In The Name of God

Sharif University of Technology
Department of Aerospace Engineering

45-766: Optimal Control II

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CH#3: Propagation of States and Covariances

Discrete-Time Systems

- $x_k = F_{k-1} x_{k-1} + G_{k-1} u_{k-1} + w_{k-1}$

Gaussian zero-mean white noise with covariance Q_{k-1}

- How does the mean of the state x_k change with time?

$$\begin{aligned} \bar{x}_k &= E(x_k) \\ &= F_{k-1}\bar{x}_{k-1} + G_{k-1}u_{k-1} \end{aligned}$$

- How does the covariance of x_k change with time?

$$P_k = E\left[(x_k - \bar{x}_k)(\cdots)^T\right]$$

Discrete-Time Systems

- $$\begin{aligned}(x_k - \bar{x}_k)(\cdots)^T &= (F_{k-1}x_{k-1} + G_{k-1}u_{k-1} + w_{k-1} - \bar{x}_k)(\cdots)^T \\ &= [F_{k-1}(x_{k-1} - \bar{x}_{k-1}) + w_{k-1}][\cdots]^T \\ &= F_{k-1}(x_{k-1} - \bar{x}_{k-1})(x_{k-1} - \bar{x}_{k-1})^T F_{k-1}^T + w_{k-1}w_{k-1}^T + \\ &\quad F_{k-1}(x_{k-1} - \bar{x}_{k-1})w_{k-1}^T + w_{k-1}(x_{k-1} - \bar{x}_{k-1})^T F_{k-1}^T\end{aligned}$$

- $$\begin{aligned}P_k &= E\left[(x_k - \bar{x}_k)(\cdots)^T\right] \\ &= F_{k-1}P_{k-1}F_{k-1}^T + Q_{k-1}\end{aligned}$$

discrete-time Lyapunov equation, or a Stein equation

Discrete-Time Systems

- **Theorem 21** *Consider the equation $P = FPF^T + Q$ where F and Q are real matrices. Denote by $\lambda_i(F)$ the eigenvalues of the F matrix.*

  1. *A unique solution P exists if and only if $\lambda_i(F)\lambda_j(F) \neq 1$ for all i, j. This unique solution is symmetric.*

  2. *Note that the above condition includes the case of stable F, because if F is stable then all of its eigenvalues are less than one in magnitude, so $\lambda_i(F)\lambda_j(F) \neq 1$ for all i, j. Therefore, we see that if F is stable then the discrete-time Lyapunov equation has a solution P that is unique and symmetric. In this case, the solution can be written as*

$$P = \sum_{i=0}^{\infty} F^i Q (F^T)^i$$

Discrete-Time Systems

3. *If F is stable and Q is positive (semi)definite, then the unique solution P is symmetric and positive (semi)definite.*

4. *If F is stable, Q is positive semidefinite, and $(F, Q^{1/2})$ is controllable, then P is unique, symmetric, and positive definite. Note that $Q^{1/2}$, the square root of Q, is defined here as any matrix such that $Q^{1/2}(Q^{1/2})^T = Q$.*

Discrete-Time Systems

- Solution of the linear system: $x_k = F_{k,0}x_0 + \sum_{i=0}^{k-1}(F_{k,i+1}w_i + F_{k,i+1}G_iu_i)$

where $F_{k,i} = \begin{cases} F_{k-1}F_{k-2}\cdots F_i & k > i \\ I & k = i \\ 0 & k < i \end{cases}$

$$\Rightarrow \quad x_k \sim N(\bar{x}_k, P_k)$$

- $x_k = F_{k-1}x_{k-1} + G_{k-1}u_{k-1} + L_{k-1}\tilde{w}_{k-1}, \qquad \tilde{w}_k \sim (0, \tilde{Q}_k)$

$$\begin{aligned} E\left[(L_{k-1}\tilde{w}_{k-1})(L_{k-1}\tilde{w}_{k-1})^T\right] &= L_{k-1}E(\tilde{w}_{k-1}\tilde{w}_{k-1}^T)L_{k-1}^T \\ &= L_{k-1}\tilde{Q}_{k-1}L_{k-1}^T \end{aligned}$$

$$\Rightarrow \quad x_k = F_{k-1}x_{k-1} + G_{k-1}u_{k-1} + w_{k-1}, \qquad w_k \sim (0, L_k\tilde{Q}_kL_k^T)$$

Discrete-Time Systems

- Measurement equation

$$y_k = H_k x_k + L_k \tilde{v}_k, \qquad \tilde{v}_k \sim (0, \tilde{R}_k)$$

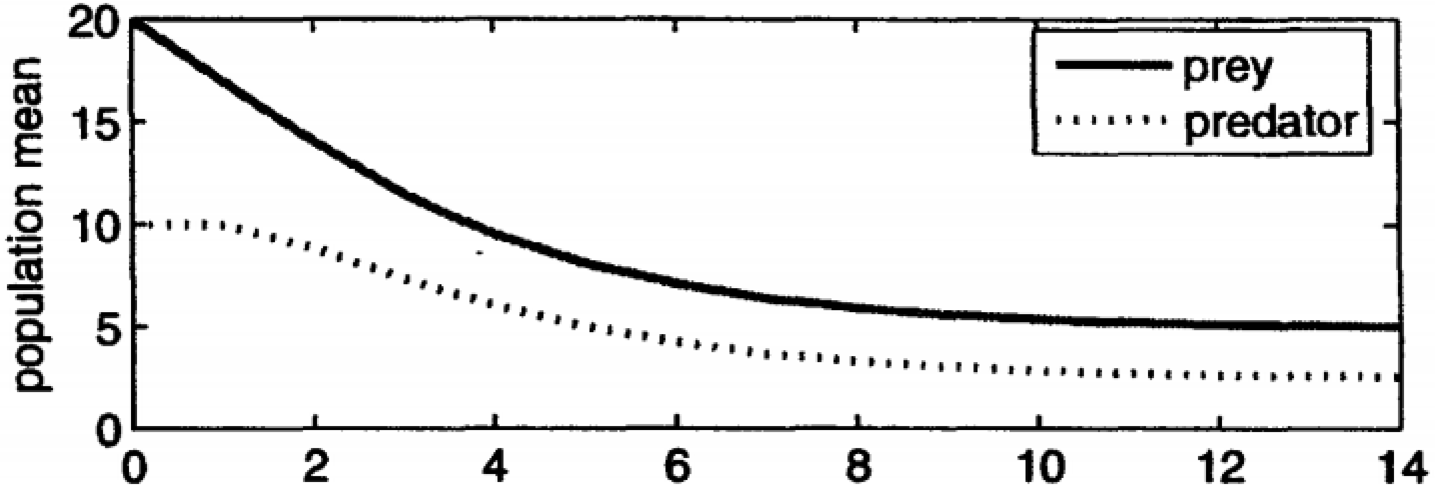
$$y_k = H_k x_k + v_k, \qquad v_k \sim (0, L_k \tilde{R}_k L_k^T)$$

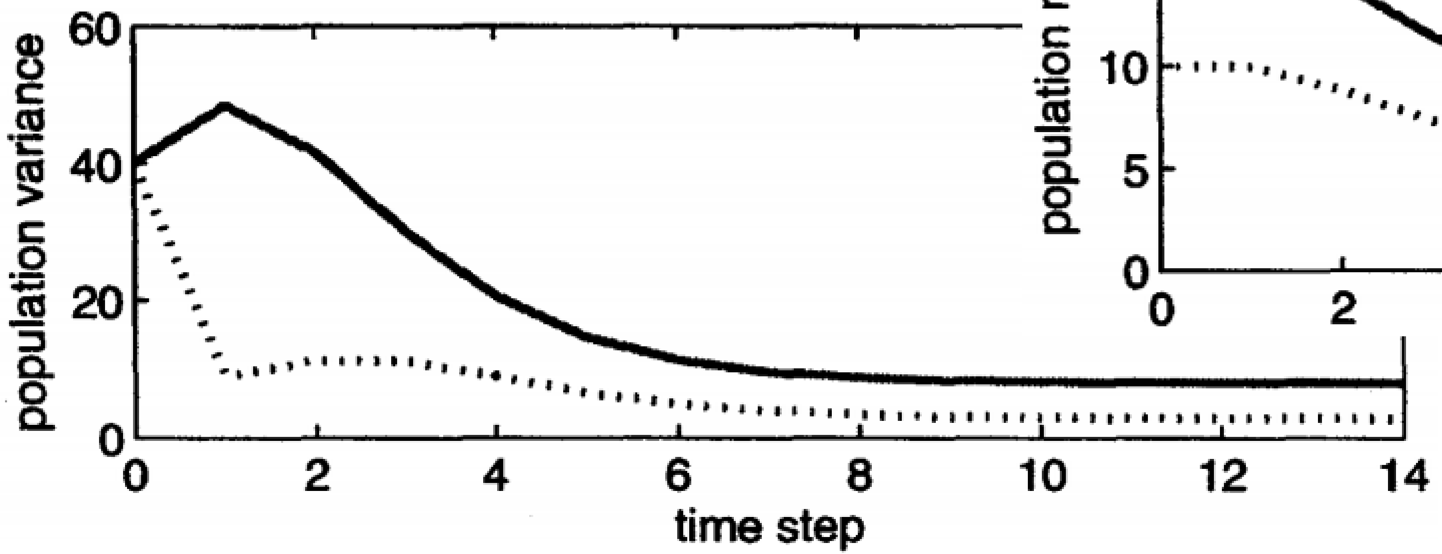
Example

- $$x_{k+1} = \begin{bmatrix} 0.2 & 0.4 \\ -0.4 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k + w_k$$

$$w_k \sim (0, Q) \quad Q = \mathrm{diag}(1, 2) \quad u_k = 1$$

$$\bar{x}_0 = \begin{bmatrix} 10 & 20 \end{bmatrix}^T \text{ and } P_0 = \mathrm{diag}(40, 40)$$





Example

- It is seen that the mean and covariance eventually reach steady-state values given by

$$
\begin{aligned}
\bar{x} &= (I-F)^{-1}Gu \\
&= \begin{bmatrix} 2.5 & 5 \end{bmatrix}^T \\
P &\approx \begin{bmatrix} 2.88 & 3.08 \\ 3.08 & 7.96 \end{bmatrix}
\end{aligned}
$$

- Note that since F for this example is stable and Q is positive definite, Theorem 21 guarantees that P has a unique positive definite steady-state solution.

Sampled-Data Systems

- A sampled-data system is a system whose dynamics are described by a continuous-time differential equation, but the input only changes at discrete time instants

$$\dot{x} = Ax + Bu + w$$

zero mean white noise

$$x(t_k) = e^{A(t_k - t_{k-1})}x(t_{k-1}) + \int_{t_{k-1}}^{t_k} e^{A(t_k-\tau)}[B(\tau)u(\tau) + w(\tau)]\, d\tau$$

$$\begin{aligned} \Delta t &= t_k - t_{k-1} \\ x_k &= x(t_k) \\ u_k &= u(t_k) \end{aligned}$$

$$x_k = \underbrace{e^{A\Delta t}}_{F_k} x_{k-1} + \underbrace{\int_{t_{k-1}}^{t_k} e^{A(t_k-\tau)}B(\tau)\, d\tau}_{G_{k-1}}\, u_{k-1} + \int_{t_{k-1}}^{t_k} e^{A(t_k-\tau)}w(\tau)\, d\tau$$

Sampled-Data Systems

- $$x_k = F_{k-1}x_{k-1} + G_{k-1}u_{k-1} + \int_{t_{k-1}}^{t_k} e^{A(t_k-\tau)}w(\tau)\,d\tau$$

- $$\begin{aligned} \bar{x}_k &= E(x_k) \\ &= F_{k-1}\bar{x}_{k-1} + G_{k-1}u_{k-1} \end{aligned}$$

- $$\begin{aligned} P_k &= E[(x_k - \bar{x}_k)(x_k - \bar{x}_k)^T] \\ &= E\left[\left(F_{k-1}x_{k-1} + G_{k-1}u_{k-1} + \int_{t_{k-1}}^{t_k} e^{A(t_k-\tau)}w(\tau)d\tau - \bar{x}_k\right)\left(\cdots\right)^T\right] \\ &= F_{k-1}P_{k-1}F_{k-1}^T + E\left[\left(\int_{t_{k-1}}^{t_k} e^{A(t_k-\tau)}w(\tau)\,d\tau\right)\left(\cdots\right)^T\right] \end{aligned}$$

Sampled-Data Systems

$$= \quad F_{k-1}P_{k-1}F_{k-1}^T + \int\int_{t_{k-1}}^{t_k} e^{A(t_k-\tau)} E\left[w(\tau)w^T(\alpha)\right] e^{A^T(t_k-\alpha)}\, d\tau\, d\alpha$$

- $E\left[w(\tau)w^T(\alpha)\right] = Q_c(\tau)\delta(\tau-\alpha)$

- $$\begin{aligned} P_k &= F_{k-1}P_{k-1}F_{k-1}^T + \int_{t_{k-1}}^{t_k} e^{A(t_k-\tau)} Q_c(\tau) e^{A^T(t_k-\tau)}\, d\tau \\ &= F_{k-1}P_{k-1}F_{k-1}^T + Q_{k-1} \end{aligned}$$

it is difficult to calculate Q_{k-1}, but for small values of $(t_k - t_{k-1})$

$$\begin{aligned} e^{A(t_k-\tau)} &\approx I \text{ for } \tau \in [t_{k-1}, t_k] \\ Q_{k-1} &\approx Q_c(t_k)\Delta t \end{aligned}$$

Example

- $$\begin{aligned} \dot{x} &= fx + w \\ E[w(t)w(t+\tau)] &= q_c \delta(\tau) \end{aligned}$$
- $$\begin{aligned} Q_{k-1} &= \int_{t_{k-1}}^{t_k} \exp[f(t_k - \tau)] q_c \exp[f(t_k - \tau)]\, d\tau \\ &= \exp(2ft_k) q_c \int_{t_{k-1}}^{t_k} \exp(-2f\tau)\, d\tau \\ &= \exp(2ft_k) q_c \left[\frac{\exp(-2ft_{k-1}) - \exp(-2ft_k)}{2f} \right] \\ &= \frac{q_c}{2f} [\exp(2f(t_k - t_{k-1})) - 1] \\ &= \frac{q_c}{2f} [\exp(2f\Delta t) - 1] \end{aligned}$$

Example

- For small values of Δt, we can expand the above equation in a Taylor series around $\Delta t = 0$ to obtain

$$\begin{aligned} Q_{k-1} &= \frac{q_c}{2f}\left[\exp(2f\Delta t) - 1\right] \\ &= \frac{q_c}{2f}\left[\left(1 + 2f\Delta t + \frac{(2f\Delta t)^2}{2!} + \cdots\right) - 1\right] \\ &\approx \frac{q_c}{2f}\left[1 + 2f\Delta t - 1\right] \\ &= q_c\Delta t \end{aligned}$$

- $$\begin{aligned} \bar{x}_k &= F_{k-1}\bar{x}_{k-1} + G_{k-1}u_{k-1} \\ &= \exp\left[f(t_k - t_{k-1})\right]\bar{x}_{k-1} + 0 \\ &= \exp(f\Delta t)\bar{x}_{k-1} \\ &= \exp(kf\Delta t)\bar{x}_0 \end{aligned}$$

- $$\begin{aligned} P_k &= F_{k-1}P_{k-1}F_{k-1}^T + Q_{k-1} \\ &\approx (1 + 2f\Delta t)P_{k-1} + q_c\Delta t \end{aligned}$$

Continuous-time Systems

- $\dot{x} = Ax + Bu + w$
- $w(t)$ is zero-mean white noise with a covariance of $E[w(t)w^T(\tau)] = Q_c\delta(t-\tau)$
- $\dot{\bar{x}} = A\bar{x} + Bu$
- The above eq. can also be obtained from the solution of sampled-data system

$$\bar{x}_k = e^{A\Delta t}\bar{x}_{k-1} + \int_{t_{k-1}}^{t_k} e^{A(t_k-\tau)}B(\tau)u(\tau)\,d\tau$$

$$\begin{aligned} F &= e^{A\Delta t} \\ &= I + A\Delta t + \frac{(A\Delta t)^2}{2!} + \cdots \approx I + A\Delta t \end{aligned}$$

Continuous-time Systems

- $$\bar{x}_k = (I + A\Delta t)\bar{x}_{k-1} + \int_{t_{k-1}}^{t_k} e^{A(t_k-\tau)} B(\tau) u(\tau)\, d\tau$$

$$\frac{\bar{x}_k - \bar{x}_{k-1}}{\Delta t} = A\bar{x}_{k-1} + \frac{1}{\Delta t}\int_{t_{k-1}}^{t_k} e^{A(t_k-\tau)} B(\tau) u(\tau)\, d\tau$$

$$\begin{aligned} \lim_{\Delta t \to 0} \frac{\bar{x}_k - \bar{x}_{k-1}}{\Delta t} &= \dot{\bar{x}} \\ \lim_{\Delta t \to 0} e^{A(t_k-\tau)} &= I \text{ for } \tau \in [t_{k-1}, t_k] \end{aligned}$$

Continuous-time Systems

- $P_k = F_{k-1}P_{k-1}F_{k-1}^T + Q_{k-1}$

$$\begin{aligned} P_k &\approx (I + A\Delta t)P_{k-1}(I + A\Delta t)^T + Q_{k-1} \\ &= P_{k-1} + AP_{k-1}\Delta t + P_{k-1}A^T\Delta t + AP_{k-1}A^T(\Delta t)^2 + Q_{k-1} \end{aligned}$$

$$\frac{P_k - P_{k-1}}{\Delta t} = AP_{k-1} + P_{k-1}A^T + AP_{k-1}A^T\Delta t + \frac{Q_{k-1}}{\Delta t}$$

Recall that for small Δt: $Q_{k-1} \approx Q_c(t_k)\Delta t$ $\longrightarrow$ $\frac{Q_{k-1}}{\Delta t} \approx Q_c(t_k)$

$\Rightarrow$ $\dot{P} = AP + PA^T + Q_c$

continuous-time Lyapunov equation, also called a Sylvester equation

Continuous-time Systems

- **Theorem 22** *Consider the equation* $AP + PA^T + Q_c = 0$ *where* A *and* Q_c *are real matrices. Denote by* $\lambda_i(A)$ *the eigenvalues of the* A *matrix.*

  1. *A unique solution* P *exists if and only if* $\lambda_i(A) + \lambda_j(A) \neq 0$ *for all* i, j. *This unique solution is symmetric.*

  2. *Note that the above condition includes the case of stable* A, *because if* A *is stable then all of its eigenvalues have real parts less than 0, so* $\lambda_i(A)+\lambda_j(A) \neq 0$ *for all* i, j. *Therefore, we see that if* A *is stable then the continuous-time Lyapunov equation has a solution* P *that is unique and symmetric. In this case, the solution can be written as*

$$P = \int_0^\infty e^{A^T \tau} Q_c e^{A\tau} \, d\tau$$

Continuous-time Systems

3. *If A is stable and Q_c is positive (semi)definite, then the unique solution P is symmetric and positive (semi)definite.*

4. *If A is stable, Q_c is positive semidefinite, and $\left[A, (Q_c^{1/2})^T\right]$ is controllable, then P is unique, symmetric, and positive definite. Note that $Q_c^{1/2}$, the square root of Q_c, is defined here as any matrix such that $Q_c^{1/2}(Q_c^{1/2})^T = Q_c$.*

Example

- $$\begin{aligned} \dot{x} &= fx + w \\ E[w(t)w(t+\tau)] &= q_c\delta(\tau) \end{aligned}$$

- $\dot{\bar{x}} = f\bar{x} \longrightarrow \bar{x}(t) = \exp(ft)\bar{x}(0)$

- $\dot{P} = 2fP + q_c \longrightarrow P(t) = \left(P(0) + \frac{q_c}{2f}\right)\exp(2ft) - \frac{q_c}{2f}$