Home Work #3

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1 Question 1

This homework used the below equation to simulate the position and velocity of the Hubble space telescope.

$$
\begin{aligned}
&\ddot{x} - 2n\dot{y} - 3n^2 x = f_x \\
&\ddot{y} + 2n\dot{x} = f_y \\
&\ddot{z} + n^2 z = f_z
\end{aligned}
$$

assumed that:

$$
\begin{aligned}
f_x &= 0 \\
f_y &= 0 \\
f_z &= 0
\end{aligned}
$$

where:

$$
n = \sqrt{\frac{\mu}{r^3}}, \quad \mu = 398600.4418 \text{ km}^3 \text{ s}^{-2}, \quad r = r_{altitude} + r_{earth} = 590 + 6378 = 6968_{km}
$$

and initial conditions:

$$
r_{relative} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T, \quad v_{relative} = \begin{bmatrix} -0.1 & -0.04 & -0.02 \end{bmatrix}^T_{m/s}
$$

Figure 1: position of the Hubble space telescope

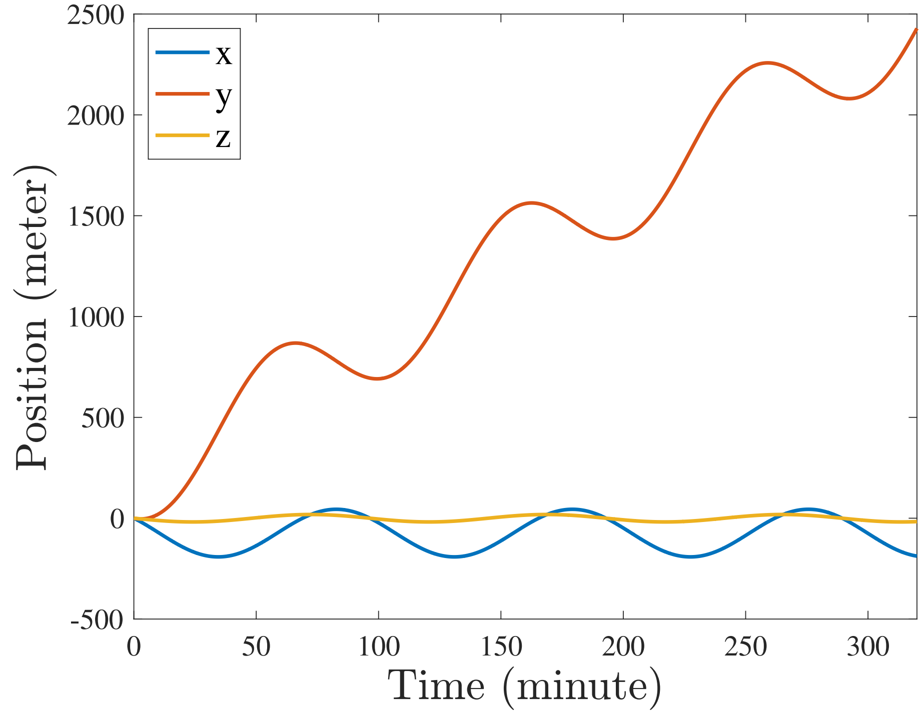
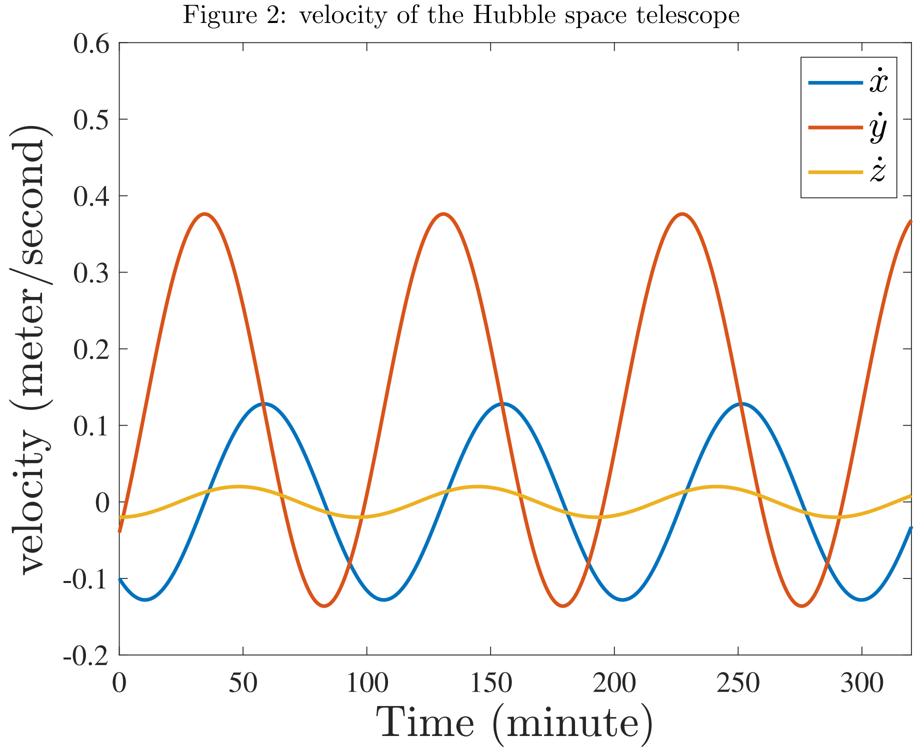


Figure 2: velocity of the Hubble space telescope



2 Question 2

Used below equations to find the orbital elements.

$$r = \begin{bmatrix}1600 & 5310 & 3800\end{bmatrix}^T_{km}, \quad v = \begin{bmatrix}-7.35 & 0.46 & 2.47\end{bmatrix}^T_{km/\sec}$$

2.1 part a

$$\boldsymbol{h} = \boldsymbol{r} \times \boldsymbol{v}$$
$$v_r = \frac{\boldsymbol{r}\boldsymbol{v}}{r}$$
$$\boldsymbol{e} = \frac{\boldsymbol{v} \times \boldsymbol{h} - \mu \dfrac{\boldsymbol{r}}{r}}{\mu}$$
$$a = \frac{h^2}{\mu(1-e^2)}$$
$$\boldsymbol{N} = \begin{bmatrix}0 & 0 & 1\end{bmatrix}^T \times \boldsymbol{h}$$
$$\theta = \begin{cases} \arccos\left(\dfrac{\boldsymbol{e}\cdot\boldsymbol{r}}{er}\right), & v_r >= 0 \\ 2\pi - \arccos\left(\dfrac{\boldsymbol{e}\cdot\boldsymbol{r}}{er}\right), & v_r < 0 \end{cases}$$
$$\Omega = \begin{cases} \arccos\left(\dfrac{\boldsymbol{N}(1)}{N}\right), & \boldsymbol{N}(2) >= 0 \\ 2\pi - \arccos\left(\dfrac{\boldsymbol{N}(1)}{N}\right), & \boldsymbol{N}(2) < 0 \end{cases}$$
$$\omega = \begin{cases} \arccos\left(\dfrac{\boldsymbol{N}\cdot\boldsymbol{e}}{Ne}\right), & \boldsymbol{e}(3) >= 0 \\ 2\pi - \arccos\left(\dfrac{\boldsymbol{N}\cdot\boldsymbol{e}}{Ne}\right), & \boldsymbol{e}(3) < 0 \end{cases}$$
$$i = \arccos\left(\frac{\boldsymbol{h}(3)}{h}\right)$$

From the above equations, initial conditions will find. The below equation shows the force of solar radiation.

$$P_{SRP} = \nu \frac{S}{c} C_R \frac{A_s}{m}$$

ν calculates if the satellite is in the earth's shadow or not. Then used the below equations for rate changes.

$$\begin{aligned}
\frac{dh}{dt} &= -p_{SR} r u_s \\
\frac{de}{dt} &= -p_{SR}\left(\frac{h}{\mu}\sin(\theta)u_r + \frac{1}{\mu h}\left((h^2+\mu r)\cos(\theta)\mu e r\right)u_s\right) \\
\frac{d\theta}{dt} &= \frac{h}{r^2} - \frac{p_{SR}}{eh}\left(\frac{h^2}{\mu}\cos(\theta)u_r - \left(r + \frac{h^2}{\mu}\right)\sin(\theta)u_s\right) \\
\frac{d\Omega}{dt} &= -p_{SR}\frac{r}{h\sin(i)}\sin(\omega+\theta)u_w \\
\frac{di}{dt} &= -p_{SR}\frac{r}{h}\cos(\omega+\theta)u_w \\
\frac{d\omega}{dt} &= -p_{SR}\left(\frac{1}{eh}\left(\frac{h^2}{\mu}\cos(\theta)u_r - \left(r+\frac{h^2}{\mu}\right)\sin(\theta)u_s\right) - \frac{r\sin(\omega-\theta)}{h\tan(i)}u_w\right)
\end{aligned}$$

For this purpose, example 10.9 was used, the Gauss planetary equations for solar radiation pressure (Equations 10.106). The script file is Q2.m.

Figure 3: h changes in 5 years

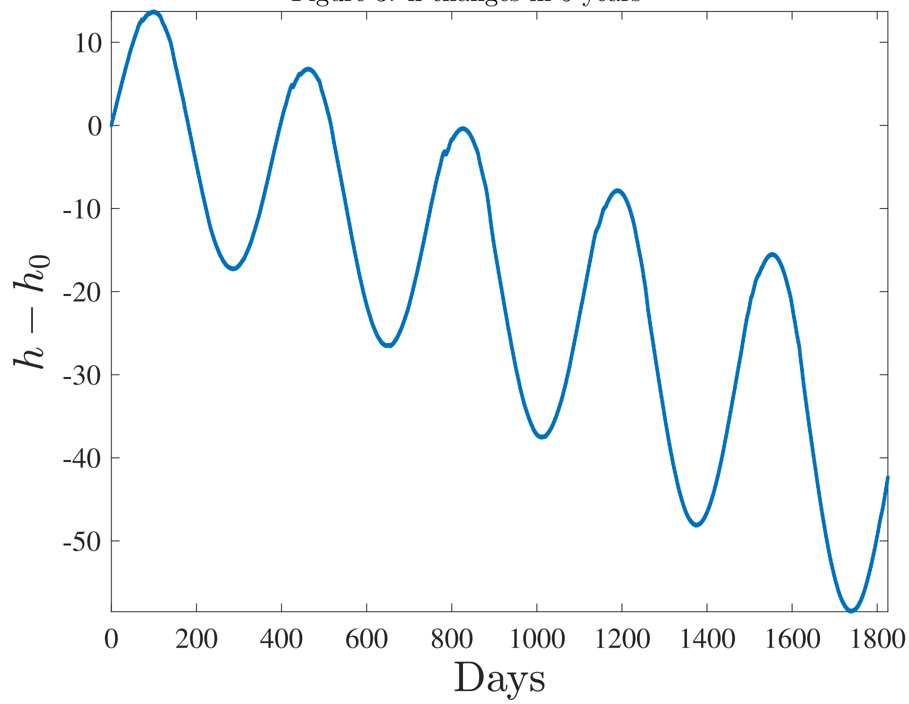


Figure 4: e changes in 5 years

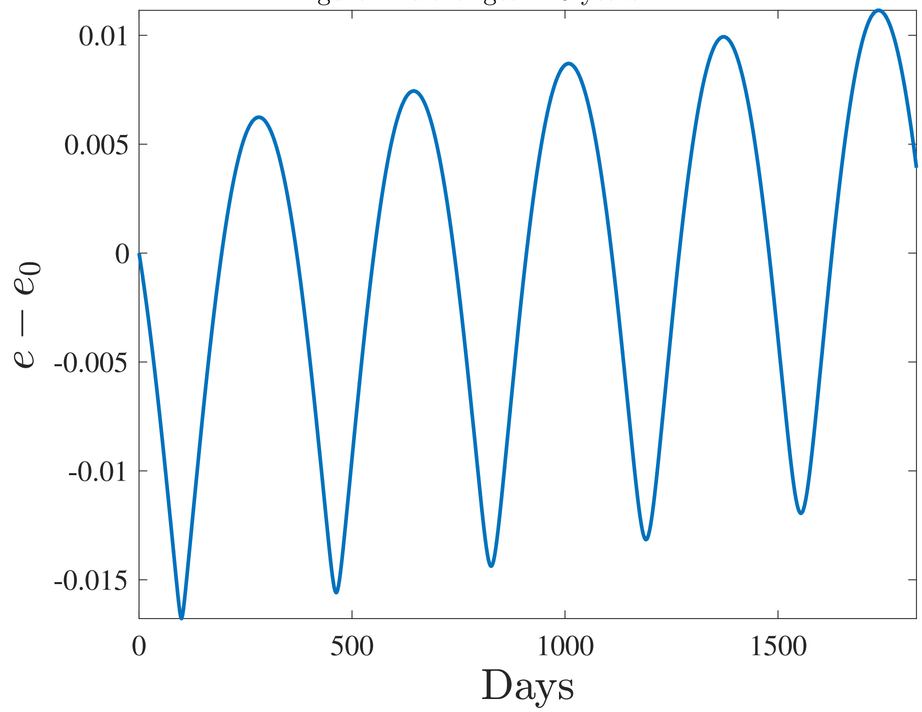


Figure 5: θ changes in 5 years (the satellite has a short period of 5 years of changes)

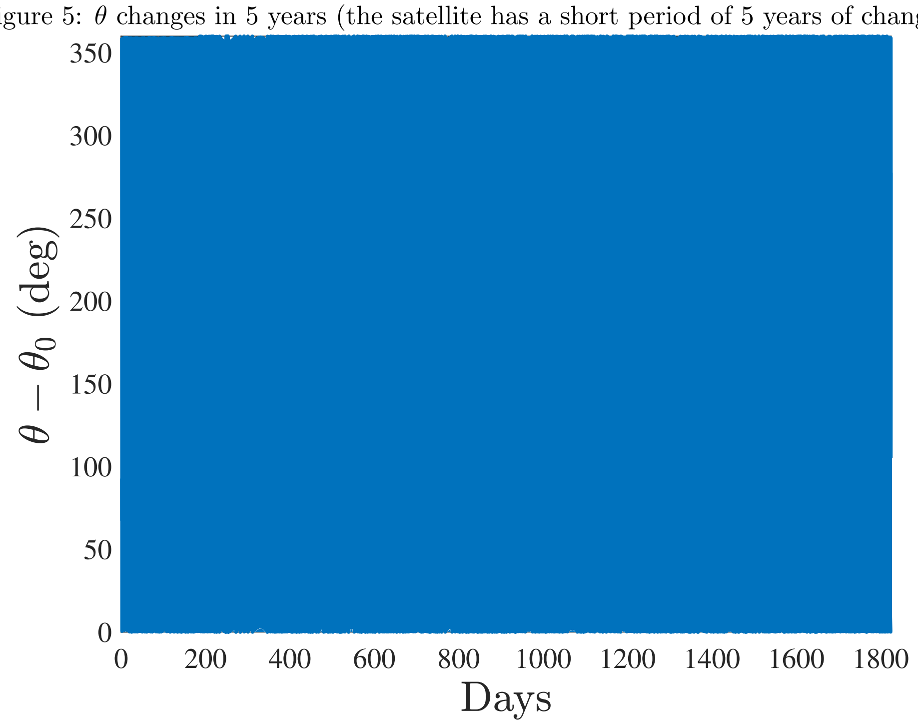


Figure 6: Ω changes in 5 years (the satellite has a short period of 5 years of changes)

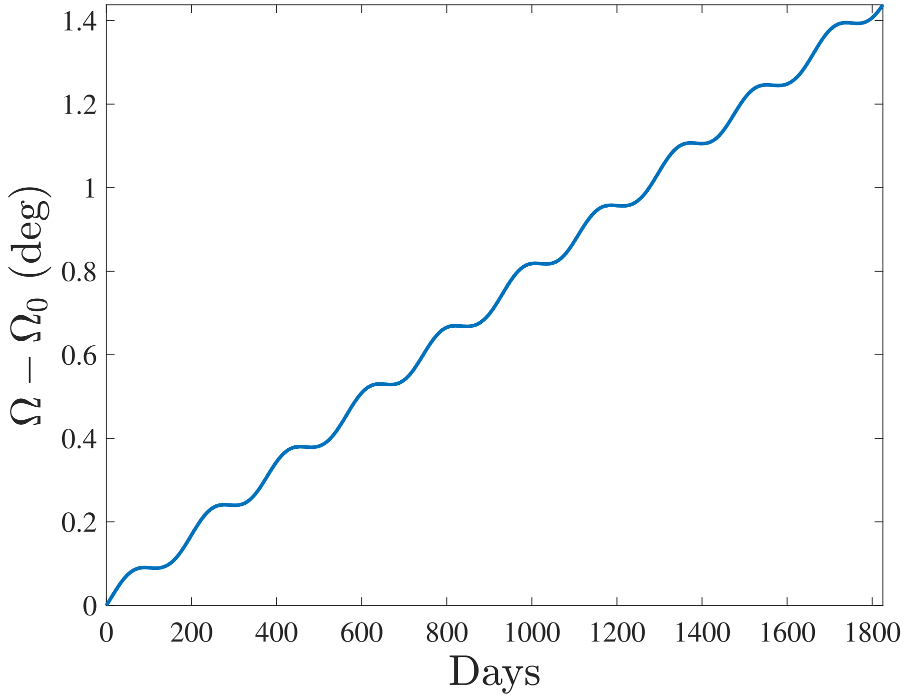


Figure 7: i changes in 5 years (the satellite has a short period of 5 years of changes)

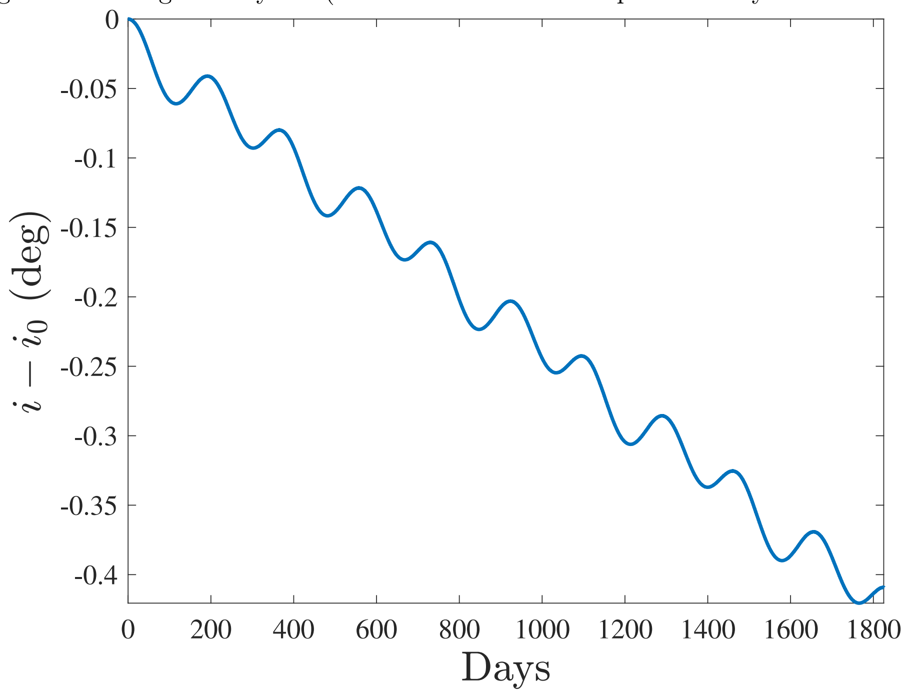
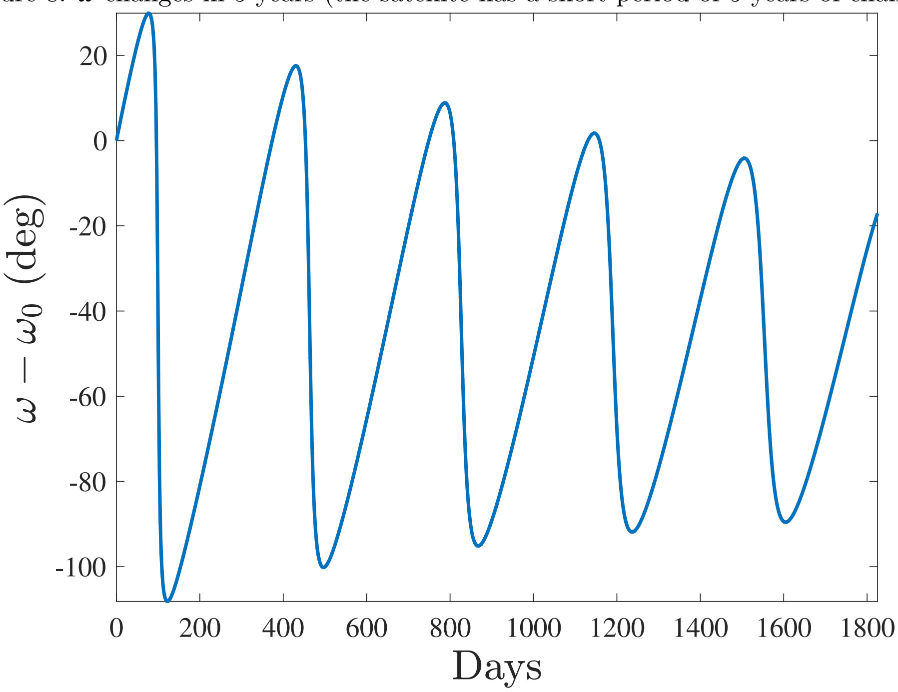


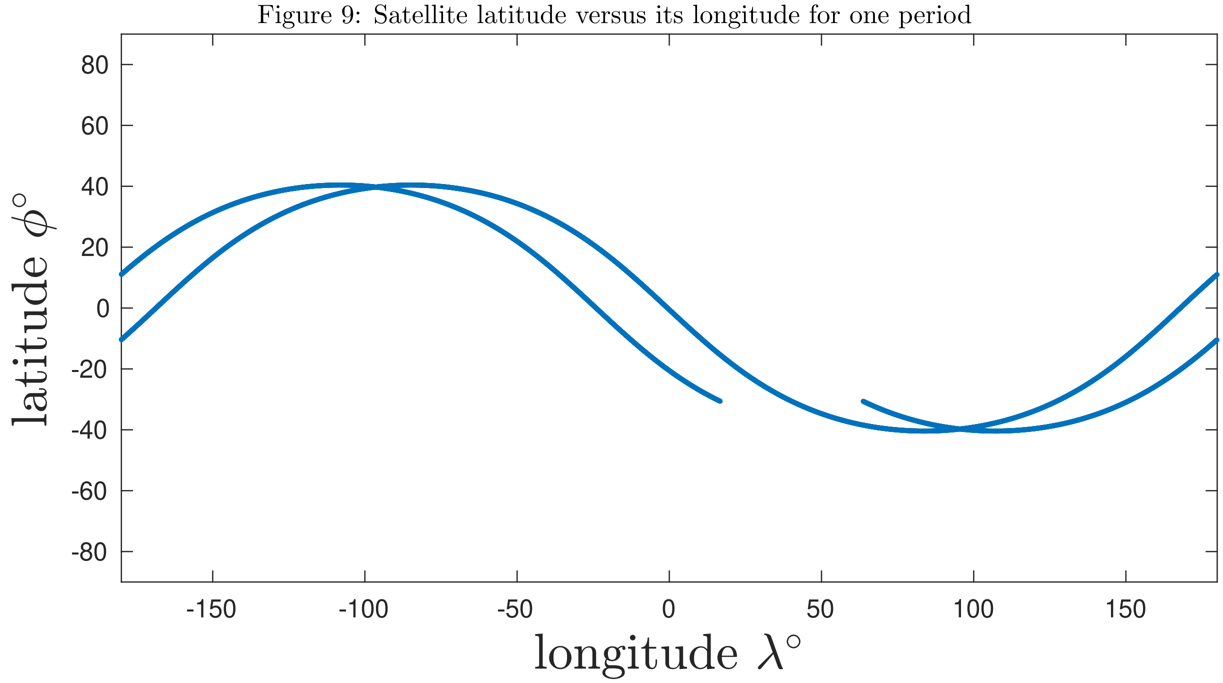
Figure 8: ω changes in 5 years (the satellite has a short period of 5 years of changes)



2.2 part b

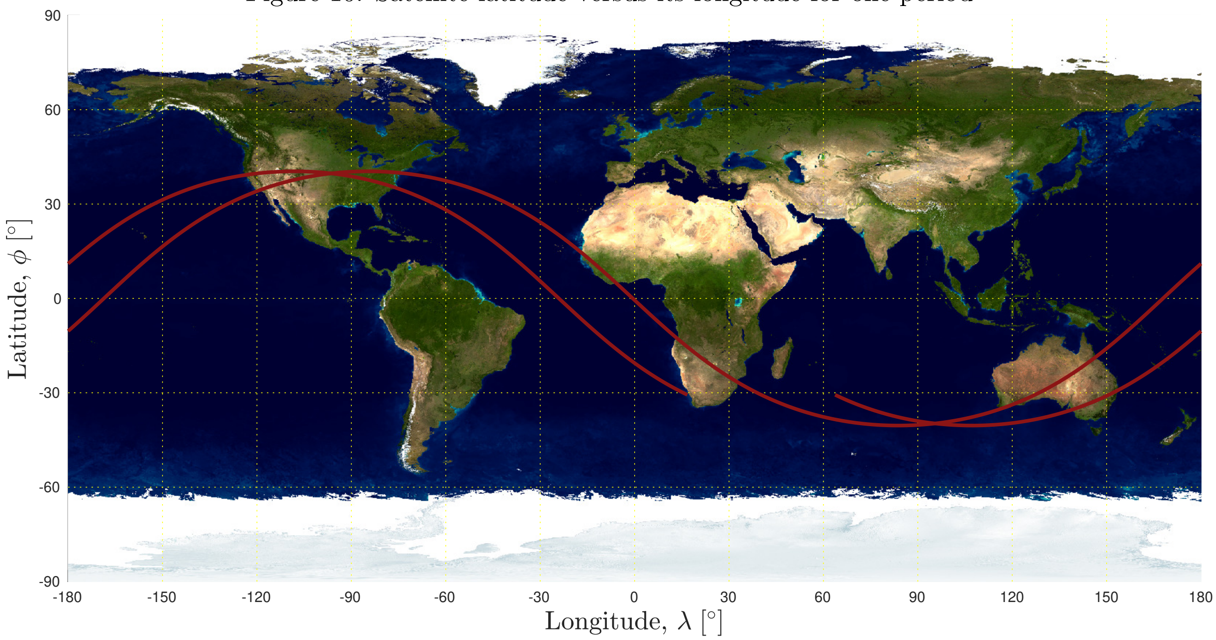
Used orbital elements from above code and sv_from_coe function from Curtis book to get satellite position and used Q1 short project to plot ground track.

Figure 9: Satellite latitude versus its longitude for one period



Below the figure drawn provided by tamaskis, please click here to see the source code. Please use mentioned library to run code or skip part on earth fig.

Figure 10: Satellite latitude versus its longitude for one period



3 Question 3

$$\Phi = -\frac{J_3\, R^3\, \mu \left(3\,\cos\left(\phi\right) - 5\cos\left(\phi\right)^3\right)}{2\, r^4}$$

$$\frac{\partial \Phi}{\partial x} = \frac{xz}{r^3 \sin(\phi)}, \quad \frac{\partial \Phi}{\partial y} = \frac{yz}{r^3 \sin(\phi)}, \quad \frac{\partial \Phi}{\partial x} = \frac{\sin(\phi)}{r}$$

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \Phi}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial \Phi}{\partial \phi}\frac{\partial \phi}{\partial x}$$

$$\frac{\partial \Phi}{\partial y} = \frac{\partial \Phi}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial \Phi}{\partial \phi}\frac{\partial \phi}{\partial y}$$

$$\frac{\partial \Phi}{\partial z} = \frac{\partial \Phi}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial \Phi}{\partial \phi}\frac{\partial \phi}{\partial z}$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\frac{\partial \Phi}{\partial x} = \frac{2\, J\, R^3\, \mu\, x \left(3\,\cos\left(\phi\right) - 5\cos\left(\phi\right)^3\right)}{r^6}$$

$$\frac{\partial \Phi}{\partial y} = \frac{2\, J\, R^3\, \mu\, y \left(3\,\cos\left(\phi\right) - 5\cos\left(\phi\right)^3\right)}{r^6}$$

$$\frac{\partial \Phi}{\partial z} = \frac{2\, J\, R^3\, \mu\, z \left(3\, \cos\left(\phi\right) - 5 \cos\left(\phi\right)^3\right)}{r^6}$$

using the fact that $\cos(\phi) = \dfrac{z}{r}$ leads to the following expressions for the gradient of perturbing potential Φ:

$$\frac{\partial \Phi}{\partial x} = -\frac{2\, J\, R^3\, \mu\, x \left(\frac{5\, z^3}{r^3} - \frac{3\, z}{r}\right)}{r^6}$$

$$\frac{\partial \Phi}{\partial y} = -\frac{2\, J\, R^3\, \mu\, y \left(\frac{5\, z^3}{r^3} - \frac{3\, z}{r}\right)}{r^6}$$

$$\frac{\partial \Phi}{\partial z} = -\frac{2\, J\, R^3\, \mu\, z \left(\frac{5\, z^3}{r^3} - \frac{3\, z}{r}\right)}{r^6}$$

$$\left[\mathbf{Q}\right]_{Xr} = \begin{bmatrix} -\sin(\Omega)\cos(i)\sin(u) + \cos(\Omega)\cos(u) & \cos(\Omega)\cos(i)\sin(u) + \sin(\Omega)\cos(u) & \sin(i)\sin(u) \\ -\sin(\Omega)\cos(i)\sin(u) - \cos(\Omega)\cos(u) & \cos(\Omega)\cos(i)\sin(u) - \sin(\Omega)\cos(u) & \sin(i)\sin(u) \\ \sin(\Omega)\sin(i) & -\cos(\Omega)\sin(i) & \cos(i) \end{bmatrix}$$

$$\begin{bmatrix} p_r \\ p_s \\ p_w \end{bmatrix} = \left[\mathbf{Q}\right]_{Xr} \begin{bmatrix} p_X \\ p_Y \\ p_Z \end{bmatrix}$$

$$\begin{bmatrix} p_r \\ p_s \\ p_w \end{bmatrix} = \left[\mathbf{Q}\right]_{Xr} \begin{bmatrix} -\frac{2\, J\, R^3\, \mu\, x \left(\frac{5\, z^3}{r^3} - \frac{3\, z}{r}\right)}{r^6} \\ -\frac{2\, J\, R^3\, \mu\, y \left(\frac{5\, z^3}{r^3} - \frac{3\, z}{r}\right)}{r^6} \\ -\frac{2\, J\, R^3\, \mu\, z \left(\frac{5\, z^3}{r^3} - \frac{3\, z}{r}\right)}{r^6} \end{bmatrix}$$

$$\frac{de}{dt} = \frac{\left(1 - e^2\right)^{1/2}}{na} \left(\sin(\theta) p_r + (\cos(\theta) + \cos(E)) p_s\right)$$

where:

$$n = \sqrt{\frac{\mu}{a^3}}$$

4 Question 4

$$\boldsymbol{r}(0) = \begin{bmatrix} 0.994 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{v}(0) = \begin{bmatrix} 0 & -2.001585106 & 0 \end{bmatrix}$$

$$m_{earth} = 5.974e24_{kg}, \quad m_{moon} = 7.348e22_{kg}, \quad r_{12} = 3.844e5_{km}$$

4.1 part a

In Canonical units, jacobi constant is defined as:

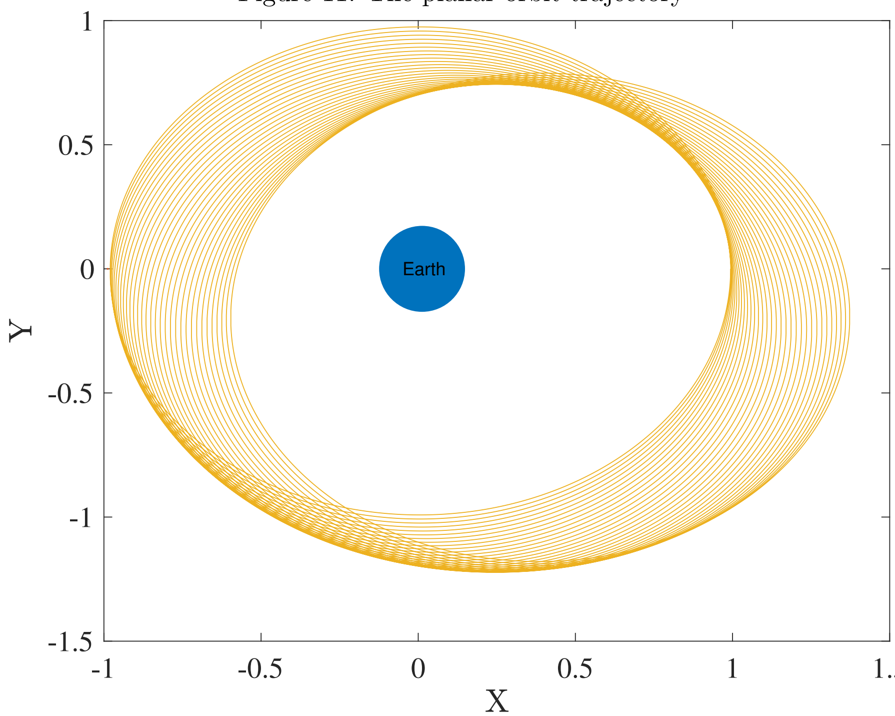
$$C = \frac{1}{2}v^2 - \frac{1}{2}(x^2 + y^2) - \mu = 1.4970$$

4.2 part b

Solve the equation of motion for the moon earth system in canonical units:

$$\begin{aligned}
\ddot{x} &= 2\dot{y} + x - \frac{1-\mu}{r_1^3}(x-\mu) - \frac{\mu}{r_2^3}(x+1-\mu) \\
\ddot{y} &= -2\dot{x} + xy - \frac{1-\mu}{r_1^3}y - \frac{\mu}{r_2^3}y \\
\ddot{z} &= -\frac{1-\mu}{r_1^3}z - \frac{\mu}{r_2^3}z
\end{aligned}$$

Figure 11: The planar orbit trajectory



4.3 part c

From Bong Wie space vehicle dynamics, we have:

$$\begin{aligned}
U_{XX} &= \frac{\partial^2 U}{\partial X^2}\bigg|_{X=X_0} = 1 - \left((1-\mu)\left(\frac{1}{r_1^3} - 3\frac{(X_0-\mu)^2}{r_1^5}\right) + \mu\left(\frac{1}{r_2^3} - 3\frac{(X_0+1-\mu)^2}{r_2^5}\right)\right) \\
U_{YY} &= \frac{\partial^2 U}{\partial U^2}\bigg|_{Y=Y_0} = 1 - \left((1-\mu)\left(\frac{1}{r_1^3} - 3\frac{Y_0^2}{r_1^5}\right) + \mu\left(\frac{1}{r_2^3} - 3\frac{Y_0^2}{r_2^5}\right)\right)
\end{aligned}$$

$$\lambda^4 + (4 - U_{XX} - U_{YY})\lambda^2 + U_{XX}U_{YY} = 0$$

Above equation solved in Q4.m MATLAB file in part c section.

$$\lambda_{1,2} = 0.3460 \pm 0i$$
$$\lambda_{3,4} = 0 \pm 1.0803i$$

The in-plane motion has a divergent mode as well as an oscillatory mode with a nondimensional frequency $\omega_{xy} = 1.0803$ and the period of the in-plane oscillatory mode is 25.25 days.

$$\omega_{xy} = 1.0803 \times \omega_{earth-moon} = 2.8794e - 6 \rightarrow \tau = \frac{2\pi}{\omega_{xy}} = 5.4553e5_{\text{sec}} = 25.2559_{day}$$

5 Bonus

General equation of mottion is:

$$\delta\ddot{x} - 3n^2\delta x - 2n\delta\dot{y} = 0$$
$$\delta\ddot{y} + 2n\delta\dot{x} = 0$$
$$\delta\ddot{z} + 2n\delta\dot{z} = 0$$

where:

$$n = \sqrt{\frac{\mu}{r^3}}$$

above differential equation can be solved in curtis book and final answer is:

$$\delta x = 4\delta x_0 + \frac{2}{n}\delta\dot{y}_0 + \frac{\delta\dot{x}_0}{n}\sin(nt) - \left(3\delta x_0 + \frac{2}{n}\delta\dot{y}_0\right)\cos(nt)$$
$$\delta y = \delta y_0 - \frac{2}{n}\delta\dot{x}_0 - 3(2n\delta x_0 + \delta\dot{y}_0)t + 2\left(3\delta x_0 + \frac{2}{n}\delta\dot{y}_0\right)\sin(nt) + \frac{2}{n}\delta\dot{x}_0\cos(nt)$$
$$\delta z = \frac{1}{n}\delta\dot{z}_0\sin(nt) + \delta z_0\cos(nt)$$

$$\delta\boldsymbol{r}(t) = \begin{bmatrix}\delta x(t)\\ \delta y(t)\\ \delta z(t)\end{bmatrix}, \quad \delta\boldsymbol{v}(t) = \begin{bmatrix}\delta u(t)\\ \delta v(t)\\ \delta w(t)\end{bmatrix}$$

$$\delta\boldsymbol{r}_0 = \begin{bmatrix}\delta x_0\\ \delta y_0\\ \delta z_0\end{bmatrix}, \quad \delta\boldsymbol{v}_0 = \begin{bmatrix}\delta u_0\\ \delta v_0\\ \delta w_0\end{bmatrix}$$

$$\delta\boldsymbol{r}(t) = \left[\boldsymbol{\Phi}_{rr}(t)\right]\delta\boldsymbol{r}_0 + \left[\boldsymbol{\Phi}_{rv}(t)\right]\delta\boldsymbol{v}_0$$
$$\delta\boldsymbol{v}(t) = \left[\boldsymbol{\Phi}_{vr}(t)\right]\delta\boldsymbol{r}_0 + \left[\boldsymbol{\Phi}_{vv}(t)\right]\delta\boldsymbol{v}_0$$

$$\boldsymbol{\Phi}_{rr}(t) = \begin{bmatrix} 4 - 3\cos(nt) & 0 & 0\\ 6(\sin(nt) - nt) & 1 & 0\\ 0 & 0 & \cos(nt)\end{bmatrix}$$

$$\boldsymbol{\Phi}_{rv}(t) = \begin{bmatrix} \frac{1}{n}\sin(nt) & \frac{2}{n}(1 - \cos(nt)) & 0\\ \frac{2}{n}(\cos(nt) - 1) & \frac{1}{n}(4\sin(nt) - 3nt) & 0\\ 0 & 0 & \frac{1}{n}\sin(nt)\end{bmatrix}$$

$$\boldsymbol{\Phi}_{vr}(t) = \begin{bmatrix} 3n\sin(nt) & 2n\sin(nt) & 0 \\ 6n(\cos(nt)-1) & 0 & 0 \\ 0 & 0 & -n\sin(nt) \end{bmatrix}$$

$$\boldsymbol{\Phi}_{vv}(t) = \begin{bmatrix} \cos(nt) & 2\sin(nt) & 0 \\ -2n\sin(nt) & 4n\cos(nt)-3 & 0 \\ 0 & 0 & \cos(nt) \end{bmatrix}$$

The Clohessy-Wiltshire matrices, for $t_f = 8_h$ and $n = 0.0011_{rad/\sec}$.

$$\boldsymbol{\Phi}_{rr} = \begin{bmatrix} 1.0361 & 0 & 0 \\ -188.4918 & 1.0000 & 0 \\ 0 & 0 & 0.9880 \end{bmatrix}$$

$$\boldsymbol{\Phi}_{rv} = \begin{bmatrix} 142 & 22 & 0 \\ -22 & -86970 & 0 \\ 0 & 0 & -142 \end{bmatrix}$$

$$\boldsymbol{\Phi}_{vr} = \begin{bmatrix} -0.0005 & -0.0003 & 0 \\ -0.0001 & 0 & 0 \\ 0 & 0 & 0.0070 \end{bmatrix}$$

$$\boldsymbol{\Phi}_{vv} = \begin{bmatrix} 0.9880 & -0.3092 & 0 \\ 0.0003 & -2.9957 & 0 \\ 0 & 0 & 0.9880 \end{bmatrix}$$

At time t_f, Hubble arrives at spaceX, at the origin of the CW frame, which means $\delta \boldsymbol{r}_f = \delta r_f = \mathbf{0}$. At t_f we find:

$$\mathbf{0} = \boldsymbol{\Phi}_{rr}(t_f)\delta\boldsymbol{r}_0 + \boldsymbol{\Phi}_{rv}(t_f)\delta\boldsymbol{v}_0^+ \to \delta\boldsymbol{v}_0^+ = -\boldsymbol{\Phi}_{rr}(t_f)^{-1}\boldsymbol{\Phi}_{rr}(t_f)\delta\boldsymbol{r}_0$$

$$\delta\boldsymbol{v}_0^+ = \begin{bmatrix} 0.9923 \\ -0.3086 \\ 0.1232 \end{bmatrix}_{m/\sec}$$

$$\delta\boldsymbol{v}_f^- = \boldsymbol{\Phi}_{vr}(t_f)\delta\boldsymbol{r}_0 + \boldsymbol{\Phi}_{vv}(t_f)\delta\boldsymbol{v}_0^+ = \boldsymbol{\Phi}_{vr}(t_f)\delta\boldsymbol{r}_0 + \boldsymbol{\Phi}_{vv}(t_f)\left(-\boldsymbol{\Phi}_{rr}(t_f)^{-1}\boldsymbol{\Phi}_{rr}(t_f)\delta\boldsymbol{r}_0\right)$$

$$\delta\boldsymbol{v}_f^- = \begin{bmatrix} 0.9577 \\ 0.9136 \\ 0.2465 \end{bmatrix}_{m/\sec}$$

So the change of velocity of Hubble is $|\delta\boldsymbol{v}_0^- - \delta\boldsymbol{v}_0^+|$ and the change of velocity of SpaceX is $|\delta\boldsymbol{v}_f^-|$. The velocity change in this maneuver is not too big. The system is unstable with LQR controller.

Figure 12: Three dimensional orbit trajectory

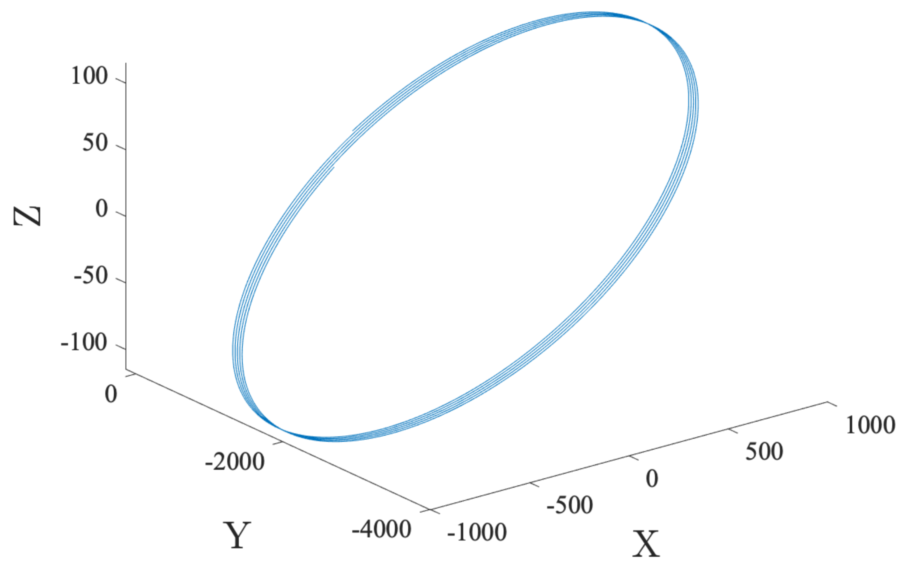
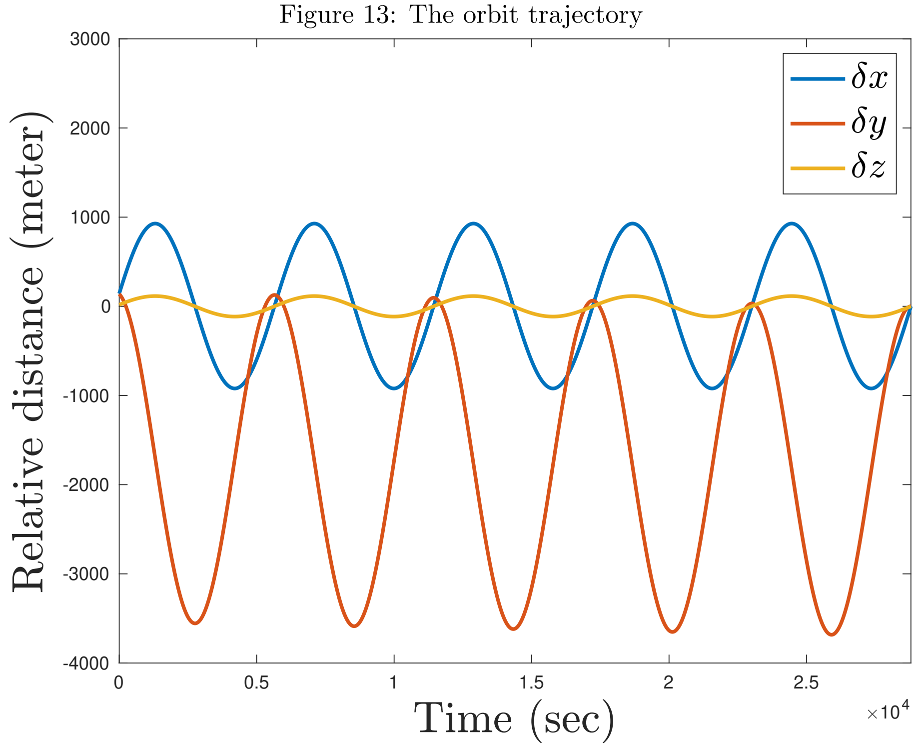


Figure 13: The orbit trajectory



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