . .	4
2	Number of Cost calculation compare between four methods . . . . .	4
3	Number of Gradient calculation compare between four methods . . . . .	4
4	Time compare between four methods . . . . .	7
5	Number of Cost calculation compare between four methods . . . . .	7
6	Number of Gradient calculation compare between four methods . . . . .	7
7	Time compare between four methods . . . . .	10
8	Number of Cost calculation compare between four methods . . . . .	10
9	Number of Gradient calculation compare between four methods . . . . .	10
10	four iteration and gradient tolerance compare . . . . .	11
11	Time compare between four methods . . . . .	16
12	Number of Cost calculation compare between four methods . . . . .	16
13	Number of Gradient calculation compare between four methods . . . . .	16