

Home Work #1

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1 Question 1

$$f_X(x) = \frac{ab}{b^2 + x^2}, \quad b > 0$$

1.1 part a

$$\int_{-\infty}^{+\infty} f(x)dx = 1 \rightarrow \int_{-\infty}^{+\infty} \frac{ab}{b^2 + x^2} dx = 1 \rightarrow a \arctan\left(\frac{x}{b}\right) \Big|_{-\infty}^{+\infty} = 1 \rightarrow a\pi = 1 \rightarrow a = \frac{1}{\pi}$$
$$f_X(x) = \frac{1}{\pi} \frac{b}{b^2 + x^2}, \quad b > 0$$

1.2 part b

$$E(X) = \mu_X = \int_{-\infty}^{+\infty} xf(x)dx$$

Because $xf(x)$ is an odd function, the result of the integrator between ∞ and $-\infty$ is zero.

$$\int_{-\infty}^{+\infty} xf(x)dx = 0 \rightarrow \mu_X = 0$$

$$\sigma_X^2 = E((X - \mu)^2) = \int_{-\infty}^{+\infty} (x - \mu)^2 f(x)dx = \int_{-\infty}^{+\infty} x^2 f(x)dx = \frac{b}{\pi} (x - b \arctan\left(\frac{x}{b}\right)) \Big|_{-\infty}^{+\infty} \neq \text{finite}$$

2 Question 3

A positive test is A event: $P(A)$, Having the flu is B event: $P(B) = 0.05$

2.1 part a

The probability of a positive test if someone has flu:

$$P(A|B) = 0.99$$

The probability of a positive test if someone doesn't have flu:

$$P(A|\bar{B}) = 0.01$$

$$P(A) = P(A \cap B) + P(A \cap \bar{B}) = P(A|B)P(B) + P(A|\bar{B})P(\bar{B}) = 0.99 \times 0.05 + 0.01 \times 0.95 = 0.059$$

$$P(A|B)P(B) = P(B|A)P(A) \rightarrow P(B|A) = \frac{P(A|B)P(B)}{P(A)} = 0.84$$

2.2 part b

C is the event when two positive tests happen. The probability of two positive tests if someone has flu:

$$P(B)P(A|B)P(A|B) = 0.049$$

The probability of two positive tests if someone doesn't have flu:

$$P(\bar{B})P(A|\bar{B})P(A|\bar{B}) = 9.5 \times 10^{-5}$$

$$P(C) = P(B)P(A|B)P(A|B) + P(\bar{B})P(A|\bar{B})P(A|\bar{B}) = 0.0491$$

$$P(B|C) = \frac{P(B \cap C)}{P(C)} = 0.998$$

3 Question 4

Assumed that the random variables A and ϕ are independent and ϕ is uniform in the interval $[0, 2\pi)$, mean and variance of ϕ is 0 and σ^2 , respectively.

$$x(t) = A \cos(\omega t + \phi)$$

3.1 part a

$$E[x(t)] = E[A \cos(\omega t + \phi)] \xrightarrow{\text{uncorrelated}} E[x(t)] = E[A]E[\cos(\omega t + \phi)] = 0 \xrightarrow{E[A]=0} E[x(t)] = 0$$

3.2 part b

$$\begin{aligned} R_X(t_1, t_2) &= E[x(t_1)x(t_2)] = \frac{1}{2}E[A^2]E[\cos \omega(t_1 - t_2) + \cos(\omega t_1 + \omega t_2 + 2\phi)] \\ E[\cos(\omega t_1 + \omega t_2 + 2\phi)] &= \frac{1}{2\pi} \int_0^{2\pi} \cos(\omega t_1 + \omega t_2 + 2\phi) d\phi = 0 \\ R_X(t_1, t_2) &= \frac{\sigma^2}{2} \cos \omega(t_1 - t_2) \end{aligned}$$

3.3 part c

$$A[x(t)] = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t) dt \rightarrow A[x(t)] = \frac{1}{2\pi} \int_0^{2\pi} A \cos(\omega t + \phi) dt = \frac{A \sin(\phi + \omega t)}{\omega} \Big|_0^{2\pi}$$

$$A[x(t)] = \frac{A \sin(\phi + 2\pi\omega)}{\omega}$$

3.4 part d

$$R[x(t), \tau] = A[x(t)x(t+\tau)] = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t)x(t+\tau) dt = \frac{1}{2\pi} \int_0^{2\pi} A^2 \cos(\omega t + \phi) \cos(\omega(t+\tau) + \phi) dt$$

$$\int \cos(\omega t + \phi) \cos(\omega(t+\tau) + \phi) dt = \begin{cases} A^2 t \cos(\phi)^2 & \text{if } \omega = 0 \\ \frac{A^2 t \cos(\omega\tau)}{2} + \frac{A^2 \sin(2\phi + 2\omega t + \omega\tau)}{4\omega} & \text{if } \omega \neq 0 \end{cases}$$

$$\int_0^{2\pi} \cos(\omega t + \phi) \cos(\omega(t+\tau) + \phi) dt = \begin{cases} 2\pi A^2 \cos(\phi)^2 & \text{if } \omega = 0 \\ \frac{A^2 (\sin(2\phi + 4\pi\omega + \omega\tau) - \sin(2\phi + \omega\tau))}{4\omega} + \pi A^2 \cos(\omega\tau) & \text{if } \omega \neq 0 \end{cases}$$

$$R[x(t), \tau] = \begin{cases} 1 A^2 \cos(\phi)^2 & \text{if } \omega = 0 \\ 0.5 A^2 \cos(\omega\tau) - \frac{0.04 A^2 (1 \sin(2\phi + \omega\tau) - \sin(12.5\omega + 2\phi + \omega\tau))}{\omega} & \text{if } \omega \neq 0 \end{cases}$$

3.5 part e

For a WSS process:

$$R_X(0) = E[x(t)x(t)]$$

$$R_X(\tau) = R_X(-\tau)$$

where $R_X(\tau)$:

$$R_X(\tau) = E \left[x(t + \frac{\tau}{2}) x^*(t - \frac{\tau}{2}) \right]$$

That is true in this stochastic process.

An ergodic process is a stationary random process for which:

$$A[x(t)] = E[x(t)]$$

$$R[x(t, \tau)] = R_X(\tau)$$

$$A[x(t)] \neq E[x(t)]$$

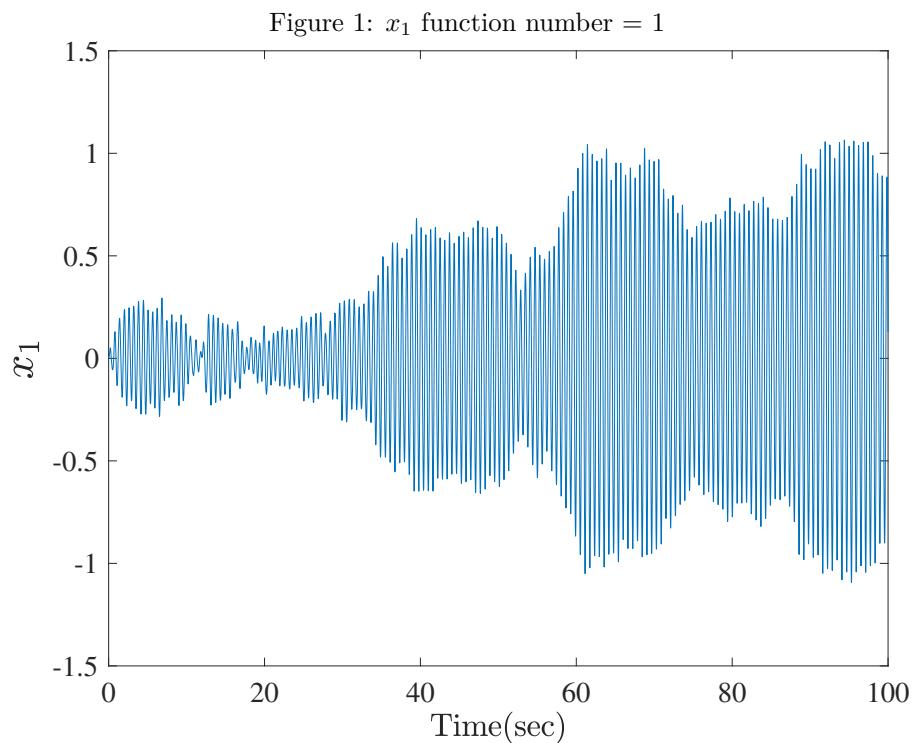
so this stochastic process is not ergodic.

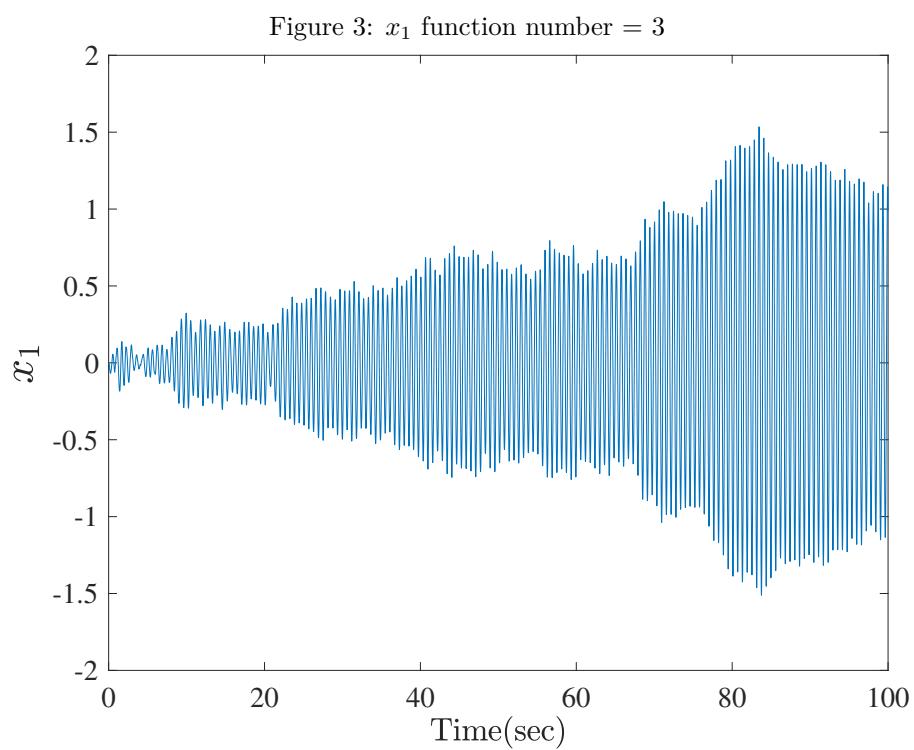
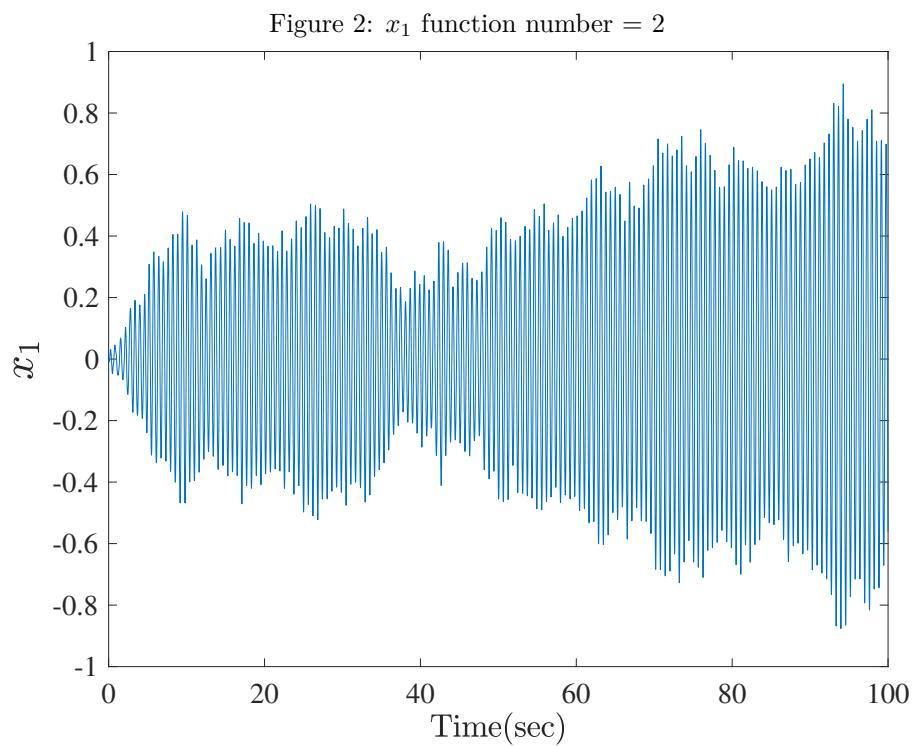
4 Question 5

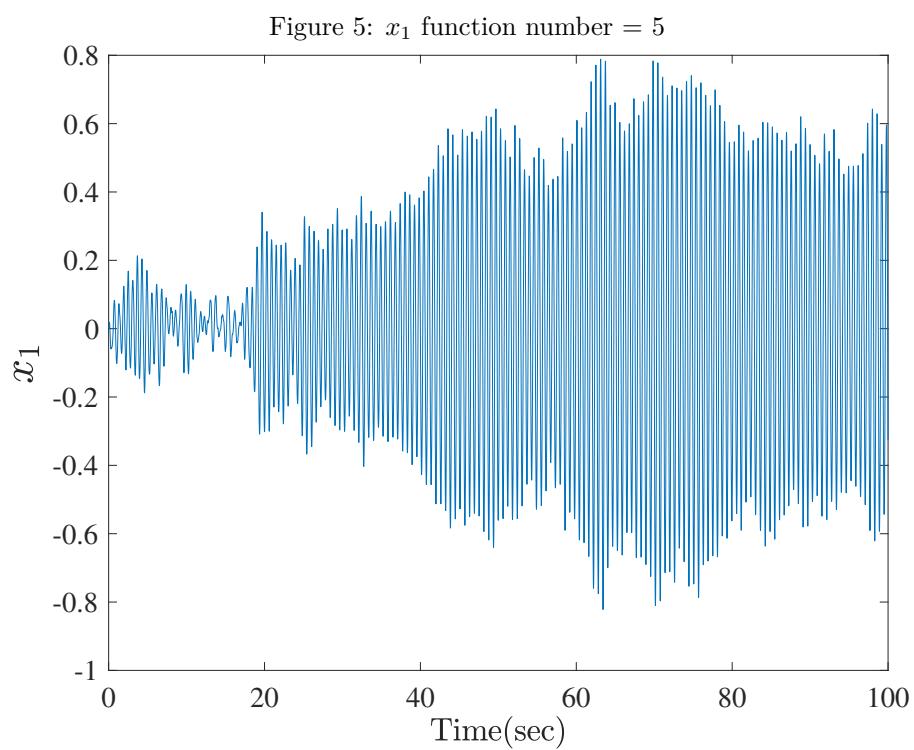
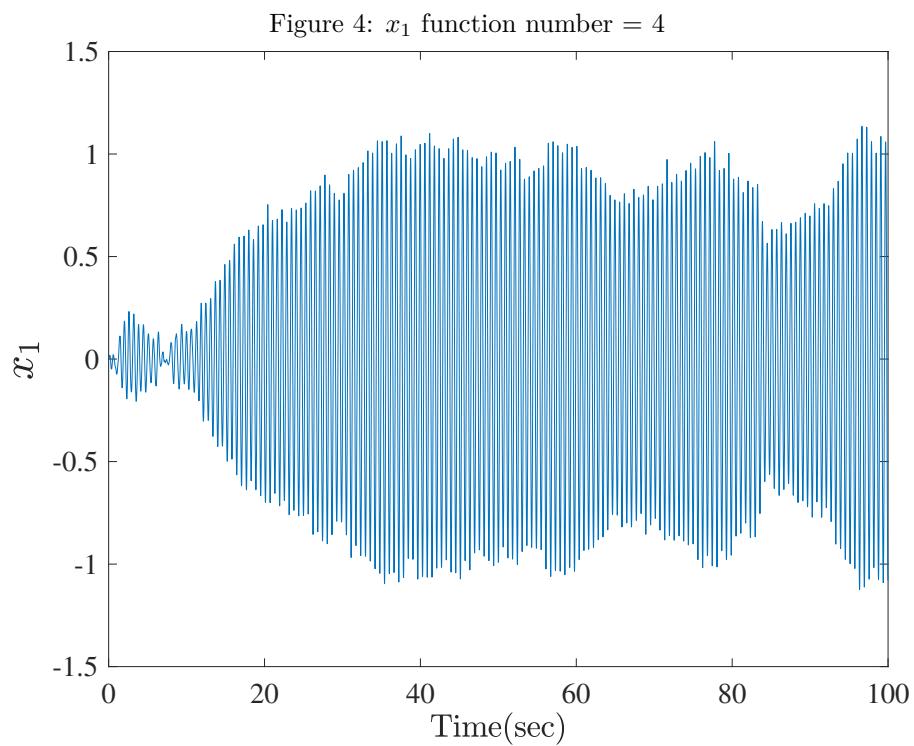
$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -\omega_n^2 x_1(t) - 2\zeta\omega_n x_2(t) + w(t) \end{cases}$$

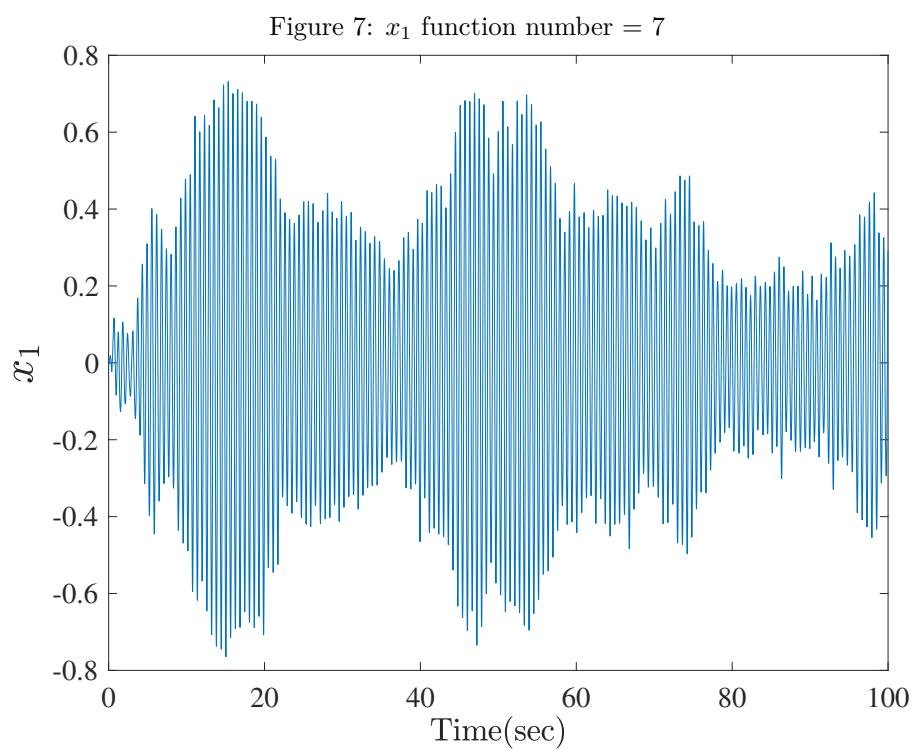
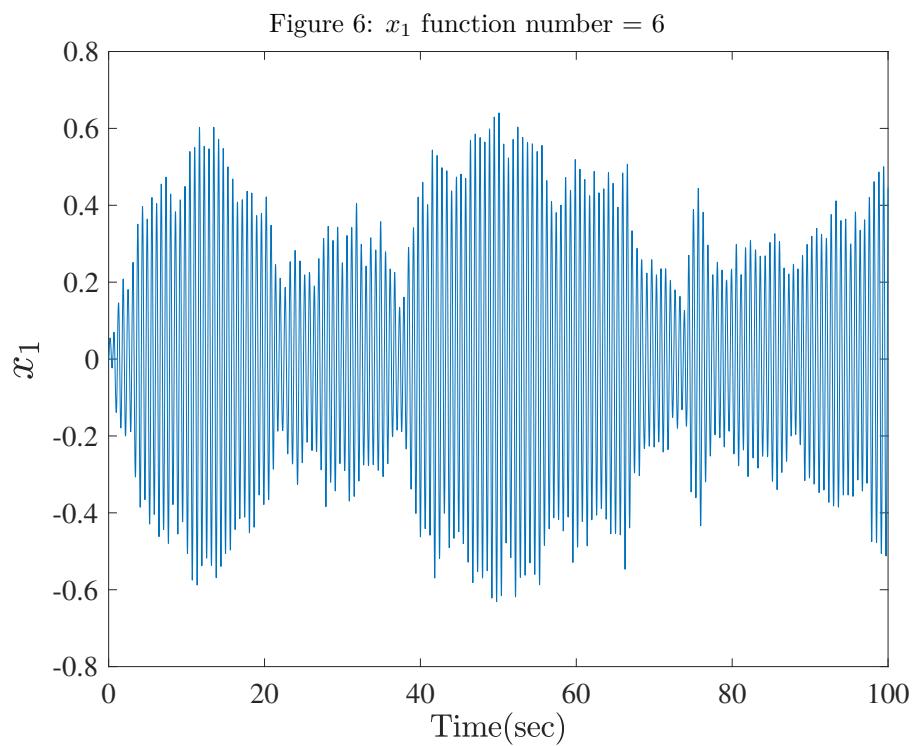
where $\omega_n = 10$, $\zeta = 0.3$ and $w(t)$ is with power σ^2 and step time 0.01 sec ($\sigma^2 = 1$).

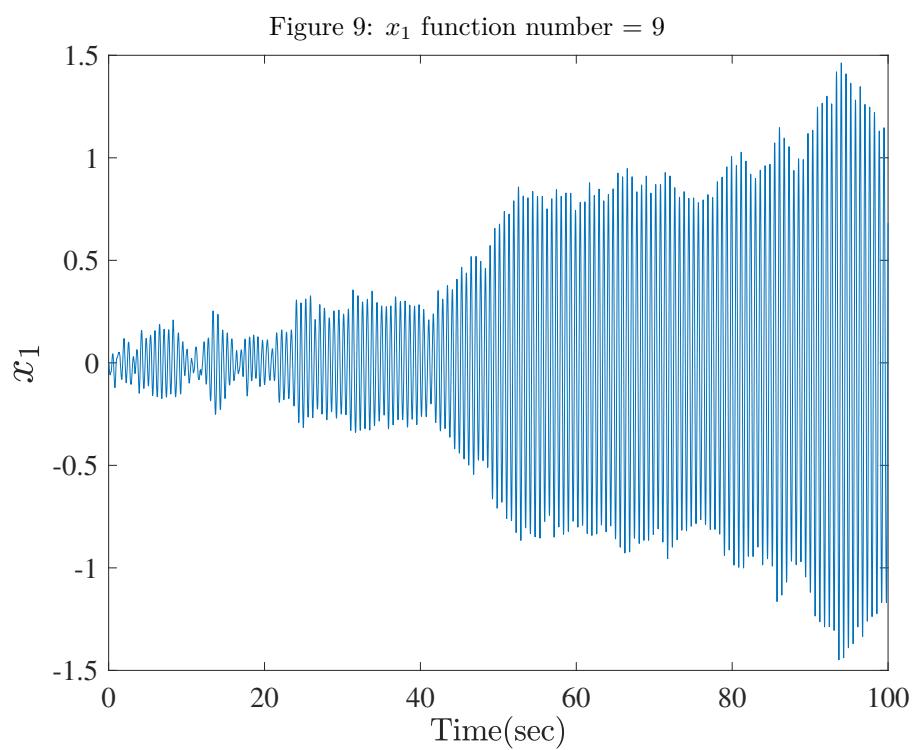
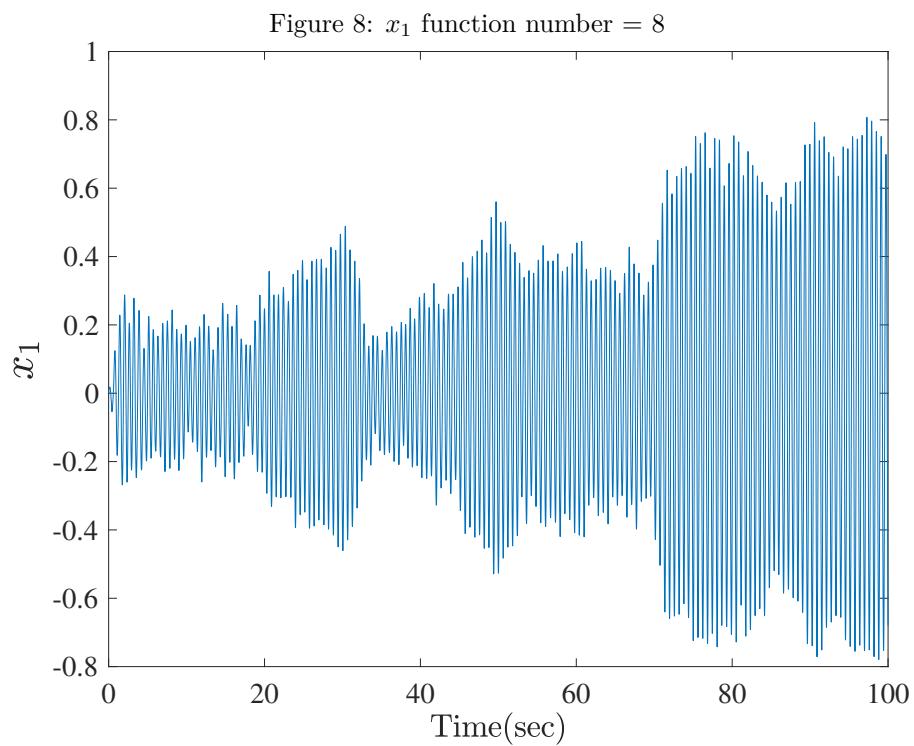
4.1 part a











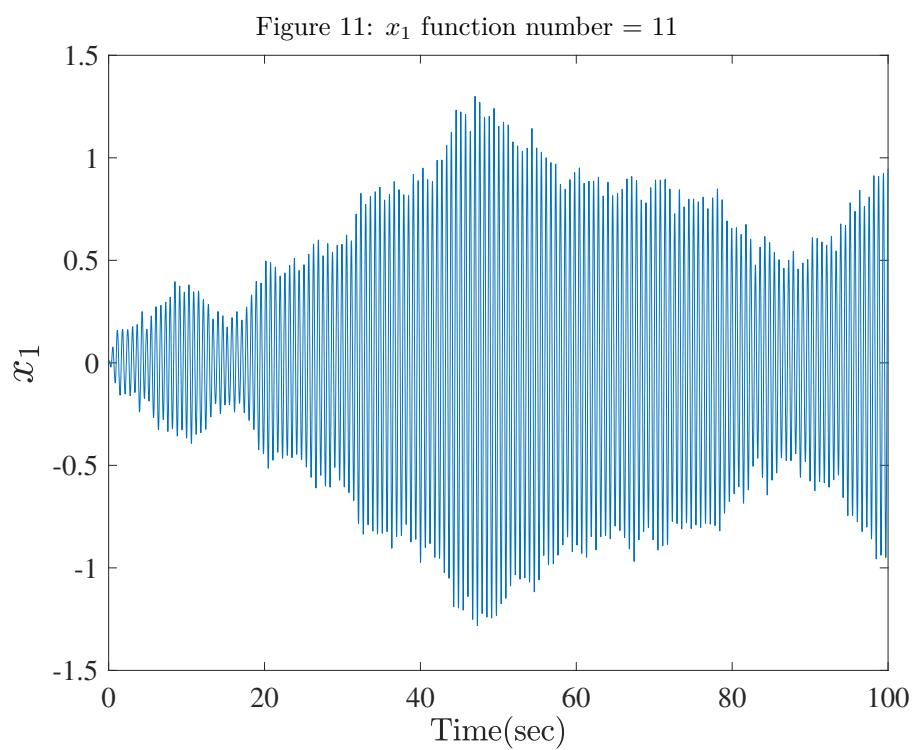
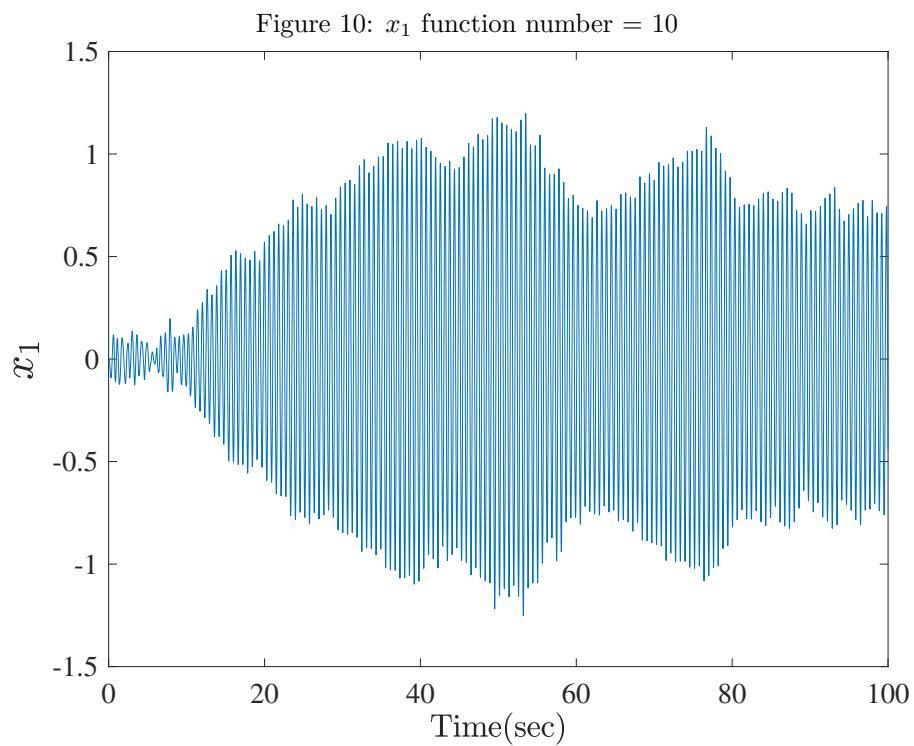
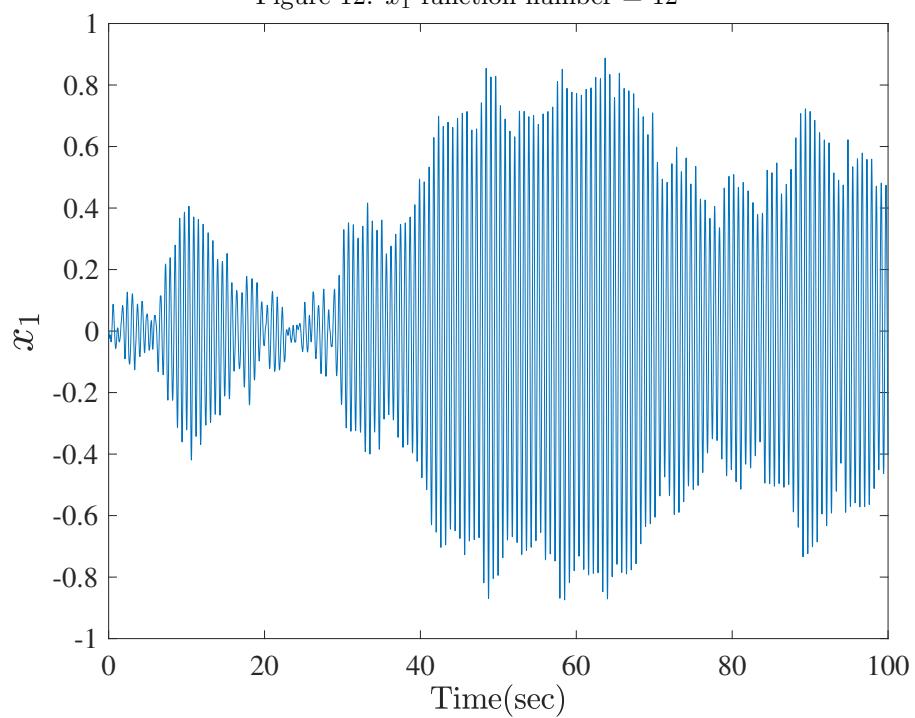
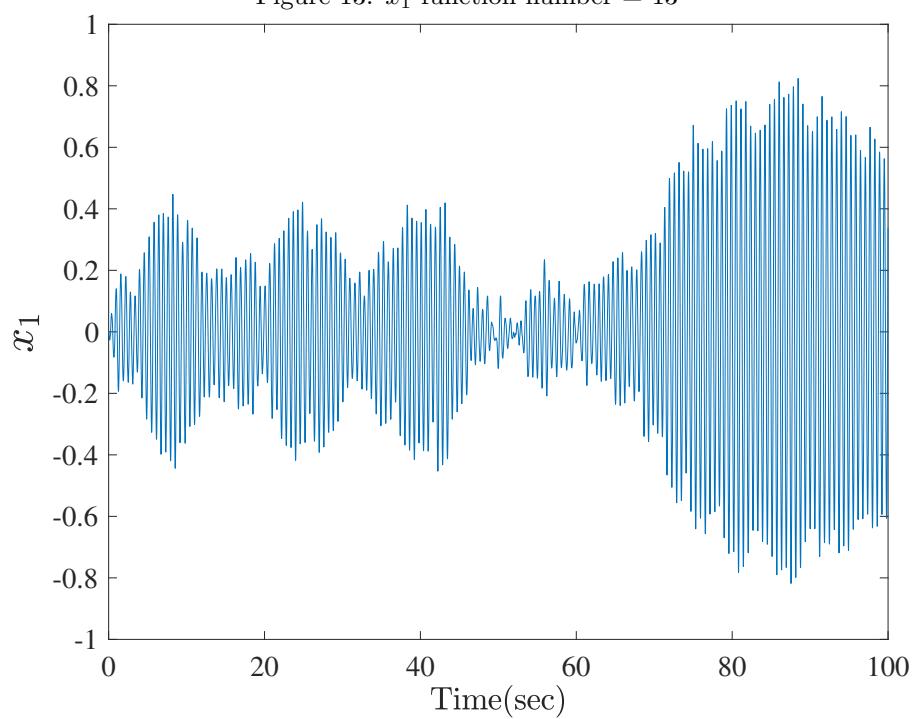
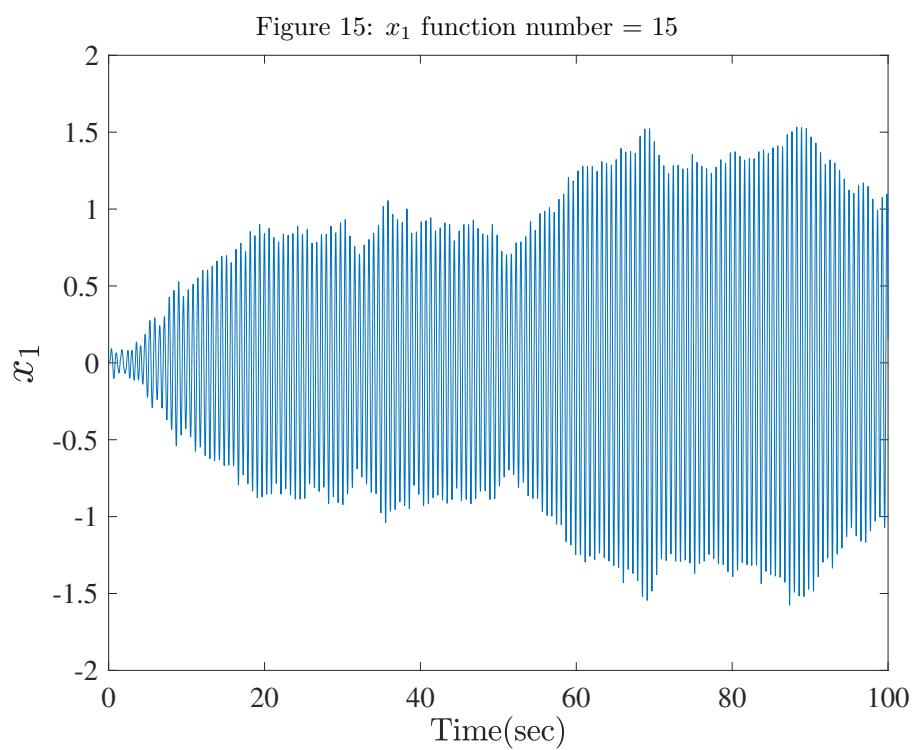
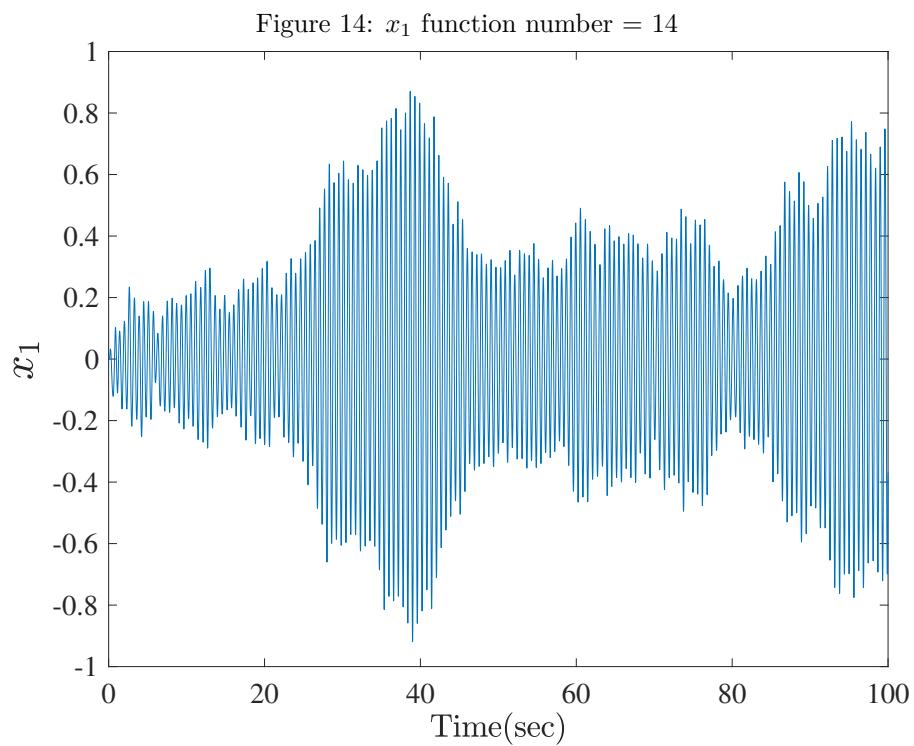


Figure 12: x_1 function number = 12Figure 13: x_1 function number = 13



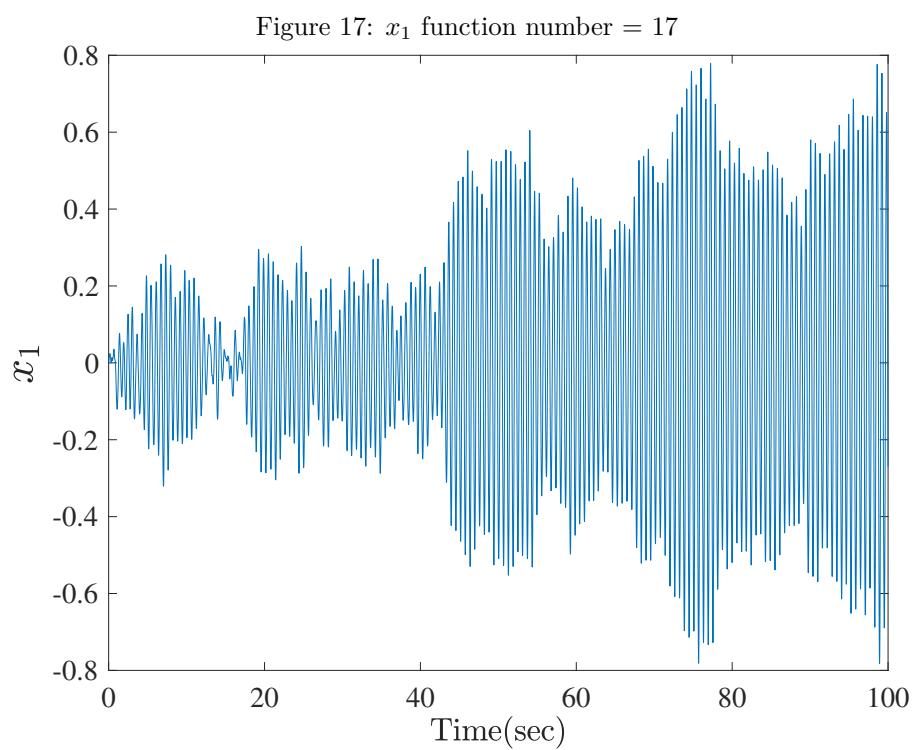
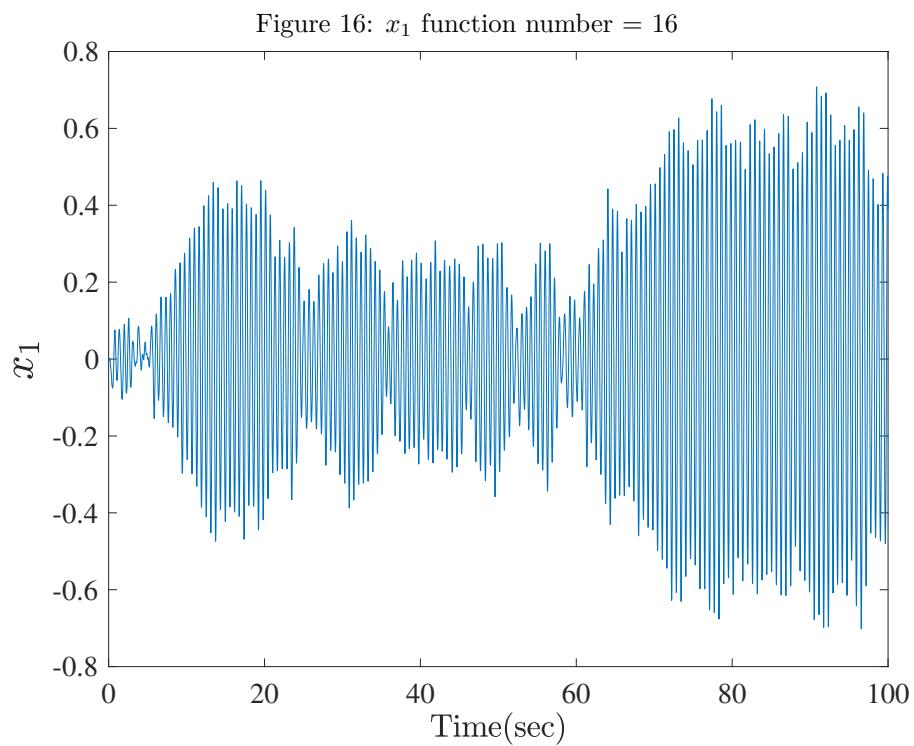
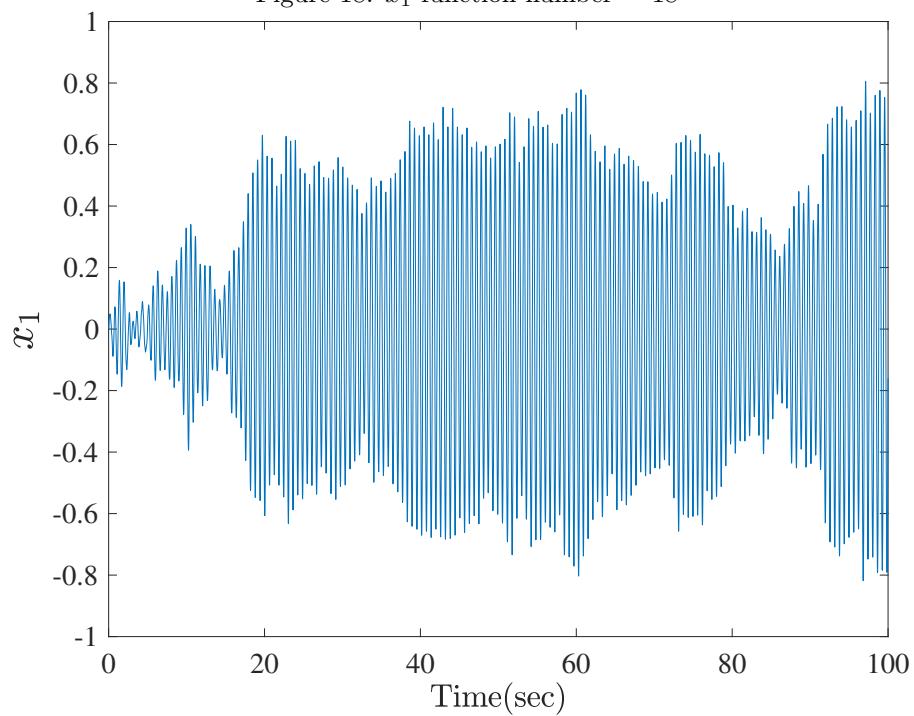
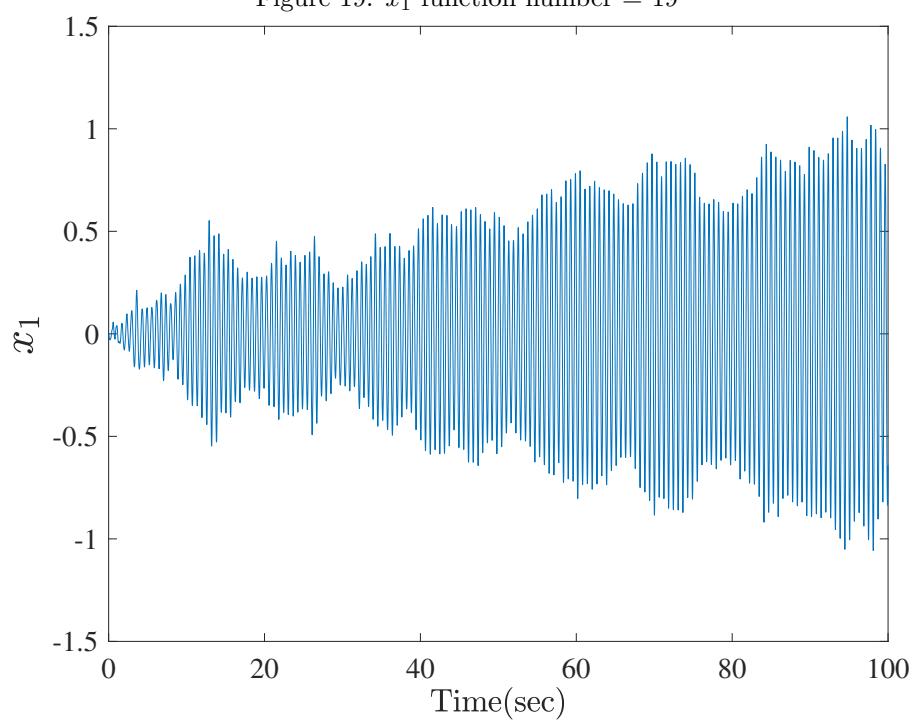
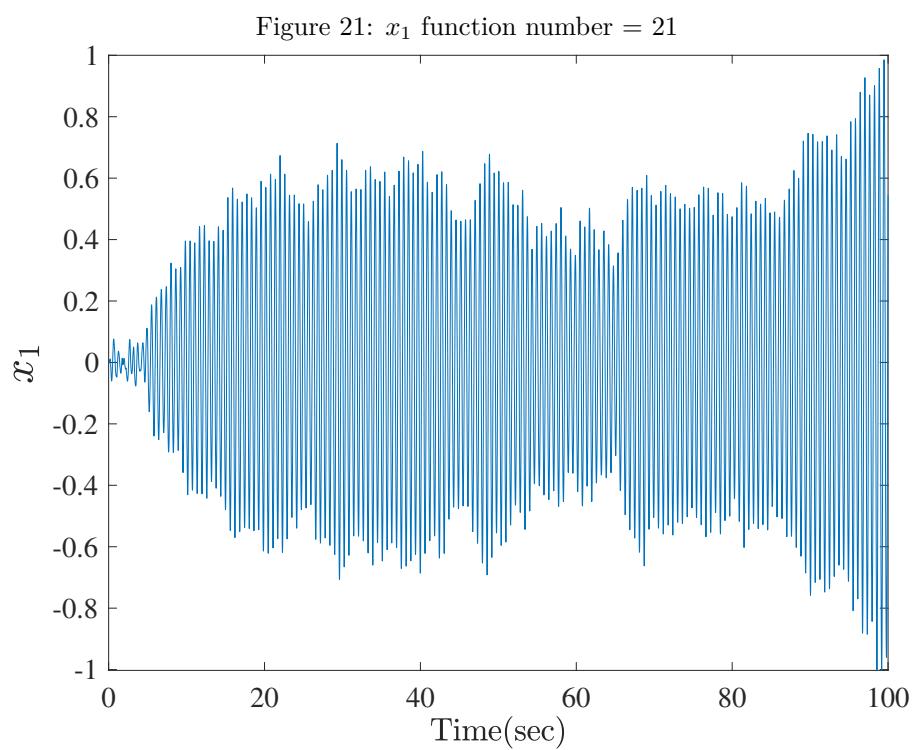
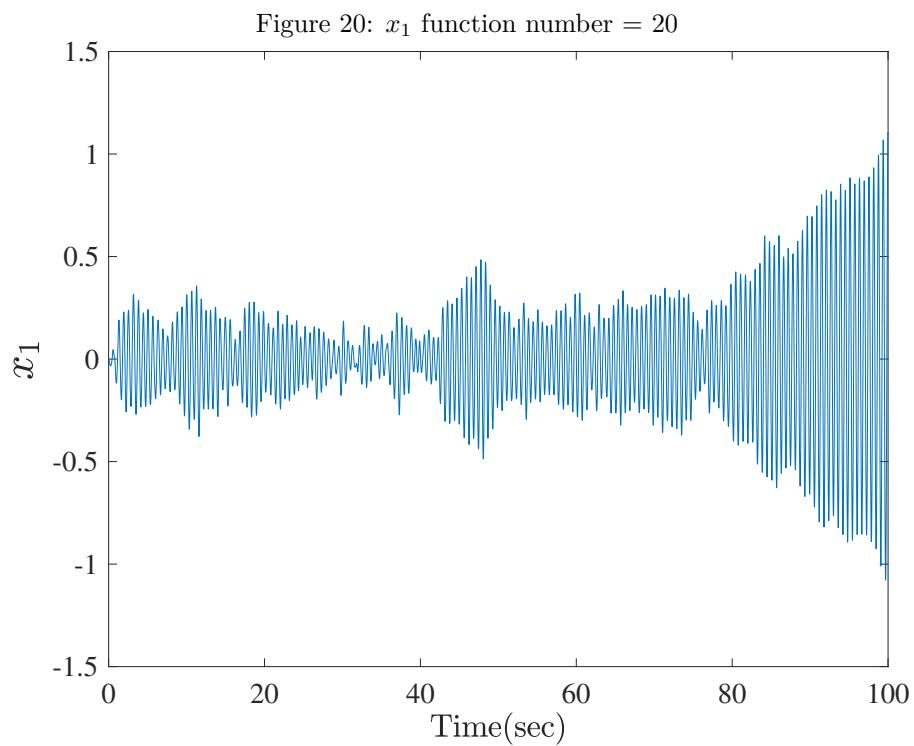
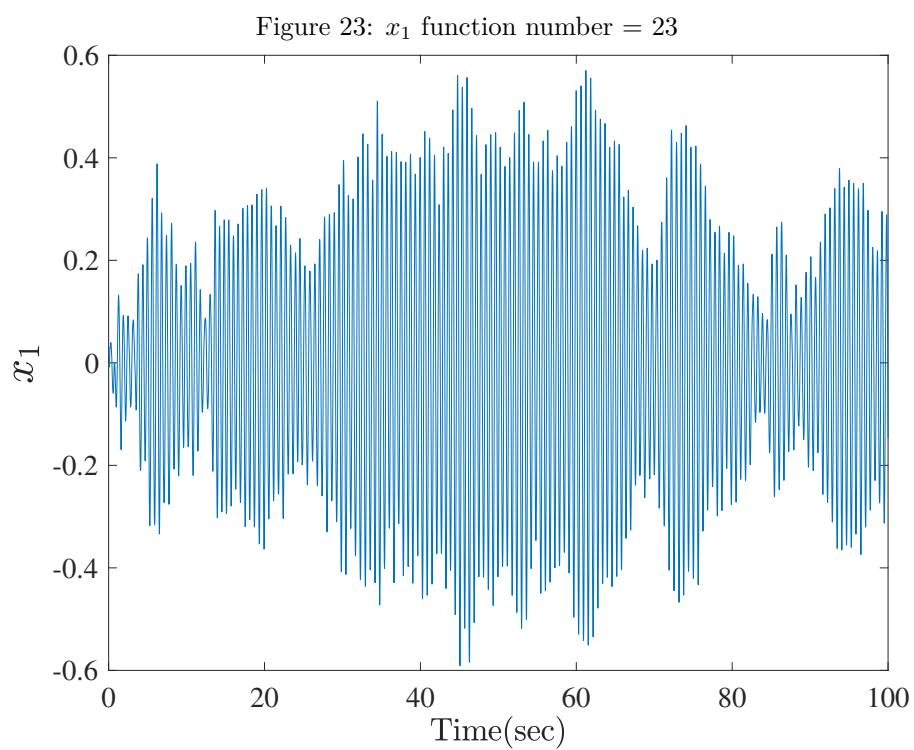
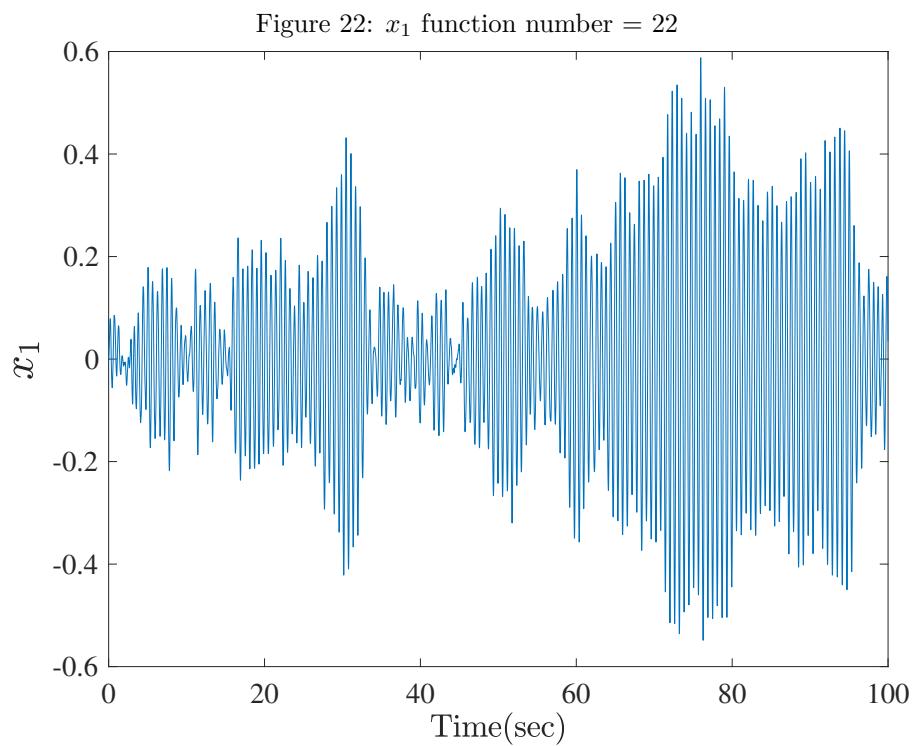
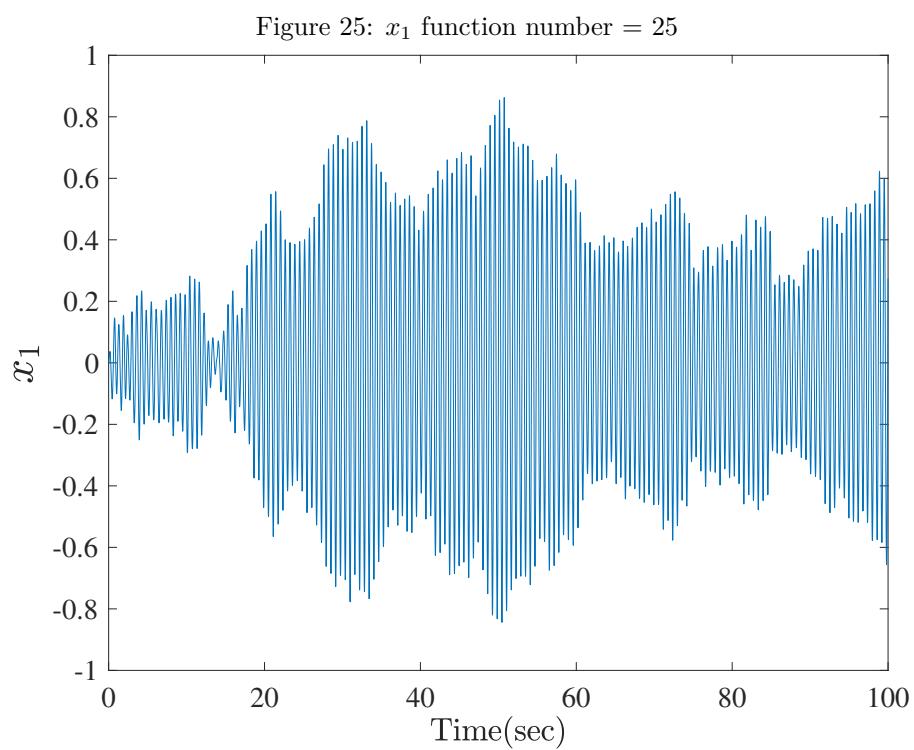
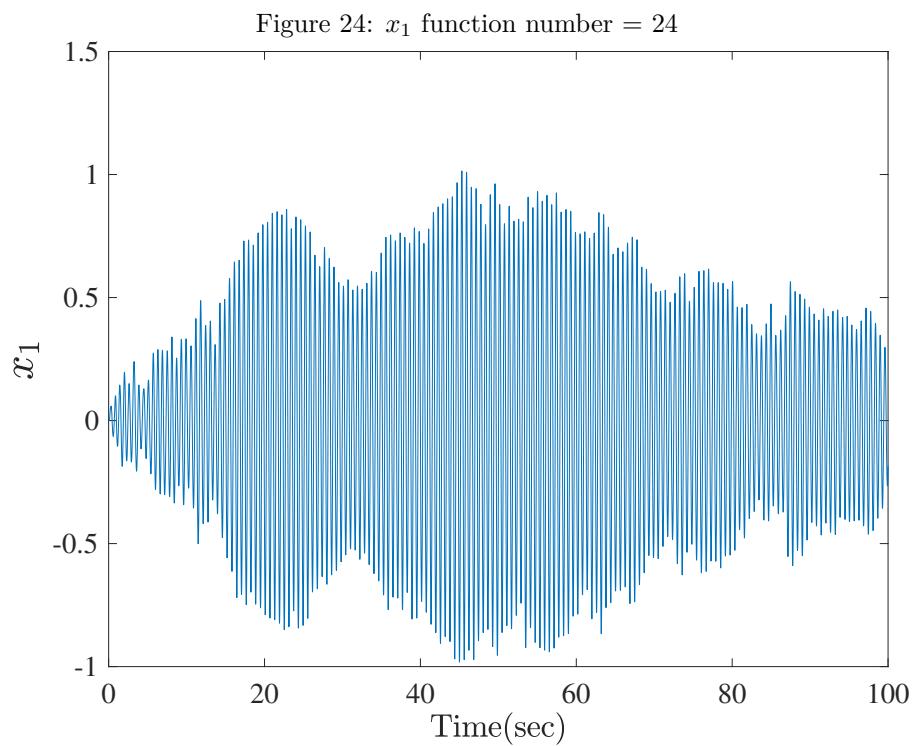
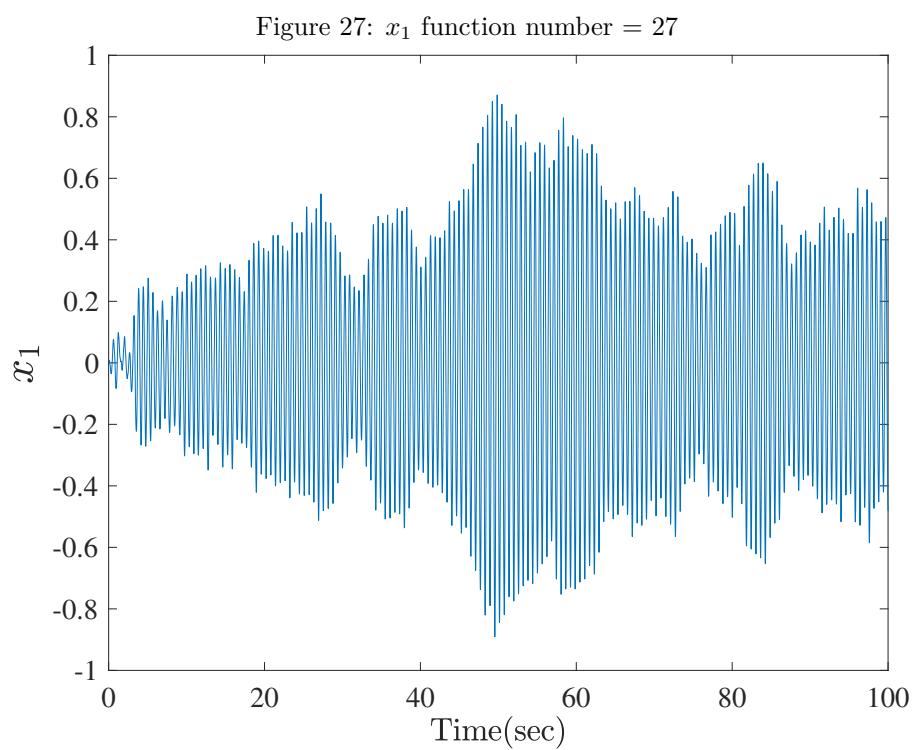
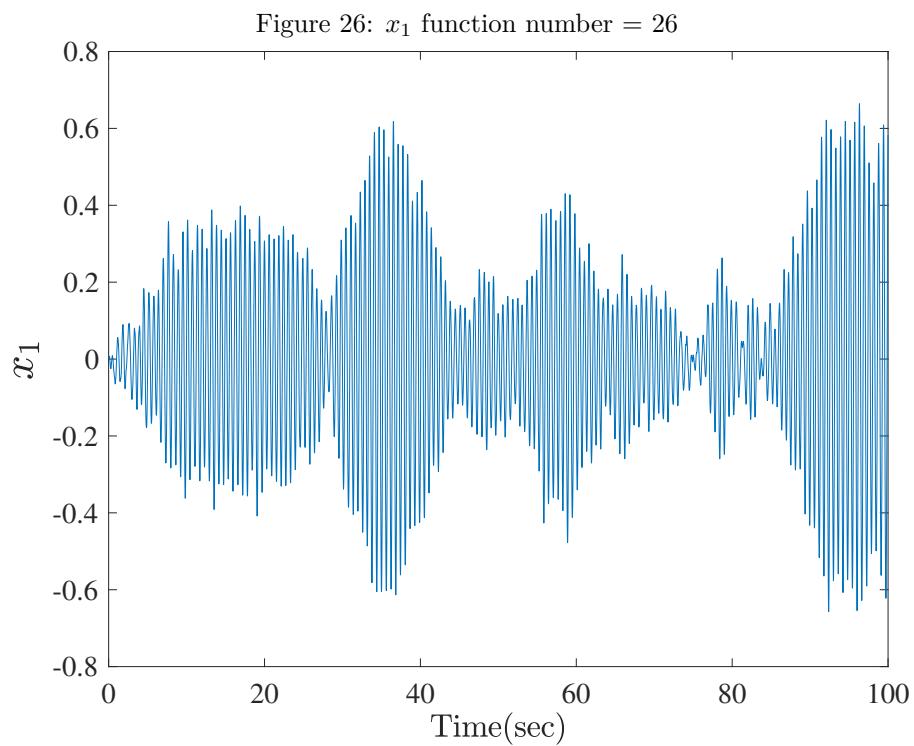


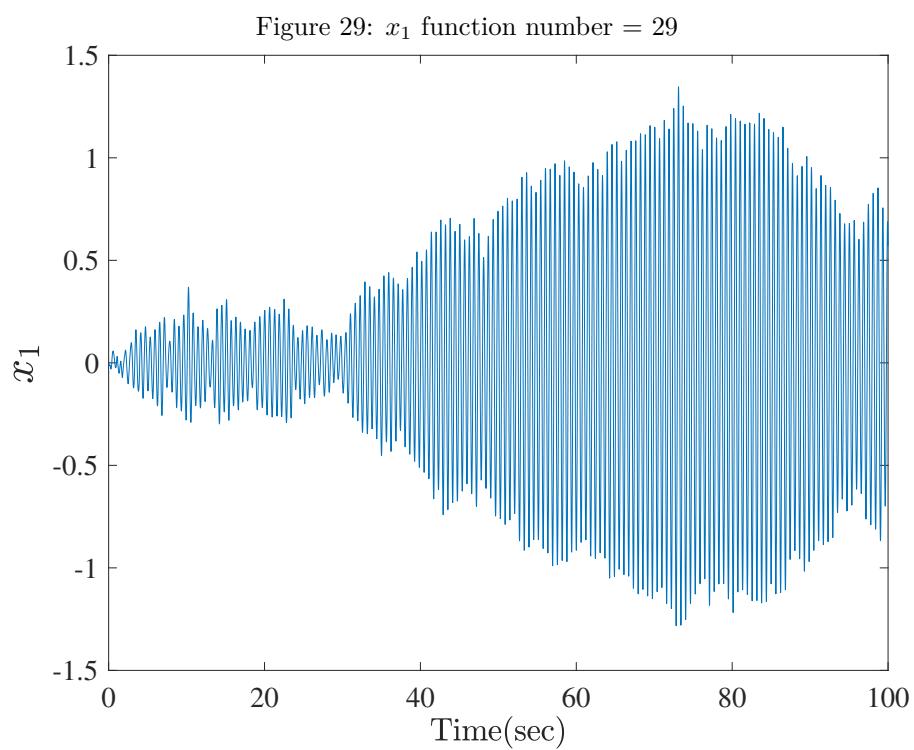
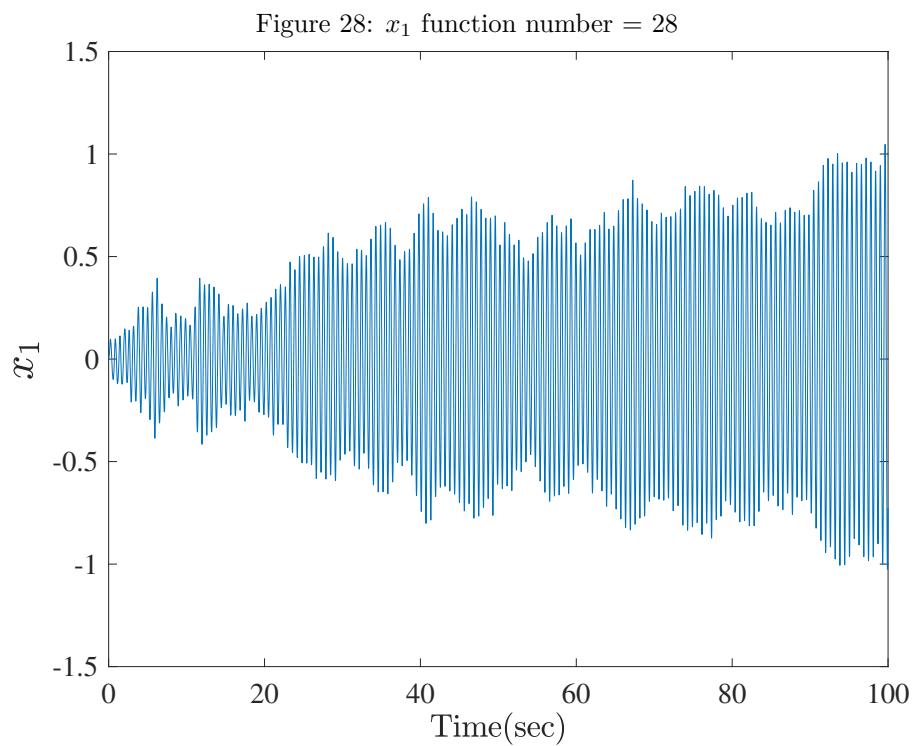
Figure 18: x_1 function number = 18Figure 19: x_1 function number = 19

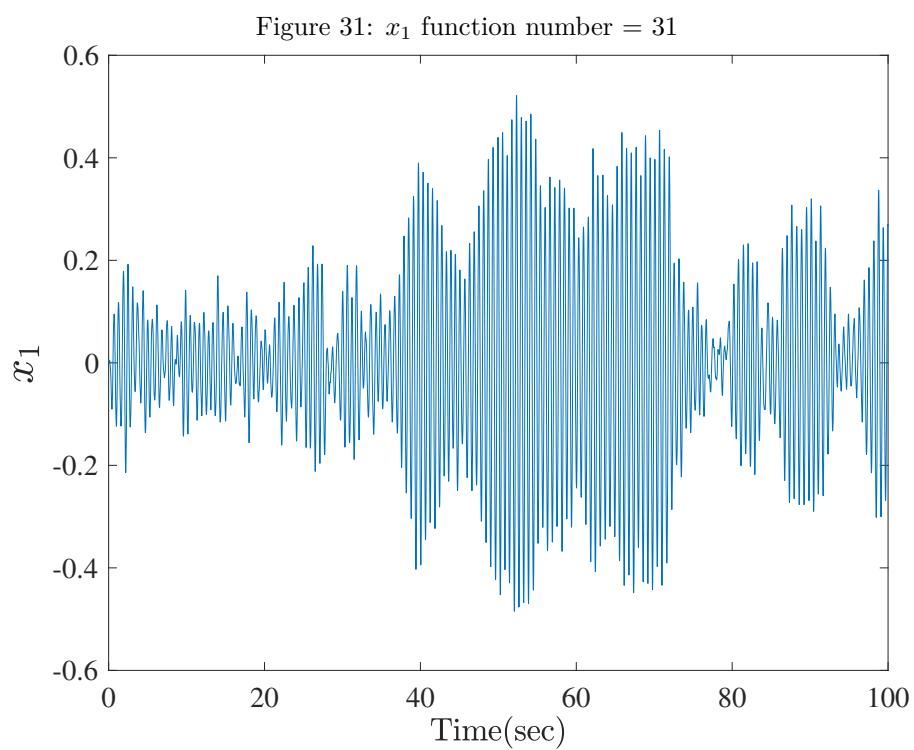
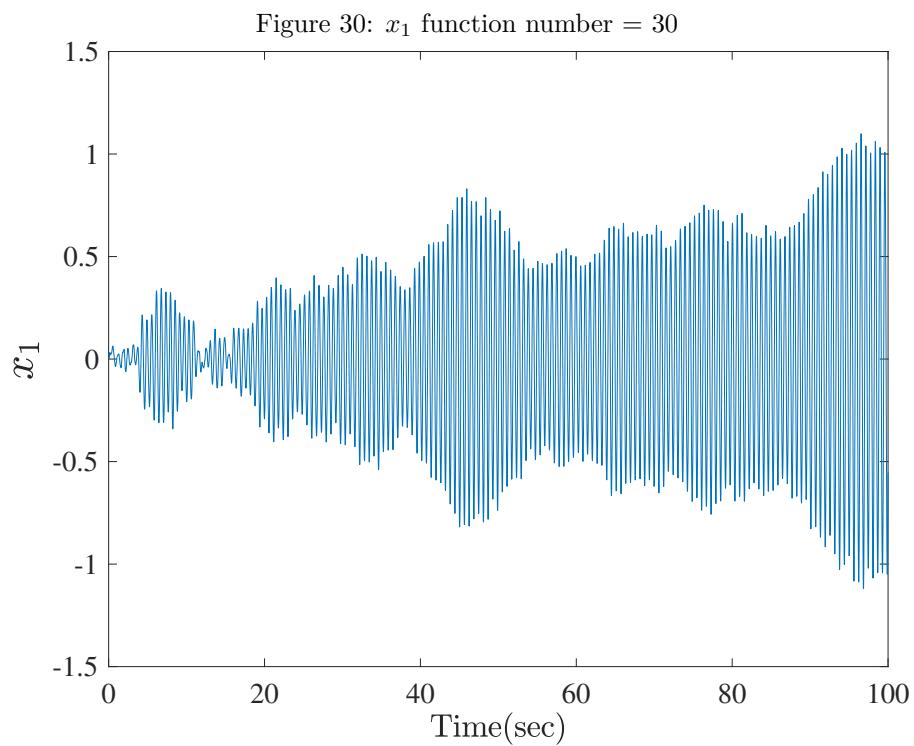


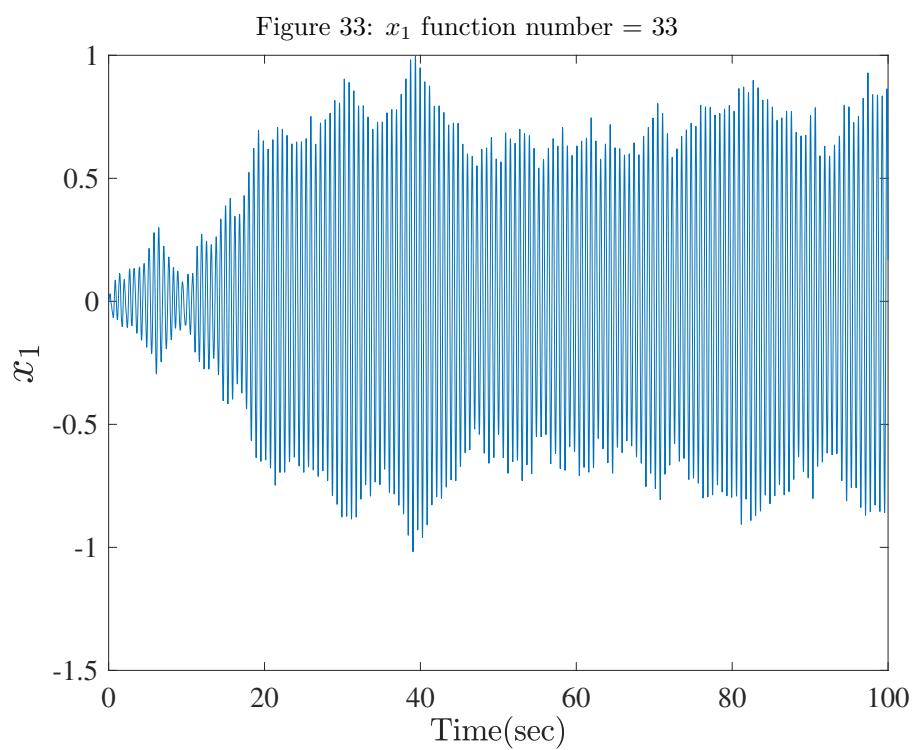
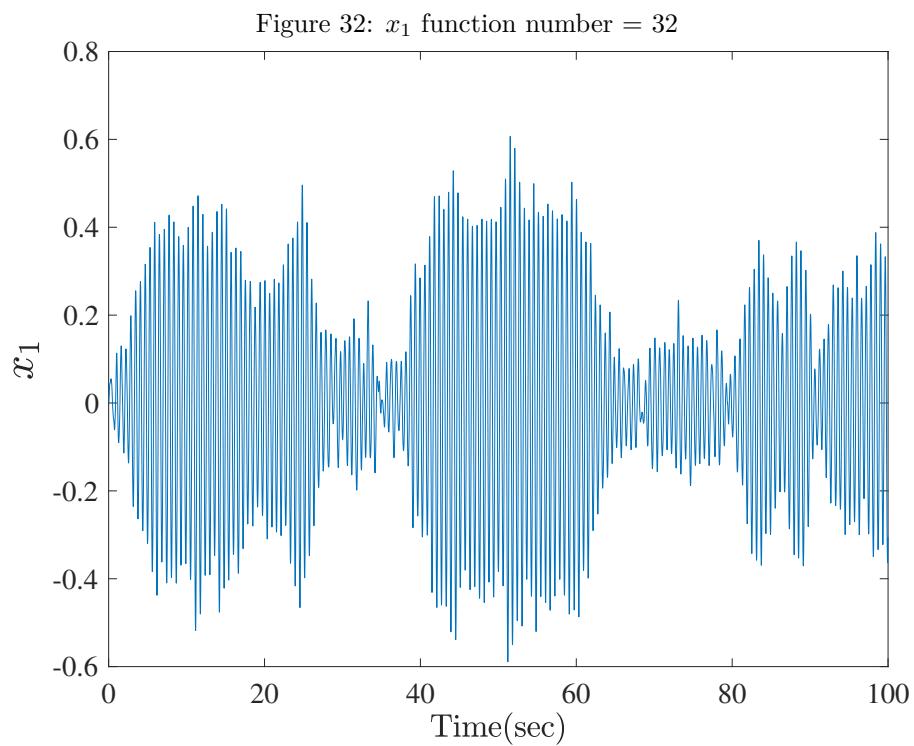












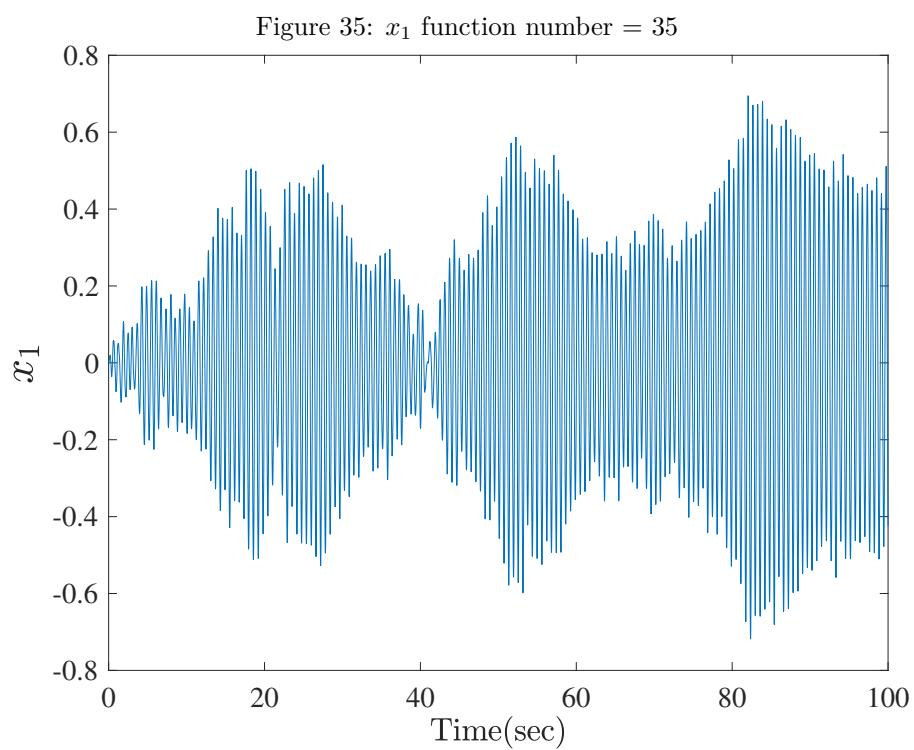
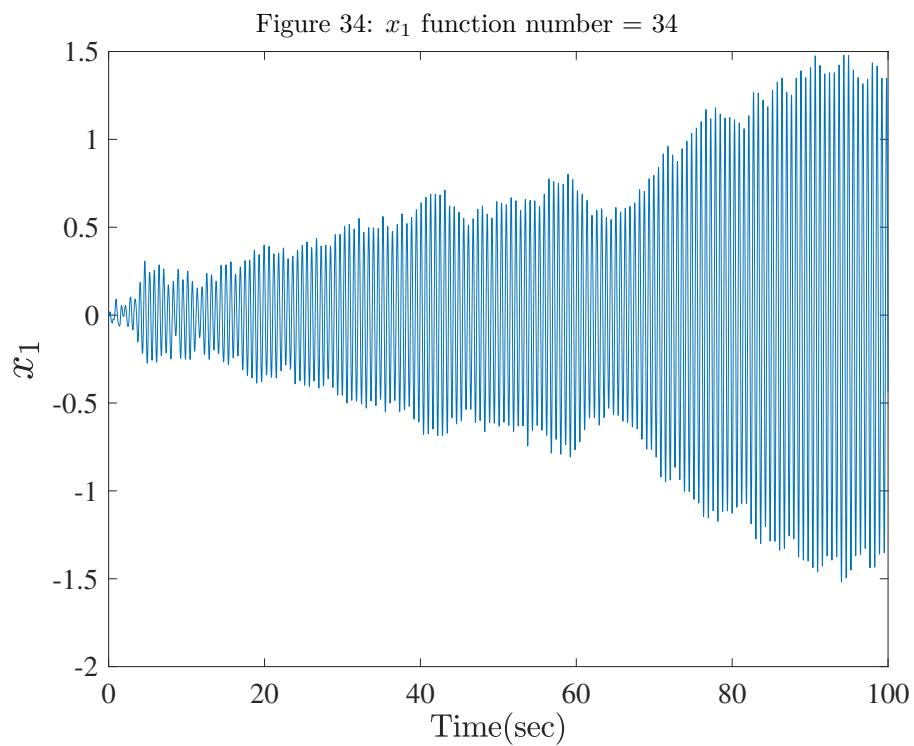


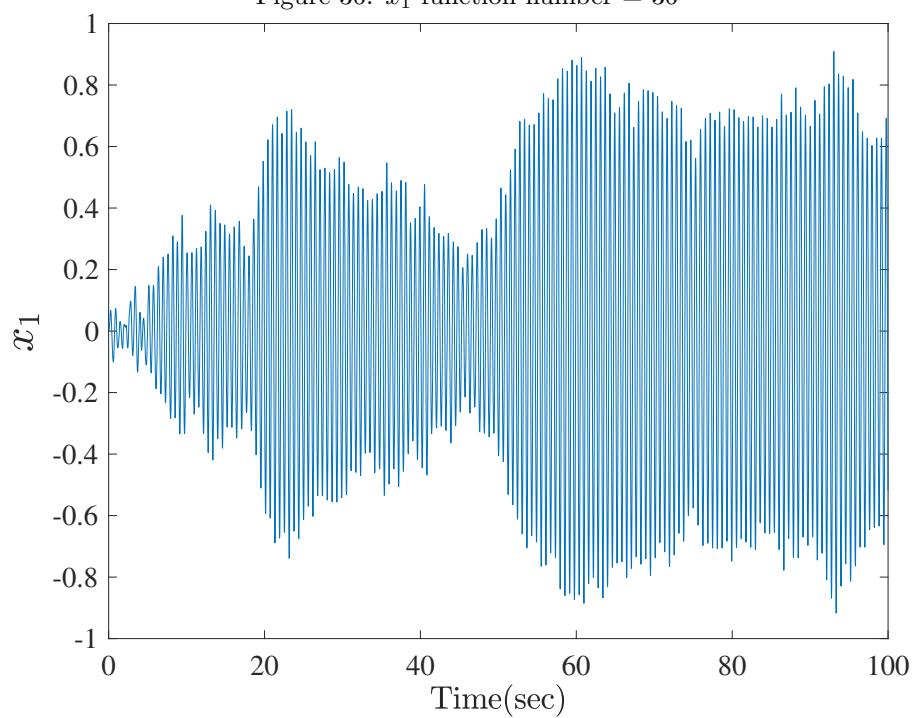
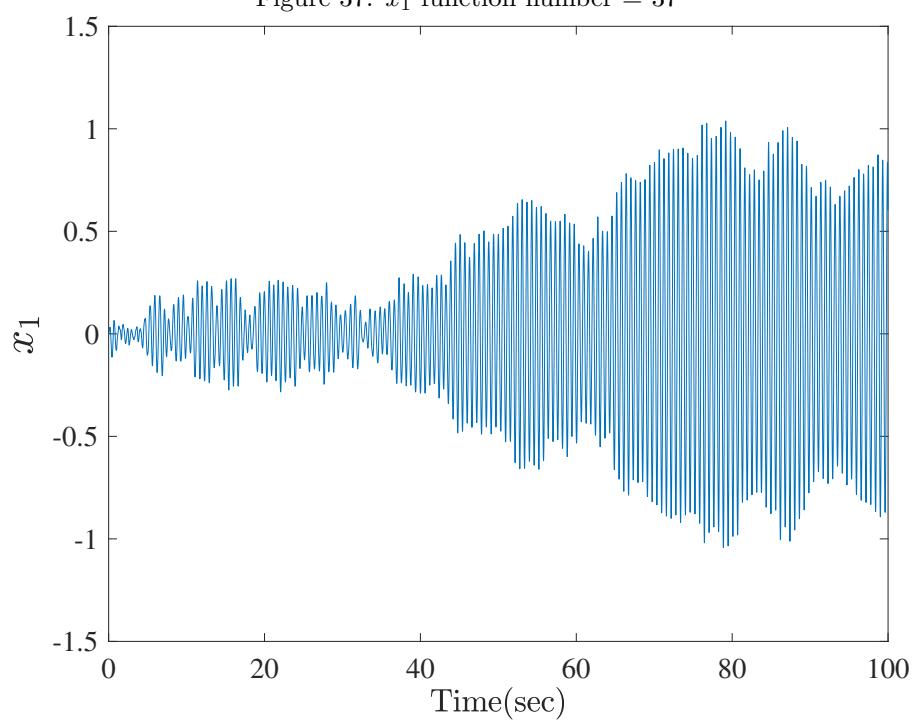
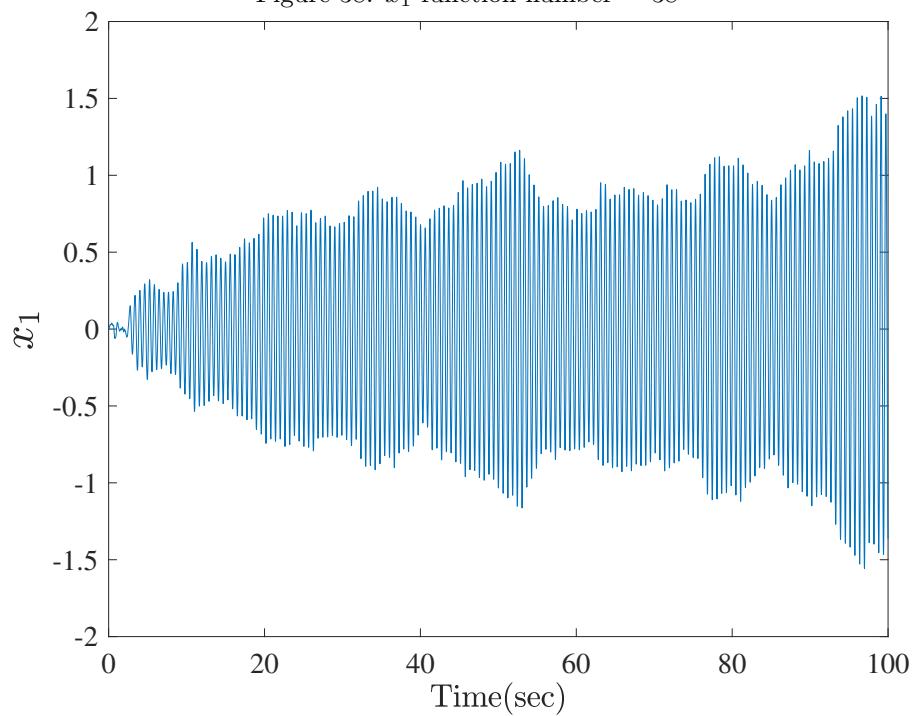
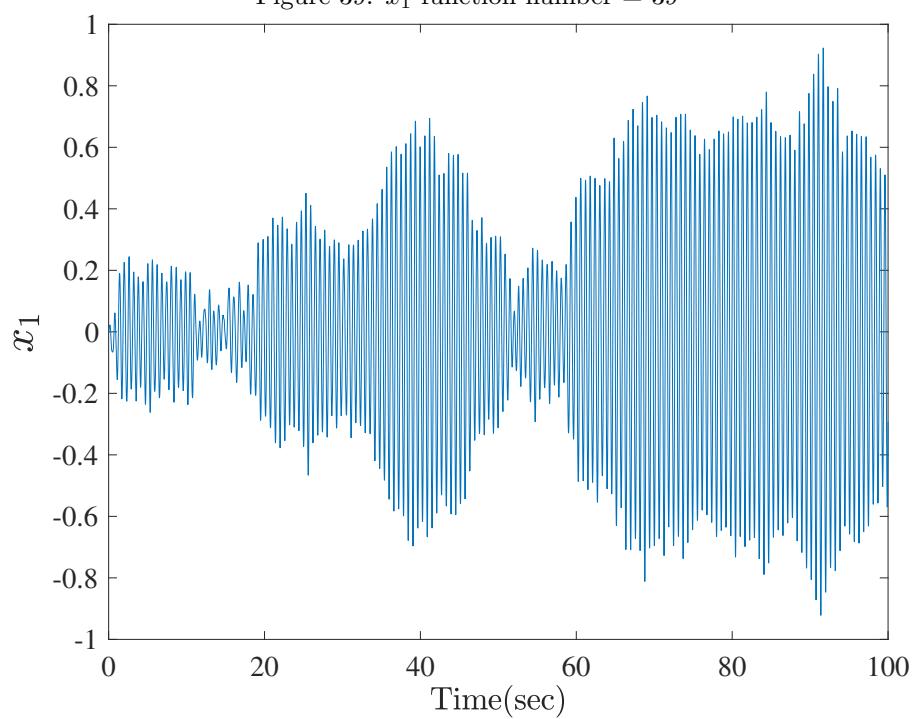
Figure 36: x_1 function number = 36Figure 37: x_1 function number = 37

Figure 38: x_1 function number = 38Figure 39: x_1 function number = 39

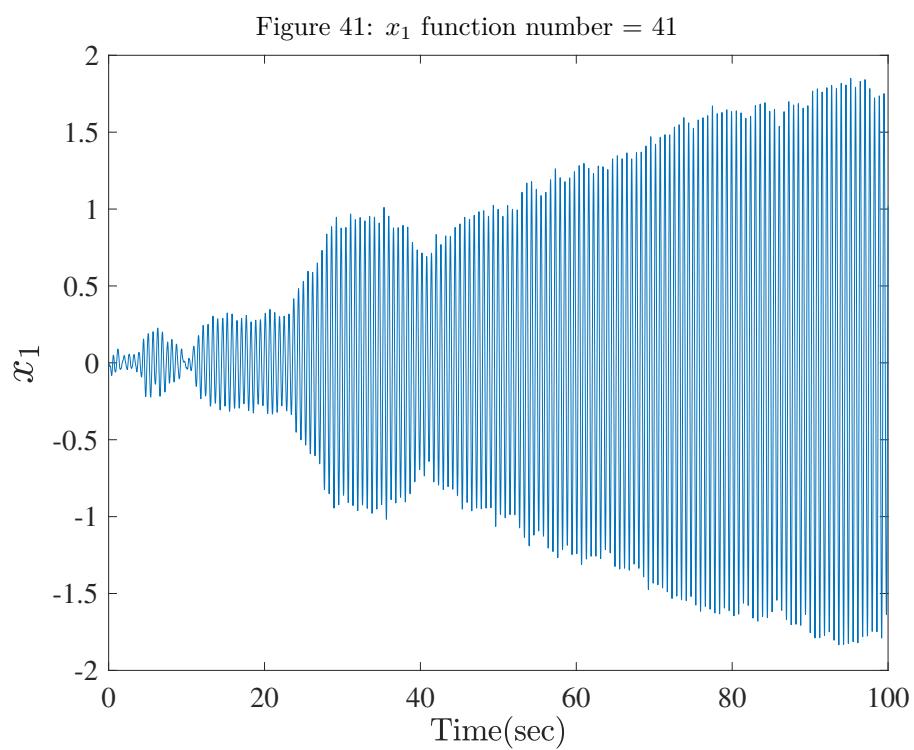
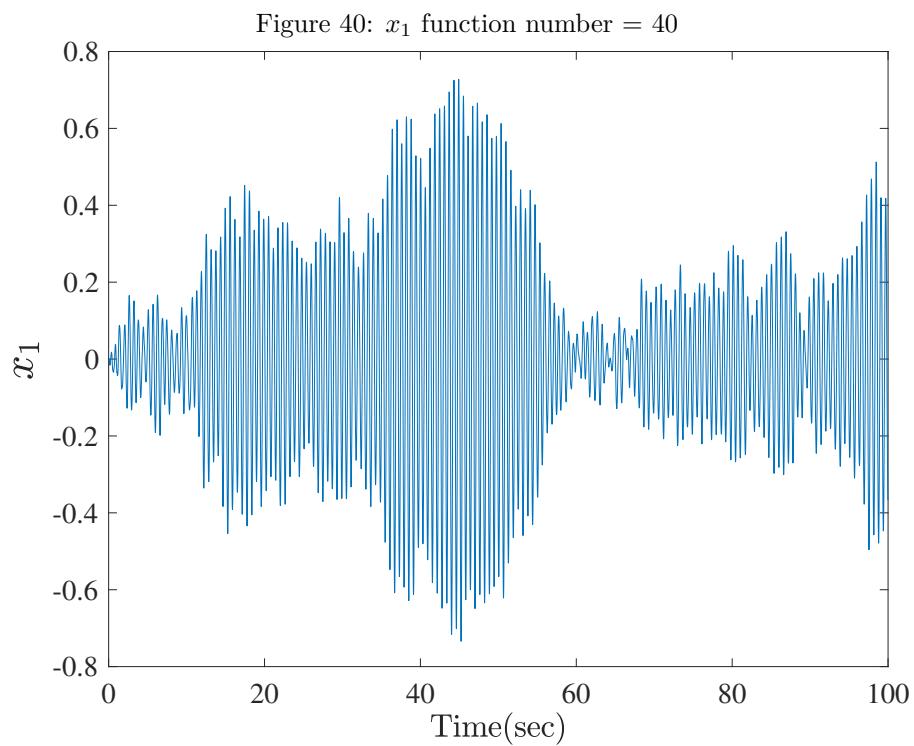


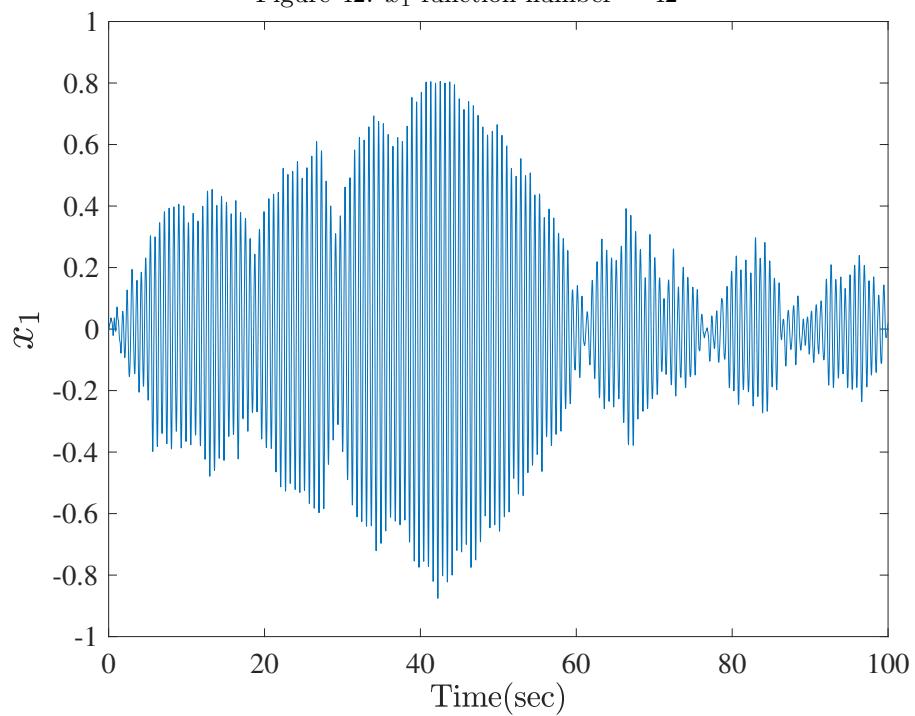
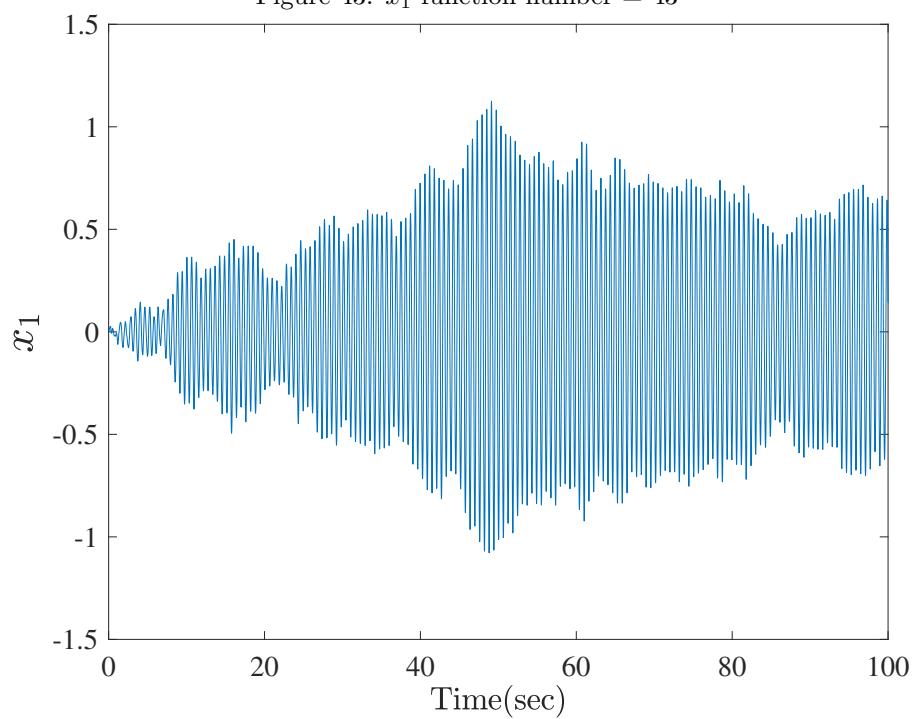
Figure 42: x_1 function number = 42Figure 43: x_1 function number = 43

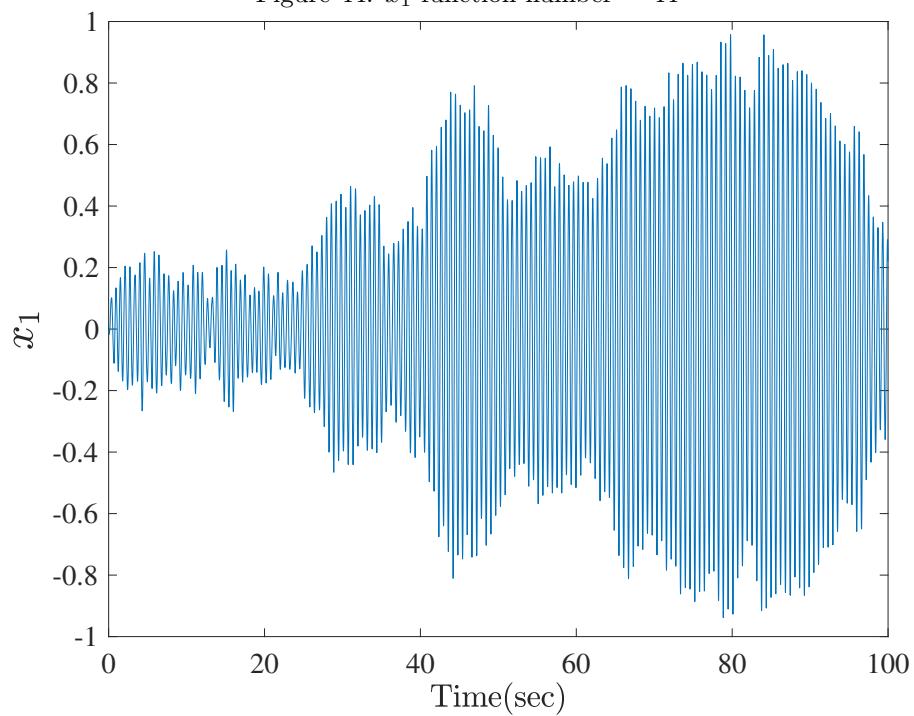
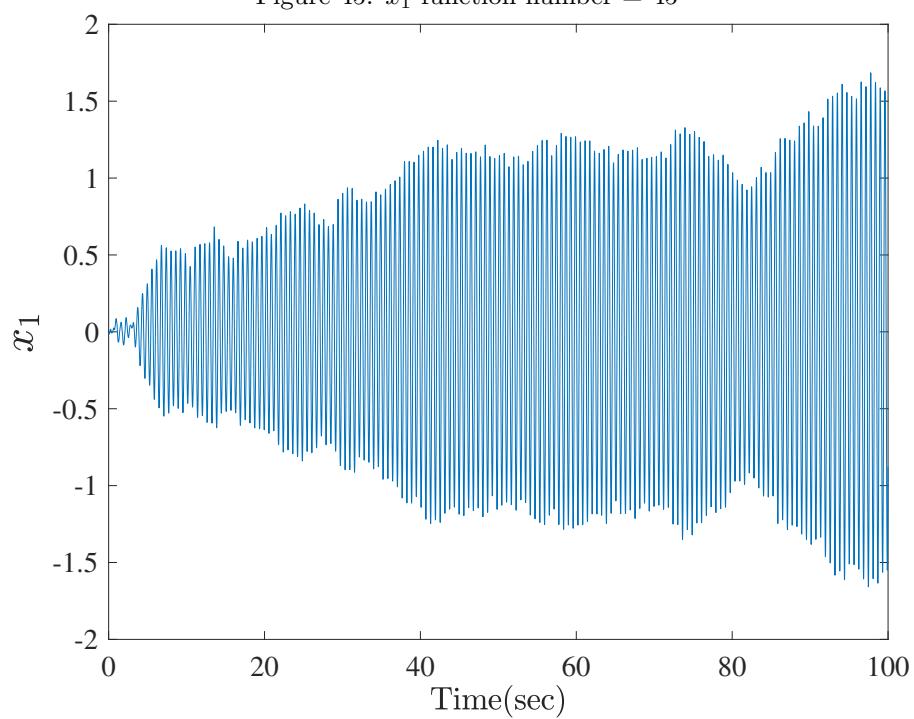
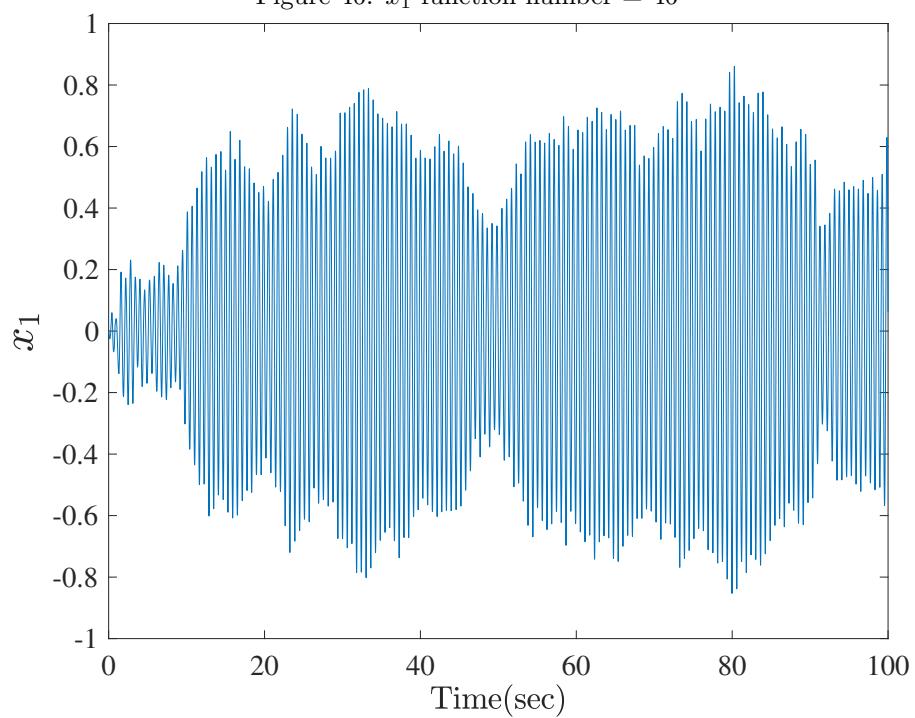
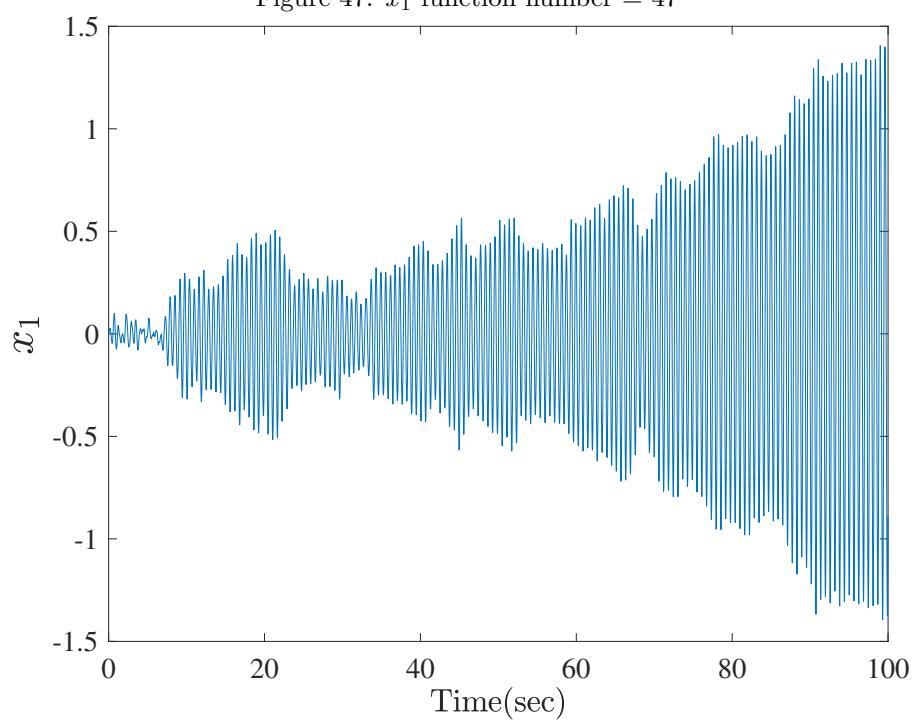
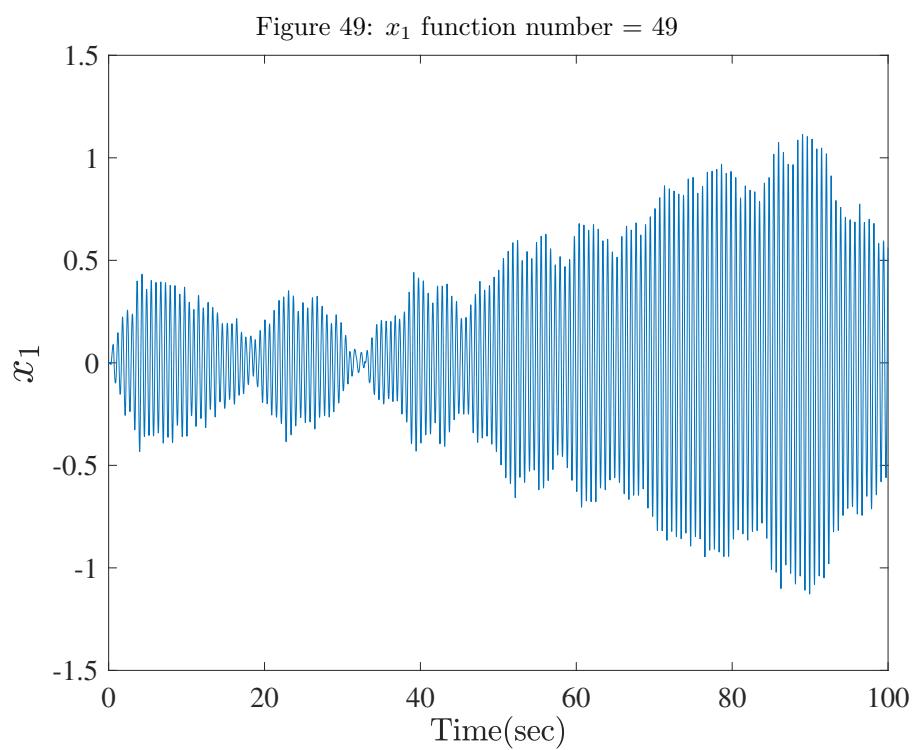
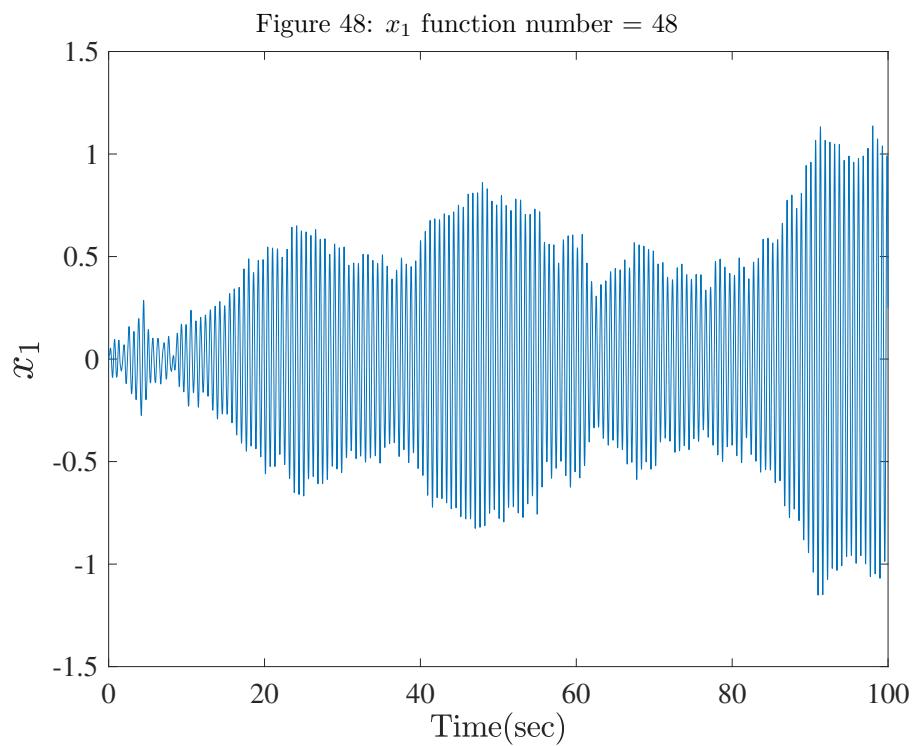
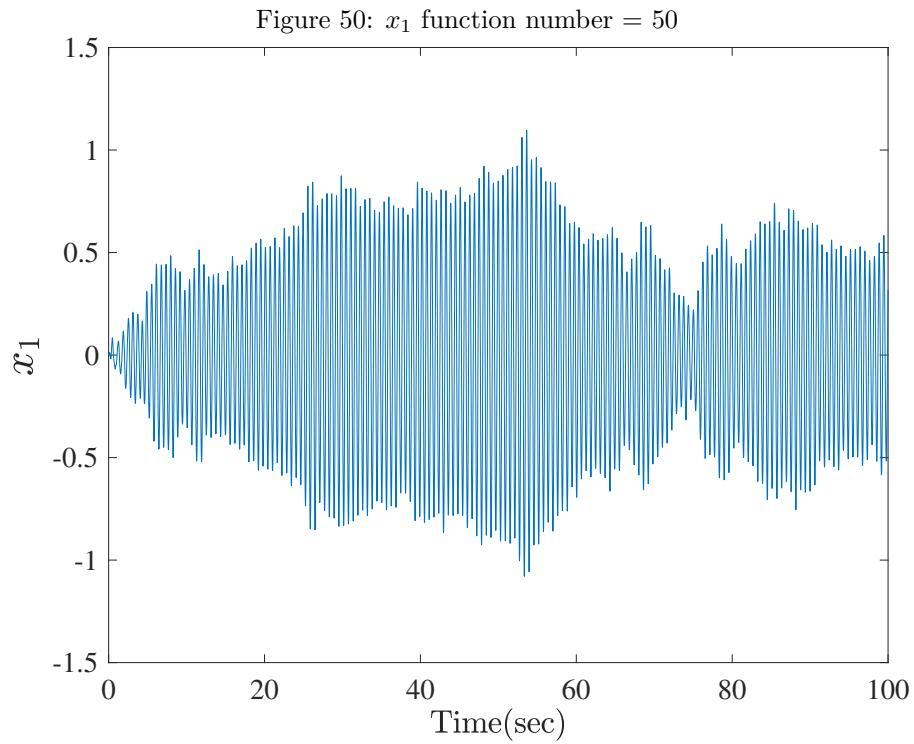
Figure 44: x_1 function number = 44Figure 45: x_1 function number = 45

Figure 46: x_1 function number = 46Figure 47: x_1 function number = 47





4.2 part b

Assume:

$$A[x(t)] = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t) dt = \frac{1}{2T} \sum_{i=0}^n x(t_i) \Delta t = 0.0008$$

4.3 part c

All parameters are calculated in the code (Q5/c.m).

$$\text{mean} = -0.0373$$

$$\text{variance} = 0.1255$$

4.4 part d

All parameters are calculated in the code (Q5/d.m).

$$\text{mean} = -0.0964$$

$$\text{variance} = 0.2509$$

4.5 extra

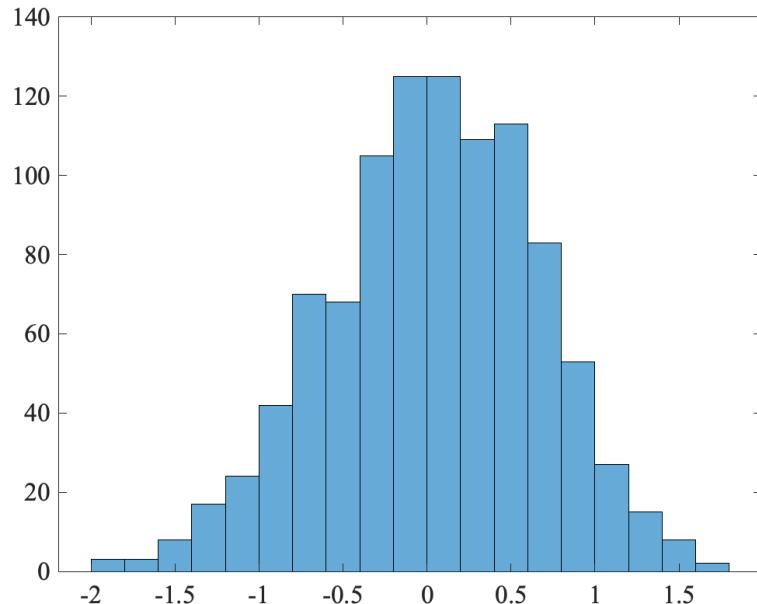
In this section, the function was simulated 1000 times instead of 50 times to see the difference and improvement.

- part c extra

mean = -0.0023

variance = 0.1489

Figure 51: histogram for $t = 80_{\text{sec}}$

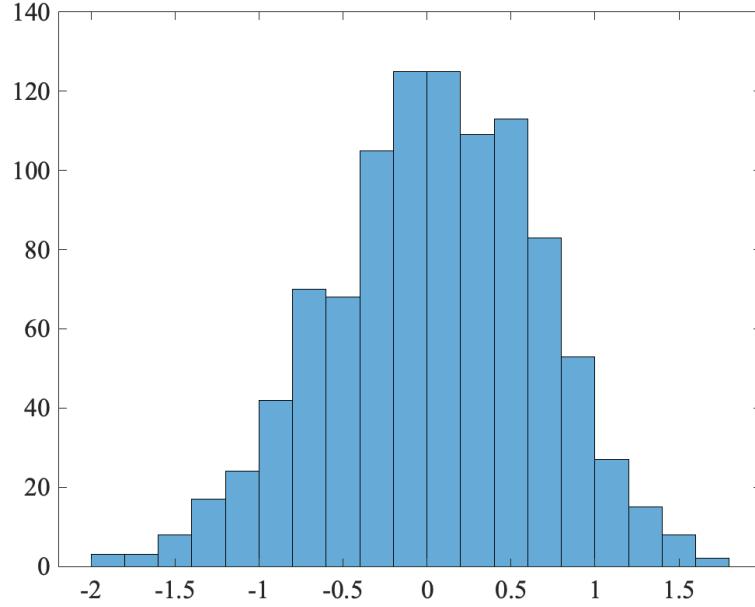


- part d extra

mean = 0.0361

variance = 0.3924

Figure 52: histogram for $t = 80_{\text{sec}}$



5 Question 6

$$\begin{bmatrix} \Delta \dot{x}_1 \\ \Delta \dot{x}_2 \\ \Delta \dot{x}_3 \\ \Delta \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3n^2 & 0 & 0 & 2nr_n \\ 0 & 0 & -\frac{2n}{r_n} & 0 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \\ \Delta x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & -\frac{1}{r_n} \end{bmatrix} \begin{bmatrix} w_r \\ w_\theta \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} r \\ \theta \\ \dot{r} \\ \dot{\theta} \end{bmatrix}$$

5.1 part a

$$C = [1 \ 0 \ 0 \ 0]$$

$$\text{rank} \left(\begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} \right) = 3$$

System is not observable.

5.2 part b

$$C = [0 \ 1 \ 0 \ 0]$$

$$\text{rank} \begin{pmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{pmatrix} = 4$$

System is observable.

5.3 part c

$$C = [1 \ 1 \ 0 \ 0]$$

$$\text{rank} \begin{pmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{pmatrix} = 4$$

System is observable.

5.4 part d

For subsection 5.2 we have:

$$\text{Singular Value} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

For subsection 5.3 we have:

$$\text{Singular Value} = \begin{bmatrix} 15.3452 \\ 1.4142 \\ 0.9979 \\ 0 \end{bmatrix}$$

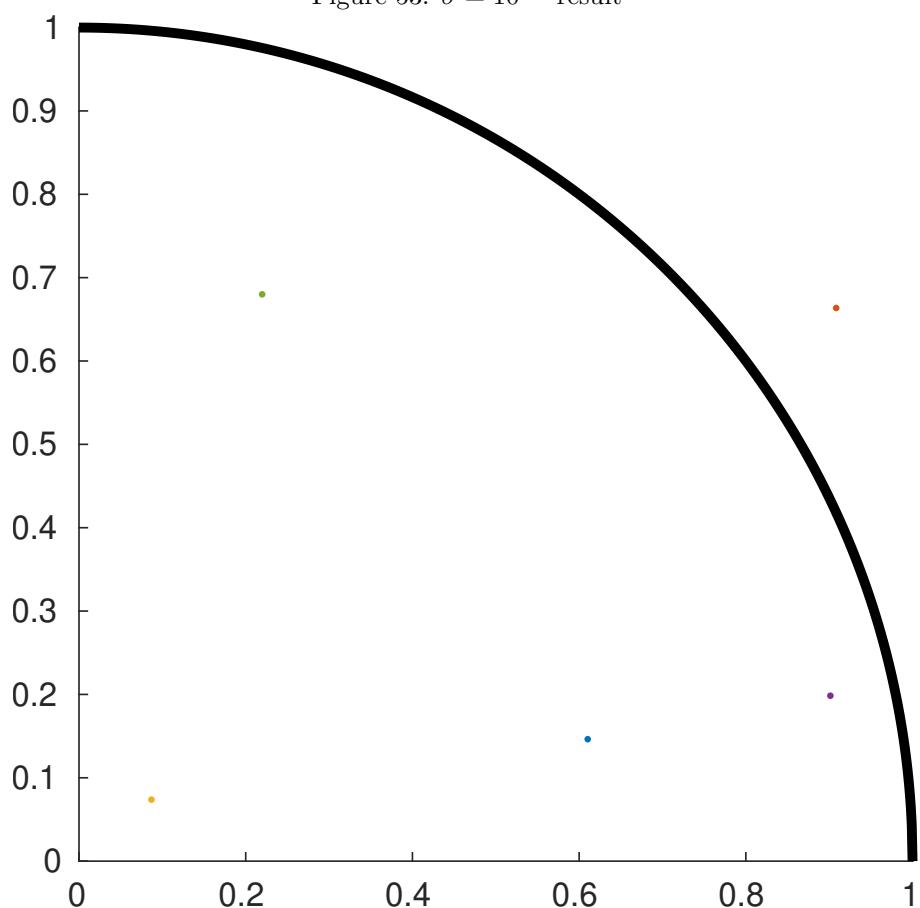
Observability is more straightforward in case 2 because the singular value is higher.

6 Bonus

This section uses the variance Criterion to find the π number. Here is the result of the π estimation with different variances:

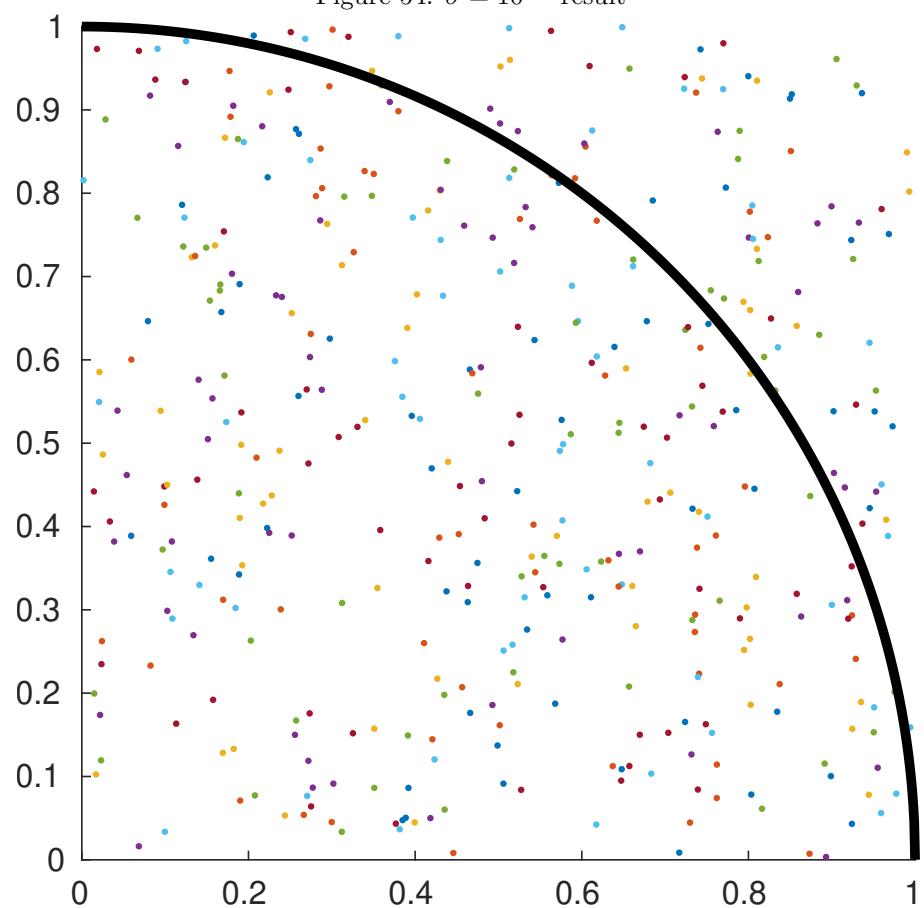
- $\sigma = 10^{-4}$

Figure 53: $\sigma = 10^{-4}$ result



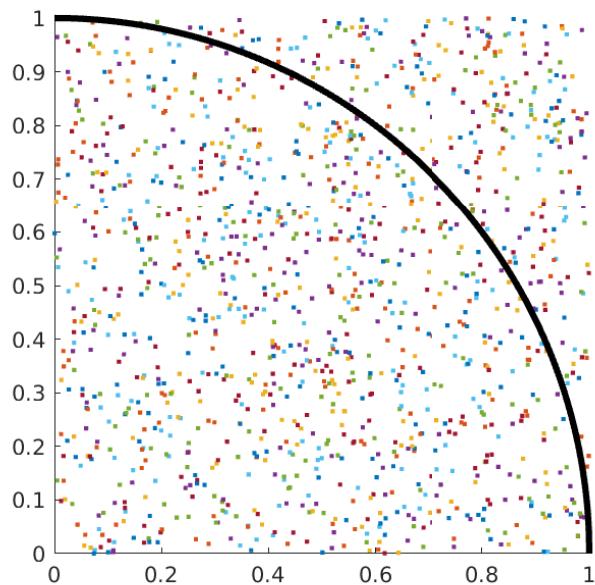
- $\sigma = 10^{-5}$

Figure 54: $\sigma = 10^{-5}$ result



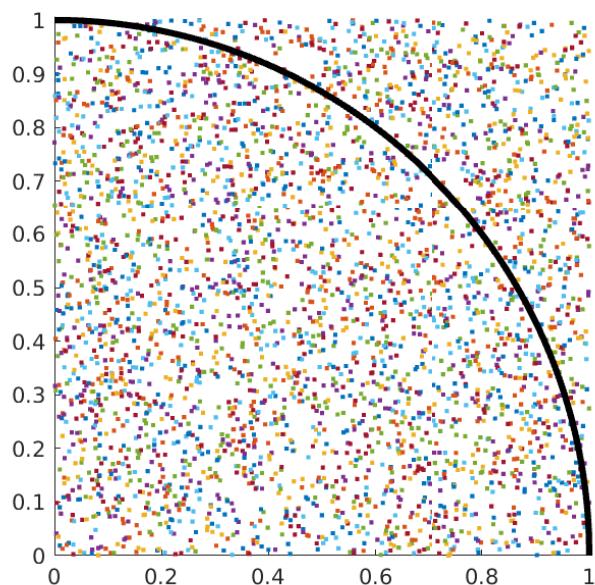
- $\sigma = 10^{-6}$

Figure 55: $\sigma = 10^{-6}$ result



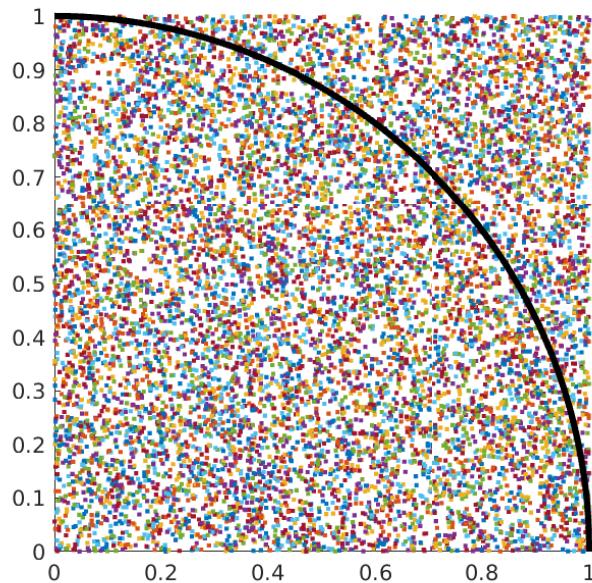
- $\sigma = 10^{-7}$

Figure 56: $\sigma = 10^{-7}$ result



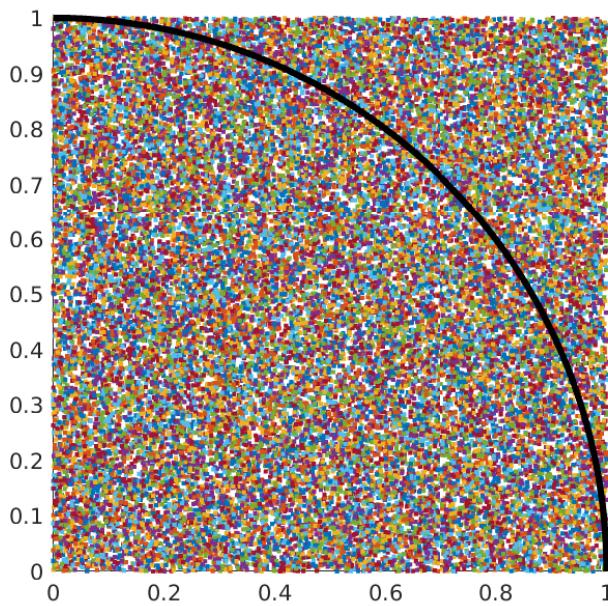
- $\sigma = 10^{-8}$

Figure 57: $\sigma = 10^{-8}$ result



- $\sigma = 10^{-9}$

Figure 58: $\sigma = 10^{-9}$ result



Contents

1 Question 1	1
1.1 part a	1
1.2 part b	1
2 Question 3	1
2.1 part a	2
2.2 part b	2
3 Question 4	2
3.1 part a	2
3.2 part b	2
3.3 part c	3
3.4 part d	3
3.5 part e	3
4 Question 5	4
4.1 part a	4
4.2 part b	29
4.3 part c	29
4.4 part d	29
4.5 extra	30
5 Question 6	31
5.1 part a	31
5.2 part b	31
5.3 part c	32
5.4 part d	32
6 Bonus	32

List of Figures

1	x_1 function number = 1	4
2	x_1 function number = 2	5
3	x_1 function number = 3	5
4	x_1 function number = 4	6
5	x_1 function number = 5	6
6	x_1 function number = 6	7
7	x_1 function number = 7	7
8	x_1 function number = 8	8
9	x_1 function number = 9	8
10	x_1 function number = 10	9
11	x_1 function number = 11	9
12	x_1 function number = 12	10
13	x_1 function number = 13	10
14	x_1 function number = 14	11
15	x_1 function number = 15	11
16	x_1 function number = 16	12
17	x_1 function number = 17	12
18	x_1 function number = 18	13
19	x_1 function number = 19	13
20	x_1 function number = 20	14
21	x_1 function number = 21	14
22	x_1 function number = 22	15
23	x_1 function number = 23	15
24	x_1 function number = 24	16
25	x_1 function number = 25	16
26	x_1 function number = 26	17
27	x_1 function number = 27	17
28	x_1 function number = 28	18
29	x_1 function number = 29	18
30	x_1 function number = 30	19
31	x_1 function number = 31	19
32	x_1 function number = 32	20
33	x_1 function number = 33	20
34	x_1 function number = 34	21
35	x_1 function number = 35	21
36	x_1 function number = 36	22
37	x_1 function number = 37	22
38	x_1 function number = 38	23
39	x_1 function number = 39	23
40	x_1 function number = 40	24
41	x_1 function number = 41	24
42	x_1 function number = 42	25
43	x_1 function number = 43	25
44	x_1 function number = 44	26
45	x_1 function number = 45	26
46	x_1 function number = 46	27
47	x_1 function number = 47	27
48	x_1 function number = 48	28
49	x_1 function number = 49	28
50	x_1 function number = 50	29

51	histogram for $t = 80_{\text{sec}}$	30
52	histogram for $t = 80_{\text{sec}}$	31
53	$\sigma = 10^{-4}$ result	33
54	$\sigma = 10^{-5}$ result	34
55	$\sigma = 10^{-6}$ result	35
56	$\sigma = 10^{-7}$ result	35
57	$\sigma = 10^{-8}$ result	36
58	$\sigma = 10^{-9}$ result	36

List of Tables