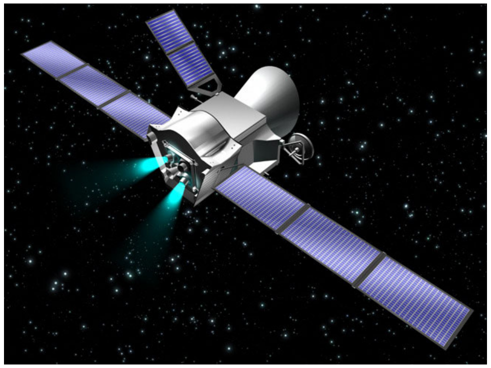
In The Name of God



AE 45780: Spacecraft Dynamics and Control

Fall 1401

3- The Three Body Problem

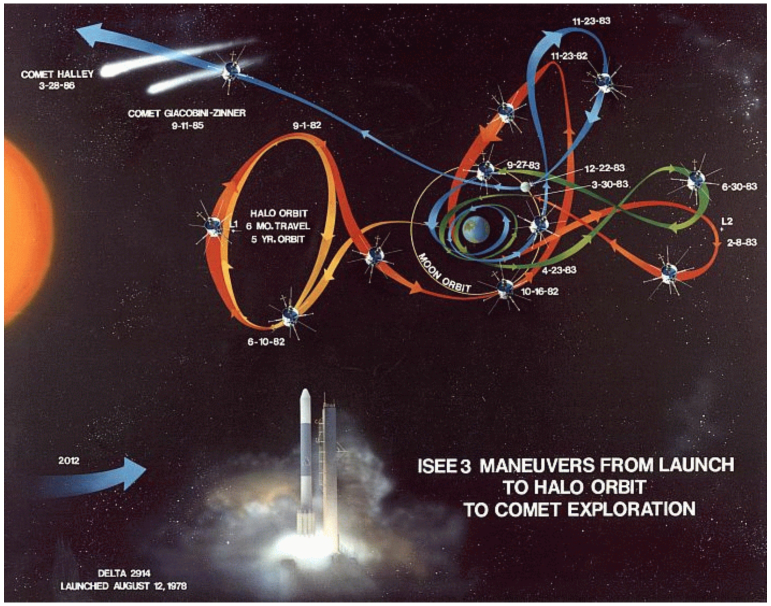
Seid H. Pourtakdoust

Introduction to 3BP

One of the byproducts of the R3BP investigation is related to determination of the equilibrium points of this system. These locations denoted by **Lagrange points (LP)** are and have been of interest in several space missions. Since some of these equilibrium points are stable, it is possible to develop periodic motions around them with small amount of energy for station keeping. LPs can be utilized to establish bases for the Sun, Earth or Moon observation.

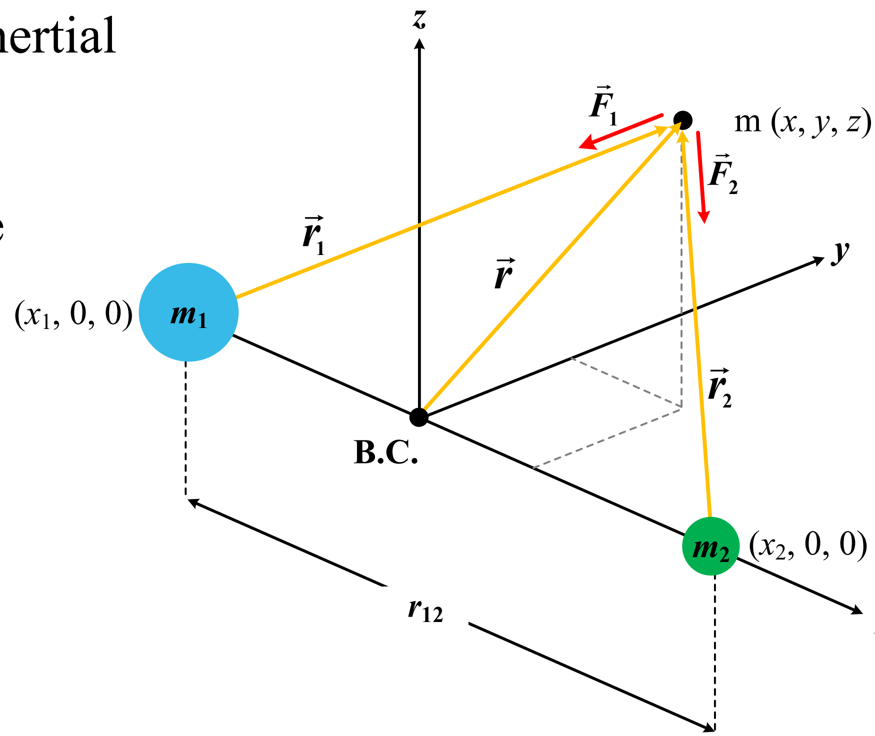
There are several Sun- Earth LP related missions:

- International Sun-Earth Explorer 3 (ISSE3), mission to L1, Aug. 1978.
  - ISEE-3 was the first spacecraft to be placed in a **halo** orbit at the L1 Earth-Sun Lagrange point.
- Solar and Heliocentric Observatory (SOHO), mission to L1, Dec. 1995.
- Advanced Composition Explorer (ACE), mission to L1, Aug. 1997.
- Genesis (The first book of the Christian bible), mission to L1, Aug. 2001.



CRTBP Equations of Motion (EOM)

Consider two main masses m_1 and m_2, which have circular Keplerian orbital motion (under their mutual gravitation) around their center of mass (barycenter or BC). We will investigate the spacecraft motion in a rotating non-inertial coordinate frame centered at the BC. The frame rotates about z axis, so that m_1 and m_2 appear to be **stationary** from a fixed observer point of view in the rotating xy plane. Thus we have:



$$T = 2\pi \frac{r_{12}^{3}}{\mu^{\frac{1}{2}}} \quad ; \quad \mu = G\left(m_1 + m_2\right) \approx GM$$

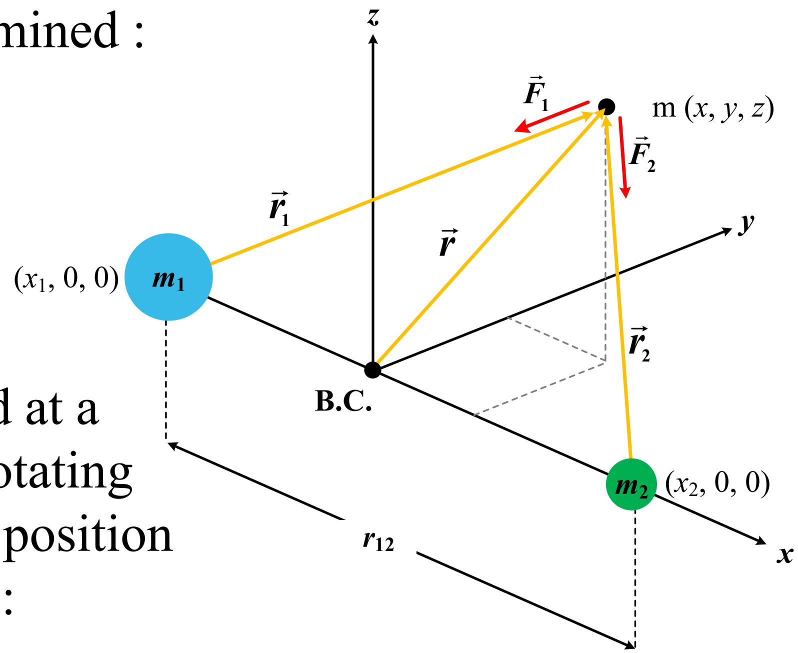
$$m_1 \triangleq \textit{larger of the two primary bodies}$$

$$\omega = \frac{2\pi}{T} \quad ; \quad \vec{\omega} = \omega\vec{k} \quad ; \quad \omega = \sqrt{\frac{\mu}{r_{12}^{3}}} \triangleq n$$

CRTBP Equations of Motion

Since m_1 and m_2 are in the xy plane, their y and z coordinate components will always be zero in the rotating frame. According to the definition of the center of mass, the position of the two main bodies on the x axis can be determined :

$$\begin{cases} m_1 x_1 + m_2 x_2 = 0 \\ x_2 - x_1 = r_{12} \end{cases} \Rightarrow \begin{cases} x_1 = -\pi_2 r_{12} \\ x_2 = \pi_1 r_{12} \end{cases} \quad ; \quad \begin{cases} \pi_1 = \dfrac{m_1}{m_1 + m_2} \\ \pi_2 = \dfrac{m_2}{m_1 + m_2} \end{cases}$$



Now we can introduce the spacecraft (SC) mass, m located at a distance r from the BC whose motion is of interest in the rotating frame where the two primary bodies seem to be fixed. The position vectors as well as the SC position in the rotating frame are :

$$\vec{r}_1 = (x - x_1)\vec{i} + y\vec{j} + z\vec{k} = (x + \pi_2 r_{12})\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{r}_2 = (x - \pi_1 r_{12})\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

CRTBP Equations of Motion

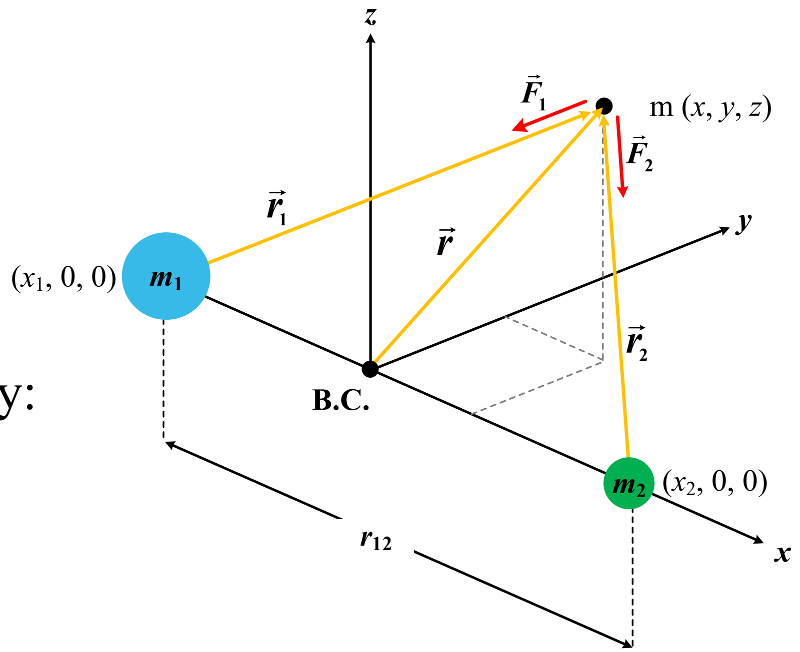
Subsequently, the inertial velocity of the third body (SC) can be determined via utility of the Coriolis law:

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\dot{\vec{r}} = \dot{\vec{r}}_b + \dot{\vec{r}}_{BC} + \vec{\omega} \times \vec{r}_b$$

$$\dot{\vec{r}}_b = \dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k} = \vec{v}_{rel}\left(relative\ velocity\right)$$

$$\dot{\vec{r}}_{BC} = \vec{V}_{BC} \triangleq inertial\ velocity\ vector\ of\ Barycenter$$



Similarly, one can find the inertial acceleration of third body:

$$\ddot{\vec{r}}_i = \ddot{\vec{r}}_{BC} + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}\right) + 2\vec{\omega} \times \dot{\vec{r}}_b + \ddot{\vec{r}}_b$$

$$\ddot{\vec{r}}_b = \vec{a}_{rel} = \ddot{x}\vec{i} + \ddot{y}\vec{j} + \ddot{z}\vec{k}$$

$$Now\ since\ \ddot{\vec{r}}_{BC} = 0, and\ \dot{\vec{\omega}} = 0 \Rightarrow \ddot{\vec{r}}_i = \vec{\omega} \times \left(\vec{\omega} \times \vec{r}\right) + 2\vec{\omega} \times \dot{\vec{r}}_b + \vec{a}_{rel}$$

So, we can have the SC EOM expressed in the body frame:

$$\ddot{\vec{r}} = \left(\ddot{x} - 2\omega\dot{y} - \omega^2 x\right)\vec{i} + \left(\ddot{y} + 2\omega\dot{x} - \omega^2 y\right)\vec{j} + \ddot{z}\vec{k}$$

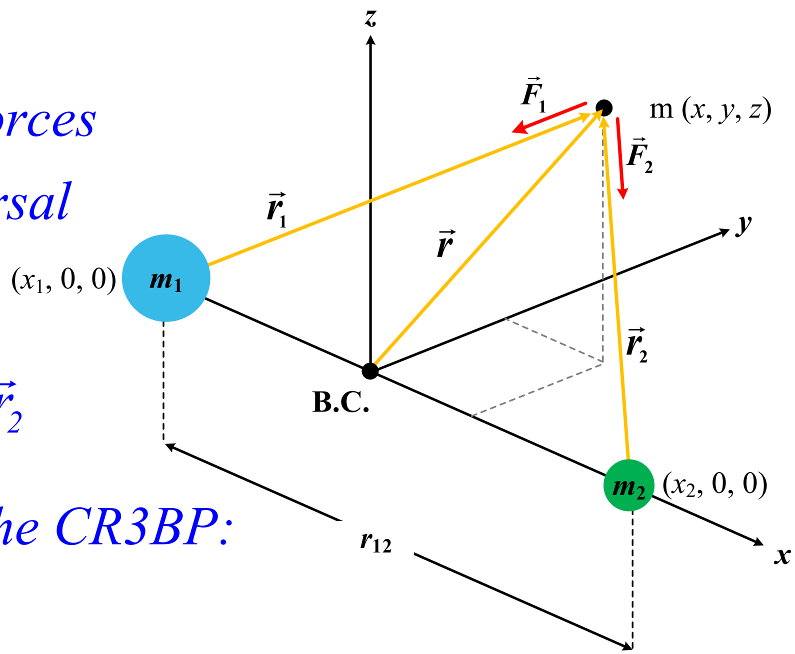
CRTBP Equations of Motion (EOM)

$$\ddot{\vec{r}} = \left(\ddot{x} - 2\omega\dot{y} - \omega^2 x\right)\vec{i} + \left(\ddot{y} + 2\omega\dot{x} - \omega^2 y\right)\vec{j} + \ddot{z}\vec{k}$$

In other words, we now have an expression for the inertial acceleration of the SC based on quantities measured in the rotating frame. According to the Newton's 2nd laws, the SC EOM will now be as follows in the body (rotating) frame:

$$m\ddot{\vec{r}} = \vec{F}_T = \vec{F}_1 + \vec{F}_2$$

Where, in this equation, $\vec{F}_1$ and $\vec{F}_2$ are the attractive forces applied to m by m_1 and m_2, which are known via universal law of gravitation.



$$\vec{F}_1 = -\frac{Gm_1m}{r_1^2}\vec{u}_{\vec{r}_1} = -\frac{\mu_1 m}{r_1^3}\vec{r}_1 \quad ; \quad \vec{F}_2 = -\frac{Gm_2m}{r_2^2}\vec{u}_{\vec{r}_2} = -\frac{\mu_2 m}{r_2^3}\vec{r}_2$$

So, we finally have the desired form of the SC EOM for the CR3BP:

$$\ddot{\vec{r}} = -\frac{\mu_1}{r_1^3}\vec{r}_1 - \frac{\mu_2}{r_2^3}\vec{r}_2$$

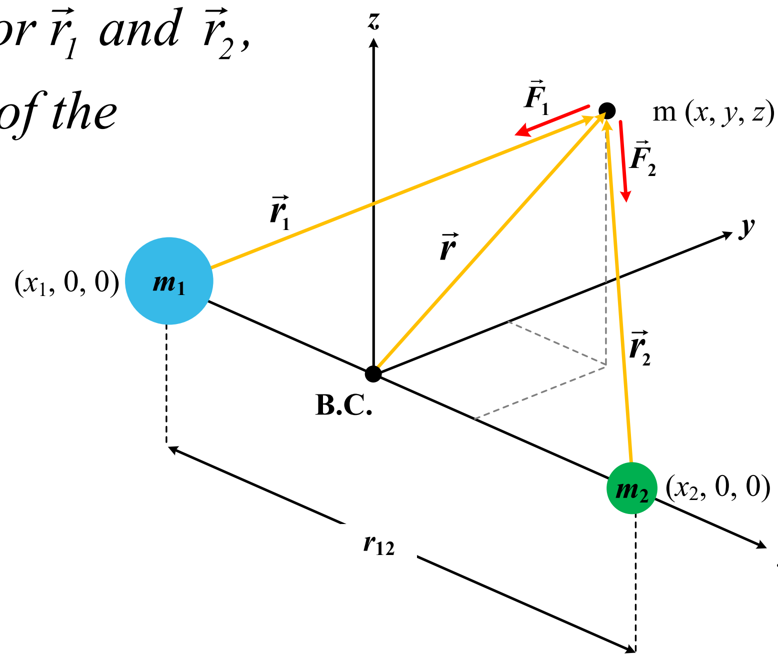
CRTBP Equations of Motion

SC EOM in the CR3BP :

$$\ddot{\vec{r}} = -\frac{\mu_1}{r_1^3}\vec{r}_1 - \frac{\mu_2}{r_2^3}\vec{r}_2$$

As the above EOM is in vector form, by sunstituting for $\vec{r}_1$ and $\vec{r}_2$, one can decompose it along each of the xyx direction of the moving frame , to get its equivalent scalar form :

$$\begin{cases} \ddot{x} - 2\omega\dot{y} - \omega^2 x = -\frac{\mu_1}{r_1^3}\left(x + \pi_2 r_{12}\right) - \frac{\mu_2}{r_2^3}\left(x - \pi_1 r_{12}\right) \\ \ddot{y} + 2\omega\dot{x} - \omega^2 y = -\frac{\mu_1}{r_1^3} y - \frac{\mu_2}{r_2^3} y \\ \ddot{z} = -\frac{\mu_1}{r_1^3} z - \frac{\mu_2}{r_2^3} z \end{cases}$$



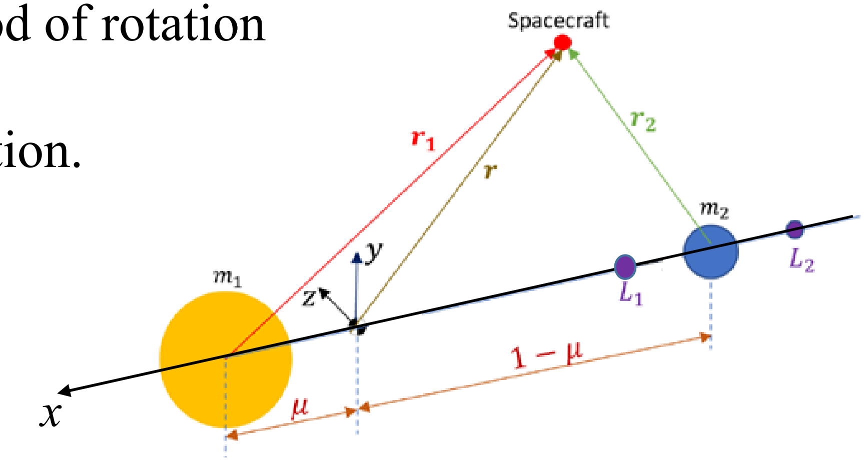
CRTBP Equations of Motion (Alternate Formulation)

There are alternative formulations for the CRTBP in some references that use Canonical units for simplification, and to have EOM in a **non-dimensional** form. In this sense:

i. The unit length is taken as the distance between the two primaries, $r_{12} = a_{12} = 1$ D.U.

ii. The unit mass unit is taken as the sum of the two primary masses (assuming $m_2 = \mu$) $m_1 + m_2 = 1$ M.U.

iii. The time unit is chosen in such a way that the angular velocity $\omega = 2\pi/\tau$ (of the primaries around the Barycenter) is equal to 1 per unit of time. In other words, the period of rotation of the two main bodies in their orbits around BC, will be equal to 2π *T.U* where $G=1$ in the Kepler equation.



$$\mu = mass\ ratio = \frac{m_2}{m_1 + m_2} = m_2$$

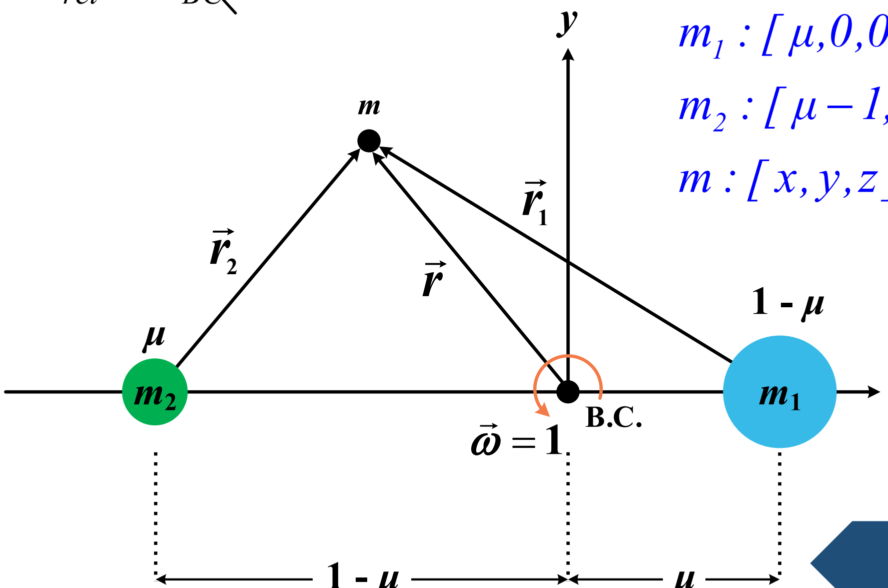
$$\tau = 2\pi \frac{a_{12}^{3/2}}{G\left(m_1 + m_2\right)^{1/2}} = 2\pi\ T.U.$$

CRTBP Equations of Motion (Alternate Formulation)

Based on the previous definitions, the alternative formulation still considers the BC at the center of the rotating frame. Looking from the top, the larger mass is now placed on the positive part of the *x*-axis. Based on this geometry, we can develop EOM in body frame as :

$$\begin{cases} m\vec{a}_g = \vec{F}_1 + \vec{F}_2 = -Gm\left[\dfrac{m_1}{r_1^3}\vec{r}_1 + \dfrac{m_2}{r_2^3}\vec{r}_2\right];\ \vec{r}_b = \vec{r}_{rel} = x\vec{i} + y\vec{j} + z\vec{k} \\ \vec{a}_g = \ddot{\vec{r}} = \ddot{\vec{r}}_i = \ddot{\vec{r}}_{rel} + \cancel{\dot{\vec{\omega}}} \times \vec{r}_{rel} + \vec{\omega}\times\left(\vec{\omega}\times\vec{r}_{rel}\right) + 2\vec{\omega}\times\dot{\vec{r}}_{rel} + \cancel{\ddot{\vec{r}}_{BC}} \end{cases} \Rightarrow \vec{a}_g = -\frac{(1-\mu)}{r_1^3}\vec{r}_1 - \frac{\mu}{r_2^3}\vec{r}_2$$

$$\begin{cases} \ddot{x} - 2\dot{y} - x = -\dfrac{1-\mu}{r_1^3}(x-\mu) - \dfrac{\mu}{r_2^3}(x+1-\mu) \\ \ddot{y} + 2\dot{x} - y = -\dfrac{1-\mu}{r_1^3}y - \dfrac{\mu}{r_2^3}y \\ \ddot{z} = -\dfrac{1-\mu}{r_1^3}z - \dfrac{\mu}{r_2^3}z \end{cases}$$



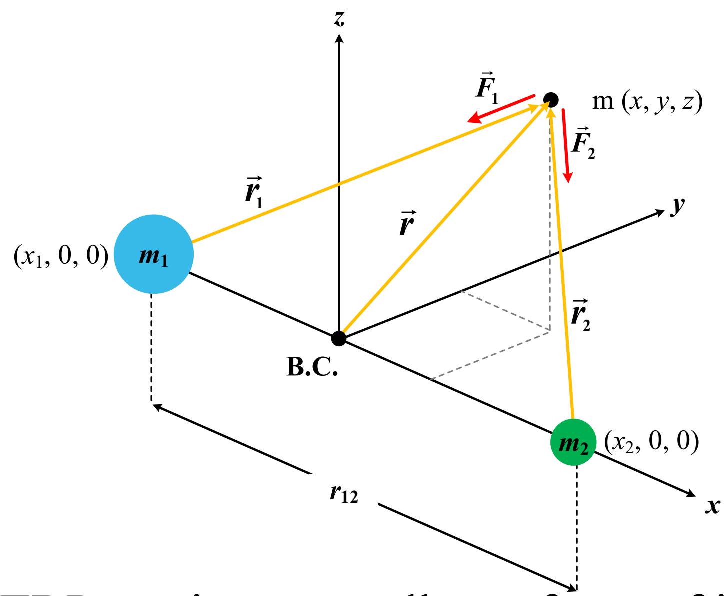
$m_1 : [\,\mu, 0, 0\,];$

$m_2 : [\,\mu - 1, 0, 0\,]$

$m : [\,x, y, z\,]$

CRTBP Equations of Motion

$$\begin{cases} \ddot{x} - 2\omega\dot{y} - \omega^2 x = -\dfrac{\mu_1}{r_1^3}\left(x + \pi_2 r_{12}\right) - \dfrac{\mu_2}{r_2^3}\left(x - \pi_1 r_{12}\right) \\ \ddot{y} + 2\omega\dot{x} - \omega^2 y = -\dfrac{\mu_1}{r_1^3} y - \dfrac{\mu_2}{r_2^3} y \\ \ddot{z} = -\dfrac{\mu_1}{r_1^3} z - \dfrac{\mu_2}{r_2^3} z \end{cases}$$

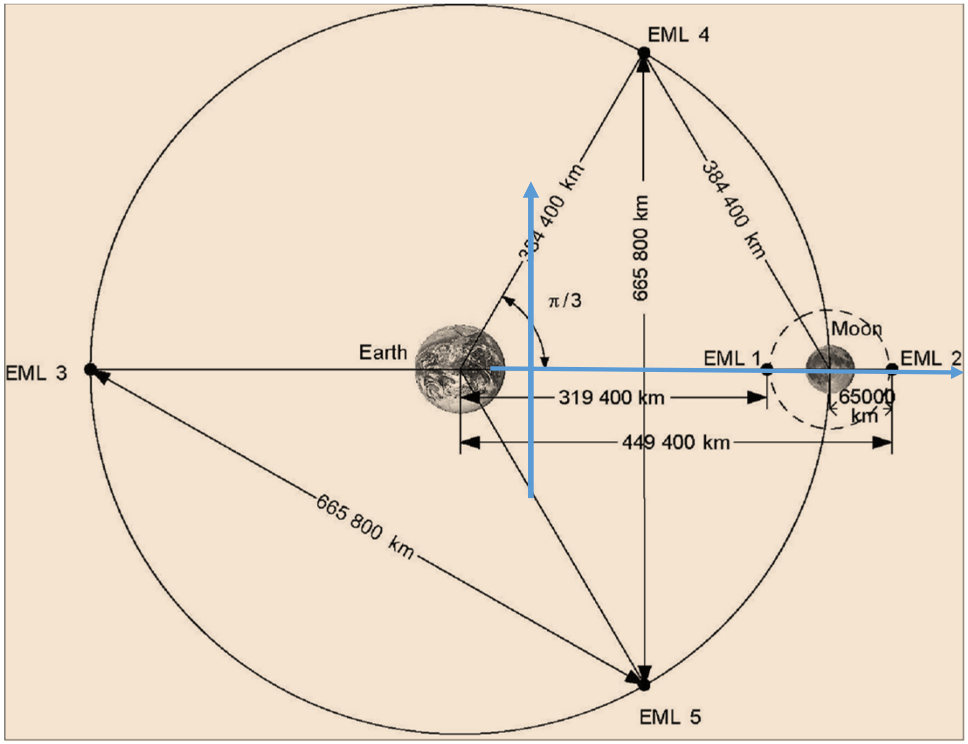


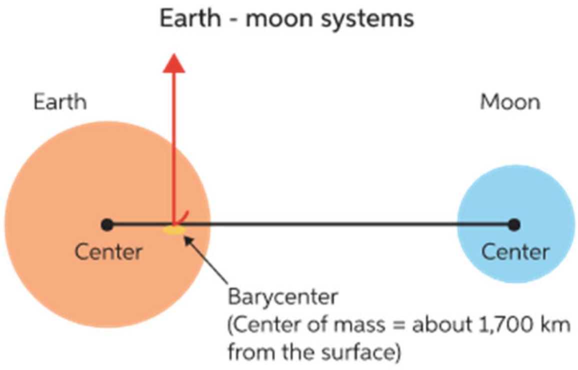
The derived differential equations obtained for the CRTBP motion, regardless of any of its alternatives and 3B systems considered, are usually nonlinear and coupled. Although its order is the same as the 2BP, **its analytical solution has not been determined** over the past 200 years. In other words, there are **no general closed-form solution** to the 3BP that can be expressed in terms of a finite number of standard mathematical operations. Of course, this does not mean that we have not gained anything about the solution and behavior of the third body problem.

Digress : Lagrangian (or Liberation) Points as Equilibrium Points

For Earth-Moon system:

$$\begin{cases} L1 : \xi = 0.8369 \Rightarrow x = 3.217 \times 10^5 \ km \\ L2 : \xi = 1.156 \Rightarrow x = 4.444 \times 10^5 \ km \\ L3 : \xi = -1.005 \Rightarrow x = -3.863 \times 10^5 \ km \\ L4 : x = 1.875 \times 10^5 \ km; \ y = 3.329 \times 10^5 \ km \\ L5 : x = 1.875 \times 10^5 \ km; \ y = -3.329 \times 10^5 \ km \end{cases}$$





Moon Radius : 1,737.4 km

Lagrange points, are attributed to **Euler** and **Lagrange** who discovered them, are some of the most interesting points in the 3-body problem. These points are distinguished by the fact that their time derivatives of position are zero in the rotating reference frame. That means velocity and acceleration are both zero.

Towards derivation of the Lagrangian Points

One of the appropriate issues, when dealing with complex dynamics such as the 3BP system, is the question of equilibrium points, i.e points where the third body (in the rotating frame RF) has zero velocity and acceleration. That is to say, if one can manage to place an object or SC at these points, they will be stationary with respect to the primaries in the RF. While they are in fact rotating in the inertial frame.

To determine the equilibrium conditions, velocity and acceleration are set to zero in the 3BP EOM to obtain the coordinates of theses stationary points.

$$Let : \dot{x} = \dot{y} = \dot{z} = 0 \quad ; \quad \ddot{x} = \ddot{y} = \ddot{z} = 0$$

$$\begin{cases} \ddot{x} - 2\omega\dot{y} - \omega^2 x = -\dfrac{\mu_1}{r_1^3}\left(x + \pi_2 r_{12}\right) - \dfrac{\mu_2}{r_2^3}\left(x - \pi_1 r_{12}\right) \\ \ddot{y} + 2\omega\dot{x} - \omega^2 y = -\dfrac{\mu_1}{r_1^3} y - \dfrac{\mu_2}{r_2^3} y \\ \ddot{z} = -\dfrac{\mu_1}{r_1^3} z - \dfrac{\mu_2}{r_2^3} z \end{cases} \Rightarrow \begin{cases} -\omega^2 x = -\dfrac{\mu_1}{r_1^3}\left(x + \pi_2 r_{12}\right) - \dfrac{\mu_2}{r_2^3}\left(x - \pi_1 r_{12}\right) \\ -\omega^2 y = -\dfrac{\mu_1}{r_1^3} y - \dfrac{\mu_2}{r_2^3} y \\ 0 = -\dfrac{\mu_1}{r_1^3} z - \dfrac{\mu_2}{r_2^3} z \end{cases}$$

Towards derivation of the Lagrangian Points

$$\begin{cases} -\omega^2 x = -\frac{\mu_1}{r_1^3}\left(x+\pi_2 r_{12}\right) - \frac{\mu_2}{r_2^3}\left(x-\pi_1 r_{12}\right) \\ -\omega^2 y = -\frac{\mu_1}{r_1^3} y - \frac{\mu_2}{r_2^3} y \\ 0 = -\frac{\mu_1}{r_1^3} z - \frac{\mu_2}{r_2^3} z \end{cases}$$

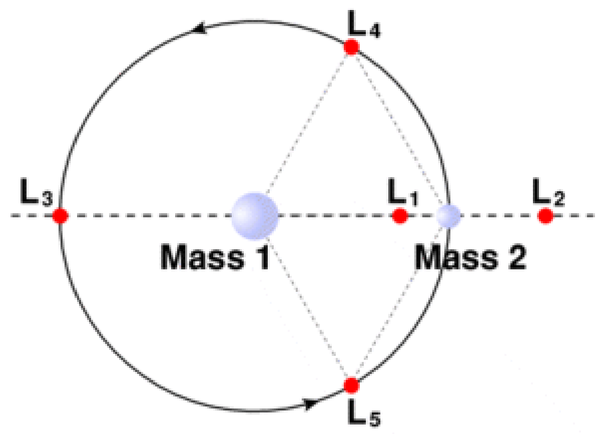
According to the third equation, and the positivity of both μ_i/r_i^3 terms, **the equilibrium points must be in the same orbital plane as the primaries** or $z = 0$.
To obtain the x, y coordinates, the remaining x *and* y equations and definitions are utilized.

$$we\ defined:\ \pi_1 = \frac{m_1}{m_1+m_2} \quad ; \quad \pi_2 = \frac{m_2}{m_1+m_2} \Rightarrow \pi_1 = 1-\pi_2$$

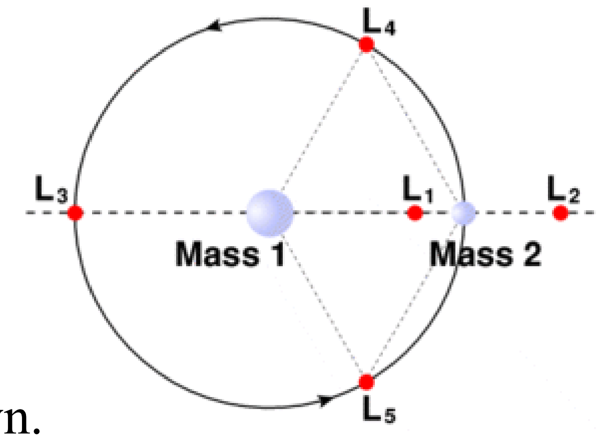
$$in\ addition: \xrightarrow{\pi_1 \times \frac{G}{G}} \pi_1 = \frac{\mu_1}{\mu} \quad ; \xrightarrow{\pi_2 \times \frac{G}{G}} \quad \pi_2 = \frac{\mu_2}{\mu};\quad \omega^2 = \frac{\mu}{r_{12}^3},$$

$$we\ can\ rewrite\ x,y\ equations\ as\ \left(assuming\ y \neq 0\right):$$

$$\begin{cases} \frac{x}{r_{12}^3} = \left(1-\pi_2\right)\left(x+\pi_2 r_{12}\right)\frac{1}{r_1^3} + \pi_2\left(x+\pi_2 r_{12} - r_{12}\right)\frac{1}{r_2^3} \\ \frac{1}{r_{12}^3} = \left(1-\pi_2\right)\frac{1}{r_1^3} + \pi_2\frac{1}{r_2^3} \end{cases}$$

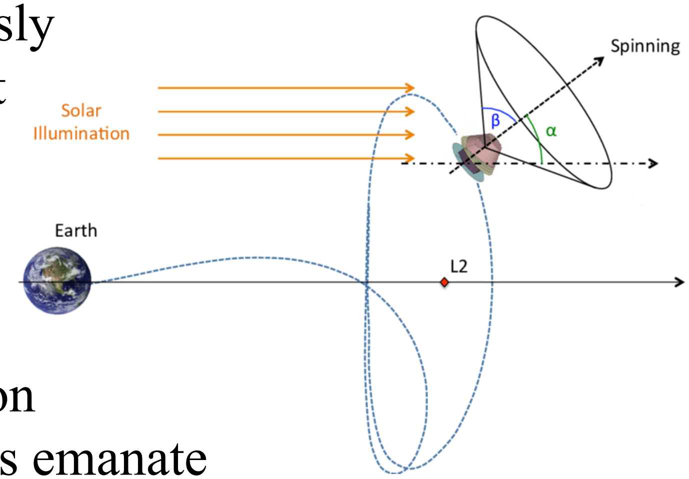


Towards derivation of the Lagrangian Points



$$\begin{cases} \dfrac{x}{r_{12}^3} = (1-\pi_2)(x+\pi_2 r_{12})\dfrac{1}{r_1^3} + \pi_2(x+\pi_2 r_{12} - r_{12})\dfrac{1}{r_2^3} \\ \dfrac{1}{r_{12}^3} = (1-\pi_2)\dfrac{1}{r_1^3} + \pi_2\dfrac{1}{r_2^3} \end{cases}$$

The resulting equations are of similar pattern, while r_{12} is known. So one should seek solution for x,y coordinates that simultaneously satisfy the two above relations (two linear equations with respect to $1/r_1^3$ and $1/r_2^3$. One such conditions occur when:



$$\frac{1}{r_{12}^3} = \frac{1}{r_1^3} = \frac{1}{r_2^3} \quad \text{or when } r_1 = r_2 = r_{12}$$

The latter can also be reached, by multiplying the second equation with x and adding it to the first one. Thus, two equilibrium points emanate from this conclusion that are located at the vertices of two equilateral triangles, which are called L4 and L5 in honor of Lagrange. Euler later discovered three other equilibrium points, which lie on the x-axis, and are called **collinear equilibrium** points.

Derivation of the Lagrangian Points

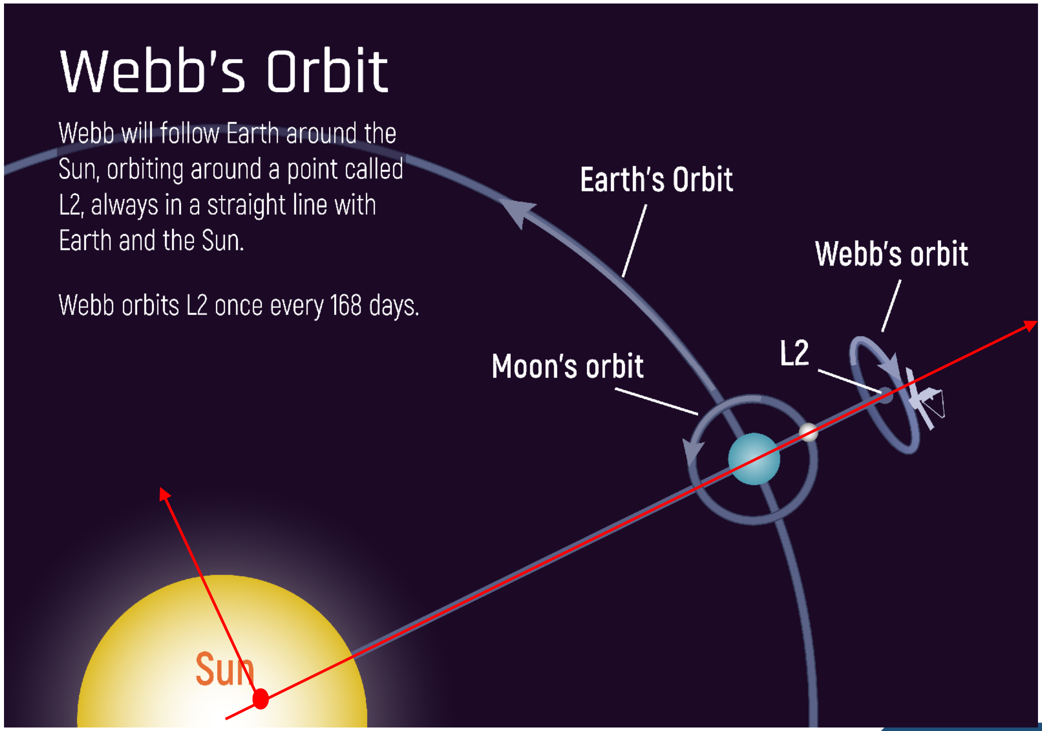
Now, according to the previous equations and the recent intuitive result ($r_1 = r_2 = r_{12}$), and according to the fact that the z-coordinate of the **triangular** Lagrange points are zero with the fact that $\pi_1 = 1 - \pi_2$, the (x, y)coordinate of **L4 and L5** will be determined :

$$\begin{cases} r_1^2 = r_{12}^2 = \left(x + \pi_2 r_{12}\right)^2 + y^2 \\ r_2^2 = r_{12}^2 = \left(x + \pi_2 r_{12} - r_{12}\right)^2 + y^2 \end{cases}$$

First equate the two equations to find the x coordinate and subsequently find the y coordinate via substitution, as :

L_4, L_5 Coordination:

$$x = \frac{r_{12}}{2} - \pi_2 r_{12} \quad ; \quad y = \pm\frac{\sqrt{3}}{2} r_{12} \quad ; \quad z = 0$$



Other Collinear Liberation Points

The other three equilibrium points were discovered by Euler in 1765, about 7 years before Lagrange found the L4 and L5 Liberation point. The former are obtained by simultaneous consideration of $y = z = 0$ on the EOMs that satisfy the last two equations. This will update :

$$\begin{cases} -\omega^2 x = -\dfrac{\mu_1}{r_1^3}\left(x+\pi_2 r_{12}\right) - \dfrac{\mu_2}{r_2^3}\left(x-\pi_1 r_{12}\right) \\ -\omega^2 y = -\dfrac{\mu_1}{r_1^3} y - \dfrac{\mu_2}{r_2^3} y \\ 0 = -\dfrac{\mu_1}{r_1^3} z - \dfrac{\mu_2}{r_2^3} z \end{cases}$$

$$r_1 = \left|x+\pi_2 r_{12}\right| \quad ; \quad r_2 = \left|x+\pi_2 r_{12} - r_{12}\right|$$

That will help to find the collinear coordinates, by substituting all information into the x part of the EOM and simplifying.

$$\omega^2 = \frac{\mu}{r_{12}^3} \quad ; \quad \pi_1 = 1-\pi_2 \quad ; \quad \mu_1 = \pi_1\mu \quad ; \quad \mu_2 = \pi_2\mu$$

$$\Downarrow$$

$$\frac{1-\pi_2}{\left|x+\pi_2 r_{12}\right|^3}\left(x+\pi_2 r_{12}\right) + \frac{\pi_2}{\left|x+\pi_2 r_{12}-r_{12}\right|^3}\left(x+\pi_2 r_{12}-r_{12}\right) - \frac{1}{r_{12}^3}\left(x\right) = 0$$

Other Collinear Liberation Points

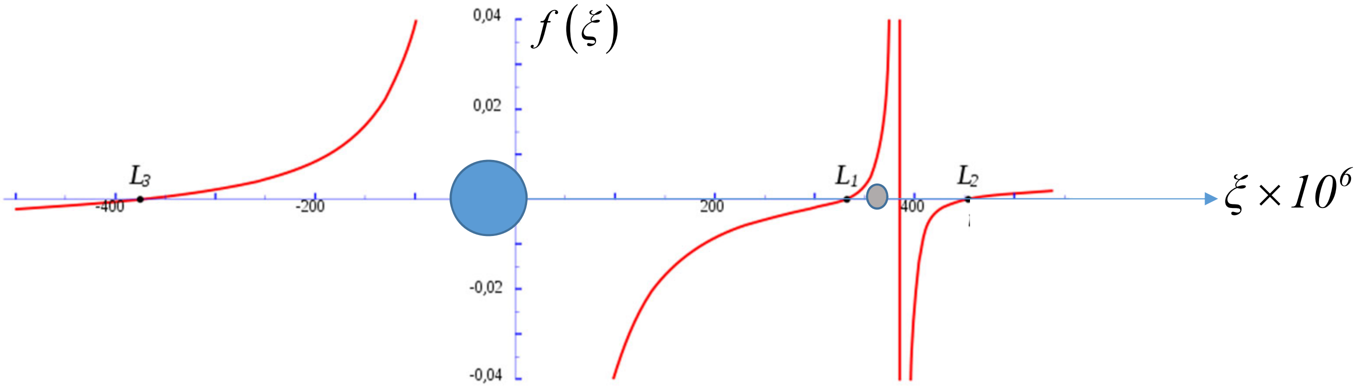
Defining new dimensionless variable, ξ, such that: $\xi = x/r_{12}$:the above equation becomes:

$$\frac{1-\pi_2}{\left|x+\pi_2 r_{12}\right|^3}\left(x+\pi_2 r_{12}\right)+\frac{\pi_2}{\left|x+\pi_2 r_{12}-r_{12}\right|^3}\left(x+\pi_2 r_{12}-r_{12}\right)-\frac{1}{r_{12}^3}\left(x\right)=0$$

Of course, the above should be numerically solved (e.g. Newton's method) for any R3B system. For example, the collinear Lagrangian points for the Earth-Moon system are:

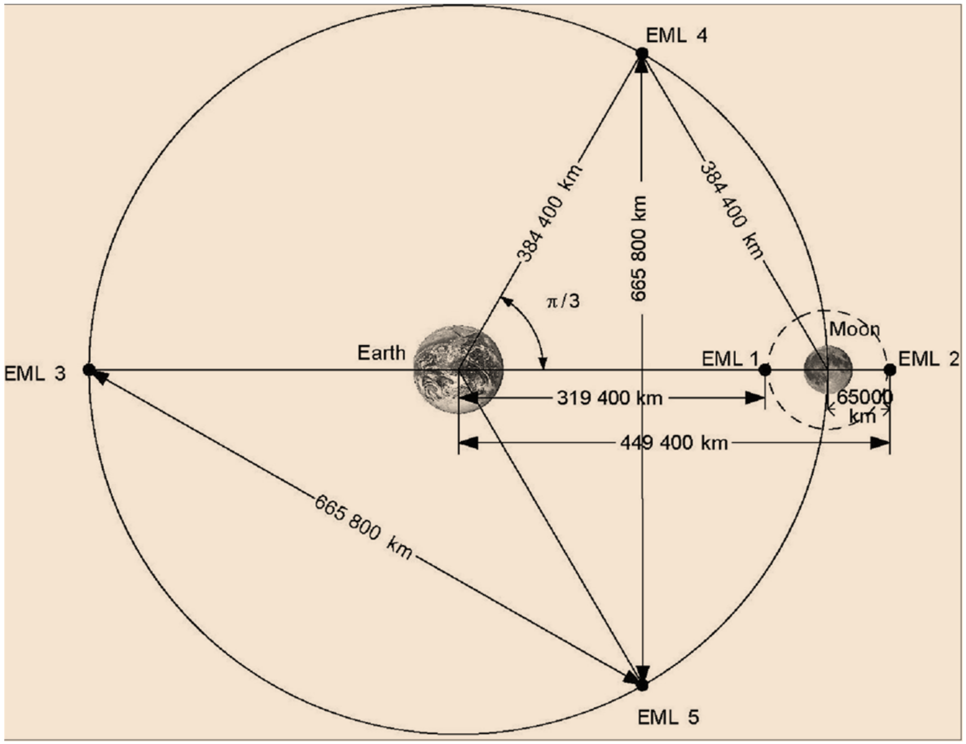
$$f\left(\xi\right)=0\Rightarrow\frac{1-\pi_2}{\left|\xi+\pi_2\right|^3}\left(\xi+\pi_2\right)+\frac{\pi_2}{\left|\xi+\pi_2-1\right|^3}\left(\xi+\pi_2-1\right)-\xi=0$$

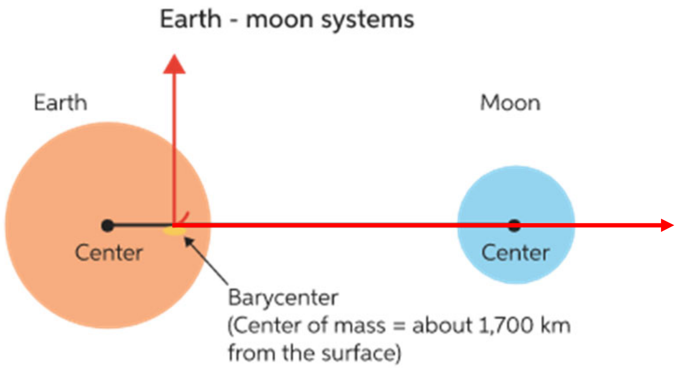
$$m_E=5.974\times10^{24}\ kg;\ m_{moon}=7.348\times10^{22}\ kg;\ r_{12}=3.844\times10^{5}\ km;\Rightarrow\pi_2=0.01215$$



Lagrangian Points for the Earth-Moon System

$$\begin{cases} L1: \xi = 0.8369 \Rightarrow x = 3.217\times 10^5\ km \\ L2: \xi = 1.156 \Rightarrow x = 4.444\times 10^5\ km \\ L3: \xi = -1.005 \Rightarrow x = -3.863\times 10^5\ km \\ L4: x = 1.875\times 10^5\ km;\ y = 3.329\times 10^5\ km \\ L5: x = 1.875\times 10^5\ km;\ y = -3.329\times 10^5\ km \end{cases}$$





Lagrangian Points (LPs) in the Alternate Formulation

By analogy, LPs in the alternate formulation can be derived, by first removing the derivatives.

$$\begin{cases} -x = -\dfrac{1-\mu}{r_1^3}(x-\mu) - \dfrac{\mu}{r_2^3}(x+1-\mu) \\ -y = -\dfrac{1-\mu}{r_1^3}y - \dfrac{\mu}{r_2^3}y \\ 0 = -\dfrac{1-\mu}{r_1^3}z - \dfrac{\mu}{r_2^3}z \end{cases}$$

1-LPs z coordinates is zero via the third equation,

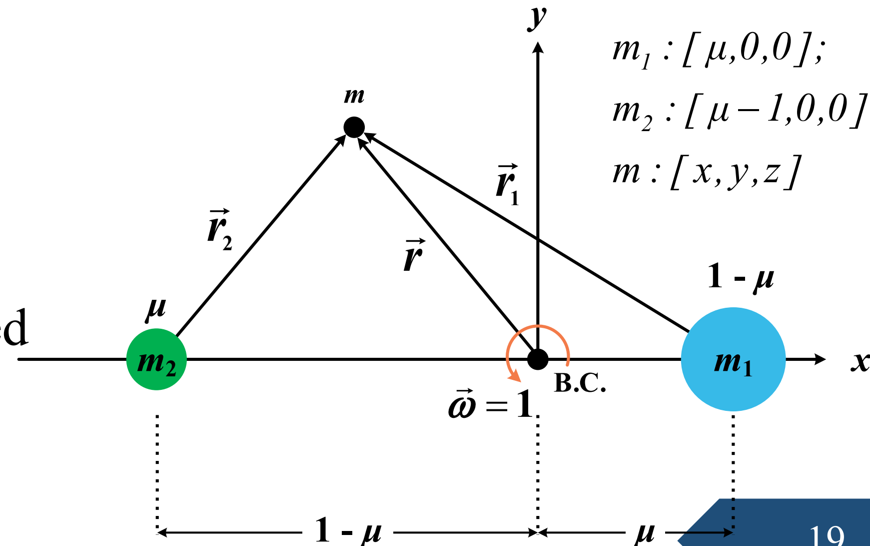
2- $r_1 = r_2 = r_{12}$ satisfies the 2nd equation for L4 and L5,

3-As y = z = 0, satisfies the last two equations, the first equations gives the collinear points by replacing for r_1 and r_2 in terms of μ or the mass ratio:

$$r_1 = \sqrt{(x-\mu)^2} \quad ; \quad r_2 = \sqrt{(x-\mu+1)^2}$$

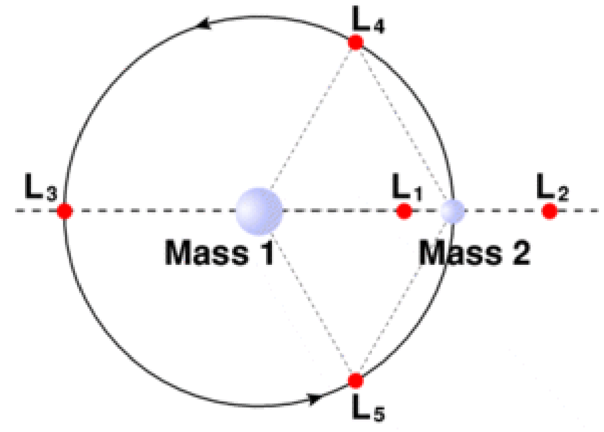
$$\Rightarrow x - \frac{1-\mu}{|x-\mu|^{3/2}}(x-\mu) - \frac{\mu}{|x+1-\mu|^{3/2}}(x+1-\mu) = 0$$

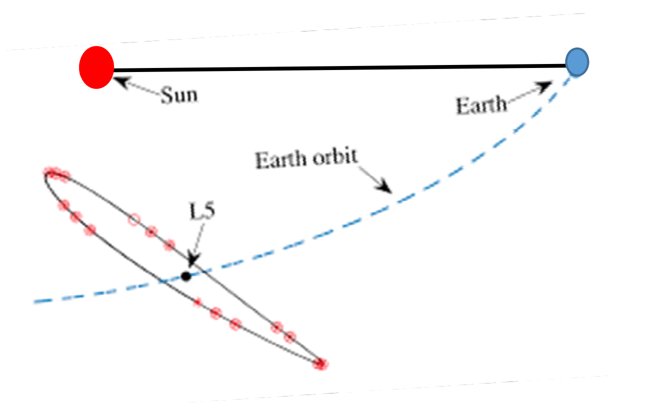
Where the last equation can be again numerically solved for the location of the collinear LPs.

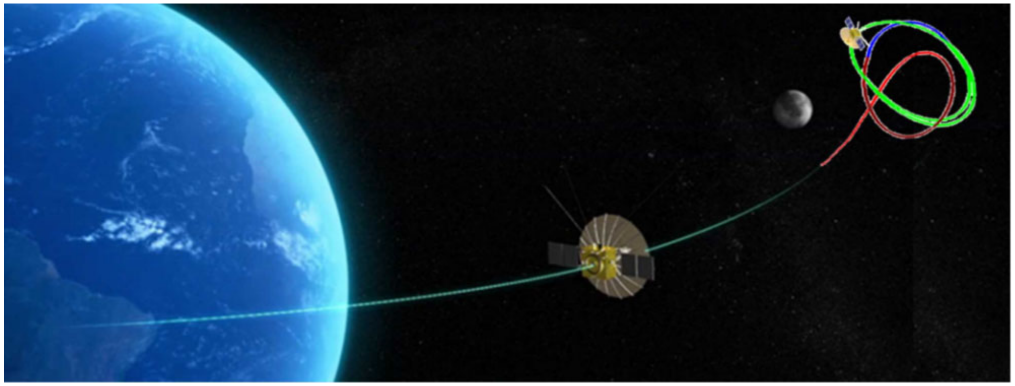


Additional Characteristics of the Lagrangian Points (LPs)

We will subsequently discuss the stability of the equilibrium points (EPs) further. Obviously, if the EPs are stable an spacecraft placed at these points will stay there as resist perturbations. It turns out that L4 and L5 are sufficiently stable and so space probes at the triangular points form closed Halo orbits without much station-keeping effort. Unfortunately, L4 and L5 EPs are usually destabilized by the Sun in the Earth-Moon system. So, usually L1 and L2 Are utilized in this system that require more station-keeping.







Halo orbit used for Lunar Relay communication at L_2

Halo orbits in the restricted three body problem (RTBP) are spatial periodic solutions formed usually around the libration points

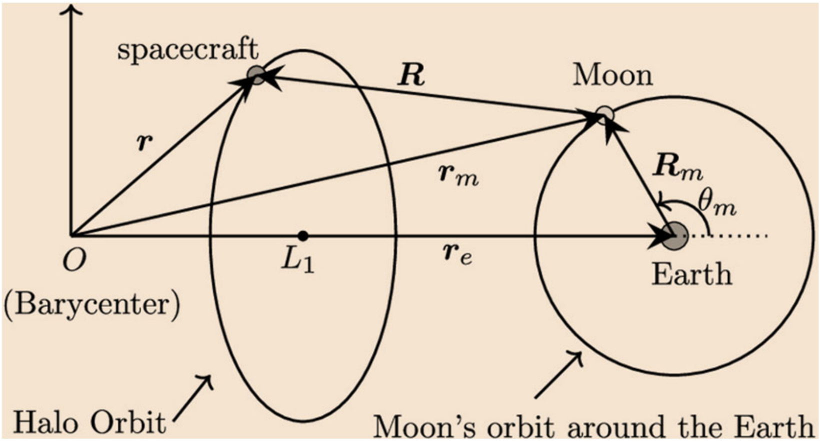
Jacobi Integral

$$\begin{cases} \ddot{x} - 2\omega\dot{y} - \omega^2 x = -\dfrac{\mu_1}{r_1^3}\left(x + \pi_2 r_{12}\right) - \dfrac{\mu_2}{r_2^3}\left(x - \pi_1 r_{12}\right) \\ \ddot{y} + 2\omega\dot{x} - \omega^2 y = -\dfrac{\mu_1}{r_1^3} y - \dfrac{\mu_2}{r_2^3} y \\ \ddot{z} = -\dfrac{\mu_1}{r_1^3} z - \dfrac{\mu_2}{r_2^3} z \end{cases}$$

Recall that we initially introduced motion constants in the 2BP, upon orbital motion could be described or identified. These quantities included *specific energy specific angular and angular momentum* as standard conservation quantities, besides the *eccentricity vector in the inertial space*.

It is of interest to know if such motion constants exists for the 3BP. Due to the complexities of 3BP, there is so far one such law derived that is known as the **Jacobi integral** or **Jacobi constant**. In this respect, a similar mathematical operation (like 2BP) is used in which the *x,y,y* EOMs are multiplied by $\dot{x}, \dot{y}, \dot{z}$ respectively and summed up. Please note that:

$$\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z} = \frac{1}{2}\frac{d}{dt}\left(\dot{x}^2 + \dot{y} + \dot{z}^2\right) = \frac{1}{2}\frac{d}{dt}\left(V^2\right)$$



Halo Orbit around L_1 Point of the Sun-Earth System

Jacobi Integral

It follows that:

$$\begin{cases} \dot{x}\times\left[\ddot{x}-2\omega\dot{y}-\omega^2 x=-\dfrac{\mu_1}{r_1^3}\left(x+\pi_2 r_{12}\right)-\dfrac{\mu_2}{r_2^3}\left(x-\pi_1 r_{12}\right)\right] \\ +\dot{y}\times\left[\ddot{y}+2\omega\dot{x}-\omega^2 y=-\dfrac{\mu_1}{r_1^3}y-\dfrac{\mu_2}{r_2^3}y\right] \\ +\dot{z}\times\left[\ddot{z}=-\dfrac{\mu_1}{r_1^3}z-\dfrac{\mu_2}{r_2^3}z\right] \end{cases}$$

$$\boxed{\begin{aligned} &\Rightarrow \dot{x}\ddot{x}+\dot{y}\ddot{y}+\dot{z}\ddot{z}-\omega^2\left(x\dot{x}+y\dot{y}\right)=-\frac{\mu_1}{r_1^3}\times A-\frac{\mu_2}{r_2^3}\times B \\ &where: \\ &A=x\dot{x}+y\dot{y}+z+\dot{z}+\pi_2 r_{12}\dot{x} \\ &B=x\dot{x}+y\dot{y}+z\dot{z}-\pi_1 r_{12}\dot{x} \end{aligned}}$$

In addition there is a relation between A and B with rate of change of distances as well. In other words,

$$\vec{r}_1=\left(x-x_1\right)\vec{i}+y\vec{j}+z\vec{k}=\left(x+\pi_2 r_{12}\right)\vec{i}+y\vec{j}+z\vec{k}$$

$$\vec{r}_2=\left(x-\pi_1 r_{12}\right)\vec{i}+y\vec{j}+z\vec{k}$$

$$\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$$

Jacobi Integral

So, one can differentiate the magnitude squared of r_1 and r_2 to get:

$$r_1^2 = \left(x + \pi_2 r_{12}\right)^2 + y^2 + z^2$$

$$taking\ time\ derivate \Rightarrow\ 2r_1 \frac{dr_1}{dt} = 2\left(x + \pi_2 r_{12}\right)\dot{x} + 2y\dot{y} + 2z\dot{z} \Rightarrow \frac{dr_1}{dt} = \frac{1}{r_1}\left(\pi_2 r_{12}\dot{x} + x\dot{x} + y\dot{y} + z\dot{z}\right)$$

$$\xrightarrow{\times -\frac{1}{r_1^2}} -\frac{1}{r_1^2}\left[\frac{dr_1}{dt} = \frac{1}{r_1}\left(\pi_2 r_{12}\dot{x} + x\dot{x} + y\dot{y} + z\dot{z}\right)\right] \Rightarrow \frac{d}{dt}\left(\frac{1}{r_1}\right) = -\frac{1}{r_1^3}\left(x\dot{x} + y\dot{y} + z\dot{z} + \pi_2 r_{12}\dot{x}\right) = -\frac{1}{r_1^3}A$$

$$similarly: \frac{d}{dt}\left(\frac{1}{r_2}\right) = -\frac{1}{r_2^3}\left(x\dot{x} + y\dot{y} + z\dot{z} - \pi_1 r_{12}\dot{x}\right) = -\frac{1}{r_2^3}B$$

$$So\ we\ had:\ \dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z} - \omega^2\left(x\dot{x} + y\dot{y}\right) = -\frac{\mu_1}{r_1^3}\times A - \frac{\mu_2}{r_2^3}\times B\ or$$

$$\Rightarrow \frac{1}{2}\frac{dV^2}{dt} - \frac{1}{2}\omega^2\frac{d}{dt}\left(x^2 + y^2\right) = \mu_1\frac{d}{dt}\left(\frac{1}{r_1}\right) + \mu_2\frac{d}{dt}\left(\frac{1}{r_2}\right) \Rightarrow \frac{d}{dt}\left[\frac{1}{2}V^2 - \frac{1}{2}\omega^2\left(x^2 + y^2\right) - \frac{\mu_1}{r_1} - \frac{\mu_2}{r_2}\right] = 0$$

Jacobi Constant (JC)

The last expression indicates that the following is true in the 3BP:

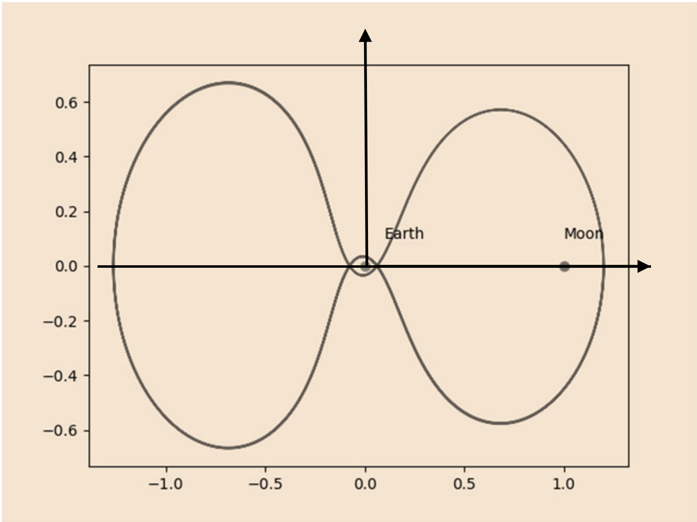
$$\frac{1}{2}V^2 - \frac{1}{2}\omega^2\left(x^2 + y^2\right) - \frac{\mu_1}{r_1} - \frac{\mu_2}{r_2} = C$$

The above equation can be considered as the first integral or one of the motion constants for the 3BP, known as the Jacobi constant. Note that JC is constant for each orbit of the third body (SC), but the constant could be different for different orbits. JC was initially discovered in 1836 by the German mathematician Carl Jacobi. It represents the total energy of the secondary mass (SC) in the rotating frame (not the total energy of the system in the inertial space), as the velocity used in the derivation was in the rotating frame. Note:

$V^2 \triangleq$ *Kinetic energy (specific) relative to rotating frame*

μ_1 / r_1 *and* $\mu_2 / r_2 \triangleq$ *Gravitational potential of the primaries on SC*

$U = -\frac{1}{2}\omega^2\left(x^2 + y^2\right) \triangleq$ *Centrifugal Potential energy due to rotation of reference frame;* $\vec{F}_C = \omega^2\left(x\vec{i} + y\vec{j}\right); F_i = -\frac{\partial U}{\partial x_i}$



Sample planar orbit in 3BP

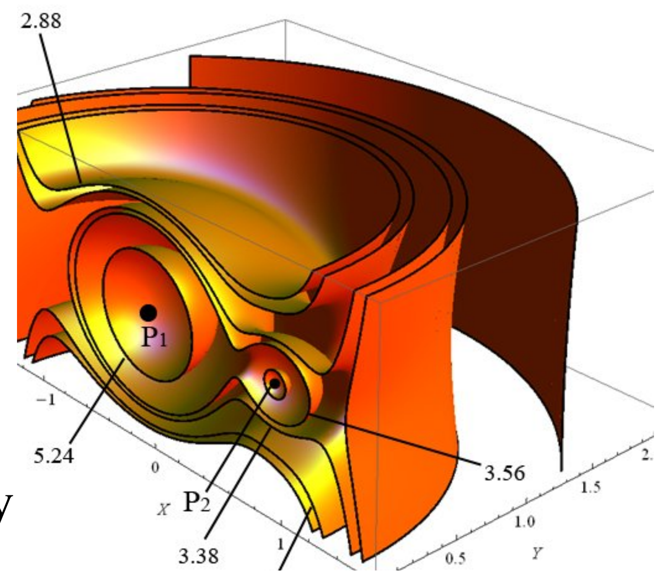
Jacobi Constant and Interpretations

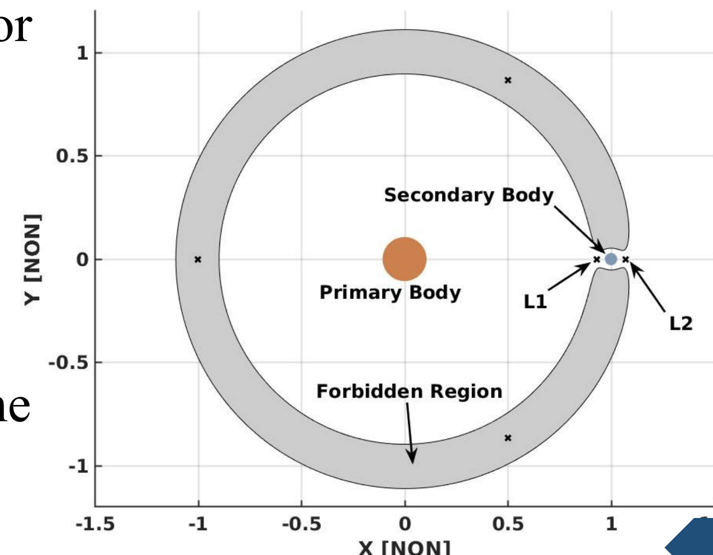
$$\frac{1}{2}V^2 - \frac{1}{2}\omega^2\left(x^2 + y^2\right) - \frac{\mu_1}{r_1} - \frac{\mu_2}{r_2} = C$$

If one solves the above equation for the velocity, V we will have :

$$V^2 = \omega^2\left(x^2 + y^2\right) + \frac{2\mu_1}{r_1} + \frac{2\mu_2}{r_2} + 2C \geq 0$$

That for each JC (or energy level) gives the relative velocity in three-dimensional space, as a function of the secondary mass position . An interesting observation is realized, that for each Jacobi constant (for which $V^2 \geq 0$) the relation describes feasible trajectories (x,y,z coordinates) in the rotating frame. In other words Trajectories that violate the above inequality are not possible. *These regions* are identified as *forbidden regions*, that in general can be three-dimensional in nature. However, if we limit the motion in the plane of the primaries ($z = 0$), their two-dimensional geometries can be visualized more easily.

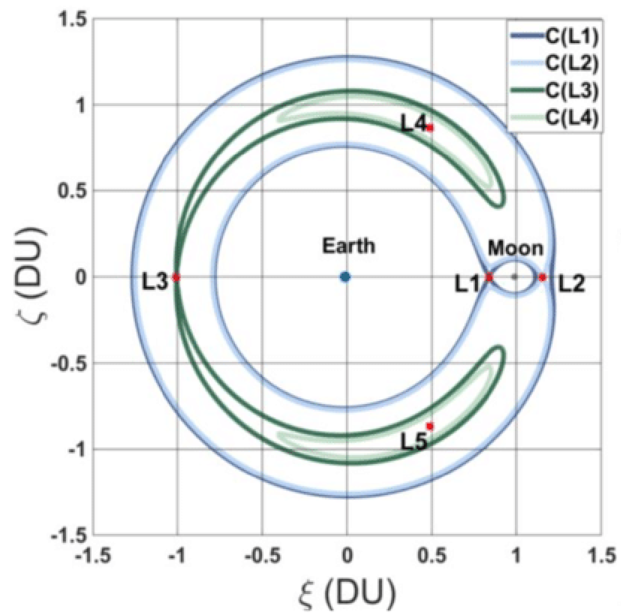




Jacobi Constant and Interpretations

In addition, the *boundary between the forbidden and allowed regions* can be obtained via setting $V^2 = 0$ to obtain coordinates that for each JC can not be crossed by the spacecraft (SC).

$$\omega^2\left(x^2+y^2\right)+\frac{2\mu_1}{r_1}+\frac{2\mu_2}{r_2}+2C=0$$



These boundaries also called **zero velocity curves (ZVC)**, define closed (small or large) geometries that are indicative of impossibl regions of spacecraft's motion within the context of 3BP.

$$V^2=\omega^2\left(x^2+y^2\right)+\frac{2\mu_1}{r_1}+\frac{2\mu_2}{r_2}+2C\geq 0$$

In addition, as the first three terms in the above equation are all positive, it is obvious that the ZVCs are associated with negative values of the Jacobi constant (C < 0). Hence, for large **negative values of C**, the spacecraft is either far from the center (large $x^2 + y^2$) or very close to one of the primaries (small r_1 and r_2). Also note that the unit of C is the same as specific energy , i.e V^2 (km^2/s^2).

Earth-Moon Example

To better understand this problem, we will consider the Earth-Moon system (EM 3BP). The goal is to investigate the relation between energy level and the zero velocity curves. The following data are easily available:

zero velocity curve relation:

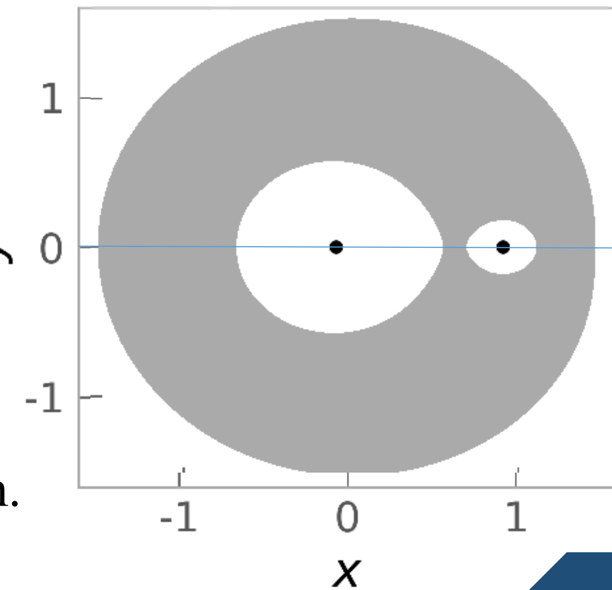
$$\omega^2\left(x^2+y^2\right)+\frac{2\mu_1}{r_1}+\frac{2\mu_2}{r_2}+2C=0$$

$$\mu_1=\mu_E=3.986\times10^5\ km^3/s^2$$

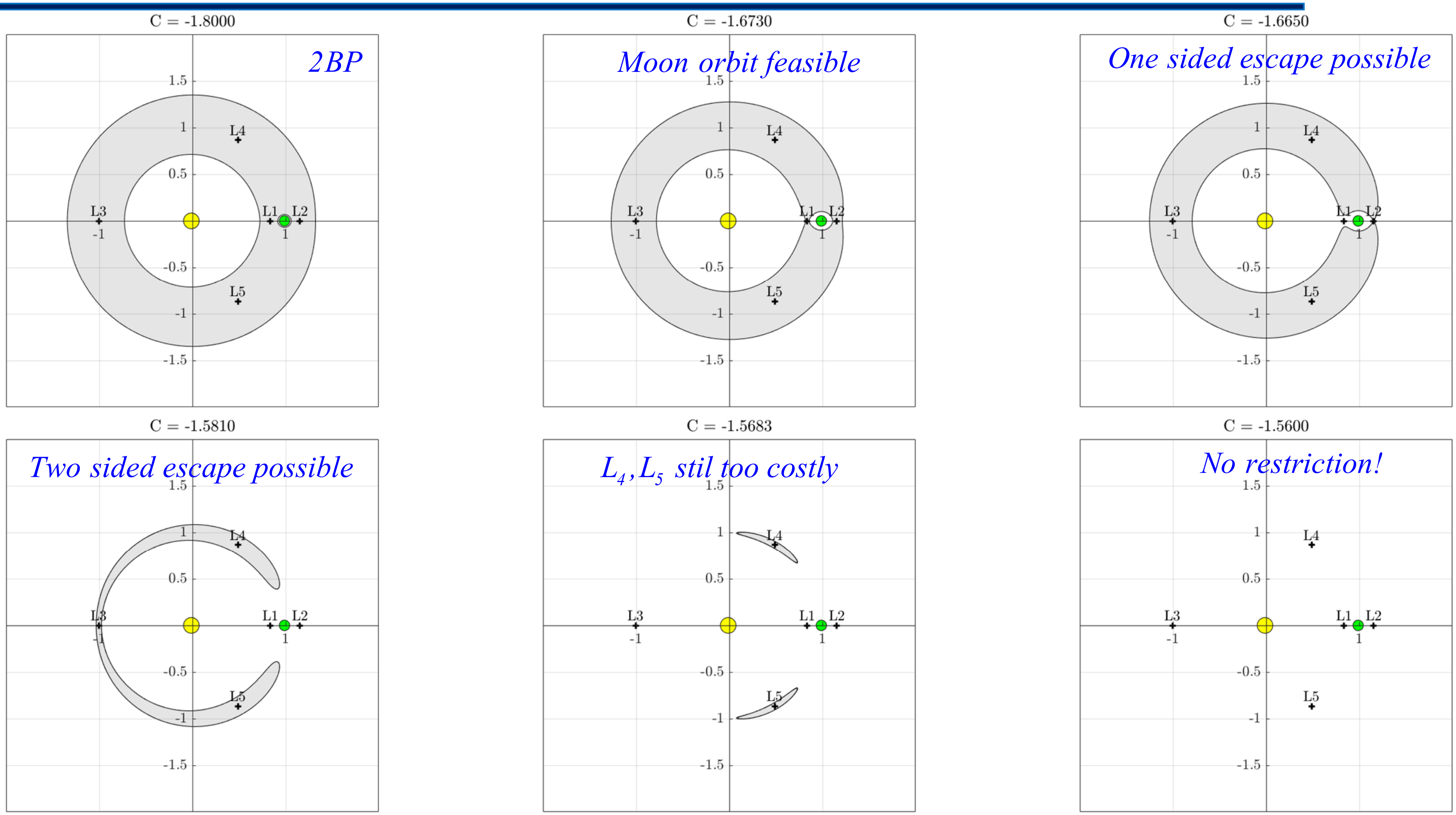
$$\mu_2=\mu_M=4.903\times10^3\ km^3/s^2$$

$$r_{12}=384400\ km;\ \omega=\sqrt{\frac{\mu_1+\mu_2}{r_{12}^3}}=2.66538\times10^{-6}\ rad/s$$

Using the ZVC relation, one can obviously obtain various curves for different Jacobi constants. A sample cases is discussed for $C_0=-1.82$ *(km^2/s^2)*. As seen by its associated ZCV, and the forbidden regions. In this condition, a launched spacecraft (with this amount of energy) cannot reach the Moon. Other cases are also shown in the next slide.



2D Forbidden Regions for the Earth-Moon R3BP



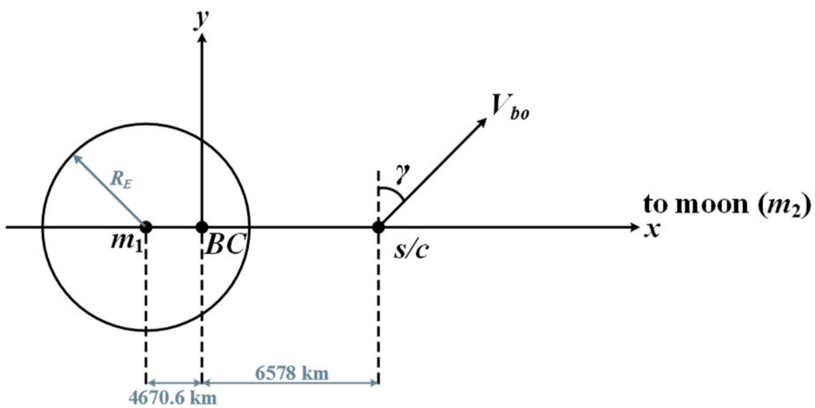
Escape velocity Example within the Context of 3BP

Again using the Earth-Moon RTBP, one can compute the spacecraft velocity at different levels of energy for stay or escape from the Earth-Moon system. If the spacecraft leaves at an altitude of *200 km* from the Earth's surface, we will have:

$$r_{sc} = 6378 + 200 = 6578\ km$$

$$x_{sc} = 6578 - 4670.6 = 1907.3\ km$$

$$y_{sc} = z_{sc} = 0$$



Energy Level	V_{bo} (km/s)
C_0 = -1.8	10.845
C_1 = -1.6735	10.857
C_2 = -1.6649	10.858
C_3 = -1.581	10.866
C_4 = -1.5683	10.867
C_5 = -1.56	10.868

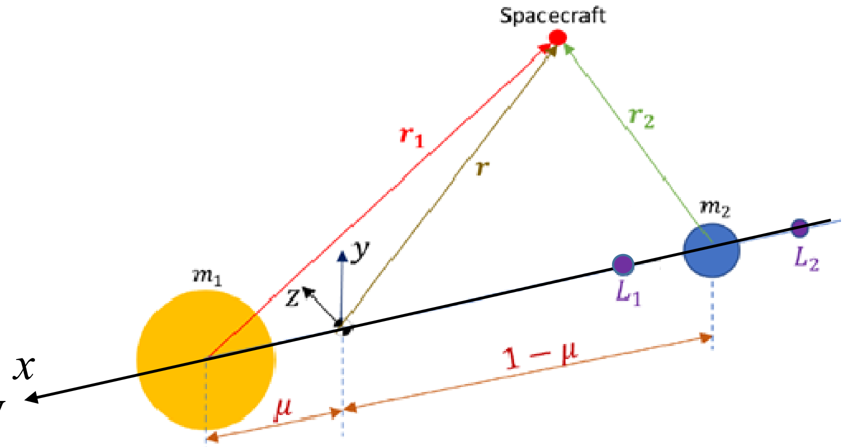
Interestingly, the speeds at various energy levels are not significantly different. However, the difference of about 0.023 km/s (about 20 m/s) can influence the access to the regions adjacent to the Moon-Earth. In addition, it is not bad to compare the 2BP escape velocity with those obtained in the table.

$$V_{esc} = \sqrt{\frac{2\mu_E}{r}} \xrightarrow[r=6578\ km]{\mu_E=3.986\times 10^5} V_{esc} = 11.01\ km$$

Stability of the Lagrangian Points

As indicated before, there are equilibrium (stationary) points (EPs) derived for the CRTBP with respect to the rotating frame. However, of one observes EPs from an inertial frame, the Lagrangian points will have circular orbits with the same period as the Moon-Earth system. The next question after existence of EPs is regarding their stability. Of course theoretically, if a spacecraft is placed exactly at these points with zero velocity, it will stay there. In other word, the discussion of their detection and existence is different from their stability.

In general, there are various criteria and definitions for stability for dynamic systems. Here we use the concept of **linear stability theory** that is often useful for initial investigation of *linearized systems* over the alternate formulation of the RTBP in canonical coordinates. For this purpose, assume the LPs coordinates $(x_e, y_e, z_e = 0)$ are available and their velocity components must also be zero, otherwise it will not be an equilibrium point. Next, use the linearized version of the spacecraft EOMs in their vicinity (perturbed) to investigate the stability of each of the five LPs.



Stability of Lagrangian Points

The development of perturbed EOMs around EPs can be found in many classical control text books, where each variable in the EOM is initially replaced by summation of two terms as follows and after the expansion, the nonlinear terms are eliminated while the steady state conditions are removed. In our case : $x = x_e + \delta x; y = y_e + \delta y; z = \delta z$

Additionally, for simplicity we restrict our analysis in the *xy* plane where the primaries are located. The resulting perturbed EOM (about EPs) will be :

$$\begin{cases} \delta\ddot{x} - 2\delta\dot{y} - \delta x = -A\delta x + B\delta y \\ \delta\ddot{y} + 2\delta\dot{x} - \delta y = C\delta x - D\delta y \end{cases} ;\ where : \begin{cases} A = (1-\mu)\left[r_{1e}^{-3} - 3A_1\right] + \mu\left[r_{2e}^{-3} - 3A_2\right] \\ B = 3(1-\mu)B_1 + 3\mu B_2 \\ C = 3(1-\mu)B_1 + 3\mu B_2 \\ D = (1-\mu)\left[r_{1e}^{-3} - 3C_1\right] + \mu\left[r_{2e}^{-3} - 3CA_2\right] \end{cases}$$

$$A_1 = \frac{(x_e - \mu)^2}{r_{1e}^5}; A_2 = \frac{(x_e + 1 - \mu)^2}{r_{2e}^5}; B_1 = \frac{(x_e - \mu)y_e}{r_{1e}^5}; B_2 = \frac{(x_e + 1 - \mu)y_e}{r_{2e}^5}; C_1 = \frac{y_e^2}{r_{1e}^5}; C_2 = \frac{y_e^2}{r_{2e}^5}$$

As expressed above, the constants *A*, *B*, *C* and *D* can be determined (based on the *EPs coordinates*) and the *three body system* under consideration.

Stability of Lagrangian Points

I. Stability of the triangular points (L4 and L5).

In the canonical system, the L4 coordinates are given below. Putting this information in the perturbed EOMs yields:

$$\begin{cases} \delta\ddot{x} - 2\delta\dot{y} - \delta x = -A\delta x + B\delta y \\ \delta\ddot{y} + 2\delta\dot{x} - \delta y = C\delta x - D\delta y \end{cases}$$

$$\Downarrow$$

$$L4 = \begin{cases} x_e = \mu - 0.5 \\ y_e = \dfrac{\sqrt{3}}{2} \end{cases} \quad ; \quad \begin{cases} r_{1e} = 1 \\ r_{2e} = 1 \end{cases}$$

$$\begin{cases} \delta\ddot{x} - 2\delta\dot{y} - \dfrac{3}{4}\delta x - \dfrac{3\sqrt{3}}{2}\left(\mu - \dfrac{1}{2}\right)\delta y = 0 \\ \delta\ddot{y} + 2\delta\dot{x} - \dfrac{3\sqrt{3}}{2}\left(\mu - \dfrac{1}{2}\right)\delta x - \dfrac{9}{4}\delta y = 0 \end{cases}$$

Using matrix algebra and the defining $\delta\vec{r} = \begin{bmatrix}\delta x & \delta y\end{bmatrix}^T$ the above equation can be written as follows:

$$R_1\delta\ddot{\vec{r}} + R_2\delta\dot{\vec{r}} + R_3\delta\vec{r} = 0$$

where R_1, R_2, and R_3 in the above vector (matrix) equation will be constant matrices (in terms of μ) as given below:

$$R_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad ; \quad R_2 = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \quad ; \quad R_3 = \begin{bmatrix} -\dfrac{3}{4} & -\dfrac{3\sqrt{3}}{2}\left(\mu - \dfrac{1}{2}\right) \\ -\dfrac{3\sqrt{3}}{2}\left(\mu - \dfrac{1}{2}\right) & -\dfrac{9}{4} \end{bmatrix}$$

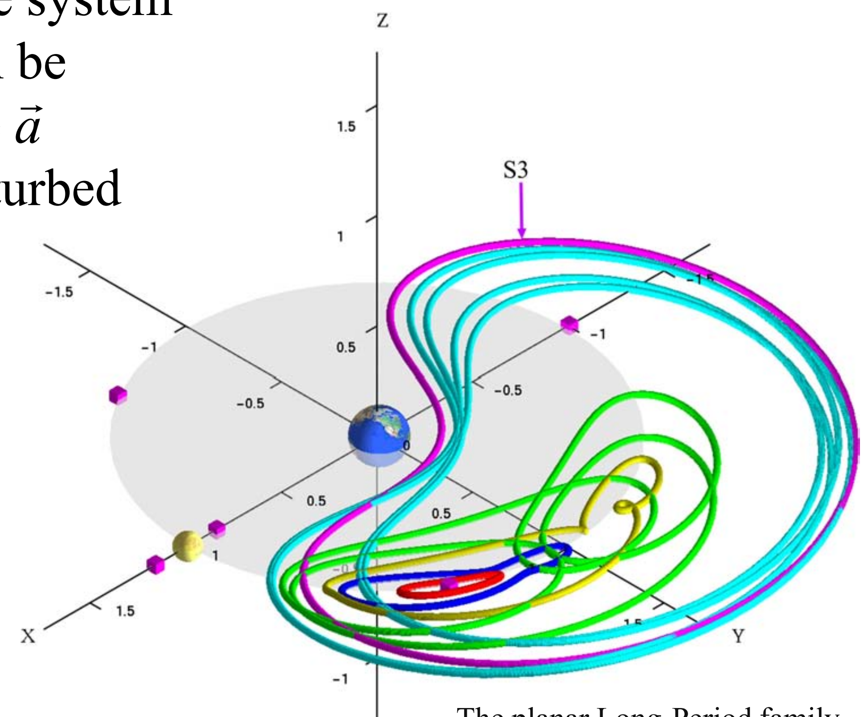
Stability of Lagrangian Points

$$R_1\delta\ddot{\vec{r}} + R_2\delta\dot{\vec{r}} + R_3\delta\vec{r} = 0$$

One can assume an exponential solution to determine the system characteristic equation, out of which the eigenvalues can be determined for stability analysis. So let $\delta\vec{r} = \vec{a}e^{\lambda t}$ where $\vec{a}$ is a constant matrix. Substituting this solution in the perturbed EOMs for L4, will give:

$$\begin{vmatrix} \lambda^2 - \frac{3}{4} & 2\lambda - \frac{3\sqrt{3}}{2}\left(\mu - \frac{1}{2}\right) \\ 2\lambda - \frac{3\sqrt{3}}{2}\left(\mu - \frac{1}{2}\right) & \lambda^2 - \frac{9}{4} \end{vmatrix} \vec{a} = 0$$

$$R\vec{a}_i = \lambda_i \vec{a}$$



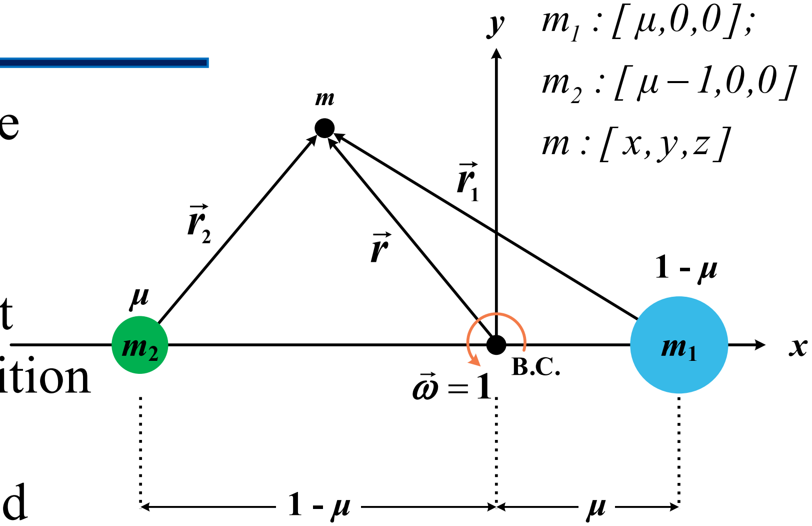
The planar Long-Period family L4 of the Earth–Moon system

The expression is an eigenvalue /eigenvector problem.

After simplifying, the determinant of the above equation one would get :

$$\lambda^4 + \lambda^2 - \frac{27}{4}\mu(\mu - 1) = 0 \xrightarrow{\text{Solving for } \lambda^2} \lambda^2 = \frac{-1 \pm \sqrt{1 - 27\mu(\mu - 1)}}{2}; \quad 0 \le \mu \le 1$$

Stability of Lagrangian Points



Neutral stability (most we can expect), requires all λ's to be imaginary, as it is not possible to obtain four roots with negative real parts for L4 (why?) . Therefore for the λ's to be pure imaginary, the discriminant (Δ) of latter equation equation must be positive. This condition renders two critical values for μ, as bounds for which L4 stability condition occurs. These critical values are obtained by solving $1\text{-}27\mu(1-\mu)=0$. In conclusion, the L4 will be stable for systems whose μ value is in the following intervals, otherwise it will be unstable.

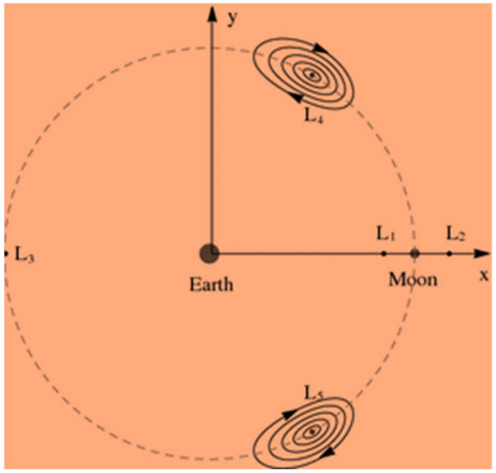
$$\mu < \mu_1 = 0.03852 \quad ; \quad \mu > \mu_2 = 0.96148$$

Fortunately, the largest μ value in the solar system is $\mu = 0.01213$, which occurs for the Earth-Moon system, and therefore **all the triangular Lagrangian points (LPs) are stable for all three bodies within our solar system**.

Stability of Lagrangian Points

Of course, it should be noted that due to the elimination of the effect of the Sun in the Earth-Moon CRTBP problem, these points do not have sufficient stability without some maintenance or station-keeping. To complete this discussion, the eigenvalues and their normalized eigenvectors can be calculated for the Earth-Moon system (μ = 0.01213).

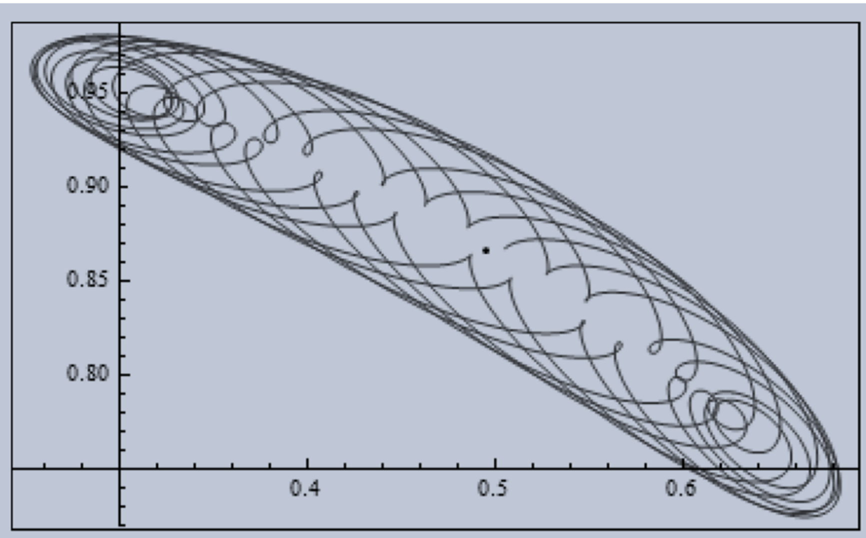
$$\lambda^2 = -0.08875, -0.911236 \Rightarrow \begin{cases} \lambda_{1,2} = \pm 0.29793i,\ \vec{a}_{1,2} = \left[0.77641 \pm 0.365i \quad 0.5137\right]^{\top} & \textit{Long period mode} \\ \lambda_{3,4} = \pm 0.954587i,\ \vec{a}_{3,4} = \left[0.4487 \pm 0.6745i \quad 0.5869\right]^{\top} & \textit{Short period mode} \end{cases}$$



The short period mode will a period of approximately
the same as the Moon's period (about a month).
On the other hand, the period of the long period mode will be about three months (when a complete circuit/trajectory is closed). Of course, keep in mind that these linear solutions were obtained assuming a small amplitude motion around the Lagrangian points, and the behavior will be different as the size of the oscillations increases. An example is shown in the next slide.

Stability of Lagrangian Points

Via utility of $\lambda_i s$ and $a_i s$, general motion around L4 can be determined by superposition. In other words, $\delta\vec{r}(t)=\sum_{i=1}^{4}\alpha_i\vec{a}_i e^{\lambda_i t}$, where the constants $\alpha_i s$ are determined based on initial perturbed conditions of position and velocity. As noted before, due to Sun's effect these points are usually not utilized or remain empty after a short time for the Earth-Moon system.



Quasi-Periodic Orbit Around L_4

Stability of Lagrangian Points

Such motions have been naturally observed for Sun-Jupiter asteroids captured around L4, though not for the Earth-Moon system due to the Sun's stronger effect.

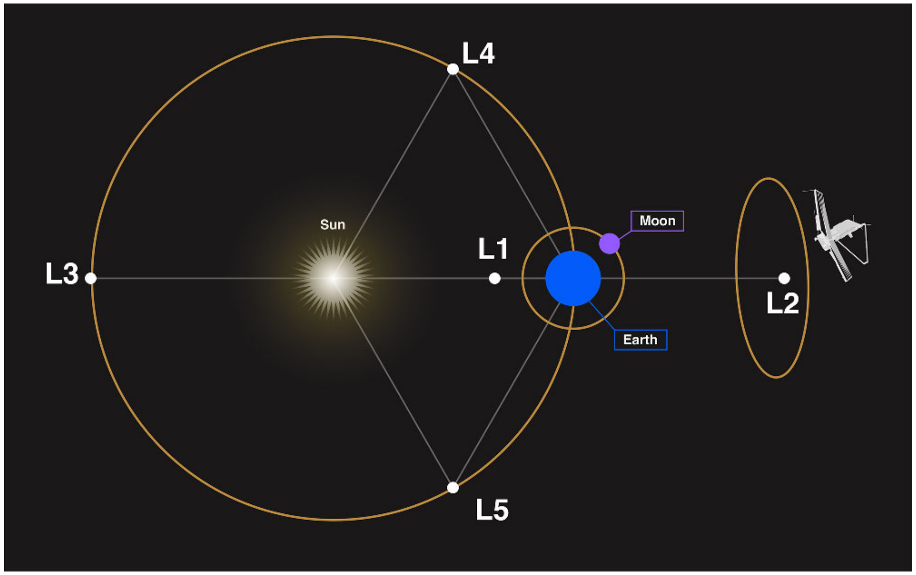
II. Stability of the collinear points (L1,L2,L3)

Similarly, one can determine the associated characteristic equation for these points, that have the basic property $(x_e, y_e = 0, z_e = 0)$, to be as follows [Bong Wie]:

$$\lambda^4 - (\sigma - 2)\lambda^2 - (2\sigma + 1)(\sigma - 1) = 0$$

Where, the parameter σ in this equation will be determined based on mass ratio and coordinates of the particular collinear point of interest in dimensionless form:

$$\sigma = \frac{1-\mu}{|x_e - \mu|^3} + \frac{\mu}{|x_e + 1 - \mu|^3}$$



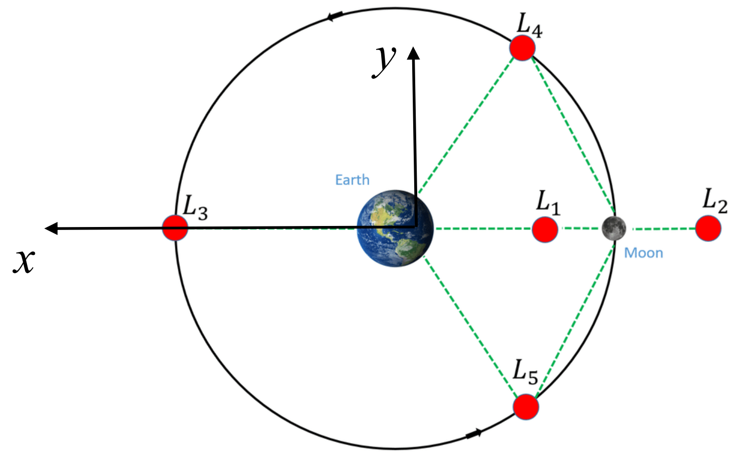
Stability of Lagrangian Points

Also, note that the perturbed equations of motion will be for $\delta x; \delta y; \delta z$ that corresponds to two in plane (*x*y) and one out plane (*z)* components of the perturbed motion around any of the collinear LPs.

For example, for **L2** with x_e = *-1.156* and σ = *3.1904* in the Earth-Moon system:

- In-plane characteristic equation has 4 roots: $\lambda_{1,2} = \pm 2.159, \lambda_{3,4} = \pm 1.863i$
  that is indicative of oscillatory and divergent behavior.
- Out-of-plane characteristic equation: $\lambda^2 + \sigma = 0 \Rightarrow \lambda_{1,2} = \pm 1.786i$
  is indicative of a simple harmonic behavior with a non-dimensional frequency of $\sqrt{\sigma} = 1.786$

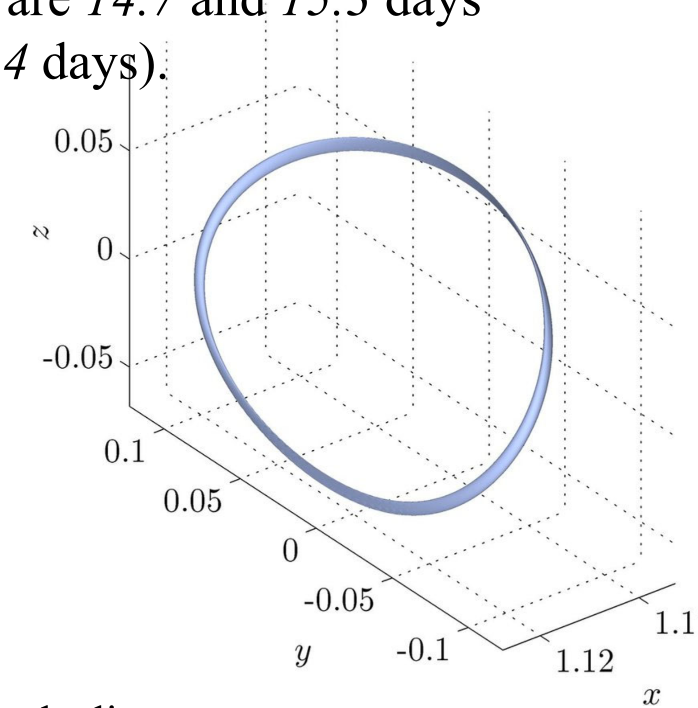
Of course, it turns out that other Earth-Moon collinear points L1 and L3 will also be unstable, but despite being unstable, L1 and L2 are being used for telecommunication and space relay stations close to the Moon.



Stability of Lagrangian Points

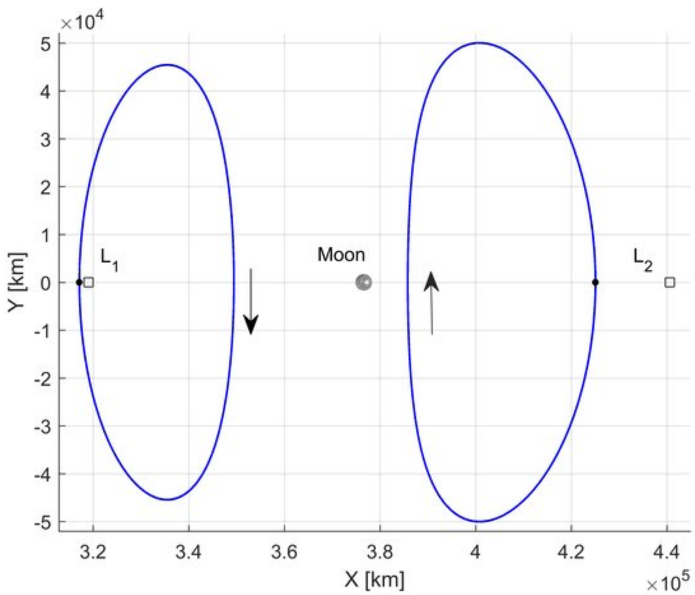
To sum up, the L2 equilibrium point in-plane motion will include a divergent and an oscillating orbit with a frequency of *1.863*, where its out-of-plane motion is a simple harmonic with a frequency of *1.786*. Note that the frequencies are in Canonical *(non-dimensional)* system, thus it can be shown that the period of the above oscillatory motions are *14.7* and *15.3* days respectively (compared to the lunar period of approximately *27.4* days).

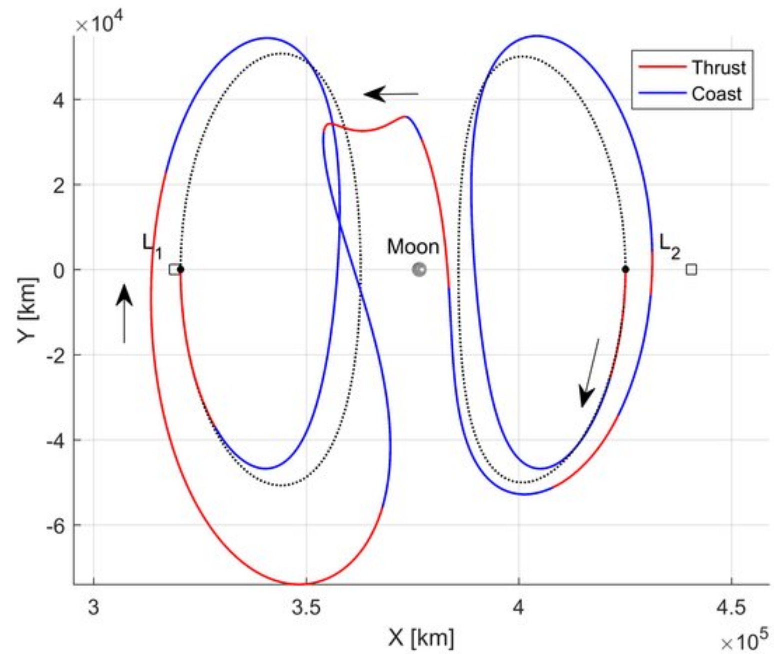
Thus:



- By choosing the right initial conditions, one can achieve periodic motions,
- The difference between the in-plane and out-of-plane frequencies create, the so called Lissajous quasi-periodic orbits,
- Usually, Lissajous paths do not close for irrational frequency ratios.
- Periodic paths have been created for rational frequency ratios, which are called **halo** orbits.
- For most initial conditions, the solution of linearized equations is not periodic, and some control effort is required to equal in-plane and out-of-plane frequencies. This is often called period or frequency control in the literature.

Low-thrust Halo orbital transfer in the Earth-Moon CR3BP





Low-Thrust Transfer Design Based on Collocation Techniques: Applications in the Restricted Three-Body Problem
Spacecraft initial mass :500 kg, Engine T max = 100 mN with an I sp = 2000 sec. , where the time of flight is 47.5 days.

Aug 2017AAS/AIAA Astrodynamics Specialist Conference

Stability of Lagrangian Points