

Home Work #4

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January 25, 2023

1 Question 1

We know inertia matrix is symmetric so we have:

$$I_{xy} = I_{yx} = 10, \quad I_{xz} = I_{zx} = 0, \quad I_{yz} = I_{zy}$$

$$\mathbf{I} = \begin{bmatrix} 30 & -10 & 0 \\ -10 & 20 & -I_{yz} \\ 0 & -I_{yz} & 30 \end{bmatrix}$$

1.1 part a

We know that:

$$\mathbf{I} \times \boldsymbol{\omega} = \mathbf{h} \quad (1)$$

$$\boldsymbol{\omega} = [10 \quad 10 \quad 10]_{RPS}^T = [10 \times 2\pi \quad 10 \times 2\pi \quad 10 \times 2\pi]_{rad/sec}^T, \quad \mathbf{h} = [200 \quad 200 \quad 400]_{kg.m^2/s}^T$$

If we use radian per second instead of revolution per second $\mathbf{I} \times \boldsymbol{\omega} \neq \mathbf{h}$ would happen.

$$\mathbf{I} \times \boldsymbol{\omega} = \begin{bmatrix} 200 \\ 100 - 10I_{yz} \\ 300 - 10I_{yz} \end{bmatrix} = \begin{bmatrix} 200 \\ 200 \\ 400 \end{bmatrix} \rightarrow I_{yz} = -10 \rightarrow \mathbf{I} = \begin{bmatrix} 30 & -10 & 0 \\ -10 & 20 & 10 \\ 0 & 10 & 30 \end{bmatrix}$$

$$T_{Rotational} = \frac{1}{2} \boldsymbol{\omega}^T \times \mathbf{I} \times \boldsymbol{\omega} = \frac{1}{2} [10 \quad 10 \quad 10] \times \begin{bmatrix} 30 & -10 & 0 \\ -10 & 20 & 10 \\ 0 & 10 & 30 \end{bmatrix} \times \begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix} = 4000 \quad (2)$$

1.2 part b

$$T_{Rotational} = \frac{1}{2} I_{\xi} \omega^2 \rightarrow I_{\xi} = \frac{2T_{Rotational}}{\omega^2} = \frac{2 \times 4000}{300} = 26.67 \quad (3)$$

Rotation matrix calculated via eigen vector of inertia matrix (used MATLAB to calculate).

$$\mathbf{A} = \text{eig}(\mathbf{I}) = \begin{bmatrix} -0.4082 & -0.7071 & -0.5774 \\ -0.8165 & -0.0000 & 0.5774 \\ 0.4082 & -0.7071 & 0.5774 \end{bmatrix} \quad (4)$$

$$\mathbf{I}' = \mathbf{A}^T \times \mathbf{I} \times \mathbf{A} = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 40 \end{bmatrix}, \quad \boldsymbol{\omega}' = \mathbf{A}^T \boldsymbol{\omega} = \begin{bmatrix} -8.1650 \\ -14.1421 \\ 5.7735 \end{bmatrix}$$

$$h = |\mathbf{I} \times \boldsymbol{\omega}| = 489.8979$$

We know that above matrix shows the direction cosines of the principal axes with the primary body axes. Used MATLAB function (dcm2angle) to calculate euler angles between two corinate system.

$$\begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} = \begin{bmatrix} 45.00^\circ \\ 35.26^\circ \\ -120.00^\circ \end{bmatrix}$$

1.3 part c

Ellipsoid of inertia calculated as folow:

$$\frac{X^2}{\left(\sqrt{\frac{1}{I_x}}\right)^2} + \frac{Y^2}{\left(\sqrt{\frac{1}{I_y}}\right)^2} + \frac{Z^2}{\left(\sqrt{\frac{1}{I_z}}\right)^2} = \frac{X^2}{\left(\sqrt{\frac{1}{10}}\right)^2} + \frac{Y^2}{\left(\sqrt{\frac{1}{30}}\right)^2} + \frac{Z^2}{\left(\sqrt{\frac{1}{40}}\right)^2} = 1 \quad (5)$$

Figure 1: elipsoid of inertia

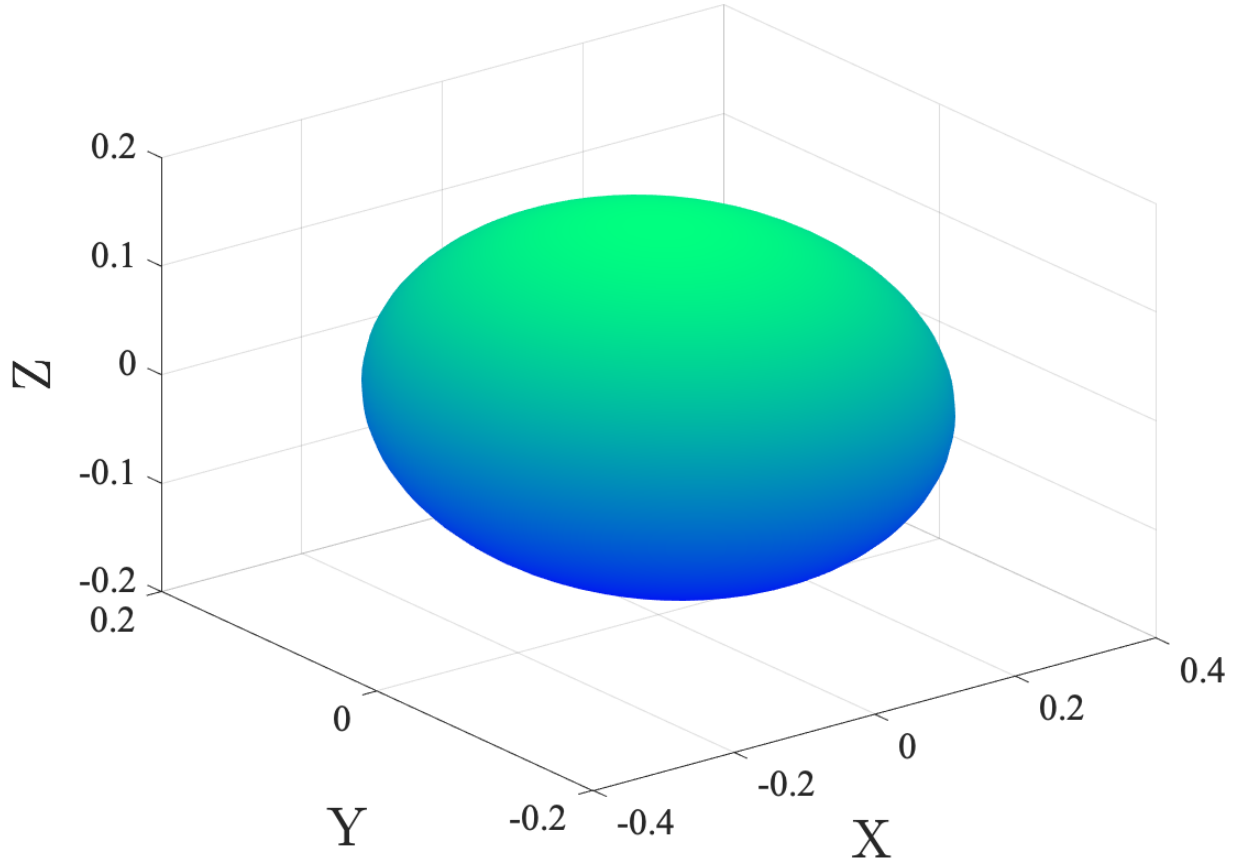


Figure 2: elipsoid of inertia in zx plane

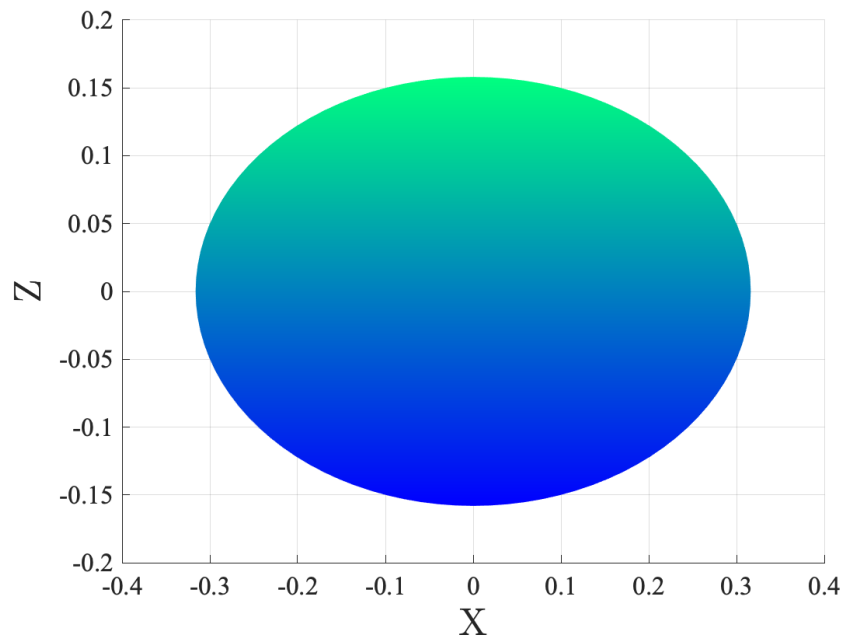


Figure 3: elipsoid of inertia zy plane

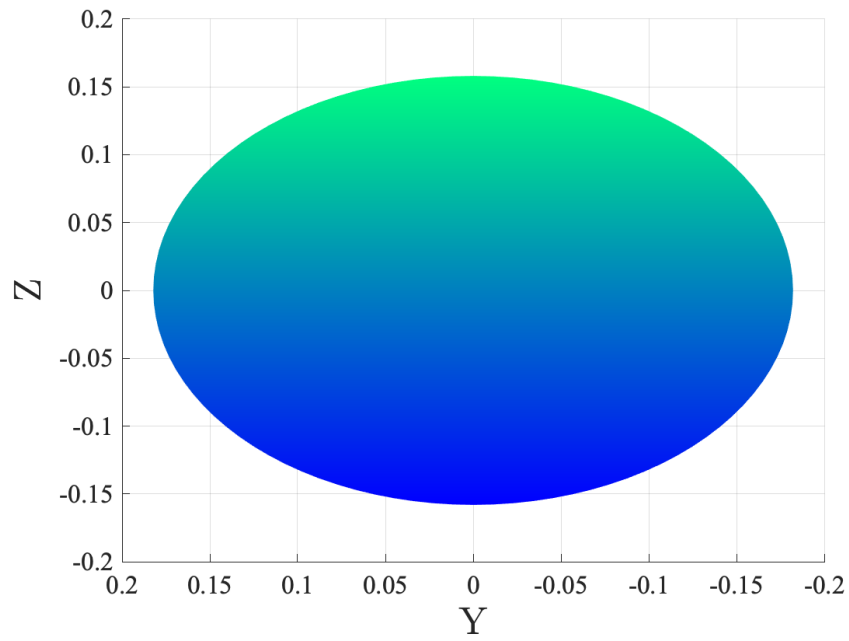
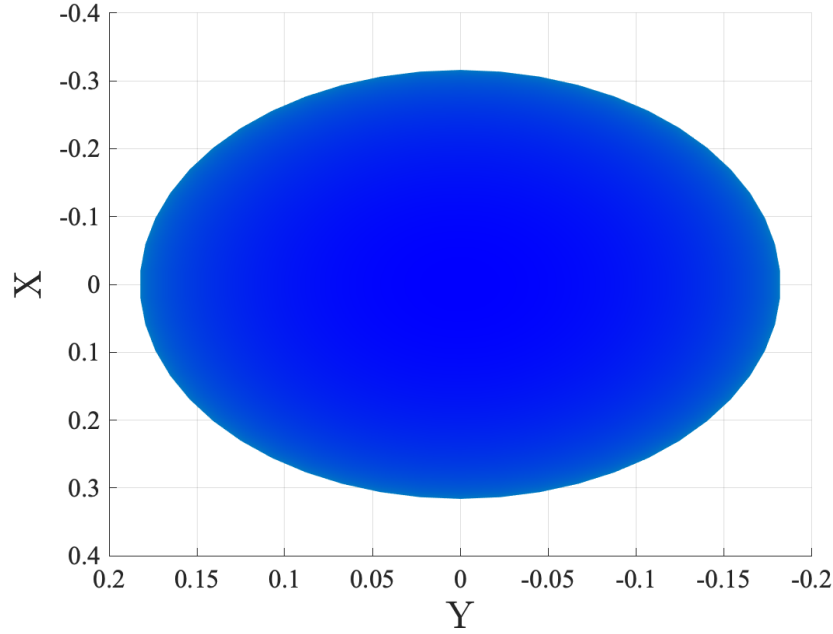


Figure 4: elipsoid of inertia in xy plane



1.4 part d

Angular momentum and rotational kinetic energy ellipsoid calculated as follow:

$$\frac{\omega_x^2}{\left(\frac{h}{I_x}\right)^2} + \frac{\omega_y^2}{\left(\frac{h}{I_y}\right)^2} + \frac{\omega_z^2}{\left(\frac{h}{I_z}\right)^2} = \frac{\omega_x^2}{\left(\frac{490}{10}\right)^2} + \frac{\omega_y^2}{\left(\frac{490}{30}\right)^2} + \frac{\omega_z^2}{\left(\frac{490}{40}\right)^2} = 1 \quad (6)$$

$$\frac{\omega_x^2}{\left(\sqrt{\frac{2T}{I_x}}\right)^2} + \frac{\omega_y^2}{\left(\sqrt{\frac{2T}{I_y}}\right)^2} + \frac{\omega_z^2}{\left(\sqrt{\frac{2T}{I_z}}\right)^2} = \frac{\omega_x^2}{\left(\sqrt{\frac{2 \times 4000}{10}}\right)^2} + \frac{\omega_y^2}{\left(\sqrt{\frac{2 \times 4000}{30}}\right)^2} + \frac{\omega_z^2}{\left(\sqrt{\frac{2 \times 4000}{40}}\right)^2} = 1 \quad (7)$$

Ellipsoid parameter calculated before, now, use them to draw the ellipsoid.

Figure 5: Angular momentum and rotational kinetic energy ellipsoid of inertia

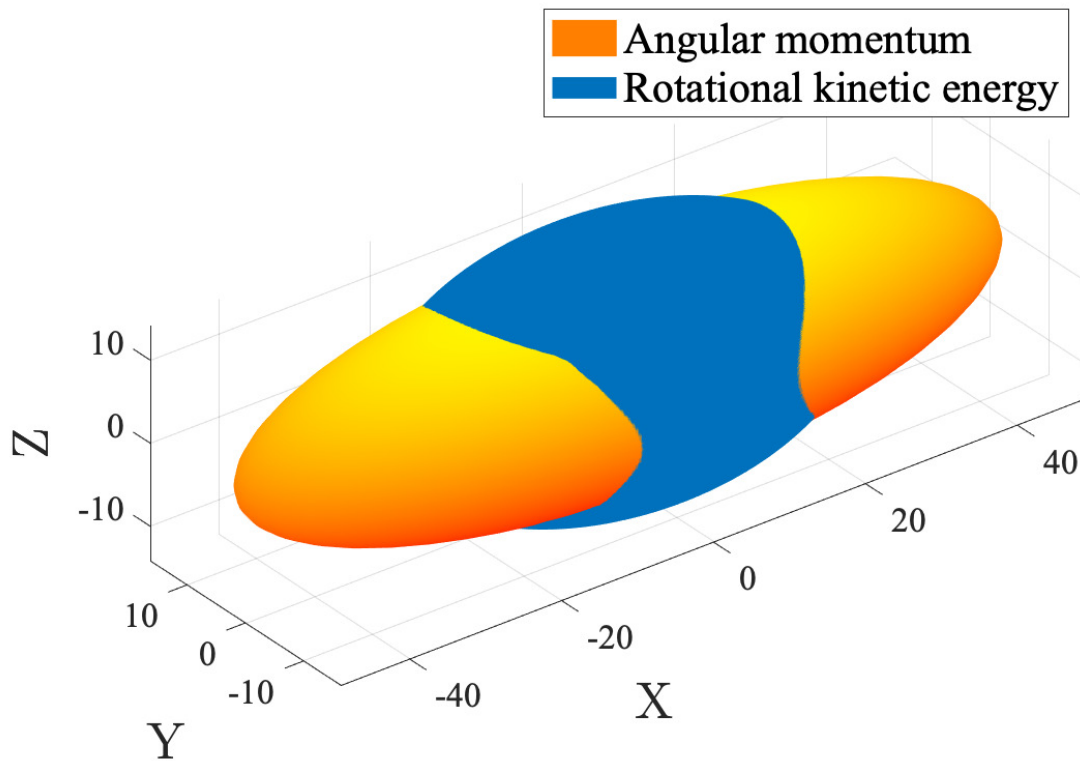


Figure 6: Angular momentum and rotational kinetic energy ellipsoid of inertia in zx plane

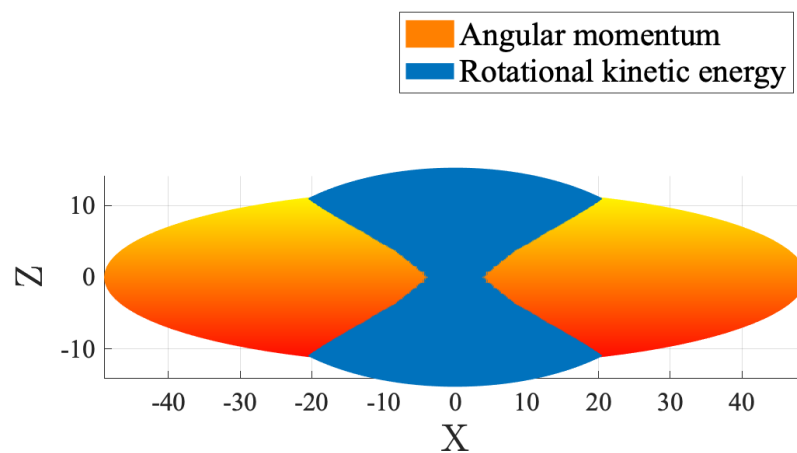


Figure 7: Angular momentum and rotational kinetic energy ellipsoid of inertia zy plane

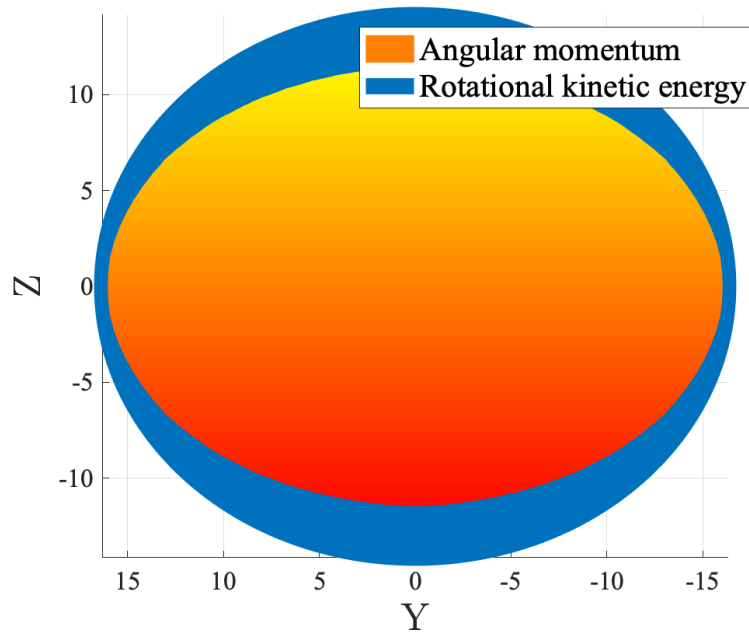
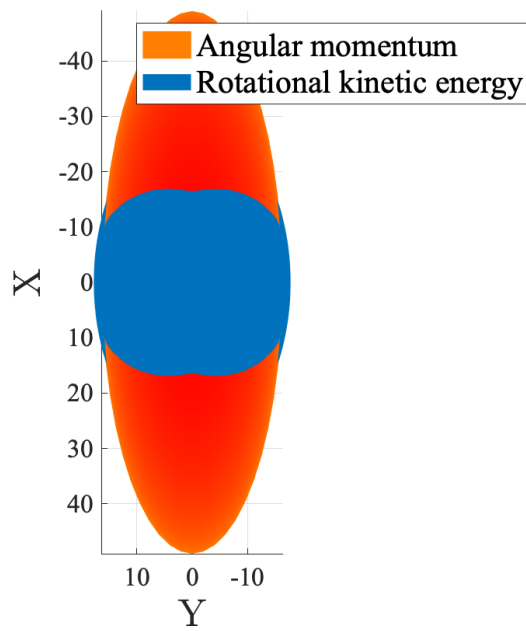


Figure 8: Angular momentum and rotational kinetic energy ellipsoid of inertia in xy plane



2 Question 2

Data of the question:

$$I_x = 8_{kg.m^2}, \quad I_y = 10_{kg.m^2}, \quad I_z = 14_{kg.m^2}$$

$$\mathbf{q}(0) = [0.3 \quad 0.2 \quad 0.5 \quad 0.7874]^T$$

$$\boldsymbol{\omega}(0) = [2 \quad 3 \quad -5]^T \times 10_{rps}^{-3} = [2 \times 2\pi \quad 3 \times 2\pi \quad -5 \times 2\pi]^T \times 10_{rad/sec}^{-3}$$

2.1 part a

Used MATLAB function(quat2eul) to calculate initial euler angles.

$$\begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} = \begin{bmatrix} 137.74^\circ \\ -0.86^\circ \\ 65.16^\circ \end{bmatrix}$$

2.2 part b

Equation of motion with gravity gradient:

$$I_x \dot{\omega}_x + \omega_y \omega_z (I_z - I_y) = G_x \quad (8)$$

$$I_y \dot{\omega}_y + \omega_x \omega_z (I_x - I_z) = G_y \quad (9)$$

$$I_z \dot{\omega}_z + \omega_x \omega_y (I_y - I_x) = G_z \quad (10)$$

$$(11)$$

$$\dot{\omega}_x = (G_x + \omega_y \omega_z (I_y - I_z)) / I_x$$

$$\dot{\omega}_y = (G_y + \omega_x \omega_z (I_z - I_x)) / I_y$$

$$\dot{\omega}_z = (G_z + \omega_y \omega_x (I_x - I_y)) / I_z$$

$$\begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} = \begin{bmatrix} p \\ q \\ r \end{bmatrix} + \mathbf{C}_R^b \begin{bmatrix} 0 \\ -\omega_0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} + \mathbf{C}_R^b \begin{bmatrix} 0 \\ \omega_0 \\ 0 \end{bmatrix} \quad (12)$$

where \mathbf{C}_R^b is transformation matrix aand $\omega_0 = \sqrt{\frac{\mu}{R_0^3}}$.

$$\mathbf{C}_b^R = \begin{bmatrix} \cos(\theta) \cos(\psi) & -\cos(\phi) \sin(\psi) + \sin(\phi) \sin(\theta) \cos(\psi) & \sin(\phi) \sin(\psi) + \cos(\phi) \sin(\theta) \cos(\psi) \\ \cos(\theta) \sin(\psi) & \cos(\phi) \cos(\psi) + \sin(\phi) \sin(\theta) \sin(\psi) & -\sin(\phi) \cos(\psi) + \cos(\phi) \sin(\theta) \sin(\psi) \\ -\sin(\theta) & \sin(\phi) \cos(\theta) & \cos(\phi) \cos(\theta) \end{bmatrix} \quad (13)$$

Using euler propagation:

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin(\phi) \tan(\theta) & \cos(\phi) \tan(\theta) \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \frac{\sin(\phi)}{\cos(\theta)} & \frac{\cos(\phi)}{\cos(\theta)} \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \quad (14)$$

In linearized equation of motion we assume:

$$\mathbf{C}_b^R = \begin{bmatrix} 1 & -\psi & \theta \\ \psi & 1 & -\phi \\ -\theta & \phi & 1 \end{bmatrix}, \quad \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \quad (15)$$

For the transformation from reference to body use the transpose of the described matrix above.

The equations of motions solved in MATLAB with ode45 function. Below is the figure of euler angles simulation in 50 and 1000 seconds.

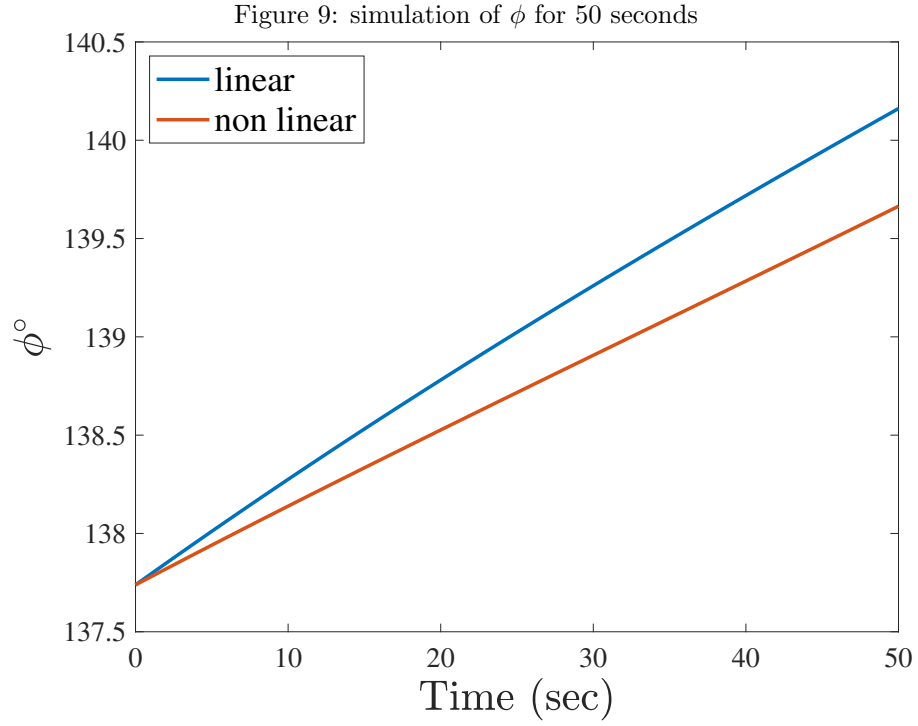
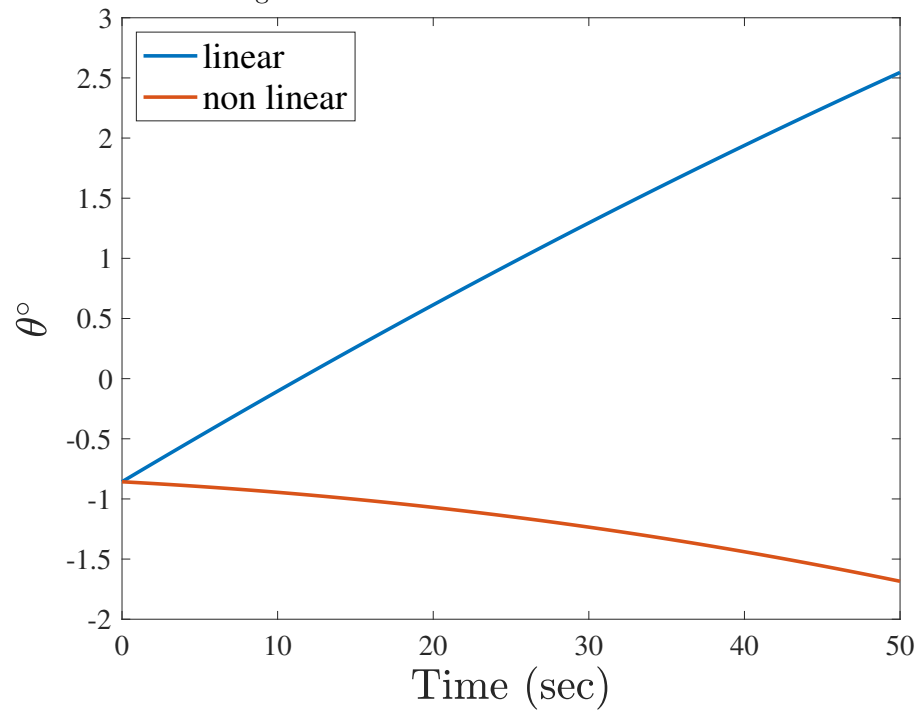
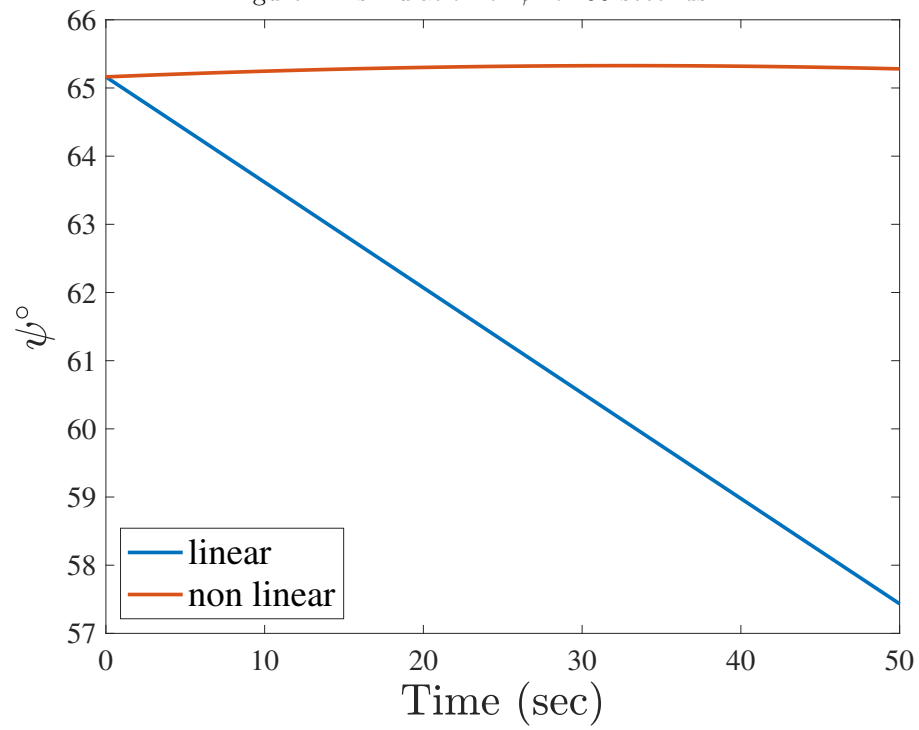
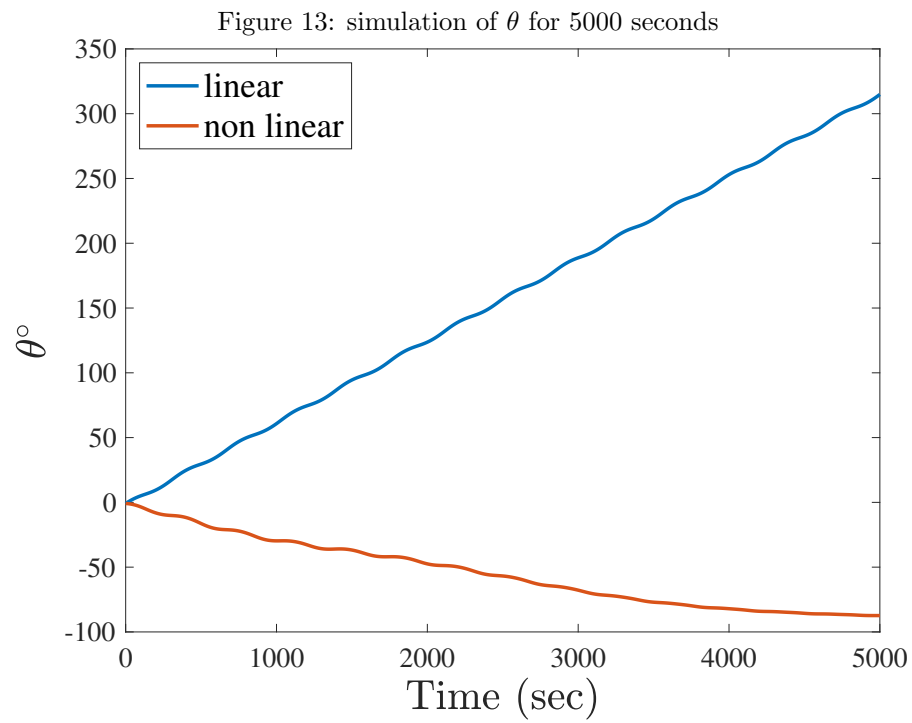
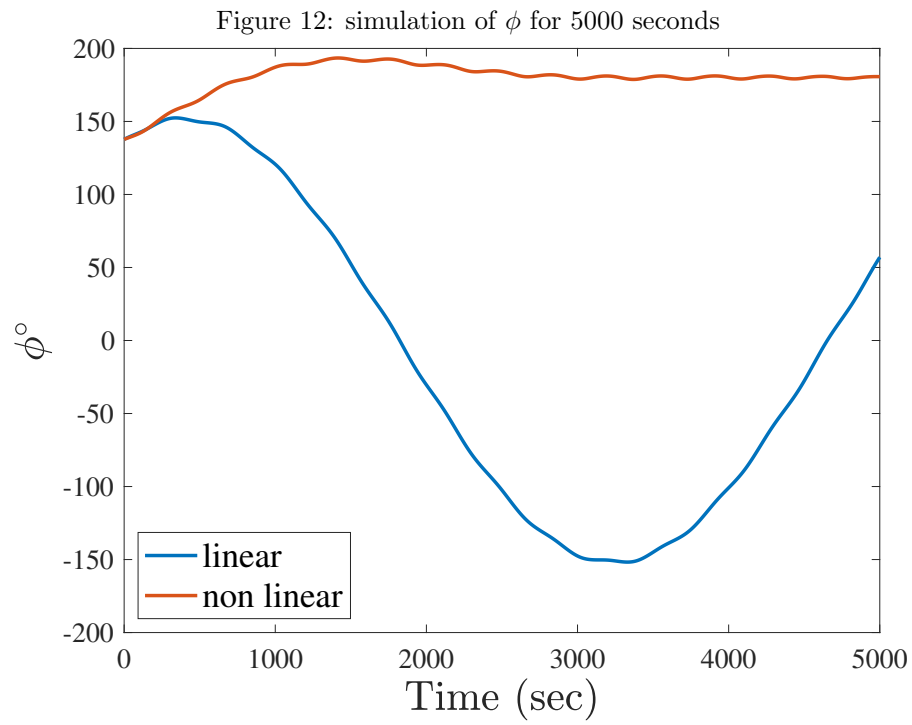
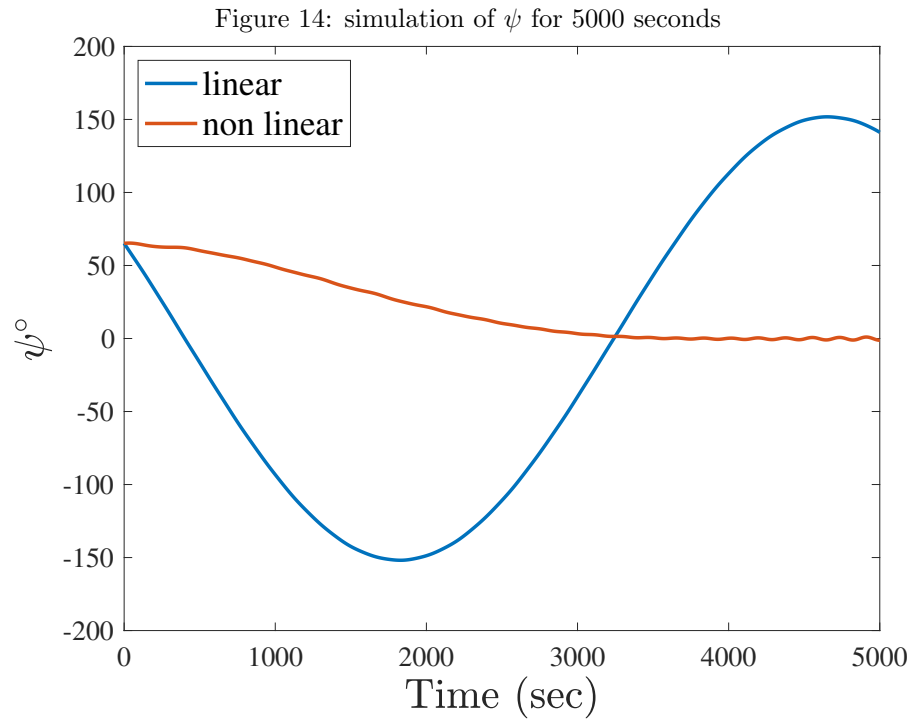


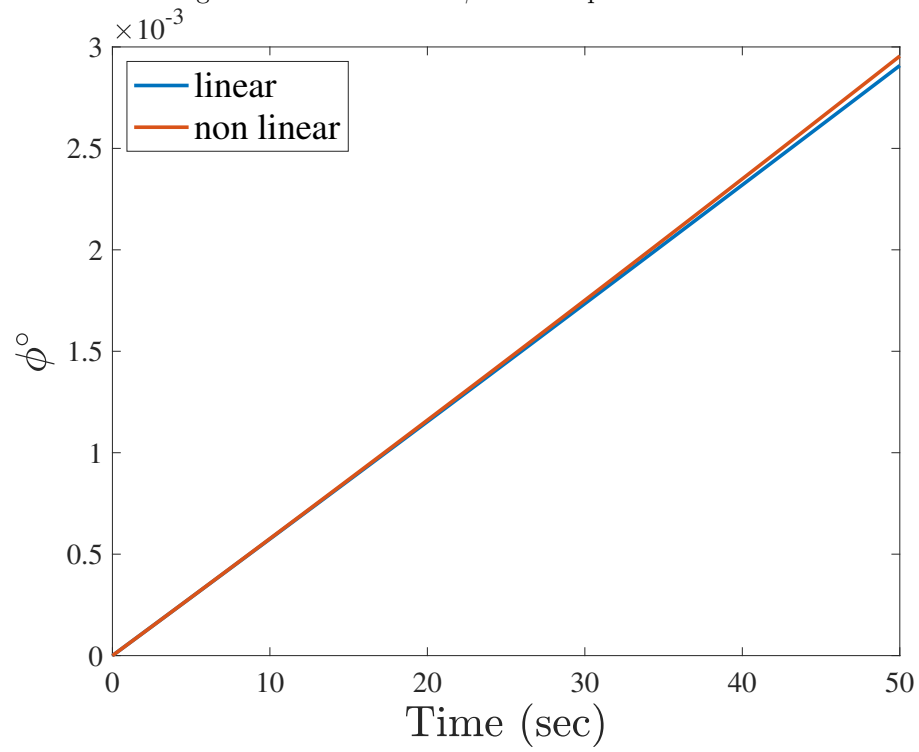
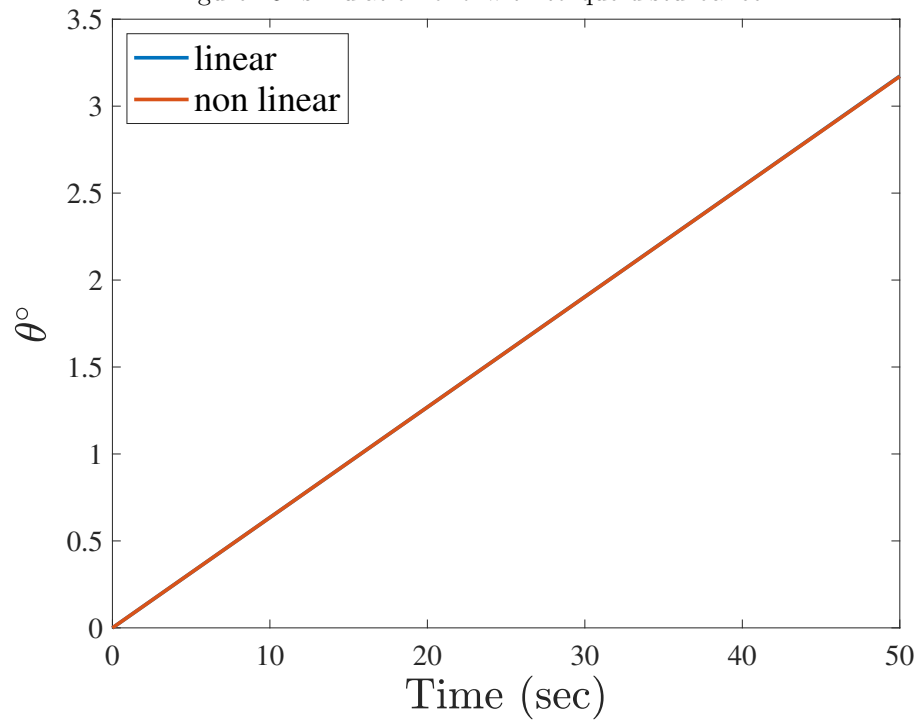
Figure 10: simulation of θ for 50 secondsFigure 11: simulation of ψ for 50 seconds

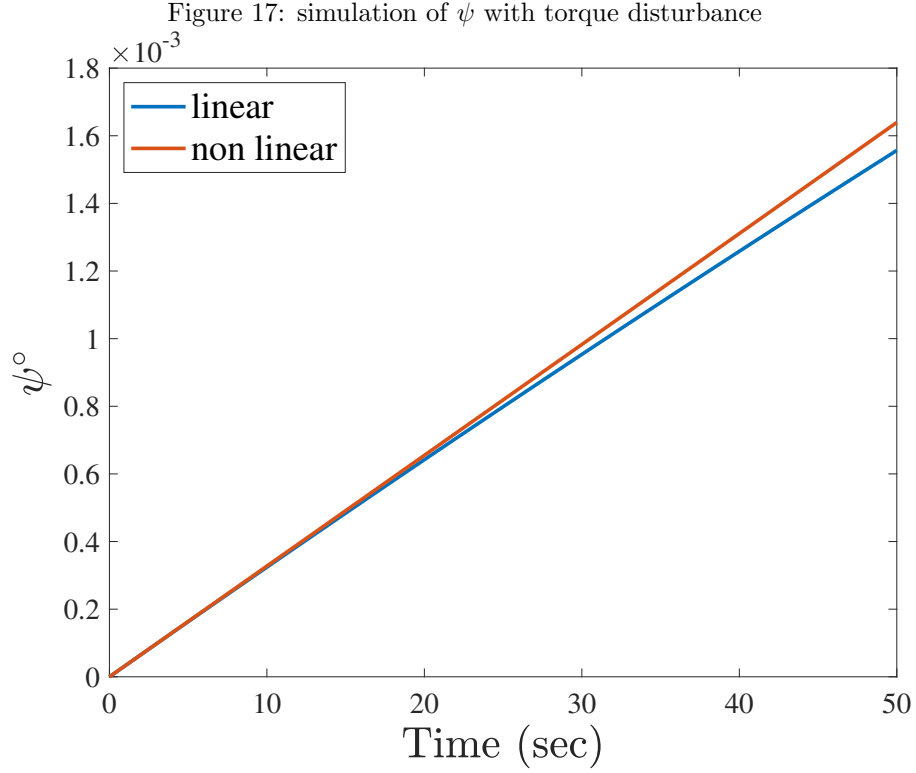




2.3 part d

Used above equation with different initial conditions and add initial torque disturbance.

Figure 15: simulation of ϕ with torque disturbanceFigure 16: simulation of θ with torque disturbance



3 Question 3

Equation of motion:

$$T_{Dx} = I_x \ddot{\phi} + 4\omega_0^2 (I_y - I_z) \phi - \omega_0 (I_x + I_y - I_z) \dot{\psi} \xrightarrow{\mathcal{L}} 4\phi (I_y - I_z) \omega_0^2 - \psi (I_x + I_y - I_z) \omega_0 s + I_x \phi s^2 = 0$$

$$T_{Dz} = I_z \ddot{\psi} + \omega_0^2 (I_z + I_x - I_y) \dot{\phi} + \omega_0^2 (I_y - I_x) \psi \xrightarrow{\mathcal{L}} \phi (I_x - I_y + I_z) \omega_0^2 s - \psi (I_x - I_y) \omega_0^2 + I_z \psi s^2 = 0$$

Using below terms to simplify the equations.

$$\sigma_x = \frac{I_y - I_z}{I_x}, \quad \sigma_z = \frac{I_y - I_x}{I_z}$$

$$\begin{aligned} 4\phi \sigma_x \omega_0^2 - \psi (1 - \sigma_x) \omega_0 s + \phi s^2 &= 0 \rightarrow \phi [s^2 + 4\sigma_x \omega_0^2] + \psi [-\omega_0 s (1 - \sigma_x)] = 0 \\ -\phi (\sigma_z - 1) \omega_0^2 s - \psi \sigma_z \omega_0^2 + \psi s^2 &= 0 \rightarrow \phi [\omega_0 s (1 - \sigma_z)] + \psi [s^2 + \omega_0^2 \sigma_z] = 0 \end{aligned} \quad (16)$$

$$\begin{bmatrix} s^2 + 4\sigma_x \omega_0^2 & -\omega_0 s (1 - \sigma_x) \\ \omega_0 s (1 - \sigma_z) & s^2 + \omega_0^2 \sigma_z \end{bmatrix} \begin{bmatrix} \phi \\ \psi \end{bmatrix} = \mathbf{A} \begin{bmatrix} \phi \\ \psi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

To avoid the trivial solution ($\phi = \psi = 0$), the determinant of the coefficient matrix must be zero.

$$\begin{aligned} \det \mathbf{A} &= \omega_0^2 s^2 + s^4 + 4\omega_0^4 \sigma_x \sigma_z + 3\omega_0^2 s^2 \sigma_x + \omega_0^2 s^2 \sigma_x \sigma_z \\ &= s^4 + \omega_0^2 (3s^2 \sigma_x + \sigma_x \sigma_y + 1) + 4\omega_0^4 \sigma_x \sigma_z = 0 \end{aligned}$$

4 Question 4

Inertia:

$$I_x = 8, \quad I_y = 10, \quad I_z = 14$$

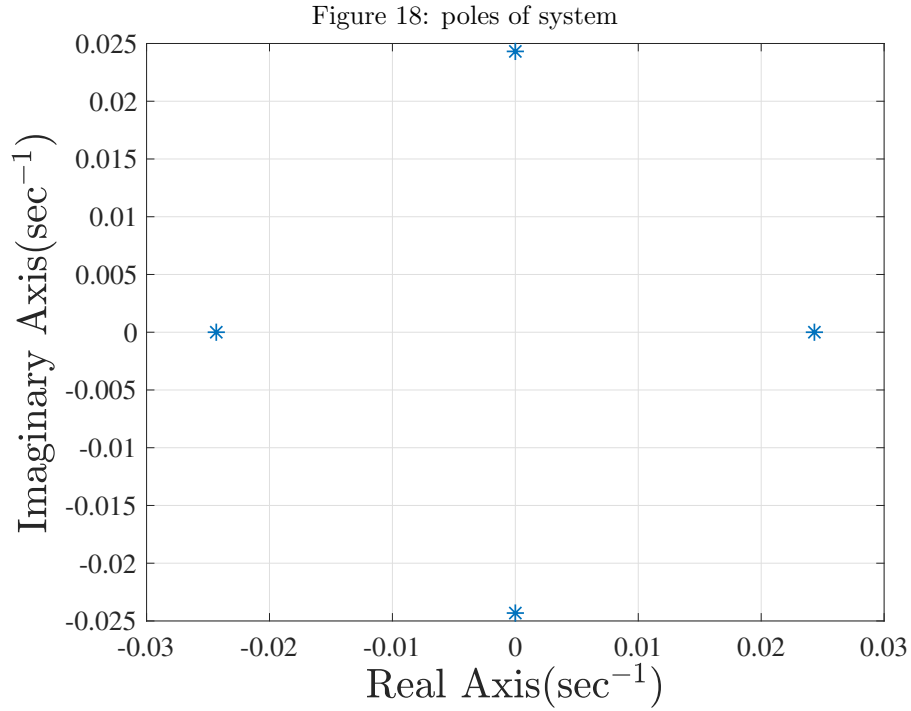
$$\sigma_x = \frac{I_y - I_z}{I_x} = -0.5, \quad \sigma_z = \frac{I_y - I_x}{I_z} = 0.1429$$

$$\omega_0 = \sqrt{\frac{\mu}{R_0^3}} = 0.0011$$

The characteristic equation of system is:

$$s^4 + \omega_0^2(3s^2\sigma_x + \sigma_x\sigma_y + 1) + 4\omega_0^4\sigma_x\sigma_z = s^4 - 6.998 \times 10^{-07}s^2 - 3.499 \times 10^{-7} \quad (17)$$

poles of the system plotted if figure 4.



Beacuse system has poles in right side of real axis, system is unstable and we have seen it in question 2.

Contents

1	Question 1	1
1.1	part a	1
1.2	part b	1
1.3	part c	2
1.4	part d	4
2	Question 2	7
2.1	part a	7
2.2	part b	7
2.3	part d	11
3	Question 3	13
4	Question 4	14

List of Figures

1	elipsoid of inertia	2
2	elipsoid of inertia in zx plane	3
3	elipsoid of inertia zy plane	3
4	elipsoid of inertia in xy plane	4
5	Angular momentum and rotational kinetic energy elipsoid of inertia	5
6	Angular momentum and rotational kinetic energy elipsoid of inertia in zx plane	5
7	Angular momentum and rotational kinetic energy elipsoid of inertia zy plane	6
8	Angular momentum and rotational kinetic energy elipsoid of inertia in xy plane	6
9	simulation of ϕ for 50 seconds	8
10	simulation of θ for 50 seconds	9
11	simulation of ψ for 50 seconds	9
12	simulation of ϕ for 5000 seconds	10
13	simulation of θ for 5000 seconds	10
14	simulation of ψ for 5000 seconds	11
15	simulation of ϕ with torque disturbance	12
16	simulation of θ with torque disturbance	12
17	simulation of ψ with torque disturbance	13
18	poles of system	14