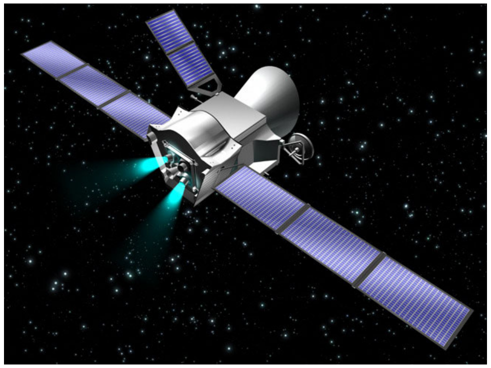
In The Name of God



AE 45780: Spacecraft Dynamics and Control

Fall 1401

2- Perturbation Effects and the Relative Motion

Seid H. Pourtakdoust

The N-Body Problem (NBP)

- Equation of motion for m_i under the influence of others.

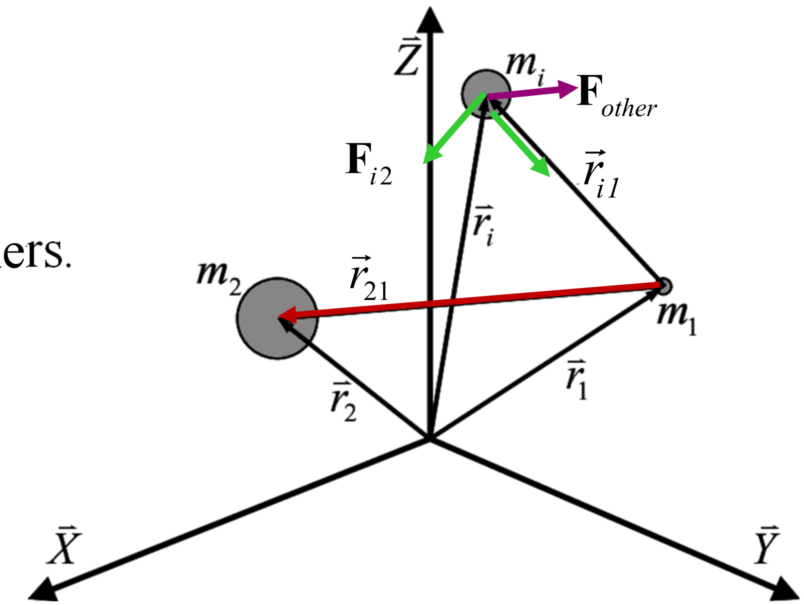
$$\mathbf{F}_{ij} = -G\frac{m_i m_j}{r_{ij}^3}\mathbf{r}_{ij} \qquad where \qquad \mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$$

$$\mathbf{F}_g = \sum_{\substack{j=1 \\ j\neq i}}^{n} \mathbf{F}_{ij} = -Gm_i \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{m_j}{r_{ij}^3}\mathbf{r}_{ij}$$

$$\mathbf{F}_{total} = \mathbf{F}_g + \mathbf{F}_{other}$$

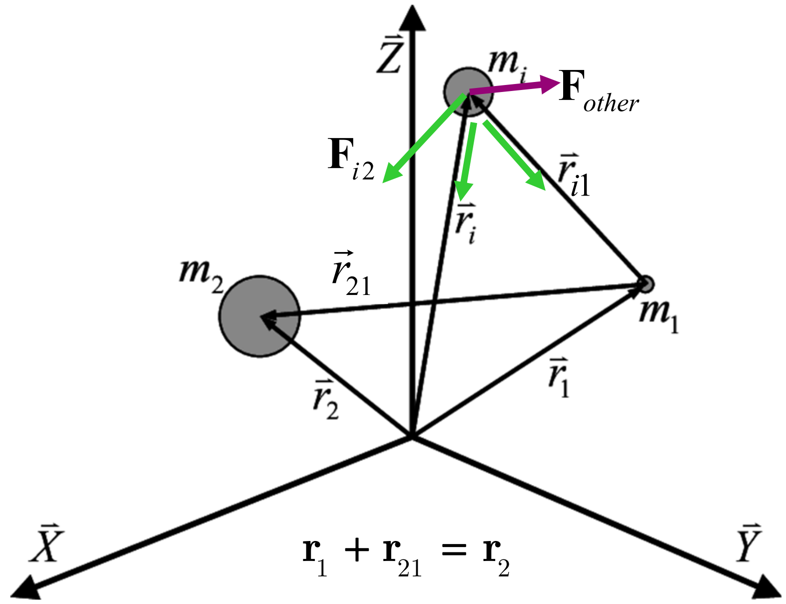
$$\frac{d}{dt}(m_i\mathbf{v}_i) = \dot{m}_i\mathbf{v}_i + m_i\ddot{\mathbf{r}}_i = \mathbf{F}_{total}$$

- Simplifying assumptions:
  - Constant mass
  - Ignoring forces except gravitational ones.



The N-Body Problem (NBP)

$$m_i\ddot{\mathbf{r}}_i = -Gm_i\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{m_j}{r_{ij}^3}\mathbf{r}_{ij} \rightarrow \ddot{\mathbf{r}}_i = -G\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{m_j}{r_{ij}^3}\mathbf{r}_{ij}$$



- Assume the first body is the Earth and the second body is a satellite.

$$\ddot{\mathbf{r}}_1 = -G\sum_{j=2}^{n}\frac{m_j}{r_{1j}^3}\mathbf{r}_{1j}, \qquad \ddot{\mathbf{r}}_2 = -G\sum_{\substack{j=1\\ j\neq 2}}^{n}\frac{m_j}{r_{2j}^3}\mathbf{r}_{2j}$$

- But

$$\mathbf{r}_{21} = \mathbf{r}_2 - \mathbf{r}_1 \rightarrow \ddot{\mathbf{r}}_{21} = \ddot{\mathbf{r}}_2 - \ddot{\mathbf{r}}_1 = -G\frac{m_1}{r_{21}^3}\mathbf{r}_{21} - G\sum_{j=3}^{n}\frac{m_j}{r_{2j}^3}\mathbf{r}_{2j} + G\frac{m_2}{r_{12}^3}\mathbf{r}_{12} + G\sum_{j=3}^{n}\frac{m_j}{r_{1j}^3}\mathbf{r}_{1j}$$

The N-Body Problem (NBP)

$$\ddot{\mathbf{r}}_{21} = -G\frac{m_1}{r_{21}^3}\mathbf{r}_{21} - G\sum_{j=3}^{n}\frac{m_j}{r_{2j}^3}\mathbf{r}_{2j} + G\frac{m_2}{r_{12}^3}\mathbf{r}_{12} + G\sum_{j=3}^{n}\frac{m_j}{r_{1j}^3}\mathbf{r}_{1j}$$

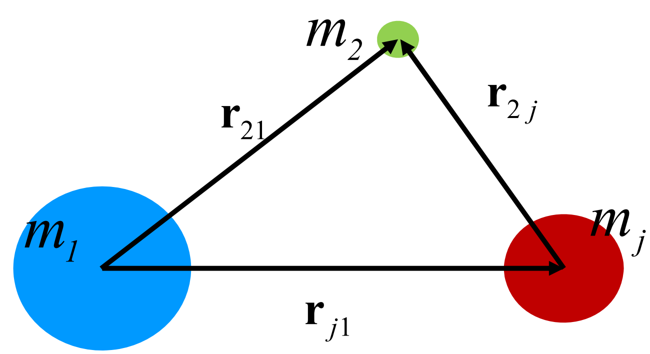
$$= -G\frac{m_1+m_2}{r_{21}^3}\mathbf{r}_{21} - \sum_{j=3}^{n} Gm_j\left(\frac{\mathbf{r}_{2j}}{r_{2j}^3} - \frac{\mathbf{r}_{1j}}{r_{1j}^3}\right)$$

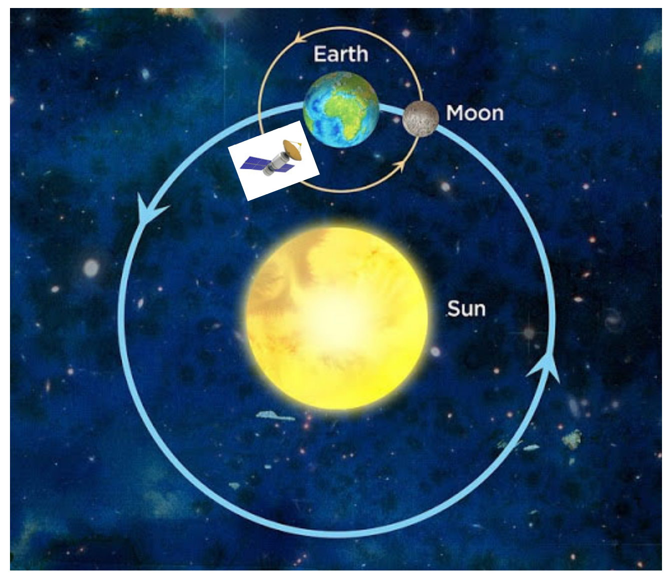
2BP

Direct effect

Indirect effect

$$\boxed{\mathbf{r}_{12} = -\mathbf{r}_{21} = \mathbf{r}}$$





Introduction to Orbital Perturbations

The Keplerian orbit (the motion of one body relative to another under their mutual gravitation) or the 2BP is actually a good approximation in some cases. However, it reflects an ideal situation, due to presence of other celestial bodies and effects in space environment.
As another alternative, one could consider the other orbit influencing effects as perturbations:

$$\ddot{\vec{r}}_i = \vec{\gamma}_K + \vec{\gamma}_P$$

$$\vec{\gamma}_P = \vec{a}_{GR} + \vec{a}_{3RD} + \vec{a}_{SRP} + \vec{a}_D + \vec{a}_{SF}$$

$$\vec{a}_{GR} \equiv Higher\ Earth\ gravity\ harmonics$$

$$\vec{a}_{3RD} \equiv Third\ \ body\ acceleration$$

$$\vec{a}_{SRP} \equiv Acceleration\ due\ to\ Solar\ radiation\ pressure$$

$$\vec{a}_D \equiv Atmospheric\ Drag\ effect$$

$$\vec{a}_{SF} \equiv Small\ forces\ effect$$

Please note that the in absence of $\vec{\gamma}_P$ *, one would have:* $\dot{\vec{\alpha}} = 0$ *with exception of* $\dot{M}=n$

while consiering the Keplerian and Perturbation acceleration: $\dot{\vec{\alpha}} \neq 0$

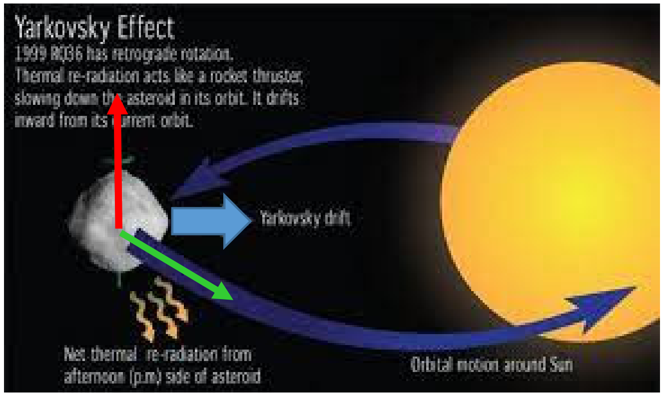
Introduction to Orbital Perturbations

Among the latter perturbative accelerations those due to special forces are really small ranging from 10^{-10} *to* $10^{-17} km.s^{-2}$ for *700 to 18600 Km* orbits. In turn they can be due to the following effects to be considered when high precision orbit propagation is required.

$$\vec{a}_{SF} = \vec{a}_{Tidal} + \vec{a}_{Reletivistic} + \vec{a}_{Infrared} + \vec{a}_{Albedo} + \vec{a}_{Yarkovsky}$$

The Yarkovsky effect is a force acting on a rotating body in space , due to the Sun and Earth Radiation, causing anisotropic emission of thermal photons that carry momentum. It is usually considered in relation to meteoroids or asteroids as its influence is most significant for these bodies.

The force (3) is perpendicular to the plane containing the asteroid rotation vector (1) and its flight direction (2) in orbit.



Introduction to Orbital Perturbations

There are in general two schemes to consider the perturbation effects:

1- Special Perturbation (SP), using numerical methods to integrate $\ddot{\vec{r}}_i = \vec{\gamma}_K + \vec{\gamma}_P$.

2- General Perturbation (GP), using analytical techniques for solving $\dot{\vec{\alpha}} = \vec{f}(\vec{\alpha}, t)$.

SP: *Encke's and Cowell's* formulations are considered among the SP methods.

Encke's formulations : Solved iteratively for $\delta\vec{r}$ (Kaplan), knowing $\vec{\gamma}_P$

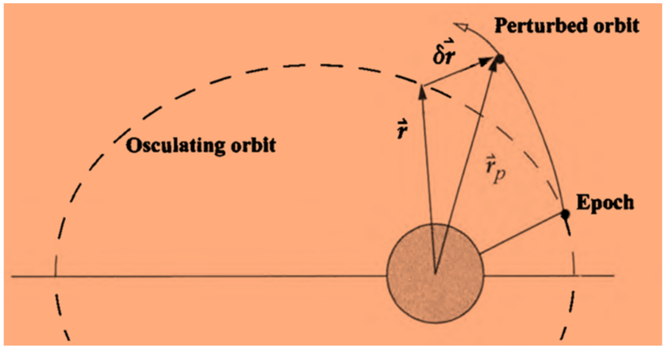
$$\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r};\quad \ddot{\vec{r}}_p = -\frac{\mu}{r_P^3}\vec{r}_P + \vec{\gamma}_P \Rightarrow \delta\ddot{\vec{r}} = \vec{\gamma}_P + \frac{\mu}{r^3}\left\{(1 - \frac{r^3}{r_P^3})\vec{r}_P - \delta\vec{r}\right\} \Rightarrow \vec{r}_p = \vec{r} + \delta\vec{r};\ \vec{v}_p = \vec{v} + \delta\vec{v}$$

Cowells's formulation :

$$\ddot{\vec{r}}_i = \vec{\gamma}_K + \vec{\gamma}_P$$

$$Define : \vec{X}^T = [\vec{r}, \vec{v}]$$

$$\Rightarrow \dot{\vec{X}} = \begin{bmatrix} \vec{v} \\ -\frac{\mu}{r^3}\vec{r} + \vec{\gamma}_P \end{bmatrix}$$



Encke's technique relies on the difference between the osculating (two-body) orbit and the actual perturbed motion. The true orbit is updated in short time spans by integrating this difference.

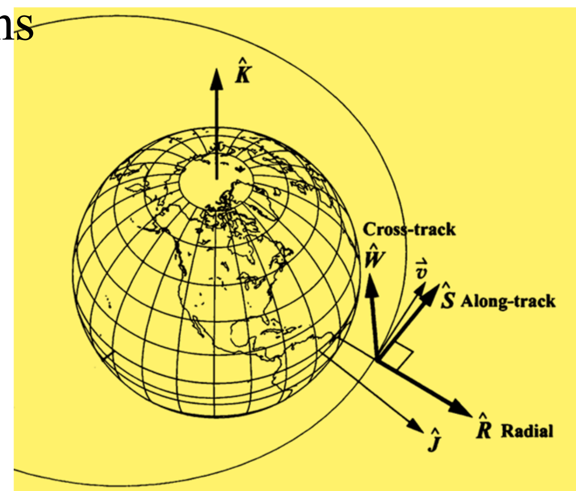
Introduction to Orbital Perturbations

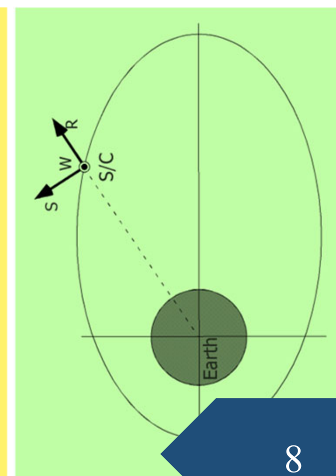
GP: Depending on the nature of the perturbative accelerations (PA) , there are two sets of equation that are utilized for the propagation of the orbital elements:

Gauss Planetary Equations (GPE) , when the PA emanate out of non-conservative or arbitrary perturbing forces.

Lagrange Planetary Equation (LPE), when the PA emanate out of conservative forces that are derivable from a potential function, $\boldsymbol{U}$. Gravitational, elastic spring, electrostatic forces between two electric charges and magnetic force between two magnetic poles are among conservative forces.

For *GPE*, the perturbative accelerations (due to each source) are usually derived in the RSW orbit reference coordinate system. Some derivations can be found in Chobotov, Kaplan, Vallado,Escobal…. Sample derivation is given in the following slides, followed by the GPE for all orbital elements.





RSW Coordination for Peturbative (specific) Accelerations:

$$\vec{\gamma}_P = R\vec{R} + S\vec{S} + W\vec{W}$$

Where : R,S,W are magnitudes of $\vec{\gamma}_P$ in each directions.

Introduction to Orbital Perturbations (*GPE*)

Rate of change of the orbit semi major axis (SMA) due to perturbative force

Noting $r=r(\theta)$, the position and velocity vectors in RSW can be writen as:

$$\vec{r} = r\vec{R};\ \vec{v} = \dot{r}\vec{R} + r\dot{\theta}\vec{S} \Rightarrow \vec{v} = \dot{\theta}\left(\frac{dr}{d\theta}\vec{R} + r\vec{S}\right)$$

In 2BP, the energy is constant, but the **specific** perturbing force cause time rate-of-change of energy from the work done by the perturbing force and the distance traveled. In other words:

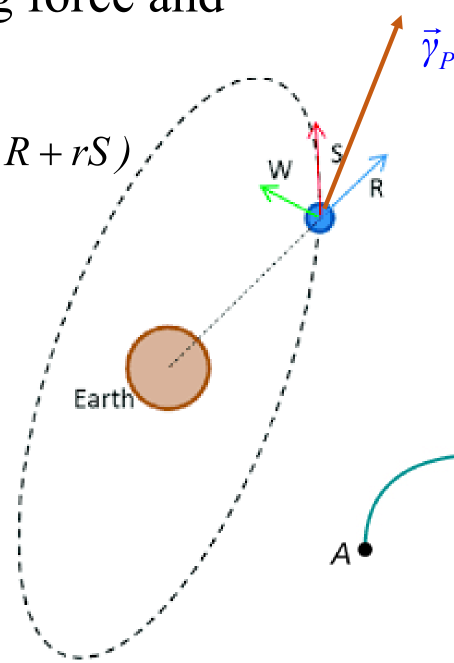
$$Let: \vec{\gamma}_P = R\vec{R} + S\vec{S} + W\vec{W} \Rightarrow d(work) = d\varepsilon = \vec{\gamma}_P.d\vec{r} = \vec{\gamma}_P.\vec{v}dt \Rightarrow \frac{d\varepsilon}{dt} = \vec{\gamma}_P.\vec{v} = \dot{\theta}\left(\frac{dr}{d\theta}R + rS\right)$$

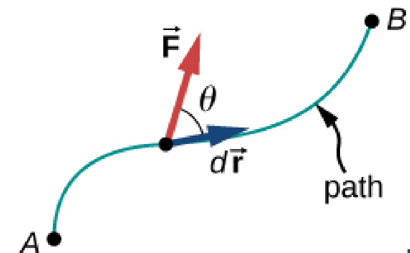
$$On\ the\ other\ hand,\ \varepsilon = -\frac{\mu}{2a}\ \ or\ a = -\frac{\mu}{2\varepsilon} \Rightarrow \frac{da}{dt} = \frac{da}{d\varepsilon}\frac{d\varepsilon}{dt} = \frac{\mu}{2\varepsilon^2}\frac{d\varepsilon}{dt}$$

$$Now, one\ can\ easily\ show\ that\ \ \frac{dr}{d\theta} = \frac{re\sin\theta}{1+e\cos\theta}\ \ and\ \dot{\theta} = \frac{na^2}{r^2}(1-e^2)^{1/2}$$

$$Thus: \frac{da}{dt} = \frac{\mu}{2\varepsilon^2}\frac{na^2}{r^2}(1-e^2)^{1/2}\left\{\frac{re\sin\theta}{1+e\cos\theta}R + rS\right\}\ \ or;$$

$$\frac{da}{dt} = \frac{2}{n(1-e^2)^{1/2}}\left\{e\sin\theta\ R + (1+e\cos\theta)S\right\}$$





Introduction to Orbital Perturbations (*GPE*)

Rate of change of the orbit semi major axis (SMA) due to perturbative force

One can show that :

$$1:\ \frac{dr}{d\theta}=\frac{re\sin\theta}{1+e\cos\theta}\ ;\ r=\frac{P}{1+e\cos\theta}\Rightarrow\frac{dr}{d\theta}=\frac{Pe\sin\theta}{(1+e\cos\theta)^2}=\frac{re\sin\theta}{1+e\cos\theta}$$

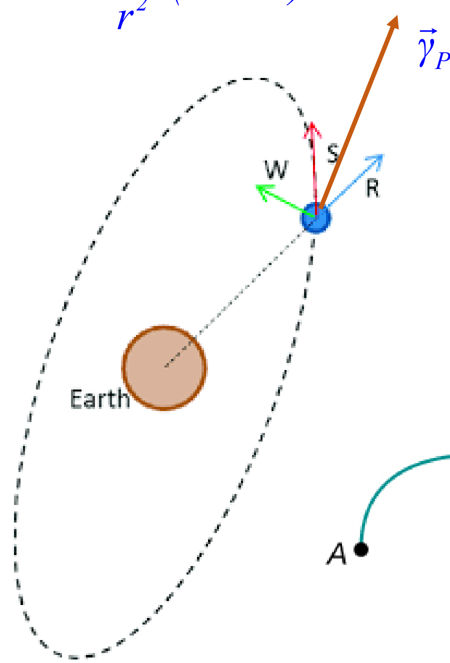
$$2:\ \dot{\theta}=\frac{na^2}{r^2}(1-e^2)^{1/2};\dot{\theta}=\frac{h}{r^2}=\frac{(\mu P)^{1/2}}{r^2}=\frac{(\mu a(1-e^2))^{1/2}}{r^2}=\frac{(na^{3/2})a^{1/2}(1-e^2)^{1/2}}{r^2}=\frac{na^2}{r^2}(1-e^2)^{1/2}$$

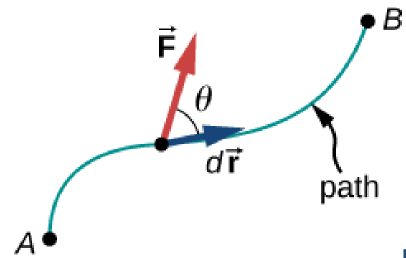
$$\text{Since: } n=\frac{2\pi}{\tau}=\sqrt{\frac{\mu}{a^3}}\ \text{ or } \sqrt{\mu}=na^{3/2}$$

$$3:\ \frac{da}{dt}=\frac{\mu}{2\varepsilon^2}\frac{na^2}{r^2}(1-e^2)^{1/2}\left\{\frac{re\sin\theta}{1+e\cos\theta}R+rS\right\}$$

$$\Downarrow$$

$$\frac{da}{dt}=\frac{2}{n(1-e^2)^{1/2}}\{e\sin\theta\ R+(1+e\cos\theta)S\}$$





Introduction to Orbital Perturbations (*GPE*)

Example: Drag Effect on SMA

For LEO satellites, the atmosphere is dense enough to create a drag (and lift) force. However, for circular orbits the lift component will have no effect on the energy change!
So, the drag related specific force can be considered to compute the rate of change of the SMA as follows:

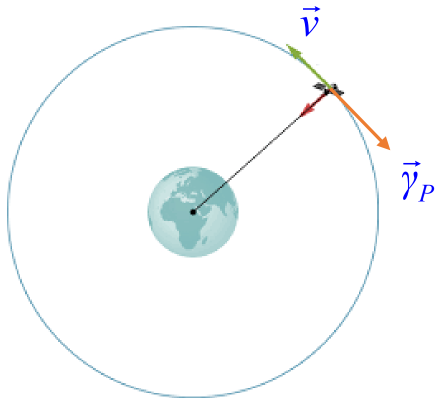
$$\frac{d\varepsilon}{dt} = \vec{\gamma}_P . \vec{v}; \quad \frac{da}{dt} = \frac{2}{n(1-e^2)^{1/2}}\{e\sin\theta\, R + (1+e\cos\theta)S\}$$

$$\vec{\gamma}_P = \frac{\vec{D}}{m_S} = -\frac{D}{m_S}\vec{S}; \quad D=\frac{1}{2}\rho v^2 sC_D; \quad \Rightarrow S=-\frac{1}{2m_S}\rho v^2 sC_D \text{ and } e=0; \; v^2 = \frac{\mu}{a}; \; n=\sqrt{\frac{\mu}{a^3}}$$

$$\frac{da}{dt} = \frac{2}{n(1-\cancel{e^2})^{1/2}}\{\cancel{e\sin\theta}\, R + (1+\cancel{e\cos\theta})S\} = -\rho\frac{sC_D}{nm_S}\frac{\mu}{a} = -\rho\frac{sC_D}{\sqrt{\frac{\mu}{a^3}}m_S}\frac{\mu}{a}$$

$$\Rightarrow \frac{da}{dt} = -\rho(\mu a)^{1/2}\frac{sC_D}{m_S}$$

Where : $\frac{sC_D}{2m_S} = \text{Ballistic Coefficient}$



Introduction to Orbital Perturbations (GPE) [Sidi]

Following similar approaches, one can develop a series of DE for Gauss Planetary Equations.

$$\frac{da}{dt} = \frac{2}{n(1-e^2)^{1/2}}\left\{e\sin\theta\ R + (1+e\cos\theta)S\right\}$$

$$\frac{de}{dt} = \frac{(1-e^2)^{1/2}}{na}\left\{\sin\theta\ R + (\cos\theta + \cos E)S\right\}$$

$$\frac{di}{dt} = \frac{r}{na^2(1-e^2)^{1/2}}\left\{\cos(\theta+\omega)W\right\}$$

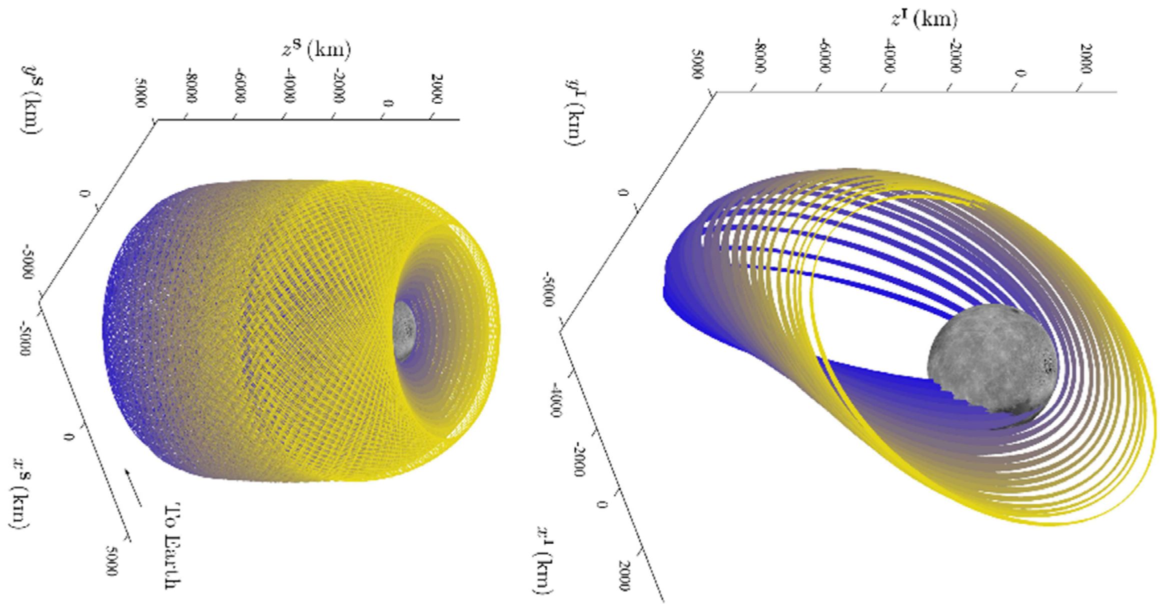
$$\frac{d\Omega}{dt} = \frac{r}{na^2(1-e^2)^{1/2}}\left\{\frac{\sin(\theta+\omega)}{\sin i}W\right\}$$

$$\frac{d\omega}{dt} = \frac{(1-e^2)^{1/2}}{nae}\left\{-\cos\theta\ R + (1+\frac{1}{1+e\cos\theta})\sin\theta)S - \dot{\Omega}\cos i\ W\right\}$$

$$\frac{dM}{dt} = n + \frac{1-e^2}{nae}\left\{(\frac{-2e}{1+e\cos\theta} + \cos\theta)\ R - (1+\frac{1}{1+e\cos\theta})\sin\theta)S\right\}$$

Note : GPE can be propagated for OE for each specific perturbative force $\vec{\gamma}_P$
given an initial condition $\vec{\alpha}(t_0)$.

Application of GPE to Lunar south pole coverage due to multiple bodies and gravity harmonics



Family of Solutions in the Inertial (Right) and the Rotating (Left) Frames

Introduction to Orbital Perturbations (*LPE*)

Lagrange Planetary equations are developed for cases in which the orbit is acted upon by non-conservative specific perturbative forces. In these cases, the perturbing acceleration are drivable from a potential function [Escobal]. In other words, $\vec{\gamma}_P = -\vec{\nabla} U(\vec{r})$.

$$\frac{da}{dt} = \frac{2}{na}\frac{\partial U}{\partial M}$$

$$\frac{de}{dt} = \frac{1-e^2}{na^2 e}\frac{\partial U}{\partial M} - \frac{(1-e^2)^{1/2}}{na^2 e}\frac{\partial U}{\partial \omega}$$

$$\frac{di}{dt} = \frac{-1}{na^2(1-e^2)^{1/2}\sin i}\left\{\frac{\partial U}{\partial \Omega} + \cos i \frac{\partial U}{\partial \omega}\right\}$$

$$\frac{d\Omega}{dt} = \frac{1}{na^2(1-e^2)^{1/2}\sin i}\frac{\partial U}{\partial i}$$

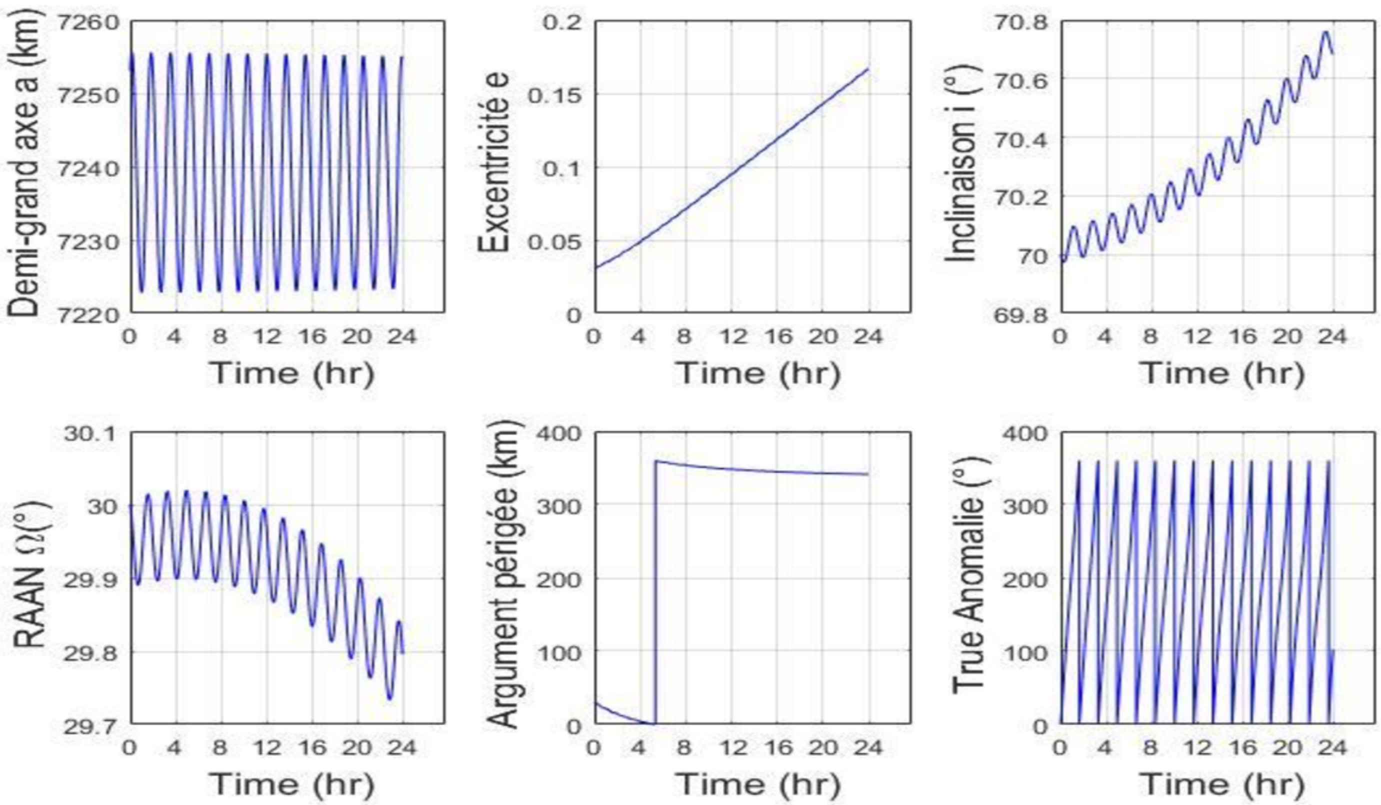
$$\frac{d\omega}{dt} = \frac{(1-e^2)^{1/2}}{na^2 e}\frac{\partial U}{\partial e} - \frac{\cos i}{na^2(1-e^2)^{1/2}\sin i}\frac{\partial U}{\partial i}$$

$$\frac{dM}{dt} = n - \frac{2}{na}\frac{\partial U}{\partial a} - \frac{1-e^2}{na^2 e}\frac{\partial U}{\partial e}$$

LPE is propagated for OE upon conservative $\vec{\gamma}_P$ *starting from an initial condition* $\vec{\alpha}(t_0)$.

Introduction to Orbital Perturbations (*LPE*)

Sample Perturbation effects in orbital elements of CubeSats

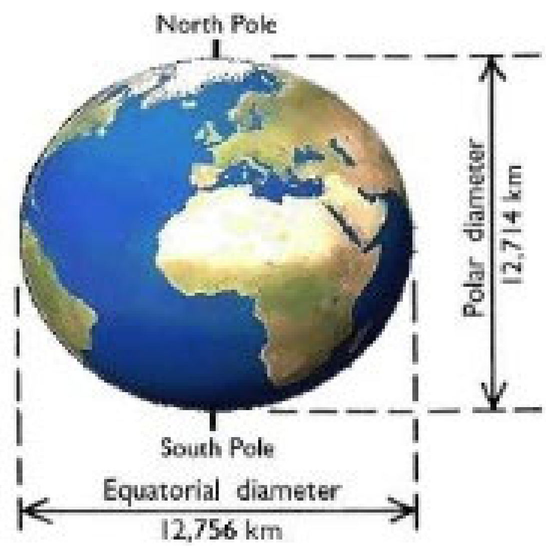


Earth Gravity Potential and the Oblateness effect

Earth's topography recognized by its variety of land forms and water areas, is usually of concern to topo-hydrographers and geophysicists. Exact mathematical modeling of the Earth surface with irregularities is extremely complicated. Since the Earth is flattened at the poles and bulges at the Equator, a common approximation to its shape is oblate spheroid or oblate ellipsoid using spherical harmonics to approximate what is called the geoid. Flattening or oblateness caused by perturbations from the Sun and the moon is a measure of the compression of a sphere along a diameter to form an ellipsoid of revolution. The combination of the Earth oblateness, nonhomogeneous mass distribution and geometry in fact has caused the Earth gravity potential to be a function of altitude, latitude and longitude as well. The aforementioned effects causes a complex model for the Earth in terms of zonal, tesseral and sectoral harmonics. [Sidi]

$$\ddot{\vec{r}}_i = \vec{\gamma}_K + \vec{\gamma}_P$$

$$\vec{\gamma}_P = \vec{a}_{GR} + \vec{a}_{3RD} + \vec{a}_{SRP} + \vec{a}_D + \vec{a}_{SF}$$



$$U(\mathbf{r}) = -\frac{\mu}{r} + \frac{\mu}{r}\left\{\sum_{n=2}^{N_z}\left[(\frac{R_E}{r})^n J_n P_n(\sin\varphi') + \sum_{m=1}^{n}(\frac{R_E}{r})^n P_{nm}(\sin\varphi')(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda)\right]\right\}$$

The zonal terms : $\quad J_n P_n(\sin\varphi') \quad n = 2,\ldots$

The tesseral terms : $\quad P_{nm}(\sin\varphi')C_{nm}\cos m\lambda\,, \quad 1 \le m \le n$

The sectoral terms : $\quad P_{nm}(\sin\varphi')S_{nm}\sin m\lambda\,, \quad 1 \le m \le n$

Earth Gravity Potential and the Oblateness effect

$$U(r) = -\frac{\mu}{r} + \frac{\mu}{r}\left\{\sum_{n=2}^{N_z}\left[\left(\frac{R_E}{r}\right)^n J_n P_n(\sin\varphi') + \sum_{m=1}^{n}\left(\frac{R_E}{r}\right)^n P_{nm}(\sin\varphi')(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda)\right]\right\}$$

r : geocentric distance of point P; R_E : Mean Equatorial radius;

φ', λ : geocentric latitude and geographical longitude;

R_E - mean equatorial radius of the earth;

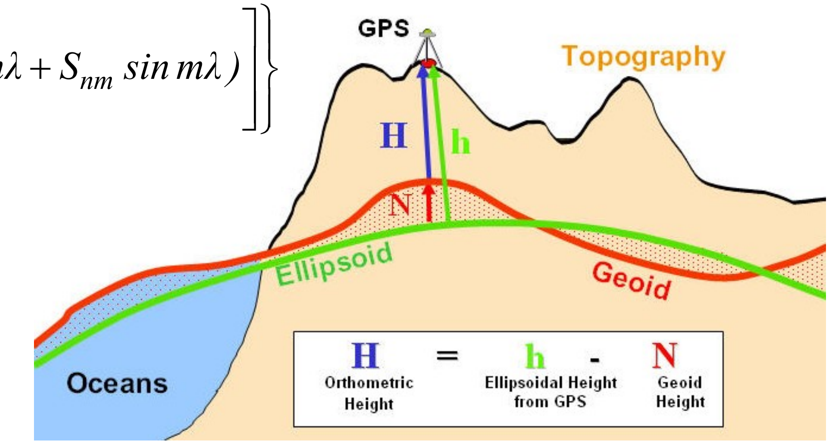
J_n : Zonal harmonics;

$P_n(\sin\varphi')$: Legendre polynomial of degree n and order 0;

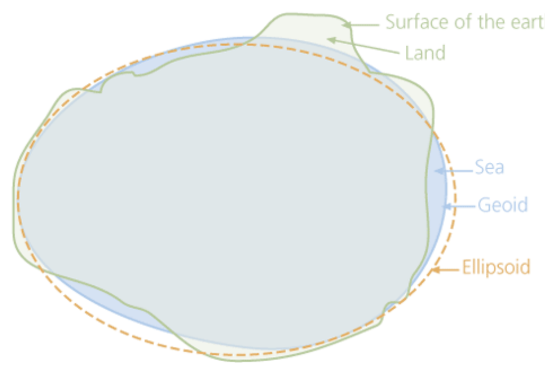
$P_{nm}(\sin\varphi')$: Legendre polynomial of degree n and order m;

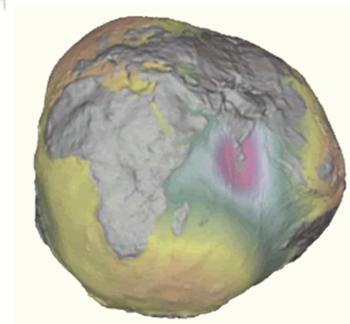
C_{nm} , S_{nm} - Tesseral and Sectoral harmonic coefficients for n & m;

$\cos m\lambda, \sin m\lambda$: harmonics in λ.



Model of the Earth





Zonal Harmonics account for most of the Earth's gravitational departure from a perfect sphere.
Sectoral harmonics take into account the extra mass distribution in the Earth longitudinal regions.
Tesseral harmonics attempt to model specific regions on the Earth which depart from a perfect sphere.

The **Geoid** is a model that is everywhere normal to the direction of gravity and coincides with the global mean sea level used to measure precise surface elevations.

Earth Oblateness Effect

It turns out that the most dominating term (after the term $-\mu/r$) in the potential function is related to the second zonal harmonic , "$J2$ term", that changes the gravitation pattern by extra terms and is related to the Earth oblateness effect . The J2 effect in turn produces average variations in some of the OE as a function of time.

In the spherical coordinate system :

$$F_\lambda = -\frac{1}{r}\frac{\partial U}{\partial \lambda} = -J_2 \frac{1}{r^4} 3\cos\lambda \sin\lambda$$

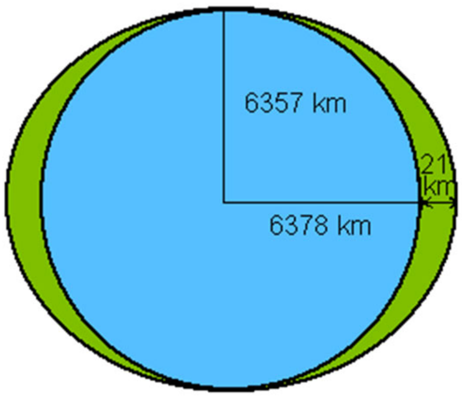
$$F_r = -\frac{\partial U}{\partial r} = J_2 \frac{1}{r^4}\frac{3}{2}\left(3\sin^2\lambda - 1\right)$$

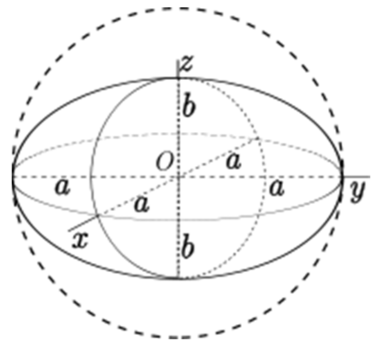
In the Cartesian ECI coordinate system :

$$F_x = -\frac{\partial U}{\partial x} = J_2 \frac{x}{r^7}\left(6z^2 - \frac{3}{2}(x^2+y^2)\right)$$

$$F_y = -\frac{\partial U}{\partial y} = J_2 \frac{y}{r^7}\left(6z^2 - \frac{3}{2}(x^2+y^2)\right)$$

$$F_z = -\frac{\partial U}{\partial z} = J_2 \frac{z}{r^7}\left(3z^2 - \frac{9}{2}(x^2+y^2)\right)$$





flattening= (a − b)/a

$$Oblateness = \frac{Equatorial\ \ radius\text{-}Polar\ \ radius}{Equatorail\ \ radius} = 0.003353$$

J_2=1082.63e-06; J_3=-2.53e-06; J_3=-1.61e-06;
C_{21} = S_{21} = 0, C_{22}= 1.57e-06;
S_{22}=-0.9 e-06, C_{31}= 2.19e-06, S_{31}= 0.27e-06;

Orbit Propagation under the J_2 effect via SP Scheme

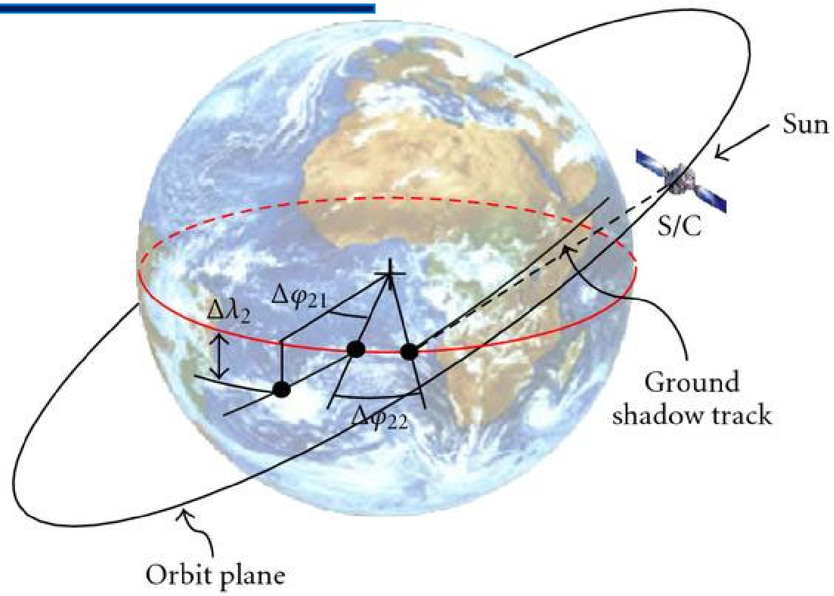
$$\ddot{\vec{r}}_i = \vec{\gamma}_K + \vec{\gamma}_{J_2}$$

$$\ddot{\vec{r}} = -\frac{\mu}{r^3}\begin{pmatrix} \left\{1+1.5J_2\left(R_E/r\right)^2\left[1-5\left(r_z/r\right)^2\right]\right\}r_x \\ \left\{1+1.5J_2\left(R_E/r\right)^2\left[1-5\left(r_z/r\right)^2\right]\right\}r_y \\ \left\{1+1.5J_2\left(R_E/r\right)^2\left[3-5\left(r_z/r\right)^2\right]\right\}r_z \end{pmatrix}$$

$$Define: \vec{X}^T = [\vec{r},\vec{v}] \Rightarrow \dot{\vec{X}} = \begin{bmatrix} \vec{v} \\ -\frac{\mu}{r^3}\vec{r} + \vec{\gamma}_{J_2} \end{bmatrix}$$

$$\Downarrow$$

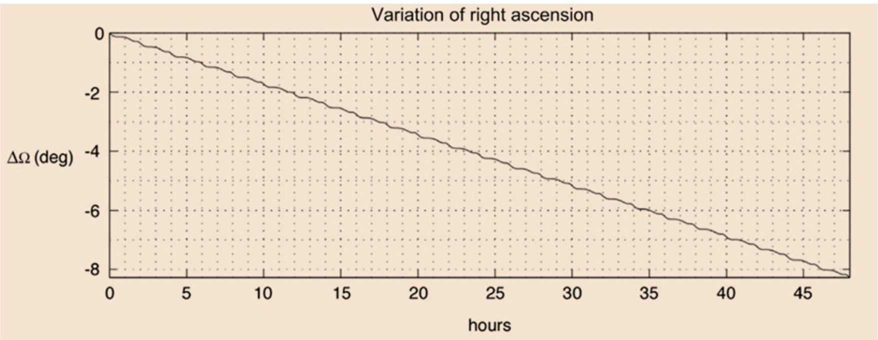
$$\vec{r},\vec{v} \rightarrow \vec{\alpha}^T(t) = [a(t), e(t), i(t), \Omega(t), \omega(t), \theta(t)]$$

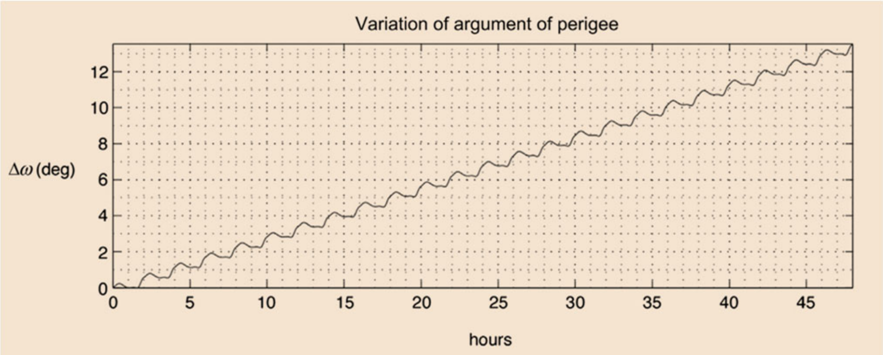


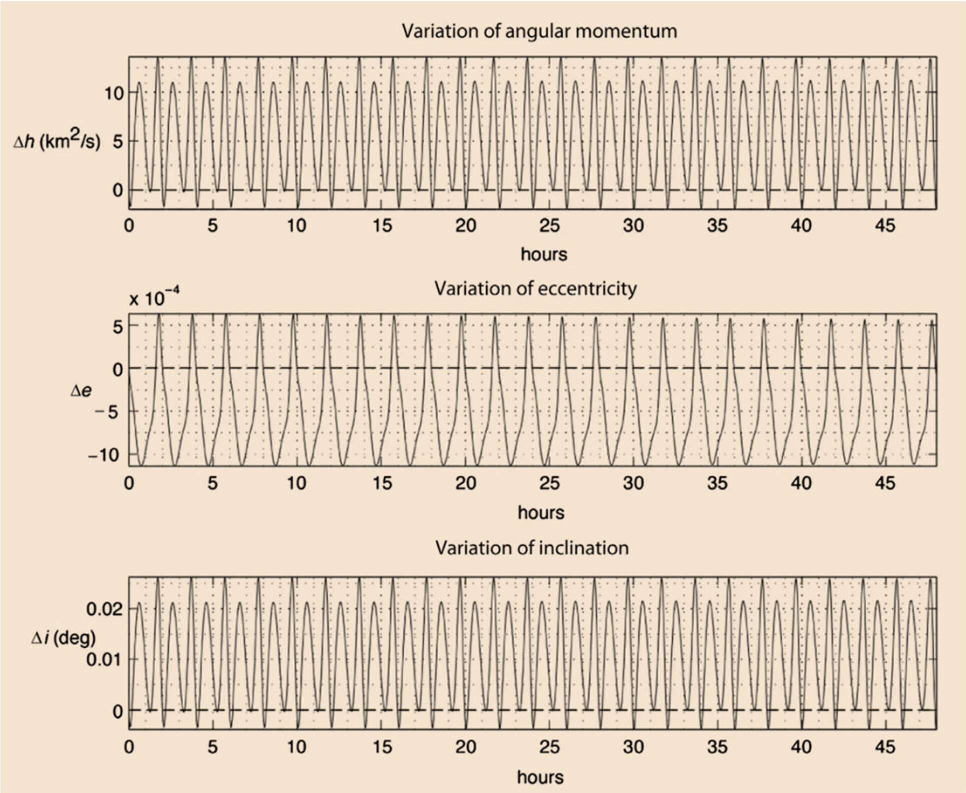
Oblateness and Second Zonal Harmonics		
Planet	**Oblateness**	**J_2**
Mercury	0.000	60×10^{-6}
Venus	0.000	4.458×10^{-6}
Earth	0.003353	1.08263×10^{-3}
Mars	0.00648	1.96045×10^{-3}
Jupiter	0.06487	14.736×10^{-3}
Saturn	0.09796	16.298×10^{-3}
Uranus	0.02293	3.34343×10^{-3}
Neptune	0.01708	3.411×10^{-3}
(Moon)	0.0012	202.7×10^{-6}

Orbit Propagation under the J_2 effect via SP Scheme

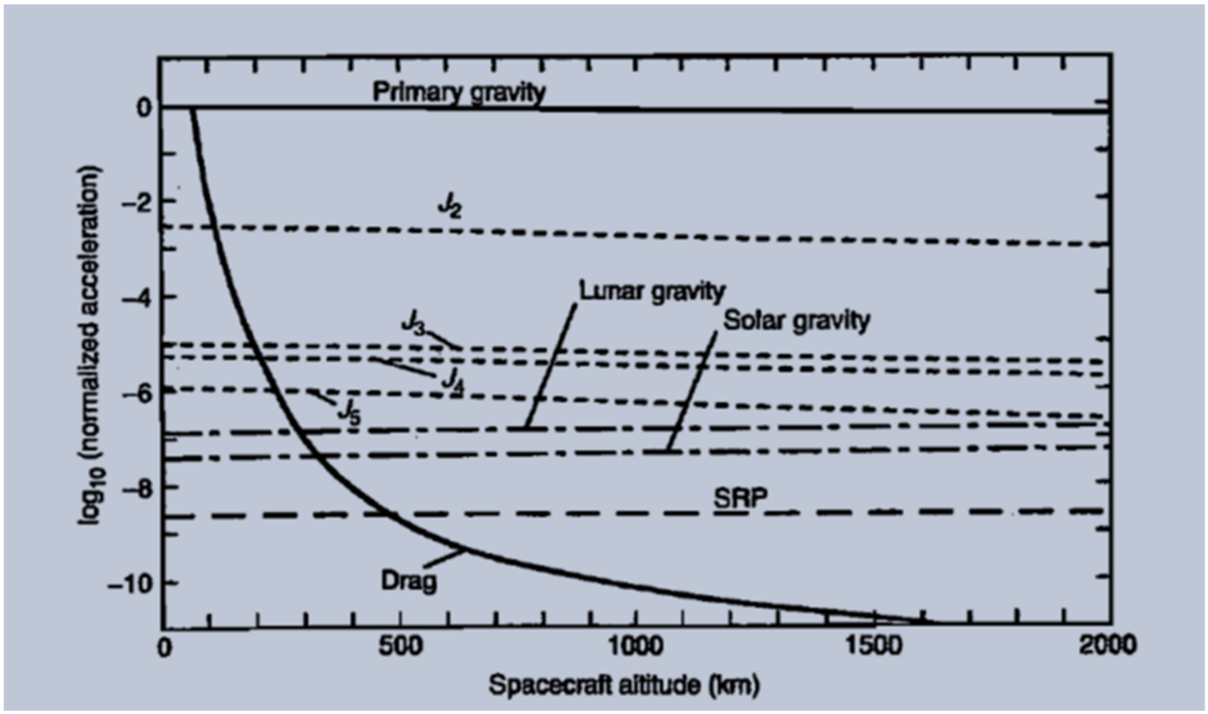
Sample simulation:

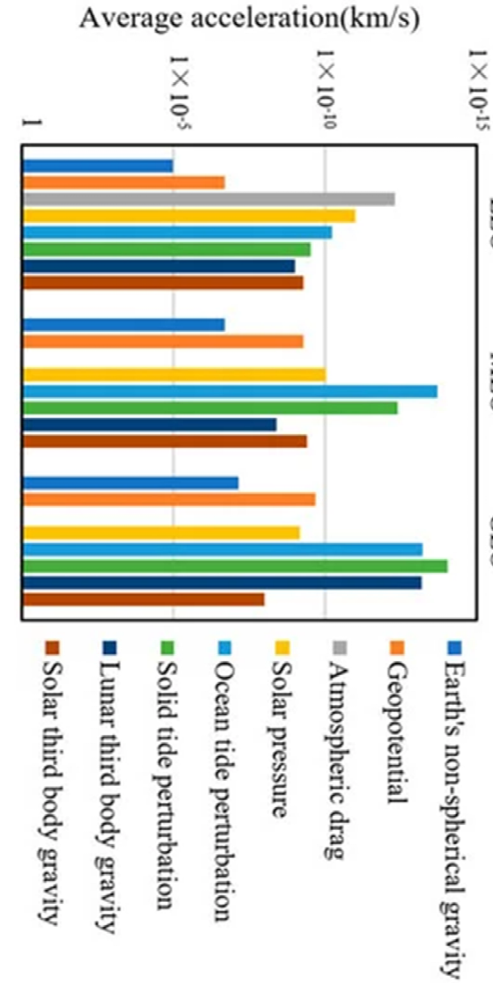






Relative Effects of Various Perturbations

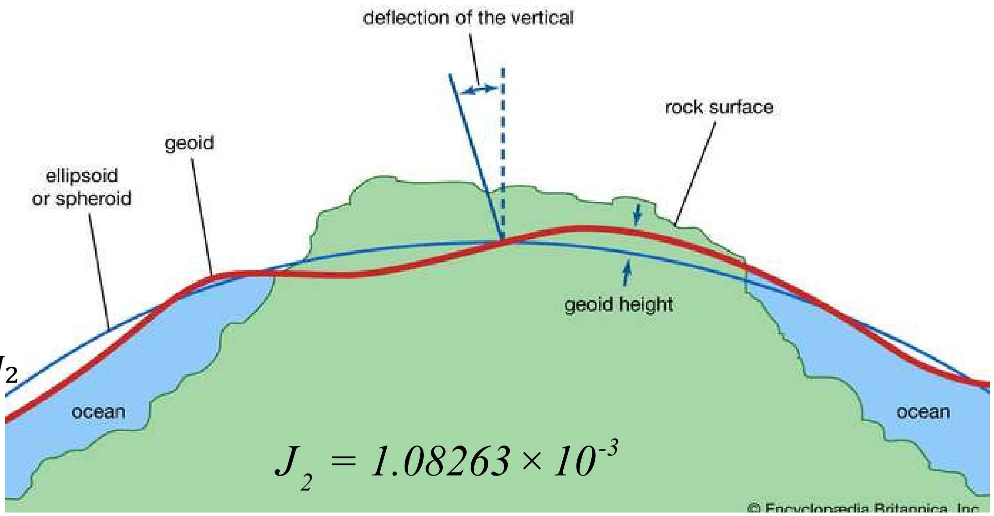




The magnitude of the perturbation forces on satellites in different orbits.

Earth Oblateness and the J2 Effects on OE

As mentioned before, the Earth complex potential can be simplified. Assuming axi-symmetric mass (Pear shape) distribution along the polar axis removes the Sectoral and Tesseral harmonics and only leaves the Zonal terms. In addition, considering the J2 effect as key orbit perturbation causes average periodic time variations in a, e and i to be zero. Thus, application of the LPE to simplified potential U_{J_2} yields the following changes to remaining OE, with some positive consequences .



$$J_2 = 1.08263 \times 10^{-3}$$

1- Sun Synchronous Orbits
2- Polar Orbits
3- Stationary Apse line Orbits

$$\dot{\Omega} = -\left[\frac{3}{2}\frac{\sqrt{\mu}J_2R_E^2}{(1-e^2)^2a^{7/2}}\right]\cos i \Rightarrow \begin{cases} 0 \le i < 90° : \dot{\Omega} < 0 \\ 90° < i \le 180° : \dot{\Omega} > 0 \\ i = 90° : \dot{\Omega} = 0 \end{cases}$$

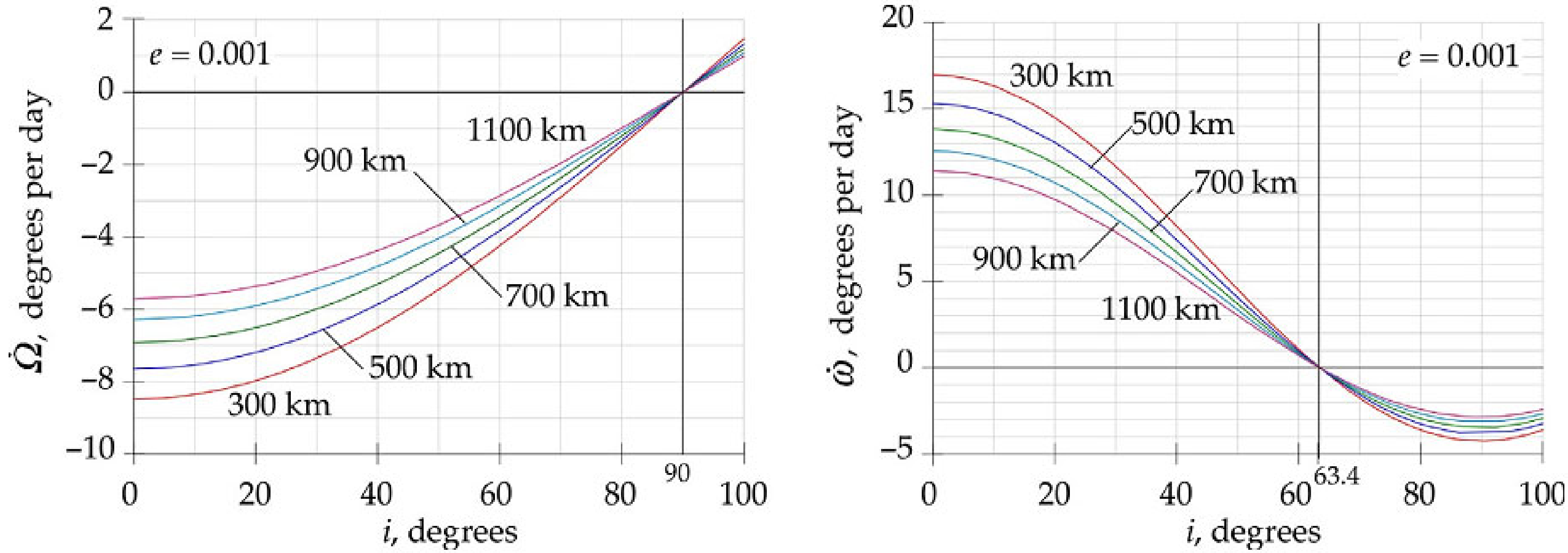
$$\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r} + \vec{\gamma}_{J_2}$$

$$\dot{\omega} = -\left[\frac{3}{2}\frac{\sqrt{\mu}J_2R_E^2}{(1-e^2)^2a^{7/2}}\right]\left(\frac{5}{2}\sin^2 i - 2\right) \Rightarrow \begin{cases} 0 \le i < 63.4° \quad or \quad 116.6 < i \le 180° : \dot{\omega} > 0 \\ 63.4° < i < 116.6° : \dot{\omega} < 0 \\ i = 63.4° \quad or \quad i = 116.6 : \dot{\omega} = 0 \end{cases}$$

$$\dot{M} = n + \left[\frac{3nJ_2}{4}\frac{[3\cos^2 i - 1]}{(1-e^2)^{3/2}}\right]\left[\frac{R_E^2}{a^2}\right]$$

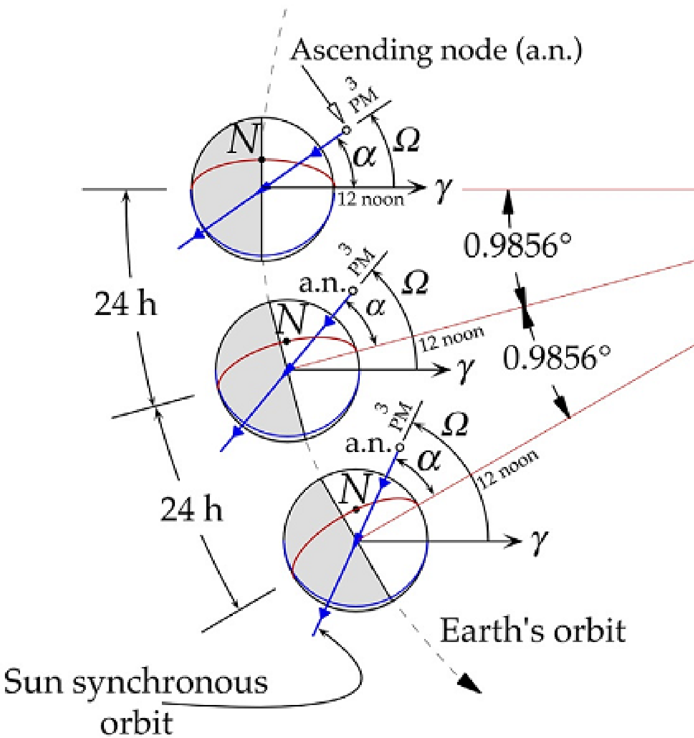
Effects of the Earth's Oblateness

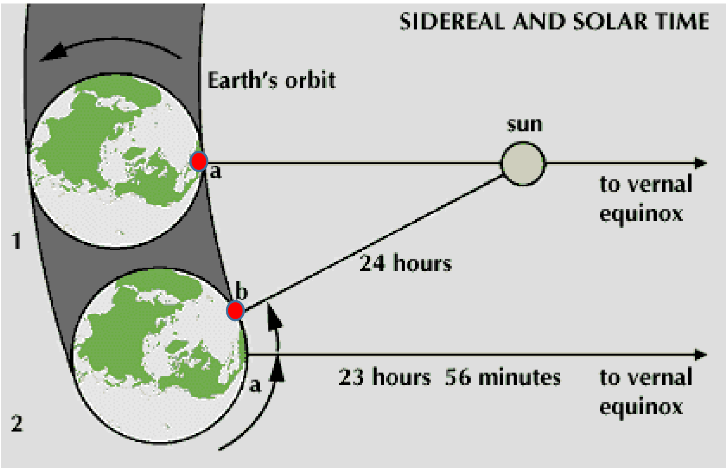
The second zonal harmonics ($J_2 = 1.08263e-3$) effects



Utility of Oblateness Effect for SSO

Sun-Synchronous Orbit: A Sun-synchronous orbit (SSO), is a geocentric orbit whose rate of change of its right ascension is equal to the Earth rotation rate around the Sun. This causes a constant radial angle between the Sun and the SSO orbital plane that produces a constant perspective of the orbital plane with the Sun in such a way that the satellite passes over any given point of the planet's surface at the same local solar time.





1 sidereal day = 23 hours, 56 minutes, 4.1 seconds

$$\dot{\Omega}_{ss} = \frac{2\pi \ rad}{365.26 \times 24 \times 3600 \ s}$$

$$= 1.991 \times 10^{-7} \quad rad / s = 0.9856^{\circ} / day$$

Example 1 - Oblateness Effect

A satellite is to be launched into a sun-synchronous circular orbit with period of 100 minutes. Determine the required altitude and inclination of its orbit.

$$\dot{\Omega}_{ss} = \frac{2\pi \ rad}{365.26 \times 24 \times 3600 \ s} = 1.991 \times 10^{-7} \quad rad/s$$

$$\tau = \frac{2\pi}{\sqrt{\mu}}\left(R_E + z\right)^{3/2} \Rightarrow 100 \times 60 = \frac{2\pi}{\sqrt{398,600}}(6378 + z)^{3/2} \Rightarrow z = 758.63\text{km} \rightarrow \text{altitude}$$

$$\dot{\Omega} = -\left[\frac{3}{2}\frac{\sqrt{\mu} J_2 R_E^2}{(1-e^2)^2 a^{7/2}}\right]\cos i$$

$$1.991 \times 10^{-7} = -\left[\frac{3}{2}\frac{\sqrt{398,600} \times 0.00108263 \times 6378^2}{\left(1-0^2\right)^2 (6378 + 758.63)^{7/2}}\right]\cos i \Rightarrow \cos i = -0.14658$$

$$\boxed{i = \cos^{-1}(-0.14658) = 98.43^{\circ}}$$

Example 2 - Oblateness Effect

The space shuttle is in a 280 km by 400 km orbit with an inclination of 51.43 ° . Find the **rates** of node regression and perigee advance due to J_2 effect.

$$\dot{\Omega} = -\left[\frac{3}{2}\frac{\sqrt{\mu}J_2R^2}{(1-e^2)^2 a^{7/2}}\right]\cos i$$

$$\dot{\omega} = -\left[\frac{3}{2}\frac{\sqrt{\mu}J_2R^2}{(1-e^2)^2 a^{7/2}}\right]\left(\frac{5}{2}\sin^2 i - 2\right)$$

$$r_p = 6378 + 280 = 6658\text{km} \quad r_a = 6378 + 400 = 6778\text{km}$$

$$e = \frac{r_a - r_p}{r_a + r_p} = 0.008931$$

$$a = \frac{1}{2}\left(r_a + r_p\right) = 6718\text{km}$$

$$\dot{\Omega} = -\left[\frac{3}{2}\frac{\sqrt{398,600}\times 0.0010826\times 6378^2}{\left(1-0.0089312^2\right)^2\times 6718^{7/2}}\right]\cos 51.43^\circ = -1.0465\times 10^{-6}\,\text{rad/s}$$

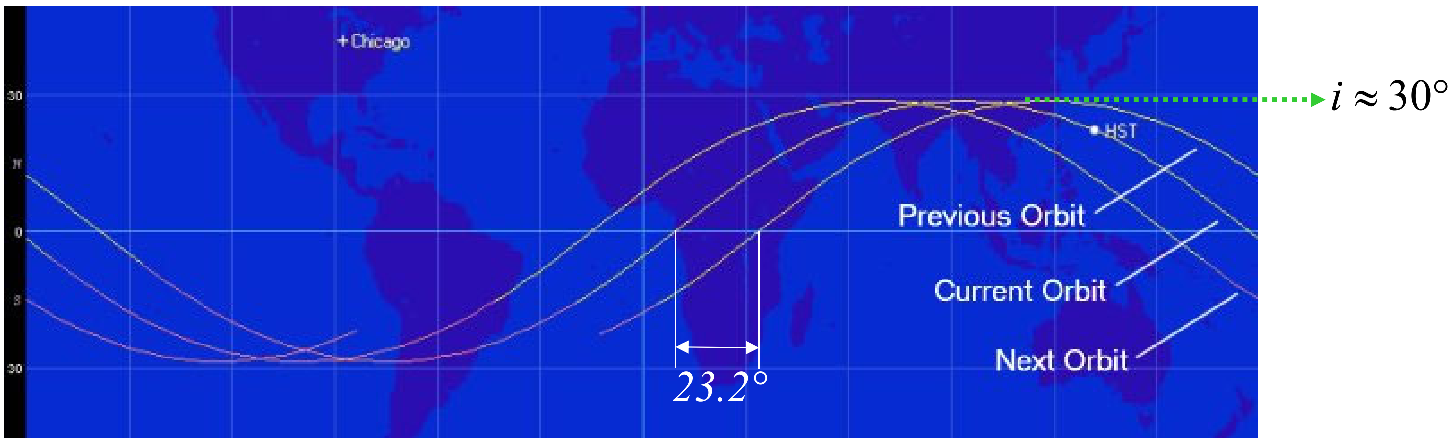
$$\dot{\Omega} = 5.181 \textit{ degrees per day to the west}$$

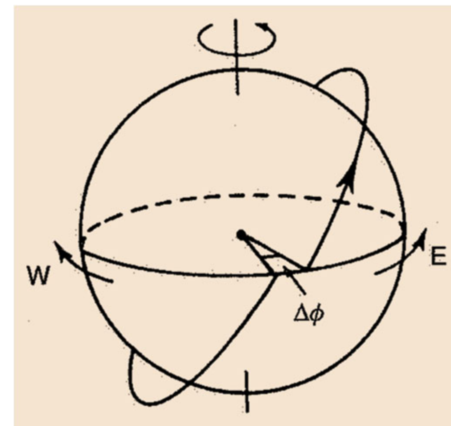
$$\dot{\omega} = -1.0465\times 10^{-6}\times\left(\frac{5}{2}\sin^2 51.43^\circ - 2\right) = +7.9193\times 10^{-7}\,\text{rad/s}$$

$$\dot{\omega} = 3.920 \textit{ degrees per day in the flight direction}$$

Example 3- Ground Track Data Extraction

Exploring orbital elements on ground tracks





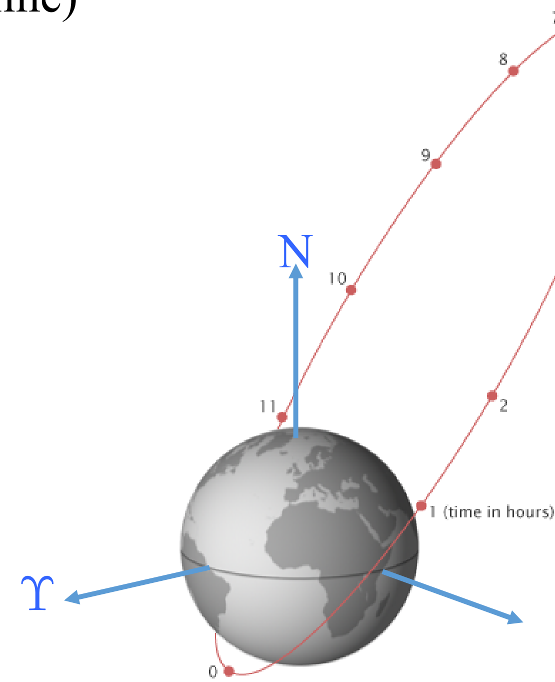
$$\Delta\varphi = \omega_E \tau_{orbit}$$

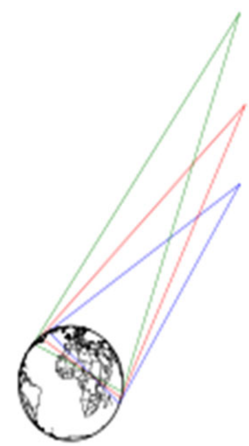
$$\begin{cases} \tau = \dfrac{23.2°}{15.04\ °/hr} = 1.54\ \ hr \quad (LEO) \longrightarrow \quad \tau = 2\pi\sqrt{\dfrac{a^3}{\mu}} \rightarrow a = 6770.3\ km \\ \tau = \dfrac{23.2°}{(15.04 - \dot{\Omega})\ °/hr} \end{cases}$$

Effects of the Earth's Oblateness

Molniya Orbit (Stationary Apse line)

- A highly elliptical orbit
- $i = 63.4\ deg$
- $\omega = -90\ deg$
- T = half of a sidereal day
- $h_a = 40{,}000\ km$





$$\dot{\omega} = -\left[\frac{3}{2}\frac{\sqrt{\mu}J_2R^2}{(1-e^2)^2a^{7/2}}\right]\left(\frac{5}{2}\sin^2 i - 2\right) \Rightarrow \begin{cases} 0 \le i < 63.4° \quad or \quad 116.6 < i \le 180° : \dot{\omega} > 0 \\ 63.4° < i < 116.6° : \dot{\omega} < 0 \\ i = 63.4° \quad or \quad i = 116.6 : \ \dot{\omega} = 0 \end{cases}$$

Example 4

Determine the period of an elliptic Sun Synchronous (SS) Earth orbiting satellite (in TU) whose eccentricity vector does not change with time (fixed apse line) , given its eccentricity of $e=0.3$.

$$SS\ orbit : \dot{\Omega} = \frac{2\pi}{365.26 \times 24 \times 3600} = 1.991e-7 \ \ rad / sec$$

$$\dot{\omega} = -\left[\frac{3}{2}\frac{\sqrt{\mu}J_2R^2}{(1-e^2)^2a^{7/2}}\right]\left(\frac{5}{2}sin^2 i - 2\right) \Rightarrow \begin{cases} 0 \le i \le 63.4^\circ \ \ or \ \ 116.6 < i \le 180^\circ : \dot{\omega} > 0 \\ 63.4^\circ < i \le 116.6^\circ : \dot{\omega} < 0 \\ i = 63.4^\circ \ \ or \ \ i = 116.6 : \ \dot{\omega} = 0 \end{cases}$$

$$i = 63.4 \ \ or \ \ 116.6 \ \ deg$$

$$\dot{\Omega} = -\left[\frac{3}{2}\frac{\sqrt{\mu}J_2R^2}{(1-e^2)^2a^{7/2}}\right]cos i \Rightarrow a = 10362.38 \ \ km$$

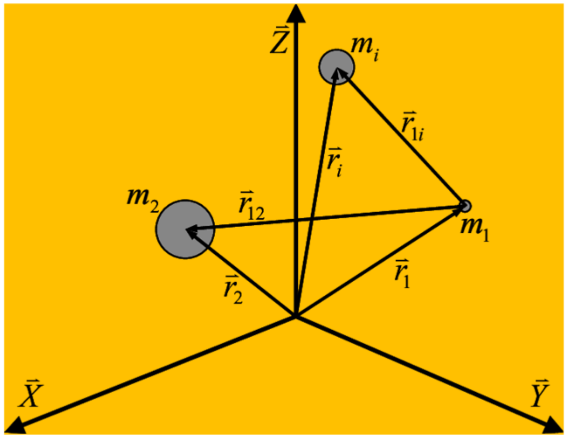
$$T = 2\pi\sqrt{\frac{a^3}{\mu}} = 2.9161 \ hr = 13.01 \ TU$$

Perturbations (Third Body Effect)

$$\ddot{\vec{r}}_i = \vec{\gamma}_K + \vec{\gamma}_P$$

$$\vec{\gamma}_P = \vec{a}_{GR} + \vec{a}_{3RD} + \vec{a}_{SRP} + \vec{a}_D + \vec{a}_{SF}$$

Third or more celestial bodies, like the Sun, Moon ...… are among the conservative perturbative forces on Keplerian orbits. If the orbital period or its application time is too long, these perturbations will change orbital elements. In this situation, one can initially consider the NBP mentioned earlier and finally simplify it for the effect of a third body.
The total sum of all forces acting on a spacecraft as the *i*-th body (due to Earth, Sun or other planets) can be written as :



$$\vec{F}_i = G\sum_{j=1}^{n}\frac{m_i m_j}{r_{ij}^3}\left(\vec{r}_j - \vec{r}_i\right) \quad ; \quad i \neq j \quad ; \quad r_{ij} = \left|\vec{r}_j - \vec{r}_i\right|$$

In the 2BP, only the first term of the above equation was used. Of course, the direction of these forces must be considered to use or apply the equation properly. Based on Newton's second law, considering the mass of the spacecraft to be constant, this force can be related to the spacecraft's acceleration:

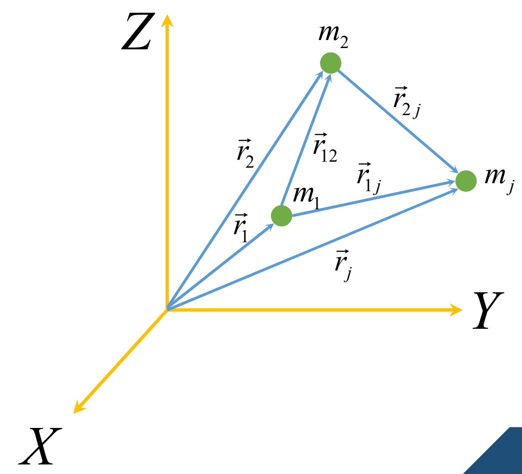
$$\vec{F}_i = m_i \ddot{\vec{r}}_i$$

Perturbations (Third Body Effect)

Replacing $\vec{F}_i$ and eliminating m_i from both sides of resulting relation, we have a relation for acceleration:

$$\ddot{\vec{r}}_i = G\sum_{j=1}^{n}\frac{m_j}{r_{ij}^3}\left(\vec{r}_j - \vec{r}_i\right) \quad ; \quad i \neq j$$

To find the effect of other bodies (as perturbations) on the orbit of the spacecraft, assume m_1 and m_2 are used for the Earth and spacecraft, respectively. Considering effect of all bodies, the equation of motion for the earth and spacecraft are:

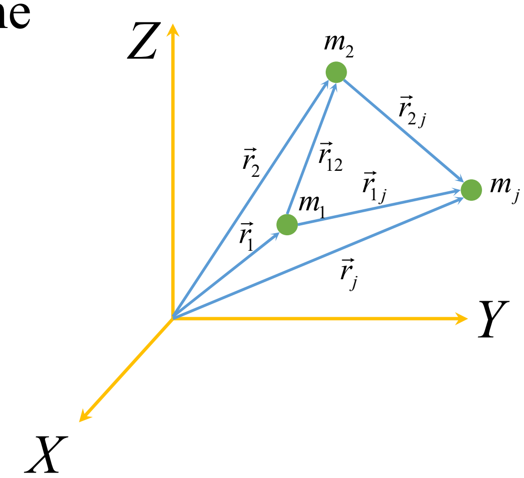


$$\ddot{\vec{r}}_1 = G\frac{m_2}{r_{12}^3}\left(\vec{r}_2 - \vec{r}_1\right) + G\sum_{j=3}^{n}\frac{m_j}{r_{1j}^3}\left(\vec{r}_j - \vec{r}_1\right)$$ Earth EOM in the inertial space

$$\ddot{\vec{r}}_2 = G\frac{m_2}{r_{21}^3}\left(\vec{r}_1 - \vec{r}_2\right) + G\sum_{j=3}^{n}\frac{m_j}{r_{2j}^3}\left(\vec{r}_j - \vec{r}_2\right)$$ Spacecraft EOM in inertial space

Perturbations (Third Body Effect)

The previous equations of motion show the motion of the Earth and the spacecraft in the inertial coordinate system. Assume that we want to represent the effect of other bodies as perturbations on the orbit of the spacecraft around the Earth. For this purpose, via introducing new parameters, we can define $\vec{\gamma}_P$ for each of the perturbing bodies (such as the 3rd body, etc.). Define the following subscripts for:



$$1 \triangleq Earth \quad ; \quad 2 \triangleq spacecraft \quad ; \quad P \triangleq other\ planets$$

Subsequently, we can develop the following equation for the relative motion of the spacecraft WRT the Earth considering $m_1 = m_E, m_2 = m_S, and\ m_j = m_{Pj}$ for the Earth, spacecraft and the j-th body mass respectively, while using ECI as the inertial coordinate system $(\vec{r}_1 = 0)$.
Subtracting the two previous equations, yields :

$$\ddot{\vec{r}} + G(m_E + m_S)\frac{\vec{r}}{r^3} = G\sum_{j=3}^{n} m_{Pj}\left[\frac{\vec{\rho}_j}{\rho_j^3} - \frac{\vec{r}_{Pj}}{r_{Pj}^3}\right]$$

$where:$

$$\vec{r} = \vec{r}_2 - \vec{r}_1 = \vec{r}_{12}$$

$$\vec{\rho}_j = \vec{r}_{2j} \ , \ \vec{r}_{Pj} = \vec{r}_{1j}$$

Perturbations (Third Body Effect)

$$\ddot{\vec{r}} + G\left(m_E + m_S\right)\frac{\vec{r}}{r^3} = G\sum_{j=3}^{n} m_{Pj}\left[\frac{\vec{\rho}_j}{\rho_j^3} - \frac{\vec{r}_{Pj}}{r_{Pj}^3}\right]$$

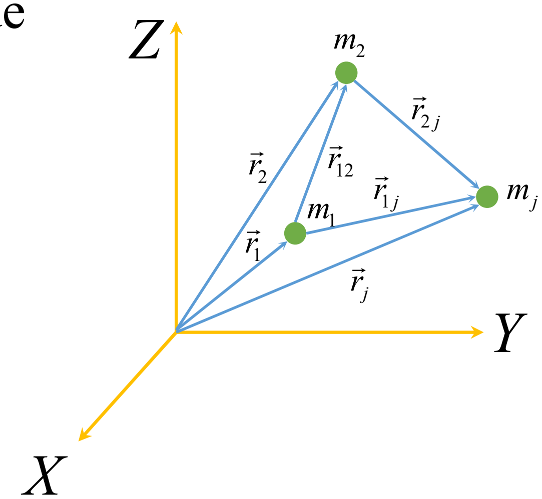
Where :

$1 \triangleq$ *Earth;* $2 \triangleq$ *spacecraft;* $P \triangleq$ *other planets*

$\vec{r} = \vec{r}_2 - \vec{r}_1 = \vec{r}_{12}$

$\vec{\rho}_j = \vec{r}_{2j}\ ,\ \vec{r}_{Pj} = \vec{r}_{1j}$

Elimination of the third body (as well as other bodies), will provide the old familiar 2BP EOM. Thus one can consider all other body *(n-2)* perturbative effects on the RHS of the above equation as below, where the other bodies gravity parameter are also defined.



$$\vec{\gamma}_P = \sum_{j=3}^{n} \mu_{Pj}\left[\frac{\vec{\rho}_j}{\rho_j^3} - \frac{\vec{r}_{Pj}}{r_{Pj}^3}\right]$$

$$\textit{Where}: \mu_{Pj} = Gm_{Pj}$$

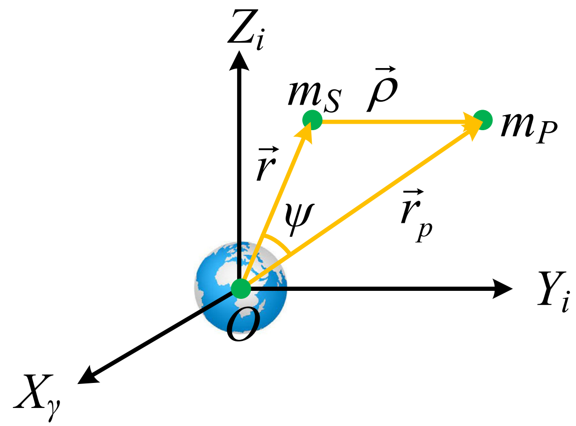
Perturbations (Third Body Effect)

$\ddot{\vec{r}}_i = \vec{\gamma}_K + \vec{\gamma}_P$

$\vec{\gamma}_P = \vec{a}_{GR} + \vec{a}_{3RD} + \vec{a}_{SRP} + \vec{a}_D + \vec{a}_{SF}$

Now as an special case, if we only consider the effect of the 3rd body on the motion of the spacecraft with respect to the Earth in ECI, by defining the angle between two vectors $\vec{r}, \vec{r}_p$ as a phase difference and show this angle with ψ :

$$\begin{cases} \vec{\gamma}_P = \mu_P \left(\dfrac{\vec{\rho}}{\rho^3} - \dfrac{\vec{r}_P}{r_P^3} \right) \\ \ddot{\vec{r}} + G\left(m_E + m_S\right)\dfrac{\vec{r}}{r^3} = \vec{\gamma}_P \end{cases}$$

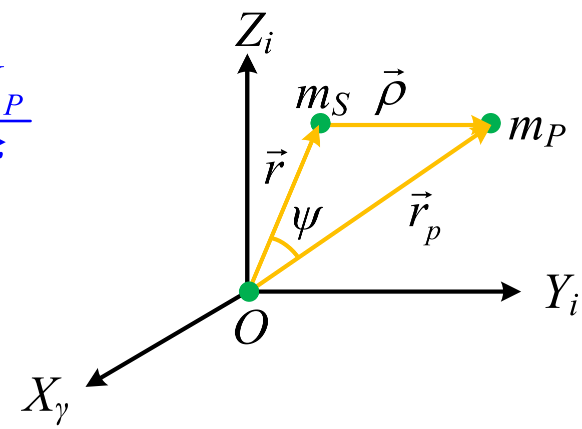


The first equation is called **effective attraction** of the third body on the spacecraft. For example, if we consider the moon as the third body, the first equation is called effective attraction of the moon on the spacecraft, which is equal to gravitational effect of the moon (first term) minus the gravitational effect of the moon on the Erath (second term).

Perturbations (Third Body Effect) [Sidi]

Note that the phase angle ψ changes as the spacecraft orbits the Earth. It is shown *[Kaplan]*, that the third body perturbative potential function can be written as a function of this changing phase angle as:

$$U_P = \mu_P\left(\frac{1}{\rho} - \frac{1}{r_P^3}\vec{r}\cdot\vec{r}_P\right); \quad where\ \vec{\gamma}_P = \vec{a}_{3RD} = -\frac{\partial U_P}{\partial \vec{r}}$$



Using triangular identity, one can show:

$$\rho^2 = r^2 + r_P^2 - 2rr_P\cos\psi \Rightarrow \frac{1}{\rho} = \frac{1}{\left(r^2 + r_P^2 - 2rr_P\cos\psi\right)^{\frac{1}{2}}}$$

Using the fact that $r/r_P \ll 1$, one can verify that U_P can be presented in the following forms via simplification [and utility of Binomial expansion or Legender polynomials $P_n(cos\psi)$].

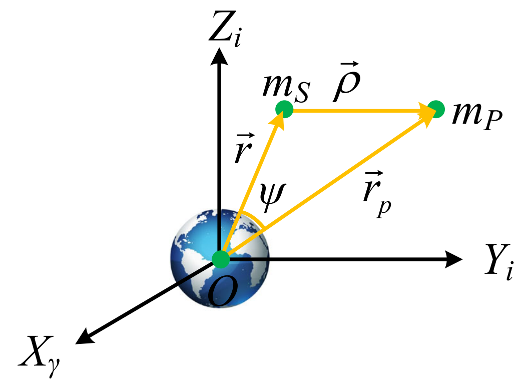
$$U_P \approx \frac{\mu_P}{r_P}\left[1 - \frac{1}{2}\left(\frac{r}{r_P}\right)^2 - \frac{3}{2}\left(\frac{r}{r_P}\right)^2\cos^2\psi + \cdots\right]$$

$$U_P \approx \frac{\mu_P}{r_P}\sum_{n=0}^{\infty}\left(\frac{r}{r_P}\right)^n P_n[\cos\psi]$$

Perturbations (Third Body Effect)

$$U_P \approx \frac{\mu_P}{r_P}\left[1-\frac{1}{2}\left(\frac{r}{r_P}\right)^2-\frac{3}{2}\left(\frac{r}{r_P}\right)^2\cos^2\psi+\cdots\right]$$

$$U_P \approx \frac{\mu_P}{r_P}\sum_{n=0}^{\infty}\left(\frac{r}{r_P}\right)^n P_n[\cos\psi] \qquad Note:\ \vec{\gamma}_P=\vec{a}_{3RD}=-\frac{\partial U_P}{\partial \vec{r}}$$



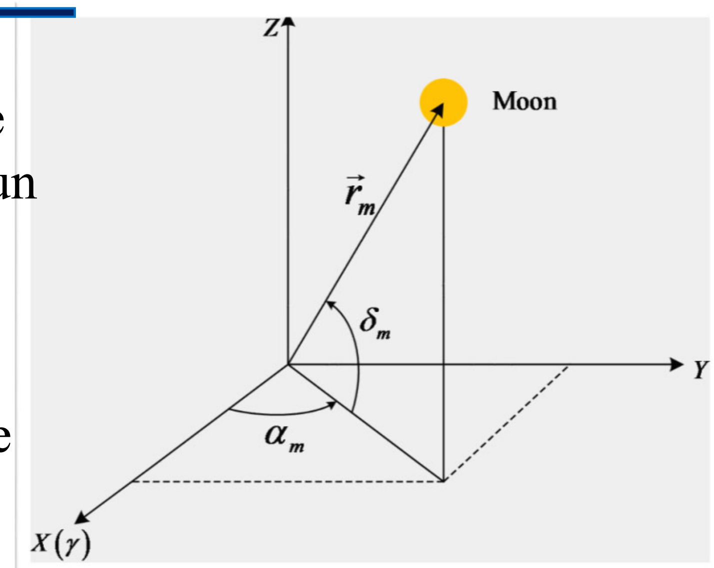
Notice that r_P and ψ will be different for the Sun or the Moon as the third perturbing body. It is also interesting to note that if we calculate the magnitude of the $\dfrac{\mu_P}{r_P^3}$ factor for the largest term of the resulting perturbative specific force, we will have:

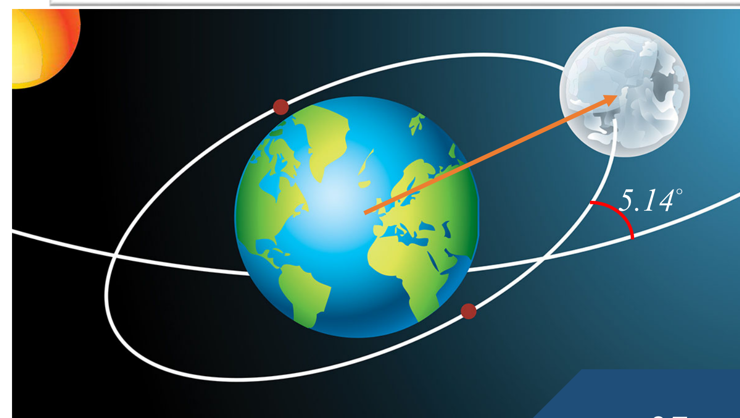
$$\left.\frac{\mu_P}{r_P^3}\right|_{\text{Moon}} \approx 8.6\times10^{-14}\,\sec^{-2} \quad ; \quad \left.\frac{\mu_P}{r_P^3}\right|_{\text{Sun}} \approx 3.96\times10^{-14}\,\sec^{-2}$$

That shows that the effect of the moon is about twice that of the sun for the earth orbiting satellites.

Perturbations (Third Body Effect)

Since utility of the above relations requires the position of the perturbing r_P (third body) planet such as the Moon and the Sun in ECI, their ephemeris will be needed that can be picked up from published by reports such as DE200 and DE400, or simulated in the celestial sphere.
In these published notes, the coordinates of the Moon and the Sun in ECI can be calculated based on their average distance from the Earth as well as their *R.A.* and δ angles (which are time/date dependent). As an example, for the Moon:





$$\vec{r}_{\text{moon}} = r_m \left[\cos\delta_m \left(\cos\alpha_m \vec{I} + \sin\alpha_m \vec{J} \right) + \sin\delta_m \vec{K} \right]$$

One can write similar equation for the Sun's orbit around the Earth.

Ephemeris is a table that provides the computed or observed positions of a celestial body for every day of a given period.

Perturbations (Third Body Effect)

$$\ddot{\vec{r}}_i = \vec{\gamma}_K + \vec{\gamma}_P$$

$$\vec{\gamma}_P = \vec{a}_{GR} + \vec{a}_{3RD} + \vec{a}_{SRP} + \vec{a}_D + \vec{a}_{SF}$$

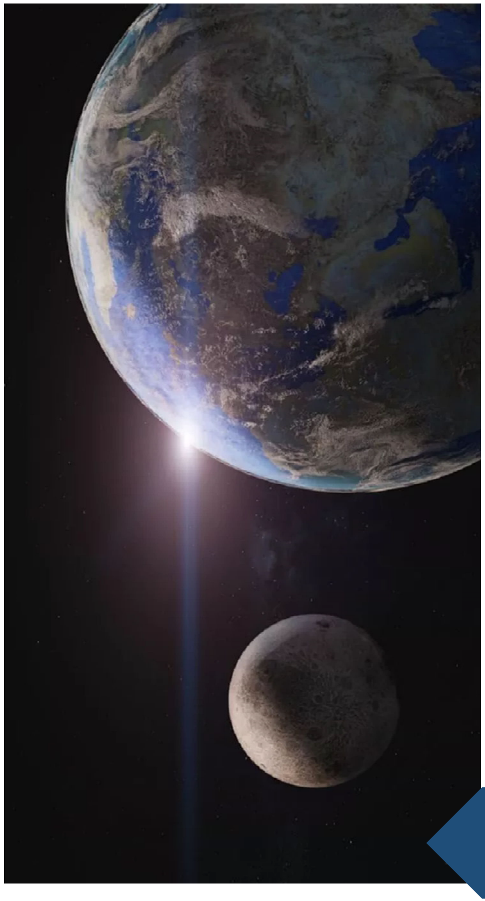
Therefore, one could use either the *GPE* or *LPE* methods to determine the propagation of OE considering the third body effect.

$$\begin{cases} \vec{\gamma}_P = \mu_P \left(\dfrac{\vec{\rho}}{\rho^3} - \dfrac{\vec{r}_P}{r_P^3} \right) \\ \ddot{\vec{r}} + G\left(m_E + m_S\right)\dfrac{\vec{r}}{r^3} = \vec{\gamma}_P \end{cases}$$

$$U_P \approx \frac{\mu_P}{r_P}\left[1 - \frac{1}{2}\left(\frac{r}{r_P}\right)^2 - \frac{3}{2}\left(\frac{r}{r_P}\right)^2 \cos^2\psi + \cdots\right]$$

$$U_P \approx \frac{\mu_P}{r_P}\sum_{n=0}^{\infty}\left(\frac{r}{r_P}\right)^n P_n\left[\cos\psi\right]$$

$$Note : \vec{\gamma}_P\Big|_{\text{moving}} = \begin{bmatrix} R & S & W \end{bmatrix}^T = C_i^m \, \vec{\gamma}_P\Big|_i$$



Perturbations (Third Body Effect)

On the other hand, there are other approximate relations for the average changes of the orbital elements due to the third body effect are as follows: *[Fortescue]*

$$\frac{da}{dt} \simeq 0$$

$$\frac{de}{dt} \simeq \frac{1}{n}\frac{-15}{2}Ke\left(1-e^2\right)^{\frac{1}{2}}\left[AB\cos 2\omega-\frac{1}{2}\left(A^2-B^2\right)\sin 2\omega\right]$$

$$\frac{d\Omega}{dt} \simeq \frac{3KC}{4n\left(1-e^2\right)^{\frac{1}{2}}\sin i}\left[5Ae^2\sin 2\omega+B\left(2+3e^2-5e^2\cos 2\omega\right)\right]$$

$$\frac{di}{dt} \simeq \frac{3KC}{4n\left(1-e^2\right)^{\frac{1}{2}}\sin i}\left[A\left(2+3e^2+5e^2\cos 2\omega\right)+5Be^2\sin 2\omega\right]$$

$$\frac{d\omega}{dt} \simeq \frac{d\Omega}{dt}\cos i+\frac{3}{2}\frac{K\left(1-e^2\right)^{\frac{1}{2}}}{n}\left[\begin{array}{c}5\left\{AB\sin 2\omega+\frac{1}{2}\left(A^2-B^2\right)\cos 2\omega\right\}\\ -1+\frac{3}{2}\left(A^2+B^2\right)+\frac{5a}{2er_P}\left\{1-\frac{5}{4}\left(A^2+B^2\right)\right\}\left(A\cos\omega+B\sin\omega\right)\end{array}\right]$$

Perturbations (Third Body Effect)

In the previous equations, there are some constants defined as :

$$K = \frac{\mu_P}{r_P^3}$$

$$A = cos(\Omega - \Omega_P)cos(u_p) + cos(i_P)sin(u_P)sin(\Omega - \Omega_P)$$

$$B = cos(i)\left[-sin(\Omega - \Omega_P)cos(u_P) + cos(i_P)sin(u_P)cos(\Omega - \Omega_P)\right] + sin(i)sin(i_P)sin(u_P)$$

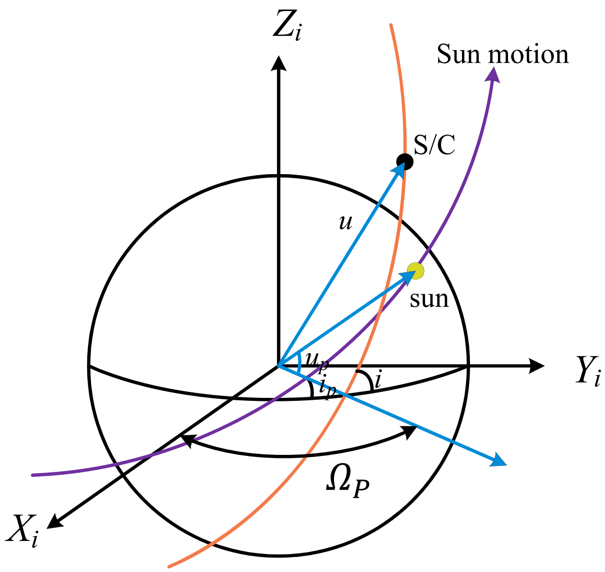
$$C = sin(i)\left[cos(u_P)sin(\Omega - \Omega_P) - cos(i_P)sin(u_P)cos(\Omega - \Omega_P)\right] + cos(i)sin(i_P)sin(u_P)$$

$\Omega_P, u_P = (\theta_P + \omega_P), i_P$ *are some of orbital elements of the 3rd disturbing body referred to the Equatorial based system.*

$i_p = i_{Sun} = \varepsilon = 23.45^\circ$ *for the Sun wrt the Equatorial plane,*

$i_{moon} = 5.14^\circ$ *with respect to the Ecliptic plane or*

$i_{moon} = 23.45 + 5.1 = 28.55^\circ$ *with respect to Equatorial plane*



Solar Radiation & Solar Wind

$$\ddot{\vec{r}}_i = \vec{\gamma}_K + \vec{\gamma}_P$$
$$\vec{\gamma}_P = \vec{a}_{GR} + \vec{a}_{3RD} + \vec{a}_{SRP} + \vec{a}_D + \vec{a}_{SF}$$

Before further investigation in solar radiation and solar wind effects on the orbits of satellites, it is appropriate to refer to some of the physical characteristics of the sun:

$$R_{\text{Sun}} = 696000\ \text{km} \quad ; \quad m_{\text{Sun}} = 2 \times 10^{30}\ \text{kg} \quad ; \quad m_E = 3 \times 10^{-6}\, m_{\text{Sun}}$$

The sun is one of the 10^{11} stars of the Milky Way galaxy, which is the closest to the Earth and the most influential on its orbital movement. After the Sun, the closest star is about 4.35 light years (Alpha Centauria) way from the Earth. In addition, the nominal energy release rate of the Sun is approximately:

$$\textit{Total Sun power output: } P = 3.85 \times 10^{26}\ Watt \left(\frac{Joules}{s} \text{ or } \frac{N \cdot m}{s} \right)$$

Binary star system Alpha Centauri AB

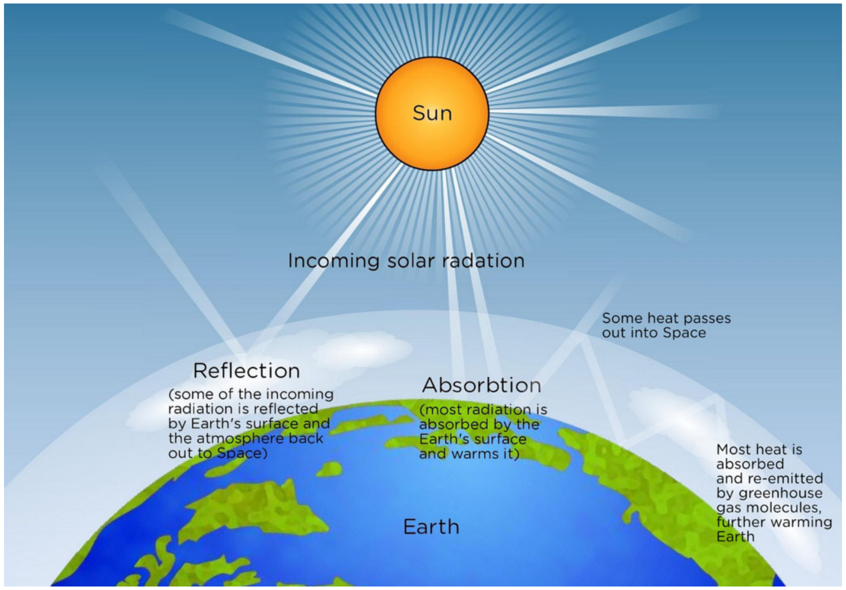
To get a better feeling about the Sun power output, one could say that to produce this amount of energy or at this rate, if all the fossil fuels on Earth were used to produce this amount of energy, they would be exhausted within 50 milliseconds.

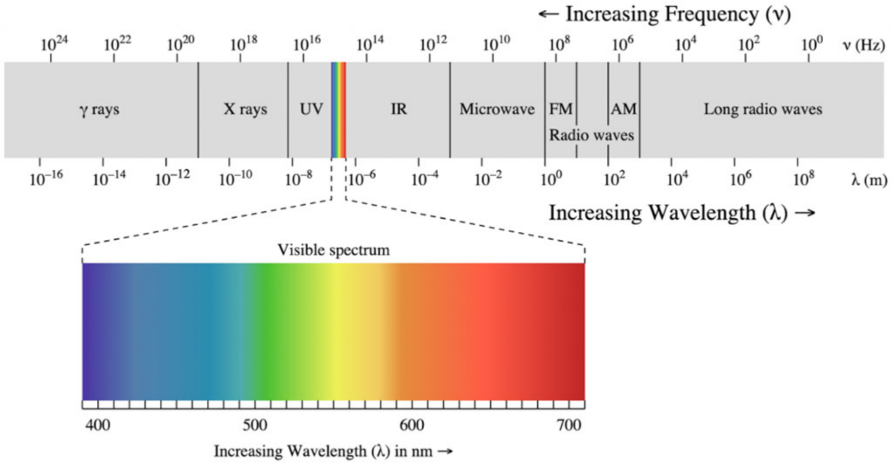
Solar Radiation(SR) & Solar Wind (SW)

Nuclear fusion reactions in the Sun nucleus is the source of the Sun's energy that produces electromagnetic radiation (EMR) at various frequencies or wavelengths:

- Infrared (IR), radiation provides heat (49% of SR),
- Visible (VI), radiation provides light (43% of SR),
- Ultraviolet (UV), radiation represent (7% of SR) can cause skin and eye problems.

EMR propagates in space at the speed of light *(299,792 km / s)*.The effect of Sun the spacecraft's motion takes place through solar radiation and solar wind. **Solar Radiation:** refers to the amount of EMR from the Sun that includes its different types of radiation.

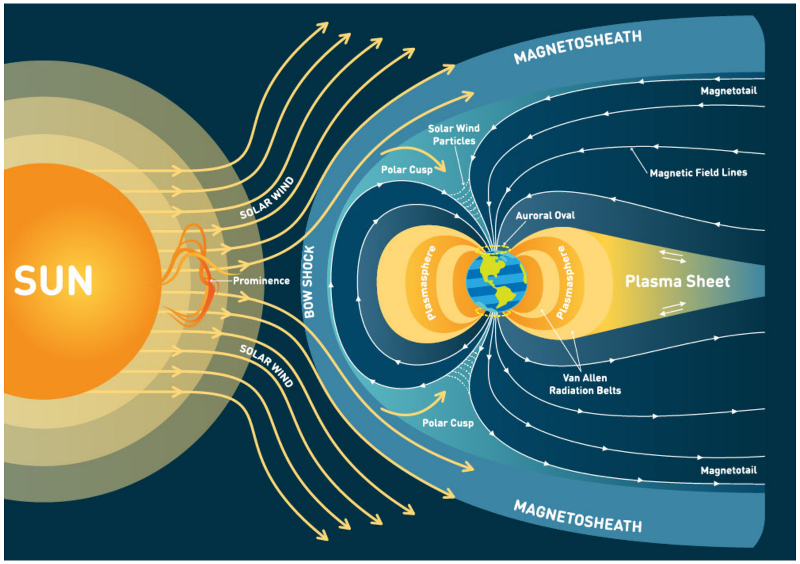


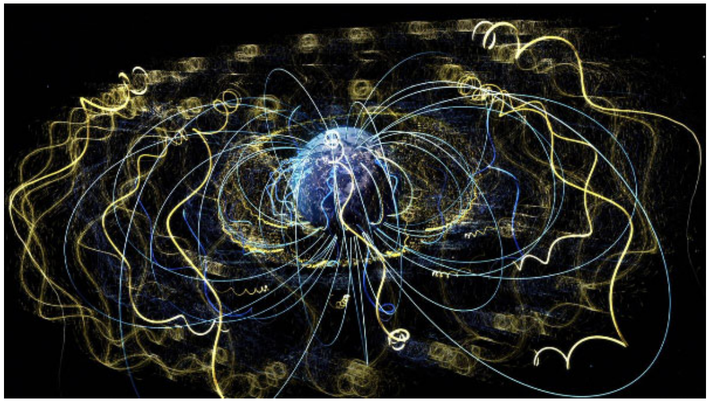


Solar Radiation(SR) & Solar Wind (SW)

Solar Wind: streams of charged particles (CP) that leave the Sun's atmosphere. These particles have a very high speed that can escape from the Sun and come towards the planets. Exploration probes have measured speeds of 300-800 km/s for these CPs with initial temperature of 100,000°C for the winds. As SW reach and encounter the upper bounds of the Earth atmosphere their speed is reduced (around Mach 9). SW are mainly ionized hydrogen as a plasma gas mixture of electrons, protons etc. References indicate that the CP density is about 8.0E17 photons per square centimeter, where each photon has a certain amount of energy. For practical purpose the combined effect of SR and SW is utilized as solar radiation constant, intensity, irradiance, or solar radiation flux ($F_S = 1367 \; to \; 1400 \; W/m^2$) to account for SRP effects on the orbital motion.

Fortunately, SWs do not directly collide with the Earth as at a distance several times bigger the radius of the Earth, they meet its magnetic field creating a **protective shock wave** shield.

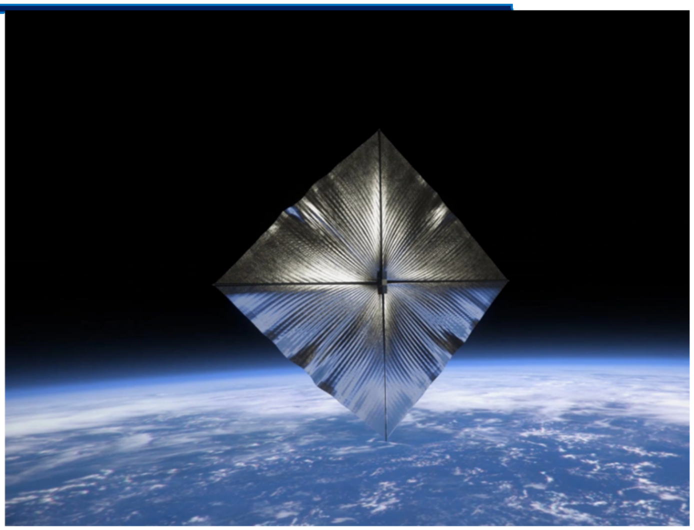


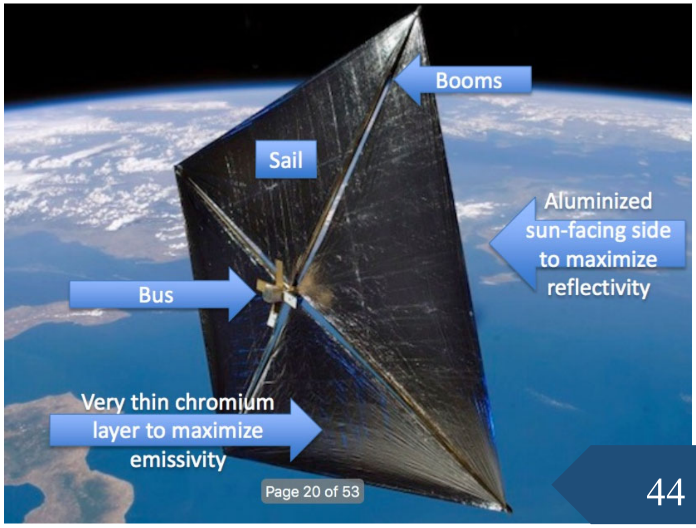


Earth's without magnetic field!. CPs bombard the planet, break down power grids and satellites. Cancer causing UV radiation increases dramatically .

Solar Radiation(SR) & Solar Wind (SW)

At 180 km near the Earth's magnetic field, the speed of these winds is in the range of 55-70 km/s with pressures up to 9.02 MPa (on fully reflecting surfaces) and 4.5 MPa (on black bodies).
As indicated before, solar radiation and solar wind create a pressure on a surface that is proportional to their momentum flux (momentum per unit area per unit time). However, the Sun's momentum flux due to radiation is stronger than its momentum flux due to its wind (between 100 and 1000 times). In this sense SW can be used as propulsion for distant space travel using large **solar sail** concept.
Due to the complexities involved, Solar radiation pressure (SRP) is considered as the combined effect of SR and SW on space probes and satellites.





Solar Radiation(SR) & Solar Wind (SW)

As mentioned earlier, SRP can be translated to a non-conservative perturbative force acting on space systems surfaces (within our solar system) causing variation of OE. In addition to direct Sun's EMR , its effect on Earth heat content and reflection can be causes of momentum exchange to satellites. Thus, we can consider the cumulative effect of SRF on satellite motion as follows:

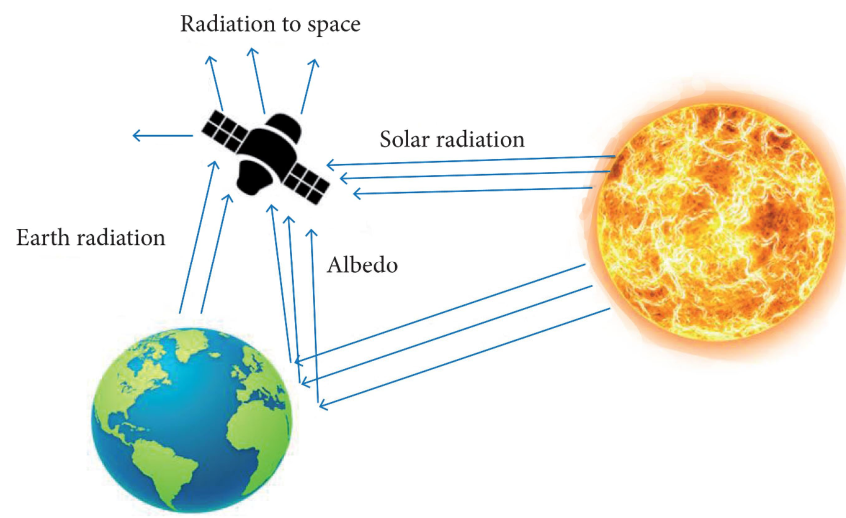
$$P_{SRP} = \frac{F_S}{c} \; in \; (\frac{N}{m^2})$$

$$F_S \triangleq Solar\ energy\ flux = J_S + J_P + J_a \approx J_S$$

$$J_S \triangleq Direct\ Sun\ radiation\ intensity = \frac{P}{4\pi d^2}$$

$$For\ d \simeq 150 \times 10^6\ km \Rightarrow J_S = F_S \simeq 1367 \sim 1400\ \frac{Watt}{m^2}$$

$$c \triangleq speed\ of\ the\ light\ (\frac{m}{s});\ (Watt = \frac{Nm}{s}\ or\ \frac{Joules}{s})$$



Solar Radiation(SR) & Solar Wind (SW)

It turns out that for an Earth orbiting satellites, the mean SRP is about 4.7×10^{-6} N/m^2. The total effect of this pressure as a perturbative force depends on the reflective properties of the spacecraft, due to indirect radiations of the Earth infrared and albedo EMR whose values are sometimes ignored in comparison the direct SRP.

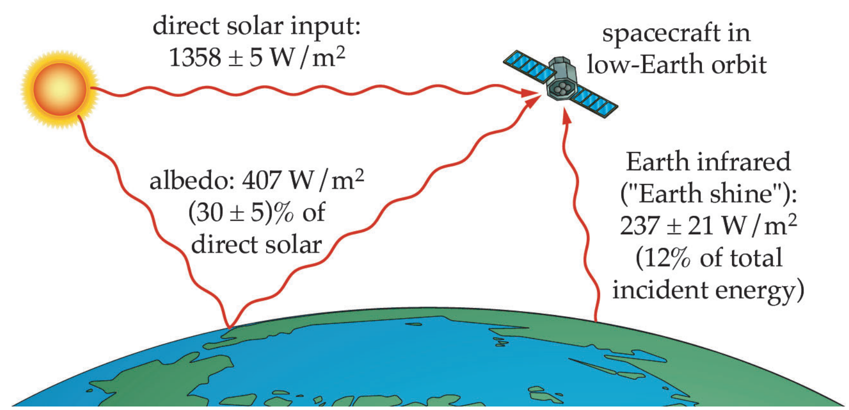
$$J_a \triangleq \text{Earth albedo radiation flux} \approx aFJ_S$$

$$a \triangleq \text{Earth or planetary albedo (different for each planet)}$$

$$a_E = 0.31 \rightarrow 0.39$$

$$F \triangleq \text{visibility factor}\left[10^{-4} \rightarrow 1 \,\&\, g\left(\beta'\right)\right]$$

$$J_P \triangleq \text{Earth or planetary radiation} = 237\left(\frac{R_E}{r}\right)^2$$



A spacecraft gets heat from the Sun, Earth (reflected and emitted energy), and from within itself

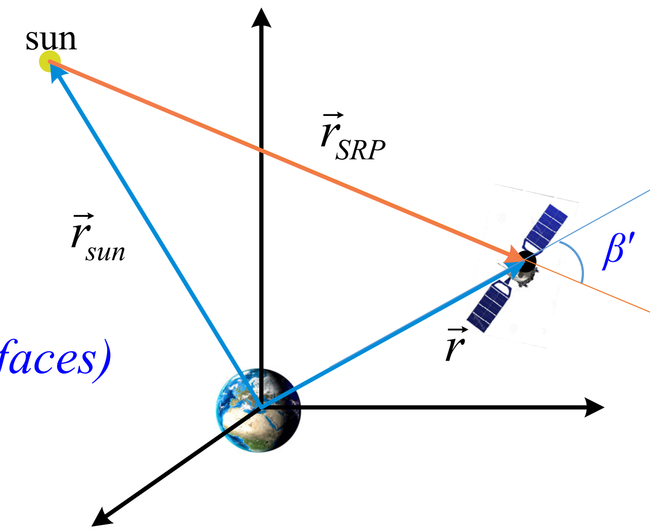
$$\text{For Example : GSO, } J_S = 1375$$

$$\Rightarrow J_a = 5.36, J_P = 8\left(\beta' = 70^\circ\right)$$

Solar Radiation(SR) & Solar Wind (SW)

$$\ddot{\vec{r}}_i = \vec{\gamma}_K + \vec{\gamma}_P$$
$$\vec{\gamma}_P = \vec{a}_{GR} + \vec{a}_{3RD} + \vec{a}_{SRP} + \vec{a}_D + \vec{a}_{SF}$$

Since J_s's effect is more significant important in the production of P_{SRP}, the **specific** perturbation force (per unit mass) of this pressure can be obtained from the following relations :



$$\vec{a}_{SRP} = S\frac{A}{m}P_{SRP}\left(\frac{R_E}{r_S}\right)^2\frac{\vec{r}_{SRP}}{r_{SRP}}$$

A/m : *Area to mass ratio of the spacecraft*

$S = 0 \to 2$ *(depends on the reflective properties of the spacecraft surfaces)*

(e.g: 1 for black body or totally absorbing, 2 for totally reflecting)

$R_E \triangleq$ *Earth mean distance to Sun*

$r_s \triangleq$ *spacecraft distance to Sun*

$\vec{r}_{SRP} = \vec{r} - \vec{r}_{Sun}$

Solar Radiation(SR) & Solar Wind (SW)

Subsequently, SRP effect can be modeled via the GPE by decomposing the specific force $\vec{a}_{SRP}$ into the RSW coordinate system.

$$R = a_{SRP} S_0(\theta) \quad ; \quad S = a_{SRP} T_0(\theta) \quad ; \quad W = a_{SRP} W_0$$

Where S_0, T_0, and W_0 are functions of the Sun's instantaneous position and orbital elements (α, λ_S, and ε) for simulation purposes (given in the next slide) .
The required parameters to convert the coordinates based on the orbital information of the Sun and the satellite can be used as follows, where for simplicity the following variables are defined :

$$a_1 = \lambda_S - u \quad ; \quad a_2 = \lambda_S + u \quad ; \quad \lambda_S \triangleq \textit{Ecliptic longitude of the Sun}$$

$$u = \theta + \omega \quad , \quad \varepsilon = \textit{obliquity of the ecliptic plane} \approx 23.45^{\circ}$$

$$b_1 = \frac{i}{2} \quad , \quad b_2 = \frac{\varepsilon}{2}$$

Solar Radiation(SR) & Solar Wind (SW)

$$S_0(\theta) = -\cos^2 b_1 \cos^2 b_2 \cos(a_1 - \Omega) - \sin^2 b_1 \sin^2 b_2 \cos(a_1 + \Omega)$$
$$-\frac{1}{2}\sin i \sin\varepsilon\left[\cos a_1 - \cos(-a_2)\right] - \sin^2 b_1 \cos^2 b_2 \cos(-a_2 + \Omega)$$
$$-\cos^2 b_1 \sin^2 b_2 \cos(-a_2 - \Omega)$$

$$T_0(\theta) = -\cos^2 b_1 \cos^2 b_2 \sin(a_1 - \Omega) - \sin^2 b_1 \sin^2 b_2 \sin(a_1 + \Omega)$$
$$-\frac{1}{2}\sin i \sin\varepsilon\left[\sin a_1 - \sin(-a_2)\right] - \sin^2 b_1 \cos^2 b_2 \sin(-a_2 + \Omega)$$
$$-\cos^2 b_1 \sin^2 b_2 \sin(-a_2 - \Omega)$$

$$W_0 = \sin i \cos^2 b_2 \sin a_1 - \sin i \sin^2 b_2 \sin a_2 - \cos i \sin\varepsilon \sin\lambda_S$$

Solar Radiation(SR) & Solar Wind (SW)

This angle is measured eastward from V.E.. The (dimensionless) vector of the sun (the direction of the earth towards the sun) in the ECI coordinate system can be considered as follows or using its ephemeries :

$$\vec{r}_{sun} = (\cos \lambda_S)\vec{I} + (\sin \lambda_S \cos \varepsilon)\vec{J} + (\sin \lambda_S \sin \varepsilon)\vec{K}$$

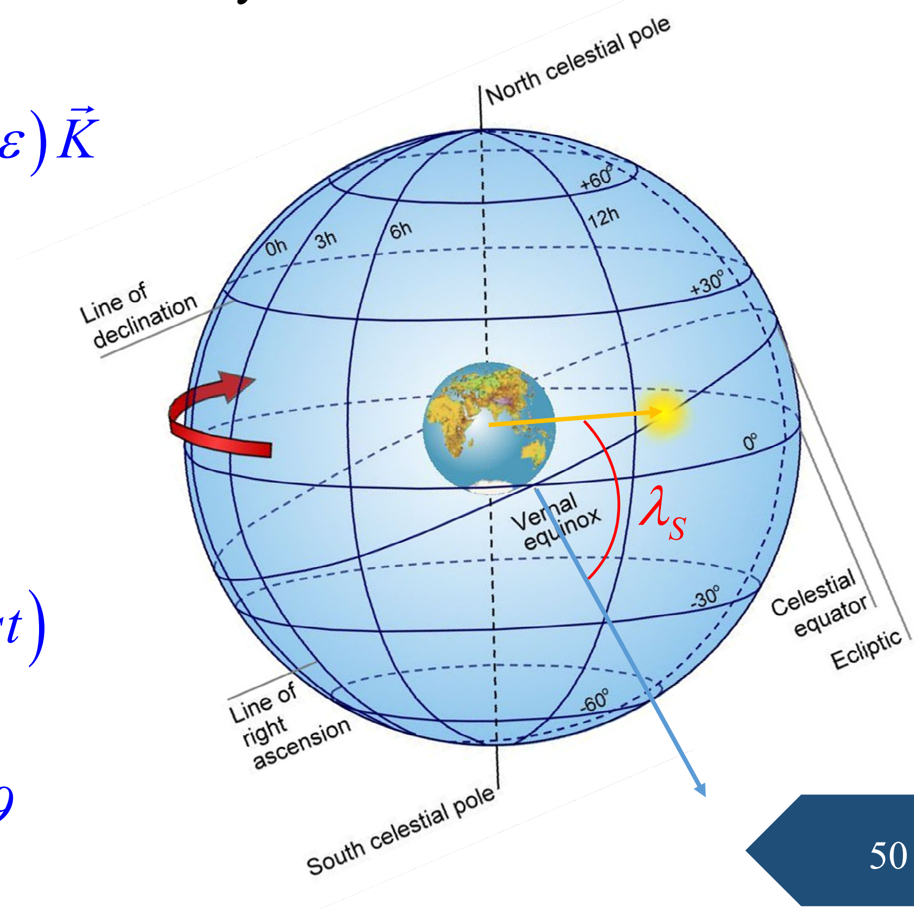
An approximate relation for λ_S (based on degree) is suggested as follows:

$$\lambda_S \approx \left(\frac{D - D_0}{365}\right) \times 360^\circ$$

$$D = Day\ number\ \left(D = 0 \triangleq 0\ hour\ on\ Jan.\ 1st\right)$$

$$D = 1.5 :\ half\text{-}day\ of\ Jan.\ 2nd$$

$$V.E.\ happens\ at\ 0\ h\ of\ March\ 21st \Rightarrow D_0 = 79$$



Some Radiation based recent research

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Satellite pose estimation using Earth radiation modeled by artificial neural networks

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Received 24 December 2021; received in revised form 13 May 2022; accepted 5 July 2022

On-Line Orbit and Albedo Estimation Using a Strong Tracking Algorithm via Satellite Surface Temperature Data

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Development of a radiation based heat model for satellite attitude determination

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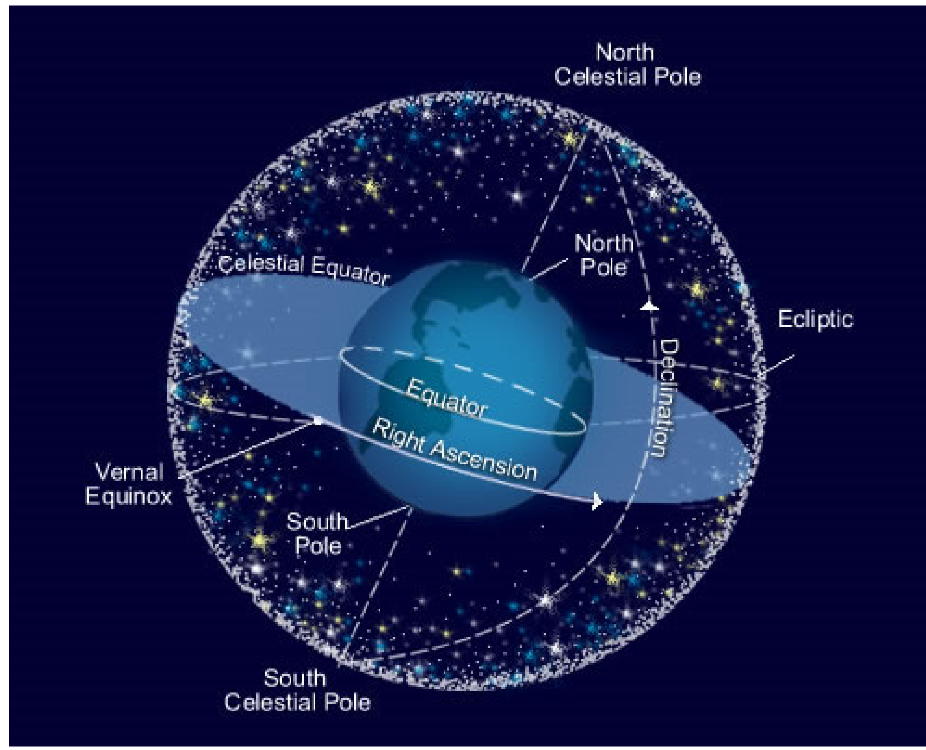
Celestial Sphere and the Coordinates of the Bodies Inside

All celestial bodies can be considered inside a infinite sphere centered at Earth.



Celestial Sphere and the Coordinates of the Bodies Inside

Angular coordinate of each body in the celestial sphere is expressed based on two angles, α (RA) and δ (declination), which can be a function of time or date. α is positive to the east with respect to the meridian passing through V.E. and δ is calculated relative to the celestial equator plane.

For objects that have mutual rotational motion with the Earth (such as the moon and the Sun), their motion in the ECI coordinates can be specified in two ways:

1) Based on α (RA) and δ:

$$\vec{u}_S = \frac{\vec{r}_S}{r_s} = \left[\cos\delta_S\left(\cos\alpha_S\vec{I} + \sin\alpha_S\vec{J} + \sin\delta_S\vec{K}\right)\right]$$

2) Based on the angle of their orbital plane and elliptical longitude within their orbital plane:

$$\left.\begin{aligned} \varepsilon &= i_S \\ \lambda_S &= \frac{D - D_0}{365}\times 360^\circ\left(D_0 = 79\right) \end{aligned}\right\} \Rightarrow \vec{u}_S = \cos\lambda_S\vec{I} + \sin\lambda_S\cos\varepsilon\vec{J} + \sin\lambda_S\sin\varepsilon\vec{K}$$

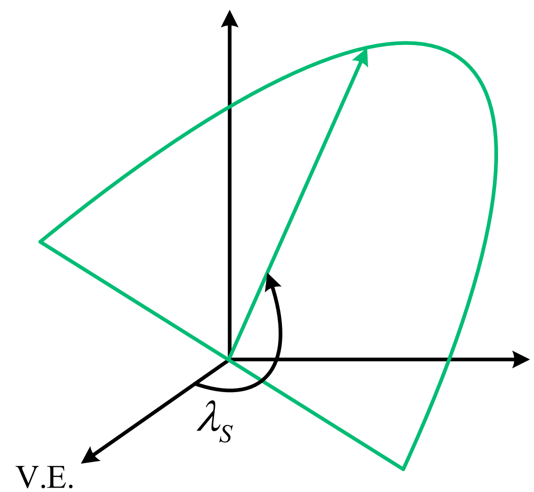
Celestial Sphere and the Coordinates of the Bodies Inside

D is measured from 0 h on the January 1st.

Pay attention that for the moon, λ_M and ε_M should be considered as follows.

$$\lambda_M = \frac{D - D_0}{28} \times 360^\circ \quad , \quad \varepsilon_M = 28.6^\circ$$

For transformation between ECI and HCI, the rotation matrix is a function of ε and the rotation is performed around the X_γ axis.



Ground Track

I) Based on the position of the satellite, we can find right ascension and declination:

$$\vec{r}^{\,i} = \begin{bmatrix} r\cos\delta\cos\alpha & r\cos\delta\sin\alpha & r\sin\delta \end{bmatrix}^T = \begin{bmatrix} x & y & z \end{bmatrix}^T$$

$$\delta = \sin^{-1}\left(\frac{z}{r}\right) \quad ; \quad \alpha = \tan^{-1}\left(\frac{y}{x}\right)$$

For a spherical Earth, these two angles can be used to determine longitude and latitude:

$$\phi = \text{latitude} = \delta \quad ; \quad \lambda = \text{longitude} = \alpha - \omega_E t - \alpha_0 = \alpha - \theta_{GST}$$

$$\begin{cases} \theta_{GST} = \alpha_0 + \omega_E t = f\left(\text{JD}\right)\left(\text{Greenwich sidereal time}\right) \\ \omega_E = 7.2921\times 10^{-5}\ \text{RPS} \end{cases}$$

II) Another way is to convert $\vec{r}^{\,i}$ to $\vec{r}^{\,ECEF}$.

Ground Track

II) Another way is to convert $\vec{r}^{\,i}$ to $\vec{r}^{\,ECEF}$.

$$\vec{r}^{\,ECEF} = \begin{bmatrix} r\cos\phi\cos\lambda & r\cos\phi\sin\lambda & r\sin\phi \end{bmatrix}^{\top} = C_i^{ECEF}(\theta)\vec{r}^{\,i} = \begin{bmatrix} x & y & z \end{bmatrix}^{\top}$$

$$r = \sqrt{x^2 + y^2 + z^2} \quad ; \quad h = r - R_E$$

$$\phi_{GC} = \sin^{-1}\left(\frac{z}{r}\right) \quad ; \quad \lambda = \tan^{-1}\left(\frac{y}{x}\right)$$

Eclipse Duration Criterion

The Earth orbiting satellites will encounter Earth shadow (Eclipse intervals) periods whose frequency and length are related to the height and inclination of the orbit.

Example 1: for low altitude equatorial orbits, 40% of each orbit.

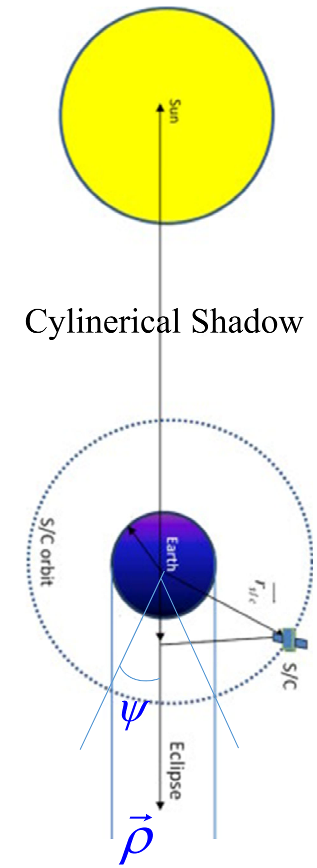
Example 2: For Sun-synchronous orbit at low altitude, we will have **eclipse free** intervals of several months.

Example 3: For equatorial geostationary (not GSO) orbits, the satellites will be exposed to sunlight most of the time.

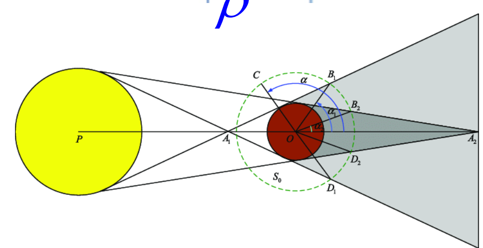
For cylindrical shadow pattern and spherical Earth, the following relations can be used to determine shadow entrance and exit conditions. In this method, the satellite will be in shadow if the angle between the Earth-satellite vector and the Sun-Earth vector line is smaller than ψ.

$$\cos^{-1}\left(\vec{\rho}\cdot\vec{r}\right)\le\psi \quad ; \quad \psi=\sin^{-1}\left(\frac{R_E}{r}\right); \; \vec{\rho}=-\vec{r}_{Earth\_Sun}$$

Similar relations can be derived for conical shadow pattern.



Cylinerical Shadow

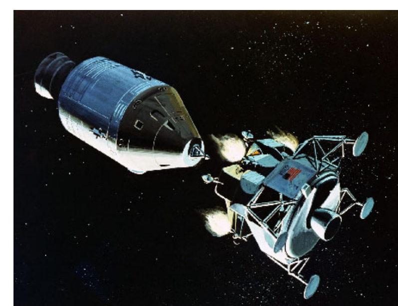

Conical Shadow

Relative Motion in Neighboring Orbits

Using the spacecraft equations of motion in inertial system that can be used for any spacecraft in any orbit, it is possible to obtain the (relative) distance between two spacecrafts orbiting in neighboring orbits. This is specially of importance when one is interested to analyze and or control the relative motion in **rendezvous** missions.

For example in the Apollo mission, returning of the lunar lander (lunar excursion) to the command service module (CSM) in the parking orbit around the moon and rendezvous with it was directly dependent on the success of this maneuver.

In general, missions involving installation and assembly of orbital stations, personnel transfer, search, retrieval and interception under perturbation effect will benefit from this type of investigation.



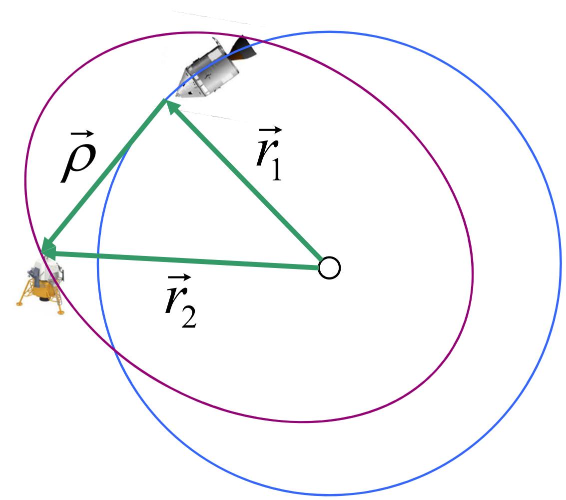
Relative Motion in Neighboring Orbits

Consider the following geometry. Suppose that at t_0 the position vectors of two spacecrafts are $\vec{r}_1, \vec{r}_2$ in inertial coordinates. Thus, the **relative** motion vector (the second body relative to the first) in a moving coordinate whose center is in the center of the first body and moves with it will be $\vec{\rho} = \vec{r}_2 - \vec{r}_1$.

Further, now suppose that the first spacecraft is in Keplerian orbit whose motion can be described by :

$$\ddot{\vec{r}}_1 = -\frac{\mu}{r_1^3}\vec{r}_1$$

$$\vec{\rho} = \vec{r}_2 - \vec{r}_1$$



Next, consider the second spacecraft with a disturbance force such as thrust or drag. Therefor its equation of motion will be:

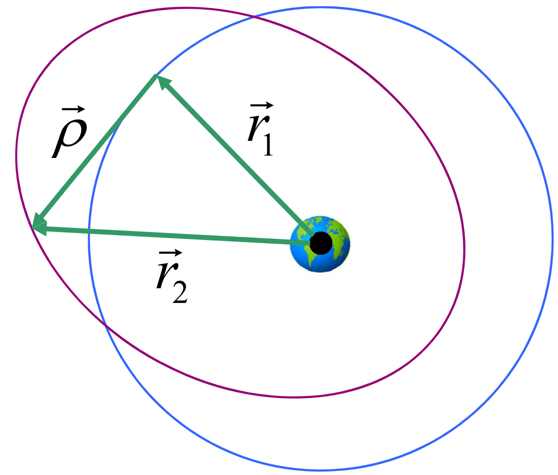
$$\ddot{\vec{r}}_2 = -\frac{\mu}{r_2^3}\vec{r}_2 + \vec{f}$$

Relative Motion in Neighboring Orbits

Based on the relation for relative positions:

$$\ddot{\vec{\rho}} = \ddot{\vec{r}}_2 - \ddot{\vec{r}}_1$$

$$= -\frac{\mu}{r_2^3}\vec{r}_2 + \vec{f} + \frac{\mu}{r_1^3}\vec{r}_1 \Rightarrow \ddot{\vec{\rho}} = \frac{\mu}{r_1^3}\left[\vec{r}_1 - \frac{r_1^3}{r_2^3}\vec{r}_2\right] + \vec{f}$$



We can simplify the expression in the bracket (use binomial expansion and keep $O(\rho^2)$ out! assuming small ρ):

$$\vec{r}_2 = \vec{r}_1 + \vec{\rho}$$

$$r_2^2 = (\vec{r}_1 + \vec{\rho}) \cdot (\vec{r}_1 + \vec{\rho})$$

$$\frac{\vec{r}_2}{r_2^3} = \frac{\vec{r}_1 + \vec{\rho}}{\left(r_1^2 + 2\vec{r}_1 \cdot \vec{\rho} + \rho^2\right)^{\frac{3}{2}}} = \frac{\vec{r}_1 + \vec{\rho}}{r_1^3}\left[1 + 2\frac{\vec{r}_1 \cdot \vec{\rho}}{r_1^2} + \frac{\rho^2}{r_1^2}\right]^{-\frac{3}{2}}$$

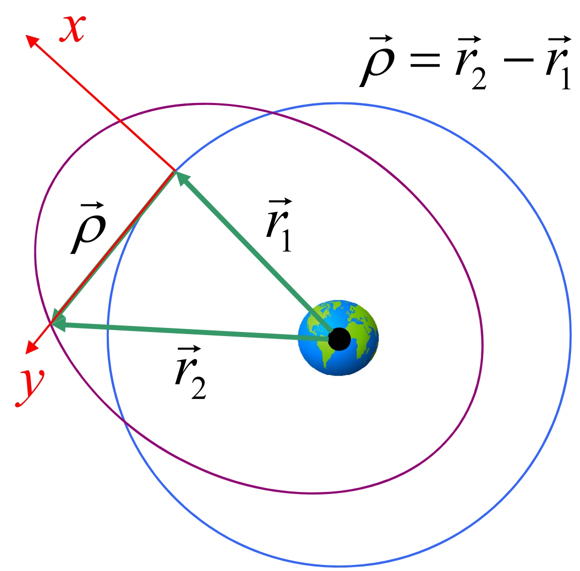
$$\Rightarrow \frac{\vec{r}_2}{r_2^3} = \frac{\vec{r}_1 + \vec{\rho}}{r_1^3}\left[1 - \frac{3}{2}(2\frac{\vec{r}_1 \cdot \vec{\rho}}{r_1^2})\right] + O\left(\rho^2\right)$$

Relative Motion in Neighboring Orbits

Substituting the approximation into the relation of the relative acceleration, and simplifying yields :

$$\ddot{\vec{\rho}} = \frac{\mu}{r_1^3}\left[-\vec{\rho} + 3\left(\frac{\vec{r}_1}{r_1}\cdot\vec{\rho}\right)\frac{\vec{r}_1}{r_1}\right] + \vec{f} + O\left(\rho^2\right)$$

Note that $\ddot{\vec{\rho}}$ is in inertial system, and its expression in the moving *(x,y,z)* frame requires application of the Coriolis law:



$$\ddot{\vec{\rho}} = \ddot{\vec{\rho}}_b + 2\vec{\omega}\times\dot{\vec{\rho}}_b + \vec{\omega}\times\left(\vec{\omega}\times\vec{\rho}\right) + \dot{\vec{\omega}}\times\vec{\rho}$$

$\vec{\omega}$: orbital angular velocity of the 1^{st} orbit (or reference orbit in body coordinate)

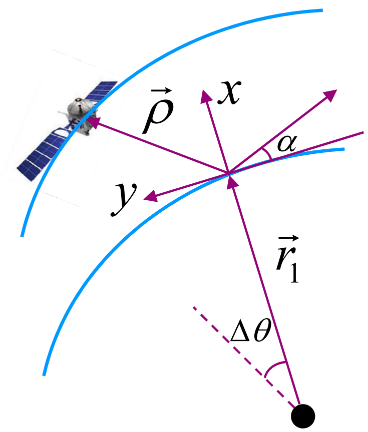
for circular reference orbits: $\dot{\vec{\omega}} = 0 \Rightarrow \begin{cases} \dot{\vec{\omega}}\times\vec{\rho} \text{ vanishes} \\ \omega = \dot{\theta} \simeq n \simeq \text{const} \Rightarrow \begin{cases} \vec{\omega} = n\vec{K} \\ n = \sqrt{\dfrac{\mu}{r_1^3}} \end{cases} \end{cases}$

Relative Motion in Neighboring Orbits

$x, y \triangleq$ in-plane distances between neighboring satellites

$z \triangleq$ off-plane distances between neighboring satellites

$\vec{i} = \dfrac{\vec{r}_1}{r_1}$; $\vec{j} \triangleq$ in dir. of local horizon ; $\hat{h} = \vec{k} = \vec{i} \times \vec{j}$

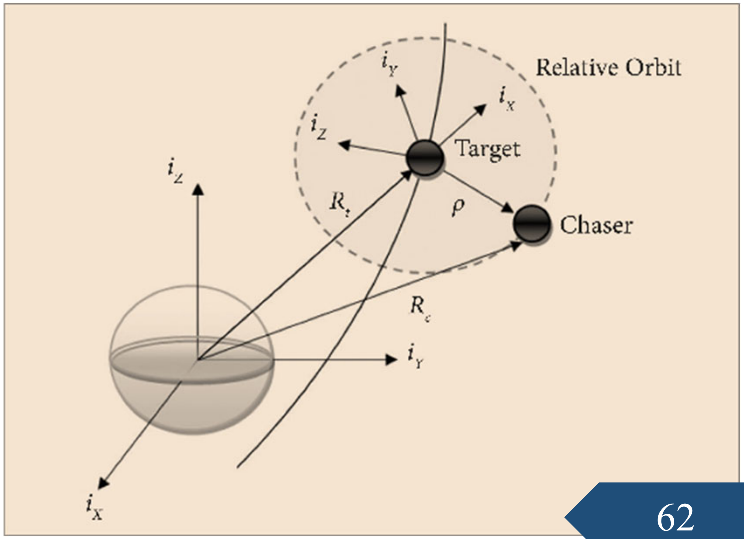


$\vec{\rho} = \begin{bmatrix} x & y & z \end{bmatrix}^{\top}$

Using above definitions and previous relations (via elimination of nonlinear terms of $O(\rho^2)$), the linearized version of the relative motion [Curtis, Kaplan] will be:

Hill's or Clohessy-Wiltshire (CW) Equations:

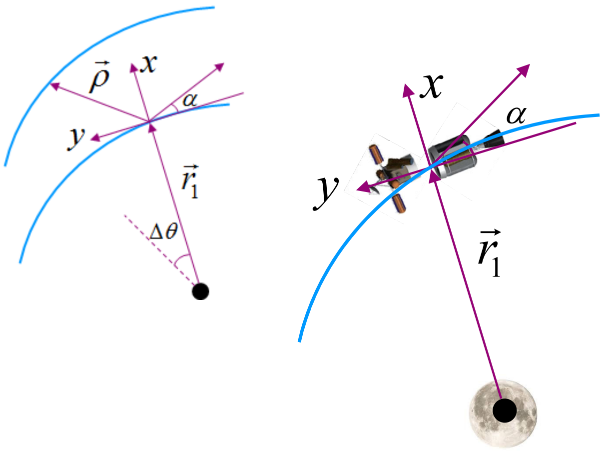
$$\begin{cases} \ddot{x} - 2n\dot{y} - 3n^2 x = f_x \\ \ddot{y} + 2n\dot{x} = f_y \\ \ddot{z} + n^2 z = f_z \end{cases} \quad ; \quad I.C.: \vec{\rho}(0) \;\&\; \dot{\vec{\rho}}(0)$$



Since y does not appear explicitly in the equation of motion, therefore its value does not need to be small. However, its approximate value based on $\vec{r}_1$ and $\Delta\theta$ is : $y \approx r_1 \Delta\theta$

Relative Motion in Neighboring Orbits

Example: Suppose a satellite is placed in its final orbit via AKM or ABM (Apogee Boost Motor), and now a separation occurs to move away the booster motor. In this regard, after initial separation, the ABM creates an initial speed opposite to the direction of movement, satellite so that $\dot{y}(0) = -V_0 cos\alpha$,where α is usually zero. This would allow for the ABM to separate from the satellite by a safe distance. Of course, perturbations or disturbances may make *α nonzero!* So that one would also have $\dot{x}(0)$. So, one goal will be to investigate the possibility of collision as a simple application of the CW EOM.



Suppose after separation, both bodies remain in the *xy* plane :

$$\dot{x}(0) \neq 0 \quad ; \quad \dot{y}(0) \neq 0 \quad ; \quad \dot{z}(0) = 0$$

$$I.C.: x(0) = y(0) = z(0) = 0$$

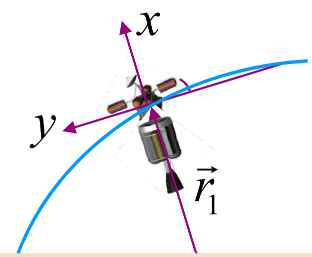
$$n = 0.001\ RPS;\ Initial\ velocities\ are\ dependent\ to\ V_0\ and\ \alpha\ .$$

$$circular\ orbit\ @\ h = 981.4\ km$$

Relative Motion in Neighboring Orbits

I) Assume ABM fires (impulsively) along the x direction, so α = 90 deg. In addition, let V_0 = 0.01 m/s. Using CW EOM, one can show that :

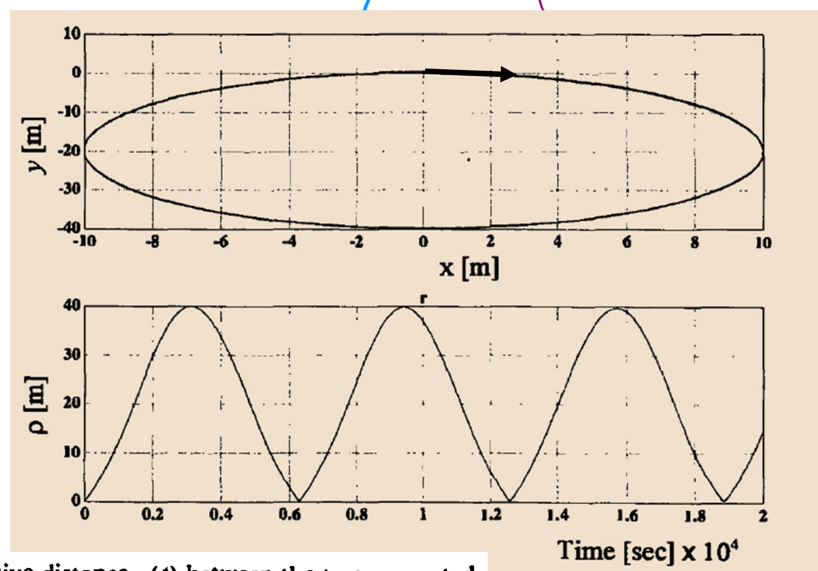
$$x(t) = V_0 \frac{\sin(nt)}{n} \quad ; \quad y(t) = \frac{2V_0}{n}\left[\cos(nt) - 1\right]$$



For a collision or possibility of collision, $\rho(t) = 0$, which must occur simultaneously for x and y:

$$\left.\begin{aligned} x(t) &= 0 \\ y(t) &= 0 \end{aligned}\right\} \Rightarrow nt = 2m\pi \quad ; \quad m = 1, 2, 3, \cdots$$

In this case, the motion of ABM around the satellite is harmonic or we could have a collision at the end of each orbital period.



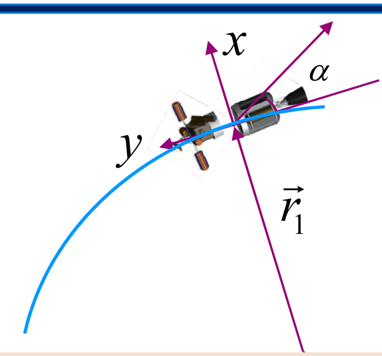
Relative distance $\rho(t)$ between the two separated bodies with initial conditions $V_0 = 0.01$ m/sec and $\alpha = 90°$.

Relative Motion in Neighboring Orbits

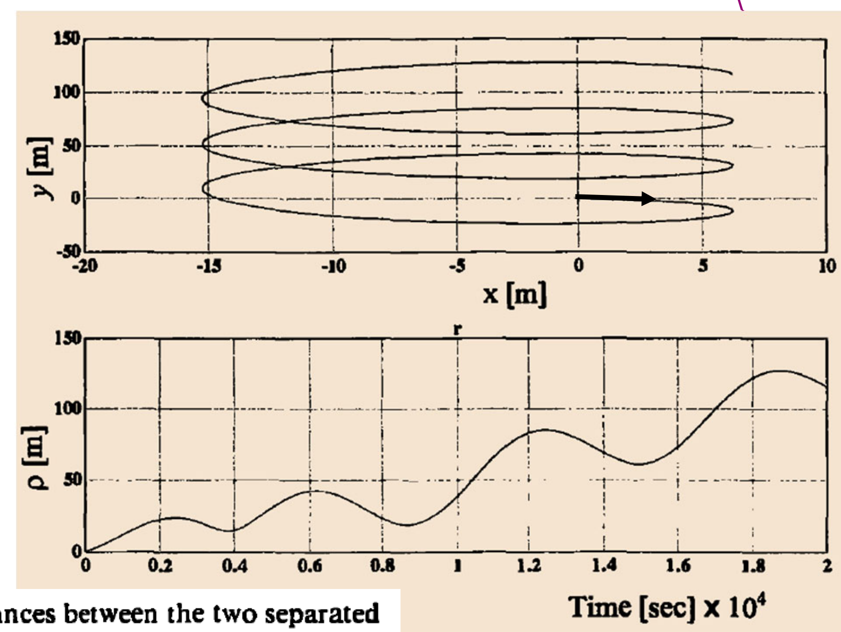
II) If $\alpha = 0$, one can similarly show that :

$$x(t) = \frac{2V_0}{n}\left[\cos(nt) - 1\right] \quad ; \quad \text{null at: } nt = 2\pi m \left(m = 1, 2, 3, \ldots\right)$$

$$y(t) = \frac{V_0}{n}\left[-4\sin(nt) + 3(nt)\right] \quad ; \quad \text{not null at } n = 2\pi m$$



In this condition, nulling occurs only at $nt = 0$ (trivial solution) and therefore, no collision occurs.



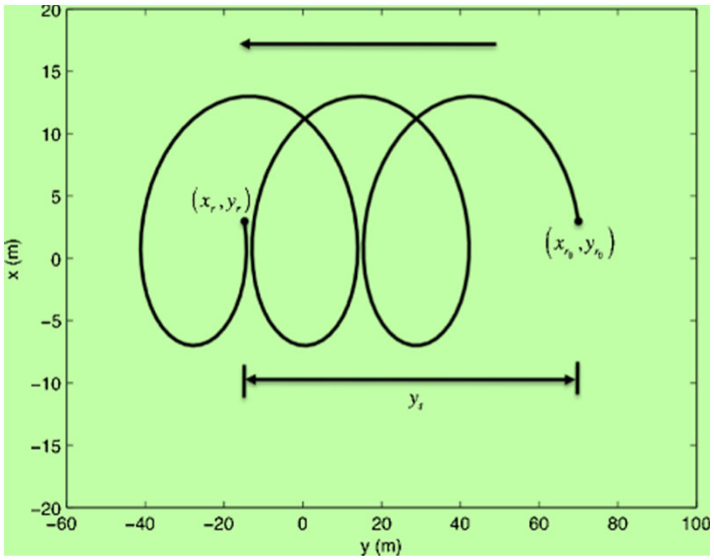
Relative distances between the two separated bodies with initial conditions $V_0 = 0.01$ m/sec and $\alpha = 77°$.

Relative Motion in Neighboring Orbits

All in all, in practice, usually in certain range of α and initial velocity different relative paths are calculated, and the possibility of collision is checked in order to make an intelligent choice in the implementation.

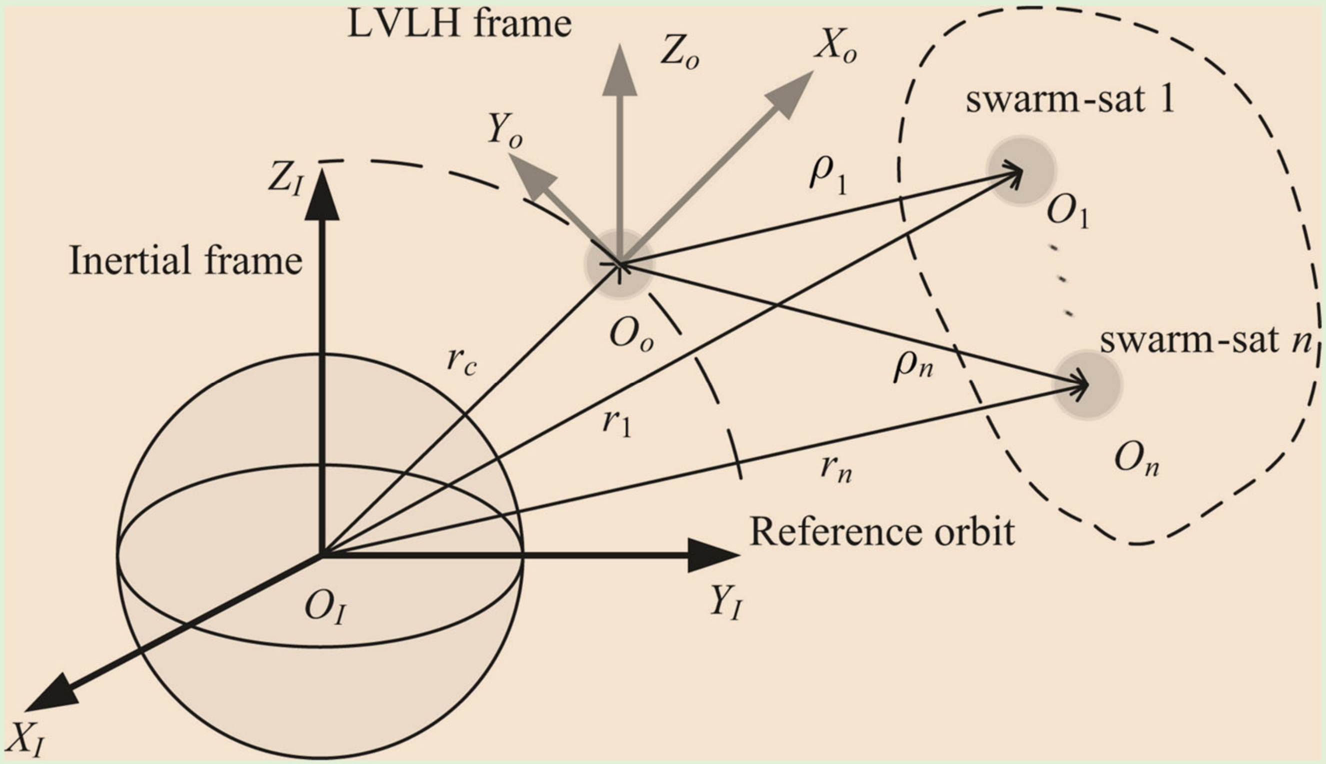
It should be noted that the calculations in this problem will be somewhat conservative, because the profile of the two objects and therefore their drag and perturbative forces have not been considered, which will increase the relative distance.

Some research work regarding CW EOM derivations and Relative motion are placed in the course EDU System for the interested readers.



Drift of chaser with respect to target figure

Relative Motion Concluded



Schematic diagram of the frames and the satellite relative positions