# Home Work #1

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## 1 Question 1

$$f_X(x) = \frac{ab}{b^2 + x^2}, \quad b > 0$$

### 1.1 part a

$$\int_{-\infty}^{+\infty} f(x)dx = 1 \quad \rightarrow \quad \int_{-\infty}^{+\infty} \frac{ab}{b^2 + x^2}dx = 1 \rightarrow a \arctan(\frac{x}{b})\Big|_{-\infty}^{+\infty} = 1 \rightarrow a\pi = 1 \rightarrow a = \frac{1}{\pi}$$

$$f_X(x) = \frac{1}{\pi}\frac{b}{b^2 + x^2}, \quad b > 0$$

### 1.2 part b

$$E(X) = \mu_X = \int_{-\infty}^{+\infty} xf(x)dx$$

Because $xf(x)$ is an odd function, the result of the integrator between $\infty$ and $-\infty$ is zero.

$$\int_{-\infty}^{+\infty} xf(x)dx = 0 \rightarrow \mu_X = 0$$

$$\sigma_X^2 = E((X - \mu)^2) = \int_{-\infty}^{+\infty} (x - \mu)^2 f(x)dx = \int_{-\infty}^{+\infty} x^2 f(x)dx = \frac{b}{\pi}(x - b\arctan(\frac{x}{b}))\Big|_{-\infty}^{+\infty} \neq \text{finite}$$

## 2 Question 2

## 3 Question 3

A positive test is A event: $P(A)$, $\quad$ Having the flu is B event: $P(B) = 0.05$

### 3.1 part a

The probability of a positive test if someone has flu:

$$P(A|B) = 0.99$$

The probability of a positive test if someone doesn't have flu:

$$P(A|\bar{B}) = 0.01$$

$$P(A) = P(A \cap B) + P(A \cap \bar{B}) = P(A|B)P(B) + P(A|\bar{B})P(\bar{B}) = 0.99 \times 0.05 + 0.01 \times 0.95 = 0.059$$

$$P(A|B)P(B) = P(B|A)P(A) \rightarrow P(B|A) = \frac{P(A|B)P(B)}{P(A)} = 0.84$$

### 3.2 part b

C is the event when two positive tests happen. The probability of two positive tests if someone has flu:

$$P(B)P(A|B)P(A|B) = 0.049$$

The probability of two positive tests if someone doesn't have flu:

$$P(\bar{B})P(A|\bar{B})P(A|\bar{B}) = 9.5 \times 10^{-5}$$

$$P(C) = P(B)P(A|B)P(A|B) + P(\bar{B})P(A|\bar{B})P(A|\bar{B}) = 0.0491$$

$$P(B|C) = \frac{P(B \cap C)}{P(C)} = 0.998$$

## 4 Question 4

Assumed that the random variables $A$ $\phi$ are independent and $\phi$ is uniform in the interval $[0, 2\pi)$, mean and variance of $\phi$ is 0 and $\sigma^2$, respectively.

$$x(t) = A\cos(\omega t + \phi)$$

### 4.1 part a

$$E\left[x(t)\right] = E\left[A\cos(\omega t + \phi)\right] \xrightarrow{uncorrelated} E\left[x(t)\right] = E[A]E[\cos(\omega t + \phi)] = 0 \xrightarrow{E[A]=0} E\left[x(t)\right] = 0$$

### 4.2 part b

$$R_X(t_1, t_2) = E[x(t_1)x(t_2)] = \frac{1}{2}E[A^2]E[\cos\omega(t_1 - t_2) + \cos(\omega t_1 + \omega t_2 + 2\phi)]$$

$$E[\cos(\omega t_1 + \omega t_2 + 2\phi)] = \frac{1}{2\pi}\int_0^{2\pi} \cos(\omega t_1 + \omega t_2 + 2\phi)d\phi = 0$$

$$R_X(t_1, t_2) = \frac{\sigma^2}{2}\cos\omega(t_1 - t_2)$$

### 4.3 part c

$$A[x(t)] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)dt \to A[x(t)] = \frac{1}{2\pi}\int_{0}^{2\pi} A\cos(\omega t+\phi)dt = \left.\frac{A\sin(\phi+\omega t)}{\omega}\right|_{0}^{2\pi}$$

$$A[x(t)] = \frac{A\sin(\phi+2\pi\omega)}{\omega}$$

### 4.4 part d

$$R[x(t),\tau] = A[x(t)x(t+\tau)] = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)x(t+\tau)dt = \frac{1}{2\pi}\int_{0}^{2\pi} A^2\cos(\omega t+\phi)\cos(\omega(t+\tau)+\phi)dt$$

$$\int \cos(\omega t+\phi)\cos(\omega(t+\tau)+\phi)dt = \begin{cases} A^2\, t\cos(\phi)^2 & \text{if } \omega = 0 \\ \dfrac{A^2\, t\, \cos(\omega\,\tau)}{2} + \dfrac{A^2\, \sin(2\,\phi+2\,\omega\,t+\omega\,\tau)}{4\,\omega} & \text{if } \omega \neq 0 \end{cases}$$

$$\int_0^{2\pi} \cos(\omega t+\phi)\cos(\omega(t+\tau)+\phi)dt = \begin{cases} 2\,\pi\,A^2\cos(\phi)^2 & \text{if } \omega = 0 \\ \dfrac{A^2\,(\sin(2\,\phi+4\,\pi\,\omega+\omega\,\tau)-\sin(2\,\phi+\omega\,\tau))}{4\,\omega} + \pi\,A^2\, \cos(\omega\,\tau) & \text{if } \omega \neq 0 \end{cases}$$

$$R[x(t),\tau] = \begin{cases} 1\,A^2\cos(\phi)^2 & \text{if } \omega = 0 \\ 0.5\,A^2\, \cos(\omega\,\tau) - \dfrac{0.04\,A^2\,(1\, \sin(2\,\phi+\omega\,\tau)-\sin(12.5\,\omega+2\,\phi+\omega\,\tau))}{\omega} & \text{if } \omega \neq 0 \end{cases}$$

### 4.5 part e

For a WSS process:

$$R_X(0) = E[x(t)x(t)]$$
$$R_X(\tau) = R_X(-\tau)$$

where $R_X(\tau)$:

$$R_X(\tau) = E\left[x(t+\frac{\tau}{2})x^*(t-\frac{\tau}{2})\right]$$

That is true in this stochastic process.

An ergodic process is a stationary random process for which:

$$A[x(t)] = E[x(t)]$$
$$R[x(t,\tau)] = R_X(\tau)$$

$$A[x(t)] \neq E[x(t)]$$

so this stochastic process is not ergodic.

# 5 Question 5

$$\begin{cases} \dot{x}_1(t) = & x_2(t) \\ \dot{x}_2(t) = & -\omega_n^2 x_1(t) - 2\zeta\omega_n x_2(t) + w(t) \end{cases}$$

where $\omega_n = 10$, $\zeta = 0.3$ and $w(t)$ is with power $\sigma^2$ and step time $0.01_{\text{sec}}$ ($\sigma^2 = 1$).

## 5.1 part a

Figure 1: $x_1$ function number = 1

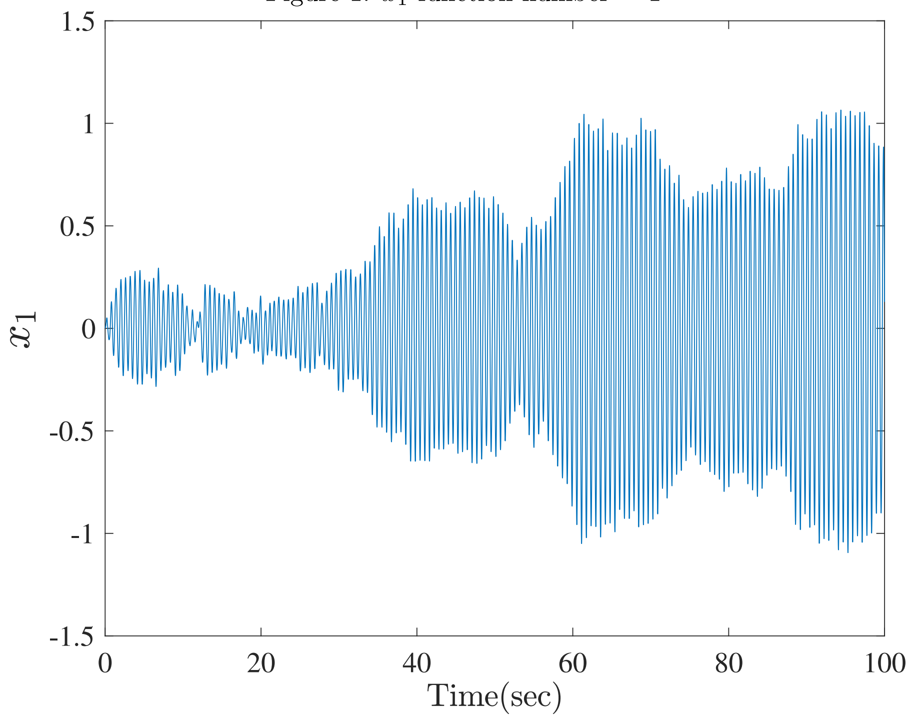


Figure 2: $x_1$ function number = 2

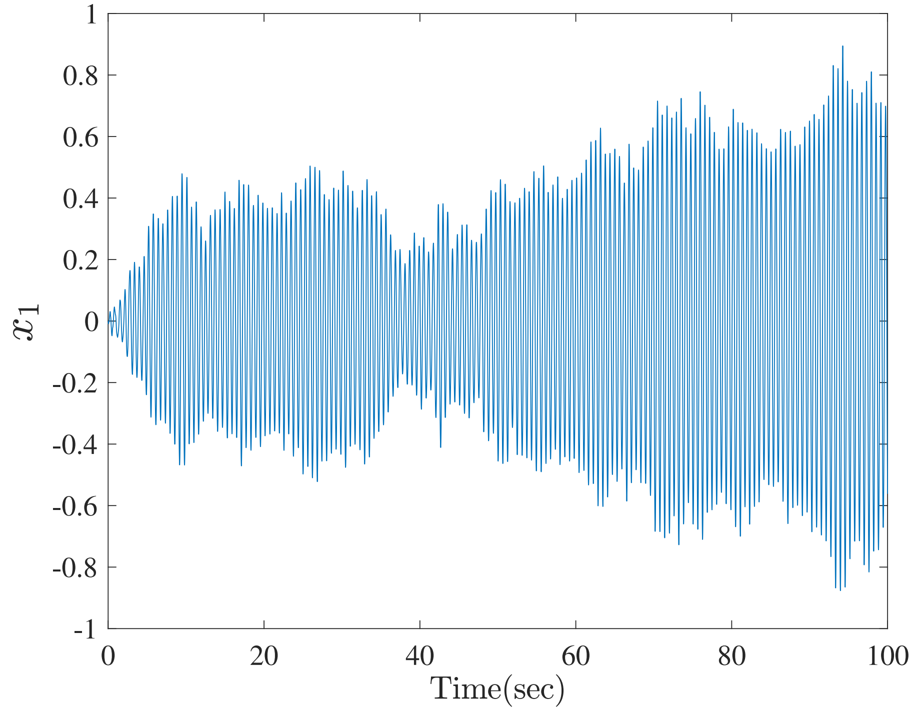



Figure 3: $x_1$ function number = 3

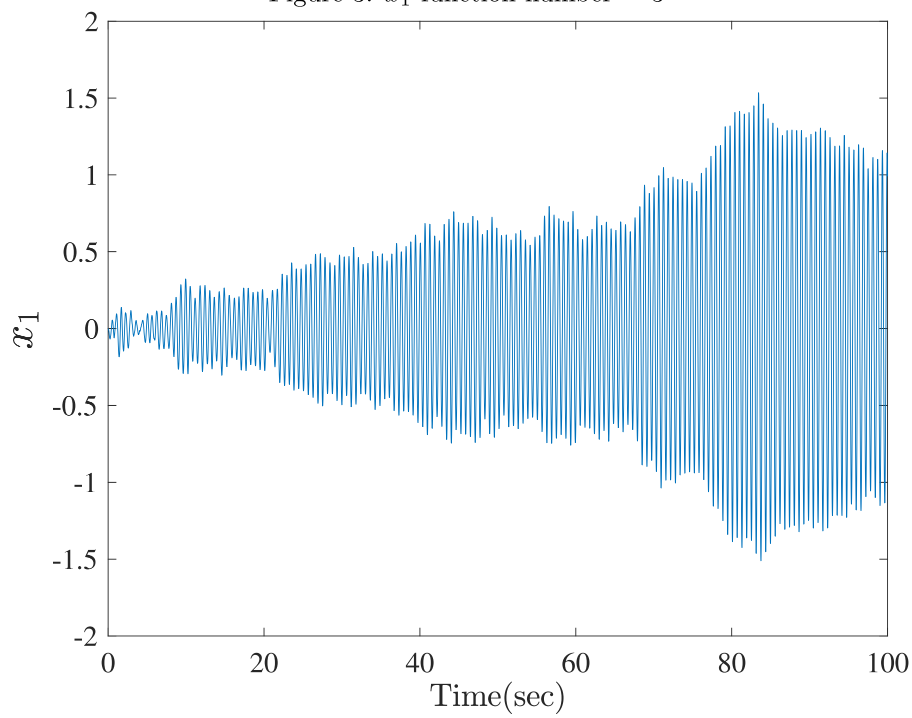


Figure 4: $x_1$ function number = 4

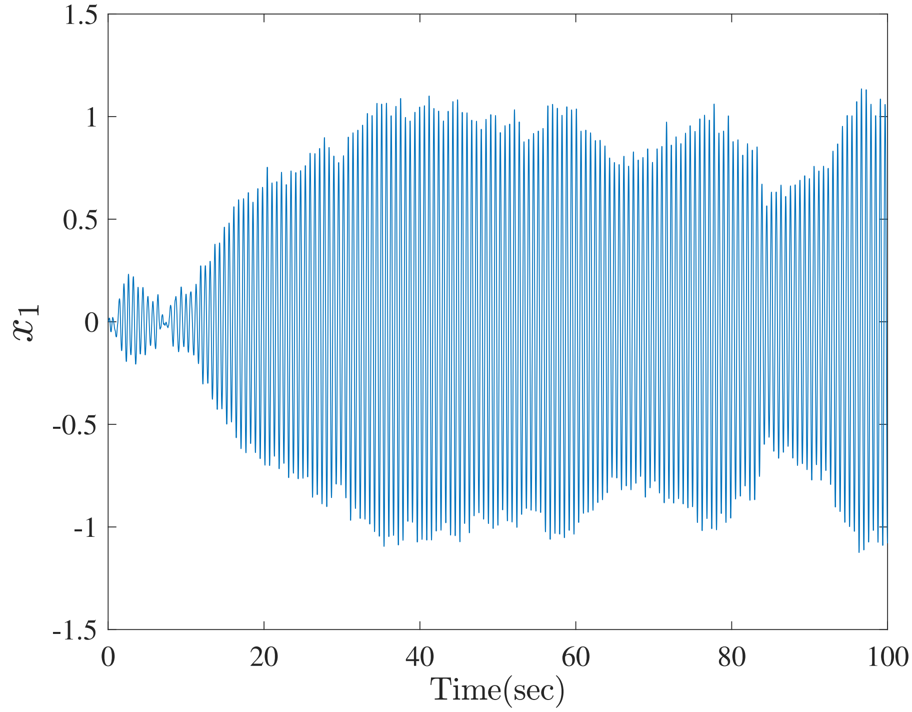



Figure 5: $x_1$ function number = 5

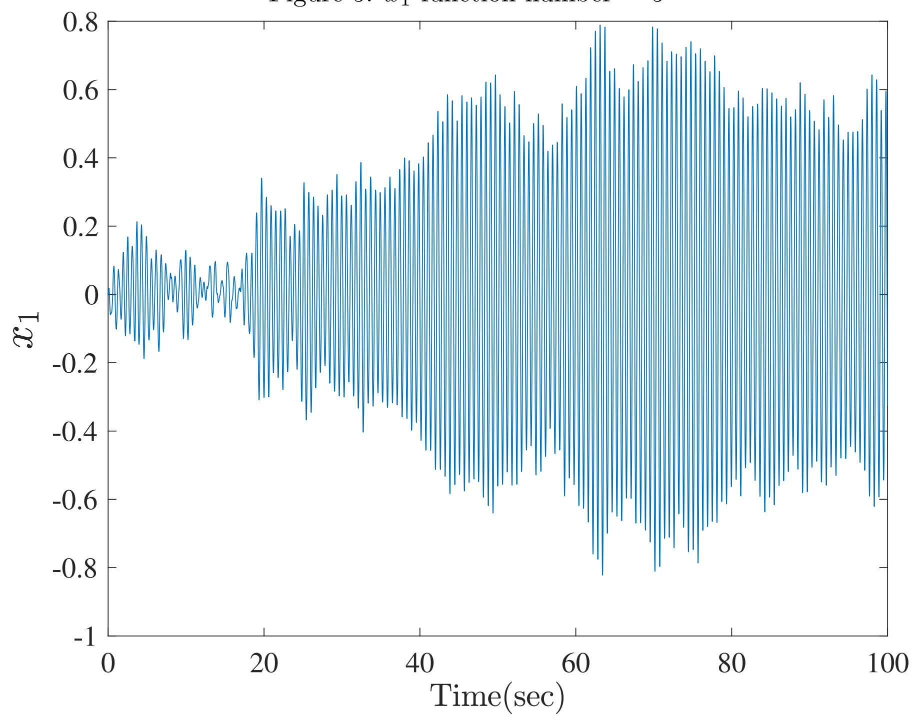


Figure 6: $x_1$ function number = 6

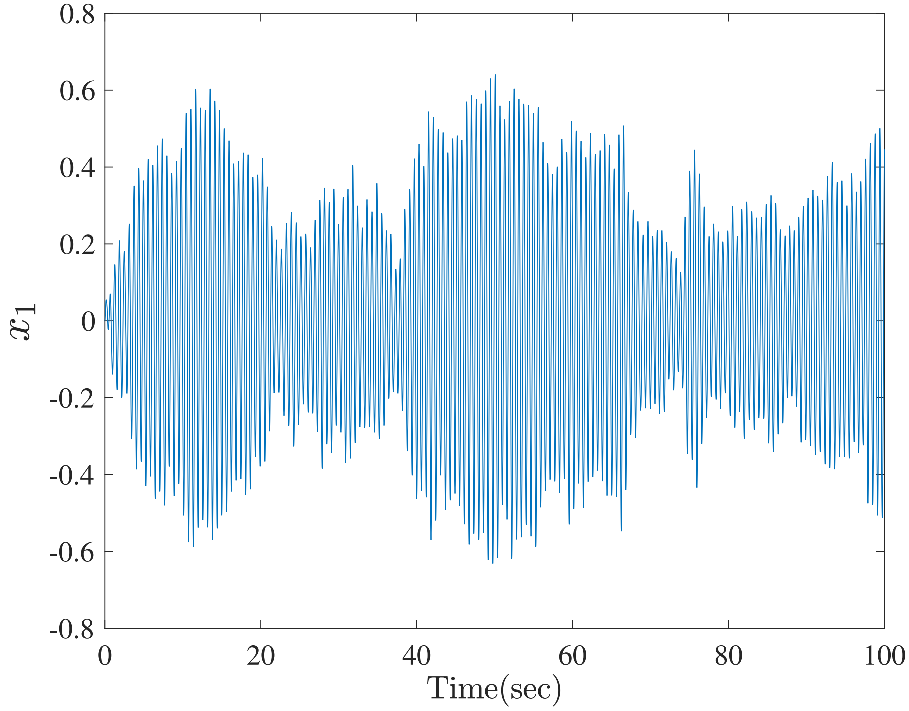



Figure 7: $x_1$ function number = 7

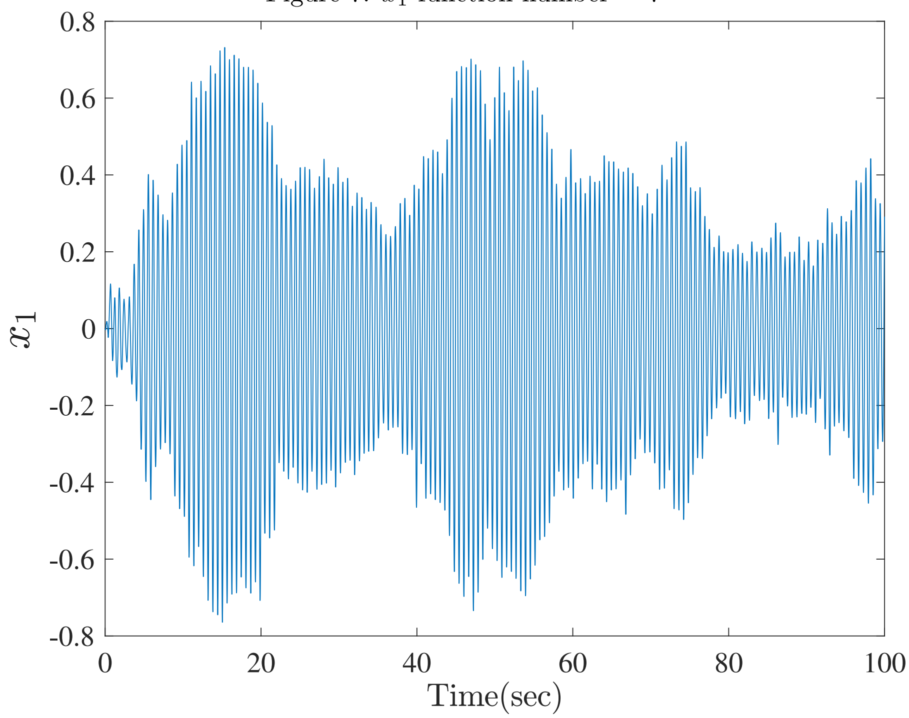


Figure 8: $x_1$ function number = 8

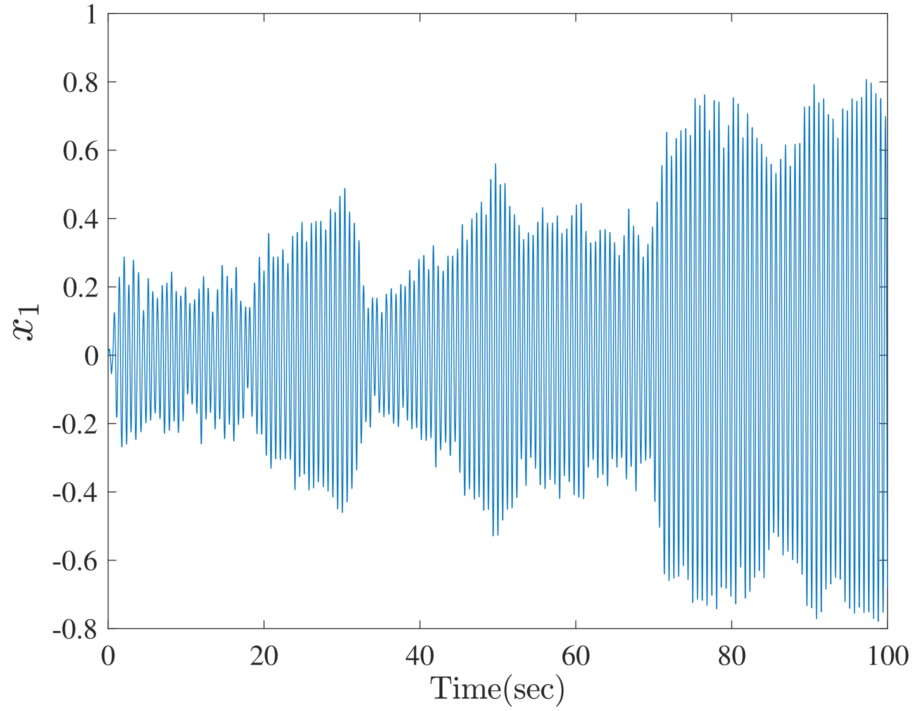



Figure 9: $x_1$ function number = 9

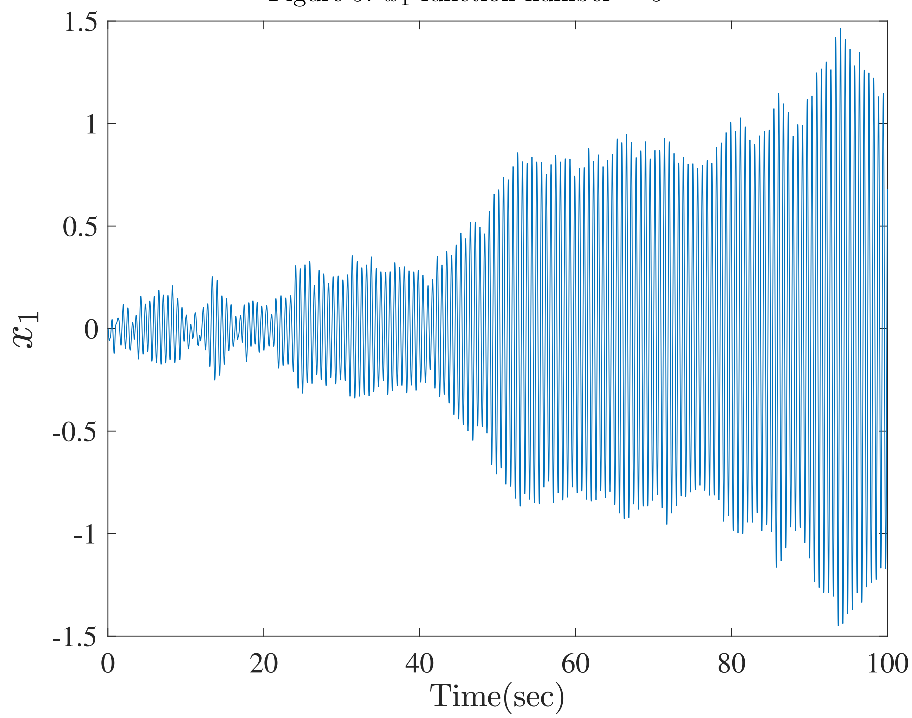


Figure 10: $x_1$ function number = 10

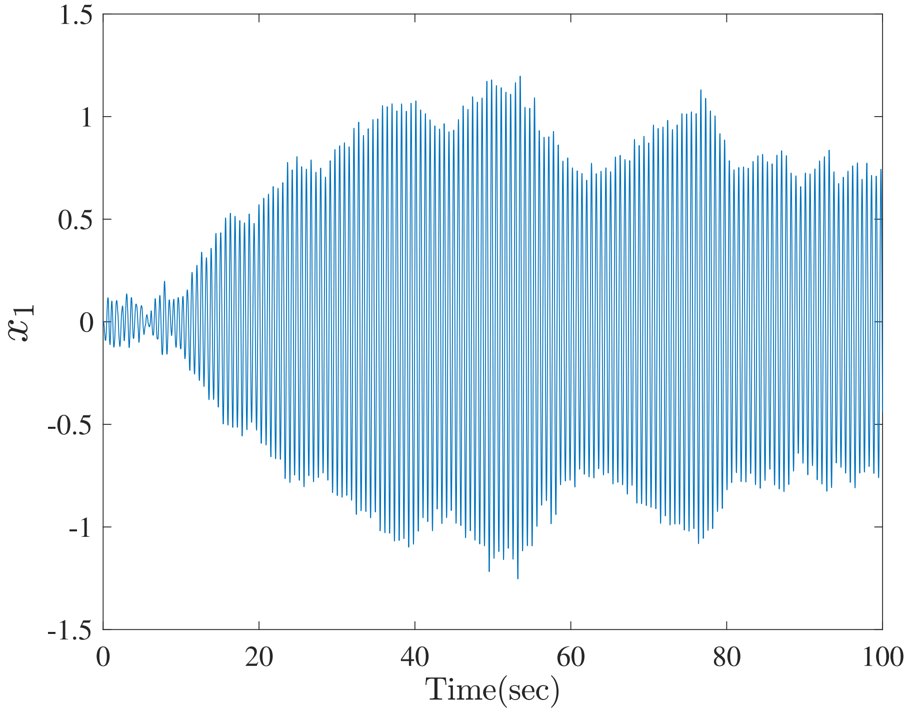



Figure 11: $x_1$ function number = 11

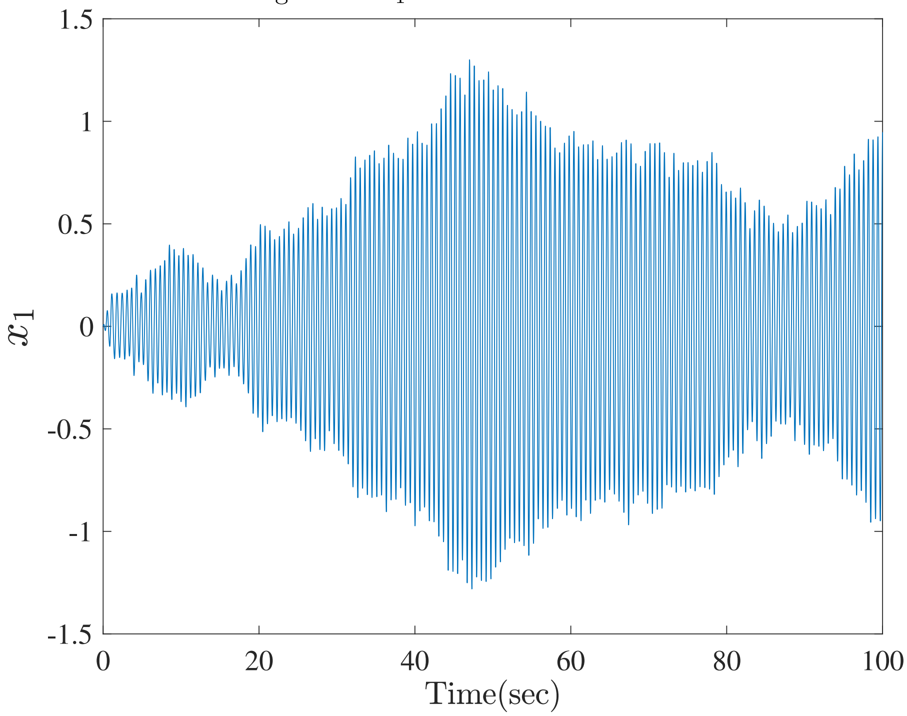


Figure 12: $x_1$ function number = 12

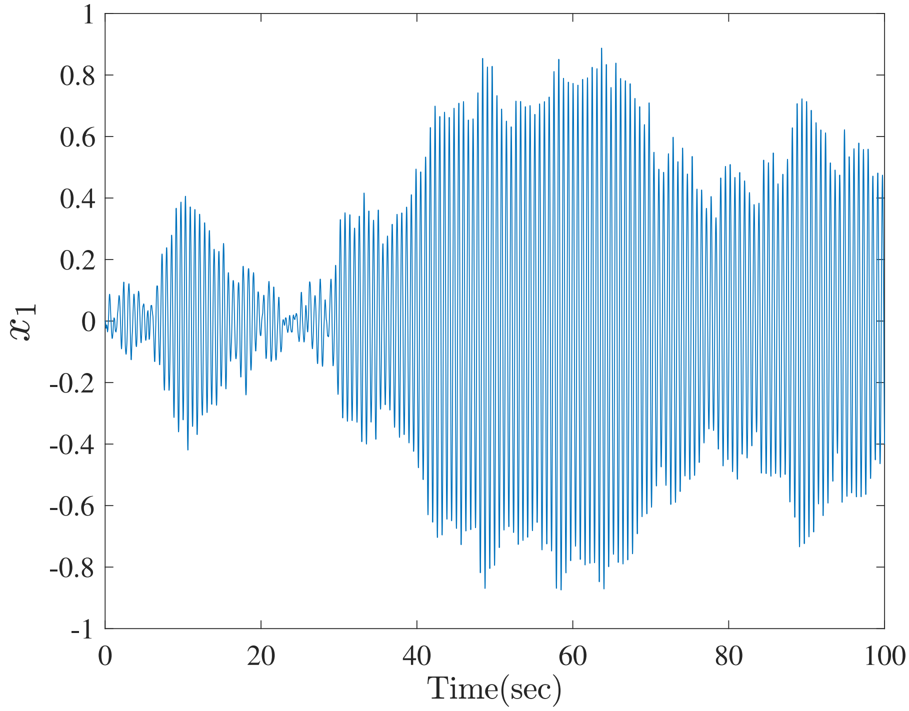



Figure 13: $x_1$ function number = 13

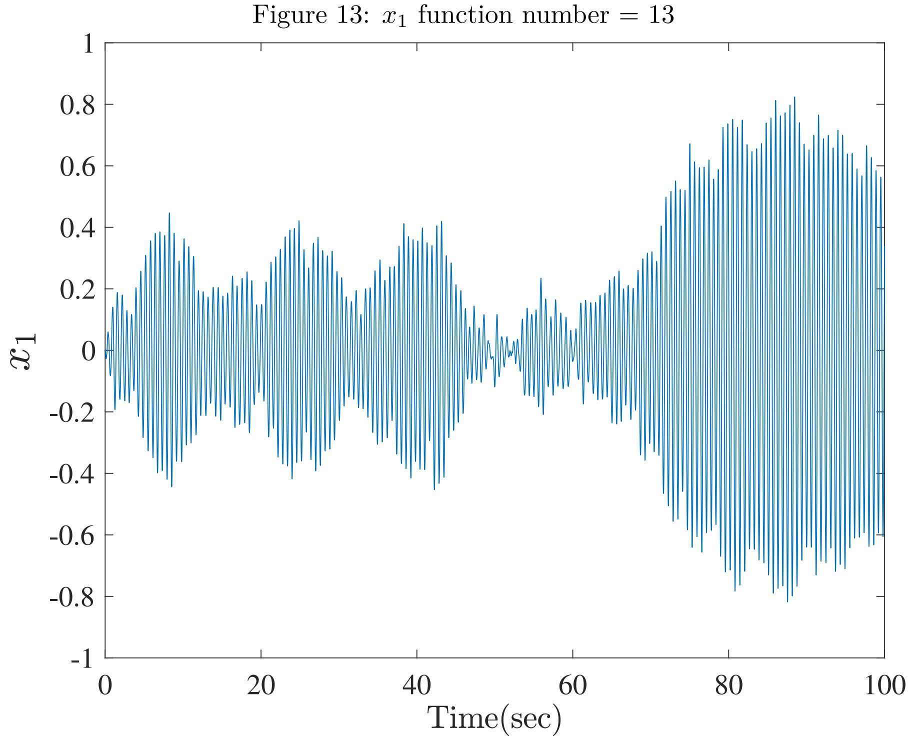


Figure 14: $x_1$ function number = 14

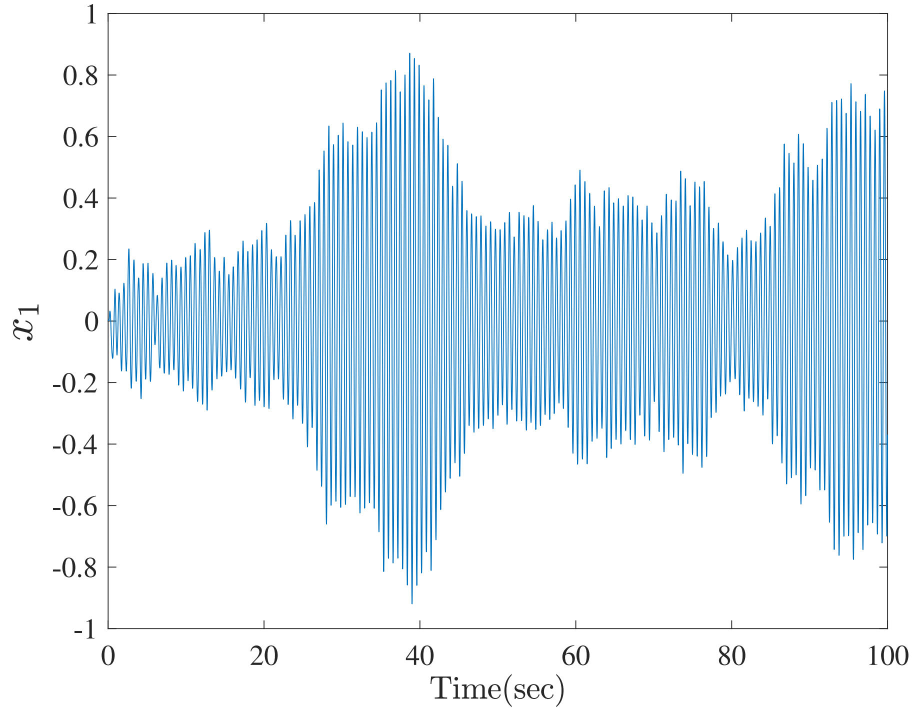



Figure 15: $x_1$ function number = 15

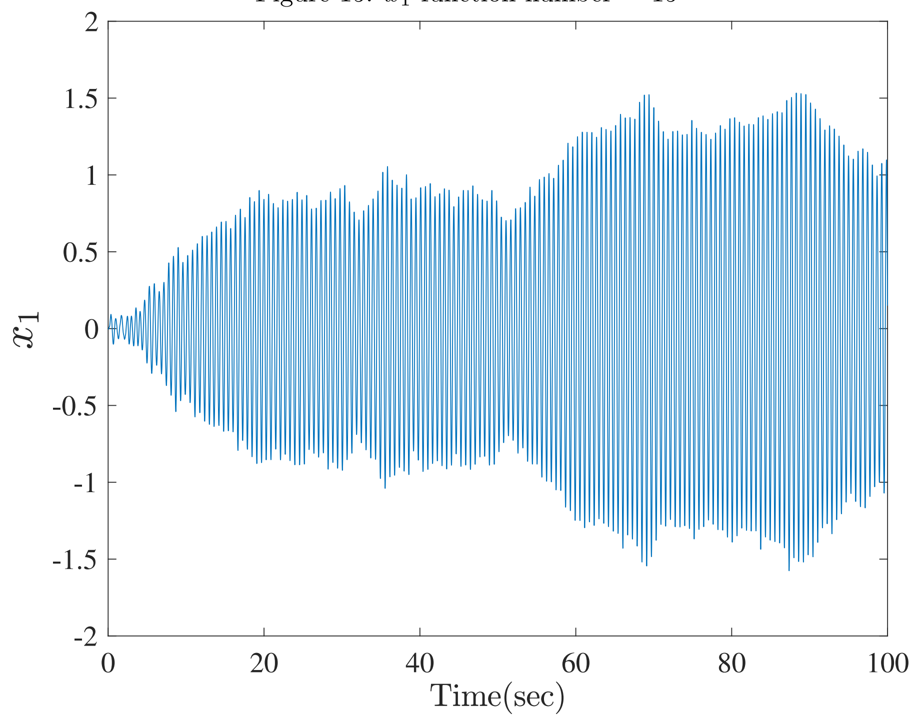


Figure 16: $x_1$ function number = 16

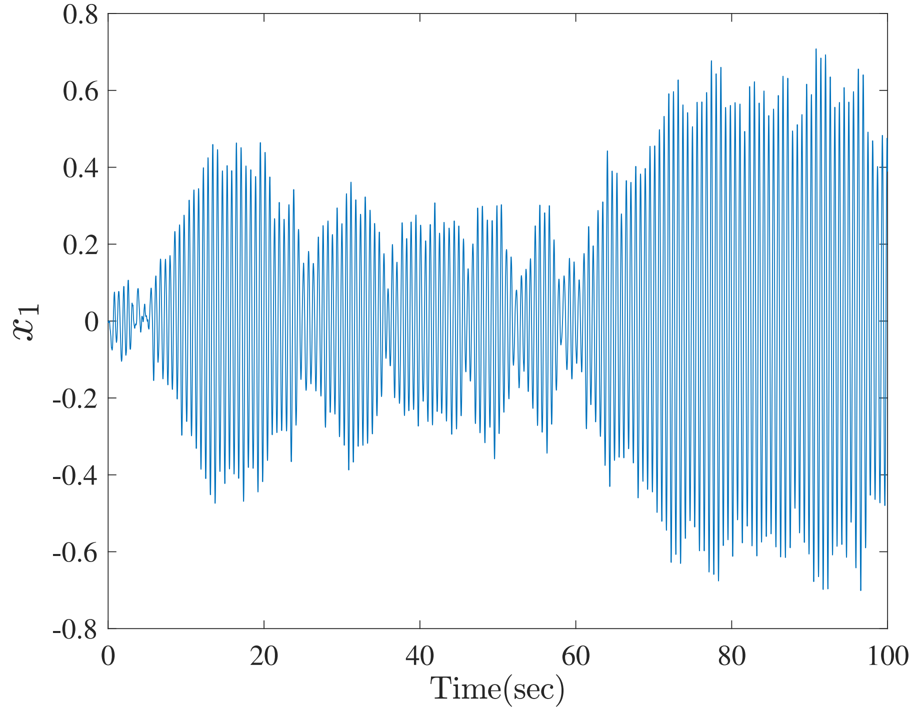



Figure 17: $x_1$ function number = 17

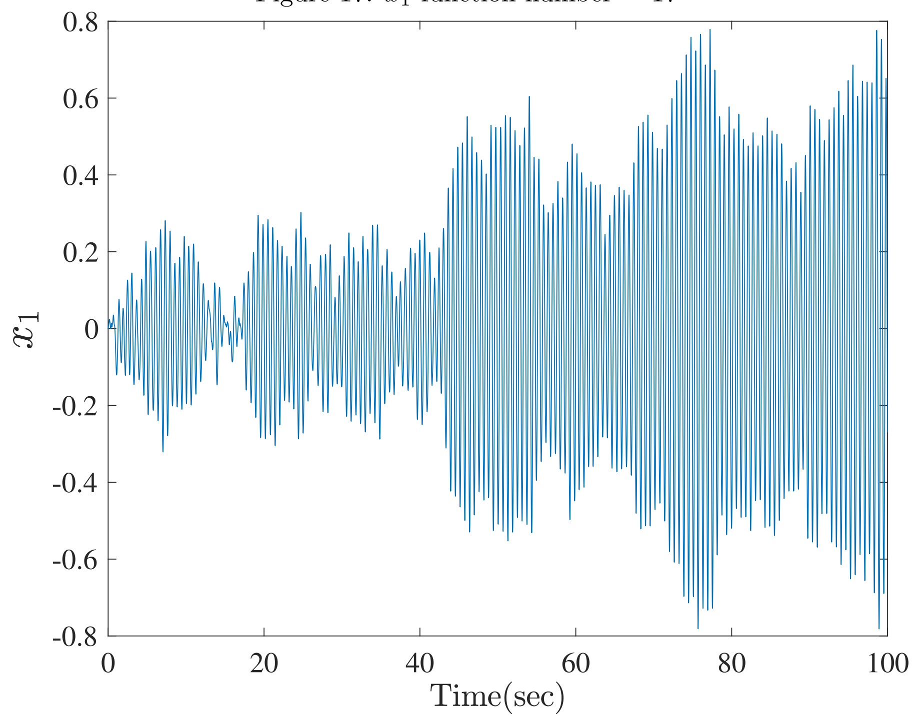


Figure 18: $x_1$ function number = 18

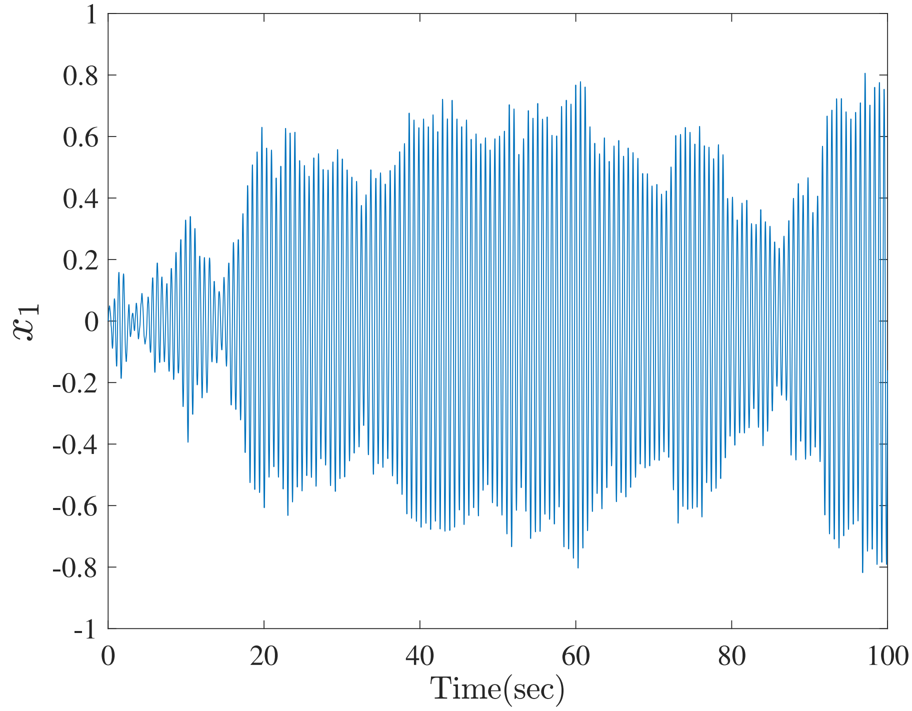



Figure 19: $x_1$ function number = 19

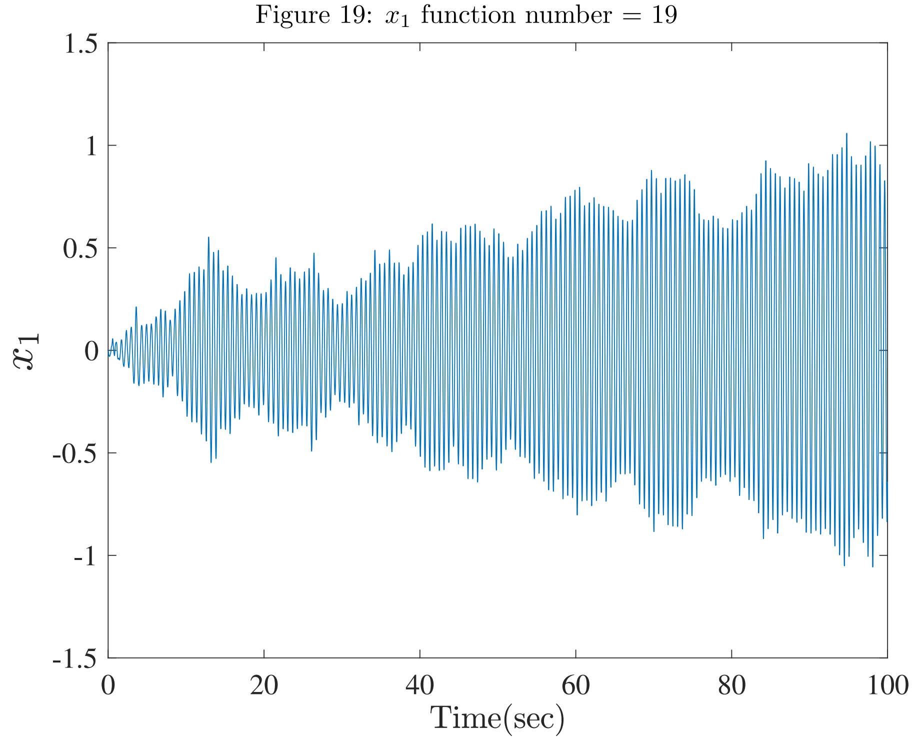


Figure 20: $x_1$ function number = 20

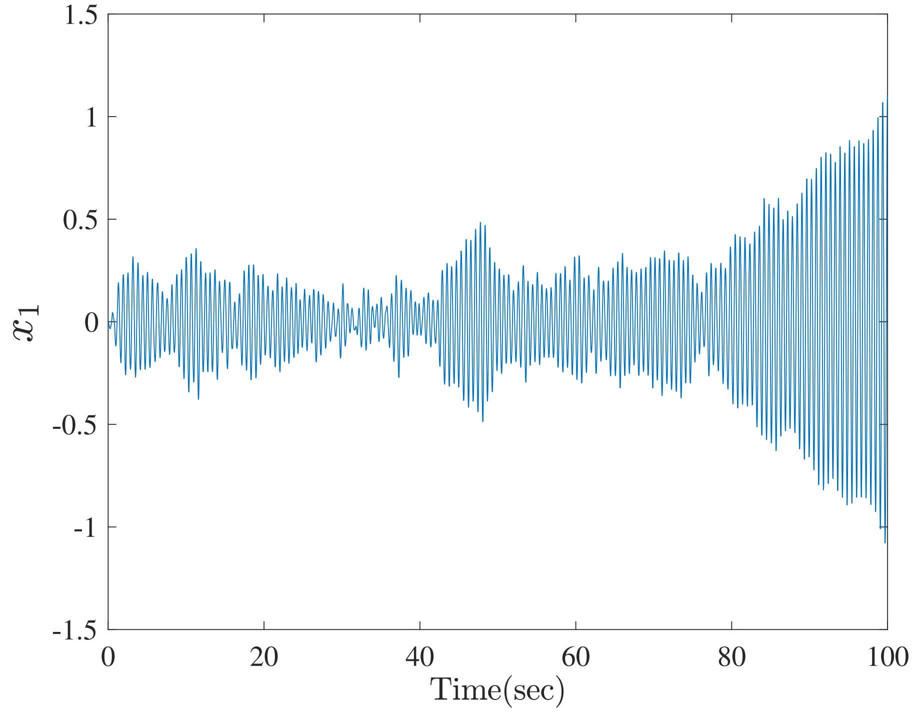



Figure 21: $x_1$ function number = 21

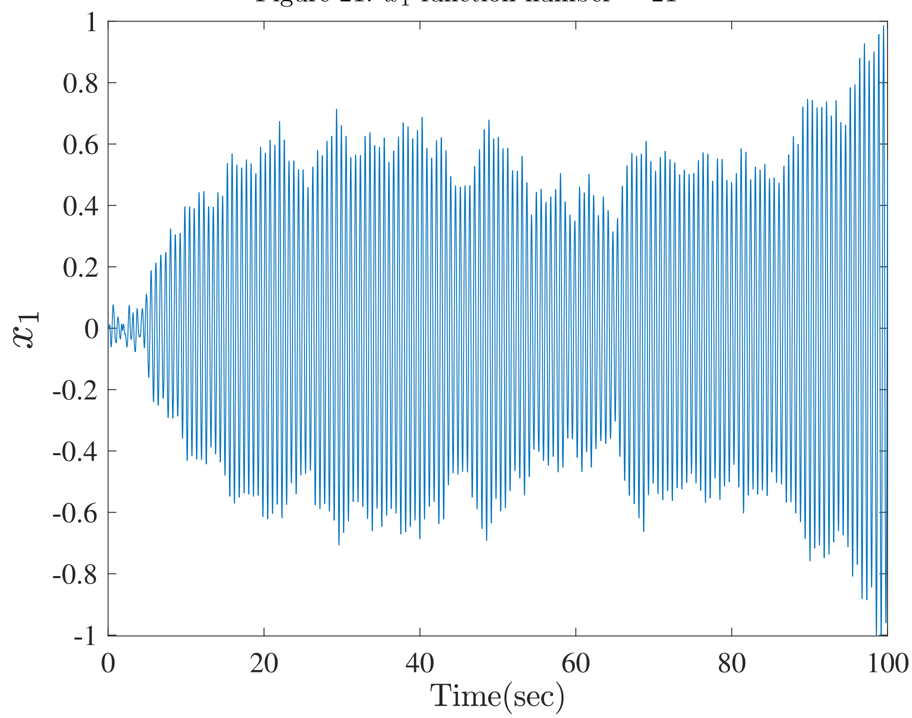


Figure 22: $x_1$ function number = 22

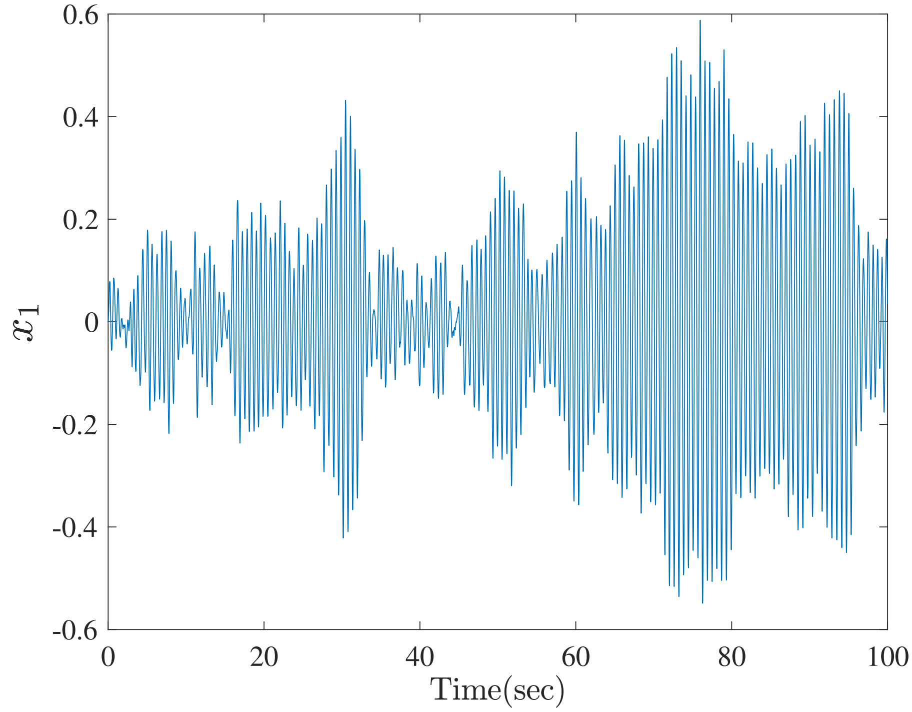



Figure 23: $x_1$ function number = 23

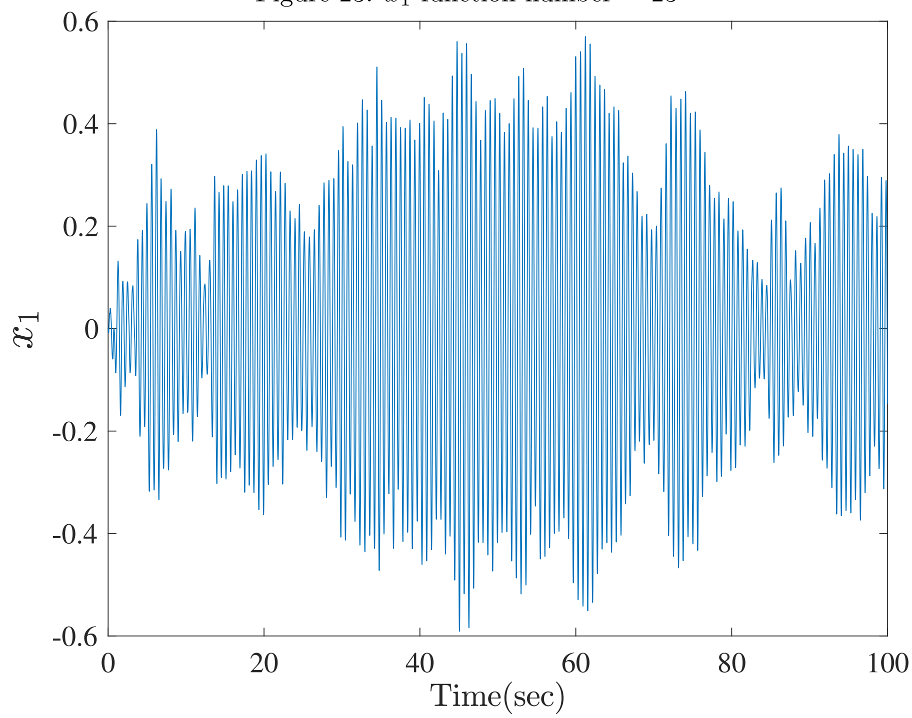


Figure 24: $x_1$ function number = 24

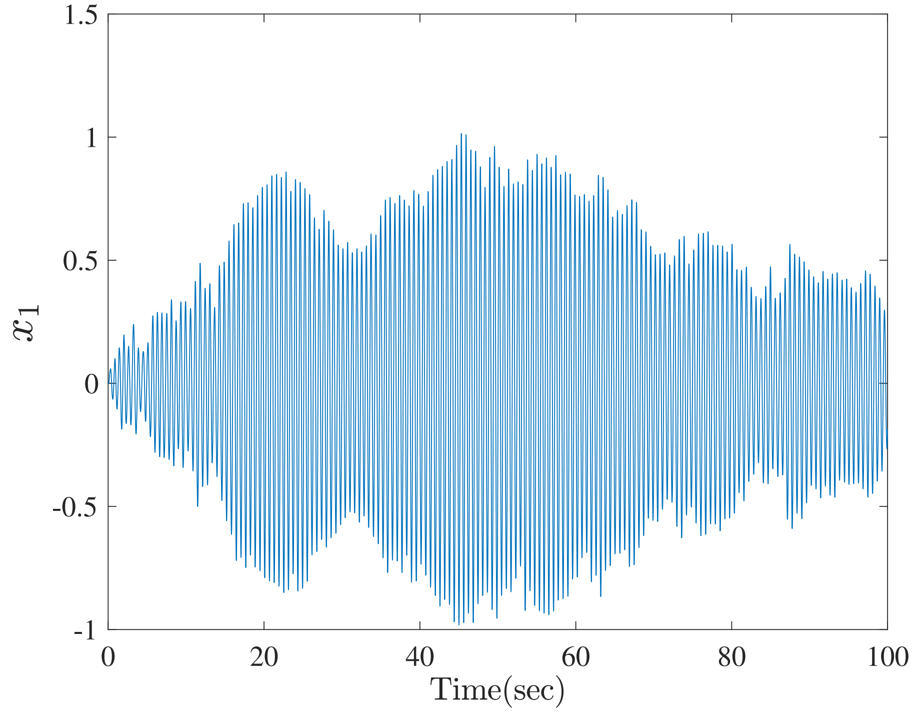



Figure 25: $x_1$ function number = 25

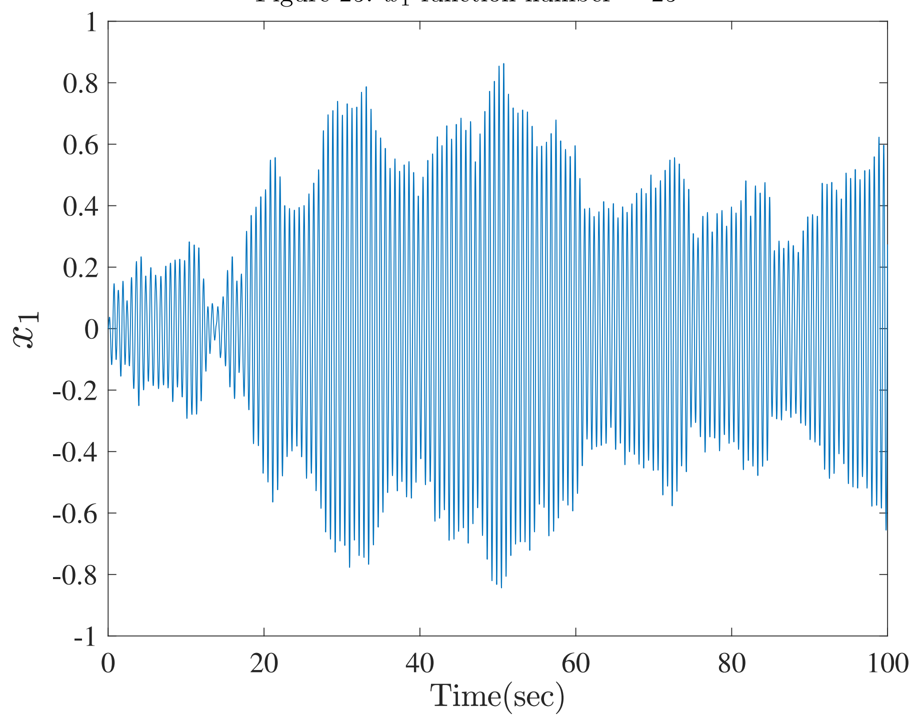


Figure 26: $x_1$ function number = 26

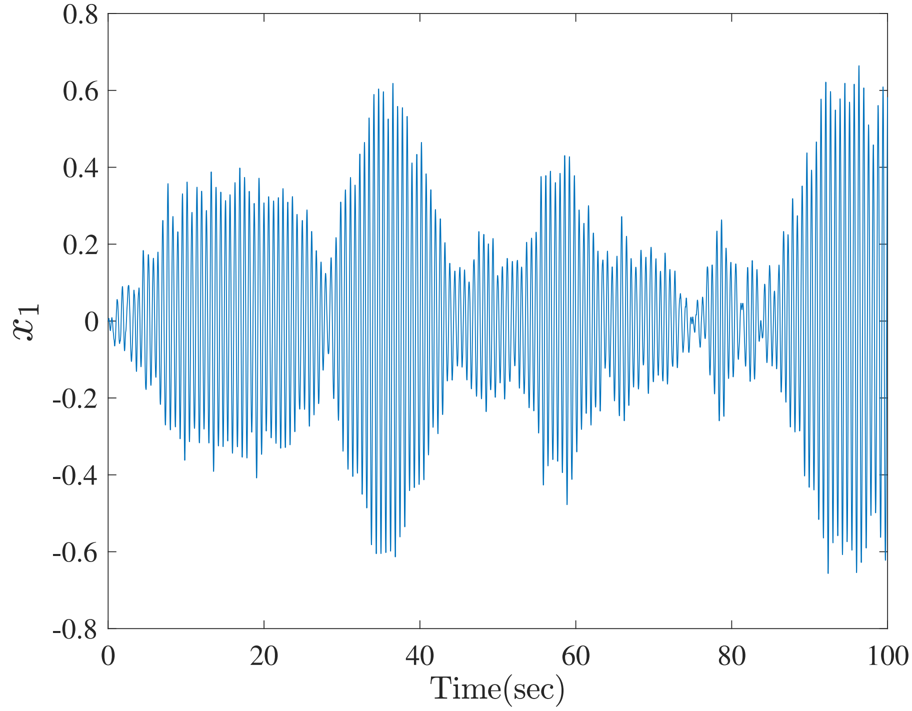



Figure 27: $x_1$ function number = 27

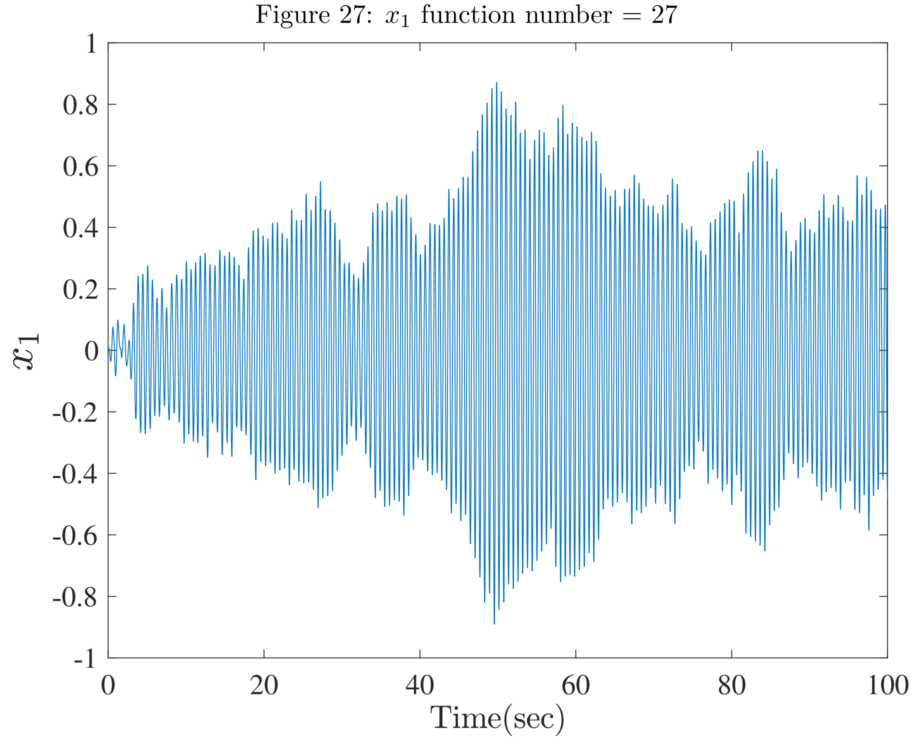


Figure 28: $x_1$ function number = 28

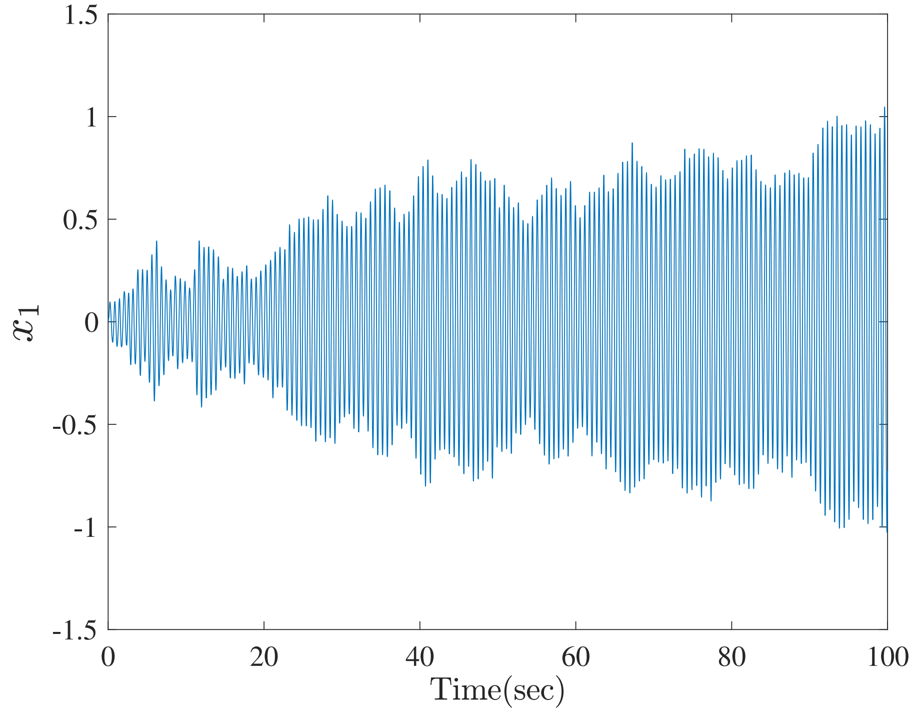



Figure 29: $x_1$ function number = 29

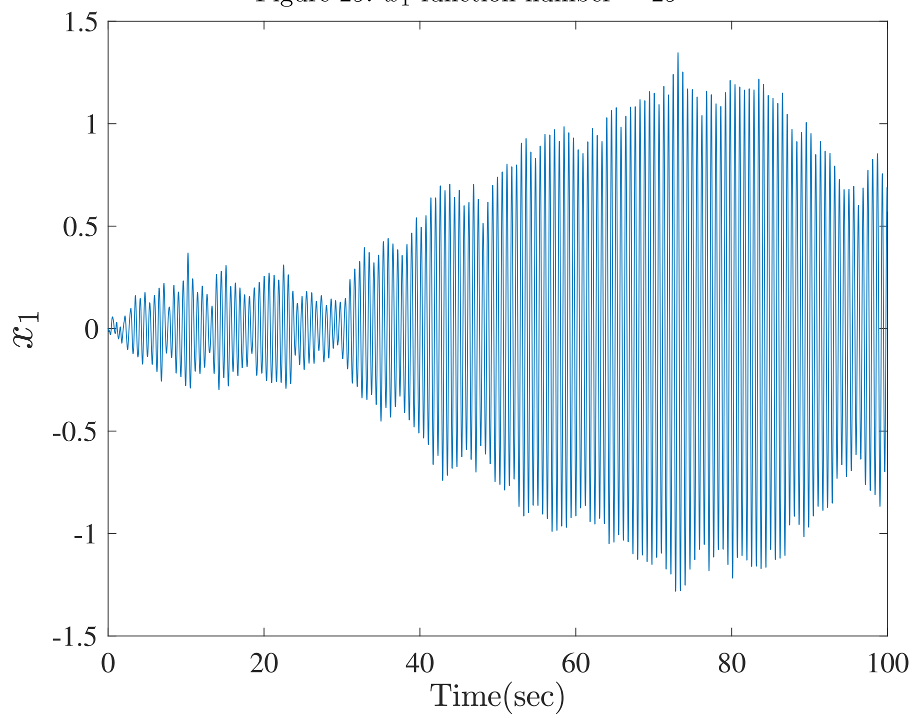


Figure 30: $x_1$ function number = 30

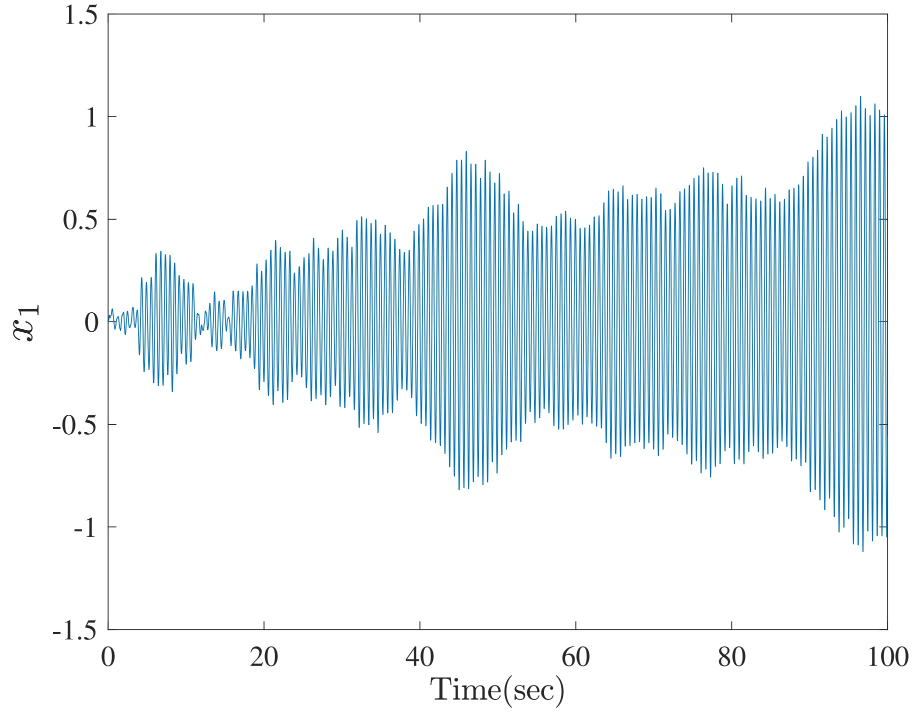



Figure 31: $x_1$ function number = 31

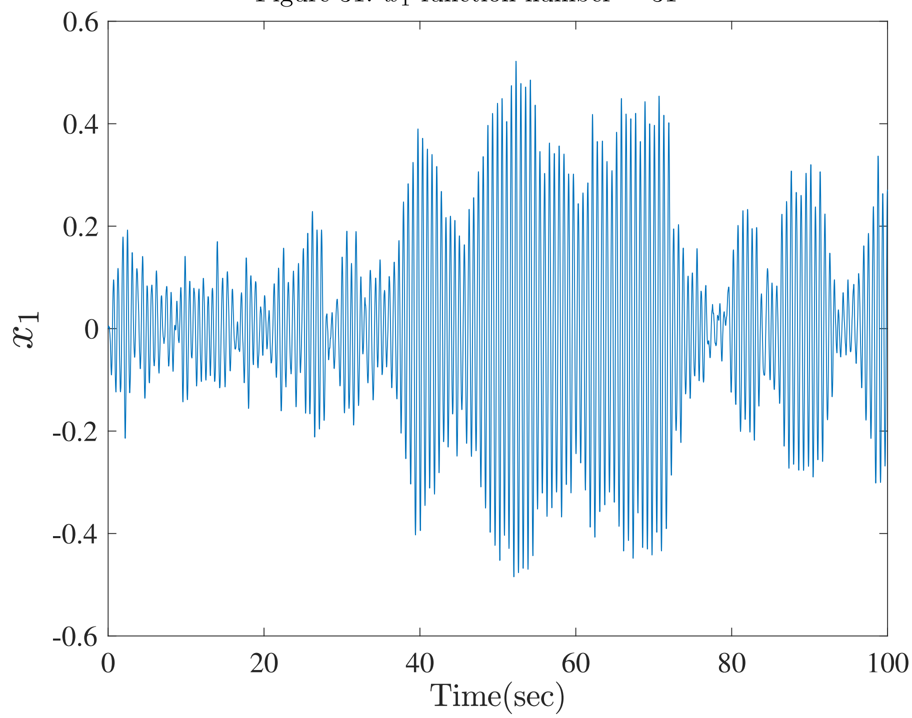


Figure 32: $x_1$ function number = 32

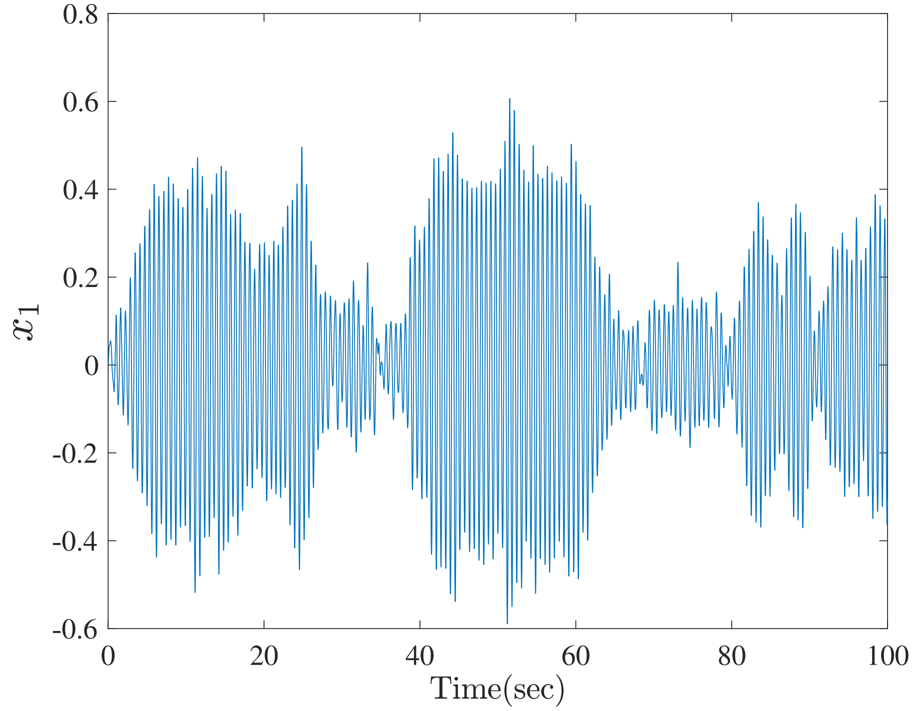



Figure 33: $x_1$ function number = 33

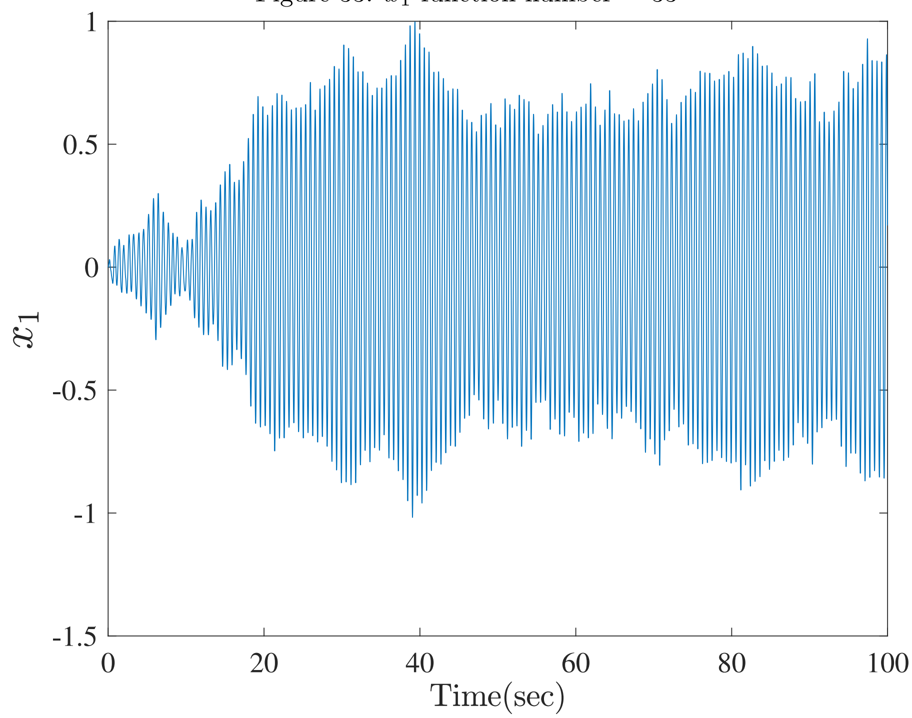


Figure 34: $x_1$ function number = 34

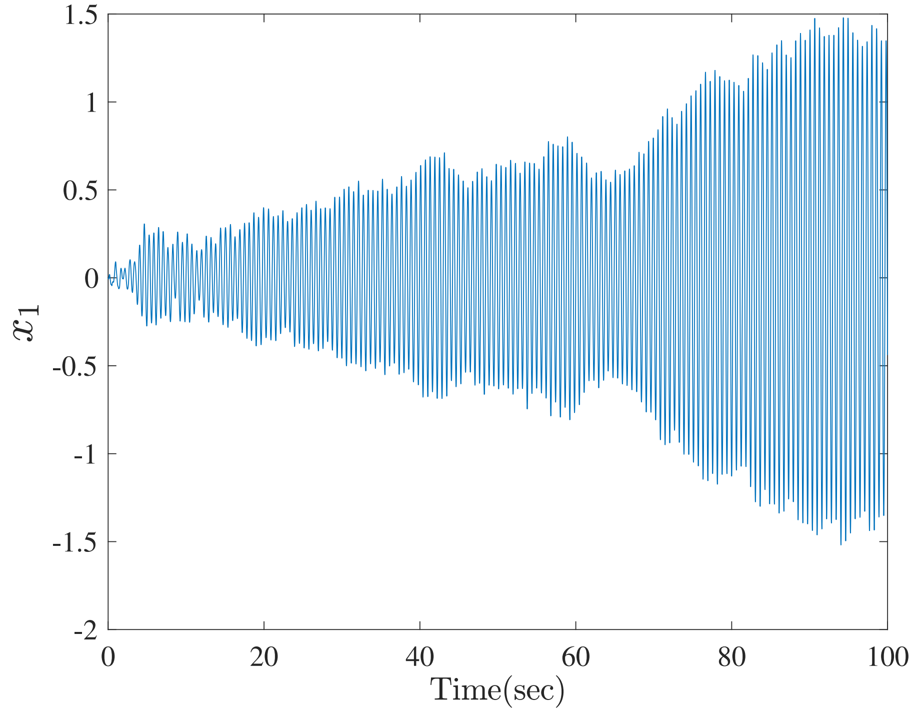



Figure 35: $x_1$ function number = 35

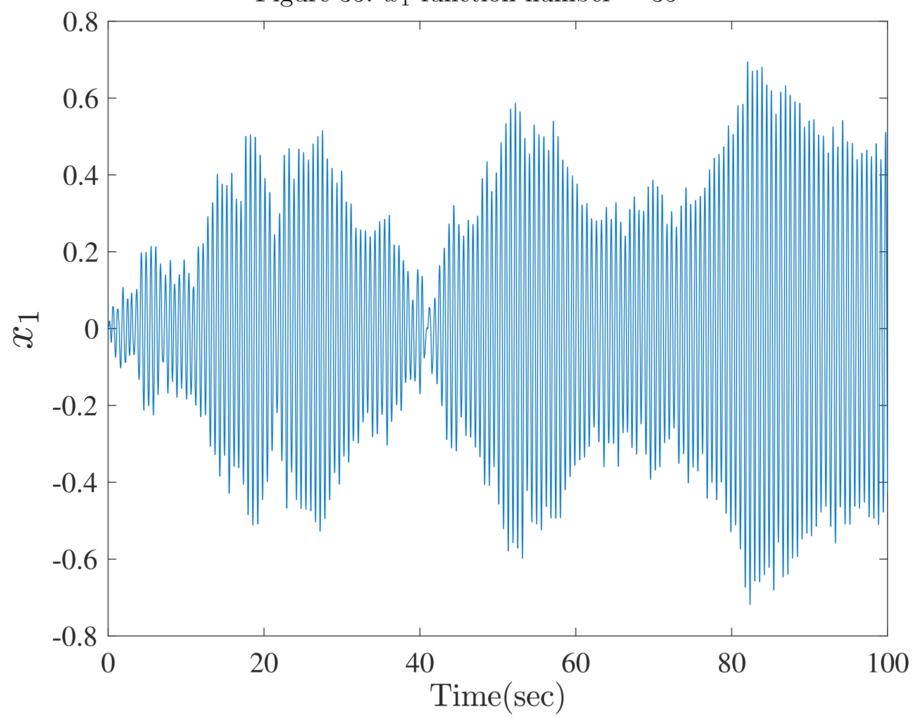


Figure 36: $x_1$ function number = 36

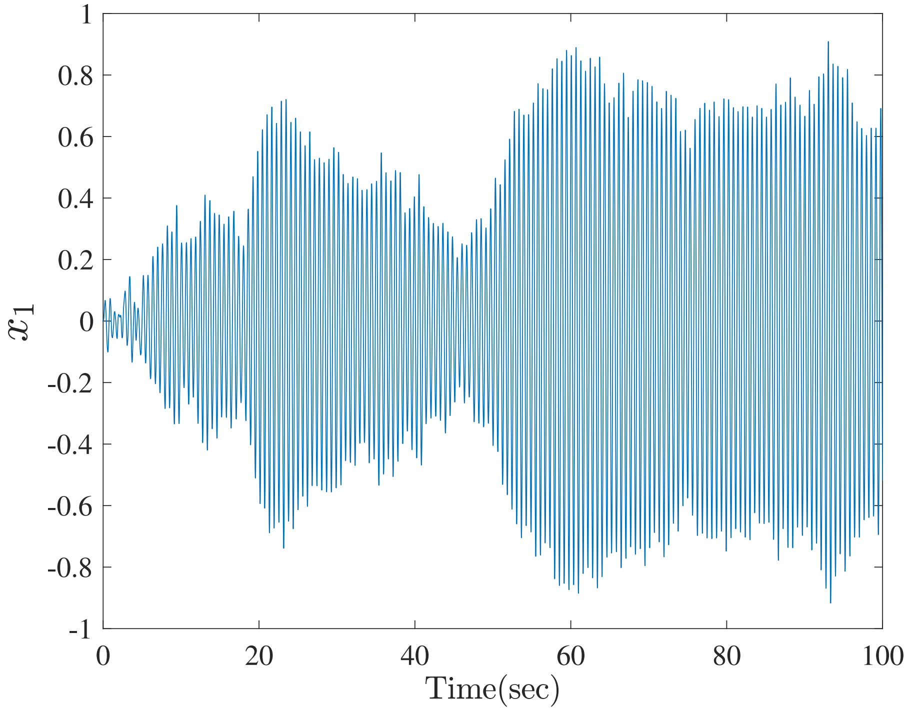



Figure 37: $x_1$ function number = 37

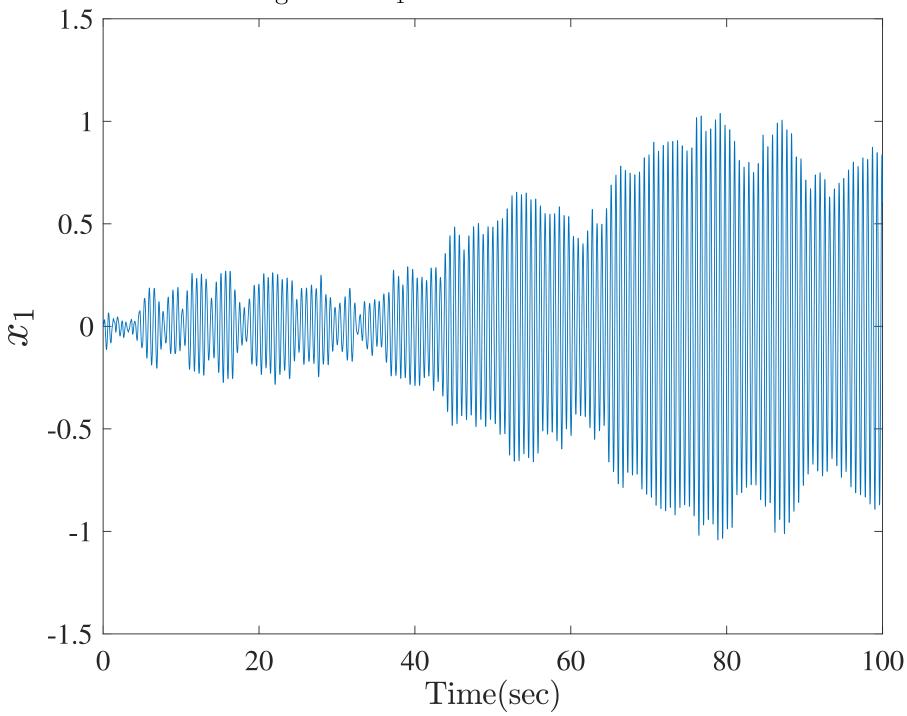


Figure 38: $x_1$ function number = 38

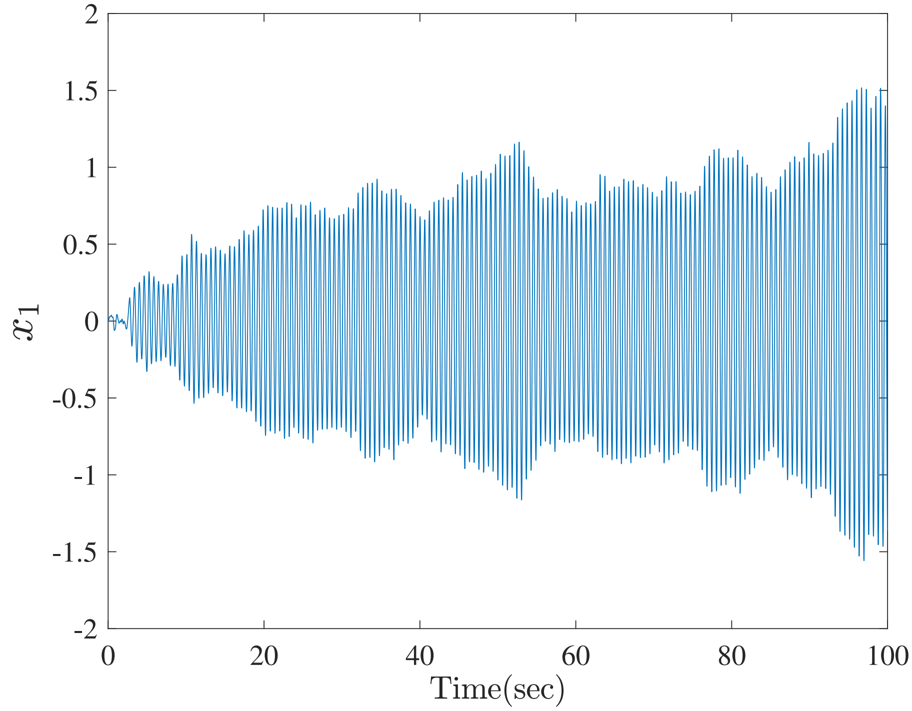



Figure 39: $x_1$ function number = 39

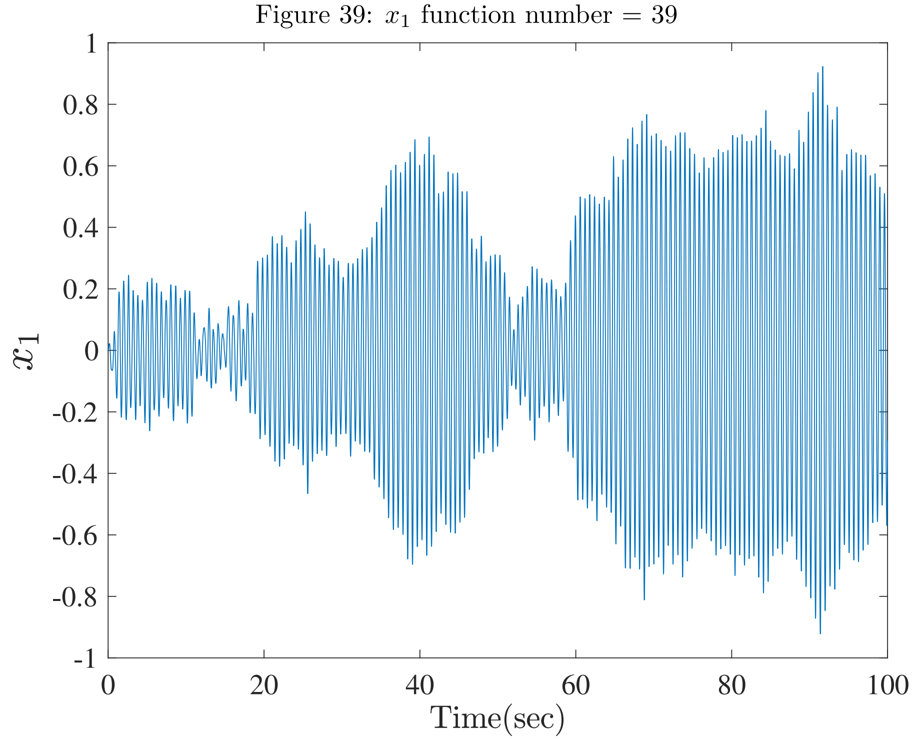


Figure 40: $x_1$ function number = 40

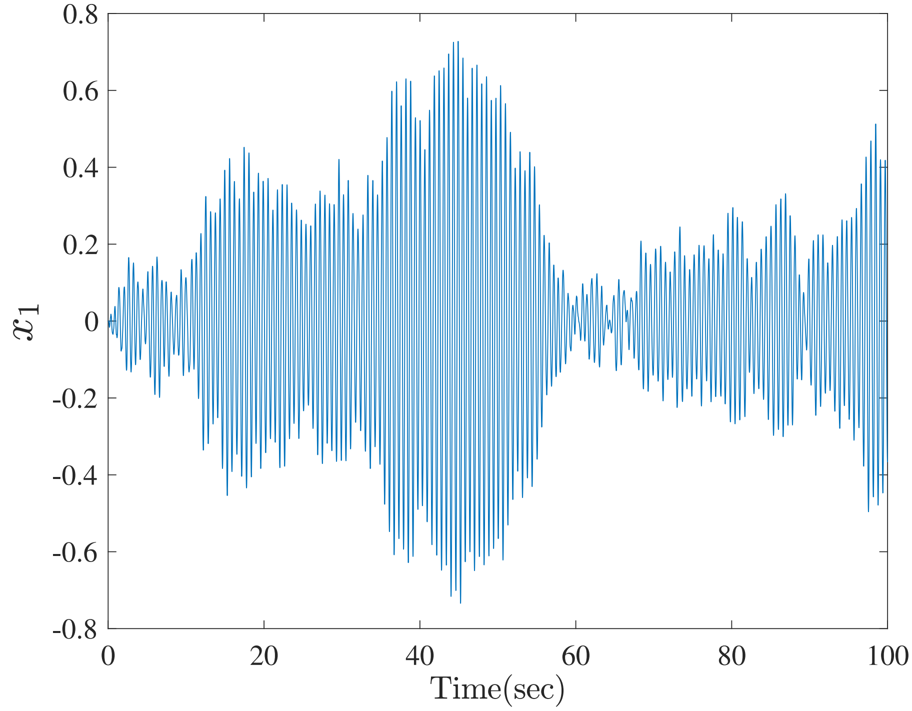



Figure 41: $x_1$ function number = 41

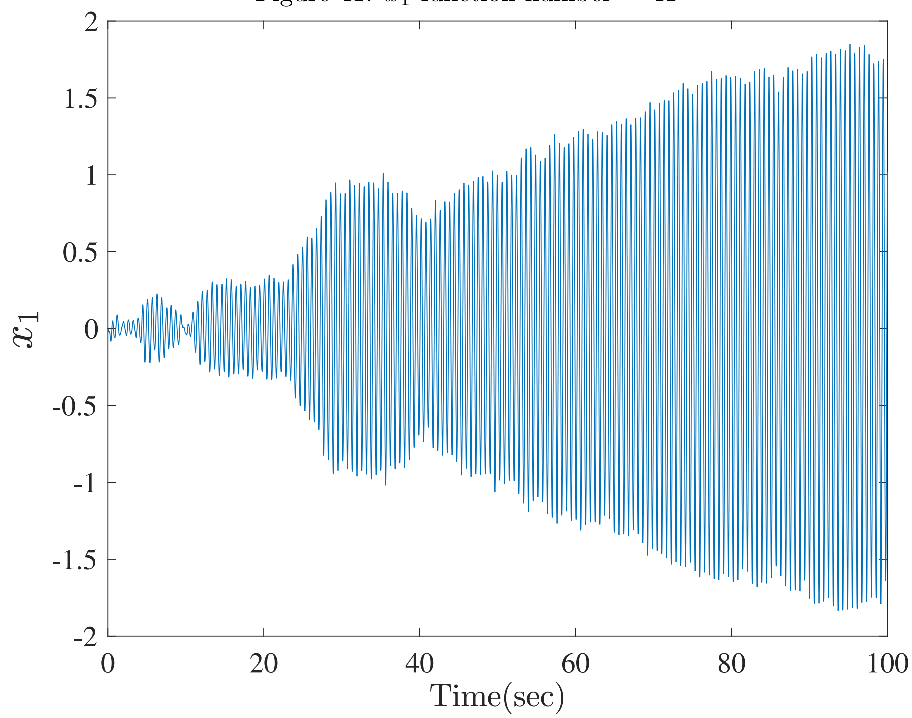


Figure 42: $x_1$ function number = 42

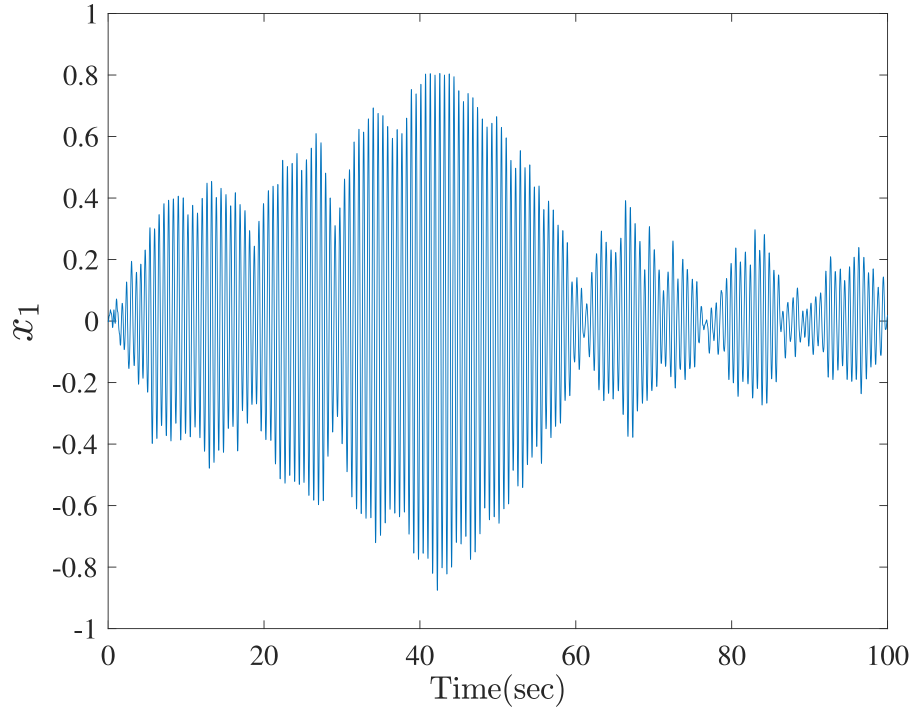



Figure 43: $x_1$ function number = 43

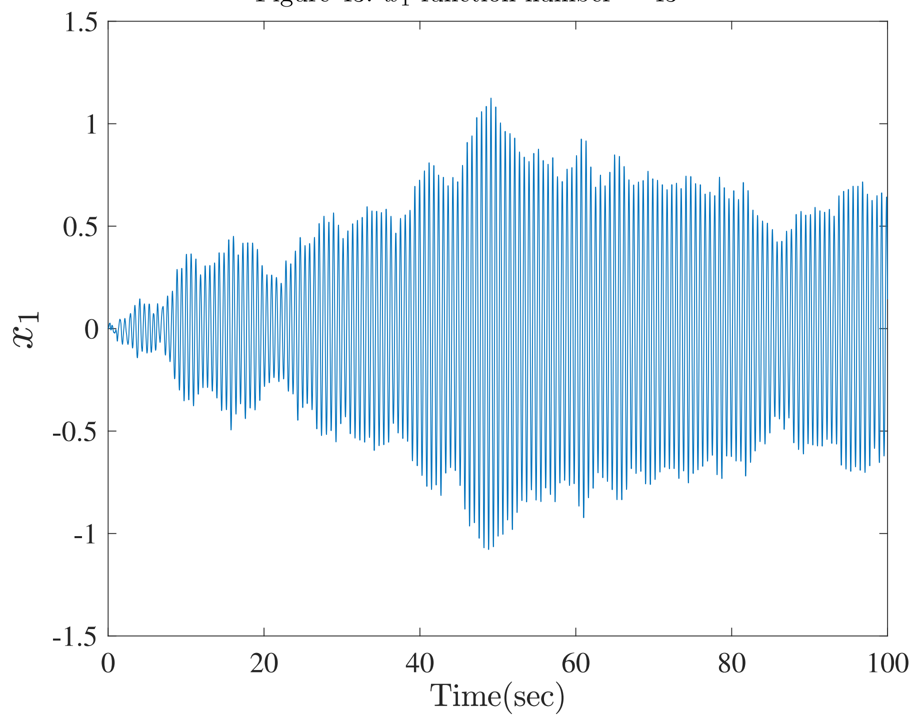


Figure 44: $x_1$ function number = 44

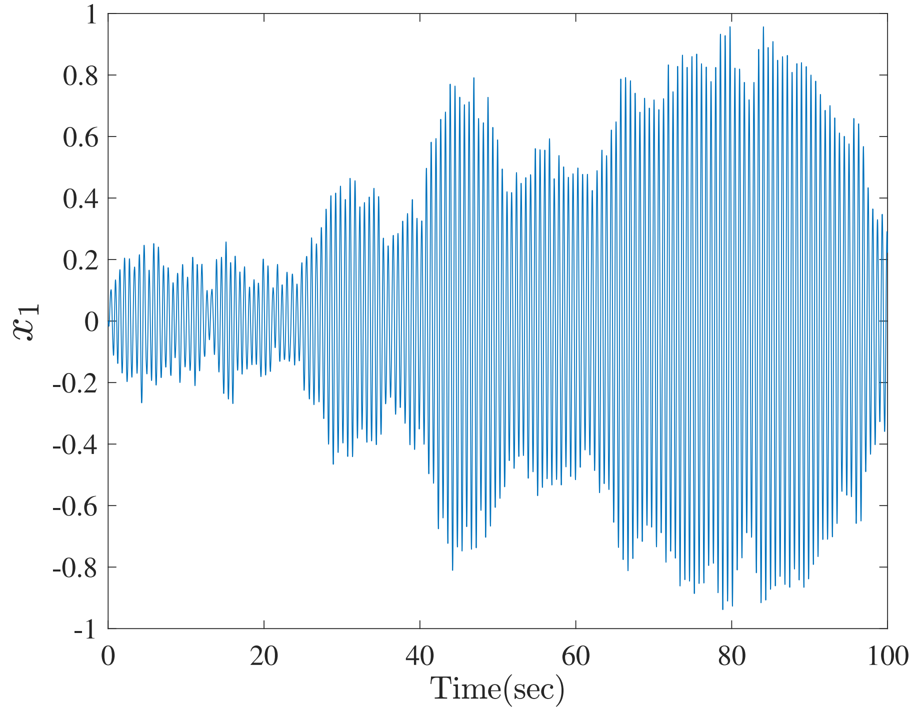



Figure 45: $x_1$ function number = 45

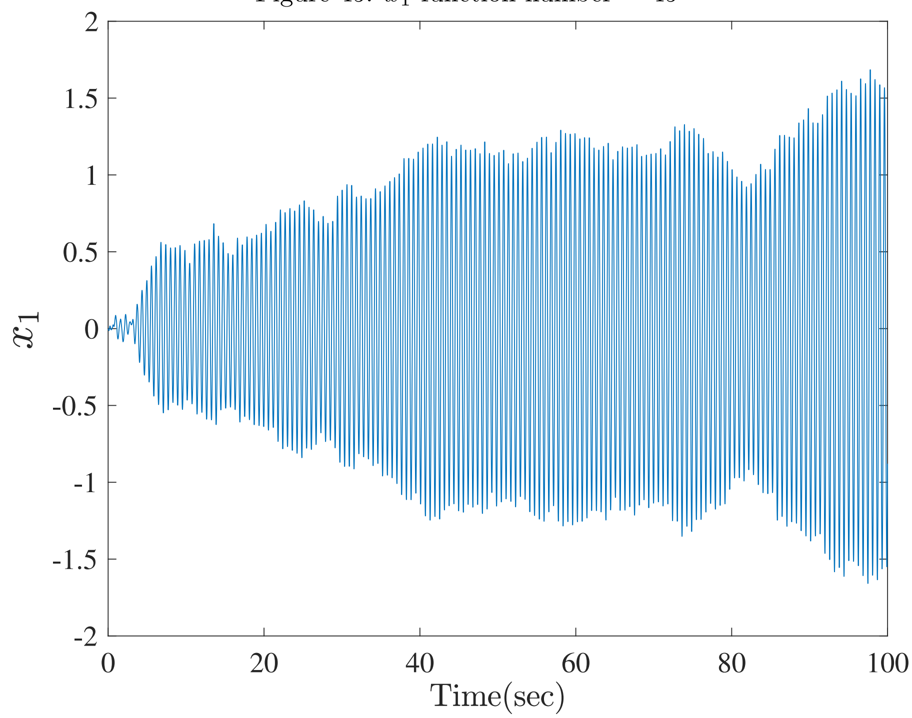


Figure 46: $x_1$ function number = 46

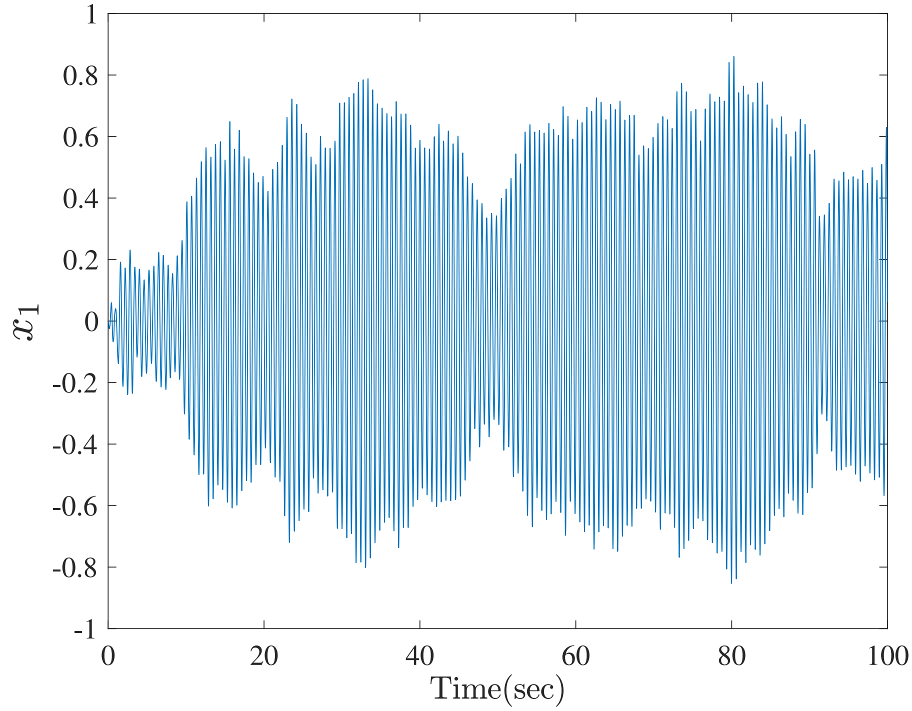



Figure 47: $x_1$ function number = 47

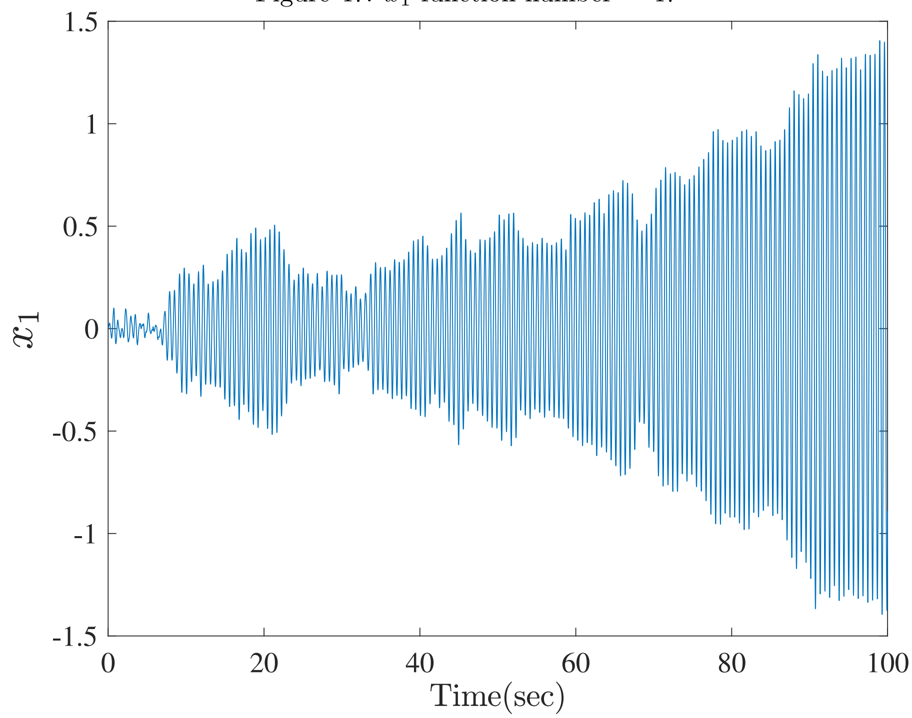


Figure 48: $x_1$ function number = 48

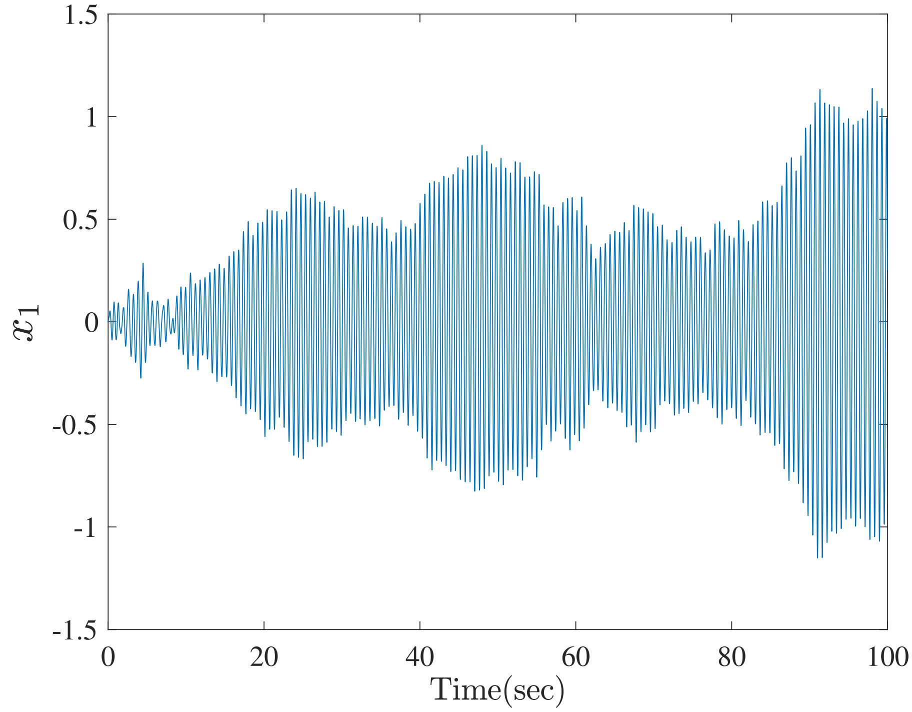



Figure 49: $x_1$ function number = 49

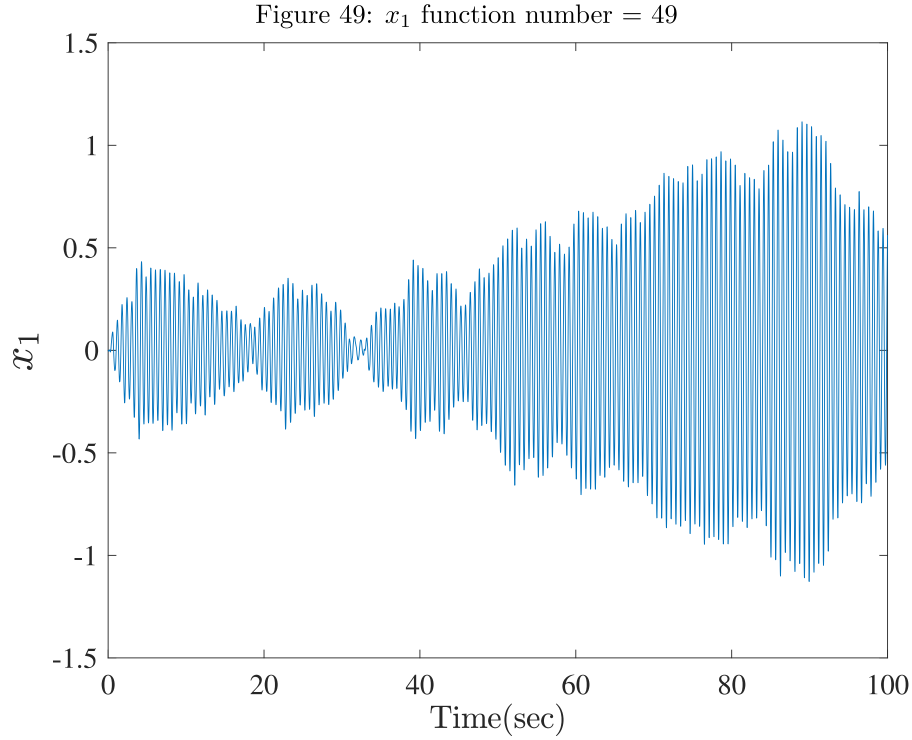
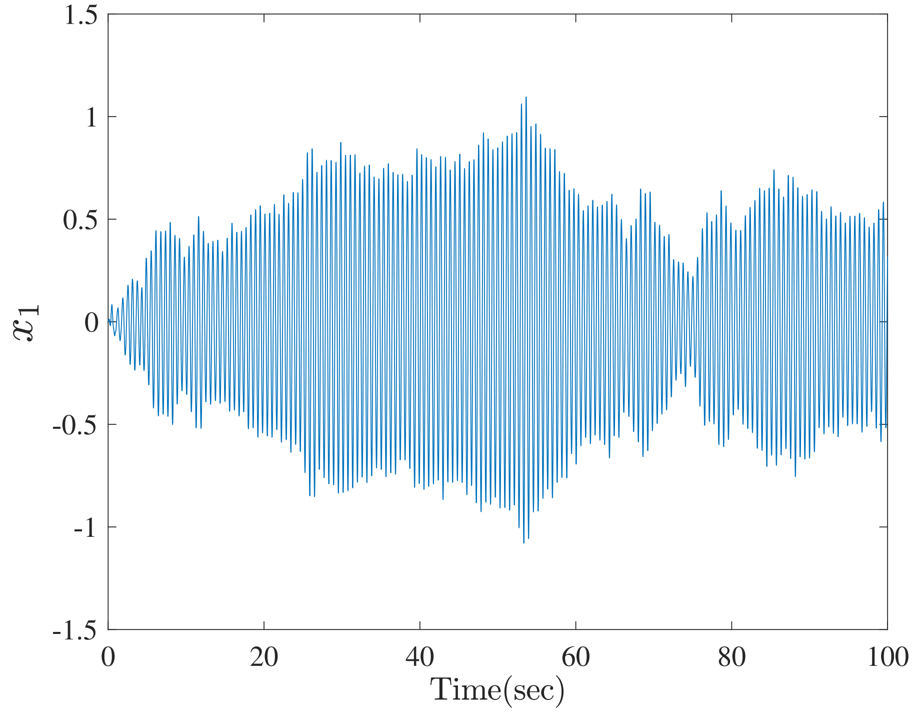


Figure 50: $x_1$ function number = 50




## 5.2 part b

Assume:

$$A[x(t)] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)dt = \frac{1}{2T}\sum_{i=0}^{n} x(t_i)\Delta t = 0.0008$$

## 5.3 part c

All parameters are calculated in the code (Q5/c.m).

$$\text{mean} = -0.0373$$

$$\text{variance} = 0.1255$$

## 5.4 part d

All parameters are calculated in the code (Q5/d.m).

$$\text{mean} = -0.0964$$

$$\text{variance} = 0.2509$$

## 5.5 extra

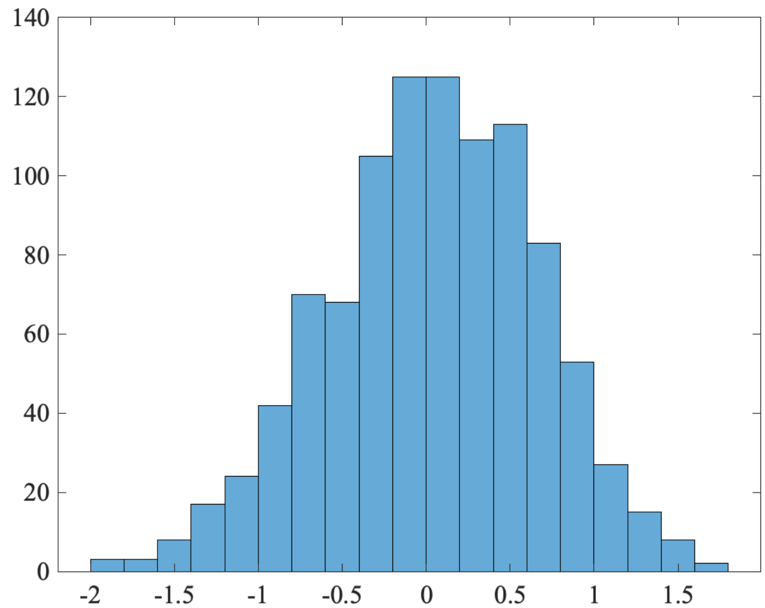
In this section, the function was simulated 1000 times instead of 50 times to see the difference and improvement.

- part c extra

$$\text{mean} = -0.0023$$

$$\text{variance} = 0.1489$$

Figure 51: histogram for $t = 80_{\text{sec}}$

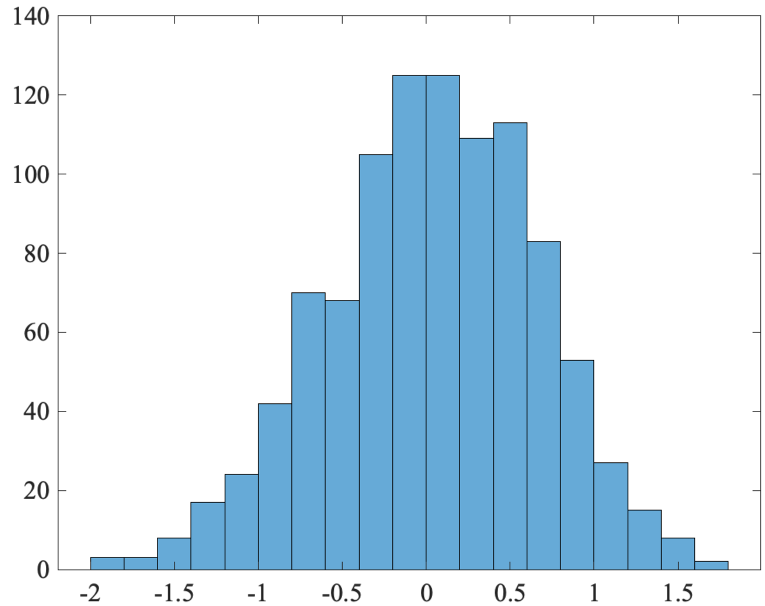


- part d extra

$$\text{mean} = 0.0361$$

$$\text{variance} = 0.3924$$

Figure 52: histogram for $t = 80_{\text{sec}}$



# 6 Question 6

$$\begin{bmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \\ \Delta\dot{x}_3 \\ \Delta\dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3n^2 & 0 & 0 & 2nr_n \\ 0 & 0 & -\dfrac{2n}{r_n} & 0 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \\ \Delta x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & -\dfrac{1}{r_n} \end{bmatrix} \begin{bmatrix} w_r \\ w_\theta \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} r \\ \theta \\ \dot{r} \\ \dot{\theta} \end{bmatrix}$$

## 6.1 part a

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{rank}\left(\begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix}\right) = 3$$

System is not observable.

## 6.2 part b

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\text{rank}\left(\begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix}\right) = 4$$

System is observable.

### 6.3 part c

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}$$

$$\text{rank}\left(\begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix}\right) = 4$$

System is observable.

### 6.4 part d

For subsection 6.2 we have:

$$\text{Singular Value} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

For subsection 6.3 we have:

$$\text{Singular Value} = \begin{bmatrix} 15.3452 \\ 1.4142 \\ 0.9979 \\ 0 \end{bmatrix}$$

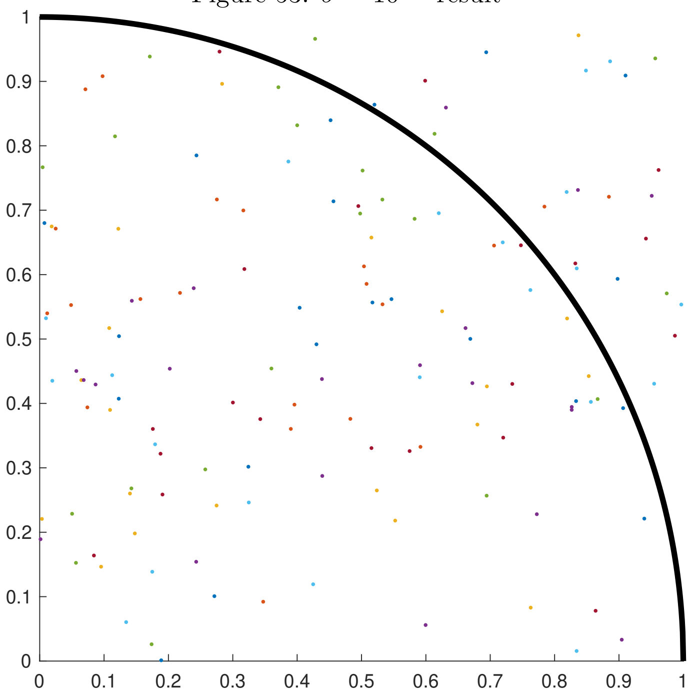
Observability is more straightforward in case 2 because the singular value is higher.

## 7 Bonus

This section uses the variance Criterion to find the $\pi$ number. Here is the result of the $\pi$ estimation with different variances:

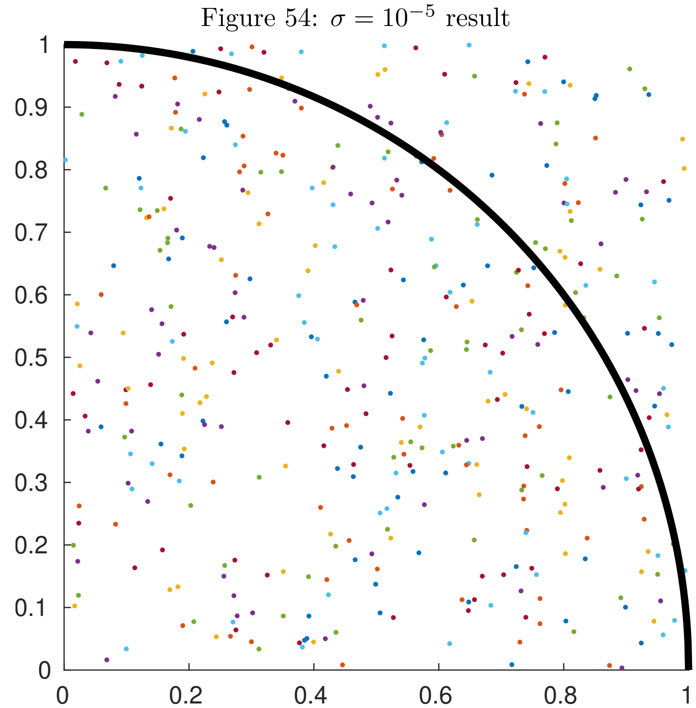
- $\sigma = 10^{-4}$, $\pi = 3.4435$

Figure 53: $\sigma = 10^{-4}$ result



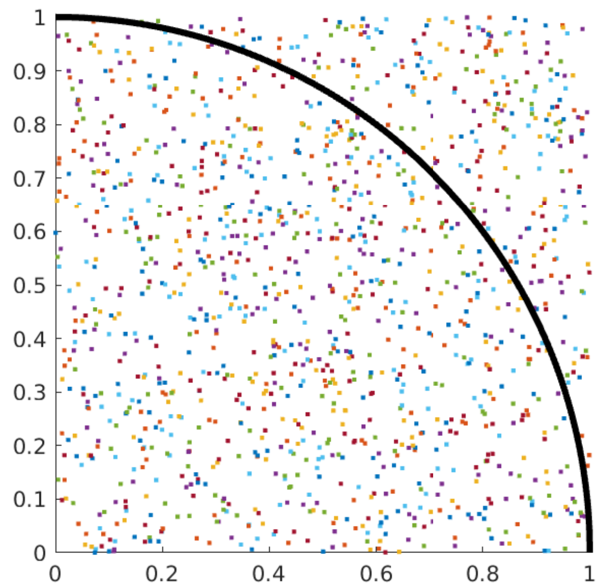
- $\sigma = 10^{-5}$, $\pi = 3.2188$

Figure 54: $\sigma = 10^{-5}$ result



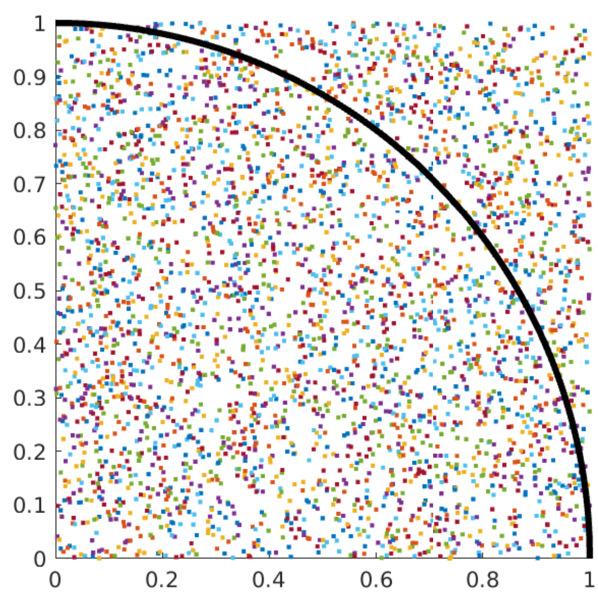
- $\sigma = 10^{-6}$, $\pi = 3.1379$

Figure 55: $\sigma = 10^{-6}$ result



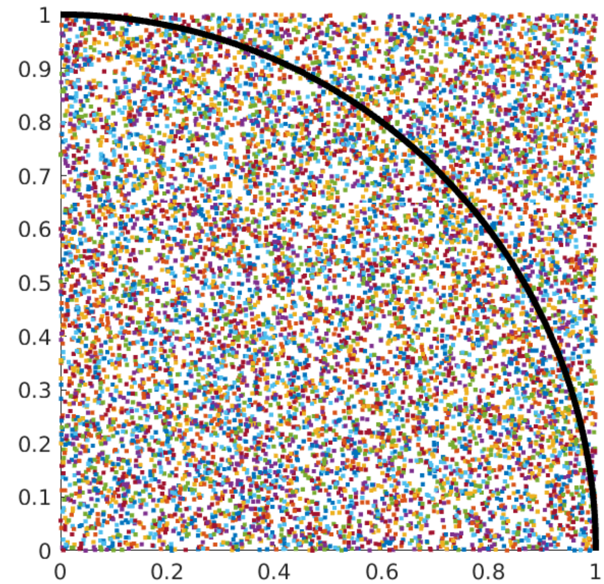

- $\sigma = 10^{-7}$, $\pi = 3.1589$

Figure 56: $\sigma = 10^{-7}$ result



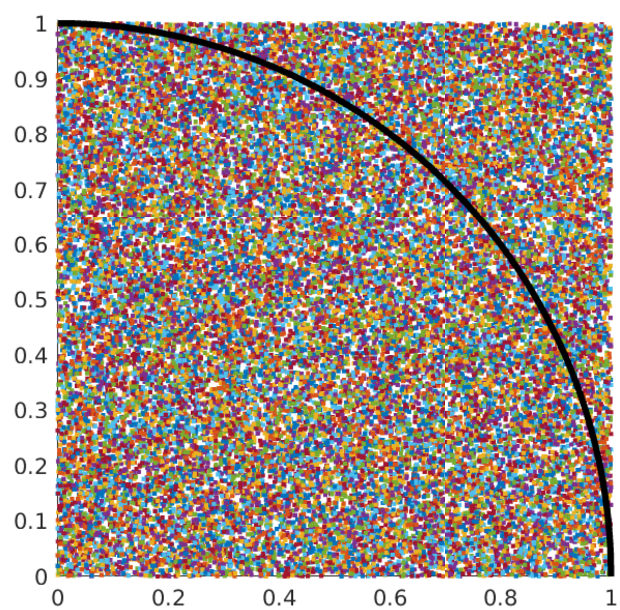
- $\sigma = 10^{-8}$, $\pi = 3.1120$

Figure 57: $\sigma = 10^{-8}$ result



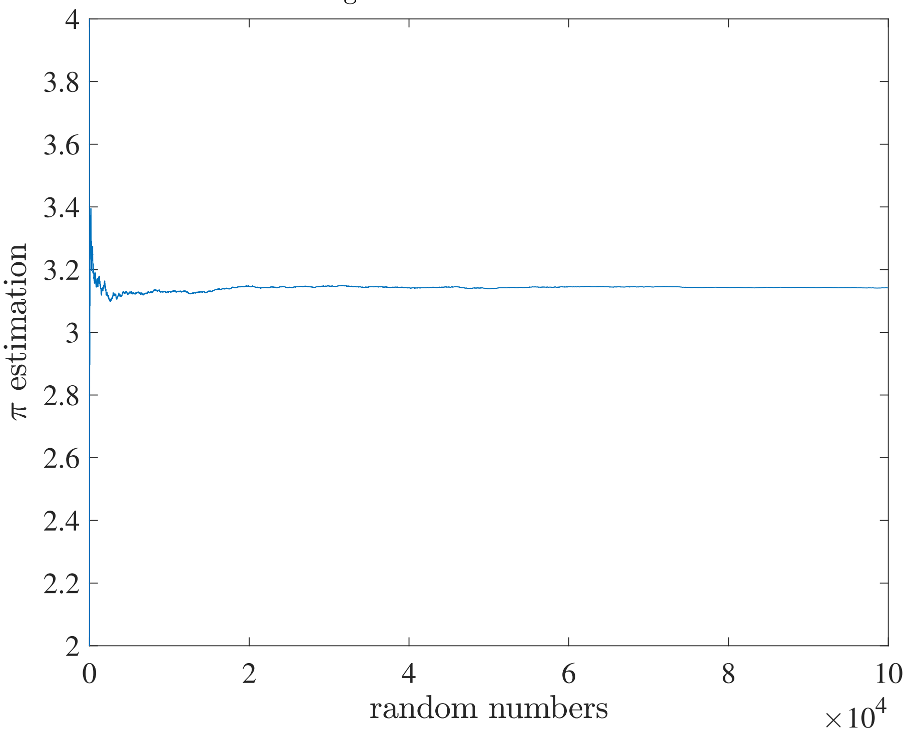

- $\sigma = 10^{-9}$, $\pi = 3.1441$

Figure 58: $\sigma = 10^{-9}$ result



Here is plot of estimated *pi* verses random number used in calculation.

Figure 59: $\pi$ estimation



# Contents

# List of Figures