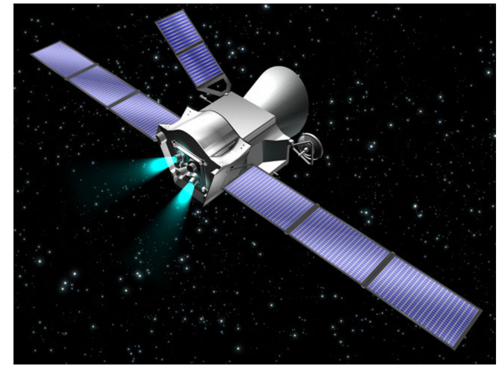
In The Name of God



AE 45780: Spacecraft Dynamics and Control

Fall 1401

3- Attitude Dynamics

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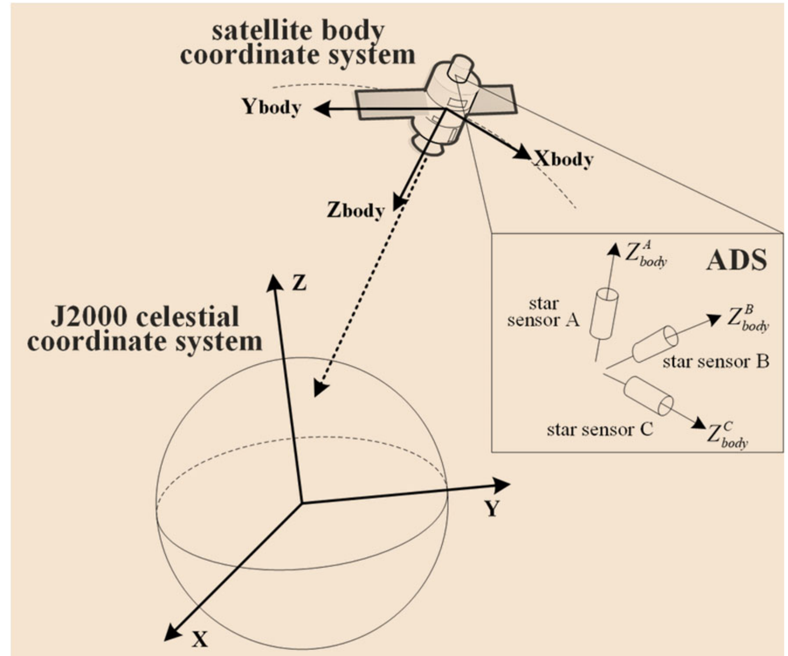
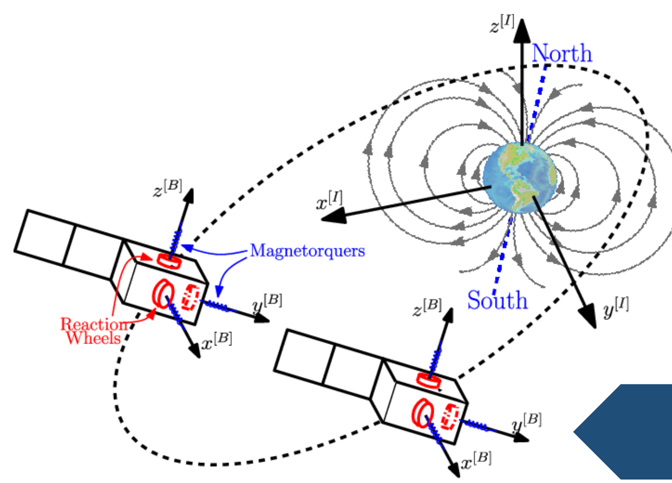
Attitude Dynamics and Control (ADC)

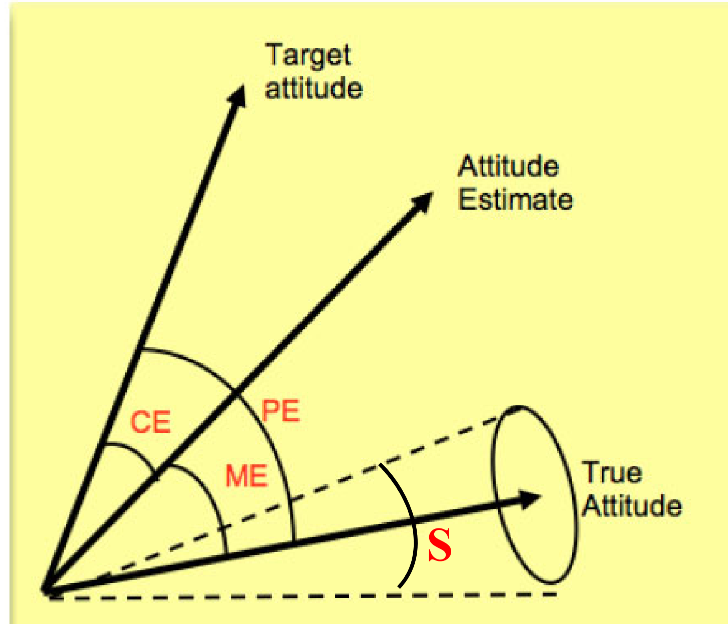
Attitude Dynamics (AD) pertains to the rotational motion of satellites and space vehicles around their center of mass. **Attitude control (AC)** is the process of controlling the orientation of a spacecraft with respect to a reference frame. Initial understanding of feasible motion dynamics of orbiting spacecraft (2BP, Perturbation effect, 3PB...) was a prerequisite to orbit transfer, maneuver and control. A similar approach is followed in the study the attitude dynamics, its propagation and kinematics, for subsequent control of SC attitude motion. In this sense, basic laws and concepts regarding angular kinetic energy and momentum will be introduced and used in the derivation of the fundamental laws of angular motion based on the Euler's equations. But, first some related terminologies are should be reviewed.





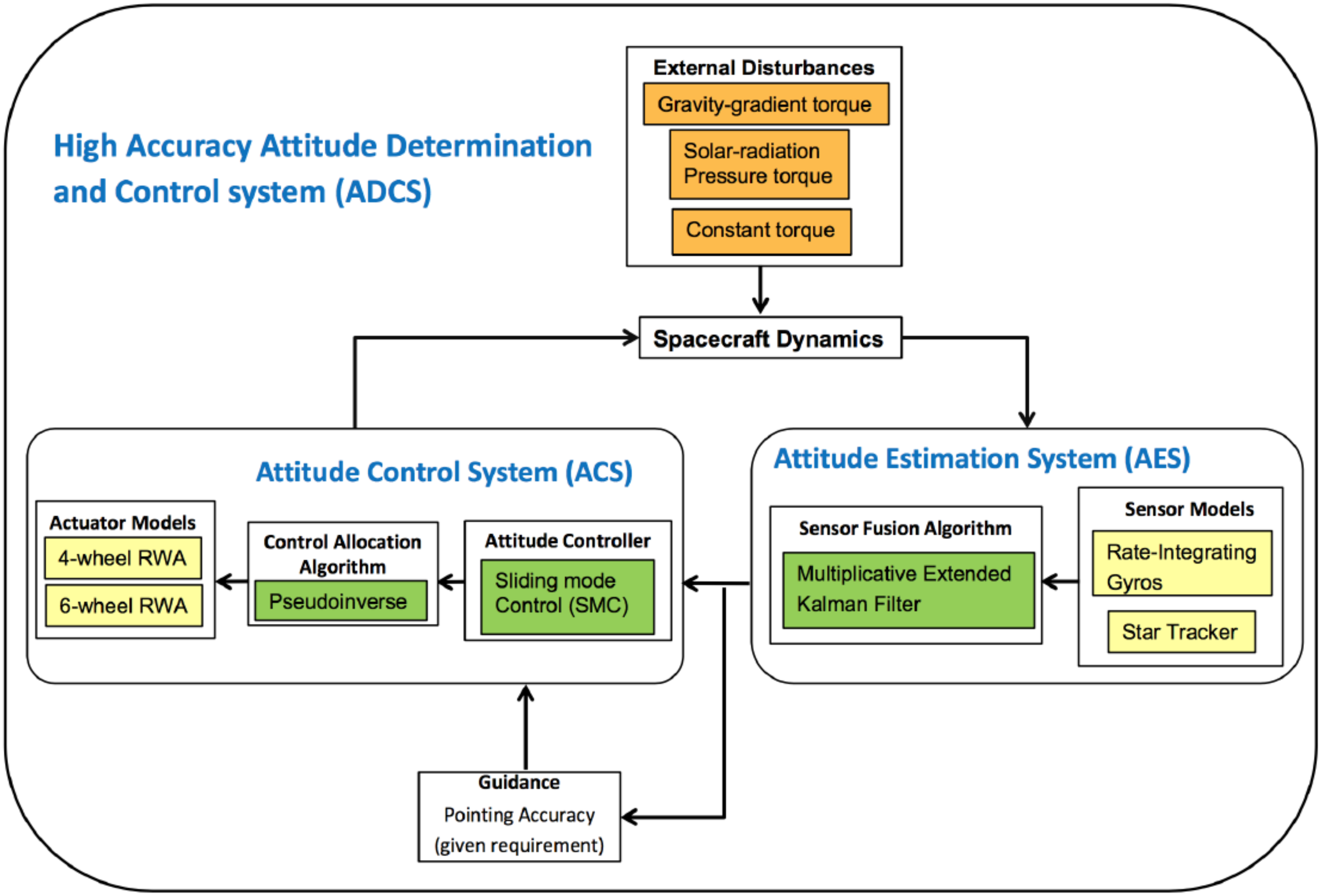
AD Related Terminologies

Attitude: Orientation of a SC body coordinate system WRT an external reference frame.
Attitude Determination: Real time knowledge or determination of SC attitude (within a given tolerance).
Attitude Control: Maintenance (keeping) of a desired specified attitude within a given tolerance.
Attitude or Pointing Error (PE) : "Low frequency" SC misalignment WRT desired attitude. Or, the amount of angular separation between the desired or commanded direction and the actual instantaneous (true) direction.
Attitude Jitter (S) : "High frequency" SC misalignment WRT desired attitude. This is usually ignored by SC attitude determination and control system (ADCS), but is important in good design for fine pointing.
Target Attitude: Desired pointing direction.
True Attitude : Actual pointing direction.
Attitude Estimate : An instantaneous estimate of the true attitude by (AES).
Measurement Error (ME) : Pointing accuracy from the attitude estimation system (AES).
Control Error (CE) : Pointing accuracy from the attitude control system(ACS).



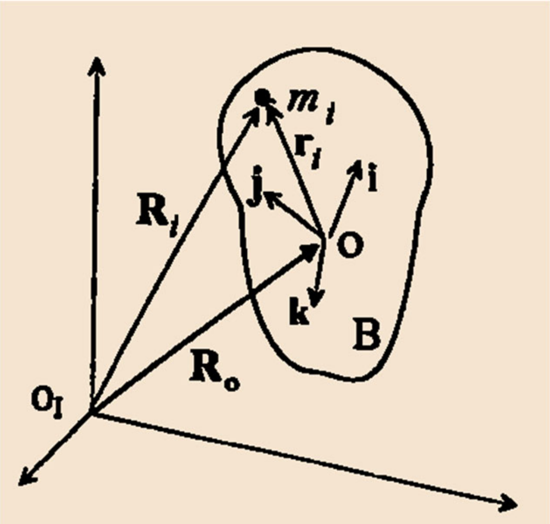
Pointing definitions

Attitude Determination and Control System (ADCS)



Components of the Closed-Loop Attitude Determination and Control System of IRASSI Spacecraft.

Angular Momentum of a Rigid Body (SC)



Suppose the origin of the SC body coordinate (moving) is located at its center of mass ,O .The angular momentum of a particle within this body about point O is: $\vec{h}_{O_i} = \vec{r}_i \times m_i \vec{V}_i$

where $\vec{V}_i$ is the inertial velocity of the particle,

which via Coriolis can be wrriten in body ccordinates as:

$$\vec{V}_i = \dot{\vec{R}}_i = \dot{\vec{R}}_O + \vec{V}_{i_b} + \vec{\omega} \times \vec{r}_i = \vec{V}_O + \dot{\vec{r}}_i + \vec{\omega} \times \vec{r}_i$$

Substituting in the relation for particle angular momentum and assuming a rigid SC :

$\vec{h}_{O_i} = \vec{r}_i \times m_i \left(\vec{V}_O + \vec{\omega} \times \vec{r}_i \right)$, where in body coordinate: $\vec{V}_{i_b} = \dot{\vec{r}}_i = 0$ is used.

If the SC body is considered as a set of particles or small masses, the total SC angular momentum will be the sum of the particle angular momentums around the point O. In other words:

$$\vec{H}_O = \sum_i h_{O_i} = \sum_i \vec{r}_i \times m_i \left(\vec{V}_O + \vec{\omega} \times \vec{r}_i \right) = \sum_i \left[\vec{r}_i \times m_i \vec{V}_O + \vec{r}_i \times m_i \vec{\omega} \times \vec{r}_i \right]$$

$$= \sum_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i) m_i - \underbrace{\vec{V}_O \times \sum_i m_i \vec{r}_i}_{=0} = \sum_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i) m_i \text{ or } \int_B \vec{r} \times (\vec{\omega} \times \vec{r}) \mathrm{d}m$$

Angular Momentum of a Rigid SC

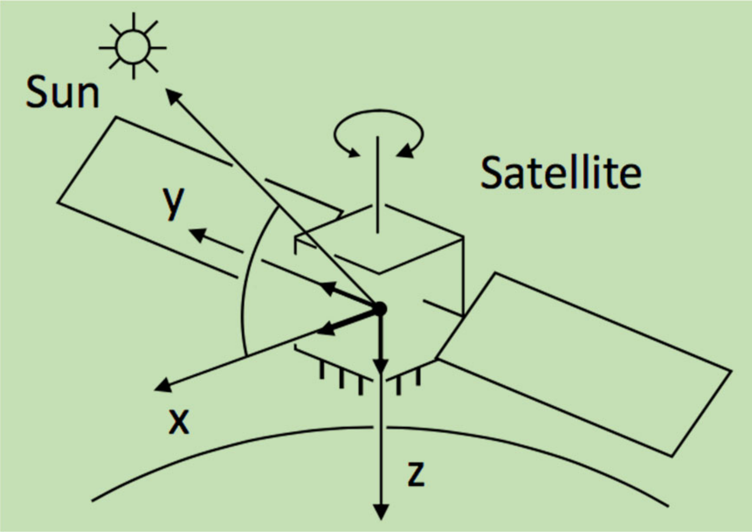
The previous relation can be presented based on the expression of the $\vec{r}_i$ and $\vec{\omega}_i$ vectors in body coordinate system. In addition using the definition of moments of inertia of the SC about the body axes will further simplify the equation:

$$\vec{h}_O = \sum_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i) m_i \text{ or } \int_B \vec{r} \times (\vec{\omega} \times \vec{r}) \mathrm{d}m$$

$$\vec{r}_i = \begin{bmatrix} x_i & y_i & z_i \end{bmatrix}^\top \quad , \quad \vec{\omega} = \begin{bmatrix} \omega_x & \omega_y & \omega_z \end{bmatrix}^\top$$

$$\vec{\omega} \times \vec{r}_i = \left(\omega_y z_i - \omega_z y_i\right)\vec{i} + \left(\omega_z x_i - \omega_x z_i\right)\vec{j} + \left(\omega_x y_i - \omega_y x_i\right)\vec{k}$$

$$\vec{r}_i \times \omega \times \vec{r}_i = \begin{Bmatrix} \left[\omega_x\left(y_i^2 + z_i^2\right) - \omega_y\left(x_i y_i\right) - \omega_z\left(x_i z_i\right)\right]\vec{i} \\ \left[-\omega_x\left(x_i y_i\right) + \omega_y\left(x_i^2 + z_i^2\right) - \omega_z\left(y_i z_i\right)\right]\vec{j} \\ \left[-\omega_x\left(x_i z_i\right) - \omega_y\left(y_i z_i\right) + \omega_z\left(x_i^2 + y_i^2\right)\right]\vec{k} \end{Bmatrix}$$

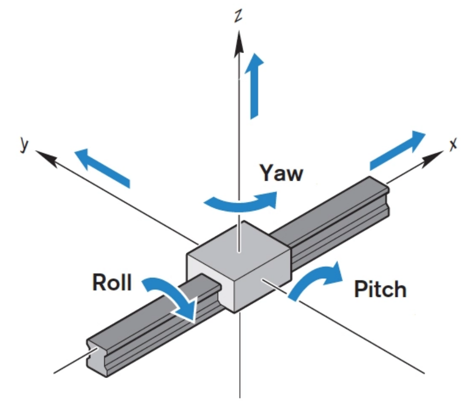


In this way, using equation of angular momentum **per unit mass** and the definition of moments of inertia of the body around its axes, yields the SC total angular momentum. Subsequently, the SC Kinematic EOM will be later derived from the basic law governing rotational dynamics via Euler EOM.

Angular Momentum of a Rigid SC

$$\begin{cases} I_{xx} = \sum_i \left(y_i^2 + z_i^2\right) m_i = \int_B \left(y^2 + z^2\right) \mathrm{d}m = I_x \\ I_{yy} = \sum_i \left(x_i^2 + z_i^2\right) m_i = \int_B \left(x^2 + z^2\right) \mathrm{d}m = I_y \\ I_{zz} = \sum_i \left(x_i^2 + y_i^2\right) m_i = \int_B \left(x^2 + y^2\right) \mathrm{d}m = I_z \end{cases} , \quad \begin{cases} I_{xy} = \sum_i \left(x_i y_i\right) m_i = \int_B (xy) \mathrm{d}m \\ I_{xz} = \sum_i \left(x_i z_i\right) m_i = \int_B (xz) \mathrm{d}m \\ I_{yz} = \sum_i \left(y_i z_i\right) m_i = \int_B (yz) \mathrm{d}m \end{cases}$$

Note that the SC moments of inertias (MOI) can be written in Tensor form and is usually **symmetric**. In addition the product moments of inertia may be negative. Using MOI tensor, and the last equation for the SC angular momentum ,one can show that:



$$I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \Rightarrow \vec{h} = I\vec{\omega} \Rightarrow \vec{h} = h_x\vec{i} + h_y\vec{j} + h_z\vec{k} \quad , \quad \begin{cases} h_x = \omega_x I_{xx} - \omega_y I_{xy} - \omega_z I_{xz} \\ h_y = -\omega_x I_{yx} + \omega_y I_{yy} - \omega_z I_{yz} \\ h_z = -\omega_x I_{zx} - \omega_y I_{zy} + \omega_z I_{zz} \end{cases}$$

Rotational Kinetic Energy of a Rigid Body

A 50 Kg Demonstration satellite with four movable solar cell paddles for high-speed satellite attitude and orbit control via paddle movement.
Source: https://www.titech.ac.jp/english/news/2021/062360

Rotational Kinetic Energy of a Rigid Body

The kinetic energy of a rotating rigid body around its center of mass is known as **rotational kinetic energy**, which can be shown to be related to its angular momentum. In this regard, let us first consider the kinetic energy of an element with a differential mass dm:

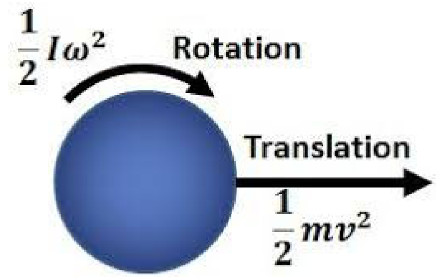
$$dT = \frac{1}{2}(dm)V^2$$

Where V is the magnitude of the absolute (inertial) velocity of dm, that can be written as $\vec{V}_i = \vec{V}_0 + \vec{\omega} \times \vec{r}_i$. Therefore: $V^2 = \vec{V}_i \cdot \vec{V}_i = V_0^2 + 2\vec{V}_O \cdot (\vec{\omega} \times \vec{r}) + (\vec{\omega} \times \vec{r}_i) \cdot (\vec{\omega} \times \vec{r}_i)$

Thus, putting back for dT, gives :

$$T = \int dT = \frac{1}{2}\int_B V^2 dm = \frac{1}{2}\int V_O^2 dm + \int \vec{V}_O \cdot (\vec{\omega} \times \vec{r})\, dm + \frac{1}{2}\int (\vec{\omega} \times \vec{r}) \cdot (\vec{\omega} \times \vec{r})\, dm$$

$$= \frac{1}{2} M V_O^2 + \vec{V}_O \cdot \underbrace{\vec{\omega} \times \int \vec{r}\, dm}_{=0} + \frac{1}{2}\int (\vec{\omega} \times \vec{r}) \cdot (\vec{\omega} \times \vec{r})\, dm = T_{Translational} + T_{Rotational}$$



In other words:

$$T_{Rotational} = \frac{1}{2}\int_B (\omega \times r) \cdot (\omega \times r)\, \mathrm{d}m$$

Rotational Kinetic Energy of Rigid SC

$$\vec{\omega}\times\vec{r}_i = \left(\omega_y z_i - \omega_z y_i\right)\vec{i} + \left(\omega_z x_i - \omega_x z_i\right)\vec{j} + \left(\omega_x y_i - \omega_y x_i\right)\vec{k}$$

$$h_x = \omega_x I_{xx} - \omega_y I_{xy} - \omega_z I_{xz}$$

$$h_y = -\omega_x I_{yx} + \omega_y I_{yy} - \omega_z I_{yz}$$

$$h_z = -\omega_x I_{zx} - \omega_y I_{zy} + \omega_z I_{zz}$$

Utilizing the previous relation we had :

$$T_{Rotational} = \frac{1}{2}\int_B (\omega\times r)\cdot(\omega\times r)\,dm$$

$$T_{Rotational} = \frac{1}{2}\int_B \left[\left(\omega_y z - \omega_z y\right)^2 + \left(\omega_z x - \omega_x z\right)^2 + \left(\omega_x y - \omega_y x\right)^2\right]dm$$

Subsequently, via expanding, integrating and utility of the MOI definitions, we will arrive at the following relation:

$$\begin{aligned}
2T_{Rotational} &= \omega_x^2 I_{xx} + \omega_y^2 I_{yy} + \omega_z^2 I_{zz} - 2\omega_x\omega_z I_{xz} - 2\omega_y\omega_z I_{yz} - 2\omega_x\omega_y I_{xy} \\
&= \omega_x\left[\omega_x I_x - \omega_y I_{xy} - \omega_z I_{xz}\right] + \omega_y\left[-\omega_x I_{yx} + \omega_y I_y - \omega_z I_{yz}\right] + \omega_z\left[-\omega_x I_{zx} - \omega_y I_{zy} + \omega_z I_z\right] \\
&= \vec{\omega}\cdot\vec{h} = \omega_x h_x + \omega_y h_y + \omega_z h_z \\
&= \vec{\omega}\cdot I\vec{\omega} \Rightarrow or \quad T_{Rotational} = \frac{1}{2}\vec{\omega}^T I\vec{\omega}
\end{aligned}$$

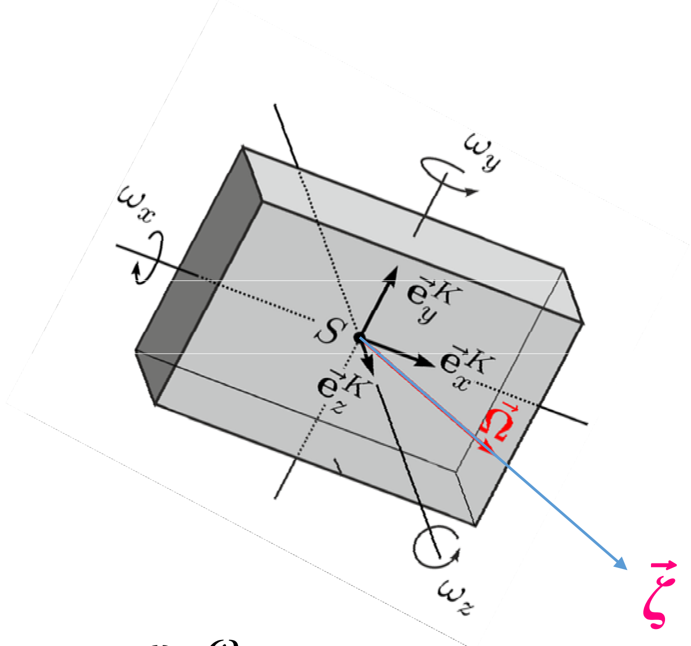
$$Note : \vec{h}_i = \frac{1}{2}\frac{\partial}{\partial\omega_i}(2T_{ROT});\ i=x,y,y$$

Moment of Inertia in Special Coordinates

The moment of inertia of a body can be defined around any axis that passes through its center of mass. In this regard, suppose an **axis is taken** parallel to the $\vec{\omega}$ vector, which passes through the SC center of mass. In this case, for $\vec{\omega}$, would only have one nonzero component . Thus it can be shown that $\vec{\omega} I \vec{\omega} = I_\xi \omega^2$. That is, if ξ is an axis along $\vec{\omega}$ and passing through the SC the center of mass, then we have:

$$T_{Rotational} = \frac{1}{2} I_\xi \omega^2$$

$$\vec{\omega} = \omega \vec{1}_\xi \ \ or\ \vec{\omega} = \begin{bmatrix} \omega_x & \omega_y & \omega_z \end{bmatrix}^\top$$



In the latter situation, notice that T_{ROT} can be written independent of the body coordinate system. Then using the definition of T_{ROT} . Also, let , $I_{ii} = I_i$ for $i = x, y, z$.

$$I_\zeta \omega^2 = \omega_x^2 I_x + \omega_y^2 I_y + \omega_z^2 I_z - 2\omega_y \omega_x I_{xy} - 2\omega_y \omega_z I_{yz} - 2\omega_z \omega_x I_{xz}$$

$$\text{or } I_\xi = \left(\frac{\omega_x}{\omega}\right)^2 I_x + \left(\frac{\omega_y}{\omega}\right)^2 I_y + \left(\frac{\omega_z}{\omega}\right)^2 I_z - 2\frac{\omega_x}{\omega}\frac{\omega_z}{\omega} I_{xz} - 2\frac{\omega_y}{\omega}\frac{\omega_z}{\omega} I_{yz} - 2\frac{\omega_x}{\omega}\frac{\omega_y}{\omega} I_{xy}$$

Moment of Inertia in Special Coordinates

$$T_{Rotational} = \frac{1}{2} I_\xi \omega^2$$

$$\vec{\omega} = \omega \vec{1}_\xi \ \ or\ \vec{\omega} = \begin{bmatrix} \omega_x & \omega_y & \omega_z \end{bmatrix}^T$$

Since $\vec{\omega} = \omega \vec{1}_\xi$, the components of this vector in body coordinate are obtained using inner product with $\vec{\imath}, \vec{\jmath}, \vec{k}$, or in other words:

$$\omega_x = \vec{\omega} \cdot \vec{i} = \omega \vec{1}_\xi \cdot \vec{i}$$

$$\omega_y = \vec{\omega} \cdot \vec{j} = \omega \vec{1}_\xi \cdot \vec{j}$$

$$\omega_z = \vec{\omega} \cdot \vec{k} = \omega \vec{1}_\xi \cdot \vec{k}$$

In addition by introducing: $a_x = \vec{1} \cdot \vec{\imath}$, $a_y = \vec{1} \cdot \vec{\jmath}$, and $a_z = \vec{1} \cdot \vec{k}$ (direction cosine angles between the $\vec{\omega}$ and each of the body coordinate axes) , that indicate the position of the axis ξ relative to the *x, y, z* axes,

one can write: $a_x = \dfrac{\omega_x}{\omega}$, $a_y = \dfrac{\omega_y}{\omega}$, $a_z = \dfrac{\omega_z}{\omega}$

That leads to the following relation in relation for I_ξ:

$$I_\xi = \left(\frac{\omega_x}{\omega}\right)^2 I_x + \left(\frac{\omega_y}{\omega}\right)^2 I_y + \left(\frac{\omega_z}{\omega}\right)^2 I_z - 2\frac{\omega_x}{\omega}\frac{\omega_z}{\omega} I_{xz} - 2\frac{\omega_y}{\omega}\frac{\omega_z}{\omega} I_{yz} - 2\frac{\omega_x}{\omega}\frac{\omega_y}{\omega} I_{xy}$$

$$I_\xi = (a_x)^2 I_x + (a_y)^2 I_y + (a_z)^2 I_z - 2a_y a_z I_{zy} - 2a_x a_z I_{xz} - 2a_x a_y I_{yx}$$

Moment of Inertia in Special Coordinates

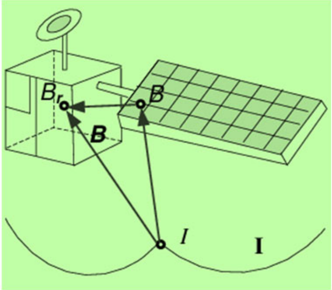
$$T_{Rotational} = \frac{1}{2} I_{\xi} \omega^2 ; \ \vec{\omega} = \omega \vec{1}_{\xi}$$

$$I_{\xi} = (a_x)^2 I_x + (a_y)^2 I_y + (a_z)^2 I_z - 2a_y a_z I_{zy} - 2a_x a_z I_{xz} - 2a_x a_y I_{yx}$$

- The given relation for T_{ROT} is very simple and attractive, but its utility requires the direction cosines as well as MOI properties in the body CS.
- I_{ζ} could be continuously changing if the SC angular velocity vector changes.

However, it would be interesting to choose a body CS in such a way that the product of inertias are zero! that further simplifies the relation. In addition, control engineers usually prefer to deal with SC with zero or negligible product of inertia that subsequently simplifies ACS designs. In these cases, small nonzero product of inertias emanating out of SC assembly and production will be considered as disturbances to check for robust ACS designs.

Inertial properties	Value	Unit
Mass	2501.45	Kg
I_{xx}	1.0225E+05	$kg \cdot m^2$
I_{yy}	1.7444E+04	$kg \cdot m^2$
I_{zz}	1.1472E+05	$kg \cdot m^2$



Principal Axes of Inertia

The problem is to convert the inertia matrix or the MOI tensor into a diagonal matrix. As this is a common routine in linear algebra, its proof is not presented here.

$$T_{Rotational} = \frac{1}{2}\vec{\omega}^T I \vec{\omega}$$

In the primary body CS, the components of $\vec{\omega}$ are ω_x , ω_y , ω_z. In order to obtain a new coordinate frame in which the MOI matrix is diagonal, a transformation or rotation matrix (TM) will be required. Lets assume that this TM is denoted by $\boldsymbol{A}$. Thus, the relationship between the components of $\vec{\omega}$ in two body CSs will be :

$$\vec{\omega} = A\vec{\omega}'; \;\; \omega' \triangleq \textit{the angular velocity in the new frame}$$

Now we can rewrite the T_{ROT} relation as :

$$2T_{rotational} = \vec{\omega}^T I \vec{\omega} \Rightarrow 2T_{rotational} = \left(A\vec{\omega}'\right)^T I\left(A\vec{\omega}'\right) = \vec{\omega}'^T A^T I A \vec{\omega}' = \vec{\omega}'^T I' \vec{\omega}' \;; where \quad \left(I' \triangleq diagonal = A^T I A\right)$$

That means that in the new frame, the inertia tensor will be diagonal. $2T_{Rotational} = \vec{\omega}'^T I' \vec{\omega}'$

The eigenvalues (λ_i) of the **inertia matrix** $\boldsymbol{I}$ will be the principal moments of inertias or in other words, the diagonal elements of I' that are obtained using $\det\left[I - \lambda[1]\right] = 0$. Likewise, the eigenvectors $I, \vec{e}_1, \vec{e}_2, \vec{e}_3$ will be the columns of the rotation matrix $\boldsymbol{A}$.

$$A = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{bmatrix}$$

Principal Axes of Inertia

$$2T_{ROT} = \omega_x^2 I_{xx} + \omega_y^2 I_{yy} + \omega_z^2 I_{zz} - 2\omega_x\omega_z I_{xz} - 2\omega_y\omega_z I_{yz} - 2\omega_x\omega_y I_{xy}$$

To generate the eigenvectors, you can use the Matlab or the solution of the following equation:

$$\lambda_i \vec{e}_i = [I]\vec{e}_i \quad , \quad i = 1,2,3$$

Conclusion: The diagonal elements of I' will be the principal moment of inertia and the new axis or frame is called the principal axes. The principal axes will contain the axis of maximum and minimum inertias.

Example**:** Suppose a SC MOI tensor is given. It is desired to diagonalize the MOI, by finding the corresponding TM or (A) as well as the new MOI in the principal axes frame (I').

$$I = \begin{bmatrix} 20 & -10 & 0 \\ -10 & 30 & 0 \\ 0 & 0 & 40 \end{bmatrix} {Nm}/{s^2} \Rightarrow 2T_{ROT} = 20\omega_x^2 + 30\omega_y^2 + 40\omega_z^2 + 20\omega_x\omega_y$$

The diagonalization process leads to:

$$I' = \begin{bmatrix} 13.82 & 0 & 0 \\ 0 & 36.18 & 0 \\ 0 & 0 & 40 \end{bmatrix} {Nm}/{s^2}; \quad A = \begin{bmatrix} 0.85066 & -0.527 & 0 \\ 0.527 & 0.85066 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\vec{\omega}' = A^T\vec{\omega} \Rightarrow \omega_1 = 0.85066\omega_x + 0.527\omega_y; \omega_2 = -0.52571\omega_x + 0.85066\omega_y; \omega_3 = \omega_z$$

$$2T_{ROT} = 13.82\omega_1^2 + 36.18\omega_2^2 + 40\omega_3^2$$

Principal Axes of Inertia : Example

1- generating eigenvalues using: $\det\left[I-\lambda[1]\right]=0$

$$\lambda^3-65\lambda^2+1025\lambda-750=0 \Rightarrow \begin{cases} \lambda_1 = 13.82 \\ \lambda_2 = 36.18 \\ \lambda_3 = 40 \end{cases}$$

2- for each λ_i, $\vec{e}_i$ is obtained by solving $[I]\vec{e}_i=\lambda_i\vec{e}_i$. For example, to determine $\vec{e}_1$:

$$\begin{array}{ccccccc} (20-13.82)e_{1x} & - & 10e_{1y} & + & 0 & = & 0 \\ -10e_{1x} & + & (30-13.82)e_{1y} & + & 0 & = & 0 \\ 0 & + & 0 & + & (40-13.82)e_{1z} & = & 0 \end{array}$$

$$\Rightarrow \begin{cases} \vec{e}_1 = [0.85066 \quad 0.527 \quad 0]^T \\ \vec{e}_2 = [-0.52571 \quad 0.85066 \quad 0]^T \\ \vec{e}_3 = [0 \quad 0 \quad 1]^T \end{cases}$$

Principal Axes of Inertia: Example

And similarly other columns of the TM can be computed:

$$A = \begin{bmatrix} e_{1x} & e_{2x} & e_{3x} \\ \vdots & \vdots & \vdots \end{bmatrix} = \begin{bmatrix} 0.85066 & -0.527 & 0 \\ 0.527 & 0.85066 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow \vec{\omega}' = A^{-1}\vec{\omega}$ or $\vec{\omega}' = A^{T}\vec{\omega}$ (duo to orthogon property of A)

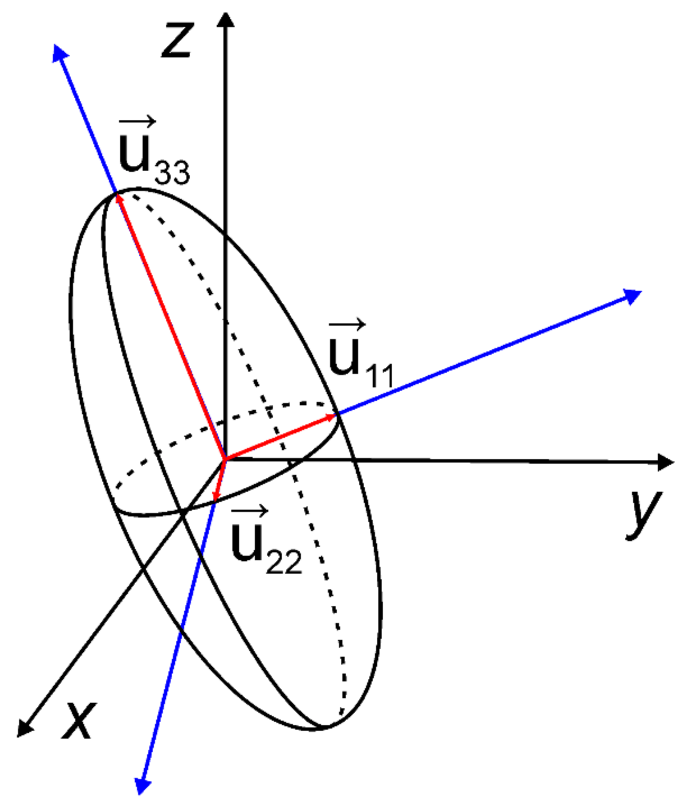
$$I' = A^{T} I A = \begin{bmatrix} 13.82 & 0 & 0 \\ 0 & 36.18 & 0 \\ 0 & 0 & 40 \end{bmatrix}$$

Note and summary: The column of the matrix A shows the direction cosines of the principal axes with the primary body axes. The third column of this matrix indicates that in reaching the principal axes system, the body CS must rotate around z_B and that is why in this process, the moment around z_B has not changed and the third column of A is as found.

Principal Axes of Inertia

Final Notes:

Fortunately for most regular geometries, the principal axes system can be determined by inspection, but the following guidelines can also be used:



- The axis of rotation for a body of revolution is always one of the principal axes. In addition, any transverse axes that passes through the center of mass will the other two principal axes.
- The plane of symmetry of a body will contain two principal axes, where third axis will be perpendicular to the symmetry plane.
- In general, the three principal axes that pass through the center of mass will include the axis of maximum and minimum inertia, " minor & major axes" respectively.

Principal Axes of Inertia

- Using the principal axes system and its associated moments simplify most of the relations. For example, components of the angular momentum vector will be:

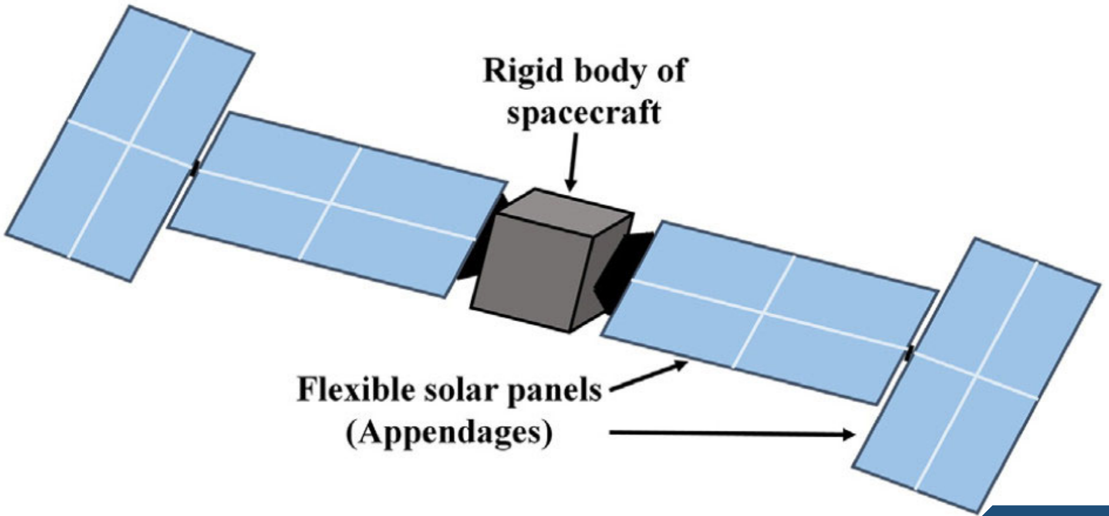
$$\begin{cases} h_1 = I_1\omega_1 \\ h_2 = I_2\omega_2 \Rightarrow \vec{h} = \left(I_1\omega_1\right)\vec{e}_1 + \left(I_2\omega_2\right)\vec{e}_2 + \left(I_3\omega_3\right)\vec{e}_3 \\ h_3 = I_3\omega_3 \end{cases}$$

- Based on previous relations, the moment of inertia around a principal spin axis (body will be:

$$I_\xi = \left(a_x\right)^2 I_1 + \left(a_y\right)^2 I_2 + \left(a_z\right)^2 I_3$$

$where : a_x, a_y, a_z$ are the direction cosines locating $\vec{\omega}$ w.r.t the principal axes

$$\text{i.e.} : \begin{cases} a_x = \vec{1}_\xi \cdot \vec{e}_1 \\ a_y = \vec{1}_\xi \cdot \vec{e}_2 \\ a_z = \vec{1}_\xi \cdot \vec{e}_3 \end{cases}$$



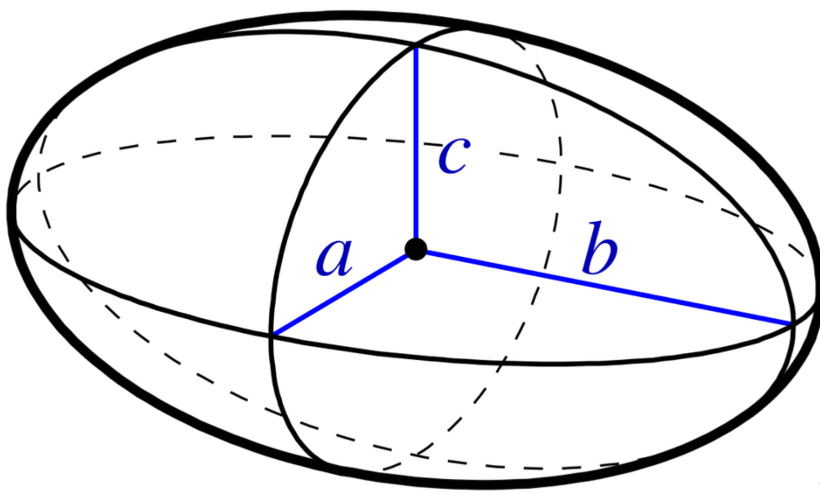
Ellipsoid of Inertia

Suppose that the principal axes of a SC are the same as the body axes frame, so according to previous equation, the moment of inertia around the spin axis or in the direction of $\vec{\omega}$ will be:

$$I_\xi = (a_x)^2 I_x + (a_y)^2 I_y + (a_z)^2 I_z \;\; ; \; where \; a_i s \; are \; direction \; cosines$$

That is the MOI about any instantaneous rotation axis (in direction of $\vec{1}_\xi$). In addition, this form suggests that a surface can be defined to describe I_ξ changes with respect to the SC instantaneous spin axes relative to the principal axes. For this purpose, let :

$$\rho = \left(\frac{1}{I_\xi}\right)^{\frac{1}{2}} ; and : X = a_x\rho, Y = a_y\rho, Z = a_z\rho$$



That allows one to rewrite I_ξ into an ellipsoidal form with axial dimensions :

$$1 = \frac{1}{\sqrt{I_x}} \quad , \quad 2 = \frac{1}{\sqrt{I_y}} \quad , \quad 3 = \frac{1}{\sqrt{I_z}}$$

Ellipsoid of Inertia

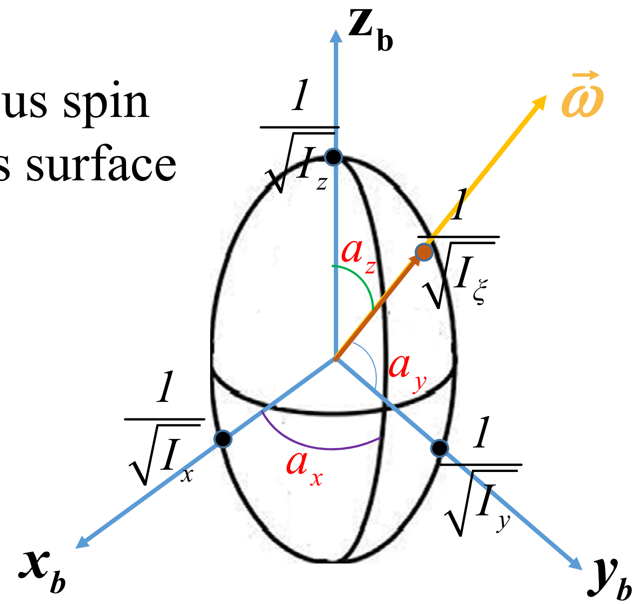
$$\rho = \left(\frac{1}{I_\xi}\right)^{\frac{1}{2}} ;and : X = a_x\rho, Y = a_y\rho, Z = a_z\rho$$

Therefore, we have:

$$I_\xi = (a_x)^2 I_x + (a_y)^2 I_y + (a_z)^2 I_z \Rightarrow 1 = X^2 I_x + Y^2 I_y + Z^2 I_z$$

$$or: \frac{X^2}{\left(\frac{1}{\sqrt{I_x}}\right)^2} + \frac{Y^2}{\left(\frac{1}{\sqrt{I_y}}\right)^2} + \frac{Z^2}{\left(\frac{1}{\sqrt{I_z}}\right)^2} = 1 \Rightarrow \text{ Axes dimensions: } 1 = \frac{1}{\sqrt{I_x}} \quad , \quad 2 = \frac{1}{\sqrt{I_y}} \quad , \quad 3 = \frac{1}{\sqrt{I_z}}$$

The above surface shows all possible values of inertia about various spin axis and is called the **ellipsoid of inertia**. Where, each point on its surface represents the value of the body's moment of inertia about a line (axis) that connects that point to the center.
Note that the angular momentum along with the rotational kinetic energy describes the dynamic state of a rotating body, therefore, using the principal axes, we have:



$$\vec{h} = I\vec{\omega} \quad , \quad h^2 = I_x^2\omega_x^2 + I_y^2\omega_y^2 + I_z^2\omega_z^2$$

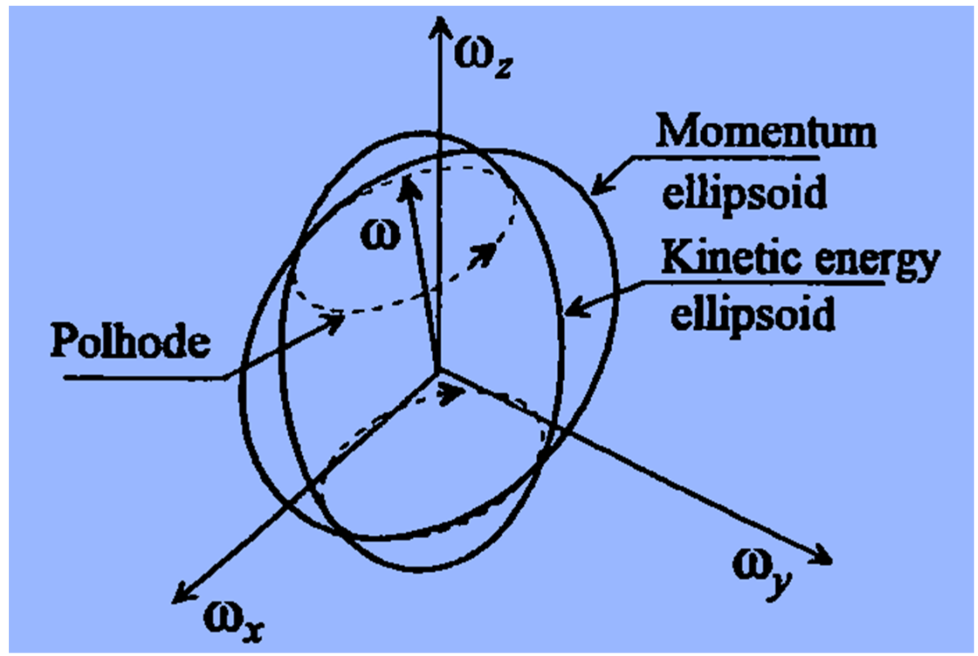
$$2T_{Rotational} = \vec{\omega}^T I \vec{\omega} = I_x^2\omega_x^2 + I_y^2\omega_y^2 + I_z^2\omega_z^2$$

Ellipsoid of Inertia

Using the previous equations, one can similarly define the **angular momentum (AM) and rotational kinetic energy (RKE) ellipsoids**, in which the components of the SC **angular velocity vector** are defined as variables. To simplify the relations, let $T_{ROT} = T$, which will now yield the T and h ellipsoids.

$$\frac{\omega_x^2}{\left(\frac{h}{I_x}\right)^2} + \frac{\omega_y^2}{\left(\frac{h}{I_y}\right)^2} + \frac{\omega_z^2}{\left(\frac{h}{I_z}\right)^2} = 1$$

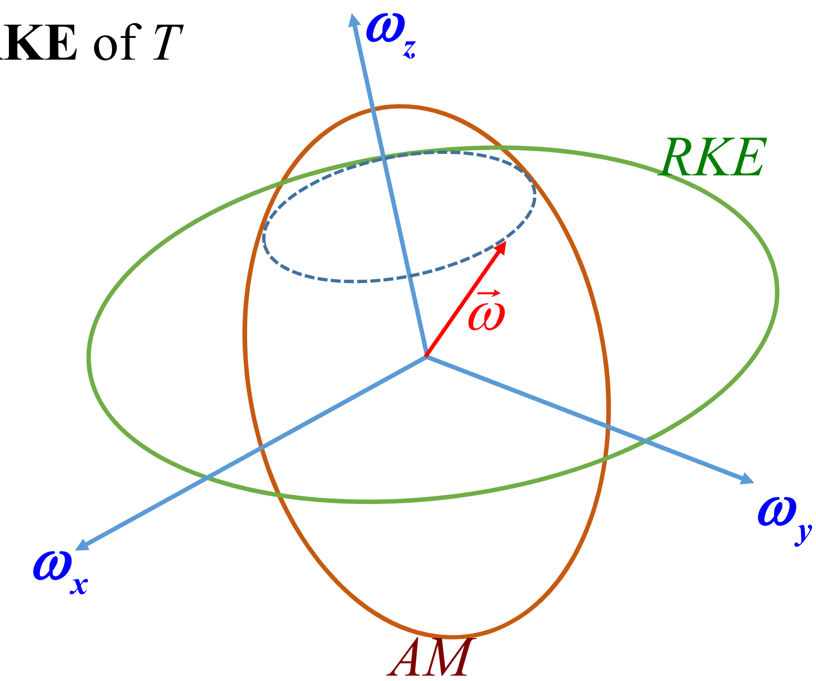
$$\frac{\omega_x^2}{\left(\sqrt{\frac{2T}{I_x}}\right)^2} + \frac{\omega_y^2}{\left(\sqrt{\frac{2T}{I_y}}\right)^2} + \frac{\omega_z^2}{\left(\sqrt{\frac{2T}{I_z}}\right)^2} = 1$$



Ellipsoid of Inertia

Note that the **RKE** ellipsoid has axial dimensions of $\sqrt{2T/I_x}$, $\sqrt{2T/I_y}$, $\sqrt{2T/I_z}$, and its surface represents all possible values of $\vec{\omega}$ that satisfy its relation. Similarly, the axial dimensions of the **AM** ellipsoid will be $\frac{h}{I_x}$; $\frac{h}{I_y}$; $\frac{h}{I_z}$ that is the geometric loci of all possible values of $\vec{\omega}$ that satisfy its relation. Therefore, for any spinning SC with **RKE** of T and **AM of** h, two ellipsoids define all possible values of angular velocities. In particular, the two ellipsoids intersect each other along the curves that satisfying both relations are of interest. These curves are known as **Polhode** and represent simultaneous solutions of both ellipsoids, as the loci of all possible values of $\vec{\omega}$ at that energy level.

Note that for different angular momentum and rotational energy, different polhodes are created.



Euler's Based Attitude Dynamics (AD)

As noted earlier, the rotational dynamic EOM of a rigid SC can be derived via the basic Euler's law of angular momentum. $\left.\frac{d\vec{h}}{dt}\right|_I = \vec{M}^I$

That led to the conservation of $\vec{h}$ in the inertial space in 2BP under pure mutual gravitational attraction and nothing else. Now, the SC can be acted upon by some torques $\vec{M}$ due to external disturbances or attitude control commands, that is : $\vec{M} = \vec{M}_D + \vec{M}_C$

As it is easier to work in the Body frame, again application of the Coriolis law gives:

$\dot{\vec{h}}_I = \dot{\vec{h}}_B + (\vec{\omega} \times \vec{h})_B$, that is called Euler's Moment Equation. By introducing the vector components of $\vec{h}$ and $\vec{\omega}$ in the body frame, we will arrive at basic AD EOM.

$\dot{\vec{h}}_B \triangleq \textit{Derivative in Body Frame; so } \dot{\vec{h}}_B = \begin{bmatrix} \dot{h}_x & \dot{h}_y & \dot{h}_z \end{bmatrix}^T \textit{;and} \quad \vec{\omega}_B = \begin{bmatrix} \omega_x & \omega_y & \omega_z \end{bmatrix}^T$

$$\begin{cases} M_x = \dot{h}_x + \omega_y h_z - \omega_z h_y \\ M_y = \dot{h}_y + \omega_z h_x - \omega_x h_z \\ M_z = \dot{h}_z + \omega_x h_y - \omega_y h_x \end{cases}$$

Euler's Moment Equation (EME)

$$\begin{cases} M_x = \dot{h}_x + \omega_y h_z - \omega_z h_y \\ M_y = \dot{h}_y + \omega_z h_x - \omega_x h_z \\ M_z = \dot{h}_z + \omega_x h_y - \omega_y h_x \end{cases}$$

The EME describes the general attitude behavior of a rigid SC that can either be converted into a set of **first order** DE for $\vec{\omega}$ using the $\vec{h}$ components derived earlier, or to a set of **second order** DE based on the Euler angles using the kinematic EOMs. This issue will be detailed again later. Now, for example assuming the BCS to be the principal axes system:

$$\vec{M} = \begin{Bmatrix} \left[\dot{h}_x + \left(\omega_y h_z - \omega_z h_y\right)\right]\vec{i} \\ \left[\dot{h}_y + \left(\omega_z h_x - \omega_x h_z\right) + \right]\vec{j} \\ \left[\dot{h}_z + \left(\omega_x h_y - \omega_y h_x\right) + \right]\vec{k} \end{Bmatrix} \Rightarrow \vec{M} = \begin{Bmatrix} \left[\dot{\omega}_x I_{xx} + \left(\omega_y \omega_z I_{zz} - \omega_z \omega_y I_{yy}\right)\right]\vec{i} \\ \left[\dot{\omega}_y I_{yy} + \left(\omega_z \omega_x I_{xx} - \omega_x \omega_z I_{zz}\right) + \right]\vec{j} \\ \left[\dot{\omega}_z I_{zz} + \left(\omega_x \omega_y I_{yy} - \omega_y \omega_x I_{xx}\right) + \right]\vec{k} \end{Bmatrix}$$

and using the Euler angles method for propagation, the above AD EOM can be directly related to Euler angles and the SC attitude due to $\vec{M}$ *as it orbits.*

$$\begin{cases} \omega_x = \dot{\phi} - \dot{\psi} \sin\theta \\ \omega_y = \dot{\theta} \cos\phi + \dot{\psi} \cos\theta \sin\phi \\ \omega_z = \dot{\psi} \cos\theta - \dot{\theta} \sin\psi \end{cases}$$

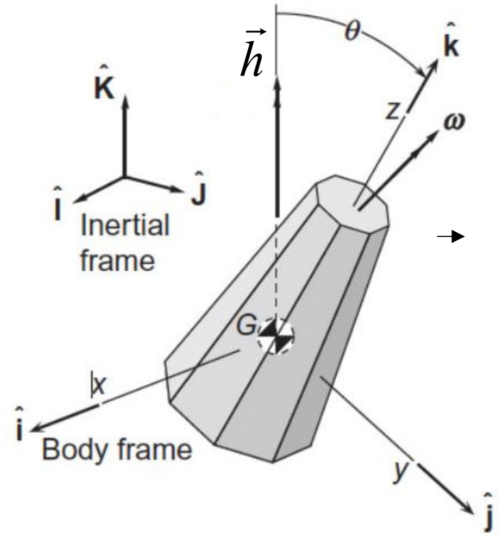
Some Observations with EME

Torque Free Motion of an Axisymmetric SC Body

Using the EME in principal axes while assuming a Torque free motion, $\vec{M} = 0$ yields:

$$\begin{cases} I_x\dot{\omega}_x + \omega_y\omega_z\left(I_z - I_y\right) = 0 \\ I_y\dot{\omega}_y + \omega_x\omega_z\left(I_x - I_z\right) = 0 \\ I_z\dot{\omega}_z + \omega_x\omega_y\left(I_y - I_x\right) = 0 \end{cases}$$

$$\begin{Bmatrix} \left[\dot{\omega}_x I_{xx} + \left(\omega_y\omega_z I_{zz} - \omega_z\omega_y I_{yy}\right)\right]\vec{i} \\ \left[\dot{\omega}_y I_{yy} + \left(\omega_z\omega_x I_{xx} - \omega_x\omega_z I_{zz}\right) + \right]\vec{j} \\ \left[\dot{\omega}_z I_{zz} + \left(\omega_x\omega_y I_{yy} - \omega_y\omega_x I_{xx}\right) + \right]\vec{k} \end{Bmatrix} = \vec{M}$$



As the above equations are nonlinear, there is no analytical solution in closed form. But, some observation can be made. Assuming the axis of symmetry in z the direction ($I_x = I_y$) , further simplifies the EME:

$$\begin{cases} I_x\dot{\omega}_x + \omega_y n\left(I_z - I_y\right) = 0 \\ I_y\dot{\omega}_y + \omega_x n\left(I_x - I_z\right) = 0 \\ I_z\dot{\omega}_z = 0 \end{cases}$$

A body is ***axisymmetric*** if any two of its three principal MOIs are equal. The axis corresponding to the third MOI is called the axis of symmetry. An axisymmetric body having the axis of symmetry as its minor principal axis is called **prolate.** An axisymmetric body is called **oblate** if its axis of symmetry is its major principal axis of inertia.

Torque Free Motion of an Axisymmetric SC

$$\begin{cases} I_x\dot{\omega}_x + \omega_y n\left(I_z - I_y\right) = 0 \\ I_y\dot{\omega}_y + \omega_x n\left(I_x - I_z\right) = 0 \\ I_z\dot{\omega}_z = 0 \end{cases}$$

$$n = \omega_z, spin\ rate$$

Let's define the parameter $\lambda = \frac{n(I_z - I_x)}{I_x}$, and reduce the governing equation further : (Note that $I_x = I_y$):

$$\begin{cases} \dot{\omega}_x + \lambda\omega_y = 0 \\ \dot{\omega}_y - \lambda\omega_x = 0 \end{cases}$$

If we multiply the previous relations by ω_x, ω_y respectively and add them together, we arrive at the following equation:

$$\omega_x\dot{\omega}_x + \omega_y\dot{\omega}_y = 0 \Rightarrow \omega_x \mathrm{d}\omega_x + \omega_y \mathrm{d}\omega_y = 0 \Rightarrow \omega_x^2 + \omega_y^2 = \text{const} = \omega_{xy}^2$$

where ω_{xy} is the component of $\vec{\omega}$ in the body *xy*-plane! Since ω_z (spin rate) is also constant, we conclude that in such conditions the SC will have an angular velocity vector with constant norm!. In other words, ($|\vec{\omega}| = const$), i.e. $\omega^2 = \omega_x^2 + \omega_y^2 + \omega_z^2$ will be constant not $\vec{\omega}$. In addition, via the simplified equations above (with λ), the initial conditions will be :

$$\dot{\omega}_x(0) = -\lambda\omega_y(0) \quad , \quad \dot{\omega}_y(0) = -\lambda\omega_x(0)$$

Torque Free Motion of an Axisymmetric SC

$$\begin{cases} \dot{\omega}_x + \lambda\omega_y = 0 \\ \dot{\omega}_y - \lambda\omega_x = 0 \end{cases}$$

Taking derivative from the $\dot{\omega}_x$ equation gives :

$$\ddot{\omega}_x + \lambda\dot{\omega}_y = \ddot{\omega}_x + \lambda^2\omega_x = 0 \Rightarrow \omega_x(s) = \frac{\dot{\omega}_x(0) + s\omega_x(0)}{s^2 + \lambda^2}$$

$$\omega_x(t) = \omega_x(0)\cos\lambda t + \frac{\dot{\omega}_x(0)}{\lambda}\sin\lambda t$$

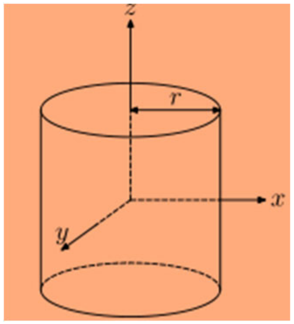
$$\omega_y(t) = -\frac{\dot{\omega}_x(t)}{\lambda} = \omega_x(0)\sin\lambda t - \frac{\dot{\omega}_x(t)}{\lambda}\cos\lambda t$$

Finally, using complex numbers notation a relation can be defined for a **in-plane** or transverse component of $\vec{\omega}$ or ω_{xy}. In other words:

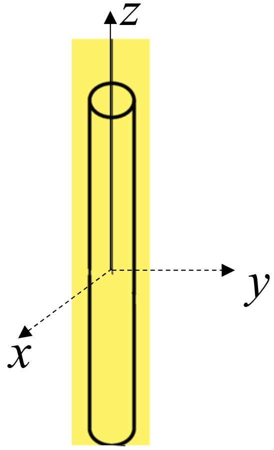
$$\omega_{xy}(t) = \omega_x + i\omega_y = \left[\omega_x(0) + i\omega_y(0)\right]\left(\cos\lambda t + i\sin\lambda t\right)$$

$$= \left[\omega_x(0) + i\omega_y(0)\right]e^{i\lambda t}$$

$$= \omega_{xy}(0)e^{i\lambda t}$$



$I_z > I_x$



$I_z < I_x$

Nutation Motion of a Spinning SC

$$\omega_{xy}(t) = \omega_{xy}(0)e^{i\lambda t}$$

The latter equation implies that ω_{xy} rotates at a rate of λ relative to the principal axes. That is, a body observer on the symmetry axis z, will see $\vec{\omega}$ changing at the rate λ, in the positive direction around z. Of course, note the this motion due to initial conditions (disturbances, $\omega_{xy}(0) \neq 0$) and there are externally applied torques.

Review of results:

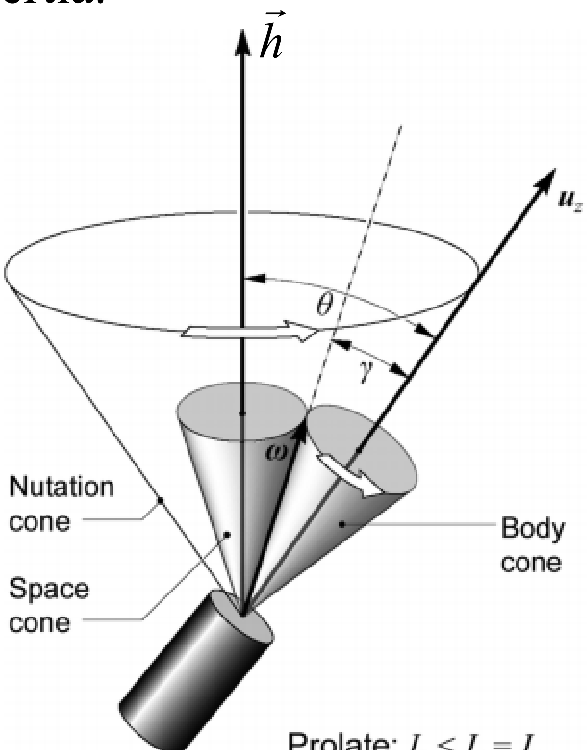
Considering the torque free motion, $\vec{M} = 0$, the angular momentum vector $\vec{h}$ will be constant in the inertial space, $\vec{h} = [I_x\omega_x \quad I_y\omega_y \quad I_z\omega_z]^T$. On the other hand, $\vec{h}$ has a component in the xy plane,

$$\vec{h}_{xy} = I_x\left(\omega_x\vec{i} + \omega_y\vec{j}\right) = I_x\vec{\omega}_{xy} \Rightarrow \vec{h} = I_x\vec{\omega}_{xy} + I_z\omega_z\vec{k}$$

Given that $\vec{\omega} = \vec{\omega}_{xy} + \omega_z\vec{k}$, It can be said that the three vectors $\vec{h}, \vec{\omega}, \vec{k}$ will always be in the same plane (despite the time behavior of ω_{xy}) and that $\vec{h}$ is **constant in the inertial space**. While usually, $\vec{h}$ *and* $\vec{\omega}$ are not collinear (or parallel, that is related to pure spin about the symmetry axis), the SC motion is described by two cones rolling against each other. This motion is graphically described in the next slide.

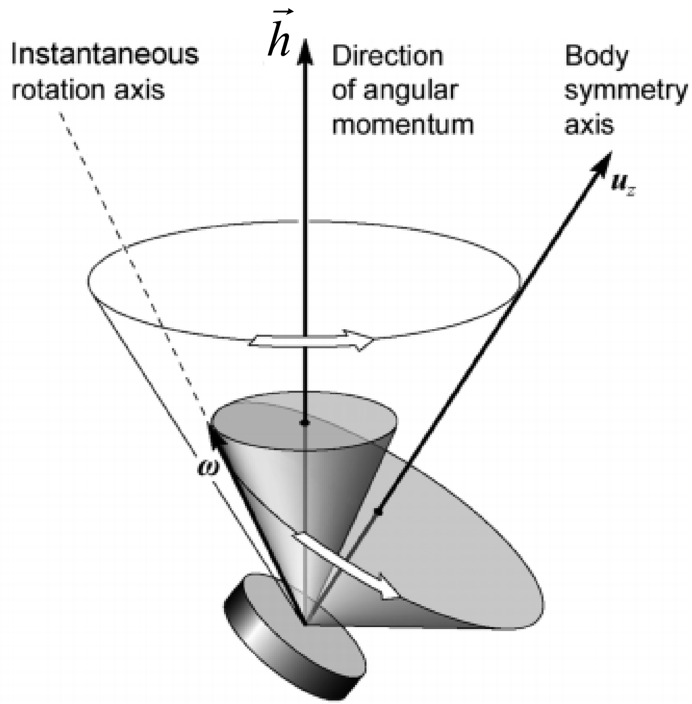
Nutation Motion of a Spinning SC

As shown in both figures, the trajectory of $\vec{\omega}$ creates a **body cone** rolling against what is called a **space cone** where $\vec{\omega}$ represents the tangent (contact) line. The body and space cones will be different based on the SC configuration (rod/disc) or relative values of the principal components of inertia.



Prolate: $I_z < I_x = I_y$

https://www.aero.iitb.ac.in/~bhat/prograde.gif



Oblate: $I_z > I_x = I_y$

https://www.aero.iitb.ac.in/~bhat/retrograde.gif

Nutation of a Spinning S/C

To fully describe the motion, one can easily determine the θ (nutation angle) and γ (the angle between $\vec{\omega}$ and $\vec{z}$). In essence, a perturbed rotating SC (could be due initial condition or injection) with no applied external torques will be **wobbling in space**. This is not a desirable behavior and will degrade the mission performance. In addition, keeping the nutation angle small is one of the important tasks of attitude control system. In addition, the following are true :

$$\vec{h} = I_x \vec{\omega}_{xy} + I_z \omega_z \vec{k}$$

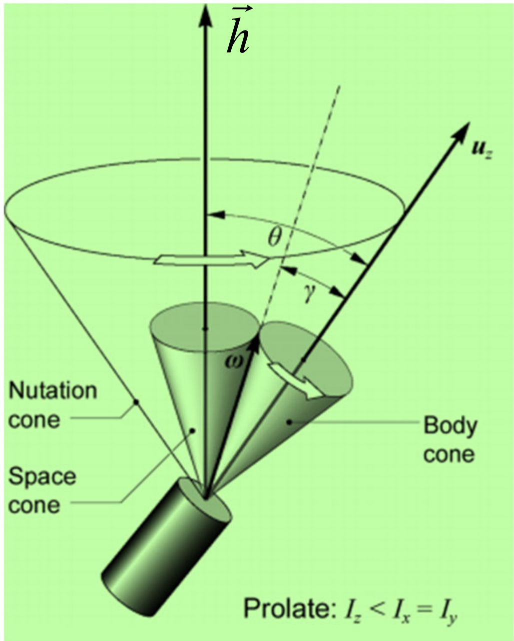
$$\vec{\omega} = \vec{\omega}_{xy} + \omega_z \vec{k}$$

1- $tan\,\theta = \dfrac{h_{xy}}{h_z} = \dfrac{I_x \omega_{xy}}{I_Z \omega_z} \quad , \quad tan\,\gamma = \dfrac{\omega_{xy}}{\omega_z}$

2- $tan\,\theta = \dfrac{I_x}{I_z} tan\,\gamma$

3- $\begin{cases} for\ a\ rod\ or\ rod\ type\ SC:\ \theta > \gamma \quad if \quad I_z < I_x \\ for\ a\ disk\ type\ SC:\ \theta < \gamma \quad if \quad I_z > I_x \end{cases}$

Note: If $\omega_{xy} = 0$,then $\theta = \gamma = 0$ and spin motion will be stable.



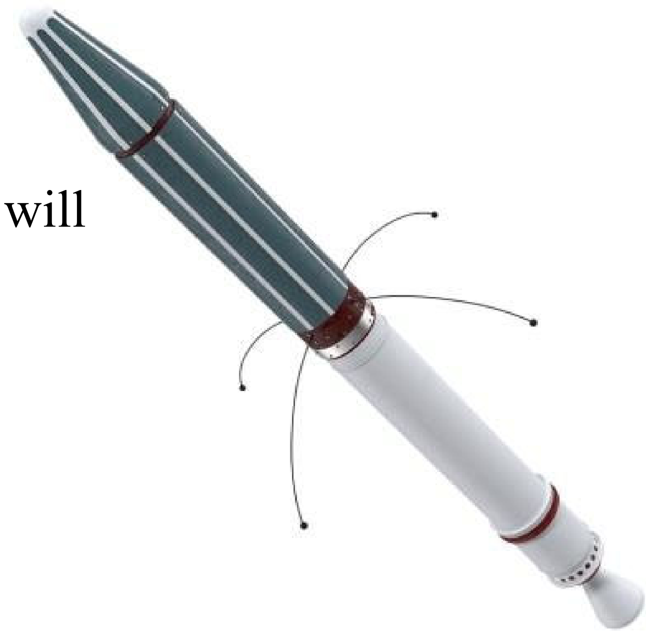
Energy Dissipation & Nutation Destabilization

In the previous parts, it was observed that the KRE around an instantaneous axis of rotation is equal to:

$$T = T_{Rot} = \frac{1}{2} I_\xi \omega^2 = \frac{1}{2}\frac{h^2}{I_\xi}$$

In the torque free motion, the rotating SC angular momentum, $\vec{h}$ will remain constant. Where I_ξ depends on the direction of the axis of rotation in the body coordinate frame. So, since h is constant, the maximum and minimum values of T will occur for the minor and major axes. In other words:

$$\begin{cases} T_{max} = \dfrac{h^2}{2I_{min}} & \textit{at the minor axis} \\ T_{min} = \dfrac{h^2}{2I_{max}} & \textit{at the major axis} \end{cases}$$



Explorer-1, the first U.S. satellite lunched in 1958

Energy Dissipation & Nutation Destabilization

$$\begin{cases} T_{max} = \dfrac{h^2}{2I_{min}} & \textit{at the minor axis} \\ T_{min} = \dfrac{h^2}{2I_{max}} & \textit{at the major axis} \end{cases}$$

According to T_{max} relation with h being constant, if the body spins about the minor axis and there is some internal **energy dissipation** mechanism that tends to decrease the RKE to its minimum, then the body will switch the spin of rotation to the major axis in order to satisfy T_{min} relation. This phenomenon was observed Explorer I, that made it rotationally unstable. For simplification, assume $I_x = I_y$:

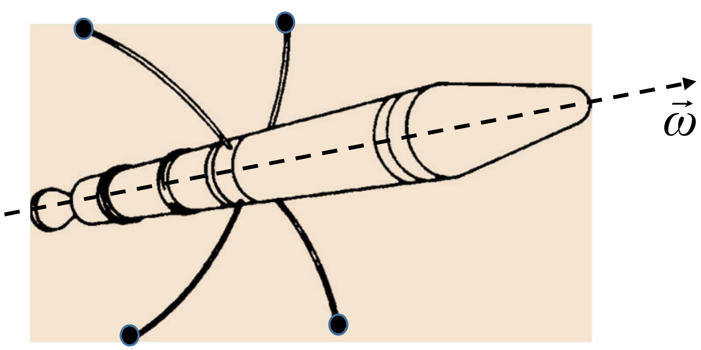
$$h^2 = \left[\omega_x^2 + \omega_y^2\right] I_x^2 + I_z^2 \omega_z^2 \xrightarrow{2T = \vec{\omega}^T I \vec{\omega}} 2T = \left[\omega_x^2 + \omega_y^2\right] I_x + \omega_z^2 I_z$$

Multiply the 2nd equation by I_x and subtract it from the first, to get:

$$h^2 - 2TI_x = \omega_z^2 \left(I_z\right)\left(I_z - I_x\right) = \omega_z^2 I_z^2 \left(\frac{I_z - I_x}{I_z}\right)$$

Noting that, $\cos\theta = \frac{h_z}{h} = \frac{I_z \omega_z}{h}$, yields:

$$2TI_x = h^2 - h^2 \cos^2\theta \left(\frac{I_z - I_x}{I_z}\right) = h^2 \sin^2\theta \left(\frac{I_z - I_x}{I_z}\right)$$



This energy dissipation was attributed to small flexible wire type antennas.

Energy Dissipation & Nutation Destabilization

$$2TI_x = h^2 \sin^2\theta\left(\frac{I_z - I_x}{I_z}\right)$$

Differentiating this equation with time yields:

$$2\dot{T}I_x = 2h^2\dot{\theta}\sin\theta\cos\theta\left(1-\frac{I_x}{I_z}\right) \Rightarrow \dot{T} = \frac{h^2}{I_x}\sin\theta\cos\theta\left(1-\frac{I_x}{I_z}\right)\dot{\theta}$$

In conclusion, one can write (simplify):

$$\dot{T} = \frac{h^2}{I_z}\sin\theta\cos\theta\left(\frac{I_z}{I_x}-1\right)\dot{\theta}$$

1- If $\dot{T} < 0$ and $I_z > I_x$ (disk type), then $\dot{\theta} < 0 \Rightarrow$ Nutation stability

2- If $\dot{T} < 0$ and $I_z < I_x$ (rod type), then $\dot{\theta} > 0 \Rightarrow$ Nutation instability

In other words, in presense of a dissipation mechanism, a spinning SC would have nutation stability if the spin axes is along the major axis.

Inertial stabilization via spinning is a common, cheap and efficient solution that was used for pioneer satellites. Modern SC can achieve stabilization via active 3-axis-stabilization methods.

Stability of Rotation about Principal Axes with no Precondition

In the previous discussion, an axisymmetric body with equal $I_x = I_y$ was considered in terms of stability. In this section, torque free motion will be investigated without this equality requirement. In this context, the stability criteria can be investigated around each spin axis (after disturbance) from an steady spin motion. Hence, the motion will be considered stable if the perturbed quantities show bounded behavior. The existence of stable motion can be assessed by perturbing the steady motion. That is, if $\vec{\omega}$ is initially parallel any of the x, y, and z axes, the torque free equations of motion must show stability criteria for ω_x, ω_y, and ω_z. Let's assume stability condition is sought, if the SC is spinning **about the z axis.**

In other words, initially $\vec{\omega} = n\vec{k}$. $Now,$ let $\omega_z = n + \varepsilon, where\ \varepsilon\ is$ the perturbation. Since, ω_x, ω_y are initially zero, they will refer to pertubation in the following along with ε. Using the previous Torque free EOMs (as perturbed form), we can investigate SC stability without precondition :

$$\begin{cases} I_x\dot{\omega}_x + \omega_y\omega_z\left(I_z - I_y\right) = 0 \\ I_y\dot{\omega}_y + \omega_x\omega_z\left(I_x - I_z\right) = 0 \\ I_z\dot{\omega}_z + \omega_x\omega_y\left(I_y - I_x\right) = 0 \end{cases} \xrightarrow{Linrearized\ \ Form} \begin{cases} I_x\dot{\omega}_x + \omega_y n\left(I_z - I_y\right) = 0 \\ I_y\dot{\omega}_y + \omega_x n\left(I_x - I_z\right) = 0 \\ I_z\dot{\varepsilon} + \omega_x\omega_y\left(I_y - I_x\right) = 0 \end{cases}$$

Stability of Rotation about Principal Axes with no Precondition

The first two equations are linear. Differentiating, the first equation While using the second equation in the process yields:

$$\begin{cases} I_x\dot{\omega}_x + \omega_y n\left(I_z - I_y\right) = 0 \\ I_y\dot{\omega}_y + \omega_x n\left(I_x - I_z\right) = 0 \\ I_z\dot{\varepsilon} + \omega_x\omega_y\left(I_y - I_x\right) = 0 \end{cases}$$

$$I_x\ddot{\omega}_x + n\left(I_z - I_y\right)\left(\frac{-\omega_x n\left(I_x - I_z\right)}{I_y}\right) = 0$$

$$\Rightarrow \ddot{\omega}_x + n^2\frac{I_z - I_y}{I_y}\frac{I_z - I_x}{I_x}\omega_x = 0$$

$$s^2 + \beta^2 = 0 \quad , \quad \beta = n\sqrt{\left[\left(1 - \frac{I_z}{I_x}\right)\left(1 - \frac{I_z}{I_y}\right)\right]}$$

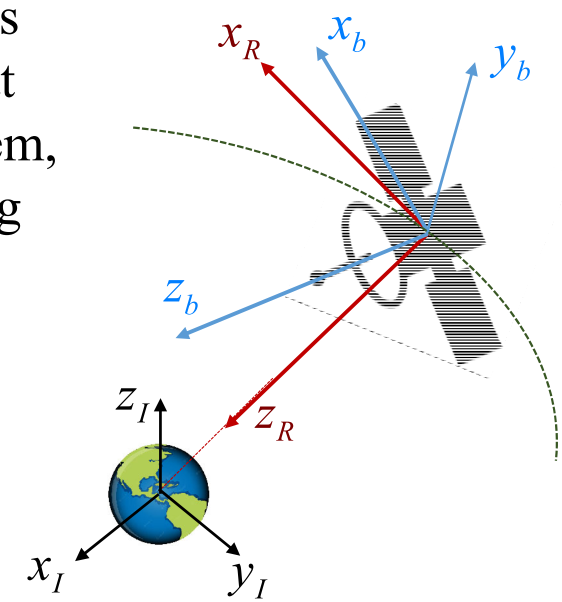
The stability condition can be detected using the characteristic equation for ω_x, the solution of which $s^2 = -\beta^2$. Thus, for stability (neutral), β is required to be real. Similar result will be obtained for ω_y .

In other words, **the stability conditions are**: $I_z > I_x, I_y$ or $I_z < I_x, I_y$

That means , for a rigid spinning SC about its maximum axis (or maximum) of moment of inertia, the angular motion WRT will be stable about other axes (spin about x and y). In addition, spinning about an intermediate axis will be unstable.

Kinematics Equation of Motion (KEOM)

Before introducing the kinematic equations of motion, it is necessary to state that the choice of a coordinate system at any stage of the satellite's operation or life is dependent on the task or conditions of the SC at that stage. For example, for an interplanetary mission within the solar system, the SC starts from a parking orbit around the Earth, for which a moving reference frame will be appropriate SC orientation relative to ECI. Midway, out of the Earth SOI, another inertial reference is utilized for attitude in space. And finally, in the vicinity of the target planet, the reference orbit around the target planet will be the best choice for attitude planning. There are in general various ways of attitude representation and propagation **(ARP)** that include:



- Direction Cosine Matrix
- Rotation Vector
- Euler Angles method
- Quaternions
- Gibbs/Rodrigues Parameters

$$\vec{i}_R = \vec{j}_R \times \vec{k}_R \quad ; \quad \vec{j}_R = \frac{\vec{v} \times \vec{r}}{|\vec{v} \times \vec{r}|} \quad ; \quad \vec{k}_R = -\frac{\vec{r}}{r}$$

Kinematics Equation of Motion (KEOM)

The attitude of the satellite relative to any reference frame (RF) is possible through utility of any of the previously mentioned ARP methods.

1- **The Euler Angles Method (EA)** ,

This is the most popular method for ARP, though it has some singularities in the propagation process. Via utility of EA, it is possible to establish a continuous relationship between the body frame and the orbital reference frame. EA is defined by three subsequent orderly rotations of the orbit RF to coincide it with the SC BF. The 3,2,1 sequence of rotations with ψ, θ, ϕ develops the following TM and ARP relations:

$$\mathbf{C}_{\mathbf{b}}^{\mathrm{R}} = \begin{bmatrix} \cos\theta\cos\psi & -\cos\varphi\sin\psi + \sin\varphi\sin\theta\cos\psi & \sin\varphi\sin\psi + \cos\varphi\sin\theta\cos\psi \\ \cos\theta\sin\psi & \cos\varphi\cos\psi + \sin\varphi\sin\theta\sin\psi & -\sin\varphi\cos\psi + \cos\varphi\sin\theta\sin\psi \\ -\sin\theta & \sin\varphi\cos\theta & \cos\varphi\cos\theta \end{bmatrix}$$

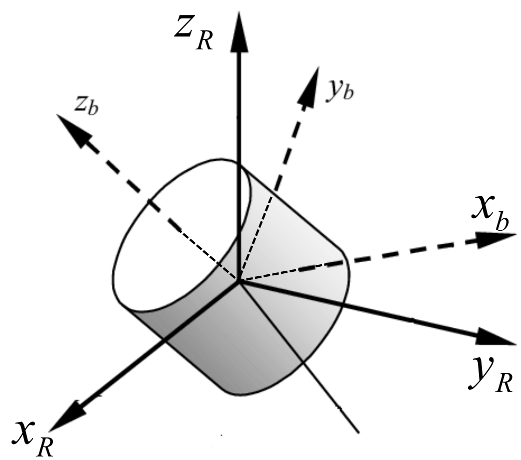
$$\begin{cases} p = \dot{\varphi} - \dot{\psi}\, sin\,\theta \\ q = \dot{\theta} cos\varphi + \dot{\psi}\, cos\,\theta sin\varphi \\ r = \dot{\psi}\, cos\,\theta cos\varphi - \dot{\theta} sin\varphi \end{cases} \quad or \quad \begin{cases} \dot{\varphi} = p + [q sin\varphi + r cos\varphi]\, tan\,\theta \\ \dot{\theta} = q cos\varphi - r sin\varphi \\ \dot{\psi} = [q sin\varphi + r cos\varphi]\, sec\,\theta \end{cases} \quad or \quad \begin{Bmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{Bmatrix} = \begin{bmatrix} 1 & sin\varphi\ tan\theta & cos\varphi\ tan\theta \\ 0 & cos\varphi & -sin\varphi \\ 0 & \dfrac{sin\varphi}{cos\theta} & \dfrac{cos\varphi}{cos\theta} \end{bmatrix} \begin{Bmatrix} p \\ q \\ r \end{Bmatrix}$$

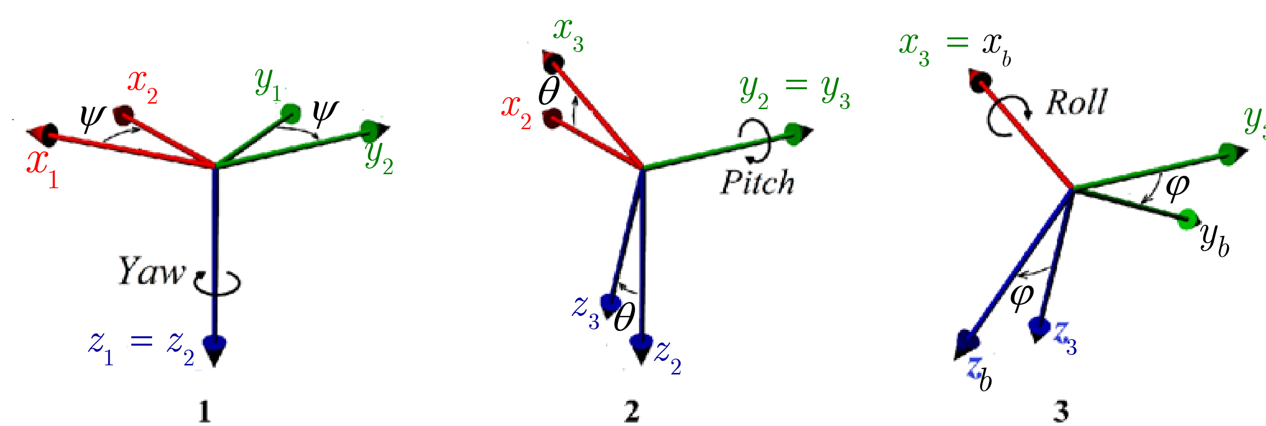
Kinematics Equation of Motion (KEOM)

The attitude of the satellite relative to any reference frame (RF) is possible through utility of any of the previously mentioned ARP methods.

1- **The Euler Angles Method (EA)** ,

This is the most popular method for ARP, though it has some singularities in the propagation process. Via utility of EA, it is possible to establish a continuous relationship between the body frame and the orbital reference frame. EA is defined by three subsequent orderly rotations of the orbit RF to coincide it with the SC BF. The 3,2,1 sequencee of rotations with ψ, θ, ϕ develops the following TM and ARP relations:





Kinematics Equation of Motion (KEOM)

Notes:

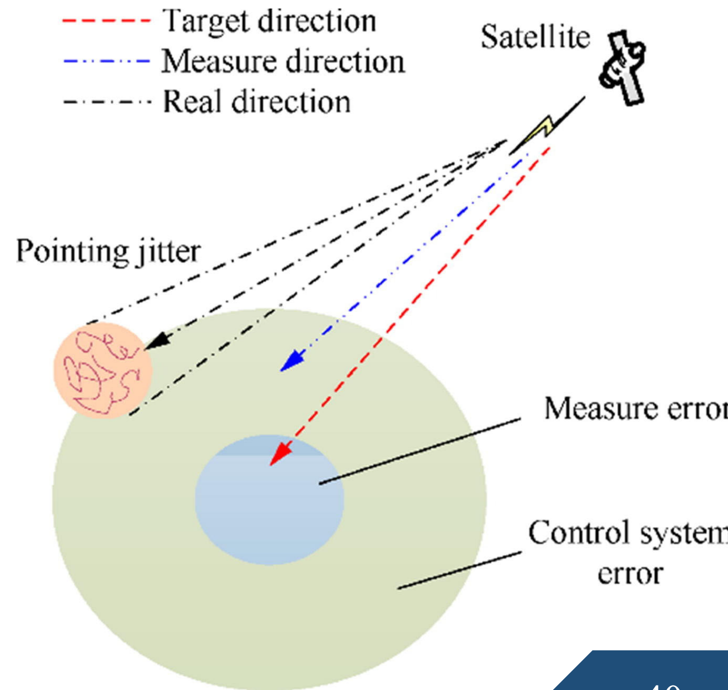
1-For very small Euler angles: $p \approx \dot{\varphi} \quad , \quad q \approx \dot{\theta} \quad , \quad r \approx \dot{\psi}$

2- $\vec{\omega}_{Rb} = \begin{bmatrix} p & q & r \end{bmatrix}^T$

3- The following summation rule can be proven between three frames: $\vec{\omega}_{ib}^{b} = \vec{\omega}_{iR}^{b} + \vec{\omega}_{Rb}^{b}$

$$\underbrace{\vec{\omega}_{ib}^{b}}_{\text{Determined or Measured}} = \underbrace{\vec{\omega}_{iR}^{b}}_{\text{Known}} + \underbrace{\vec{\omega}_{Rb}^{b}}_{\text{Determined}}$$

This way, by determining the state of the angular velocity vector through the solution of the Euler equation, $\vec{\omega}_{ib}$ and $\vec{\omega}_{Rb} = [p \quad q \quad r]^T$ are known and via the known initial conditions on the Euler angles, the APR equations can be utilized between the body and the reference frame for ACS purposes and instantaneous generation of TM matrix, $\mathbf{C}_{\mathbf{b}}^{\mathrm{R}}$. Similar procedures can be handled using DCM, Quaternions ,etc.



Kinematics Equation of Motion (KEOM)

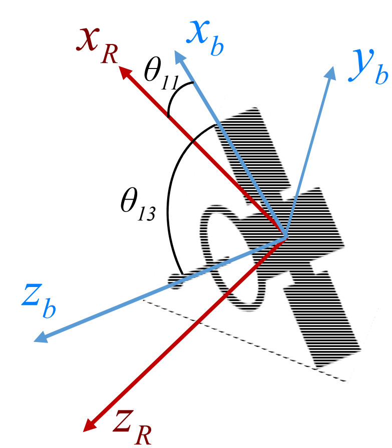
2- **Direction Cosine Matrix (DCM)** ,

DCM is another method to relate two CS, in this case the SC body and the moving reference frame. The DCM can be initially sought by taking the cosine of the angles between the unit base vectors of the two frames. In other words:

$$C_R^b = \left[cos\,\theta_{ij} \right] = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix} \Rightarrow C_R^b = \begin{bmatrix} \vec{b}_1.\vec{R}_1 & \vec{b}_1.\vec{R}_2 & \vec{b}_1.\vec{R}_3 \\ \vec{b}_2.\vec{R}_1 & \vec{b}_2.\vec{R}_2 & \vec{b}_3.\vec{R}_3 \\ \vec{b}_3.\vec{R}_1 & \vec{b}_3.\vec{R}_2 & \vec{b}_3.\vec{R}_3 \end{bmatrix}$$

$$where\ t_{ij} = cos\angle\left(b_i, R_j\right) : i, j = 1,2,3$$

$$e.g : t_{11} = cos\,\theta_{11} = cos\angle\left(b_1, R_1\right)$$



In addition, DCM can be easily propagated using the following set of ordinary DEs with no singularities.

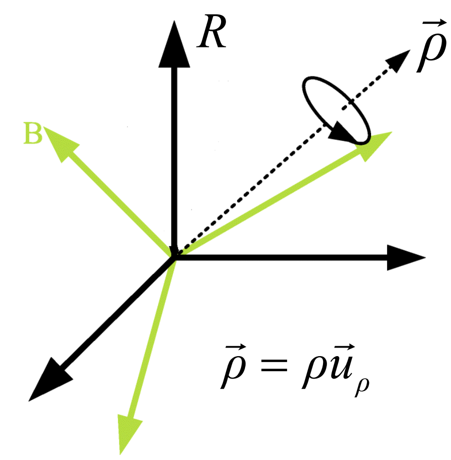
$$\dot{C}_b^R = C_b^R\, \Omega_{Rb}^b \quad where\ \Omega_{Rb}^b = \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix}$$

$$where : \Omega = \left[skew\text{-}symmetric\ form\ of\ \vec{\omega} \right]$$

Kinematics Equation of Motion

3- Quaternions,

The quaternions between two frames can be initially generated via the concept of a rotation vector (RV). RV can be thought of as a vector, $\vec{\rho} = \rho \vec{u}_\rho$ or $\vec{\varphi} = \varphi \vec{u}_\varphi$ (magnitude and direction) through which two frames coincide. Note, That there are some other alternative arrangements for Quaternions that may be found in some references.



$$\vec{q} = \begin{bmatrix} q_1 & q_2 & q_3 & q_4 \end{bmatrix}^T ,or\ \vec{q} = \begin{bmatrix} a & b & c & d \end{bmatrix}^T$$

where :

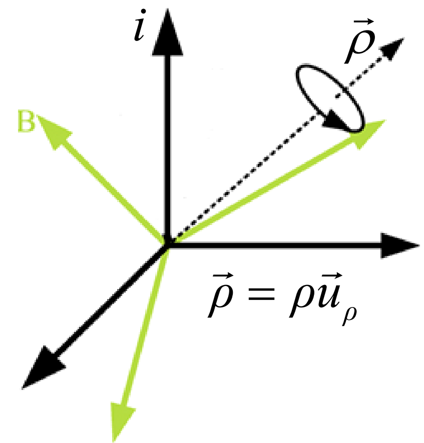
$$\vec{q} = \begin{Bmatrix} \cos\frac{\rho}{2} \\ \vec{u}_\rho \sin\frac{\rho}{2} \end{Bmatrix} = [\cos\frac{\rho}{2}; \frac{\rho_x}{\rho}\sin\frac{\rho}{2}; \frac{\rho_y}{\rho}\sin\frac{\rho}{2}; \frac{\rho_z}{\rho}\sin\frac{\rho}{2}]^T$$

$$C_b^R = \begin{bmatrix} (a^2+b^2-c^2-d^2) & 2(bc-ad) & 2(bd+ac) \\ 2(bc+ad) & (a^2-b^2+c^2-d^2) & 2(cd-ab) \\ 2(bd-ac) & 2(cd+ab) & (a^2-b^2-c^2+d^2) \end{bmatrix}$$

Kinematics Equation of Motion

3- Quaternion Propagation,

Quaternions are easily propagated with no singularities. They are mostly utilized for space utility. Here, propagation is given between the **body and inertial** CSs.



$$\dot{\vec{q}} = 0.5\, Q(\vec{q})\, p_{ib}^{b} \;\; ; \;\; where\ \vec{q} = \begin{bmatrix} a & b & c & d \end{bmatrix}^T \;\; and$$

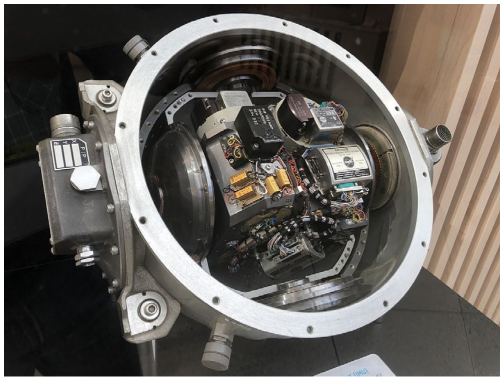
$$Q(\vec{q}) = \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix}; \quad p_{ib}^{b} = \begin{bmatrix} 0 & \vec{\omega}_{ib}^{b\,T} \end{bmatrix}^T = \begin{bmatrix} 0 & \omega_x & \omega_y & \omega_z \end{bmatrix}^T$$

$$\dot{a} = -0.5\left(b\omega_x + c\omega_y + d\omega_z\right)$$

$$\dot{b} = 0.5\left(a\omega_x - d\omega_y + c\omega_z\right)$$

$$\dot{c} = 0.5\left(d\omega_x + a\omega_y - b\omega_z\right)$$

$$\dot{d} = 0.5\left(-c\omega_x + b\omega_y + a\omega_z\right)$$



Relationship Between Different AR Methods for any Two Coordinate Frames

$$
\begin{aligned}
C_b^R &= \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \\
&= \begin{bmatrix} a^2+b^2-c^2-d^2 & 2(bc-ad) & 2(bd+ac) \\ 2(bc+ad) & a^2-b^2+c^2-d^2 & 2(cd-ab) \\ 2(bd-ac) & 2(cd+ab) & a^2-b^2-c^2+d^2 \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta\cos\psi & -\cos\varphi\sin\psi+\sin\varphi\sin\theta\cos\psi & \sin\varphi\sin\psi+\cos\varphi\sin\theta\cos\psi \\ \cos\theta\sin\psi & \cos\varphi\cos\psi+\sin\varphi\sin\theta\sin\psi & -\sin\varphi\cos\psi+\cos\varphi\sin\theta\sin\psi \\ -\sin\theta & \sin\varphi\cos\theta & \cos\varphi\cos\theta \end{bmatrix}
\end{aligned}
$$

Relationship Between Different AR Methods

Relationship Between Quaternions and DCM

$$a = \frac{1}{2}\left(1 + c_{11} + c_{22} + c_{33}\right)^{1/2} \qquad b = \frac{1}{4a}\left(c_{32} - c_{23}\right)$$

$$c = \frac{1}{4a}\left(c_{13} - c_{31}\right) \qquad d = \frac{1}{4a}\left(c_{21} - c_{12}\right)$$

$$a = \cos\frac{\phi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2} + \sin\frac{\phi}{2}\sin\frac{\theta}{2}\sin\frac{\psi}{2}$$
$$b = \sin\frac{\phi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2} - \cos\frac{\phi}{2}\sin\frac{\theta}{2}\sin\frac{\psi}{2}$$
$$c = \cos\frac{\phi}{2}\sin\frac{\theta}{2}\cos\frac{\psi}{2} + \sin\frac{\phi}{2}\cos\frac{\theta}{2}\sin\frac{\psi}{2}$$
$$d = \cos\frac{\phi}{2}\cos\frac{\theta}{2}\sin\frac{\psi}{2} + \sin\frac{\phi}{2}\sin\frac{\theta}{2}\cos\frac{\psi}{2}$$

Relationship Between Euler Angles and Quaternions

Relationship Between Euler Angles and DCM

$$\varphi = tan^{-1}\left(\frac{c_{32}}{c_{33}}\right) \qquad \theta = sin^{-1}\left(-c_{31}\right) \qquad \psi = tan^{-1}\left(\frac{c_{21}}{c_{11}}\right)$$

Kinematics Equation of Motion

As mentioned before, please note that in the propagation of Euler angles (or other ARP methods), it is required to know the angular velocity vector of the body frame with respect to the rotating frame, which was introduced with previously ($\vec{\omega}_{Rb} = [p \quad q \quad r]^T$).
According to the summation rule of $\vec{\omega}_{ib} = \vec{\omega}_{iR} + \vec{\omega}_{Rb}$, it can be seen that: $\vec{\omega}_{Rb} = \vec{\omega}_{ib} - \vec{\omega}_{iR}$
Given that the unit vectors of the reference frame R can be defined through the following simple equations: $\vec{i}_R = \vec{j}_R \times \vec{k}_R \quad ; \quad \vec{j}_R = \dfrac{\vec{v} \times \vec{r}}{|\vec{v} \times \vec{r}|} \quad ; \quad \vec{k}_R = -\dfrac{\vec{r}}{r}$

Another way to determine $\vec{\omega}_{iR}$, could be via getting the time of change of these unite vectors, i.e., the angular velocity of the orbit reference frame.

$$\frac{d\vec{i}_R}{dt} \quad ; \quad \frac{d\vec{j}_R}{dt} \quad ; \quad \frac{d\vec{k}_R}{dt}$$

$$\vec{\omega}_{iR} = \begin{bmatrix} \omega_i & \omega_j & \omega_k \end{bmatrix}$$

$$\omega_i = 0$$

$$\omega_j = \frac{-1}{r^2 |\vec{v} \times \vec{r}|} \left\{ \vec{v} \cdot \left[(\vec{r} \cdot \vec{v}) \vec{r} - (r^2 \vec{v}) \right] \right\}$$

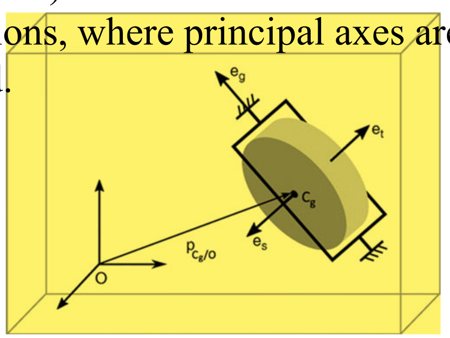
$$\omega_j = \frac{1}{|\vec{v} \times \vec{r}||\vec{r}|} \left[\ddot{\vec{r}} \times \vec{r} \right] \cdot \left[(\vec{v} \times \vec{r}) \times \vec{r} \right]$$

For Keplerian orbits, there is no out-of-plane component of $\vec{\omega}$, and therefore $\omega_k = 0$. As an example for a circular Keplerian reference orbit that has a constant angular velocity, we will have : $\omega_{iR}^R = [0 \quad -\omega_0 \quad 0]^T$

General Attitude Dynamics (AD) EOM for Non-Rotating Satellites

In the previous sections, Euler's equation was obtained for a rigid satellite, considering there no internal rotating mechanisms. However, in large satellites or SCs there could be some rotating devices such as reaction wheels, momentum wheels, and control moment gyros (CMG) that create angular momentums as well. Thus, we need to include their effect in the Euler's equation. To this end, let $\vec{T}$ now be used instead of $\vec{M}$ in the Euler's equation, to represent all external torques acting on the satellite , so the Euler's EOM will be:

These equations provide the **complete state equations for simulating the 6 degrees of freedom of the satellite** in **body Coordinates** in order to analyze the state control system. Note that angular velocity components, will either be computed (for controller design and simulations) and measured in ACS applications, where principal axes are assumed.

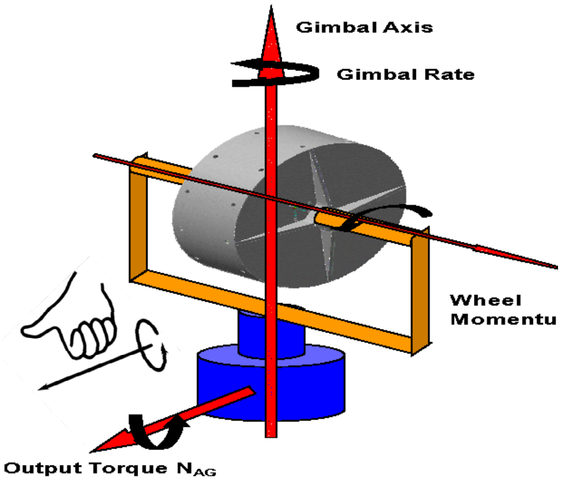


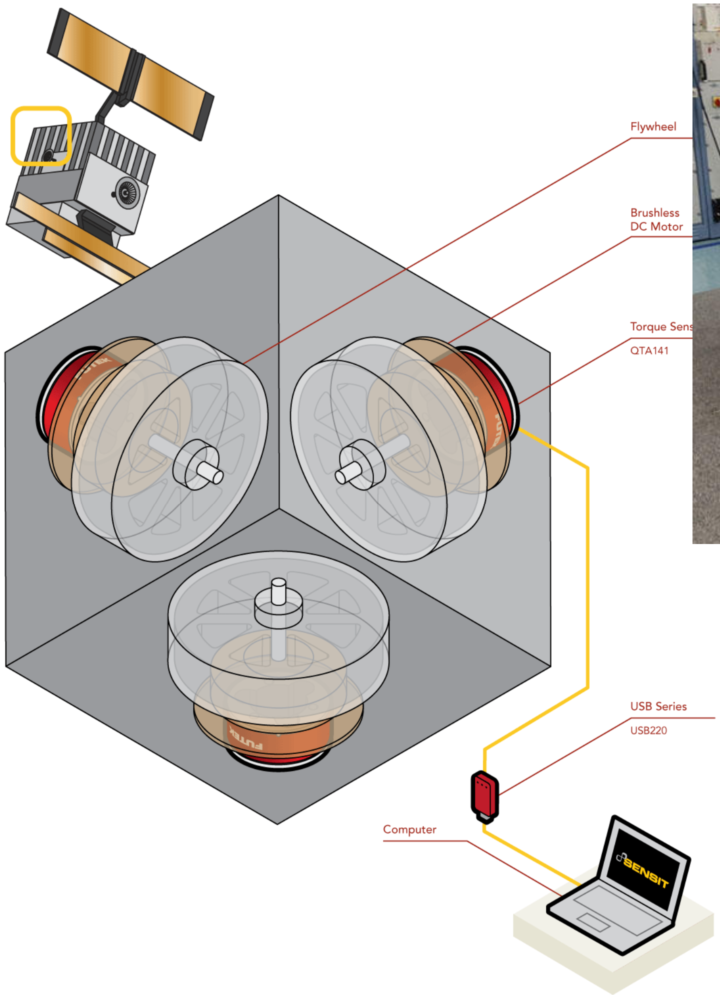
$$\vec{T} = \dot{\vec{h}}_i = \dot{\vec{h}} + \vec{\omega} \times \vec{h};\ where: \vec{T} = \vec{T}_C + \vec{T}_D$$

$$And: \vec{h} = \vec{h}_B + \vec{h}_W \quad , \quad \begin{cases} \vec{h}_B = \begin{bmatrix} h_x & h_y & h_z \end{bmatrix}^T \\ \vec{h}_W = \begin{bmatrix} h_{Wx} & h_{Wy} & h_{Wz} \end{bmatrix}^T = \sum \vec{h}_{W_i} \end{cases}$$

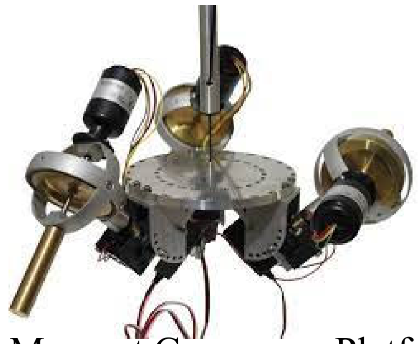
$$\vec{T} = \vec{T}_C + \vec{T}_D = \begin{cases} \left[\dot{h}_x + \dot{h}_{Wx} + \left(\omega_y h_z - \omega_z h_y \right) + \left(\omega_y h_{Wz} - \omega_z h_{Wy} \right) \right] \vec{i} \\ \left[\dot{h}_y + \dot{h}_{Wy} + \left(\omega_z h_x - \omega_x h_z \right) + \left(\omega_z h_{Wx} - \omega_x h_{Wz} \right) \right] \vec{j} \\ \left[\dot{h}_z + \dot{h}_{Wz} + \left(\omega_x h_y - \omega_y h_x \right) + \left(\omega_x h_{Wy} - \omega_y h_{Wx} \right) \right] \vec{k} \end{cases}$$

General Attitude Dynamics (AD) EOM for Non-Rotating Satellites



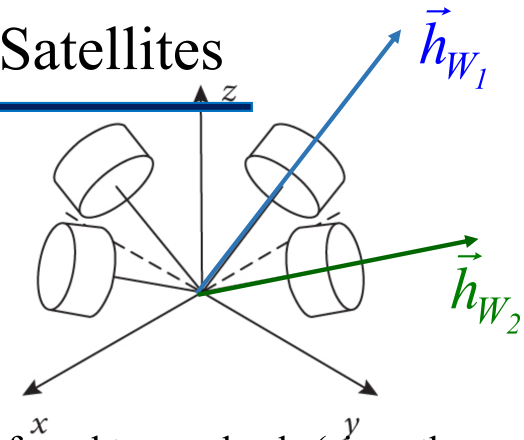


Reaction wheels ensure satellites maintain the right attitude



Control Moment Gyroscope Platform

Linearized AD EOM for Stability Analysis of Non-Rotating Satellites

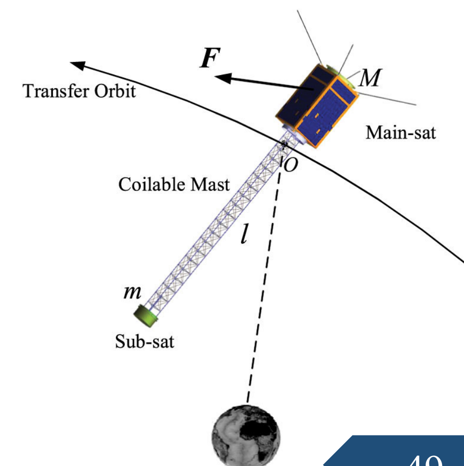


In the next step, the equations can be **linearized** with small Euler angles assumption for stability analysis and initial ACS design using various control laws. The wheels AM vector can be easily decomposed in BC based on their geometric installation.

Some notes before simplifying the AD general equations:

a) As mentioned before, there are **momentum exchange devices** in the satellite, which are referred to as wheels (since they have many variations), and these wheels, whose momentum is given in the equations, can have initial values that are called biased. For example, it is possible for the satellite to have $h_{Wy_0} \neq 0$ to create angular rigidity or stability around y_B. Though, this can done for wheels on any axis, but Bias is usually is considered only around one axis.

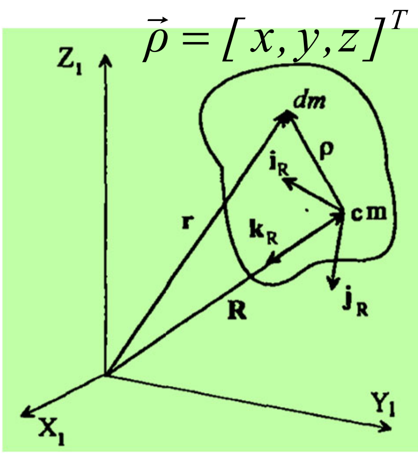
b) In general, the **rotation axis of the wheels** may not necessarily be **aligned with the body** or principal axes, and the number of wheels can be less or more than 3. For example, $\dot{\boldsymbol{h}}_{\boldsymbol{Wx}} = \boldsymbol{I}_{(\boldsymbol{Wx})}\dot{\boldsymbol{\omega}}_{\boldsymbol{Wx}} = -\boldsymbol{M}_{\boldsymbol{Wx}} = \boldsymbol{T}_{\boldsymbol{C_x}}$, or in other words, it is the negative of the angular torque that the X wheel applies on the SC around X_B .It is possible to control the angular orientation of satellite via adjustment of the wheels AM accelerations using some control laws.



c) Gravity Gradient Moment : One of the external moments on SC is due to gravity gradient (**GG**) that is initially considered as a disturbance in design and analysis of LEO satellites with a passive ACS. Usually, an asymmetric SC in the gravitational field will experience a moment that tends to align the z-axis of inertia with the direction of the gravity field. This property **can be positively utilized** for some Nadir pointing passive satellites by designing a GG boom.

Linearized AD EOM and Stability of Non-Rotating Satellites

For the GG discussion, consider a satellite at a distance R_0 from the Earth, considering the previous definitions of the reference and body frames. Accordingly, $\vec{R} = -R_0\vec{k}_R$ can be transformed in BC using Euler TM, where the effect of ψ angle will not appear due to special form of $\vec{R}$ (only z coordinate): $\vec{R}^b = C_R^b(\psi,\theta,\varphi)\vec{R}^R$, this will give:



$$\vec{R}^b = \begin{bmatrix} R_x & R_y & R_z \end{bmatrix}^T = \begin{bmatrix} -a_{13}R_0 & -a_{23}R_0 & -a_{33}R_0 \end{bmatrix}^T$$

$$where: a_{13} = -\sin\theta;\ a_{23} = \sin\varphi\cos\theta;\ a_{33} = \cos\varphi\cos\theta$$

$$d\vec{F} = -\frac{\mu dm}{r^3}\vec{r} \quad , \quad \vec{r} = \vec{R} + \vec{\rho} \Rightarrow d\vec{G} = \vec{\rho}\times d\vec{F} = \frac{\mu dm}{r^3}\vec{\rho}\times\vec{r} = \frac{\mu dm}{r^3}\vec{\rho}\times\left(\vec{R}+\vec{\rho}\right)$$

Where the gravity gradient vector is denoted by $\vec{G} = [G_x \quad G_y \quad G_z]^{\mathrm{T}}$. One can simplify the GG differential by integration over the SC body, noting that $\rho \ll R_0$, to finally show that : (Sidi:4.8.3)

$$\vec{G} = \frac{3\mu}{R_0^5}\int_m \left[\vec{R}\cdot\vec{\rho}\right]\left[\vec{\rho}\times\vec{R}\right]dm = \frac{3\mu}{2R_0^3}\left[\left(I_z - I_y\right)\sin 2\varphi\cos^2\theta;\left(I_z - I_x\right)\sin 2\theta\cos\varphi;\left(I_x - I_y\right)\sin 2\theta\sin\varphi\right]^T$$

$$\Rightarrow \vec{G} = \left[3\omega_0^2\left(I_z - I_y\right)\varphi;3\omega_0^2\left(I_z - I_x\right)\theta;0\right]^T, \text{for a circular orbit } \omega_0 = \sqrt{\frac{\mu}{R_0^3}}, \text{ with small Euler angles.}$$

Linearizing AD EOM for the Non-Rotating Satellites

In the process of linearization of the Kinematic Equations (around a local equilibrium condition), the following **assumptions** are considered:

1- The Euler angles are small (as perturbations), the wheel bias is only around y_B, so $h_{Wy_0} \neq 0$,

2- The local instantaneous osculating orbit is circular with radius R_0 , the related GG torque is accounted for (50).

3- The Summation rule is utilized $\vec{\omega}_{ib}^b = \vec{\omega}_{iR}^b + \vec{\omega}_{Rb}^b$ in the SC BCS ,

4- It is possible to extract $\vec{\omega}$ in BCS via Euler angles method of ARP using the above assumptions. In other words :

$$\vec{\omega}_{iR}^b = C_R^b(\psi,\theta,\varphi)\ \vec{\omega}_{iR}^R \text{ ; where } \vec{\omega}_{iR}^R = \begin{bmatrix} 0 & -\omega_0 & 0 \end{bmatrix}^T \text{ is the angular velocity of the reference orbit,}$$

5- The TM or rotation matrix between the reference and body frame for small Euler angles simplify to,

$$C_R^b(\psi,\theta,\varphi) = \begin{bmatrix} 1 & \psi & -\theta \\ -\psi & 1 & \varphi \\ \theta & -\varphi & 1 \end{bmatrix} \Rightarrow \vec{\omega}_{iR}^b = \begin{bmatrix} \omega_{iRx} \\ \omega_{iRy} \\ \omega_{iRz} \end{bmatrix} = \begin{bmatrix} -\psi\omega_0 \\ -\omega_0 \\ \varphi\omega_0 \end{bmatrix}$$

6- Accordingly, using 1,3 and the Euler ARP in slide 40, we will have:

$$\begin{bmatrix} \omega_x & \omega_y & \omega_z \end{bmatrix}^T = \vec{\omega}_{ib}^b = \vec{\omega}_{iR}^b + \vec{\omega}_{Rb}^b \text{ ; where: } \vec{\omega}_{Rb}^b = \begin{bmatrix} p & q & r \end{bmatrix}^T = \begin{bmatrix} \dot{\varphi} & \dot{\theta} & \dot{\psi} \end{bmatrix}^T$$

$$\begin{cases} \omega_x = \dot{\varphi} - \psi\omega_0 \\ \omega_y = \dot{\theta} - \omega_0 \\ \omega_z = \dot{\psi} + \varphi\omega_0 \end{cases} \xrightarrow{\text{time derivative}} \begin{cases} \dot{\omega}_x = \ddot{\varphi} - \dot{\psi}\omega_0 \\ \dot{\omega}_y = \ddot{\theta} \\ \dot{\omega}_z = \ddot{\psi} + \dot{\varphi}\omega_0 \end{cases}$$

Linearizing AD EOM for the Non-Rotating Satellites

Finally, considering the AD equation in BCS (slide 47) and using the stipulated assumptions, the linearized version of the attitude dynamics results as given below:

$$\mathrm{T}_x = T_{Dx} + T_{Cx} = I_x\ddot{\varphi} + 4\omega_0^2\left(I_y - I_z\right)\varphi + \omega_0\left(I_y - I_z - I_x\right)\dot{\psi} + \dot{h}_{Wx} - \omega_0 h_{Wz} - \dot{\psi} h_{Wy_0} - \varphi\omega_0 h_{Wy_0} - I_{xy}\ddot{\theta}$$
$$-I_{xz}\ddot{\psi} - I_{xz}\omega_0^2\psi + 2I_{yz}\omega_0\dot{\theta}$$

$$\mathrm{T}_y = T_{Dy} + T_{Cy} = I_y\ddot{\theta} + 3\omega_0^2\left(I_x - I_z\right)\theta + \dot{h}_{Wy} - I_{xy}\left(\ddot{\varphi} - 2\omega_0\dot{\psi} - \omega_0^2\phi\right) + I_{yz}\left(-\ddot{\psi} - 2\omega_0\dot{\varphi} + \omega_0^2\psi\right)$$

$$\mathrm{T}_z = T_{Dz} + T_{Cz} = I_z\ddot{\psi} + \omega_0\left(I_z + I_x - I_y\right)\dot{\varphi} + \omega_0^2\left(I_y - I_x\right)\psi + \dot{h}_{Wz} + \omega_0 h_{Wx} + \dot{\varphi} h_{Wy_0} - \psi\omega_0 h_{Wy_0} - I_{yz}\ddot{\theta}$$
$$-I_{xz}\ddot{\varphi} - 2\omega_0 I_{xy}\dot{\theta} - \omega_0^2 I_{xz}\varphi$$

Where h_{Wx}, h_{Wy}, h_{Wz} $(h_{Wi} = I_{Wi}\omega_{Wi})$ are the wheels momentum components with the rotation axes along the SC BCS. In addition, considering the initial bias on the y axis wheel, we have:

individual wheels moments of inertia: $\begin{cases} h_{Wx} = I_{Wx}\omega_{Wx} \\ h_{Wy} = \cancel{I_{Wy}\omega_{Wy}} + h_{Wy_0} \\ h_{Wz} = I_{Wz}\omega_{Wz} \end{cases}$, wheels angular velocity : $\begin{cases} \omega_{Wx} \\ \omega_{Wy} \\ \omega_{Wz} \end{cases}$

Linearizing AD EOM for the Non-Rotating Satellites

The above equations (or other similar alternatives for wheel biases) can be used as a **reference for various investigations** in terms of stability and design evaluation of ACS. The latter linearize AD EOM still may seem complicated, though we have assumes the **wheels axis of rotation** to be in line with the SC body axes. Otherwise the wheels gimballing (installation) angles will also be included in the equations. However, we may simplify the proposed EOM, by assuming the SC BCS to be the principal axes to remove the SC product of inertias:

$$\begin{cases} \mathrm{T}_x = I_x\ddot{\varphi} + \left[a - \omega_0 h_{Wy_0}\right]\varphi + \left[b - h_{Wy_0}\right]\dot{\psi} + \dot{h}_{Wx} - \omega_0 h_{Wz} \\ \mathrm{T}_y = I_y\ddot{\theta} + d\theta + \dot{h}_{Wy} \\ \mathrm{T}_z = I_z\ddot{\psi} + \left[-b + h_{Wy_0}\right]\dot{\varphi} + \left[c - \omega_0 h_{Wy_0}\right]\psi + \dot{h}_{Wz} + \omega_0 h_{Wx} \end{cases}, \quad \begin{cases} a = 4\omega_0^2\left(I_y - I_z\right) \\ b = -\omega_0\left(I_x + I_z - I_y\right) \\ c = \omega_0^2\left(I_y - I_x\right) \\ d = 3\omega_0^2\left(I_x - I_z\right) \end{cases}$$

The above equations are the basis for SC attitude stabilization as well as stability analysis of non-rotating SC. In addition they can be utilized to derive SC transfer functions for various purposes including initial ACS designs.

SC Attitude Control and Stability Analysis

According to the equations introduced in the previous part, now we can talk about various techniques of SC Stabilization and control for which there are many variations. The main tasks of the attitude control system (ACS) can be considered in the following cases:

1. During orbital impulsive maneuvers, the SC attitude should be maintained in the desired $\Delta\vec{v}$ direction.
2. A spin stabilized satellite must be designed in such a way that its spin axis points to a specific direction.
3. A three-axis stabilized satellite for nadir pointing (pointing towards the Earth center) must keep its three Euler angles close to zero , relative to the reference orbit such as communication satellites.
4. In Earth-surveying satellites, the ACS system is designed to allow the payload (such as a camera) to track ground (surface) targets.
5. A celestial observer satellite must aim its optical instruments (such telescope) toward target stars in the celestial sphere at a predetermined rate.

An important distinction for attitude control exists between passive and active attitude control concepts . Passive attitude control is attractive because the hardware requirement is less complicated and relatively inexpensive. Natural in this case, the physical SC design properties and its environment are used to control its attitude with no active mechanism. However, the achievable accuracies with passive attitude control methods may lower than the active control techniques.

SC Attitude Control and Stability Analysis

Another important distinction is between attitude-maneuvering and nadir-pointing (Earth-pointing) satellites in terms of hardware and the appropriate design concepts used for these two classes of SCs. For example GG passive stabilization is a reliable passive technique for the latter SC missions that is considered during the design process.

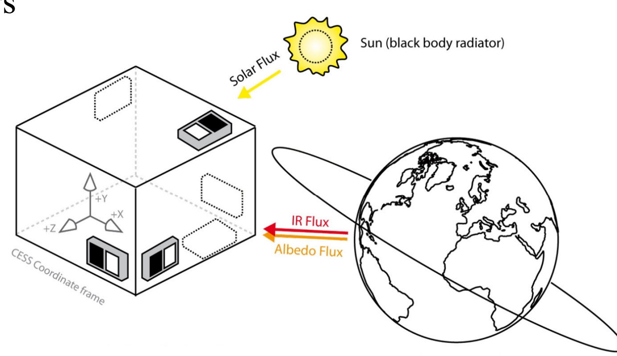
The **attitude and orbit control** of SCs are performed with the aid of hardware classified under either **attitude determination** or **attitude control** .

The **attitude determination** hardware enables direct measurement (or estimation) of the SC attitude with respect to some reference coordinate system in space. Among this type of hardware, one can refer to Earth sensors, Sun/Star sensors as well as integrating gyroscopes.

Magnetorquers

Attitude control hardware can potentially provide linear and angular accelerations such that one can change the SC orientation and angular rates as needed for the mission. Among these mechanisms, one can refer to reaction or momentum wheels, CMG, reaction thrusters and magnetic torquers to name a few.

Sun Sensor



Earth- Sun Flux Sensor

Pure Passive Control by Design

The linear equations developed in slide 53 can be used to check the stability of SC for passive stabilization and control. Note that the equations were derived with GG torque effect. However, for passive control and stability the control related such as T_{ci}, h_{Wi} should be removed. This yields:

$$T_{Dx} = I_x\ddot{\varphi} + 4\omega_0^2\left(I_y - I_z\right)\varphi - \omega_0\left(I_x + I_y - I_z\right)\dot{\psi}$$

$$T_{Dy} = I_y\ddot{\theta} + 3\omega_0^2\left(I_x - I_z\right)\theta$$

$$T_{Dz} = I_z\ddot{\psi} + \omega_0\left(I_z + I_x - I_y\right)\dot{\varphi} + \omega_0^2\left(I_y - I_x\right)\psi$$

Note the SC was initially at the equilibrium condition denoted by $\psi = \theta = \phi = 0$. Therefore, in this situation, angular stability can be evaluated via initial disturbing torques and/or initial Euler angles and rates. For simplicity, let's define the following relation to analyze the SC passive stability requirements :

$$\sigma_y = \frac{I_x - I_z}{I_y} \quad , \quad \sigma_x = \frac{I_y - I_z}{I_x} \quad , \quad \sigma_z = \frac{I_y - I_x}{I_z}$$

Pure Passive Control by Design

i: The equations indicate that pitch Stability about the y_B axis, can be independently investigated.

$$T_{Dy} = I_y\ddot{\theta} + 3\omega_0^2\left(I_x - I_z\right)\theta$$

$$\sigma_y = \frac{I_x - I_z}{I_y}$$

$$\ddot{\theta} + \left(3\omega_0^2\sigma_y\right)\theta = \frac{T_{Dy}}{I_y} \Rightarrow \text{characteristic equation: } s^2 + 3\omega_0^2\sigma_y = 0$$

According to σ_y, the above equation has one unstable root if $I_z > I_x$, so the condition for pitch stability becomes $I_x > I_z$.

ii: Stability about the two other x_B, z_B axes: keep in mind that the values of σ_x and σ_z are limited and that they are usually **smaller than 1**, as can be seen by the definition of MOIs below. So, $\sigma_y < 1, \sigma_z < 1$, and $I_y < I_x + I_z$ in order to have $\sigma_x < 1$.

$$T_{Dx} = I_x\ddot{\varphi} + 4\omega_0^2\left(I_y - I_z\right)\varphi - \omega_0\left(I_x + I_y - I_z\right)\dot{\psi}$$

$$T_{Dz} = I_z\ddot{\psi} + \omega_0\left(I_z + I_x - I_y\right)\dot{\varphi} + \omega_0^2\left(I_y - I_x\right)\psi$$

$$\sigma_x = \frac{I_y - I_z}{I_x}; \sigma_z = \frac{I_y - I_x}{I_z}$$

$$\sigma_x = \frac{I_y - I_z}{I_x} = \frac{\int\left(x^2 + z^2\right)\mathrm{d}m - \int\left(x^2 + y^2\right)\mathrm{d}m}{\int\left(y^2 + z^2\right)\mathrm{d}m} = \frac{\int\left(z^2 - y^2\right)\mathrm{d}m}{\int\left(y^2 + z^2\right)\mathrm{d}m} < 1$$

Pure Passive Control by Design

$$T_{Dx} = I_x\ddot{\varphi} + 4\omega_0^2\left(I_y - I_z\right)\varphi - \omega_0\left(I_x + I_y - I_z\right)\dot{\psi}$$

$$T_{Dz} = I_z\ddot{\psi} + \omega_0\left(I_z + I_x - I_y\right)\dot{\varphi} + \omega_0^2\left(I_y - I_x\right)\psi$$

$$\sigma_x = \frac{I_y - I_z}{I_x};\sigma_z = \frac{I_y - I_x}{I_z}$$

Since the latter equations are coupled, the characteristic equation is via Laplace :

$$s^4 + \omega_0^2\left[3\sigma_x + \sigma_x\sigma_z + 1\right]s^2 + 4\omega_0^2\sigma_x\sigma_z = 0$$

The above equation is of order 4 and thus, the solution for s^2 will be:

$$\frac{s^2}{\omega_0^2} = \frac{-\left(3\sigma_x + \sigma_x\sigma_z + 1\right) \pm \sqrt{\left(3\sigma_x + \sigma_x\sigma_z + 1\right)^2 - 16\sigma_x\sigma_z}}{2}$$

It can be seen that if s_1 is a root of the above equation, then so is $-s_1$ (both satisfy the equation). Therefore, for s_1 to be a root with no positive real part, it is necessary to have s_1 be imaginary and thus the pink term (discriminant) must be positive to have or $s^2 < 0$. Thus in this coupled situation (two othe axes), the following three conditions must be simultaneously satisfied:

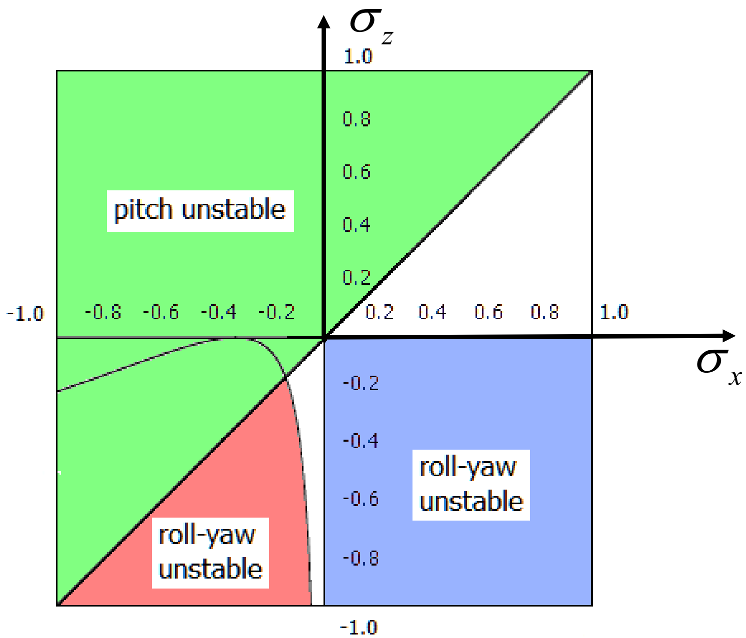
$$\begin{cases} 3\sigma_x + \sigma_x\sigma_z + 1 > 4\sqrt{\sigma_x\sigma_z} \\ \sigma_x\sigma_z > 0 \\ 3\sigma_x + \sigma_x\sigma_z + 1 > 0 \end{cases}$$

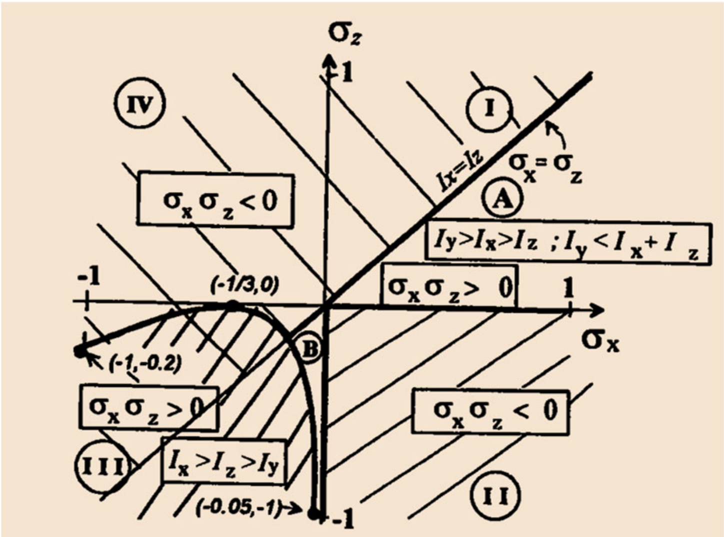
Pure Passive Control by Design

$$\begin{cases} 3\sigma_x + \sigma_x\sigma_z + 1 > 4\sqrt{\sigma_x\sigma_z} \\ \sigma_x\sigma_z > 0 \\ 3\sigma_x + \sigma_x\sigma_z + 1 > 0 \end{cases}$$

In addition, remember that for stability about the y_B axis: $I_x > I_z$.
Also we had that $\sigma_i < 1$, and for σ_x<1, needed $I_y < I_x + I_z$.
Moreover, the latter two axes stability requirements can be translated to the $\sigma_x - \sigma_z$ plane where stability regions (about the x_B and z_B axes) **are displayed** to be observed in design process.
Of course , the y_B and other stability criteria can also be included in that plot by manipulation.
Pre-multiplying , $I_y < I_x + I_z$ with $I_x - I_z > 0$, gives :

$$(I_x - I_z) I_y < I_x^2 - I_z^2$$
$$I_x^2 - I_z^2 > I_x I_y - I_y I_z$$
$$I_z(I_y - I_z) > I_x(I_y - I_x)$$
$$\frac{I_y - I_z}{I_x} > \frac{I_y - I_x}{I_z}$$
$$\Rightarrow \sigma_x > \sigma_z$$





Time Behavior of a Purely Passive GG-Stabilized Satellite

$$T_{Dx} = I_x\ddot{\varphi} + 4\omega_0^2\left(I_y - I_z\right)\varphi - \omega_0\left(I_x + I_y - I_z\right)\dot{\psi}$$
$$T_{Dy} = I_y\ddot{\theta} + 3\omega_0^2\left(I_x - I_z\right)\theta$$
$$T_{Dz} = I_z\ddot{\psi} + \omega_0\left(I_z + I_x - I_y\right)\dot{\varphi} + \omega_0^2\left(I_y - I_x\right)\psi$$

Using the linearized equations via utility of Laplace Transform (LP), one can find the SC time response around its three axes as well.

i) Time response **about the y_B pitch** axis:

This motion depends on the initial conditions only, as well as on the external disturbance T_{Dy}. It can be easily shown that:

$$\theta(s) = \frac{T_{Dy}}{I_y s\left(s^2 + 3\omega_0^2\sigma_y\right)} + \frac{s\theta(0) + \dot{\theta}(0)}{s^2 + 3\omega_0^2\sigma_y} \Rightarrow \theta(t)$$

ii) Time response about the x_B and z_B axes: Since these equations are coupled, they must be simultaneously solved using LP and initial conditions.

$$\mathcal{L}\left(f^n(t)\right) = s^n f(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - f^{n-1}(0)$$

$$\begin{cases} \left(s^2 + 4\omega_0^2\sigma_x\right)\varphi - s\omega_0\left(1-\sigma_x\right)\psi = \dfrac{T_{Dx}}{I_x} + s\varphi_0 + \dot{\varphi}_0 - \omega_0\left(1-\sigma_x\right)\psi_0 \\ \left(s^2 + \omega_0^2\sigma_z\right)\psi + s\omega_0\left(1-\sigma_z\right)\varphi = \dfrac{T_{Dz}}{I_z} + \omega_0\left(1-\sigma_z\right)\varphi_0 + s\psi_0 + \dot{\psi}_0 \end{cases} \Rightarrow \varphi(t), \psi(t)$$

Time Behavior of a Purely Passive GG-Stabilized Satellite

For constant T_{Dx} and T_{Dz}, it can be shown the time behavior of ϕ, ψ. The results show that in the best condition of satellite design and for the previous conditions of passive GG stabilization, the responses are oscillatory (with low amplitude and depend on T_i's as well as initial conditions). In addition, for $I_x = I_z$, we cannot use passive GG stabilization notion. Nowadays, some passive methods (from design) are used for some satellites and have different versions such as: boom dampers, point-mass dampers, rod dampers, external spring boom dampers, and wheel dampers.

GG Stabilization with Active Damping Using Magnetic Torques

Owing to the absence of adequate gravity gradient moments z_B, stabilization about this axis is quite difficult so time behavior of ψ may be not acceptable (yaw damping). A positive way against external disturbances is to use magnetic torque-rods, which - by interacting with the earth's magnetic field - can produce the needed moments to counteract the disturbances. If $\vec{B}$ is the earth's magnetic field vector and if $\vec{M}$ is the magnetic dipole value (produced by energizing the current-carrying coil), then the mechanical moment exerted on the body will be:

$$\vec{T} = \vec{M} \times \vec{B} = -\vec{B} \times \vec{M} = B^* \vec{M}$$

$$\vec{M} = \text{magnetic dipole vector} = \begin{bmatrix} M_x & M_y & M_z \end{bmatrix} \triangleq NAI\, \vec{e}_i$$

$$\vec{B} = \text{earth magnetic field vector} \triangleq \begin{bmatrix} B_x & B_y & B_z \end{bmatrix}$$

GG Stabilization with Active Damping Using Magnetic Torques

The above vector multiplication can also be obtained through the skew-symmetric matrix $-\vec{B}$ (the skew-symmetric matrix is singular, so producing its inverse to find $\vec{M}$ is not possible):

$$\vec{T} = \vec{T}_C = \begin{bmatrix} T_{Cx} \\ T_{Cy} \\ T_{Cz} \end{bmatrix} = \begin{bmatrix} 0 & B_z & -B_y \\ -B_z & 0 & B_x \\ B_y & -B_x & 0 \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix}$$

If $\vec{T}_{Ci}$ is calculated through the above control relation, assuming the ability to measure the attitude errors $(\Delta\phi, \Delta\psi, \Delta\theta)$, the problem that remains is the required dipole problem that must be produced by the torque generators (or the torque rod).

GG Stabilization with Active Damping Using Magnetic Torques

To implement this simple rule, it will be necessary to measure (or estimate) ψ, ϕ, and the magnetic field of the earth must be known or measured with the help of a magnetometer. Euler angles can also be estimated through the three-axis magnetometer sensor. Pay attention that from this mechanism, T_{Cy} will also be produced and applied implicitly, which should be seen as a disturbance that can be limited in the boom design.

How to Solve Inverse Matrix Problem

An approximate method for the three-axis active stabilization of the gravity gradient using magnetic torque generators along the main axes using the principle of perpendicularity has been proposed in the sources. This method uses the equality of multiplication of three vectors. Assume that we multiply the torque calculation relation by $\vec{B}$ on both sides. Therefore:

$$\vec{B} \times \vec{T}_C = \vec{B} \times \left(\vec{M} \times \vec{B}\right) = \left(\vec{B} \cdot \vec{B}\right)\vec{M} - \left(\vec{B} \cdot \vec{M}\right)\vec{B}$$

Now suppose that the magnetic dipole or its vector ($\vec{M}$ applied inside the s/c) is always perpendicular to the Earth's magnetic vector $\vec{B}$, which is not true. In this case, the above relationship can be expressed as follows:

$$\vec{B} \cdot \vec{M} = 0 \Rightarrow \vec{M} = \frac{1}{B^2}\left(\vec{B} \times \vec{T}_C\right)$$

How to Solve Inverse Matrix Problem

The above relation together with the control model for $\vec{T}_C$ and with the known $\vec{B}$ provides the required active control for stabilization, which needs to be tested through simulation in several disturbance conditions. It has been shown that this method produces accuracies of up to two orders of magnitude. Also, this method can determine the basic technical properties of torque generators ($\vec{M}$: Magnetic Dipoles).

Check out Sidi's book example on this. For other disturbance torques, maximum and minimum, as well as the selection of P, PD, and PID controllers and their coefficients in for initial conditions $\vec{\omega}$ and initial Euler angles. In the book, T_{Ci} is chosen to reach $\xi = 0.9$ and $\omega_n = 0.05\ rad/s$. The output will be the Euler angles, $\vec{\omega}$ and $\vec{M}$.

How to Solve Inverse Matrix Problem

In addition, there are several patterns for the Earth's magnetic field, and the intensity of the field can be approximated based on the following relation:

$$B = \frac{m}{R^3}\sqrt{1 + 3\sin^2\lambda} \quad , \quad m: \text{magnetic dipole strength} \approx 8\times10^5\ \text{Wb}\times\text{m}$$

λ (magnetic latitude) is the angle between the orbital plane and the plane perpendicular to the earth's magnetic dipole axis and m is a constant number. Therefore, the magnetic field for polar orbits will be approximately double and will decrease due to altitude.

Earth Magnetic Field

The nature of the Earth's magnetic field is very complex, and its parameters change with time, despite the bipolar imagination. Roughly, the Earth's magnetic field is like a tilted dipole that does not pass through the center of the Earth. The IAGA (International Association of Geomagnetism and Aeronomy) Institute publishes patterns for the Earth's magnetic model called IGRF (International Geomagnetic Reference Field), which updates the coefficients or parameters in them approximately every 5 years.

One of the famous magnetic field models is the Spherical Harmonic Model, which is defined as follows:

$\vec{B} = -\nabla V \quad , \quad V =$ magnetic potential

Earth Magnetic Field

$$V(r,\theta,\phi) = R_E \sum_{n=1}^{k} \left(\frac{R_E}{r}\right)^{n+1} \sum_{m=0}^{n} \left(g_n^m \cos(m\phi) + h_n^m \sin(m\phi)\right) P_n^m \cos\theta$$

$r \triangleq$ dist. from earth center (km)

$\theta \triangleq$ co-latitude $= 90 -$ latitude $\triangleq 90 - \phi$

$\phi \triangleq$ longitude

$g_n^m, h_n^m \triangleq$ gaussian coefficients (given by IAGA)

$P_n^m \triangleq$ Schmidt quasi-normalized Legendre function

figure

$$\vec{B} = \begin{bmatrix} B_r & B_\theta & B_\phi \end{bmatrix}^\top$$

$$B_r \triangleq -\frac{\partial V}{\partial r} : \text{outward radial positive}$$

$$B_\theta \triangleq -\frac{1}{r}\frac{\partial V}{\partial \theta} : \text{southward positive}$$

$$B_\phi \triangleq -\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi} : \text{eastward positive}$$

A Simple Approximation for Earth Magnetic Field

Using first-order spherical harmonics ($n = 1, m = 0 \; and \; 1$), a simpler model of V and thus of B will be produced:

$$V(r,\theta,\phi) = \frac{R_E^3}{r^2}\left[g_1^0 \cos\theta + g_1^1 \cos\phi \sin\theta + h_1^1 \sin\phi \sin\theta\right]$$

and using IGRF 2000:

$g_1^0 = -29619.4 \text{ nT}$

$g_1^1 = -1728.2 \text{ nT}$

$h_1^1 = 5186.1 \text{ nT}$

And in this way, a simple pattern for $\vec{B}$ (in the spherical coordinate, could be transformed to other coordinates) is obtained.

A Simple Approximation for Earth Magnetic Field

$$B_r = 2\left(\frac{R_E}{r}\right)^3 \left[g_1^0 \cos\theta + g_1^1 \cos\phi \sin\theta + h_1^1 \sin\phi \sin\theta\right]$$

$$B_\theta = \left(\frac{R_E}{r}\right)^3 \left[g_1^0 \sin\theta - g_1^1 \cos\phi \cos\theta - h_1^1 \sin\phi \cos\theta\right]$$

$$B_\phi = \left(\frac{R_E}{r}\right)^3 \left[g_1^1 \sin\phi - h_1^1 \cos\phi\right]$$

These relations could be defined in NED coordinates.

Magnetic Attitude Control

As mentioned earlier, the interaction between the magnetic torques generated inside the s/c and the Earth's magnetic field creates a mechanical torque on the s/c. in other words:

$\vec{T}_B = \vec{M} \times \vec{B}$, $\vec{T}_B$ = mechanical torque

The expansion of the above equation by substituting $\vec{T}_B$ instead of $\vec{T}_C$ in the previous equations will similarly lead to the creation of a singular matrix that cannot be inverted.

Here $\vec{T}_B = \vec{T}_C \triangleq$ Desired Torques and in other words the similar equation can be written:

$$\vec{T}_B = \begin{bmatrix} T_{Bx} \\ T_{By} \\ T_{Bz} \end{bmatrix} = \begin{bmatrix} 0 & B_z & -B_y \\ -B_z & 0 & B_x \\ B_y & -B_x & 0 \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} \quad , \quad \vec{M} : \text{generated mag. moment inside s/c}$$

Magnetic Attitude Control

Thus, for a required torque $\vec{T}_B$ (known), the magnetic torque must be calculated, which is not possible. If we replace one of the torque rods with a wheel (for example RW), the previous equation will change. Suppose we replace the torque generator around the y_B axis with a RW. In this case:

$$\begin{bmatrix} T_{Bx} \\ T_{By} \\ T_{Bz} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -B_y \\ -B_z & 1 & B_x \\ B_y & 0 & 0 \end{bmatrix} \begin{bmatrix} M_x \\ \dot{h}_{Wy} \\ M_z \end{bmatrix}$$

In this case, the matrix will have an inverse and can be solved for the control torques.

$$\begin{bmatrix} M_x \\ \dot{h}_{Wy} \\ M_z \end{bmatrix} = \frac{1}{B_y^2} \begin{bmatrix} 0 & 0 & B_y \\ B_x B_y & B_y^2 & B_y B_z \\ -B_y & 0 & 0 \end{bmatrix} \begin{bmatrix} T_{Cx} \\ T_{Cy} \\ T_{Cz} \end{bmatrix}$$

Magnetic Attitude Control

The previous equation will be the basic equation for magnetic/reaction wheel attitude control. Of course, depending on the purpose, the wheel can be replaced with a torque generator on each axis. Of course, magnetic torque generators will not be useful for every s/c, and according to the reduction of the earth field or its intensity with height, the hardware requirements of torque generators will increase.

Control with Momentum Exchange Devices

In the case of attitude control, control command law can be generated in various ways through Euler angles, DCM error matrix, Euler rotation axis or quaternion error vector. For example, using Euler's angle error, we have:

$$\begin{cases} T_{Cx} = k_x \phi_E + k_{xd}\dot{\phi} + k_{xi}\int \phi \mathrm{d}t \\ T_{Cy} = k_y \theta_E + k_{yd}\dot{\theta} + k_{yi}\int \theta \mathrm{d}t \\ T_{Cz} = k_x \psi_E + k_{xd}\dot{\psi} + k_{xi}\int \psi \mathrm{d}t \end{cases} , \quad \begin{cases} \phi_E = \phi_{comm} - \phi \\ \theta_E = \theta_{comm} - \theta \\ \psi_E = \psi_{comm} - \psi \end{cases}$$

The above rules are given assuming the existence of hardware to produce command torques.

Control with Momentum Exchange Devices

There are at least four distinct methods of generating torque for s/c control purposes:

1) Earth magnetic field: these methods provide smooth and continuous control, but their output torque levels are small (on the order of 1-10 mN.m) and are not suitable for fast attitude maneuvers. Also, the resulting torque depends on the type of orbit (height and inclination of the orbit) and can only be used for orbits around the Earth.
2) Thrusters (reaction to expulsion of gas or ion particles): these controls are non-linear. That is, they produce fixed reaction torques in a certain time and with an unlimited amount, but not smooth because of their impulsive nature.

Control with Momentum Exchange Devices

There are at least four distinct methods of generating torque for s/c control purposes:

3) Solar radiation pressure: the torques from the SRP (only when s/c sees the sun) are in a very small range (on the order of 1-10 μN.m) which is not sufficient for maneuvering and cannot effectively generate torque about any axis.
4) Momentum exchange devices (rotating bodies inside s/c): the next choice is based on the rotating components inside the s/c, which allows the exchange of momentum between the s/c parts without changing the overall inertial angular momentum.

Note: the first three methods are called inertial control, because they change the inertia $\vec{h}$ and there is no exchange.

Control with Momentum Exchange Devices

Inside the s/c of a rotating symmetric object, angular momentum can be created by acceleration (around the axis of rotation). The wheel may also have a constant initial momentum h_W, but since this momentum is internal, increasing $\vec{h}$ does not change the inertia of s/c. But it will cause it to be transferred (with the opposite sign) to s/c. Therefore, each momentum exchange device ($\dot{h}_{Wx}$) will create an angular moment about its own axis but in the opposite direction. In this way, the equations related to torque calculation give the same torques that should be obtained through the wheels, that is:

$$\begin{cases} T_{Cx} = \dot{h}_{Wx} \\ T_{Cy} = \dot{h}_{Wy} \\ T_{Cz} = \dot{h}_{Wz} \end{cases}$$

Control with Momentum Exchange Devices

which is subsequently achieved by accelerating the wheels (through their electric motors), which itself has a control loop.

Momentum Accumulation and Damping

One of the drawbacks of momentum exchange devices is that they cannot independently remove accumulated angular momentum! That is, according to Euler's equation, external disturbing torques increase the angular momentum of s/c, which without active control will lead to a change in the angular velocity of the orbit and its attitude! But with the wheel system to stabilize and control the attitude, the accumulated angular momentum will be transferred to the wheels, which will eventually lead to their saturation, so that they will no longer lead to the production of new control torque. Therefore, the unwanted accumulated momentum (more than normal) must be removed, which is known as dumping. One of the common mechanisms for dumping or emptying the wheels are magnetic torque rods (i) and reactive thrusters (ii).

Momentum Accumulation and Damping

i) The magnetic torque rods create dipole magnetic moments, whose interaction with the Earth's magnetic field provides the necessary moments to remove the extra angular momentum of the wheels. The main idea of this work is very simple, the basis control equation for emptying the extra momentum of the wheels is as follows:

$$\vec{T} = -k\left(\vec{h} - \vec{h}_N\right) = -k\Delta\vec{h}$$

$k \triangleq$ control gain

$\vec{h} \triangleq$ wheel's momentum vector

$\vec{h}_N \triangleq$ nominal wheel momentum vector

$\Delta\vec{h} \triangleq$ excess momentum to be removed

Momentum Accumulation and Damping

Note that $\vec{T} = \vec{M} \times \vec{B}$, and therefore: $-k\Delta\vec{h} = \vec{M} \times \vec{B}$

But due to the same previous problem (no torque about the local earth magnetic field vector!), the control magnetic dipole vector $\vec{M}$ cannot be obtained from the above relation, and by multiplying its vector on both sides, it can be calculated as follows:

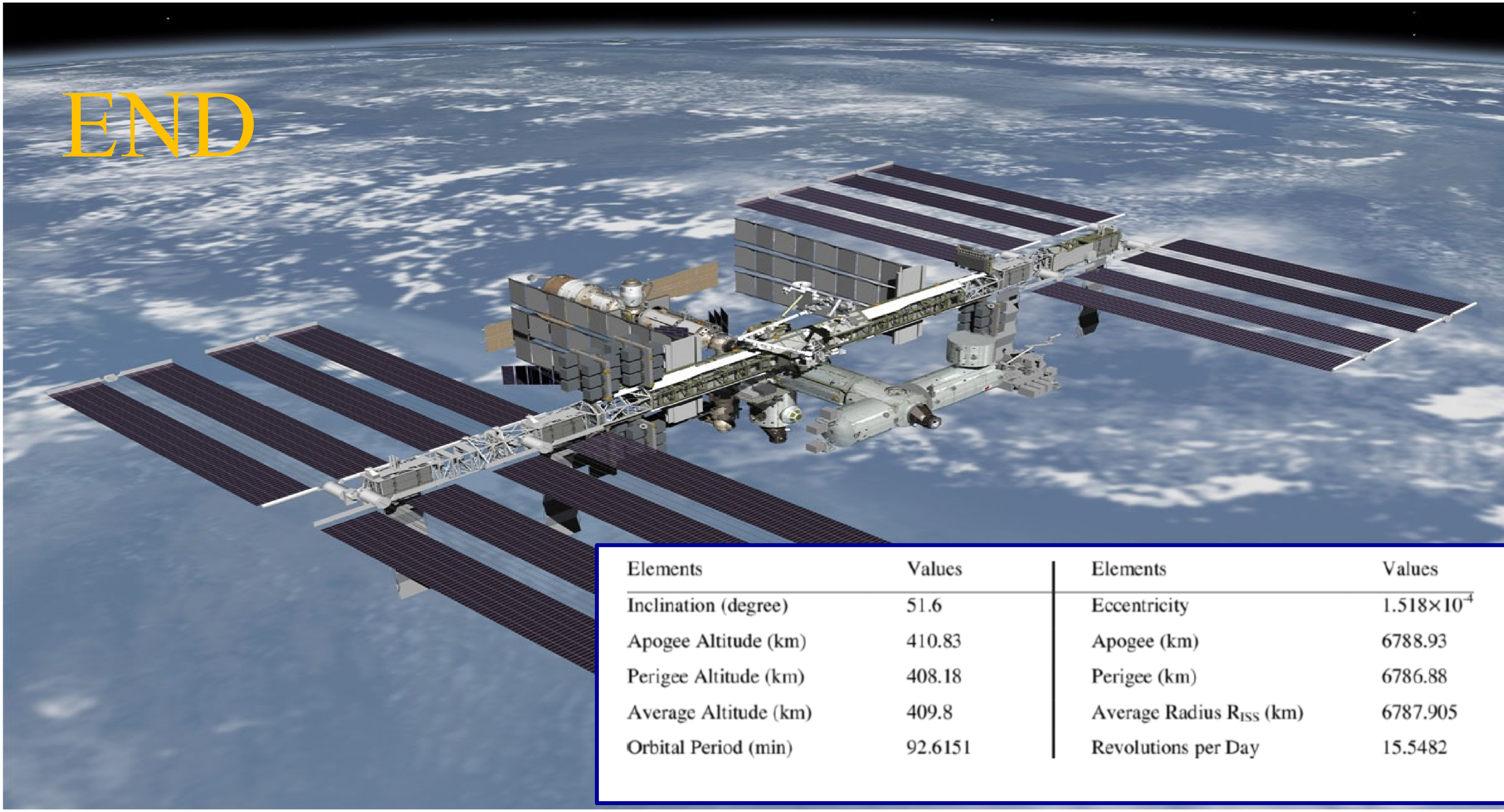
$$\vec{M} = -\frac{k}{B^2}\left(\vec{B} \times \Delta\vec{h}\right)$$

$$\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} = -\frac{k}{B^2}\begin{bmatrix} B_y\Delta h_z - B_z\Delta h_y \\ B_z\Delta h_x - B_x\Delta h_z \\ B_x\Delta h_y - B_y\Delta h_x \end{bmatrix}$$

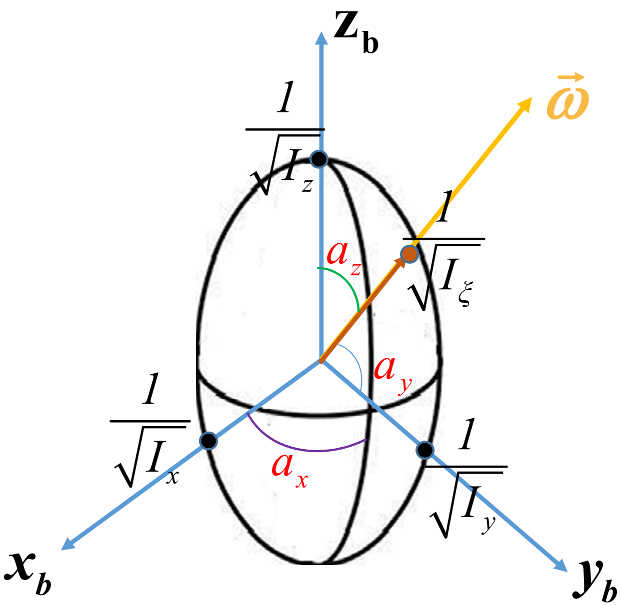
Momentum Accumulation and Damping

where B_i's are the field components in the body coordinate that will be measured or modeled through the 3-axis magnetometer! And the additional momentum is also obtained by measuring the angular speed of the wheels, which will be the required bipolar for of the torque generators.

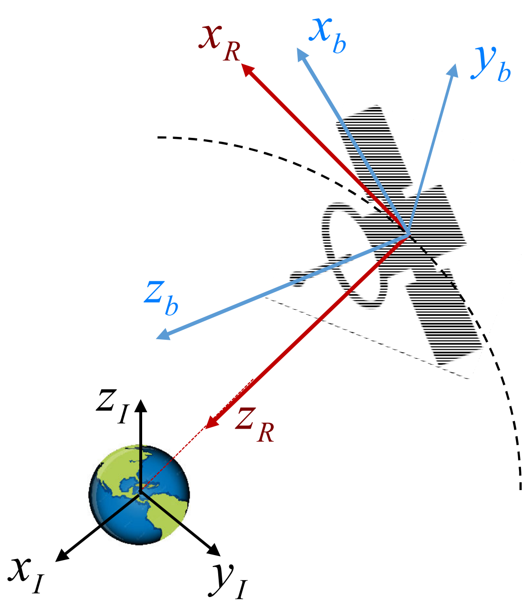
END

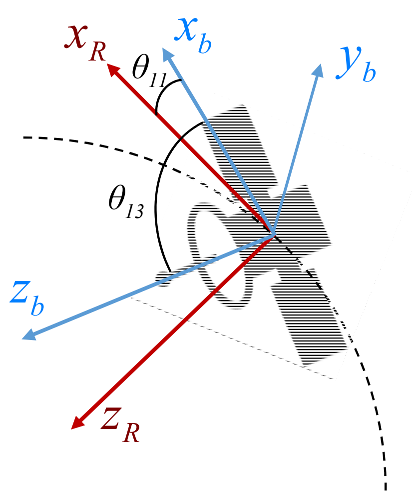


Elements	Values	Elements	Values
Inclination (degree)	51.6	Eccentricity	1.518×10^{-4}
Apogee Altitude (km)	410.83	Apogee (km)	6788.93
Perigee Altitude (km)	408.18	Perigee (km)	6786.88
Average Altitude (km)	409.8	Average Radius R_{ISS} (km)	6787.905
Orbital Period (min)	92.6151	Revolutions per Day	15.5482



z_b
$\vec{\omega}$
$\frac{1}{\sqrt{I_z}}$
$\frac{1}{\sqrt{I_\xi}}$
a_z
a_y
a_x
$\frac{1}{\sqrt{I_x}}$
$\frac{1}{\sqrt{I_y}}$
x_b
y_b


x_R
x_b
y_b
z_b
z_R
z_I
x_I
y_I



x_R
x_b
y_b
θ_{11}
θ_{13}
z_b
z_R

Kinematics Equation of Motion