Home Work #1

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1 Question 1

$$f_X(x) = \frac{ab}{b^2 + x^2}, \quad b > 0$$

1.1 part a

$$\int_{-\infty}^{+\infty} f(x)dx = 1 \quad \to \quad \int_{-\infty}^{+\infty} \frac{ab}{b^2 + x^2}dx = 1 \to a \arctan(\frac{x}{b})\Big|_{-\infty}^{+\infty} = 1 \to a\pi = 1 \to a = \frac{1}{\pi}$$

$$f_X(x) = \frac{1}{\pi}\frac{b}{b^2 + x^2}, \quad b > 0$$

1.2 part b

$$E(X) = \mu_X = \int_{-\infty}^{+\infty} xf(x)dx$$

Because $xf(x)$ is an odd function, the result of the integrator between ∞ and $-\infty$ is zero.

$$\int_{-\infty}^{+\infty} xf(x)dx = 0 \to \mu_X = 0$$

$$\sigma_X^2 = E((X-\mu)^2) = \int_{-\infty}^{+\infty} (x-\mu)^2 f(x)dx = \int_{-\infty}^{+\infty} x^2 f(x)dx = \frac{b}{\pi}(x - b\arctan(\frac{x}{b}))\Big|_{-\infty}^{+\infty} \neq \text{finite}$$

2 Question 3

A positive test is A event: $P(A)$, Having the flu is B event: $P(B) = 0.05$

2.1 part a

The probability of a positive test if someone has flu:

$$P(A|B) = 0.99$$

The probability of a positive test if someone doesn't have flu:

$$P(A|\bar{B}) = 0.01$$

$$P(A) = P(A \cap B) + P(A \cap \bar{B}) = P(A|B)P(B) + P(A|\bar{B})P(\bar{B}) = 0.99 \times 0.05 + 0.01 \times 0.95 = 0.059$$

$$P(A|B)P(B) = P(B|A)P(A) \rightarrow P(B|A) = \frac{P(A|B)P(B)}{P(A)} = 0.84$$

2.2 part b

C is the event when two positive tests happen. The probability of two positive tests if someone has flu:

$$P(B)P(A|B)P(A|B) = 0.049$$

The probability of two positive tests if someone doesn't have flu:

$$P(\bar{B})P(A|\bar{B})P(A|\bar{B}) = 9.5 \times 10^{-5}$$

$$P(C) = P(B)P(A|B)P(A|B) + P(\bar{B})P(A|\bar{B})P(A|\bar{B}) = 0.0491$$

$$P(B|C) = \frac{P(B \cap C)}{P(C)} = 0.998$$

3 Question 4

Assumed that the random variables A ϕ are independent and ϕ is uniform in the interval $[0, 2\pi)$, mean and variance of ϕ is 0 and σ^2, respectively.

$$x(t) = A\cos(\omega t + \phi)$$

3.1 part a

$$E\left[x(t)\right] = E\left[A\cos(\omega t + \phi)\right] \xrightarrow{uncorrelated} E\left[x(t)\right] = E[A]E[\cos(\omega t + \phi)] = 0 \xrightarrow{E[A]=0} E\left[x(t)\right] = 0$$

3.2 part b

$$R_X(t_1, t_2) = E[x(t_1)x(t_2)] = \frac{1}{2}E[A^2]E[\cos\omega(t_1 - t_2) + \cos(\omega t_1 + \omega t_2 + 2\phi)]$$

$$E[\cos(\omega t_1 + \omega t_2 + 2\phi)] = \frac{1}{2\pi}\int_0^{2\pi} \cos(\omega t_1 + \omega t_2 + 2\phi)d\phi = 0$$

$$R_X(t_1, t_2) = \frac{\sigma^2}{2}\cos\omega(t_1 - t_2)$$

3.3 part c

$$A[x(t)] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)dt \to A[x(t)] = \frac{1}{2\pi}\int_{0}^{2\pi} A\cos(\omega t+\phi)dt = \left.\frac{A\sin(\phi+\omega t)}{\omega}\right|_{0}^{2\pi}$$
$$A[x(t)] = \frac{A\sin(\phi+2\pi\omega)}{\omega}$$

3.4 part d

$$R[x(t),\tau] = A[x(t)x(t+\tau)] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)x(t+\tau)dt = \frac{1}{2\pi}\int_{0}^{2\pi} A^2\cos(\omega t+\phi)\cos(\omega(t+\tau)+\phi)dt$$

$$\int \cos(\omega t+\phi)\cos(\omega(t+\tau)+\phi)dt = \begin{cases} A^2\,t\cos(\phi)^2 & \text{if } \omega = 0 \\ \dfrac{A^2\,t\,\cos(\omega\,\tau)}{2} + \dfrac{A^2\,\sin(2\,\phi+2\,\omega\,t+\omega\,\tau)}{4\,\omega} & \text{if } \omega \neq 0 \end{cases}$$

$$\int_0^{2\pi} \cos(\omega t+\phi)\cos(\omega(t+\tau)+\phi)dt = \begin{cases} 2\,\pi\,A^2\cos(\phi)^2 & \text{if } \omega = 0 \\ \dfrac{A^2\,(\sin(2\,\phi+4\,\pi\,\omega+\omega\,\tau)-\sin(2\,\phi+\omega\,\tau))}{4\,\omega} + \pi\,A^2\,\cos(\omega\,\tau) & \text{if } \omega \neq 0 \end{cases}$$

$$R[x(t),\tau] = \begin{cases} 1\,A^2\cos(\phi)^2 & \text{if } \omega = 0 \\ 0.5\,A^2\,\cos(\omega\,\tau) - \dfrac{0.04\,A^2\,(1\,\sin(2\,\phi+\omega\,\tau)-\sin(12.5\,\omega+2\,\phi+\omega\,\tau))}{\omega} & \text{if } \omega \neq 0 \end{cases}$$

3.5 part e

For a WSS process:

$$R_X(0) = E[x(t)x(t)]$$
$$R_X(\tau) = R_X(-\tau)$$

where $R_X(\tau)$:

$$R_X(\tau) = E\left[x(t+\frac{\tau}{2})x^*(t-\frac{\tau}{2})\right]$$

That is true in this stochastic process.

An ergodic process is a stationary random process for which:

$$A[x(t)] = E[x(t)]$$
$$R[x(t,\tau)] = R_X(\tau)$$

$$A[x(t)] \neq E[x(t)]$$

so this stochastic process is not ergodic.

4 Question 5

$$\begin{cases} \dot{x}_1(t) = & x_2(t) \\ \dot{x}_2(t) = & -\omega_n^2 x_1(t) - 2\zeta\omega_n x_2(t) + w(t) \end{cases}$$

where $\omega_n = 10$, $\zeta = 0.3$ and $w(t)$ is with power σ^2 and step time 0.01_{sec} ($\sigma^2 = 1$).

4.1 part a

4.2 part b

Assume:

$$A[x(t)] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)dt = \frac{1}{2T}\sum_{i=0}^{n} x(t_i)\Delta t = 0.0008$$

4.3 part c

All parameters are calculated in the code (Q5/c.m).

$$\text{mean} = -0.0373$$

$$\text{variance} = 0.1255$$

4.4 part d

All parameters are calculated in the code (Q5/d.m).

$$\text{mean} = -0.0964$$

$$\text{variance} = 0.2509$$

4.5 extra

In this section, the function was simulated 1000 times instead of 50 times to see the difference and improvement.

- part c extra

$$\text{mean} = -0.0023$$

$$\text{variance} = 0.1489$$

- part d extra

$$\text{mean} = 0.0361$$

$$\text{variance} = 0.3924$$

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