Home Work #2

Ali BaniAsad 96108378

May 13, 2021

1 Question 1

System:

$$\dot{x}(t) = -0.1x(t) + u(t)$$

Subjected to $0 \leq u(t) \leq M$

1.1 part a

$$J = \int_0^{100} -x(t)dt$$

Hamiltonian matrix:

$$\mathcal{H} = g(\vec{x}(t), u(t), t) + \vec{p}(t)^T a(\vec{x}(t), u(t), t)$$
$$\mathcal{H} = -x(t) - 0.1p(t)x(t) + p(t)u(t) \tag{1}$$

Euler–Lagrange equation:

$$\dot{\vec{x}} = \frac{\partial \mathcal{H}}{\partial \vec{p}} = a(\vec{x}(t), u(t), t) \tag{2}$$
$$\dot{\vec{p}} = -\frac{\partial \mathcal{H}}{\partial \vec{x}} \tag{3}$$

Now we use above equation for solve problem.

$$-\frac{\partial \mathcal{H}}{\partial x} = 1 + 0.1p$$

There is two differential equation and two unknowns.

$$\dot{x} = -x(t) - 0.1px + pu \tag{4}$$
$$\dot{p} = 1 + 0.1p \tag{5}$$

Equation 5 solved in MATLAB(Q1_a.m) and code attached to file.

$$p(t) = C_1 \exp(t/10) - 10 \tag{6}$$

Final $x(t)$ is free so:

$$h_{\vec{x}} - \vec{p} = \vec{0}\Big|_{*,t_f} \rightarrow p(t_f) = 0$$

Use new boundry condition($p(t_f) = 0$) in equation 6 to find p function($p(t)$).

$$p(100) = C_1 \exp(100/10) - 10 = 0 \rightarrow C_1 = 10 \exp(-10)$$

$$p(t) = 10\exp\left(0.1(t-100)\right) - 10 \tag{7}$$

We know that $u(t)$ has limit so for optimization we have another condition to select $u(t)$ for every time.

$$u(t) = \begin{cases} \dfrac{\partial \mathcal{H}}{\partial u} & < 0 \quad u(t) = M \\ \dfrac{\partial \mathcal{H}}{\partial u} & = 0 \quad \mathcal{H} \text{ is not a function of } u(t) \\ \dfrac{\partial \mathcal{H}}{\partial u} & > 0 \quad u(t) = 0 \end{cases} \tag{8}$$

From equation 1 we calculate $\dfrac{\partial \mathcal{H}}{\partial u}$.

$$\frac{\partial \mathcal{H}}{\partial u} = p(t)$$

From equation 7 we know that at $t_0 \to t_f$ $p(t)$ is less than zero($p(t) < 0$), so $u(t)$ for every time is M.

1.2 part b

$$J = \int_0^{100} -x(t)dt$$

Subjected to:

$$\int_0^{100} u(t)dt = K(\text{a known constant})$$

$z(t)$ is new state:

$$z(t) = \int_0^t u(t)dt \to \frac{dz}{dt} = u(t)$$

New differential constraints:

$$\frac{dz}{dt} - u(t) = 0$$

$$g_a(x, \dot{x}, \dot{z}, t, \lambda) = g(x, \dot{x}, t) + \lambda(t) f(x, \dot{x}, \dot{z}, t)$$

$$g_a = -x(t) + \lambda(\dot{z}(t) - u(t))$$

Hamiltonian matrix:

$$\mathcal{H} = g_a(\vec{x}(t), u(t), z(t), \lambda, t) + \vec{p}(t)^T a(\vec{x}(t), u(t), t)$$

We assume $p_2 = \lambda$

$$\mathcal{H} = -x(t) - 0.1p_1(t)x(t) + p_1(t)u(t) + p_2(t)(\dot{z}(t) - u(t)) \tag{9}$$

$$\dot{\vec{p}} = -\frac{\partial \mathcal{H}}{\partial \vec{x}} = \begin{bmatrix} -\dfrac{\partial \mathcal{H}}{\partial x} \\ -\dfrac{\partial \mathcal{H}}{\partial z} \end{bmatrix} = \begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix}$$

$$\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} 0.1p_1 + 1 \\ 0 \end{bmatrix}$$

Above equation solved in previous part.

$$\begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} = \begin{bmatrix} 10\exp\left(0.1(t-100)\right) - 10 \\ C_1 \end{bmatrix} \tag{10}$$

In this part we have the same condition that described in previous in equation 8. From equation 9 we calculate $\frac{\partial \mathcal{H}}{\partial u}$.

$$\frac{\partial \mathcal{H}}{\partial u} = p_1 - p_2 = (10\exp\left(0.1(t-100)\right) - 10) - C_1$$

From equation 10 we know that C_1 is constant.

There is four scenario for this problem.

1. For all the time($t_0 \to t_f$) $\frac{\partial \mathcal{H}}{\partial u} > 0$ so $u(t) = 0$,this scenario maybe possible if $K = 0$.
2. For all the time($t_0 \to t_f$) $\frac{\partial \mathcal{H}}{\partial u} < 0$ so $u(t) = M$,this scenario maybe possible if $K = M \times t_f$.
3. For time($t_0 \to t$) $\frac{\partial \mathcal{H}}{\partial u} < 0$ and for time($t \to t_f$) $\frac{\partial \mathcal{H}}{\partial u} > 0$ so for time($t_0 \to t$), $u(t) = M$ and for time($t \to t_f$), $u(t) = 0$ and this scenario maybe possible for $0 \leq K \leq M \times t_f$.
4. For time($t_0 \to t$) $\frac{\partial \mathcal{H}}{\partial u} > 0$ and for time($t \to t_f$) $\frac{\partial \mathcal{H}}{\partial u} < 0$ so for time($t_0 \to t$), $u(t) = 0$ and for time($t \to t_f$), $u(t) = M$ and this scenario is not possible because p_1 is growing by the time and p_2 is constant all the time.

1.3 part c

$$J = -x(100)$$

Subjected to:

$$\int_0^{100} u(t)dt = K(\text{a known constant})$$

$z(t)$ is new state:

$$z(t) = \int_0^t u(t)dt \to \frac{dz}{dt} = u(t)$$

New differential constraints:

$$\frac{dz}{dt} - u(t) = 0$$

$$g_a(x, \dot{x}, \dot{z}, t, \lambda) = g(x, \dot{x}, t) + \lambda(t) f(x, \dot{x}, \dot{z}, t)$$

$$g_a = -x(t) + \lambda(\dot{z}(t) - u(t))$$

Hamiltonian matrix:

$$\mathcal{H} = g_a(\vec{x}(t), u(t), z(t), \lambda, t) + \vec{p}(t)^T a(\vec{x}(t), u(t), t)$$

We assume $p_2 = \lambda$

$$\mathcal{H} = -0.1p_1(t)x(t) + p_1(t)u(t) + p_2(t)(\dot{z}(t) - u(t)) \tag{11}$$

$$\dot{\vec{p}} = -\frac{\partial \mathcal{H}}{\partial \vec{x}} = \begin{bmatrix} -\frac{\partial \mathcal{H}}{\partial x} \\ -\frac{\partial \mathcal{H}}{\partial z} \end{bmatrix} = \begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix}$$

$$\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} 0.1p_1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} = \begin{bmatrix} C_1\exp(t/10) \\ C_2 \end{bmatrix} \tag{12}$$

Final $x(t)$ is free so:

$$h_{\vec{x}} - \vec{p} = \vec{0}\Big|_{*,t_f} \rightarrow p_1(t_f) = -1 \tag{13}$$

From equation 12 and 13 we can find $p_1(t)$ function.

$$p_1(100) = C_1 \exp(100/10) = -1 \rightarrow C_1 = -\exp(-10)$$

$$p_1(t) = -\exp(0.1(t-100))$$

This problem is like section 1.2 and have the same scenarios.

2 Question 2

2.1 part a

Dynamic of system:

$$\ddot{x}(t) = \frac{T}{M}\cos(\beta(t)), \quad \ddot{y}(t) = \frac{T}{M}\sin(\beta(t))$$

Now we define new states:

$$x_1(t) = x(t), \quad x_2(t) = \dot{x_1}(t), \quad x_3(t) = y(t), \quad x_4(t) = \dot{x_3}(t), \quad u(t) = \beta(t)$$

New dynamic of system:

$$a(\vec{x(t)}, u, t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ \frac{T}{M}\cos(u(t)) \\ x_4(t) \\ \frac{T}{M}\sin(u(t)) \end{bmatrix}$$

2.2 part b

For minimum time:

$$J = t_f - t_0 = \int_{t_0}^{t_f} dt$$

Hamiltonian matrix:

$$\mathcal{H} = g(\vec{x}(t), u(t), t) + \vec{p}(t)^T a(\vec{x}(t), u(t), t)$$

$$\mathcal{H} = 1 + \begin{bmatrix} p_1(t) & p_2(t) & p_3(t) & p_4(t) \end{bmatrix} \begin{bmatrix} x_2(t) \\ \frac{T}{M}\cos(u(t)) \\ x_4(t) \\ \frac{T}{M}\sin(u(t)) \end{bmatrix}$$

$$\mathcal{H} = 1 + p_1(t)x_2(t) + \frac{T}{M}p_2(t)\cos(u(t)) + p_3(t)x_4(t) + \frac{T}{M}p_4(t)\sin(u(t)) \tag{14}$$

Euler–Lagrange equation:

$$\dot{\vec{x}} = \frac{\partial \mathcal{H}}{\partial \vec{p}} = a(\vec{x}(t), u(t), t) \tag{15}$$

$$\dot{\vec{p}} = -\frac{\partial \mathcal{H}}{\partial \vec{x}} \tag{16}$$

$$\vec{0} = \frac{\partial \mathcal{H}}{\partial \vec{u}} \tag{17}$$

Now we use equation 14 to solve Euler-Lagrange equation.

$$\begin{bmatrix} \dot{p_1} \\ \dot{p_2} \\ \dot{p_3} \\ \dot{p_4} \end{bmatrix} = \begin{bmatrix} -\dfrac{\partial \mathcal{H}}{\partial x_1} \\ -\dfrac{\partial \mathcal{H}}{\partial x_2} \\ -\dfrac{\partial \mathcal{H}}{\partial x_3} \\ -\dfrac{\partial \mathcal{H}}{\partial x_4} \end{bmatrix} = \begin{bmatrix} 0 \\ -p_1 \\ 0 \\ -p_3 \end{bmatrix} \tag{18}$$

Answer of equation 18 is:

$$\begin{bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \\ p_4(t) \end{bmatrix} = \begin{bmatrix} C_1 \\ -C_1 t + C_2 \\ C_3 \\ -C_3 t + C_4 \end{bmatrix} \tag{19}$$

From Euler-Lagrange equation:

$$\vec{0} = \frac{\partial \mathcal{H}}{\partial \vec{u}} \rightarrow p_4 \cos(u) = p_2 \sin(u) \rightarrow u(t) = \tan^{-1}\left(\frac{p_4(t)}{p_2(t)}\right) \tag{20}$$

Boundary conditions:

$$t_0 = 0, \quad t_f = \text{free}$$

$$\vec{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{x}(t_f) = \begin{bmatrix} x_1(t_f) \\ x_2(t_f) \\ x_3(t_f) \\ x_4(t_f) \end{bmatrix} = \begin{bmatrix} \text{Free} \\ V \\ D \\ 0 \end{bmatrix} (D, V \text{ Are known constant})$$

Final time is free so:

$$(\mathcal{H} + h_t)|_{*,t_f} = 0 \rightarrow \mathcal{H}|_{*,t_f} = 0 \tag{21}$$

Final x_1 is free so:

$$\left.(\frac{\partial h}{\partial x_1} - p_1)\right|_{*,t_f} = 0 \rightarrow p_1(t_f) = 0 \tag{22}$$

2.3 part c

There is $11(2n + 3)$ equation and 11 unknowns.

Unknowns:

$$\vec{x}(t)|_{4\times 1}, \quad \vec{p}(t)|_{4\times 1}, \quad t_f, \quad x_1(t_f), \quad u(t)$$

Now use previous equation to find unknowns. From equation 22 and 30 we can find out:

$$p_1(t) = C_1 = 0, \quad p_2(t) = 0t + C_2 = C_2 \tag{23}$$

Now we use equation 23, 30, 20 and 14 to have new equation and put parameters in equation 21.

$$\mathcal{H}|_{*,t_f} = 1 + 0 \times x_2(t_f) + \frac{T}{M} C_2 \cos\left(\tan^{-1}\left(\frac{-C_3 t + C_4}{C_2}\right)\right) + p_3(t_f) \times 0 + \frac{T}{M} p_4(t) \sin\left(\tan^{-1}\left(\frac{-C_3 t + C_4}{C_2}\right)\right)$$

$$\mathcal{H}|_{*,t_f} = 1 + \frac{T}{M} C_2 \cos\left(\tan^{-1}\left(\frac{-C_3 t + C_4}{C_2}\right)\right) + \frac{T}{M} p_4(t) \sin\left(\tan^{-1}\left(\frac{-C_3 t + C_4}{C_2}\right)\right) = 0 \tag{24}$$

This far i could go to solve problem analytically. :)

2.4 part d

For maximize horizontal range we must change cost function.

$$J = -x_1(t_f)$$

Hamiltonian matrix:

$$\mathcal{H} = g(\vec{x}(t), u(t), t) + \vec{p}(t)^T a(\vec{x}(t), u(t), t)$$

$$\mathcal{H} = \begin{bmatrix} p_1(t) & p_2(t) & p_3(t) & p_4(t) \end{bmatrix} \begin{bmatrix} x_2(t) \\ \frac{T}{M}\cos(u(t)) \\ x_4(t) \\ \frac{T}{M}\sin(u(t)) \end{bmatrix}$$

$$\mathcal{H} = p_1(t)x_2(t) + \frac{T}{M}p_2(t)\cos(u(t)) + p_3(t)x_4(t) + \frac{T}{M}p_4(t)\sin(u(t)) \tag{25}$$

Euler–Lagrange equation:

$$\dot{\vec{x}} = \frac{\partial \mathcal{H}}{\partial \vec{p}} = a(\vec{x}(t), u(t), t) \tag{26}$$

$$\dot{\vec{p}} = -\frac{\partial \mathcal{H}}{\partial \vec{x}} \tag{27}$$

$$\vec{0} = \frac{\partial \mathcal{H}}{\partial \vec{u}} \tag{28}$$

In this problem like previous we must satisfy Euler-Equation equation. Now we use equation 25 to solve Euler-Lagrange equation.

$$\begin{bmatrix} \dot{p_1} \\ \dot{p_2} \\ \dot{p_3} \\ \dot{p_4} \end{bmatrix} = \begin{bmatrix} -\frac{\partial \mathcal{H}}{\partial x_1} \\ -\frac{\partial \mathcal{H}}{\partial x_2} \\ -\frac{\partial \mathcal{H}}{\partial x_3} \\ -\frac{\partial \mathcal{H}}{\partial x_4} \end{bmatrix} = \begin{bmatrix} 0 \\ -p_1 \\ 0 \\ -p_3 \end{bmatrix} \tag{29}$$

Answer of equation 18 is:

$$\begin{bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \\ p_4(t) \end{bmatrix} = \begin{bmatrix} C_1 \\ -C_1 t + C_2 \\ C_3 \\ -C_3 t + C_4 \end{bmatrix} \tag{30}$$

Boundary conditions:

$$t_0 = 0, \quad t_f = t_1$$

$$\vec{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{x}(t_f) = \begin{bmatrix} x_1(t_f) \\ x_2(t_f) \\ x_3(t_f) \\ x_4(t_f) \end{bmatrix} = \begin{bmatrix} \text{Free} \\ \text{Free} \\ D \\ \text{Free} \end{bmatrix} \text{ (D is known constant)}$$

Final x_1, x_2 and x_4 are free so:

$$(\frac{\partial h}{\partial \vec{x}} - \vec{p})\Big|_{*,t_f} = \vec{0} \rightarrow \begin{bmatrix} \frac{\partial h}{\partial \vec{x}_1} \\ \frac{\partial h}{\partial \vec{x}_2} \\ \frac{\partial h}{\partial \vec{x}_4} \end{bmatrix}\Bigg|_{*,t_f} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} p_1(t_f) \\ p_2(t_f) \\ p_4(t_f) \end{bmatrix} \tag{31}$$

3 Question 3

$$\ddot{x}(t) = -x(t) - 0.1\dot{x}(t) + u(t), \qquad x(0) = \dot{x}(0) = 1$$

Assume:

$$x_1(t) = x(t), \quad x_2(t) = \dot{x}(t) \rightarrow a(\vec{x}, u, t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -x_1(t) - 0.1x_2(t) + u(t) \end{bmatrix}$$

$$J = \frac{1}{2}x^T(t_f)Hx(t_f) + \frac{1}{2}\int_0^{t_f} \left(\alpha(x^2 + \dot{x}^2) + \beta u^2)\right) dt$$

3.1 part a

$\alpha = \beta = 1$, $t_f \rightarrow \infty$ and $H = 0$:

$$J = \frac{1}{2}\int_0^{t_f} (x^2 + \dot{x}^2 + u^2)dt$$

$$\vec{p}(t) = \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix}$$

Hamiltonian matrix:

$$\mathcal{H} = g(\vec{x}(t), u(t), t) + \vec{p}(t)^T a(\vec{x}(t), u(t), t)$$

$$\mathcal{H} = x_1^2(t) + x_2^2(t) + u^2(t) + \begin{bmatrix} p_1(t) & p_2(t) \end{bmatrix} \begin{bmatrix} x_2(t) \\ -x_1(t) - 0.1x_2(t) + u(t) \end{bmatrix}$$

$$\mathcal{H} = x_1^2(t) + x_2^2(t) + u^2(t) + p_1(t)x_2(t) - p_2(t)x_1(t) - 0.1p_2(t)x_2(t) + p_2(t)u(t)$$

Euler–Lagrange equation:

$$\dot{\vec{x}} = \frac{\partial \mathcal{H}}{\partial \vec{p}} = a(\vec{x}(t), u(t), t) \tag{32}$$

$$\dot{\vec{p}} = -\frac{\partial \mathcal{H}}{\partial \vec{x}} = \begin{bmatrix} -\frac{\partial \mathcal{H}}{\partial x_1} \\ -\frac{\partial \mathcal{H}}{\partial x_2} \end{bmatrix} \tag{33}$$

$$\vec{0} = \frac{\partial \mathcal{H}}{\partial \vec{u}} \tag{34}$$

Now we use above equation for solve problem.

$$\begin{bmatrix} -\frac{\partial \mathcal{H}}{\partial x_1} \\ -\frac{\partial \mathcal{H}}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -2x_1 + p_2 \\ -2x_2 - p_1 + 0.1p_2 \end{bmatrix} = \begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix}$$

$$\frac{\partial \mathcal{H}}{\partial \vec{u}} = 2u + p_2 = 0 \rightarrow u = -0.5p_2$$

There is four differential equation and four unknowns.

$$\dot{x}_1 = x_2 \tag{35}$$

$$\dot{x}_2 = -x_1 - 0.1x_2 - 0.5p_2 \tag{36}$$

$$\dot{p}_1 = -2x_1 + p_2 \tag{37}$$

$$\dot{p}_2 = -2x_2 - p_1 + 0.1p_2 \tag{38}$$

Contents