Home Work #1

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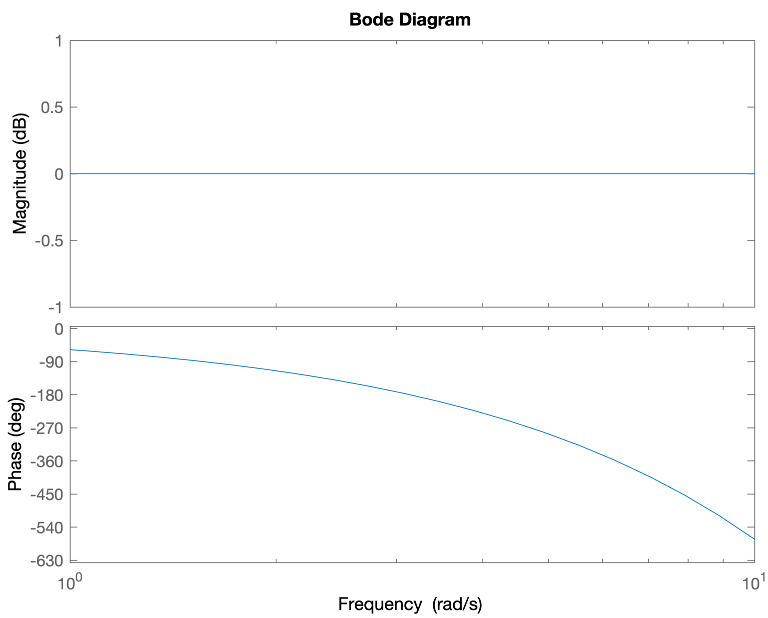
1 Question 1

$$G(s) = \exp(-\tau s), \quad \tau > 0$$

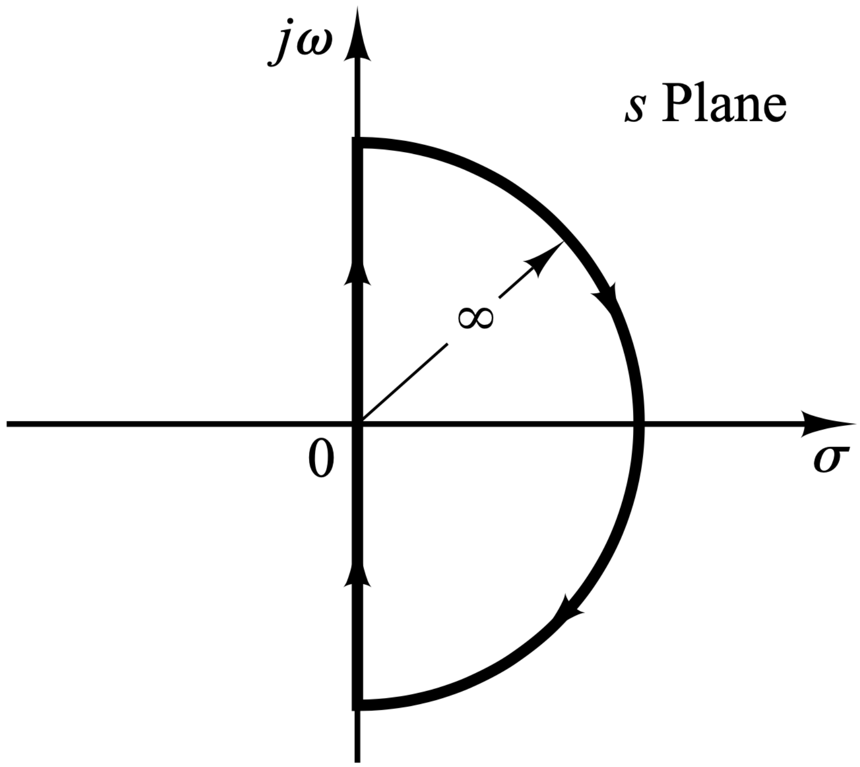
$$\angle G(j\omega) = -\tau\omega$$
$$|G(j\omega)| = 1 \rightarrow 20\log(|G(j\omega)|) = 0$$

Figure 1: Bode digram using MATLAB ($\tau = 1$)



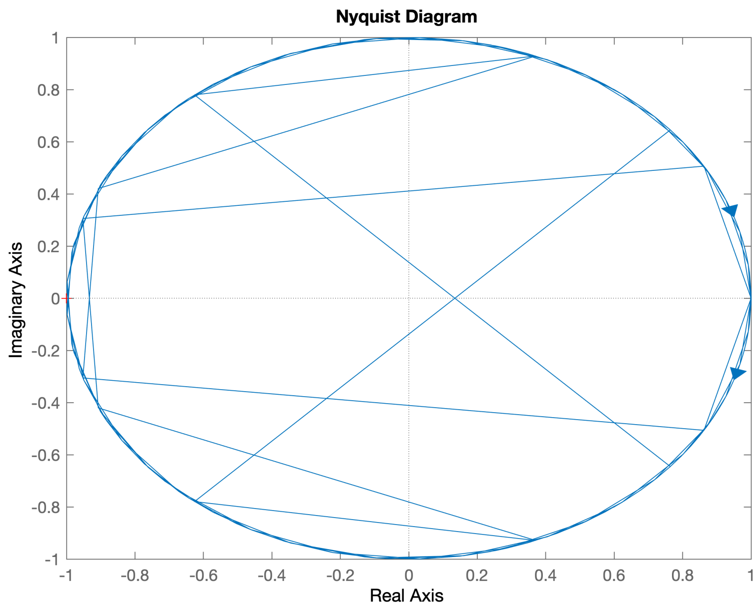
There is no pole in imaginary axis and nyquist path is from $0 \rightarrow \infty \rightarrow -\infty \rightarrow 0$.

Figure 2: Closed contour in the s plane



So the Nyquist plot is circle in orogin with radius 1 and clock wise.

Figure 3: Nyquist plot using MATLAB ($\tau = 1$)



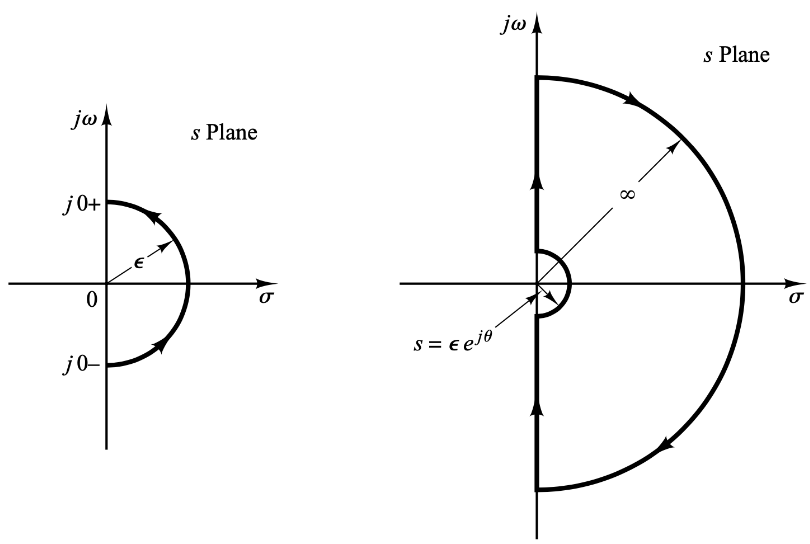
2 Question 2

$$G(s) = \frac{K}{s(s^2+s+a)}, \quad a>0, \quad K>0 \tag{1}$$

The transfer function in which s is replaced by $j\omega$ where ω is frequency.

Cause $a > 0$ we have just on pole on imaginary axis.

Figure 4: Contour near the origin of the s plane and closed contour in the s plane

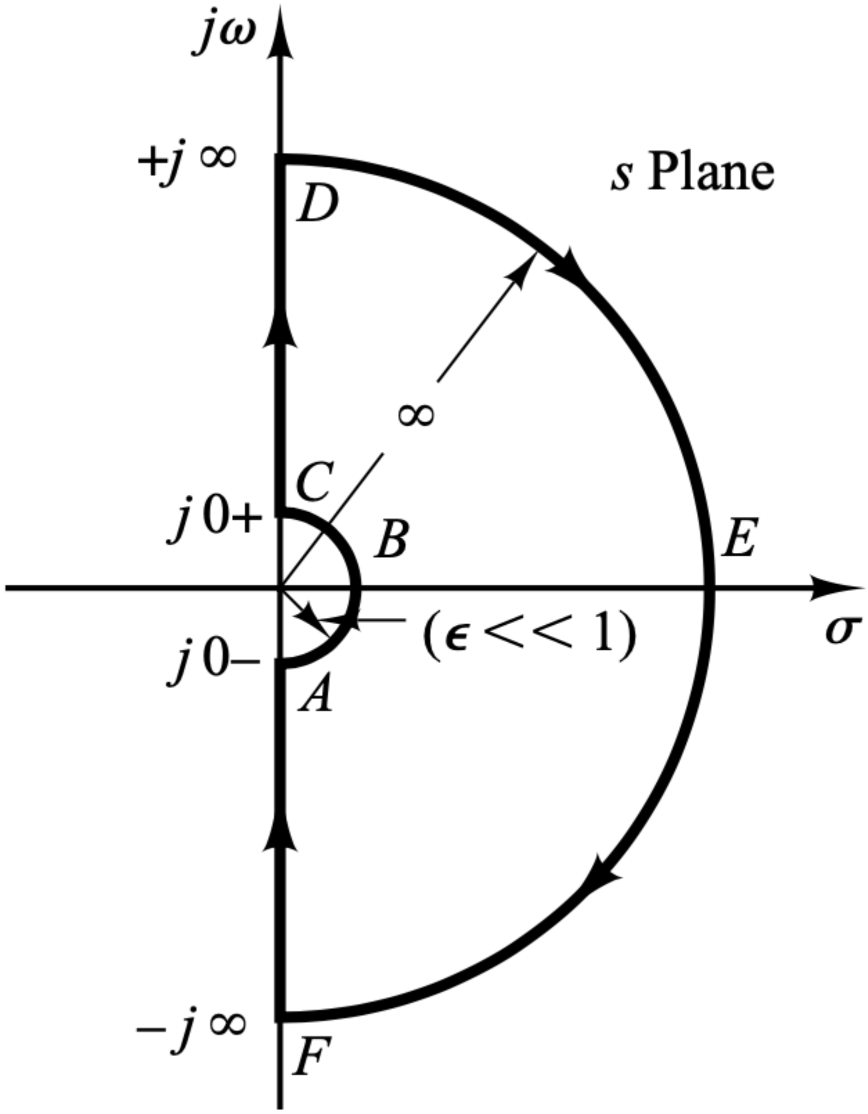


On the semicircular path with radius ϵ (where $\epsilon \ll 1$), the complex variable s can be written

$$s = \epsilon \exp(j\theta)$$

where θ varies from $-\dfrac{\pi}{2}$ to $\dfrac{\pi}{2}$.

Figure 5: s-Plane



Now we calculate $j\omega$ in A, C, D and F.
A:
$s = j\omega, \quad \omega = -\epsilon$

$$\lim_{\omega \to -\epsilon} G(j\omega) = \frac{K}{-j\epsilon((j\epsilon)^2 - j\epsilon + a)} = \frac{K}{-j\epsilon(-\epsilon^2 - j\epsilon + a)} = \frac{K}{j\epsilon^3 - \epsilon^2 - aj\epsilon}$$

$$\lim_{\epsilon \to 0} \frac{K}{j\epsilon^3 - \epsilon^2 - aj\epsilon} = \lim_{\epsilon \to 0} \frac{K}{-aj\epsilon} = \lim_{\epsilon \to 0} \frac{Kj}{a\epsilon} = \infty\angle 90^\circ$$

C:

$s = j\omega, \quad \omega = \epsilon$

$$\lim_{\omega \to \epsilon} G(j\omega) = \frac{K}{j\epsilon((j\epsilon)^2 + j\epsilon + a)} = \frac{K}{j\epsilon(-\epsilon^2 + j\epsilon + a)} = \frac{K}{j\epsilon^3 - \epsilon^2 - aj\epsilon}$$

$$\lim_{\epsilon \to 0} \frac{K}{j\epsilon^3 - \epsilon^2 + aj\epsilon} = \lim_{\epsilon \to 0} \frac{K}{-aj\epsilon} = \lim_{\epsilon \to 0} \frac{-Kj}{a\epsilon} = \infty\angle - 90^\circ$$

A to C:
$s = j\omega, \quad \omega = \epsilon \exp(j\theta), \quad \theta : -\frac{\pi}{2} \to 0 \to \frac{\pi}{2}(CCW)$

$$\lim_{\epsilon \to 0} G(\epsilon \exp(j\theta)) = \frac{K}{\epsilon \exp(j\theta)((\epsilon \exp(j\theta))^2 + \epsilon \exp(j\theta) + a)} = \frac{K}{\epsilon^3 \exp(3j\theta) + \epsilon^2 \exp(2j\theta) + a\epsilon \exp(j\theta)}$$

$$\lim_{\epsilon \to 0} \frac{K}{a\epsilon \exp(j\theta)} = \lim_{\epsilon \to 0} \frac{K \exp(-j\theta)}{a\epsilon} = \infty\angle - \theta$$

D:
$s = j\omega, \quad \omega = \infty$

$$\lim_{\omega \to \infty} G(j\omega) = \lim_{\omega \to \infty} \frac{K}{j\omega((j\omega)^2 + j\omega + a)} = \lim_{\omega \to \infty} \frac{K}{j\omega(-\omega^2 + j\omega + a)} = \lim_{\omega \to \infty} \frac{K}{-j\omega^3 - \omega^2 + aj\omega}$$

$$\lim_{\omega \to \infty} \frac{K}{-j\omega^3} = \lim_{\omega \to \infty} \frac{jK}{\omega^3} = 0\angle 90^\circ$$

F:
$s = j\omega, \quad \omega = -\infty$

$$\lim_{\omega \to -\infty} G(j\omega) = \lim_{\omega \to -\infty} \frac{K}{j\omega((j\omega)^2 + j\omega + a)} = \lim_{\omega \to -\infty} \frac{K}{j\omega(-\omega^2 + j\omega + a)} = \lim_{\omega \to -\infty} \frac{K}{-j\omega^3 - \omega^2 + aj\omega}$$

$$\lim_{\omega \to -\infty} \frac{K}{-j\omega^3} = \lim_{\omega \to -\infty} \frac{jK}{\omega^3} = 0\angle - 90^\circ$$

Figure 6: Nyquist plot using wolfram ($a = 1, K = 1$)

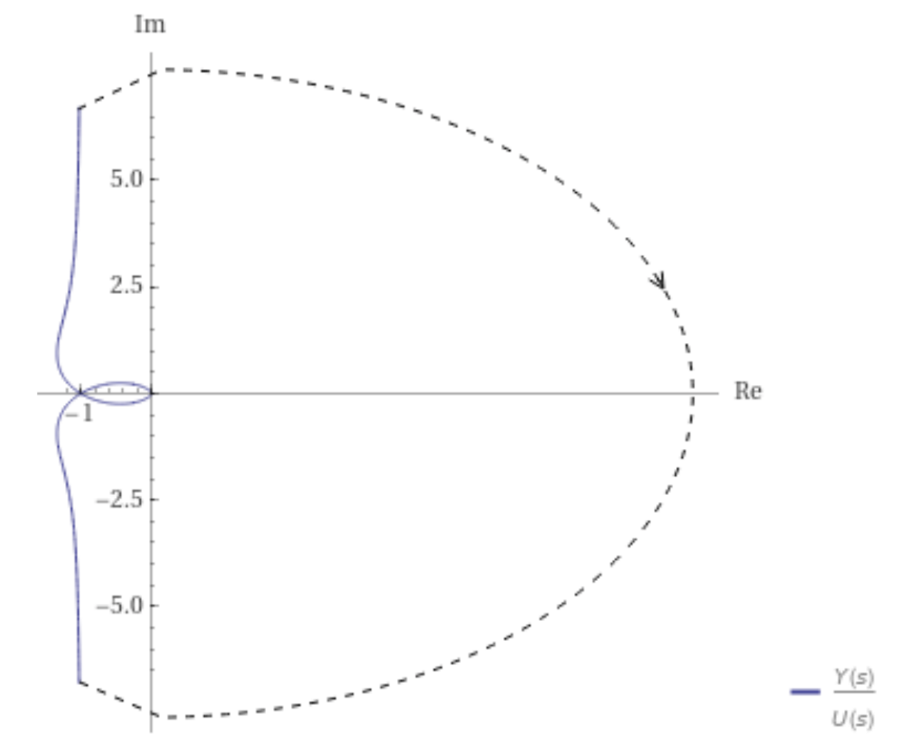
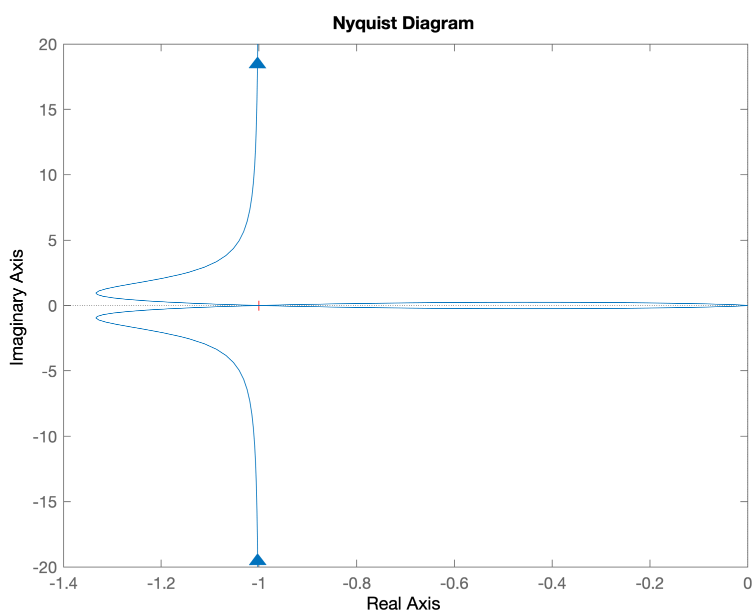


Figure 7: Nyquist plot using MATLAB ($a = 1, K = 1$)



Now we know shape of nyquist plot. we find out where it's equal to -1.

$$G(j\omega) = -1 + 0j = \frac{K}{j\omega((j\omega))^2 + j\omega + a)} = \frac{K}{-j\omega^3 - \omega^2 + aj\omega} = -1$$

$$\rightarrow j\omega^3 + \omega^2 - aj\omega = K \rightarrow j\omega^3 + \omega^2 - aj\omega - K = 0$$

Two equation and two unknowns.

$$j\omega^3 = aj\omega \rightarrow \omega^2 = a$$
$$\omega^2 = K \rightarrow \omega^2 = K$$

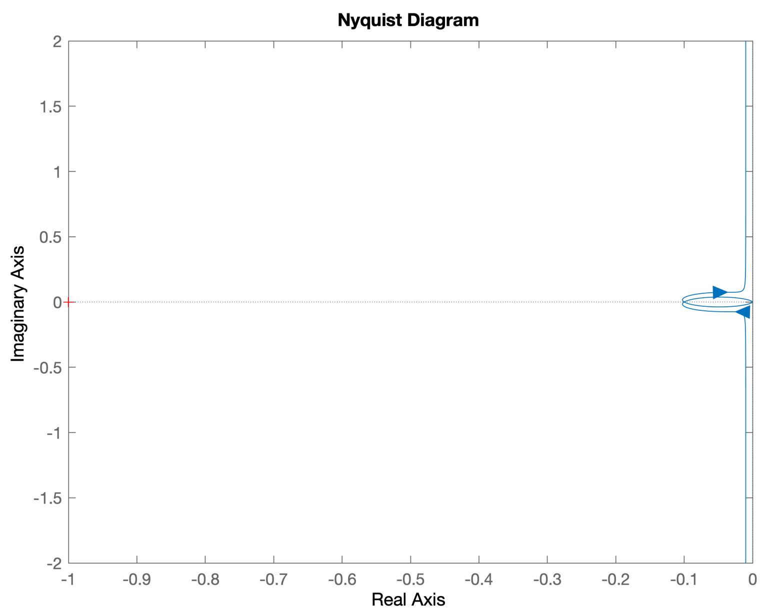
When $a = K$ the nyquist plot cross from -1 point. When $a > K$ the nyquist plot is before -1 and when $a < K$ the nyquist plot cross -1 and system is unstable.

Now we check above using MATLAB.

- $a > K$

  $a = 10, K = 1$

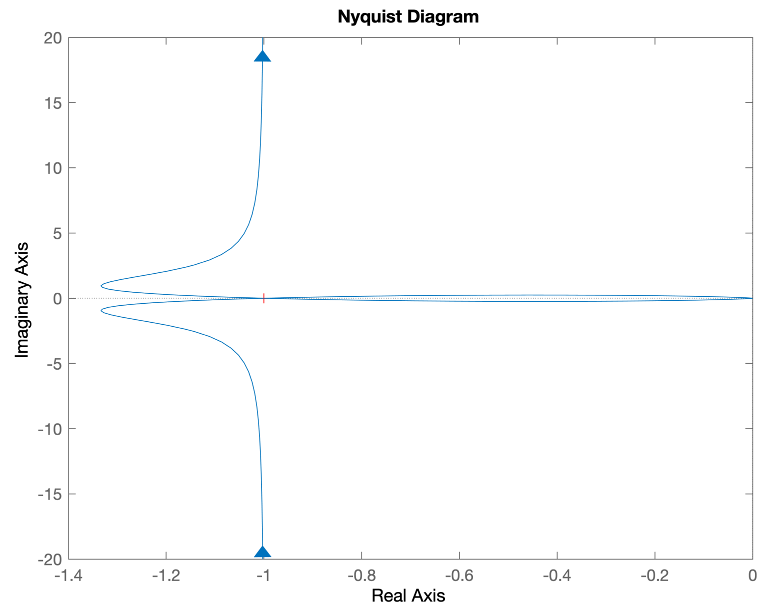
Figure 8: Nyquist plot using MATLAB $(a = 10, K = 1)$



- $a = K$

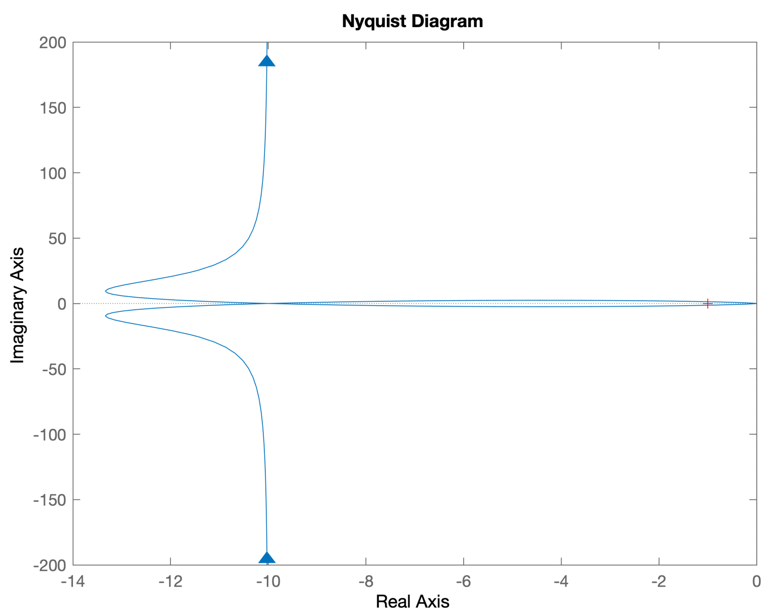
  $a = 1, K = 1$

Figure 9: Nyquist plot using MATLAB ($a = 1, K = 1$)



- $a < K$
  $a = 1, K = 10$

Figure 10: Nyquist plot using MATLAB ($a = 1, K = 10$)



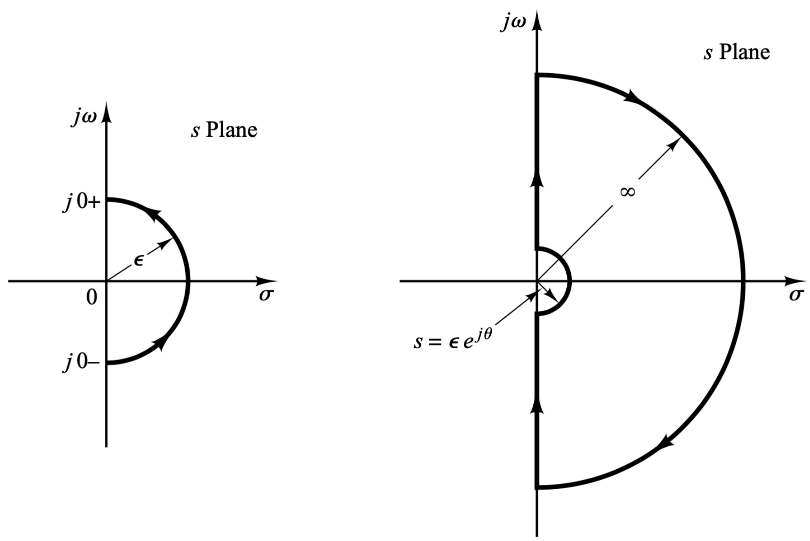
3 Question 3

$$G(s) = \frac{K\exp(-\tau s)}{s(s^2+s+a)}, \quad \tau > 0, \quad a > 0, \quad K > 0 \tag{2}$$

The transfer function in which s is replaced by $j\omega$ where ω is frequency.

Cause $a > 0$ we have just on pole on imaginary axis.

Figure 11: Contour near the origin of the s plane and closed contour in the s plane

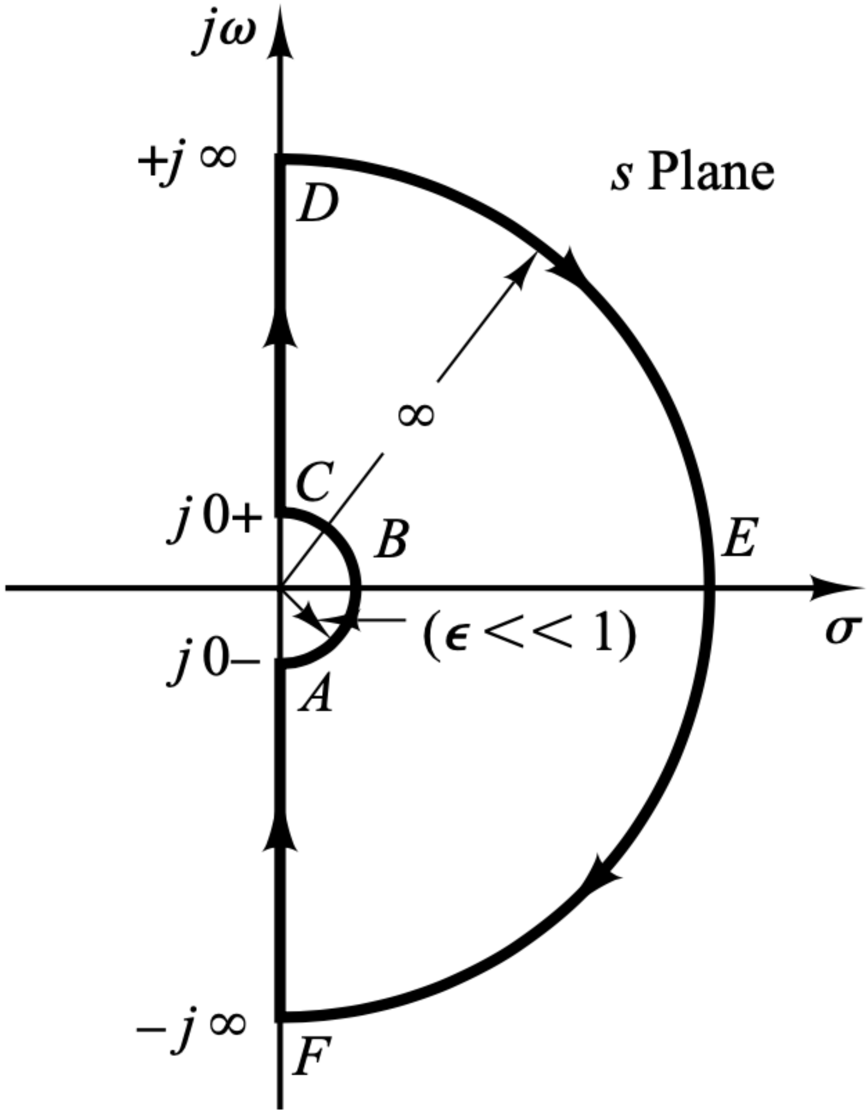


On the semicircular path with radius ϵ (where $\epsilon \ll 1$), the complex variable s can be written

$$s = \epsilon \exp(j\theta)$$

where θ varies from $-\dfrac{\pi}{2}$ to $\dfrac{\pi}{2}$.

Figure 12: s-Plane



Now we calculate $j\omega$ in A, C, D and F.
A:
$s = j\omega, \quad \omega = -\epsilon$

$$\lim_{\omega \to -\epsilon} G(j\epsilon) = \frac{K\exp(\tau j\epsilon)}{-j\epsilon((j\epsilon)^2 - j\epsilon + a)} = \frac{K\exp(\tau j\omega)}{-j\epsilon(-\epsilon^2 - j\epsilon + a)} = \frac{K\exp(\tau j\epsilon)}{j\epsilon^3 - \epsilon^2 - aj\epsilon}$$

$$\lim_{\epsilon \to 0} \frac{K\exp(\tau j\epsilon)}{j\epsilon^3 - \epsilon^2 - aj\epsilon} = \lim_{\epsilon \to 0} \frac{K\exp(\tau j\epsilon)}{-aj\epsilon} = \lim_{\epsilon \to 0} \frac{Kj\exp(\tau j\epsilon)}{a\epsilon} = \infty\angle 90^\circ$$

C:

$s = j\omega, \quad \omega = \epsilon$

$$\lim_{\omega\to\epsilon} G(j\omega) = \frac{K\exp(-\tau j\epsilon)}{j\epsilon((j\epsilon)^2 + j\epsilon + a)} = \frac{K\exp(-\tau j\epsilon)}{j\epsilon(-\epsilon^2 + j\epsilon + a)} = \frac{K\exp(-\tau j\epsilon)}{j\epsilon^3 - \epsilon^2 - aj\epsilon}$$

$$\lim_{\epsilon\to 0} \frac{K\exp(-\tau j\epsilon)}{j\epsilon^3 - \epsilon^2 + aj\epsilon} = \lim_{\epsilon\to 0} \frac{K\exp(-\tau j\epsilon)}{-aj\epsilon} = \lim_{\epsilon\to 0} \frac{-Kj\exp(-\tau j\epsilon)}{a\epsilon} = \infty\angle -90^\circ$$

A to C:
$s = j\omega, \quad \omega = \epsilon\exp(j\theta), \quad \theta : -\frac{\pi}{2} \to 0 \to \frac{\pi}{2}(CCW)$

$$\lim_{\epsilon\to 0} G(\epsilon\exp(j\theta)) = \frac{K\exp(-\tau j\epsilon)}{\epsilon\exp(j\theta)((\epsilon\exp(j\theta))^2 + \epsilon\exp(j\theta) + a)} = \frac{K\exp(-\tau j\epsilon)}{\epsilon^3\exp(3j\theta) + \epsilon^2\exp(2j\theta) + a\epsilon\exp(j\theta)}$$

$$\lim_{\epsilon\to 0} \frac{K\exp(-\tau j\epsilon)}{a\epsilon\exp(j\theta)} = \lim_{\epsilon\to 0} \frac{K\exp(-j\theta - \tau j\epsilon)}{a\epsilon} = \infty\angle -\theta$$

D:
$s = j\omega, \quad \omega = \infty$

$$\lim_{\omega\to\infty} G(j\omega) = \lim_{\omega\to\infty} \frac{K\exp(-\tau j\omega)}{j\omega((j\omega)^2 + j\omega + a)} = \lim_{\omega\to\infty} \frac{K\exp(-\tau j\omega)}{j\omega(-\omega^2 + j\omega + a)} = \lim_{\omega\to\infty} \frac{K\exp(-\tau j\omega)}{-j\omega^3 - \omega^2 + aj\omega}$$

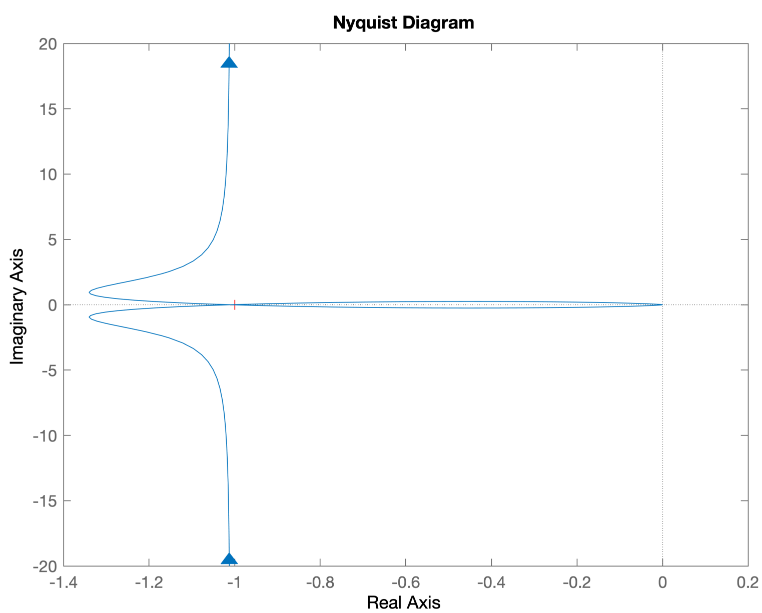
$$\lim_{\omega\to\infty} \frac{K\exp(-\tau j\omega)}{-j\omega^3} = \lim_{\omega\to\infty} \frac{jK\exp(-\tau j\omega)}{\omega^3} = 0\angle 90^\circ$$

F:
$s = j\omega, \quad \omega = -\infty$

$$\lim_{\omega\to-\infty} G(j\omega) = \lim_{\omega\to-\infty} \frac{K\exp(-\tau j\omega)}{j\omega((j\omega)^2 + j\omega + a)} = \lim_{\omega\to-\infty} \frac{K\exp(-\tau j\omega)}{j\omega(-\omega^2 + j\omega + a)} = \lim_{\omega\to-\infty} \frac{K\exp(-\tau j\omega)}{-j\omega^3 - \omega^2 + aj\omega}$$

$$\lim_{\omega\to-\infty} \frac{K\exp(-\tau j\omega)}{-j\omega^3} = \lim_{\omega\to-\infty} \frac{jK\exp(-\tau j\omega)}{\omega^3} = 0\angle -90^\circ$$

Figure 13: Nyquist plot using MATLAB ($a = 1, K = 1, \tau = 0.01$)



Now we know shape of nyquist plot. we find out where it's equal to -1.

$$G(j\omega) = -1 + 0j = \frac{K\exp(-\tau j\omega)}{j\omega((j\omega))^2 + j\omega + a)} = \frac{K\exp(-\tau j\omega)}{-j\omega^3 - \omega^2 + aj\omega} = -1$$

$$\rightarrow j\omega^3 + \omega^2 - aj\omega = K\exp(-\tau j\omega) \rightarrow j\omega^3 + \omega^2 - aj\omega - K\exp(-\tau j\omega) = 0$$

if we assume $\omega\tau \ll 1$ the problem is like previous question. Two equation and two unknowns.

$$j\omega^3 = aj\omega - Kj\omega\tau \rightarrow \omega^2 = a - K\tau$$
$$\omega^2 = K \rightarrow \omega^2 = K$$

When $a = K$ the nyquist plot cross from -1 point. When $a - \tau\omega > K$ the nyquist plot is before -1 and when $a - \tau\omega < K$ the nyquist plot cross -1 and system is unstable.

Figure 14: Stable Nyquist plot using MATLAB ($a = 2, K = 1, \tau = 0.01$)

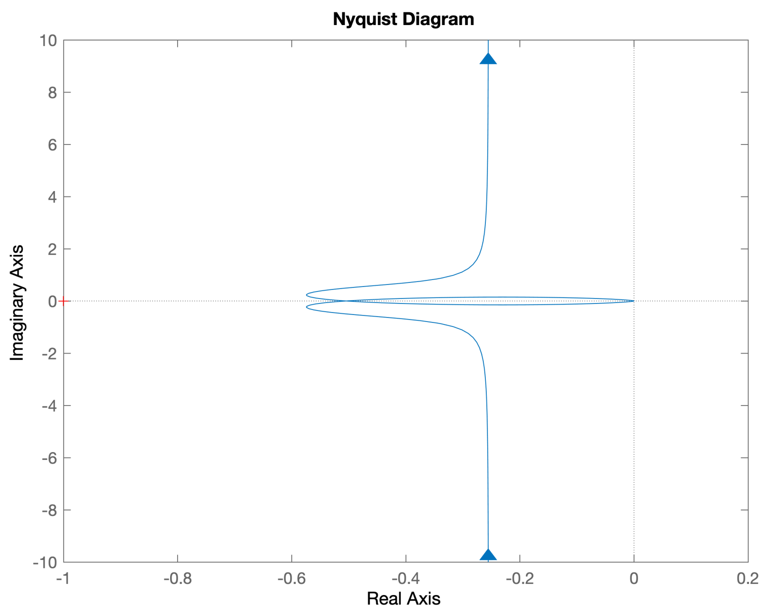
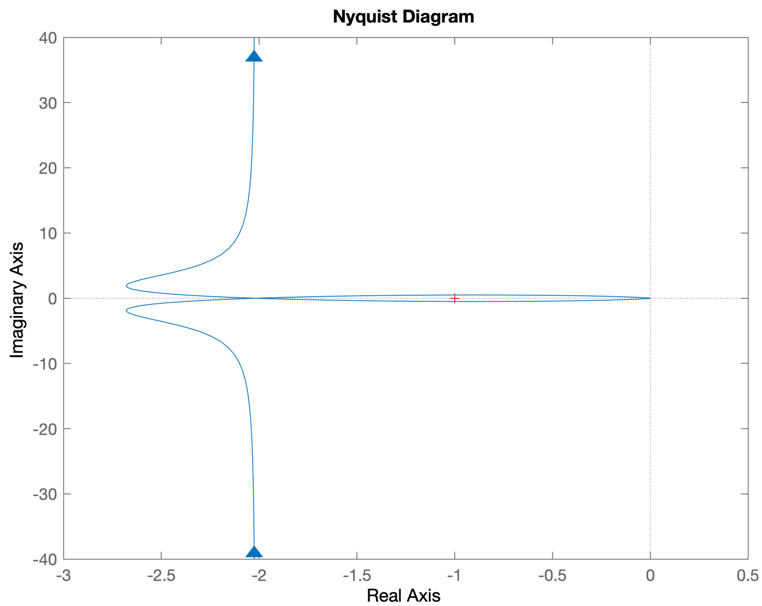


Figure 15: Unstable Nyquist plot using MATLAB ($a = 1, K = 2, \tau = 0.01$)



When $\tau \ll 1$ it's not true and we have nonlinear equation that is hard to slove.

$$j\omega^3 = aj\omega - Kj\sin(\tau\omega)$$
$$\omega^2 = K\cos(\tau\omega)$$

when τ increase we must increase a or decrease K for stable system.

Figure 16: Unstable Nyquist plot using MATLAB ($a = 1, K = 1, \tau = 2$)

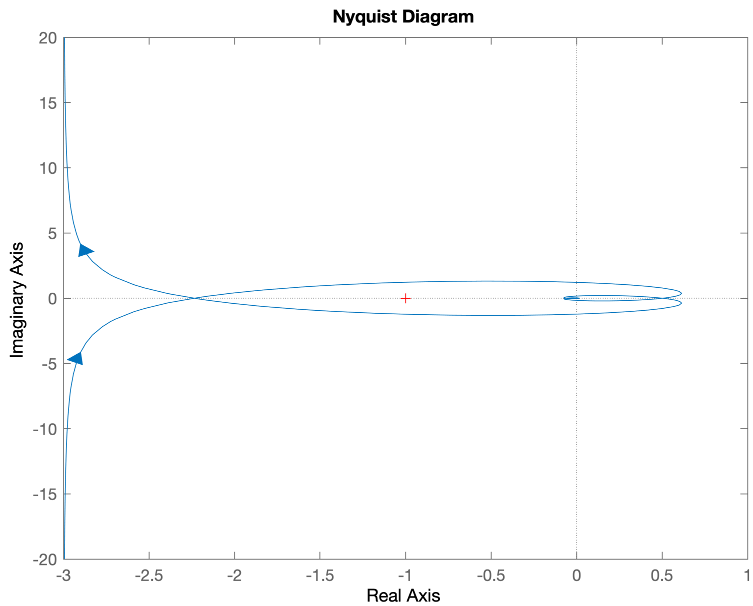
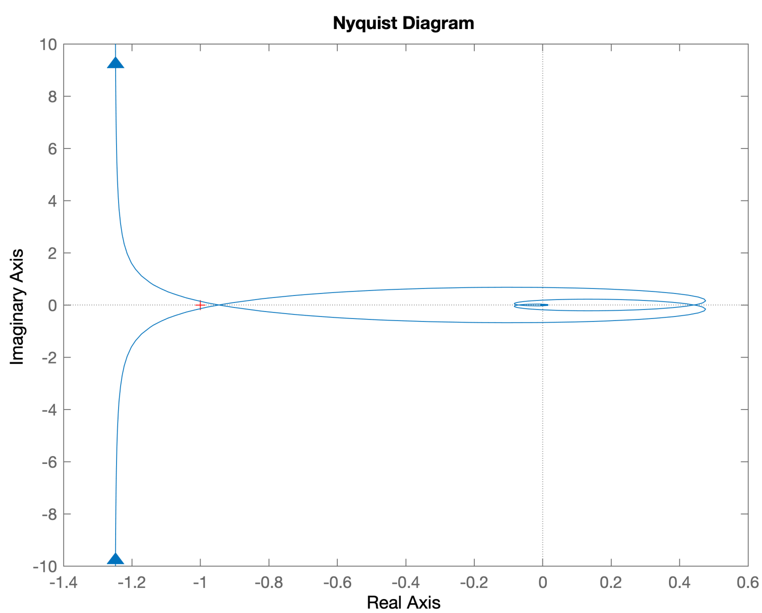


Figure 17: Stable Nyquist plot using MATLAB ($a = 2, K = 1, \tau = 2$)



When we increase τ it's make our system unstable.

Figure 18: Unstable Nyquist plot with large τ using MATLAB ($a = 2, K = 1, \tau = 100$)

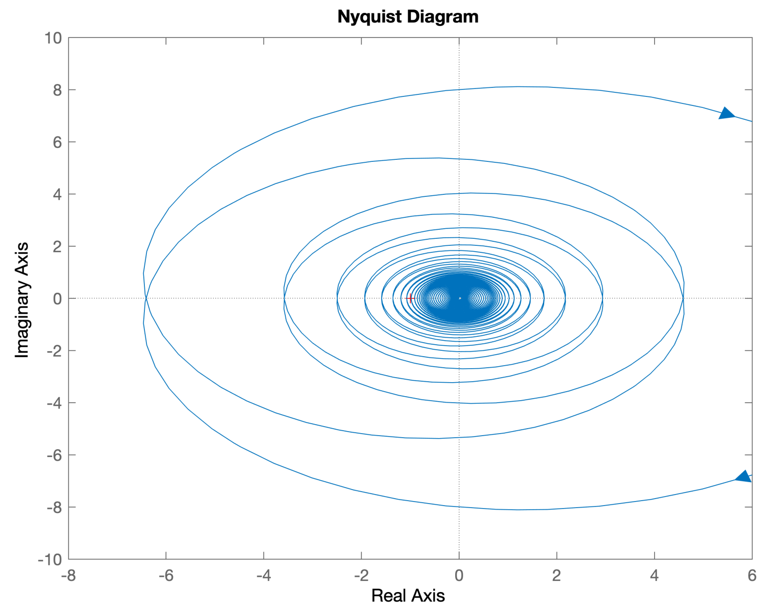
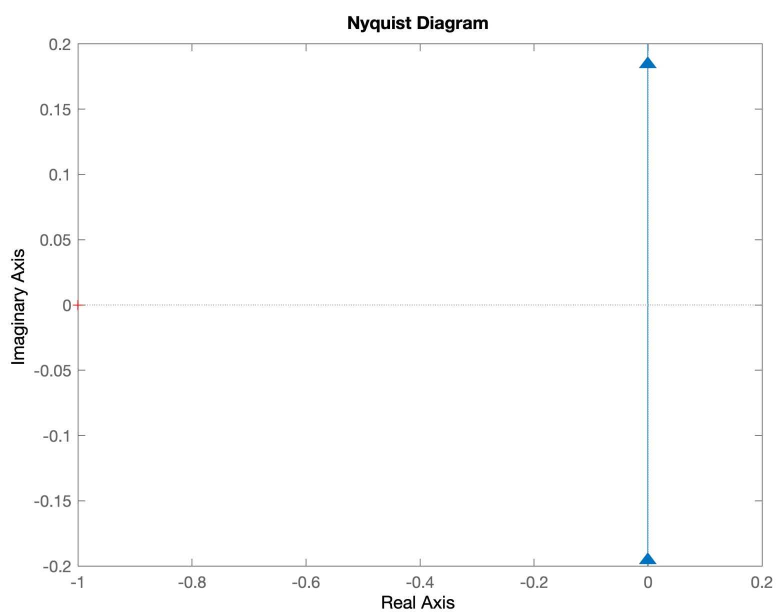


Figure 19: Stable Nyquist plot with large τ and small K using MATLAB ($a = 2, K = 0.00001, \tau = 100$)



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