Sharif University of Technology
Institute of Aerospace

Homework #1

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Course: *Optimal Control I* – Professor: *Dr. Assadian*
Due date: *March 28th, 2025*

Problem 1

(a) $z = f(x, y) = y\sin(x + y) - x\sin(x - y)$
Gradient of f(x, y):

$$\vec{\nabla} f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}$$

$$\vec{\nabla} f = \begin{bmatrix} y\cos(x + y) - \sin(x - y) - x\cos(x - y) \\ y\cos(x + y) + \sin(x + y) + x\cos(x - y) \end{bmatrix}$$

Two nonlinear equations with two unknowns. We use MATLAB to solve this equations. MATLAB file is attached. Answers are provided in table 5

Table 1: Answers

x	y
-3.41877	-1.82764
-2.88904	1.84693
-2.02875	0.00000
-1.84693	-2.88904
-1.82764	3.41877
-1.75560	0.36547
-0.36547	-1.7556
0.00000	-2.02875
0.00000	0.00000
0.00000	2.02875
0.36547	1.7556
1.75560	-0.36547
1.82764	-3.41877
1.84693	2.88904
2.02875	0.00000
2.88904	-1.84693
3.41877	1.82764

Hessian matrix:

$$H = \frac{\partial^2 f}{\partial \vec{X}^2} = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial xy} \\ \frac{\partial^2 f}{\partial yx} & \frac{\partial f}{\partial y^2} \end{bmatrix}$$

$$\vec{\nabla} f = \begin{matrix} -y\sin(x+y) - 2\cos(x-y) + x\cos(x-y) & \cos(x+y) - y\sin(x+y) + cos(x-y) - x\sin(x-y) \\ cos(x+y) - y\sin(x+y) + \cos(x-y) - x\sin(x-y) & x\sin(x-y) + 2\cos(x+y) - y\sin(x+y) \end{matrix}$$

Hessian matrix and eigenvalues have calculated in MATLAB and attached.

Table 2: Answers With Conditions

x	y	Point Condition
-3.41877	-1.82764	Maximum
-2.88904	1.84693	Saddle Point
-2.02875	0.00000	Saddle Point
-1.84693	-2.88904	Saddle Point
-1.82764	3.41877	Minimum
-1.75560	0.36547	Maximum
-0.36547	-1.7556	Minimum
0.00000	-2.02875	Saddle Point
0.00000	0.00000	Saddle Point
0.00000	2.02875	Saddle Point
0.36547	1.7556	Minimum
1.75560	-0.36547	Saddle Point
1.82764	-3.41877	Minimum
1.84693	2.88904	Saddle Point
2.02875	0.00000	Saddle Point
2.88904	-1.84693	Saddle Point
3.41877	1.82764	Maximum

Answers and conditions are provided in table 2

Figure 1: 3D figure of function

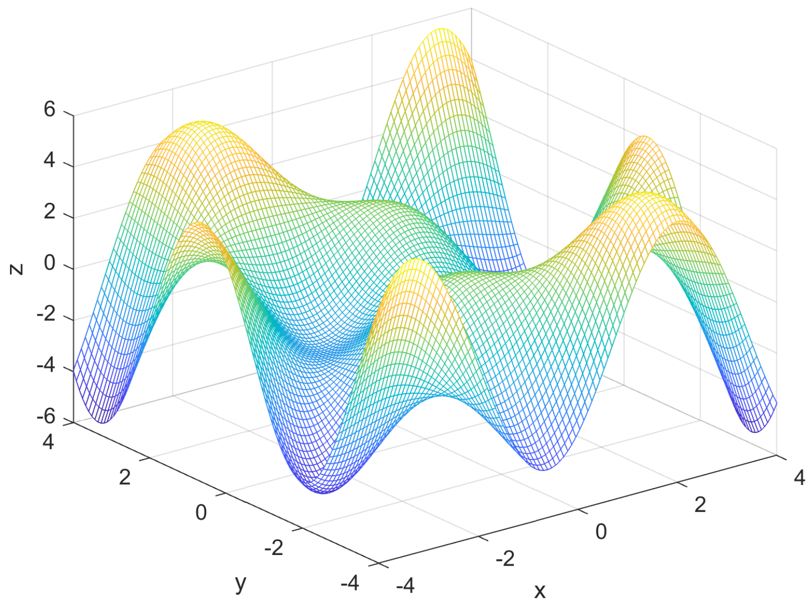


Figure 2: 3D figure of function with Points

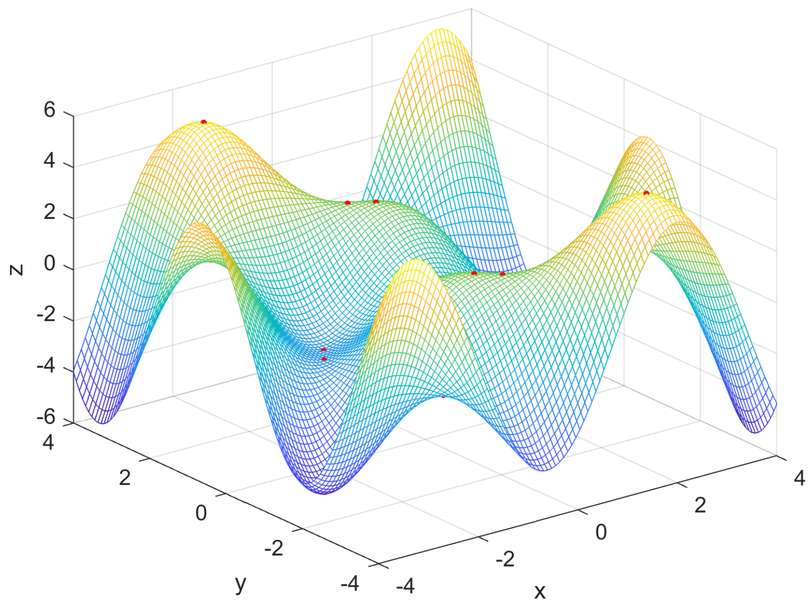


Figure 3: Contour figure of function

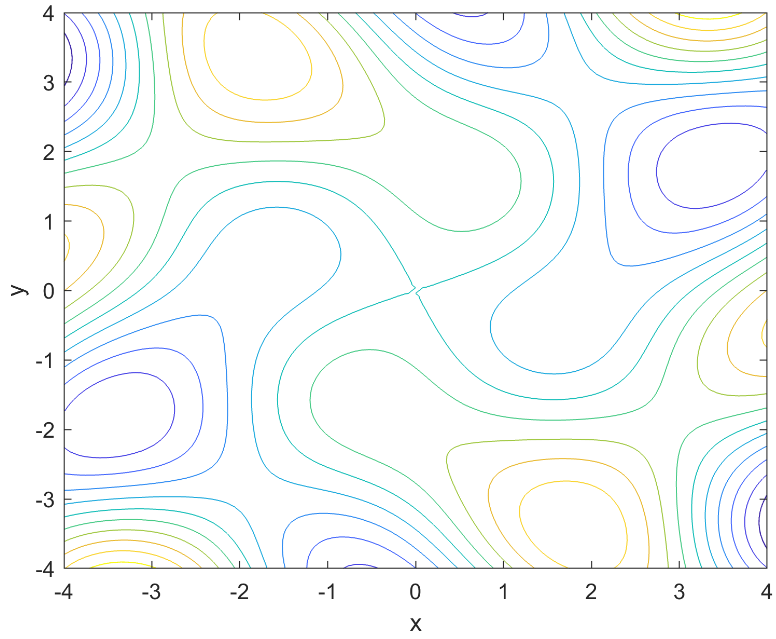
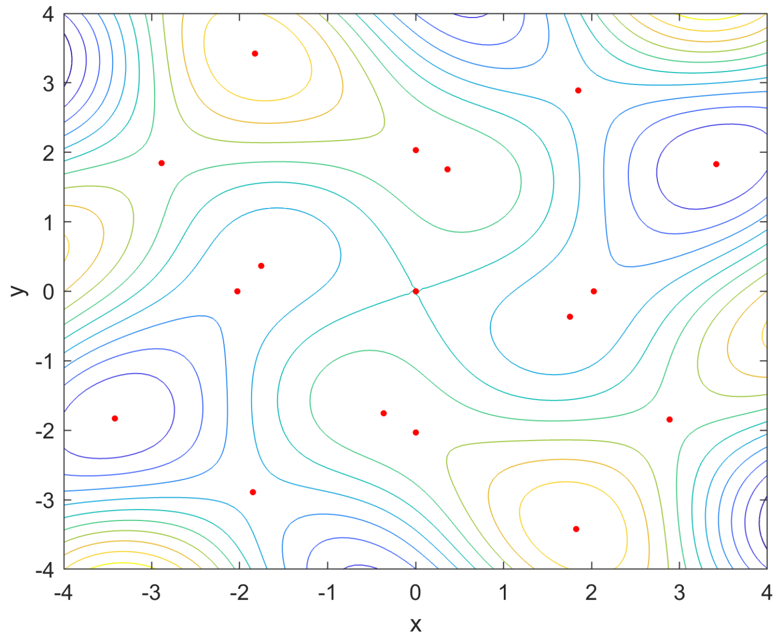


Figure 4: Contour figure of function with Points



(b) $z = f(x, y) = x^3 - 3xy^2$
Gradient of f(x, y):

$$\vec{\nabla} f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}$$

$$\vec{\nabla} f = \begin{bmatrix} 3x^2 - 3y^2 \\ -6xy \end{bmatrix}$$

Two linear equations with two unknowns.

$$3x^2 - 3y^2 = 0$$

$$-6xy = 0$$

Answers is $x = 0$ and $y = 0$.

Hessian matrix:

$$H = \frac{\partial^2 f}{\partial \vec{X}^2} = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial xy} \\ \frac{\partial^2 f}{\partial yx} & \frac{\partial f}{\partial y^2} \end{bmatrix}$$

$$H = \begin{bmatrix} 6x & -6y \\ -6y & -6x \end{bmatrix}$$

In $x = 0$ and $y = 0$ Hessian matrix in :

$$H = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

so this point is saddle point.

Figure 5: 3D figure of function

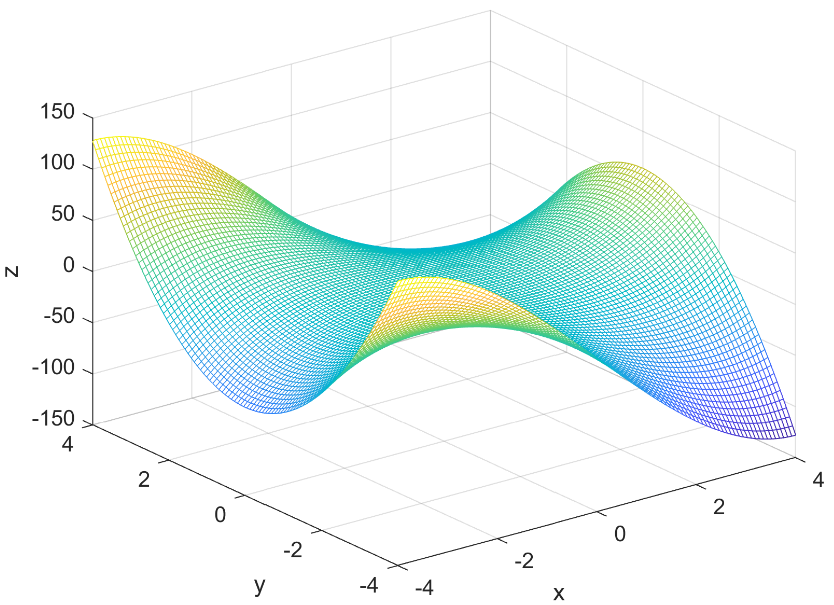


Figure 6: 3D figure of function with Points

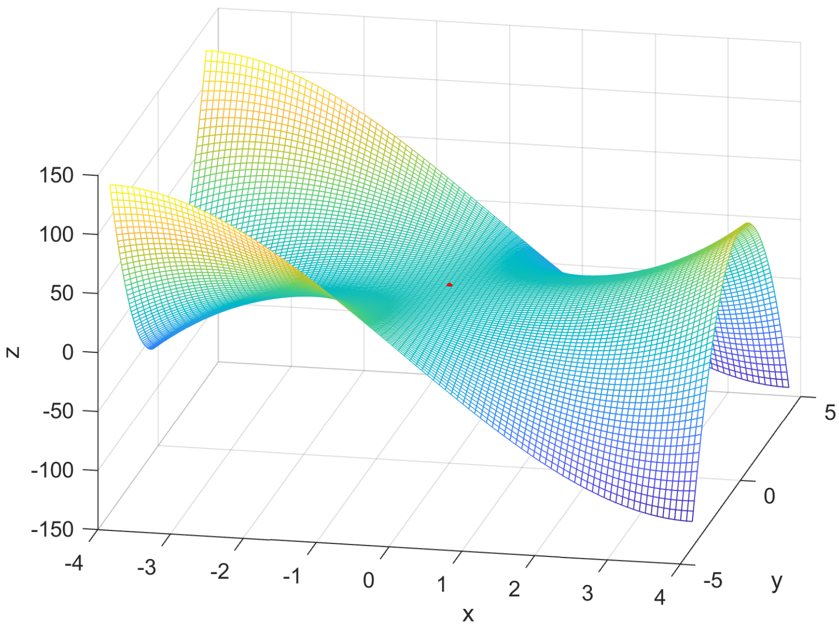


Figure 7: Contour figure of function

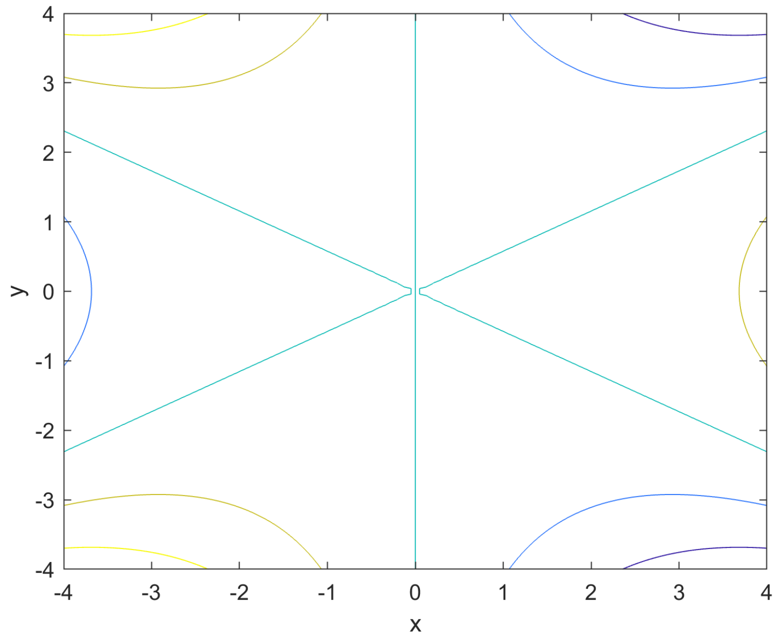
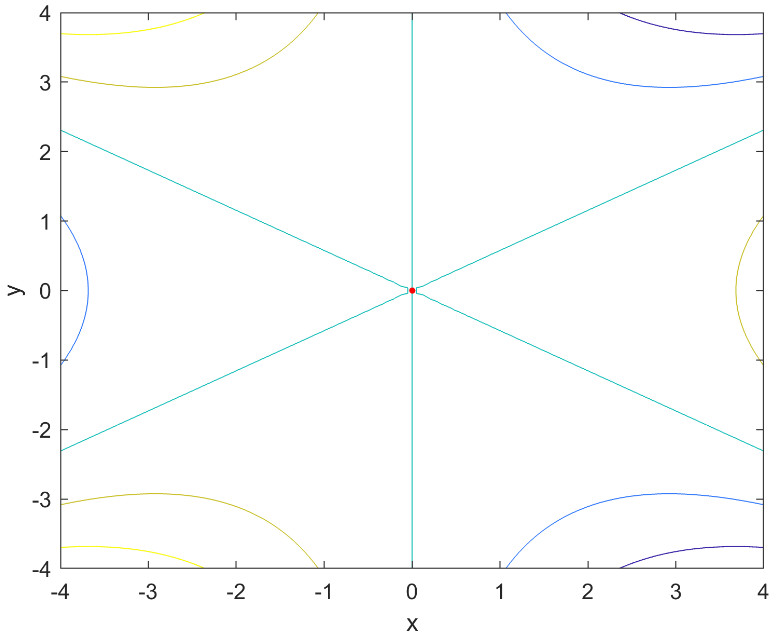


Figure 8: Contour figure of function with Points



(c) $z = f(x_1, x_2, x_3) = x_1^2 + x_1x_2 - 4x_2^2 - x_3^2 + 3x_2x_3$
Gradient of $f(x_1, x_2, x_3)$:

$$\vec{\nabla} f = \frac{\partial f}{\partial \vec{X}} = \begin{bmatrix} \dfrac{\partial f}{\partial x_1} \\ \dfrac{\partial f}{\partial x_2} \\ \dfrac{\partial f}{\partial x_3} \end{bmatrix}$$

$$\vec{\nabla} f = \begin{bmatrix} 2x_1 + x_2 \\ x_1 - 8x_2 + 3x_3 \\ 3x_2 - 2x_3 \end{bmatrix} = \vec{0}$$

Three linear equations with Three unknowns.

$$2x_1 + x_2 = 0$$

$$x_1 - 8x_2 + 3x_3 = 0$$

$$3x_2 - 2x_3 = 0$$

Answers is $x_1 = x_2 = x_3 = 0$ Hessian matrix:

$$H = \frac{\partial^2 f}{\partial \vec{X}^2} = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 x_2} & \dfrac{\partial^2 f}{\partial x_1 x_3} \\ \dfrac{\partial^2 f}{\partial x_2 x_1} & \dfrac{\partial f}{\partial x_2^2} & \dfrac{\partial^2 f}{\partial x_2 x_3} \\ \dfrac{\partial^2 f}{\partial x_3 x_1} & \dfrac{\partial f}{\partial x_3 x_2} & \dfrac{\partial^2 f}{\partial x_3^2} \end{bmatrix}$$

$$H = \begin{bmatrix} 2 & 1 & 0 \\ 1 & -8 & 3 \\ 0 & 3 & -2 \end{bmatrix}$$

All Hessian eigenvalues are:

$$eig(H) = \begin{bmatrix} -9.3182 \\ -0.8077 \\ 2.1259 \end{bmatrix}$$

So $(0, 0, 0)$ is a saddle point.

Problem 2

$$\min f(x,y,z) = x^2 + y^2 + z^2$$

subject to :

$$z = \sin(x) + \cos(y)$$

(a) Direct Substitution:

$f(x,y,z) = x^2 + y^2 + z^2 \xrightarrow{z=\sin(x)+\cos(y)} f(\vec{X}) = f(x,y) = x^2 + y^2 + (\sin(x) + \cos(y))^2$
$f(x,y) = x^2 + y^2 + \sin(x)^2 + 2\sin(x)\cos(y) + \cos(y)^2$ Gradient of f(x, y):

$$\vec{\nabla} f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}$$

$$\vec{\nabla} f = \begin{bmatrix} 2x + 2\cos(x)\cos(y) + 2\cos(x)\sin(x) \\ 2y - 2\cos(y)\sin(y) - 2\sin(x)\sin(y) \end{bmatrix}$$

$$2x + 2\cos(x)\cos(y) + 2\cos(x)\sin(x) = 0$$

$$2y - 2\cos(y)\sin(y) - 2\sin(x)\sin(y) = 0$$

Above equation solved in MATLAB and code (Q2_a.m) has attached to homework.
$x = -0.47872, \quad y = 0.0 \rightarrow z = 0.5393$

Table 3: Answers

x	y	z
-0.47872	0.000	0.5393

(b) Lagrange multipliers
$\min \mathcal{L}(\vec{X}, \vec{\lambda}) = f(\vec{X}) + \vec{\lambda}^T \vec{g}$
necessary condition:

$$\vec{\nabla}\mathcal{L} = \begin{bmatrix} \vec{\nabla}_{\vec{X}}\mathcal{L} \\ \vec{\nabla}_{\vec{\lambda}}\mathcal{L} \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathcal{L}}{\partial \vec{X}} \\ \frac{\partial \mathcal{L}}{\partial \vec{\lambda}} \end{bmatrix} = \vec{0}$$

$f(\vec{X}) = x^2 + y^2 + z^2, \quad g(\vec{X}) = \sin(x) + \cos(y) - z = 0$

$$\min \mathcal{L}(\vec{X}, \vec{\lambda}) = x^2 + y^2 + z^2 + \lambda(\sin(x) + \cos(y) - z)$$

$$\vec{\nabla}\mathcal{L} = \begin{bmatrix} \frac{\partial \mathcal{L}}{\partial x} \\ \frac{\partial \mathcal{L}}{\partial y} \\ \frac{\partial \mathcal{L}}{\partial z} \\ \frac{\partial \mathcal{L}}{\partial \lambda} \end{bmatrix} = \begin{bmatrix} 2x + \lambda\cos(x) \\ 2y - \lambda\sin(y) \\ 2z - \lambda \\ \sin(x) + \cos(y) - z \end{bmatrix}$$

Above equation solved in MATLAB and code (Q2_b.m) has attached to homework.

Table 4: Answers

x	y	z	λ
-0.47872	0.000	0.5393	1.078708

Problem 3

$$\min f(x_1, x_2, y_1, y_2) = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

subject to :

$$y_1 = x_1^2, \quad y_2 = x_2 - 1$$

(a) Direct Substitution:

$$f(x_1, x_2, y_1, y_2) = (x_1 - x_2)^2 + (y_1 - y_2)^2 \xrightarrow{y_1 = x_1^2, \quad y_2 = x_2 - 1} f(x_1, x_2) = (x_1 - x_2)^2 + (x_1^2 - (x_2 - 1)^2)^2$$

$$f(x_1, x_2) = x_1^4 - 2x_1^2 x_2^2 + 4x_1^2 x_2 - x_1^2 - 2x_1 x_2 + x_2^4 - 4x_2^3 + 7x_2^2 - 4x_2 + 1$$

Gradient of $f(x_1, x_2)$:

$$\vec{\nabla} f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix}$$

$$\vec{\nabla} f = \begin{bmatrix} 4x_1^3 - 4x_1 x_2^2 + 8x_1 x_2 - 2x_1 - 2x_2 \\ -4x_1^2 x_2 + 4x_1^2 - 2x_1 + 4x_2^3 - 12x_2^2 + 14x_2 - 4 \end{bmatrix}$$

Two nonlinear equations with two unknowns. We use MATLAB to solve this equations. MATLAB file (Q3_a.m) is attached.

$$4x_1^3 - 4x_1 x_2^2 + 8x_1 x_2 - 2x_1 - 2x_2 = 0$$

$$-4x_1^2 x_2 + 4x_1^2 - 2x_1 + 4x_2^3 - 12x_2^2 + 14x_2 - 4 = 0$$

$$x_1 = \frac{1}{2} \rightarrow y_1 = \frac{1}{4}, \quad x_2 = \frac{7}{8} \rightarrow y_2 = -\frac{1}{8}$$

Table 5: Answers

x_1	y_1	x_2	y_2
0.5	0.25	0.875	-0.125