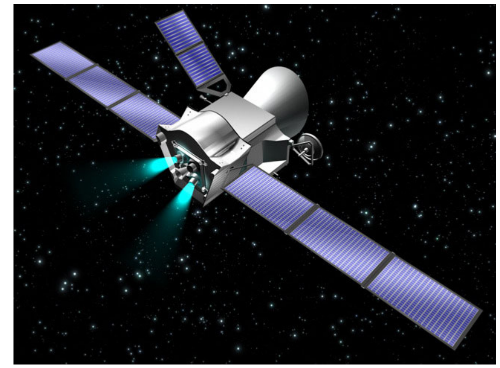
In The Name of God



AE 45780: Spacecraft Dynamics and Control

Fall 1401

Chapter One

Introduction and Review of Orbital Mechanics

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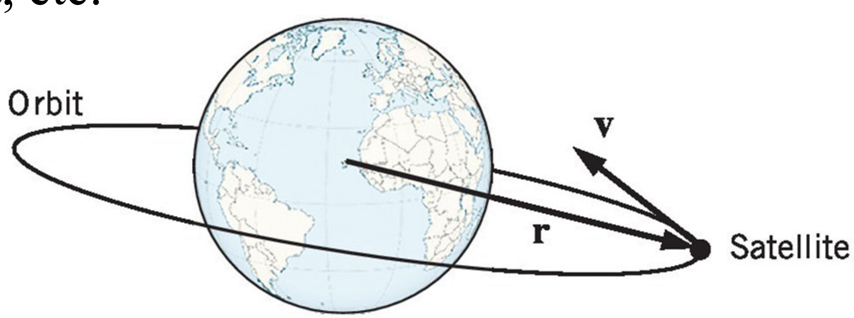
Introduction and review

Astrodynamics

- **Orbital Mechanics**
- **Attitude Dynamics**

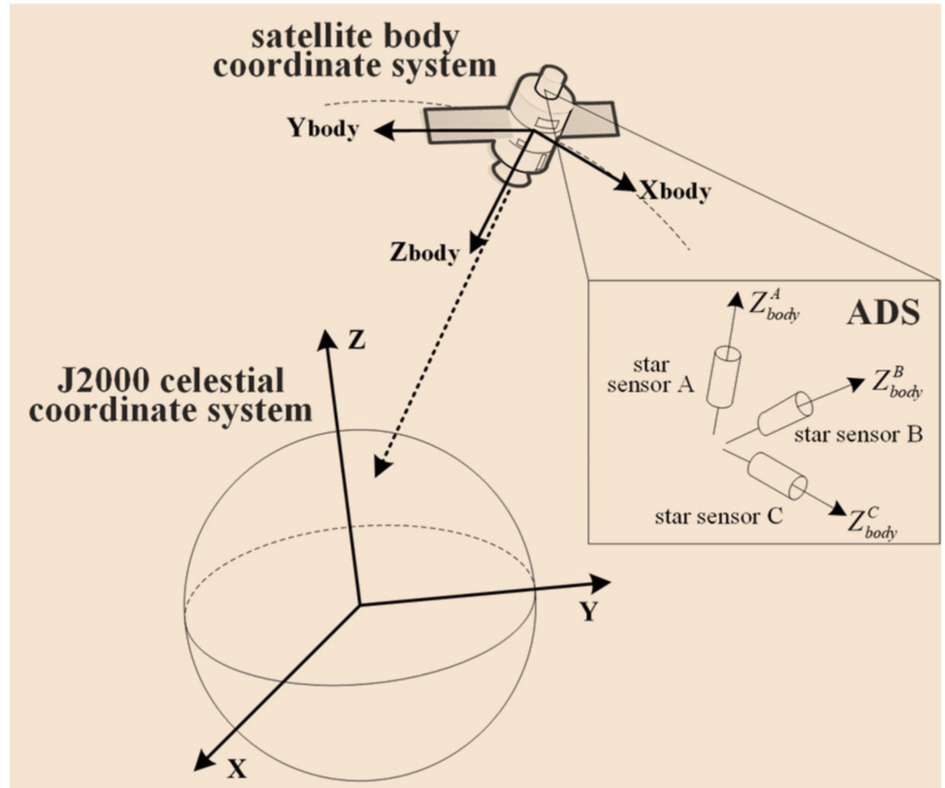
Orbital Mechanics (OM) or **Celestial Mechanics** pertains to the translational motions of artificial satellites/spacecraft and or natural bodies moving under the influence of forces such as gravity, atmosphere, third body, Solar radiation, thrust, etc.



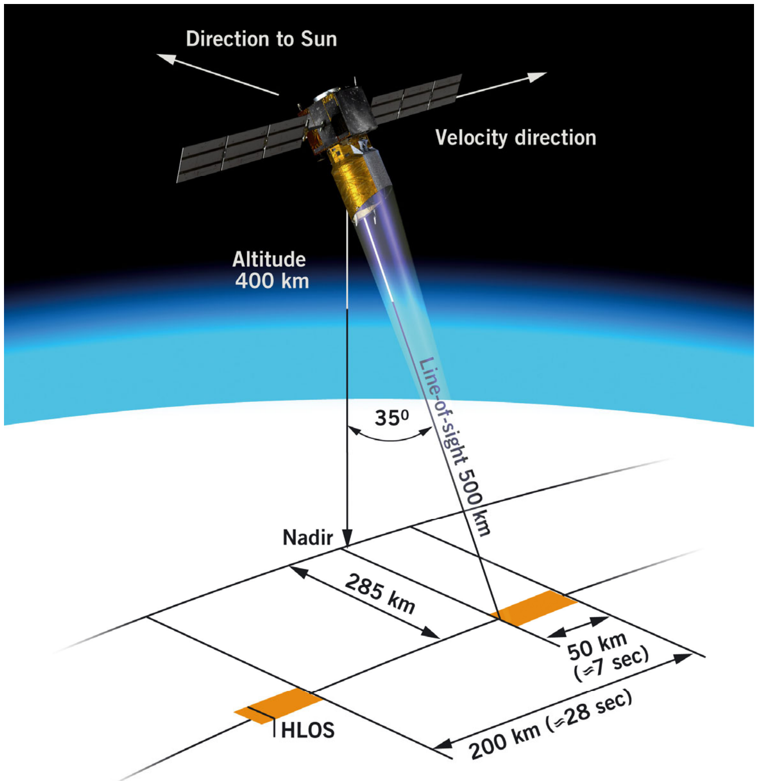
Attitude Dynamics (AD) , on the other hand is usually referred to the study of rotational motions of satellites/spacecrafts about their center of mass. This topic will be discussed later in the course.

Introduction and review

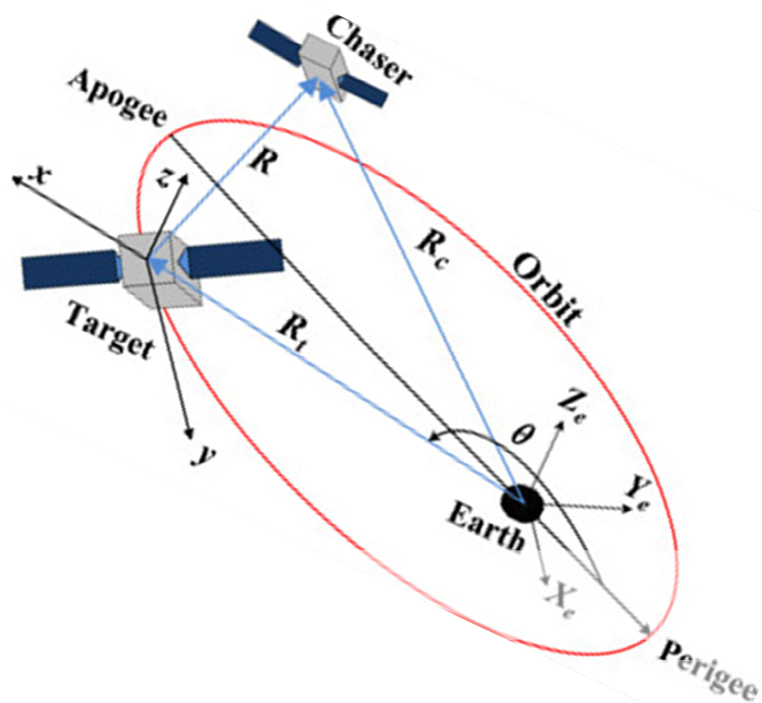
Attitude control (AC) is the process of controlling the orientation of a space vehicle with respect to a reference frame . AC requires sensors to measure vehicle orientation, actuators to apply the torques needed to orient the vehicle to a desired attitude and algorithms to compute the actuators command. The integrated field that studies the combination of sensors, actuators and algorithms is called guidance, navigation and control (GNC). AC as applicable to spacecrafts will also be introduced in this course.



Introduction and review

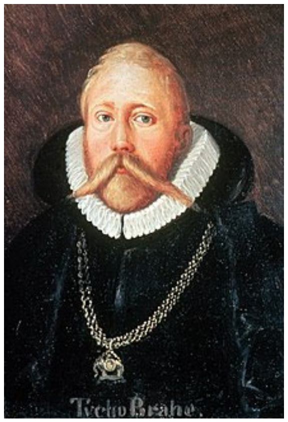


Attitude Control

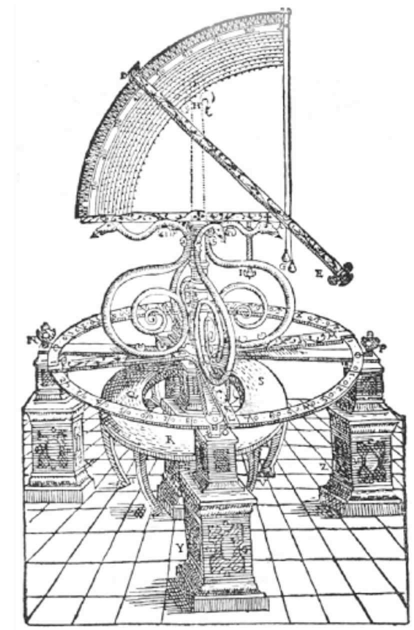


Orbit Control

Historical Background

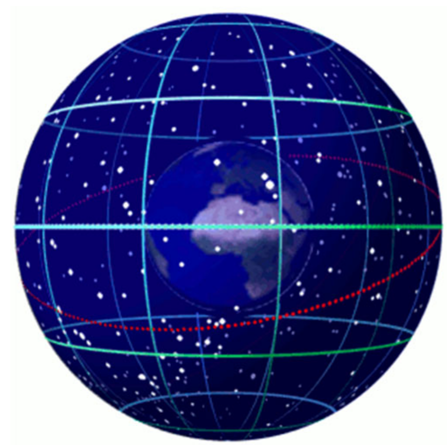


Tycho Brahe : 1546 –1601 was a Danish astronomer known for his accurate and comprehensive astronomical observations of planet orbits. His assistant, Johannes Kepler later used his astronomical data to discover **Kepler's** laws of planetary motion.



Tycho's Brass Azimuthal Quadrant

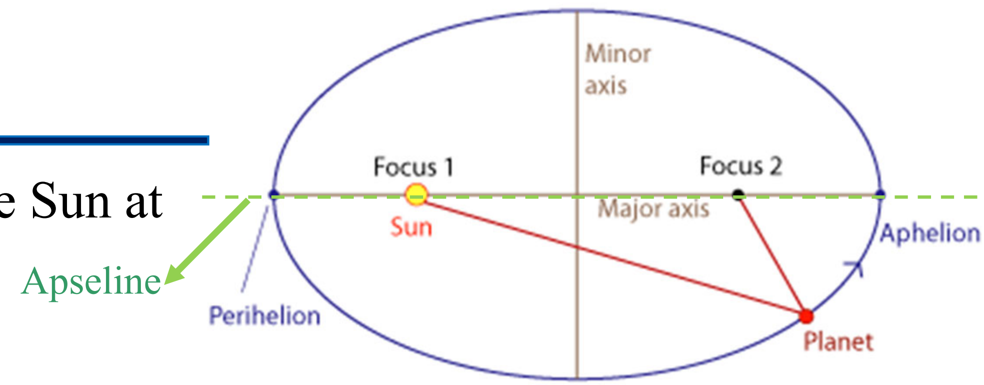
Johannes Kepler : 1571-1630 was an astrologer, mathematician and German astronomer. He is known for his three **laws of planetary motion**, as well as his books. His discoveries have formed the basis for Newton's theory of universal gravitation.



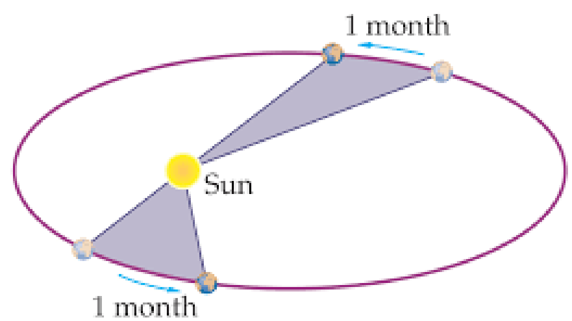
Celestial Sphere

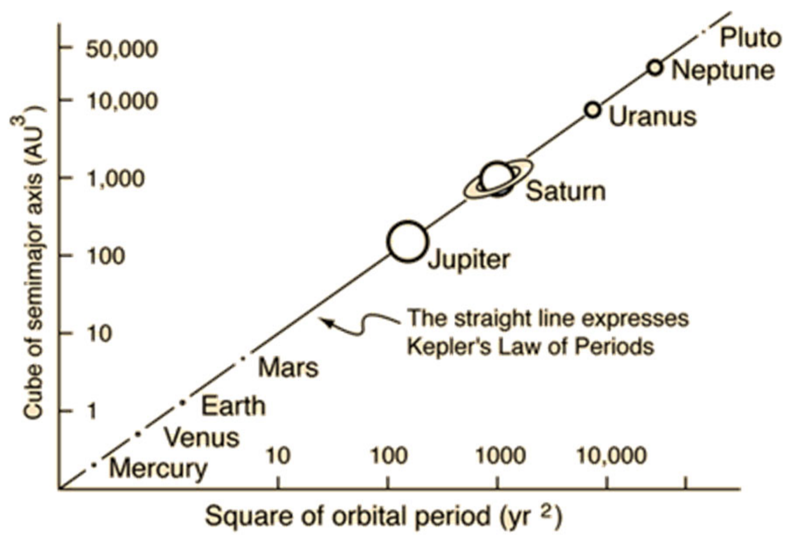
KEPLER'S LAW OF PLANETARY MOTION
(Based on Observation)

1. The orbit of a each planet is an ellipse with the Sun at one focus (1609).



2. The line joining each planet to Sun sweeps out equal areas in equal times (1609).





3. The square of the period of a planet is proportional to the cubic of its mean distance to the Sun (1619).

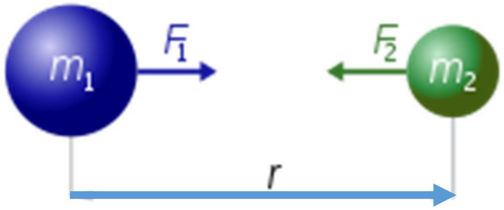
$$\tau^2 \propto r^3 \quad or \quad \tau \propto r^{\frac{3}{2}}$$

Newton's Law and Gravity Parameter

4 Law of Universal Gravitation

$$F_g = G\frac{m_1 m_2}{r^2}, \qquad \vec{F}_2 = \vec{F}_{12} = -G\frac{m_1 m_2}{r^2}\hat{u}_r = -G\frac{m_1 m_2}{r^3}\vec{r}$$

$$G = 6.6742e-11 \quad Nm^2/kg^2$$



Weight of a mass m, is considered as Gravity Pull of the Planet (with the mass M)

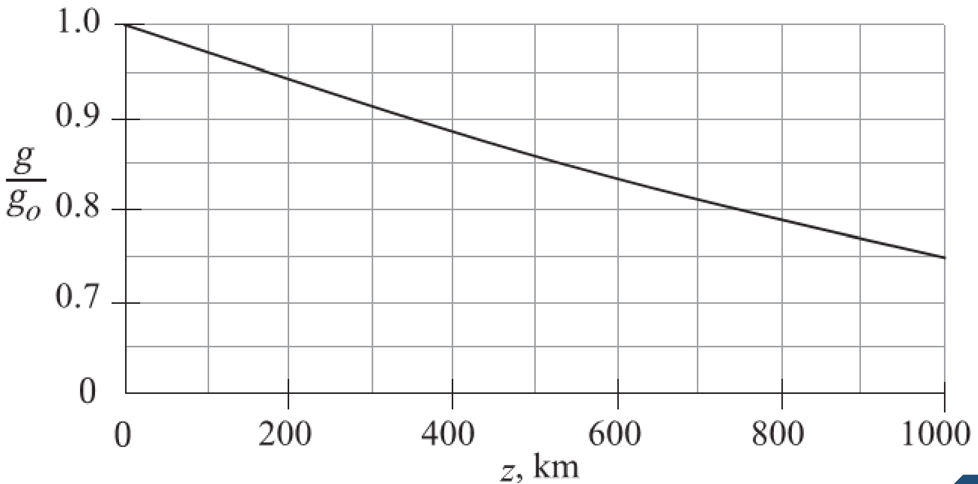
$$F_g = W = G\frac{Mm}{r^2} = m\left(\frac{GM}{r^2}\right) = mg = m\frac{\mu}{r^2}; \text{ so } \mu = GM$$

$$\text{For the Earth}: g_0 = \frac{GM}{R_E^2}; \quad r = R_E + z$$

$$g = \frac{\mu}{r^2} = \frac{\mu}{(R_E+z)^2} = \frac{\mu}{R_E^2(1+z/R_E)^2}$$

$$\text{Or}: g = \frac{g_0}{(1+z/R_E)^2}$$

$$\mu_E = \mu_\oplus = GM_\oplus \overset{EGM\ 2008}{=} 3.986004415e+5 \quad km^3/s^2$$



Newton's Laws

Digress: When the gravity pull is balanced?

$$Weight\ of\ a\ mass\ m,\ as\ Gravity\ Pull\ of\ the\ Earth\ with\ the\ mass\ M$$

$$F_g = W = G\frac{Mm}{r^2} = m\left(\frac{GM}{r^2}\right) = mg = m\frac{\mu}{r^2}$$

$$\mu = GM;\quad \frac{\mu}{r^2} \triangleq Gravitational\ Acceletation$$

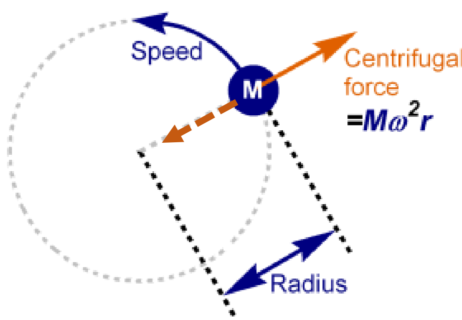
$$\mu_E = \mu_{\oplus} = GM_{\oplus} \overset{EGM\ 2008}{=} 3.986004415e+5\quad km^3/s^2$$

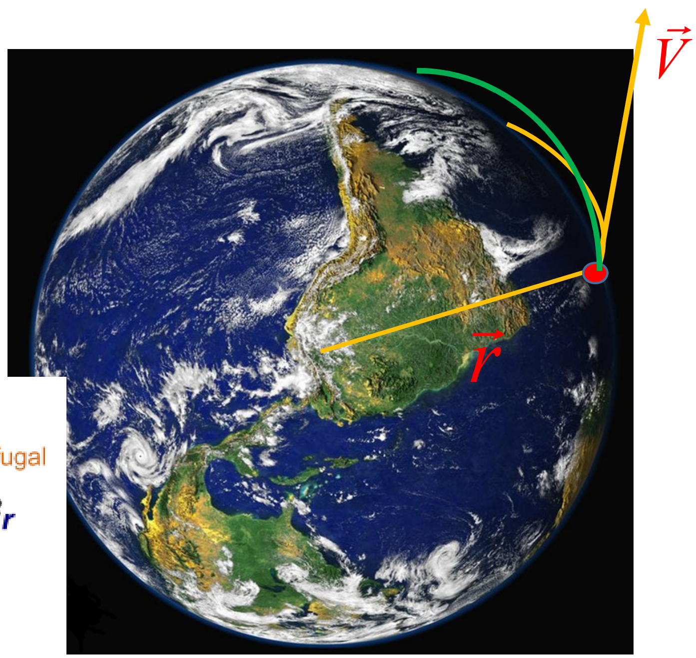
$$F_g = F_C;\ F_C \triangleq Centrifugal\ Force$$

$$m\frac{\mu}{r^2} = mr\omega^2 = m\frac{V^2}{r}$$

$$\frac{\mu}{r^2} = \frac{V^2}{r} \Rightarrow V_C = \sqrt{\frac{\mu}{r}} \triangleq Circular\ Velocity$$

$$at\ R_E \Rightarrow V_C = 25936\ fps \approx Mach = 23!$$





Orbital Mechanics and the Two Body Problem

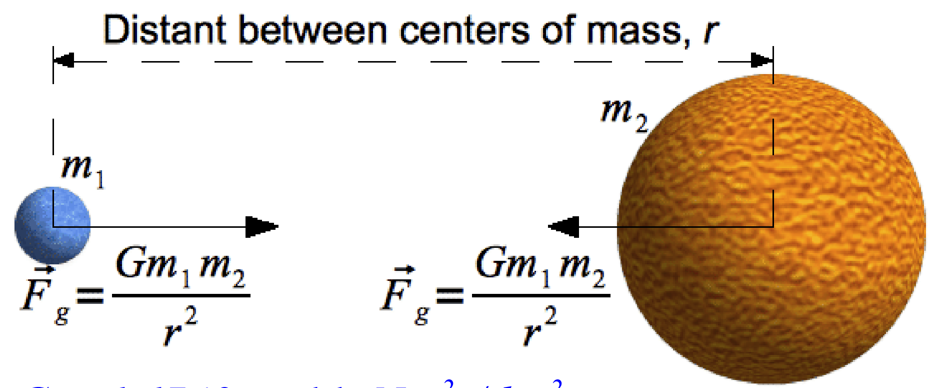
OM roots back to the 17th when Newton (1642-1727) formulated his law of **Universal Gravitation**. A consequence of this law, analytical solutions to what is called the two body problem (2BP) have evolved both in **three dimensions** or **within the orbital plane**.

2BP Governing Equation of Motion

$$\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r}$$

$$\mu_{Earth} = 398{,}600 km^3/s^2$$

$$IC: \vec{r}(0), \dot{\vec{r}}(0) \Rightarrow \vec{r}(t)$$



$$G = 6.6742e-11 \ \ Nm^2/kg^2$$

Universal Law of Gravitation :

Every particle in the universe attracts every other particle with a force along a line joining them. The force is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

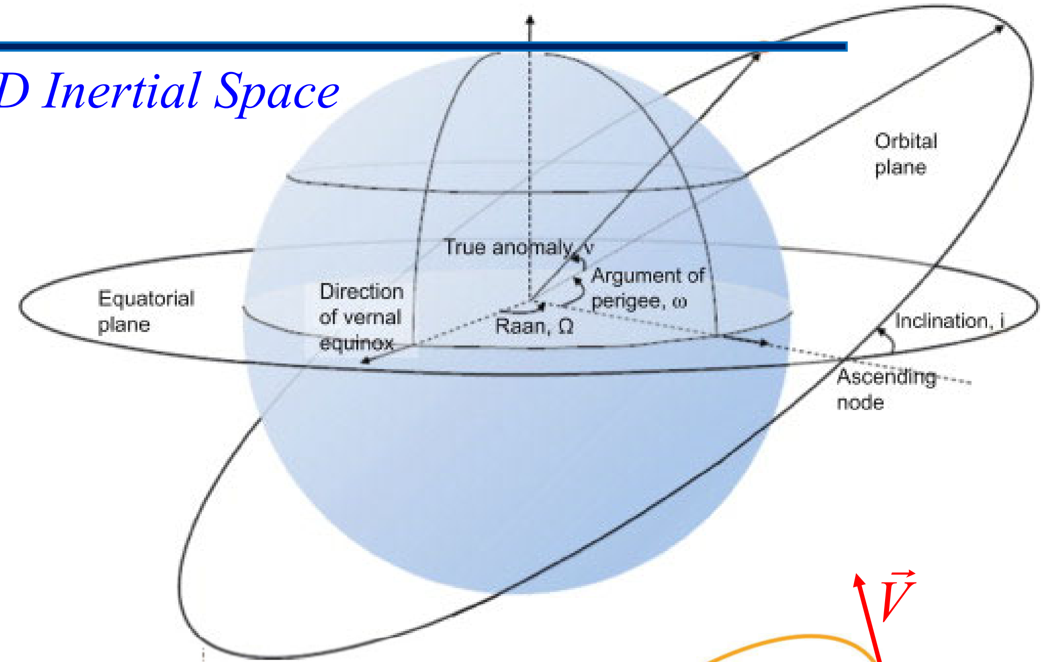
Orbital Mechanics

1. 2BP Governing Equation of Motion in 3D Inertial Space

$$\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r}$$

$$\mu_{Earth} = 398{,}600 km^3/s^2$$

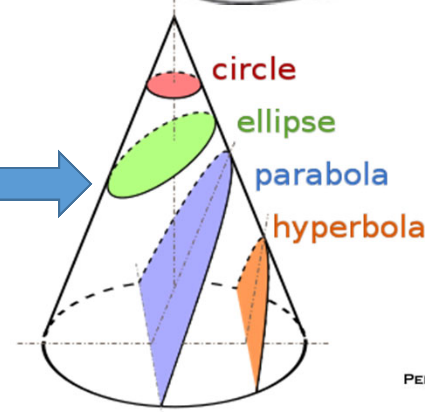
$$IC: \vec{r}(0), \dot{\vec{r}}(0) \Rightarrow \vec{r}(t)$$

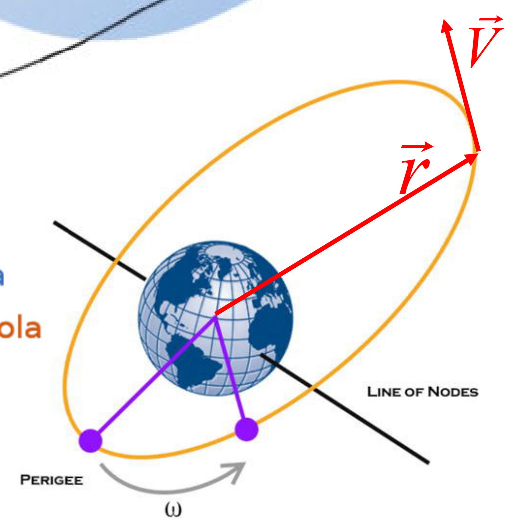


2. Planar 2BP EOM in Polar Form:

$$r = \frac{P}{1 + e\cos\theta} \quad ; \textit{Represents Conic Sections}$$

This equation shows the orbital motion in a 2D polar coordinate whose origin is centered at the center of mass of the Planet or Celestial body.



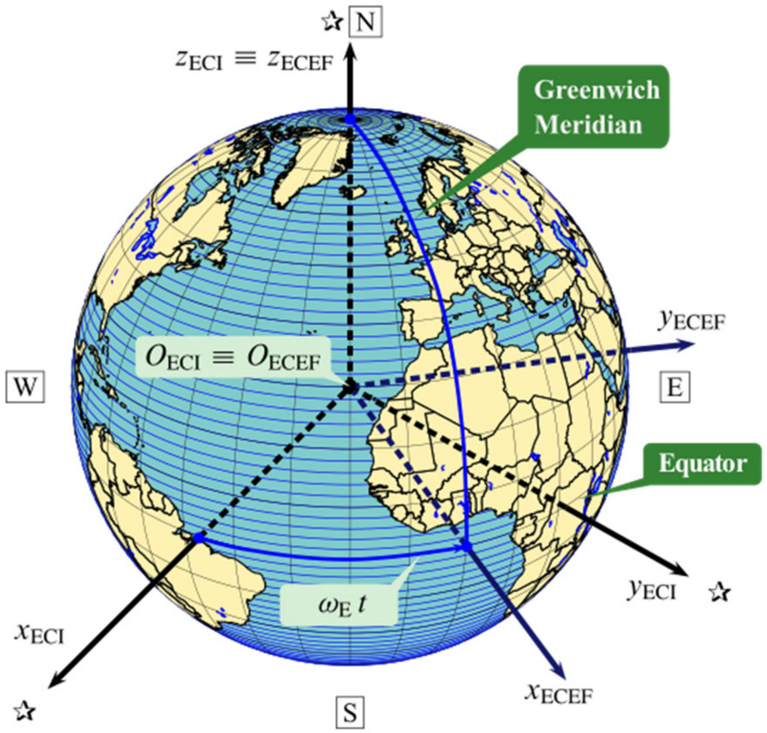


Validity of the Newton's Laws

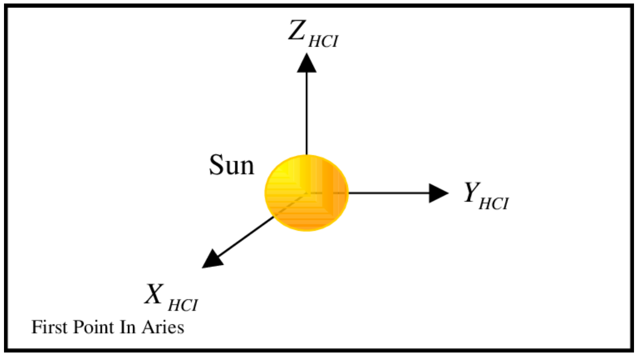
Inertial Coordinate System (*ICS*)

Requirements :

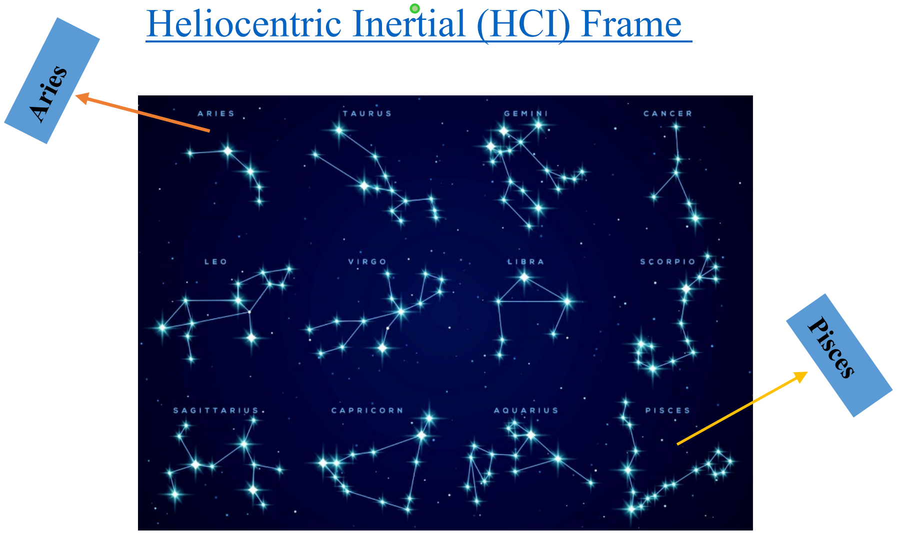
- The center must not accelerate
- It must not be rotating (axes)



Earth Centered Inertial (ECI) Frame



Heliocentric Inertial (HCI) Frame



Constellations are used as fixed points for ICS

Derivation of the Two Body Problem (2BP) EOM in 3D Space

Using the Newton's 2nd law for the two masses :

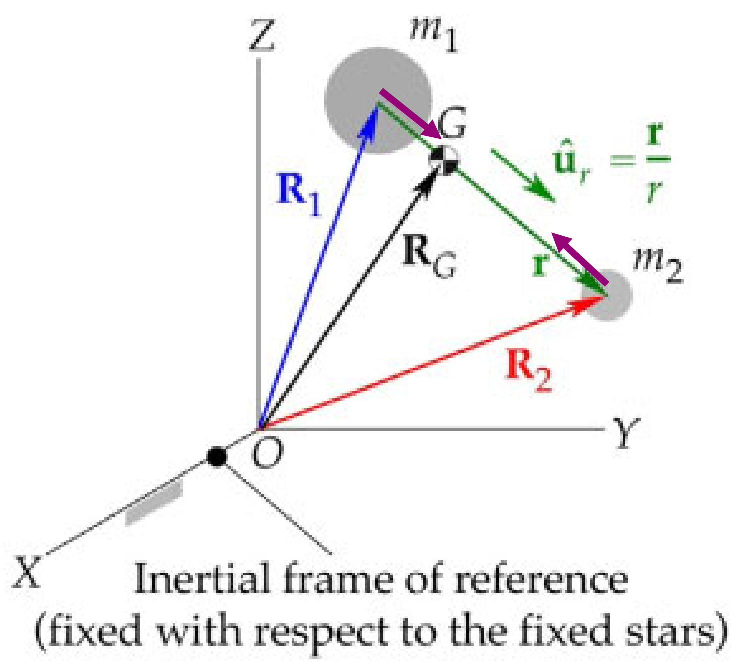
$$m_1\ddot{\boldsymbol{R}}_1 = \frac{Gm_1m_2\boldsymbol{r}}{r^3}; \quad m_2\ddot{\boldsymbol{R}}_2 = -\frac{Gm_1m_2\boldsymbol{r}}{r^3}$$

Now consider the Relative motion of m_2 WRT m_1 :

$$\boldsymbol{r} = \boldsymbol{R}_2 - \boldsymbol{R}_1 \rightarrow \ddot{\boldsymbol{r}} = \ddot{\boldsymbol{R}}_2 - \ddot{\boldsymbol{R}}_1$$

$$\Rightarrow \ddot{\boldsymbol{r}} = -G\frac{m_1}{r^3}\boldsymbol{r} - G\frac{m_2}{r^3}\boldsymbol{r} = -\,G\frac{m_1+m_2}{r^3}\boldsymbol{r}$$

$$\mu = G(m_1 + m_2) \cong Gm_1; \;\; for\ m_2 \ll m_1 (Earth\ mass)$$



$$\Rightarrow 2BP\ \ EOM\ in\ 3D\ ICS: \quad \ddot{\boldsymbol{r}} = -\frac{\mu\boldsymbol{r}}{r^3}$$

$$\mu_E = 3.986012e+5\,km^3/s^2; \;\; m_E = 5.924e+24\ kg$$

Two Body Problem (2BP) in 3D and 2D

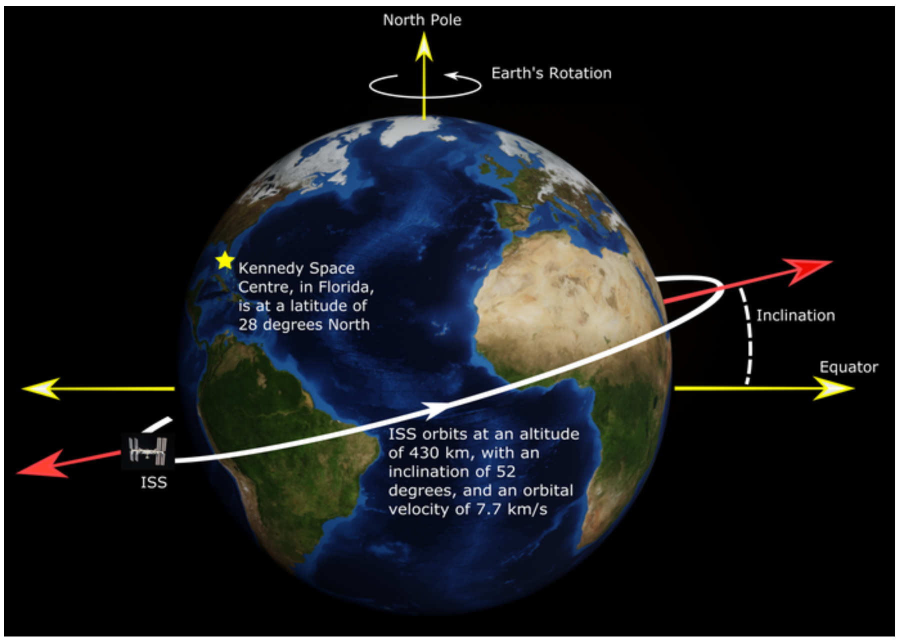
$2BP\ \ EOM\ in\ 3D\ ICS: \quad \ddot{\mathbf{r}} = -\frac{\mu \mathbf{r}}{r^3}$

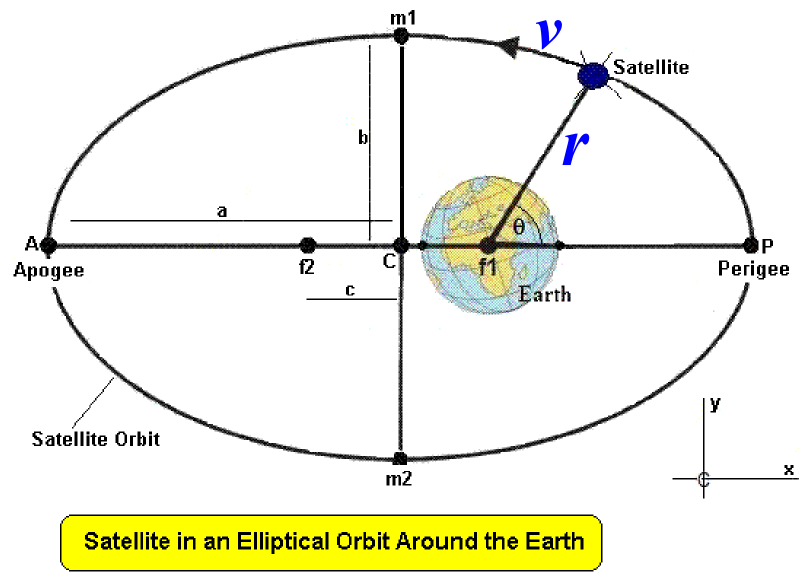
$with: \mathbf{r}(0), \dot{\mathbf{r}}(0) \Rightarrow \mathbf{r}(t)$

$\mu_E = 3.986012e+5\ km^3 / s^2$

$Planar(2D)\ 2BP\ \ EOM: r = \frac{P}{1+e\cos\theta}$

$With\ P\ and\ e\ specified \Rightarrow r\ in\ polar\ coordinate$

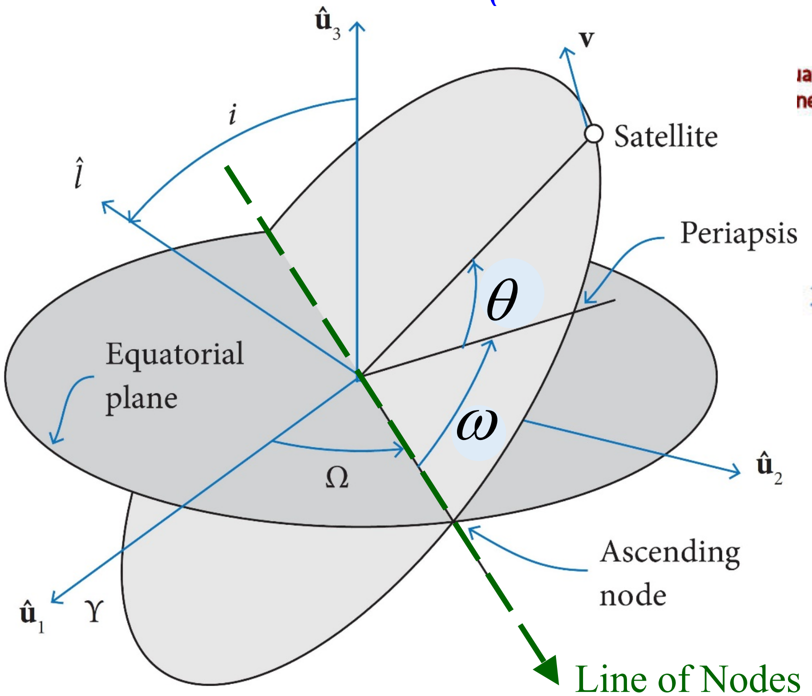


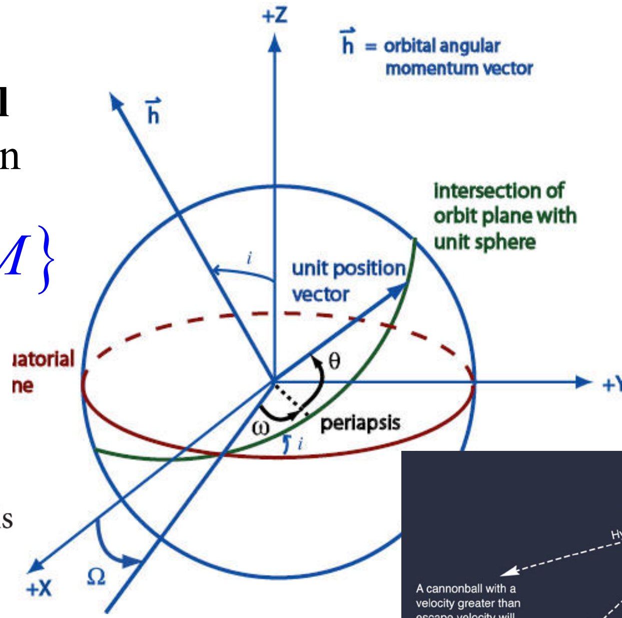


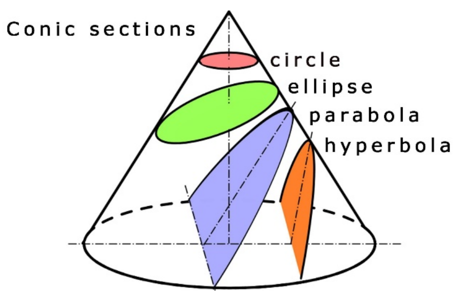
Orbital Parameters (OP)

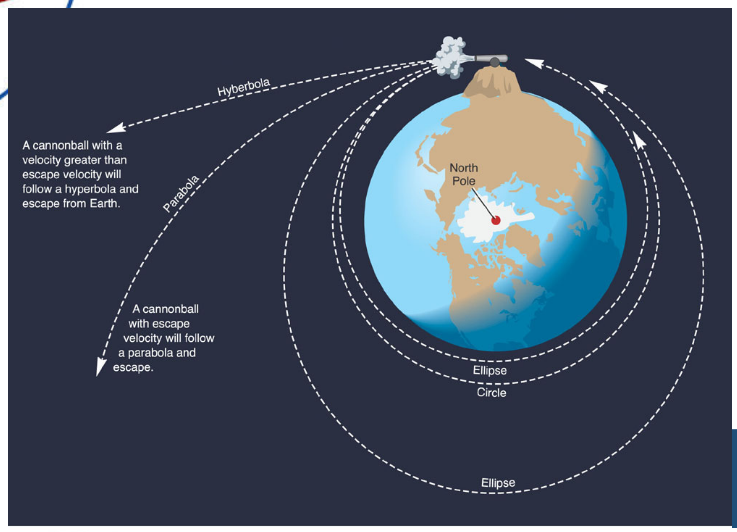
Six orbital parameters (OP) are in general needed to completely define and the **orbital plane (in 3D space)** and the current position of the satellite on it.

$$OP:\{a,e,\omega,\Omega,i,M\}$$









2BP Properties and Motion Constants (MCs)

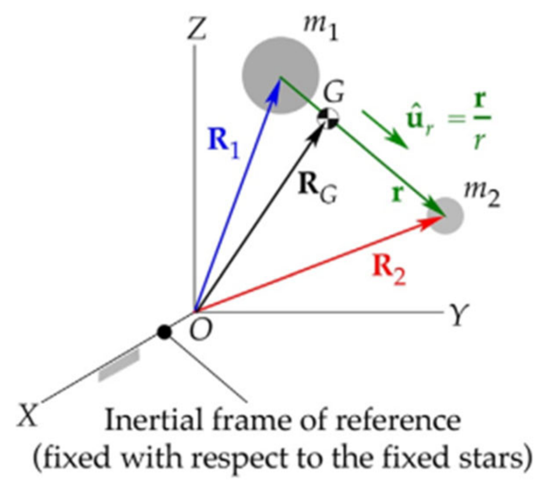
Center of mass (CM) in a 2PB:

$$\boldsymbol{R}_{cm} = \frac{m_1\boldsymbol{R}_1 + m_2\boldsymbol{R}_2}{m_1 + m_2} \Rightarrow \ddot{\boldsymbol{R}}_{cm} = \frac{m_1\ddot{\boldsymbol{R}}_1 + m_2\ddot{\boldsymbol{R}}_2}{m_1 + m_2}$$

Using the Newton's 2nd law for the two masses , we had :

$$m_1\ddot{\boldsymbol{R}}_1 = \frac{Gm_1m_2\boldsymbol{r}}{r^3}; \quad m_2\ddot{\boldsymbol{R}}_2 = -\frac{Gm_1m_2\boldsymbol{r}}{r^3} \Rightarrow \ddot{\boldsymbol{R}}_{cm} = 0 \rightarrow \boldsymbol{R}_{cm} = \boldsymbol{C}_1 t + \boldsymbol{C}_2$$

CM in 2BP does not accelerate ! so it can act as the origin of an inertial frame.



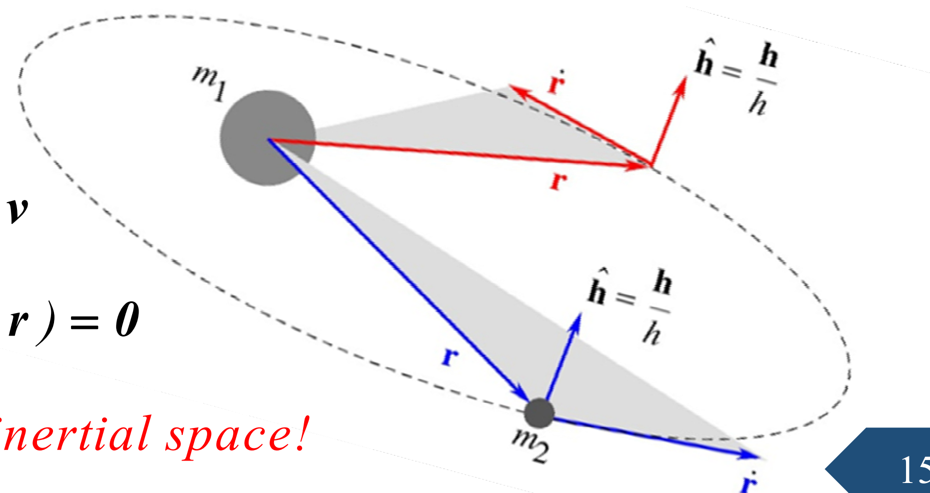
Integrals of the motion:

1- Angular momentum per unit mass

$$\textit{specific angular momentum} \triangleq \boldsymbol{h} = \boldsymbol{r} \times \dot{\boldsymbol{r}} = \boldsymbol{r} \times \boldsymbol{v}$$

$$\frac{d\boldsymbol{h}}{dt} = \dot{\boldsymbol{r}} \times \dot{\boldsymbol{r}} + \boldsymbol{r} \times \ddot{\boldsymbol{r}} = 0 + \boldsymbol{r} \times \left(-\frac{\mu}{r^3}\boldsymbol{r}\right) = -\frac{\mu}{r^3}(\boldsymbol{r} \times \boldsymbol{r}) = \boldsymbol{0}$$

$\boldsymbol{h} = Constant \Rightarrow$ *Orbital plane is fixed in the inertial space!*



Two Body Problem Motion Constants (MCs)

$$\boldsymbol{h} = \boldsymbol{r} \times \boldsymbol{v} = r\hat{\boldsymbol{u}}_r \times (v_r\hat{\boldsymbol{u}}_r + v_\perp\hat{\boldsymbol{u}}_\perp) = rv_\perp\hat{\boldsymbol{h}}$$

$$v_\perp = v\,sin\,\phi; v_r = v\,cos\,\phi$$

$$v_\perp = v\,cos\,\gamma; v_r = v\,sin\,\gamma \Rightarrow tan\,\gamma = \frac{v_r}{v_\perp}$$

ϕ = Zenith angle = $(\pi/2) - \gamma$

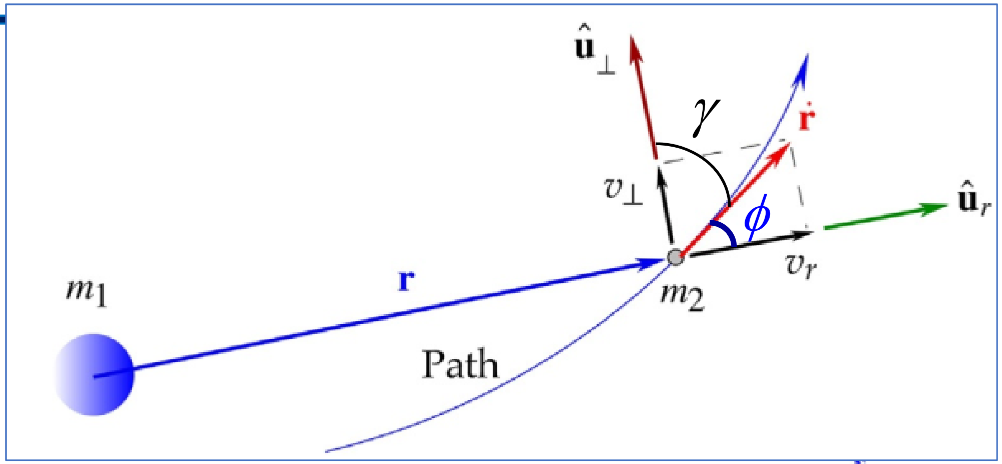
γ = flight path angle

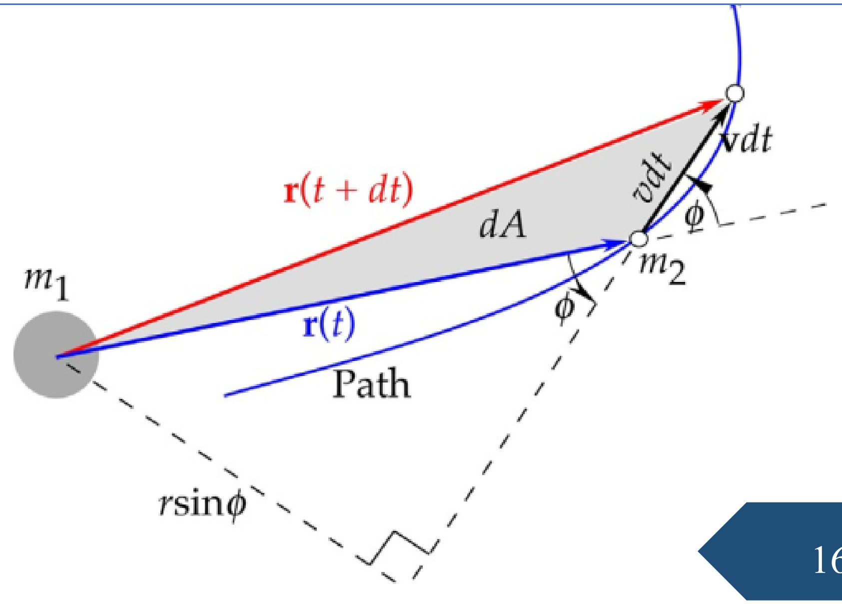
$$dA = \frac{1}{2} \times base \times altitude$$

$$= \frac{1}{2} \times vdt \times r\sin\phi$$

$$= \frac{1}{2} r(v\sin\phi)dt = \frac{1}{2} rv_\perp dt$$

$$\frac{dA}{dt} = \frac{h}{2} = Constant! \text{ (2nd Kepler's law)}$$





Two Body Problem Polar EOM (2D)

Analytic solution of 2BP

$$\ddot{\boldsymbol{r}} \times \boldsymbol{h} = -\frac{\mu}{r^3}\boldsymbol{r} \times \boldsymbol{h}$$

since $\quad d/dt\,(\dot{\boldsymbol{r}}\times\boldsymbol{h}) = \ddot{\boldsymbol{r}}\times\boldsymbol{h} + \cancel{\dot{\boldsymbol{r}}\times\dot{\boldsymbol{h}}}$

Notes:

$A \times B \times C = B(A.C) - C(A.B)$

$\dot{\boldsymbol{h}} = 0$

$\boldsymbol{r}.\dot{\boldsymbol{r}} = |\boldsymbol{r}|\;|\dot{\boldsymbol{r}}| = r\dot{r}$

$$RHS:\quad \frac{1}{r^3}\mathbf{r}\times\mathbf{h} = \frac{1}{r^3}[\mathbf{r}\times(\mathbf{r}\times\dot{\mathbf{r}})] = \frac{1}{r^3}[\mathbf{r}(\mathbf{r}.\dot{\mathbf{r}}) - \dot{\mathbf{r}}(\mathbf{r}.\mathbf{r})] = \frac{1}{r^3}[\mathbf{r}(r\dot{r}) - \dot{\mathbf{r}}r^2)]$$

$$= \frac{1}{r^3}[\mathbf{r}(r\dot{r}) - \dot{\mathbf{r}}r^2)] = \frac{\mathbf{r}\dot{r} - \dot{\mathbf{r}}r}{r^2}$$

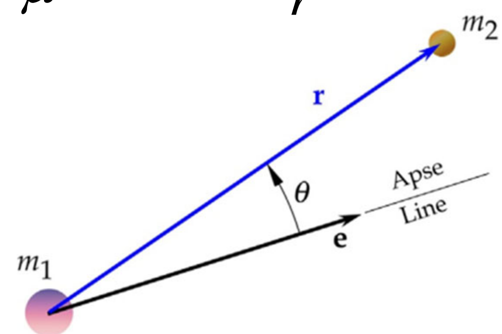
$$\text{but}\quad \frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right) = \frac{r\dot{\mathbf{r}} - \mathbf{r}\dot{r}}{r^2} = -\frac{\mathbf{r}\dot{r} - r\dot{\mathbf{r}}}{r^2} \;\longrightarrow\; \frac{d}{dt}(\dot{\mathbf{r}}\times\mathbf{h}) = \frac{d}{dt}\left(\mu\frac{\mathbf{r}}{r}\right) \;\longrightarrow\; \frac{d}{dt}\left(\dot{\mathbf{r}}\times\mathbf{h} - \mu\frac{\mathbf{r}}{r}\right) = 0$$

Two Body Problem (1st Kepler's Law)

$$\frac{d}{dt}\left(\dot{\mathbf{r}}\times\mathbf{h}-\mu\frac{\mathbf{r}}{r}\right)=0 \rightarrow \dot{\mathbf{r}}\times\mathbf{h}-\mu\frac{\mathbf{r}}{r}=\mathbf{C} \rightarrow \frac{\mathbf{r}}{r}+\mathbf{e}=\frac{\dot{\mathbf{r}}\times\mathbf{h}}{\mu} \xrightarrow{.\mathbf{r}} \frac{\mathbf{r}.\mathbf{r}}{r}+\mathbf{r}.\mathbf{e}=\frac{\mathbf{r}.(\dot{\mathbf{r}}\times\mathbf{h})}{\mu}$$

$$\rightarrow r+\mathbf{r}.\mathbf{e}=\frac{h^2}{\mu} \rightarrow r+re\cos\theta=\frac{h^2}{\mu}$$

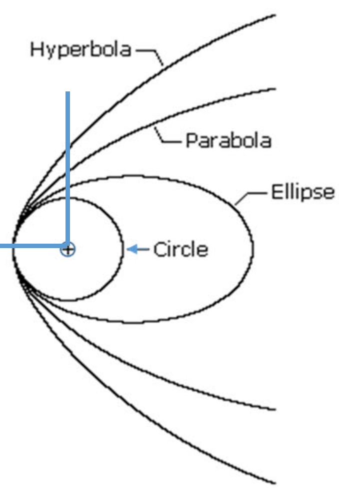
$$\rightarrow \boxed{r=\frac{P}{1+e\cos\theta} \quad \textit{Planar Orbit Equation}}$$

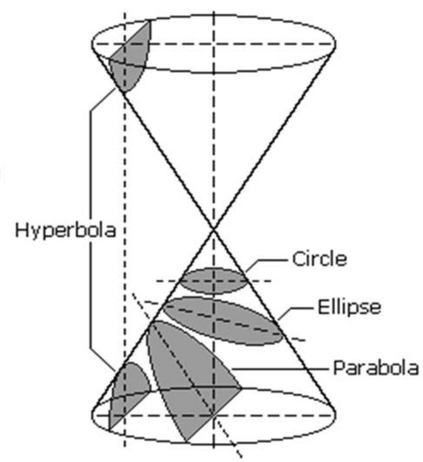


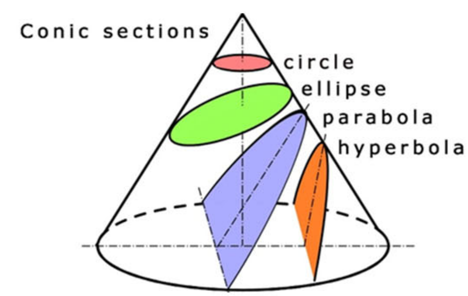
$p=h^2/\mu \triangleq$ *Semi Latuc Rectum*

$\theta\equiv f\equiv\upsilon \triangleq$ *True Anomaly*

2- $\vec{C}/\mu=\vec{e}\equiv$ *Eccentricity Vector(MC)*







Note: Another Vector Identity

$$\mathbf{A}.(\mathbf{B}\times\mathbf{C})=(\mathbf{A}\times\mathbf{B}).\mathbf{C}$$

$$\rightarrow \mathbf{r}.(\dot{\mathbf{r}}\times\mathbf{h})=(\mathbf{r}\times\dot{\mathbf{r}})\mathbf{h}=\mathbf{h}.\mathbf{h}=h^2$$

2BP Geometry in Polar Coordinate System (Planar)

$$\dot{\mathbf{r}} \times \mathbf{h} - \mu \frac{\mathbf{r}}{r} = \mathbf{C} \xrightarrow{\boldsymbol{h}.}$$

$$\cancel{\boldsymbol{h}.(\dot{\boldsymbol{r}} \times \boldsymbol{h})} + \cancel{\boldsymbol{h}.\mu \frac{\boldsymbol{r}}{r}} = \boldsymbol{h}.\mu\boldsymbol{e}$$

$$\Rightarrow \boldsymbol{h}.\mu\boldsymbol{e} = 0 \Rightarrow \boldsymbol{h} \perp \boldsymbol{e}$$

$\boldsymbol{e}$ *lies in the orbital plane*

$$\boldsymbol{r}.\boldsymbol{e} = re\cos\theta \Rightarrow \cos\theta = \frac{\boldsymbol{r}.\boldsymbol{e}}{re}$$

$p = h^2 / \mu \triangleq$ *Semi Latuc Rectum*

$\theta \equiv$ *True Anomaly*

$\vec{e} \equiv$ *Eccentricity Vector*

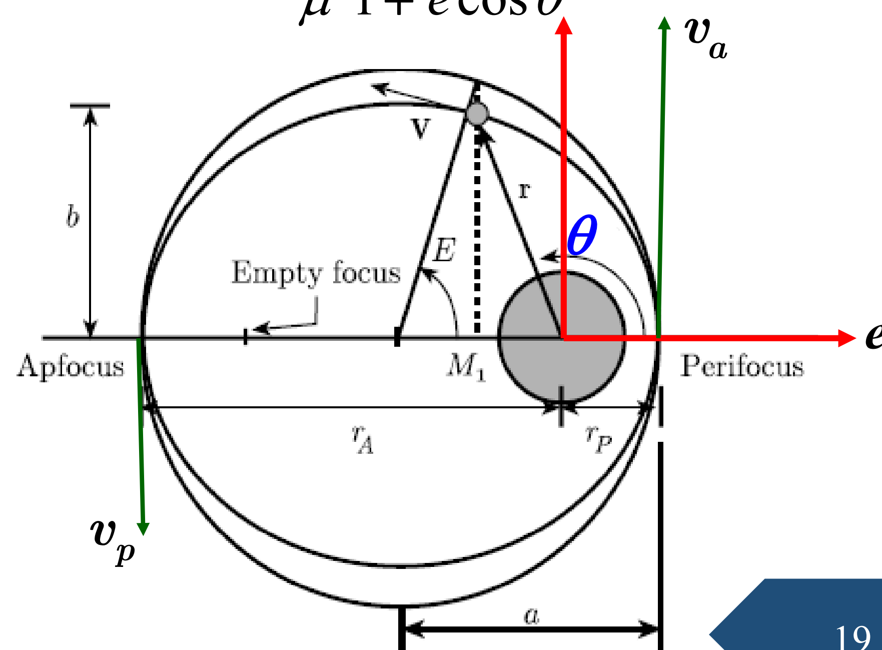
$$\begin{cases} \theta = 0^\circ \rightarrow r_{min} \\ \theta = 180^\circ \rightarrow r_{max} \end{cases}$$

Periapsis (perigee, perihelion,…)

Apoapsis (apogee, apohelion,…)

Orbit equation

$$r = \frac{h^2}{\mu} \frac{1}{1 + e\cos\theta}$$



Two Body Problem

- $v_{\perp} = r\dot{\theta} \rightarrow h = rv_{\perp} = r^2\dot{\theta}$

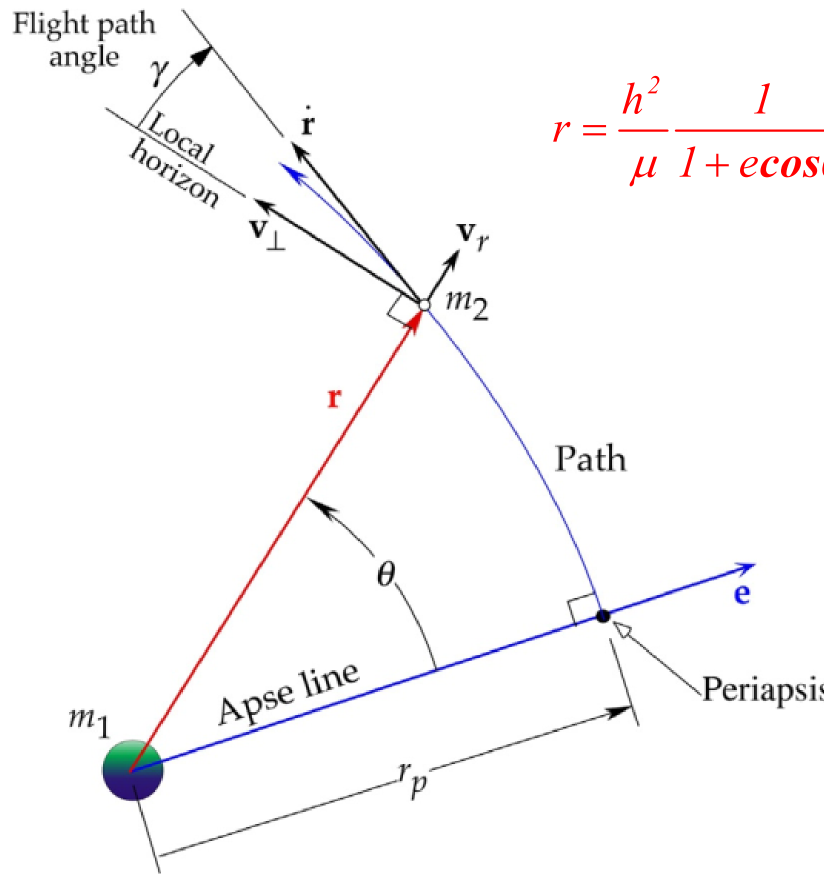
$$v_{\perp} = \frac{h}{r} = \frac{\mu}{h}(1 + e\cos\theta)$$

- $\dot{r} = \dfrac{dr}{dt} = \dfrac{d}{dt}\left[\dfrac{h^2}{\mu}\dfrac{1}{1+e\cos\theta}\right]$

$$= \frac{h^2}{\mu}\left[-\frac{e(-\dot{\theta}\sin\theta)}{(1+e\cos\theta)^2}\right]$$

$$= \frac{h^2}{\mu}\frac{e\sin\theta}{(1+e\cos\theta)^2}\frac{h}{r^2}$$

$$\Rightarrow v_r = \frac{\mu}{h} e\sin\theta$$



$$r = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta}$$

$$\tan\gamma = \frac{v_r}{v_{\perp}} = \frac{e\sin\theta}{1+e\cos\theta}$$

Specific Energy Law and Conservation of Energy

- Relative linear momentum per unit mass

$$\frac{m_2\dot{\mathbf{r}}}{m_2} = \dot{\mathbf{r}}$$

- $\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r} \xrightarrow{.\dot{\mathbf{r}}} \ddot{\mathbf{r}}.\dot{\mathbf{r}} = -\mu\frac{\mathbf{r}.\dot{\mathbf{r}}}{r^3}$

Earth Gravitational Potential Function: U(r)

$$F_g = W = G\frac{Mm}{r^2} = m\left(\frac{GM}{r^2}\right) = mg = m\frac{\mu}{r^2}$$

$$F_g / m = \frac{\mu}{r^2} = -\nabla U \Rightarrow U = -\frac{\mu}{r}$$

$$LHS: \ddot{\boldsymbol{r}}.\dot{\boldsymbol{r}} = \frac{1}{2}\frac{d}{dt}(\dot{\boldsymbol{r}}.\dot{\boldsymbol{r}}) = \frac{1}{2}\frac{d}{dt}(\boldsymbol{v}.\boldsymbol{v}) = \frac{1}{2}\frac{d}{dt}(v^2) = \frac{d}{dt}(\frac{v^2}{2})$$

$$RHS: \mu\frac{\boldsymbol{r}.\dot{\boldsymbol{r}}}{r^3} = \mu\frac{r\dot{r}}{r^3} = \mu\frac{\dot{r}}{r^2} = -\frac{d}{dt}\left(\frac{\mu}{r}\right)$$

$$\frac{d}{dt}(\frac{1}{r}) = -\frac{1}{r^2}\frac{dr}{dt}$$

$$\frac{d}{dt}\left(\frac{v^2}{2} - \frac{\mu}{r}\right) = 0$$

$\Rightarrow$ *3. Specific Energy Equation for orbits* ***(MC)***

$$\frac{v^2}{2} - \frac{\mu}{r} = \varepsilon \quad \textbf{(constant)}$$

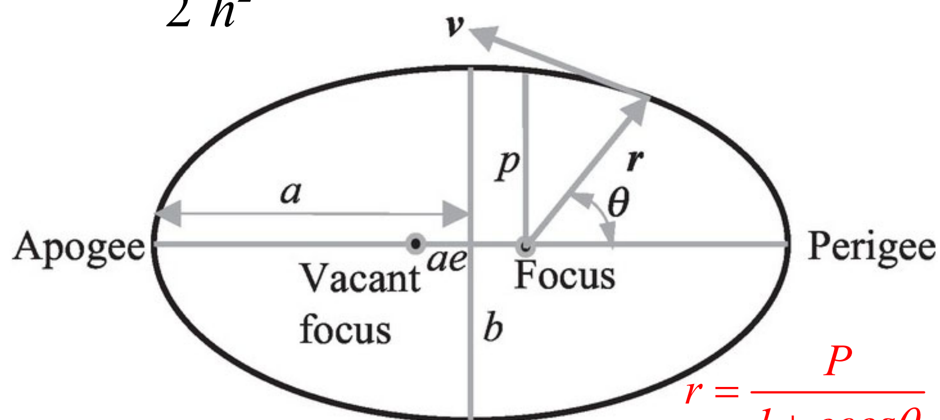
Total Orbit Energy : $E = m\varepsilon$

Relation between Orbit Energy and Eccentricity (Geometry)

$$\varepsilon = \varepsilon_p = \frac{v_p^2}{2} - \frac{\mu}{r_p} \xrightarrow{v_p = v_\perp = h/r_p} \varepsilon = \frac{1}{2}\frac{h^2}{r_p^2} - \frac{\mu}{r_p}$$

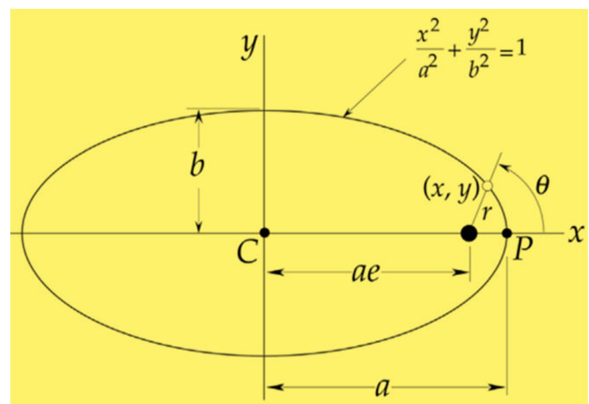
Substituting for $r_p = \dfrac{h^2/\mu}{1+e}$ *and simplifying*

$$\varepsilon = -\frac{1}{2}\frac{\mu^2}{h^2}(1-e^2)$$



$$r = \frac{P}{1 + e\cos\theta}$$

e	ε	***Orbit Geometry***
$0 < e < 1$	$\varepsilon < 0$	*Circle to Ellipse*
$e = 1$	$\varepsilon = 0$	*Parabola*
$e > 1$	$\varepsilon > 0$	*Hyperbola*



1- Circular Orbits (3rd Kepler's Law)

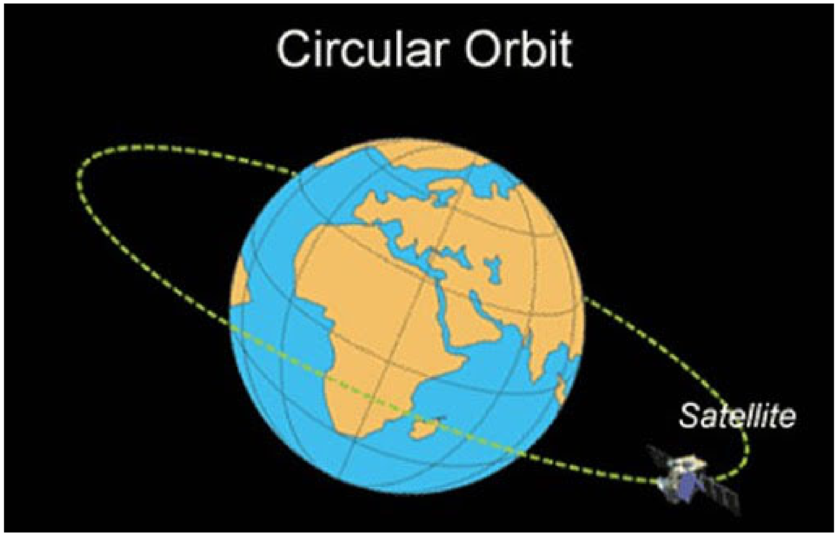
$$r = \frac{h^2/\mu}{1 + e\cos\theta} \xrightarrow{e=0} r = \frac{h^2}{\mu}$$

$$\varepsilon = -\frac{1}{2}\frac{\mu^2}{h^2}(1-e^2) < 0$$

$$\frac{v^2}{2} - \frac{\mu}{r} = \varepsilon \;(constant)$$

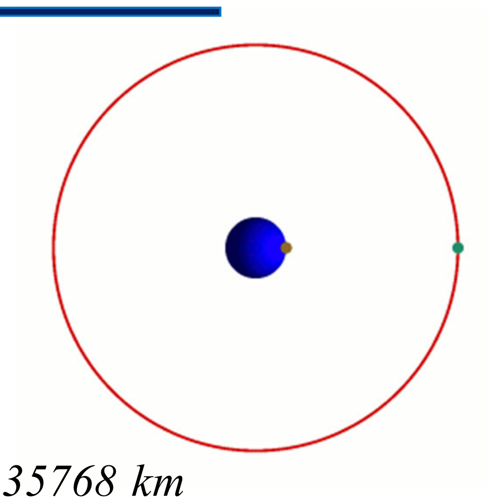
$$h = rv \rightarrow v_{circular} = v_C = \sqrt{\frac{\mu}{r}}$$

$$Period\ \tau = \frac{circumference}{speed} = \frac{2\pi r}{\sqrt{\mu/r}} = \frac{2\pi}{\sqrt{\mu}} r^{\frac{3}{2}} = 2\pi\sqrt{\frac{r^3}{\mu}}$$



$$\varepsilon = -\frac{1}{2}\frac{\mu^2}{h^2} = -\frac{\mu}{2r}$$

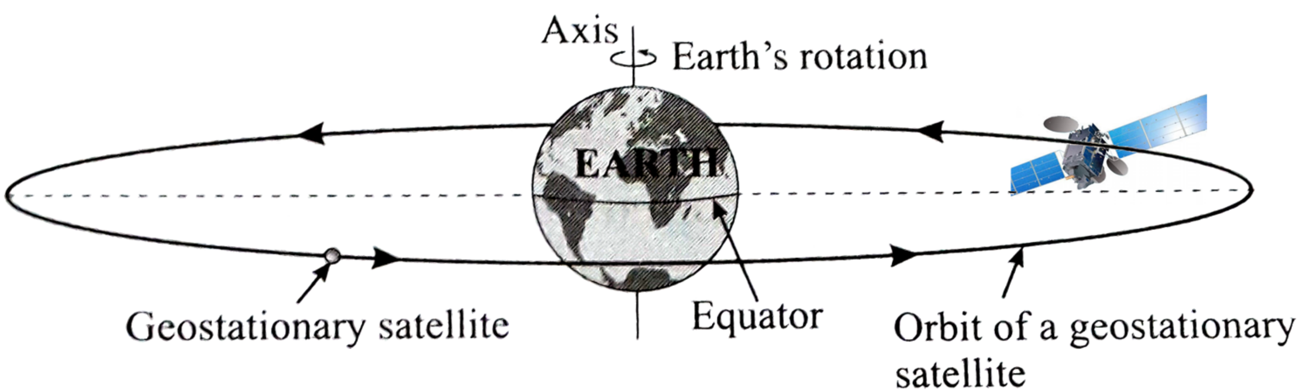
GEO Orbits and Satellites



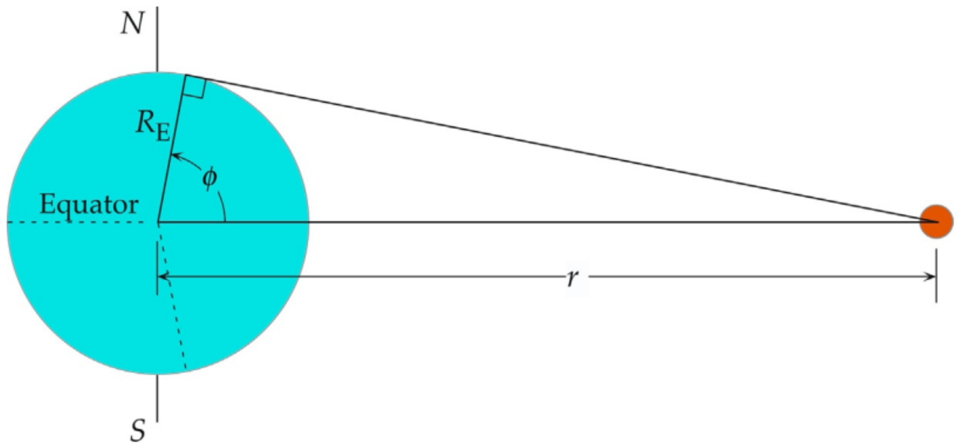
- Geostationary (Geo-Synchronous) equatorial orbits (GEO),Circular.
- Satellite angular velocity is the same as the Earth

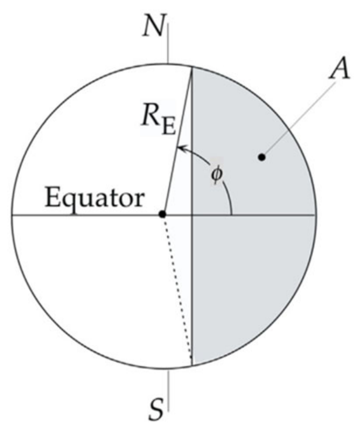
$$\omega_E = \frac{2\pi}{24 \times 3600} \approx 7.2722 \times 10^{-5}\ rad/s;\ \mu = 3.986 \times 10^{-5}\ km^3/s^2$$

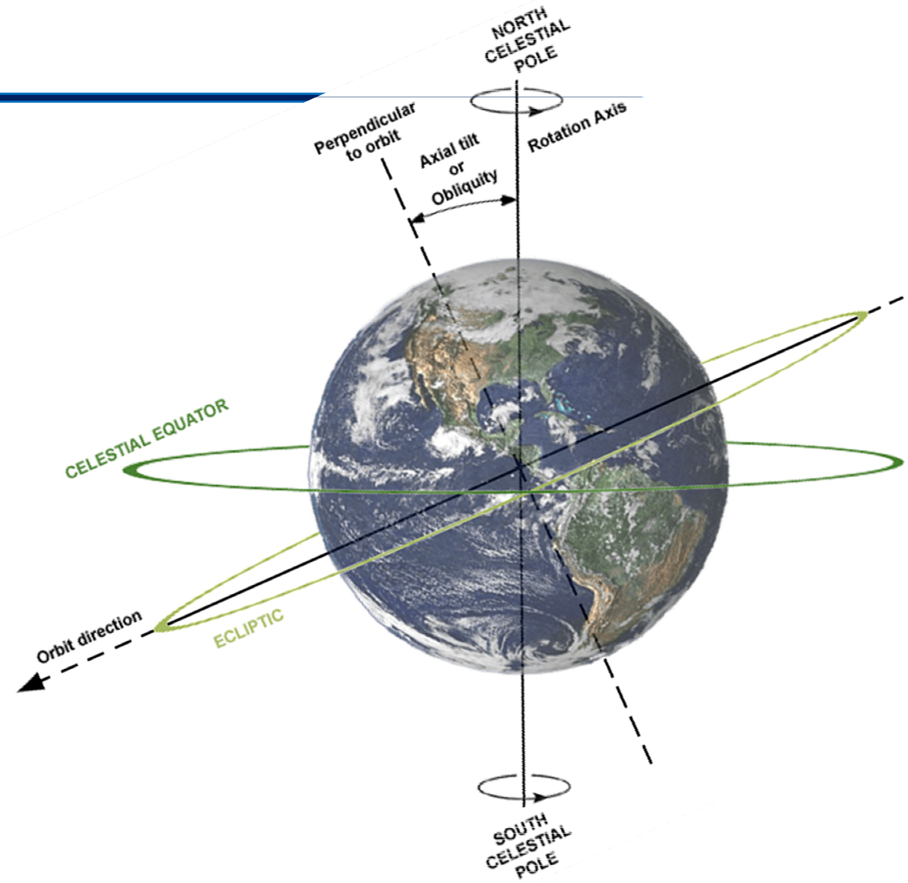
$$v_{GEO} = \sqrt{\frac{\mu}{r_{GEO}}}, \qquad v_{GEO} = \omega_E r_{GEO} \rightarrow r_{GEO} = \sqrt[3]{\frac{\mu}{\omega_E^2}} = 42,164\ km \rightarrow h_{GEO} = r_{GEO} - R_E = 35768\ km$$



GEO Orbit Coverage Area





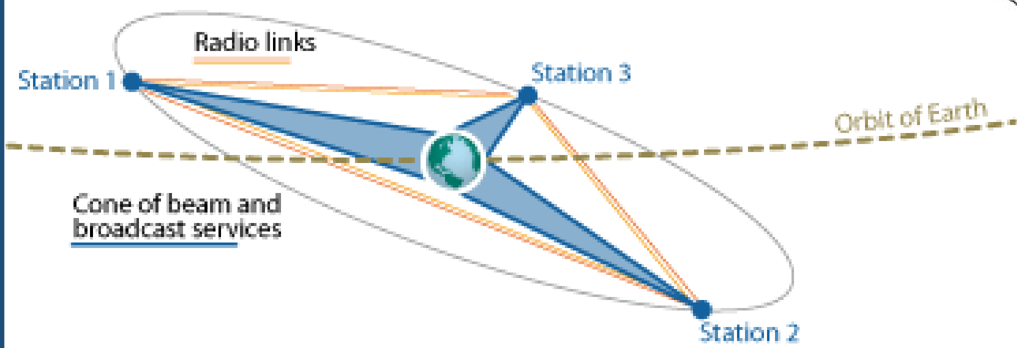


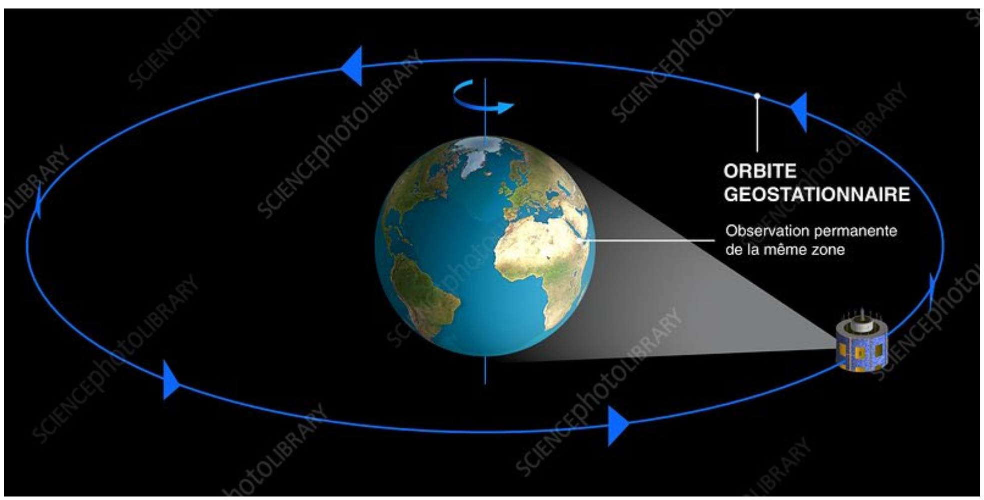
$$\varphi = cos^{-1}\frac{R_E}{r} = cos^{-1}\frac{6378}{42,164} = 81.30°$$

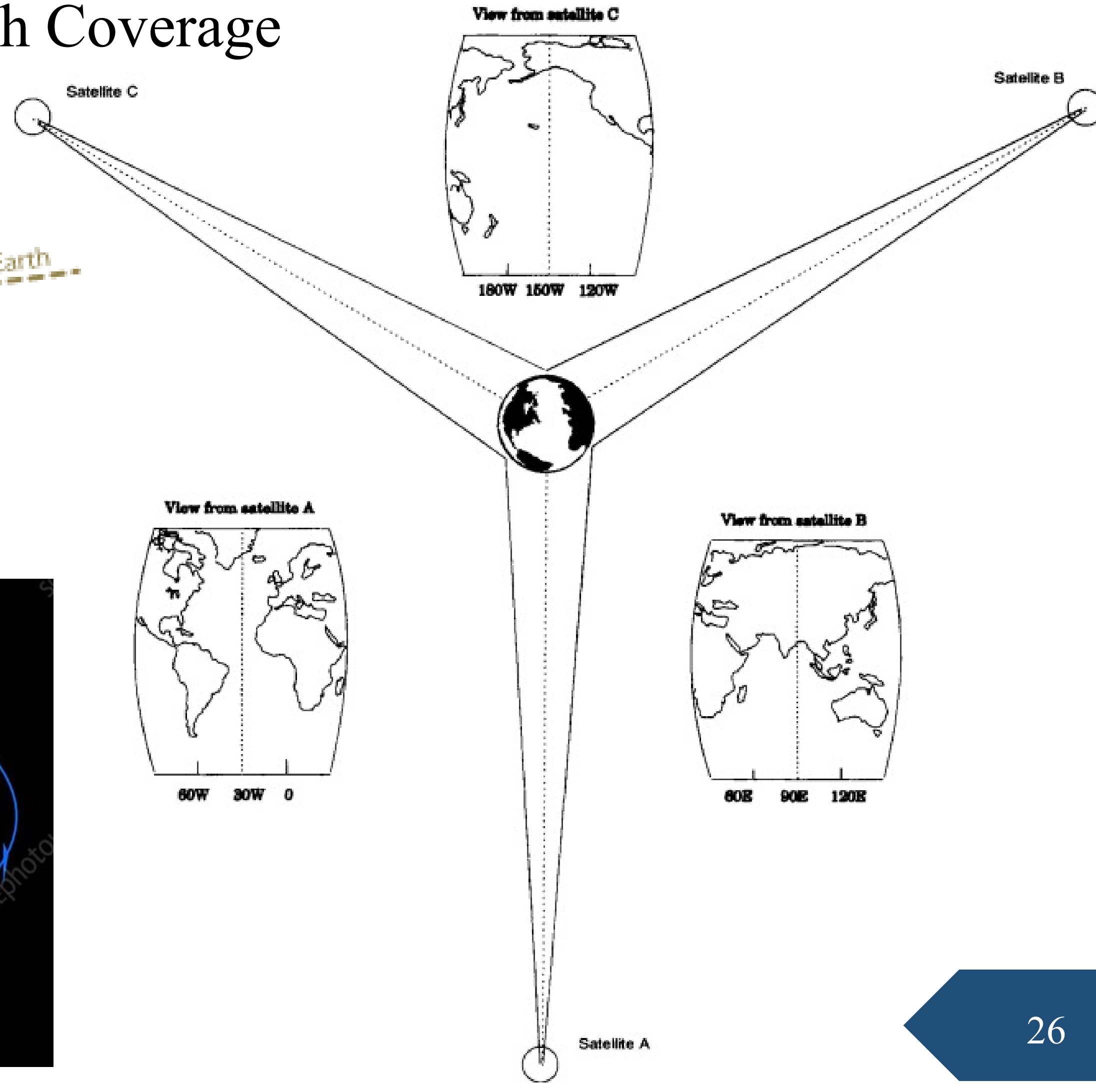
$$S = 2\pi R_E^2(1 - cos\,\varphi)$$

$$\frac{S}{2\pi R_E^2} \times 100 = 84.9\ \%$$

Three GEO Satellites Give Full Earth Coverage







2- Elliptical Orbit in Polar Coordinates

$0 < e < 1;\ \varepsilon < 0$

$$r = \frac{h^2/\mu}{1 + e\cos\theta} = \frac{P}{1 + e\cos\theta}$$

$$r_p = r_{\theta=0} = \frac{P}{1+e};\quad r_a = r_{\theta=180} = \frac{P}{1-e}$$

$$2a = r_p + r_a = \frac{2P}{1-e^2} \Rightarrow P = \frac{h^2}{\mu} = a(1-e^2)$$

$$r_p = a(1-e);\ r_a = a(1+e)$$

$$CF = C = a - FP = a - r_p = a - a(1-e) = ae$$

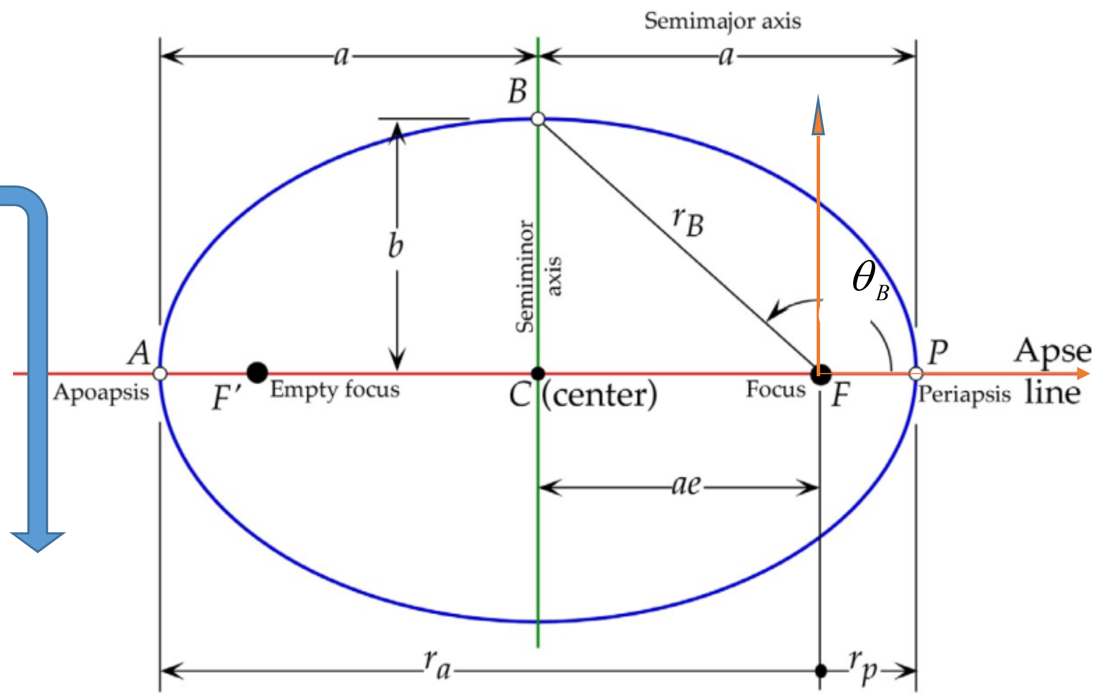
$$b^2 = a^2 - CF^2 = a^2 - a^2e^2 = a^2(1-e^2)$$

$$b = a\sqrt{1-e^2}$$

$$r_B^2 = C^2 + b^2 = (ae)^2 + a^2(1-e^2) = a^2$$

$$r_B = a\frac{1-e^2}{1+e\cos\theta_B} = a \Rightarrow \cancel{1} - e^2 = \cancel{1} + e\cos\theta_B$$

$$\Rightarrow \cos\theta_B = -e \rightarrow \theta_B = \cos^{-1}(-e)$$



$$r = \frac{P}{1 + e\cos\theta}$$

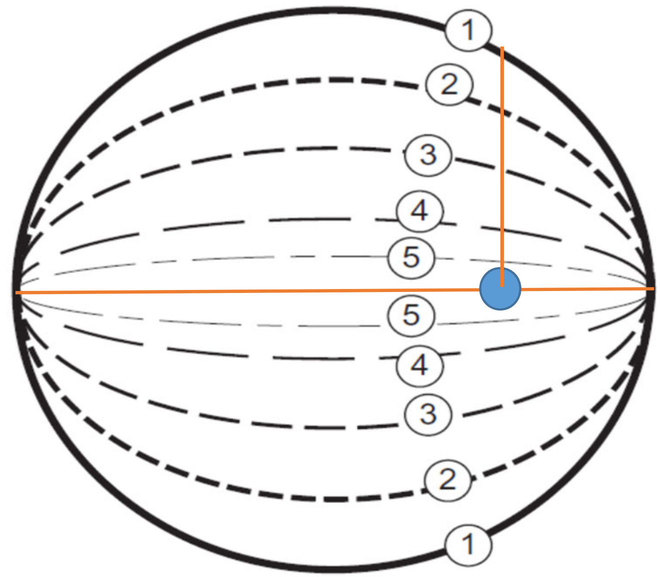
Elliptical Orbits

$$0 < e < 1;\ \varepsilon < 0$$

- Specific energy of an elliptical orbit

$$\varepsilon = -\frac{1}{2}\frac{\mu^2}{h^2}(1-e^2) < 0$$

$$\varepsilon = \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}$$



- Orbital Period τ

$$\frac{dA}{dt} = \frac{h}{2} \rightarrow \Delta A = \frac{h}{2}\Delta t \Rightarrow For\ one\ period$$

$$\Delta t = \tau,\quad \Delta A = \pi ab = \frac{h}{2}\tau \Rightarrow \tau = \frac{2\pi ab}{h}$$

$$\tau = \frac{2\pi}{h}a^2\sqrt{1-e^2} = \frac{2\pi}{\sqrt{\mu a(1-e^2)}}a^2\sqrt{1-e^2} = 2\pi\sqrt{\frac{a^3}{\mu}}$$

$$\boldsymbol{\tau = 2\pi\sqrt{\frac{a^3}{\mu}}}$$

Example 1

A meteorite has been spotted to approach the Earth at an **altitude** of 37000 km with a speed of 8 km/s. if at the time of initial observation its flight path is -65 degrees, investigate the possibility of its encounter with the Earth or the closest approach (pass) distance.

Other related questions: Where does it hit the Earth (2D,3D)?
When will it hit the Earth?

$$\varepsilon = \frac{v^2}{2} - \frac{\mu}{r} = 22.81 > 0 \Rightarrow Hyperbolic\ Trajectory$$

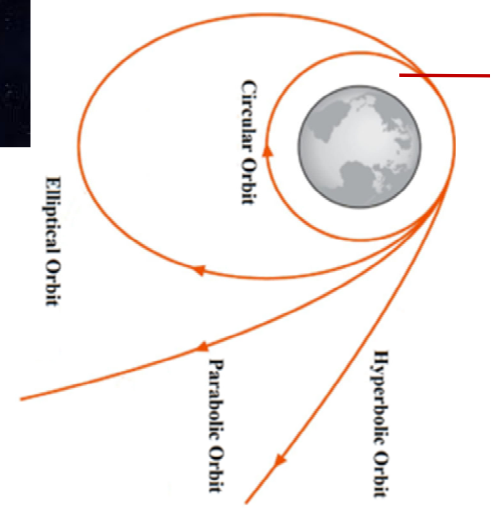
$$h = rv\cos\gamma = 1.467e+5\ km^2/s$$

$$\varepsilon = -\frac{1}{2}\frac{\mu^2}{h^2}(1-e^2) \Rightarrow e = 2.678;\ P = \frac{h^2}{\mu} = 53961$$

$$r_p = P/(1+e) = 14671.2 > R_E \Rightarrow No\ Encounter!$$

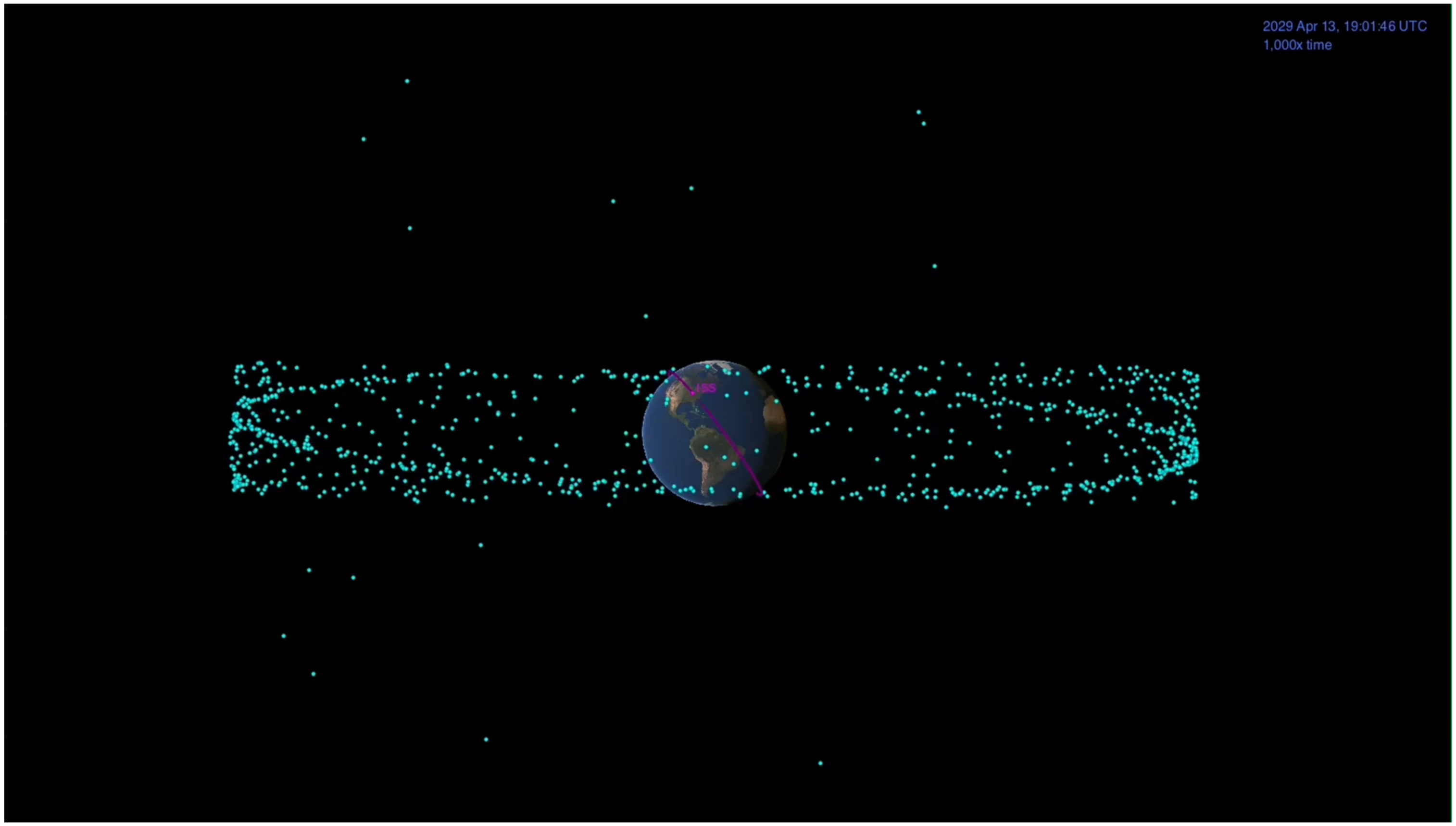
$$Closest\ Approach = r_p - R_E = 5605\ km$$

$$r = \frac{53961}{1+2.678\cos\theta}$$



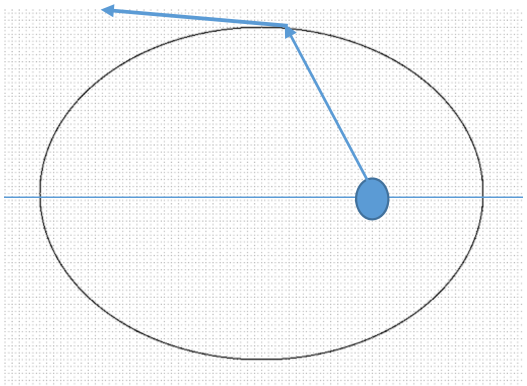
Potential Asteroid Hazards

Animation shows Apophis' 2029 path compared to the swarm of satellites orbiting Earth.



Example 2

An Earth satellite has a perigee altitude of 6.22 km with a maximum velocity of 8.6038 km/s. Find the orbital period and velocity of satellite when the true anomaly is 120 degrees.



$$a)\ \varepsilon = \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}$$

$$\frac{8.6038^2}{2} - \frac{398600}{6378 + 6.22} = -\frac{\mu}{2a} \rightarrow a = 7839.5\ km$$

$$\tau = 2\pi\sqrt{\frac{a^3}{\mu}} = 6907.9\ \ sec = 1.91\ hr$$

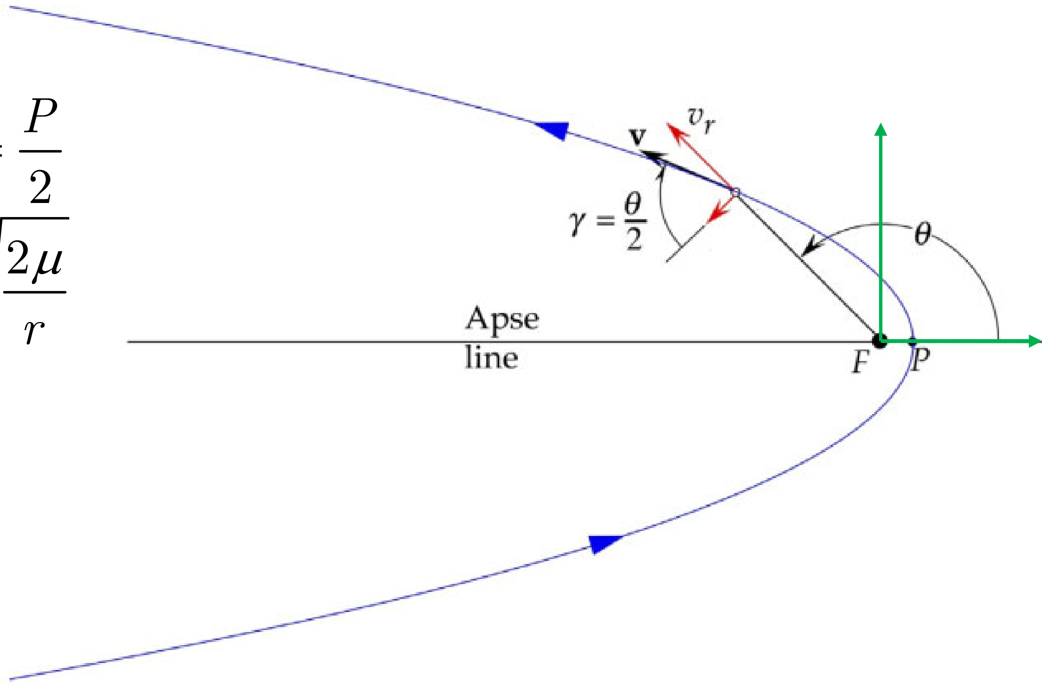
$$b)\ r = \frac{h^2 / \mu}{1 + e\cos\theta},\quad h = r_p v_p = 5.4929e{+}04 \Rightarrow h^2 = \mu P = \mu a(1 - e^2) \rightarrow e = 0.18$$

$$r_{\theta=120} = \frac{h^2 / \mu}{1 + 0.18\cos 120} = 8343.6\ km$$

$$\varepsilon = \frac{v_{\theta=120}^2}{2} - \frac{\mu}{r_{\theta=120}} \rightarrow v_{\theta=120} = 6.6859\ km / s$$

3- Parabolic Trajectory $e = 1;\ \varepsilon = 0$

Parabolic orbits are not usually observed in nature! They are however important as the boundary between open and closed orbits.



$$r = \frac{h^2}{\mu}\frac{1}{1+\cos\theta}; \quad \theta = \pi \to r \to \infty \quad ;\theta = 0 : r_p = \frac{P}{2}$$

$$\varepsilon = -\frac{1}{2}\frac{\mu^2}{h^2}(1-e^2) \xrightarrow{e=1} \frac{v^2}{2} - \frac{\mu}{r} = 0 \Rightarrow v = \sqrt{\frac{2\mu}{r}}$$

$$\Rightarrow EscapeVelocity = v_{ESC} = \sqrt{\frac{2\mu}{r}} = \sqrt{2}v_C$$

$$\tan\gamma = \frac{v_r}{v_\perp} = \frac{e\sin\theta}{1+e\cos\theta}$$

$$\tan\gamma = \frac{\sin\theta}{1+\cos\theta} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \tan\frac{\theta}{2} \Rightarrow \gamma = \frac{\theta}{2}$$

4- Hyperbolic Trajectory $e > 1;\ \varepsilon > 0$

Meteorite and asteroids are usually on a hyperbolic path. Hyperbolic trajectories (HT) are also utilized for interplanetary missions due to their positive specific energy.
Net effect of a HT flyby will be a change in the velocity direction denoted by an angle δ, which is the angle between The asymptotes! Other HT properties are as follows:

1. *Asymptotes (defined for $r \to \infty$)*

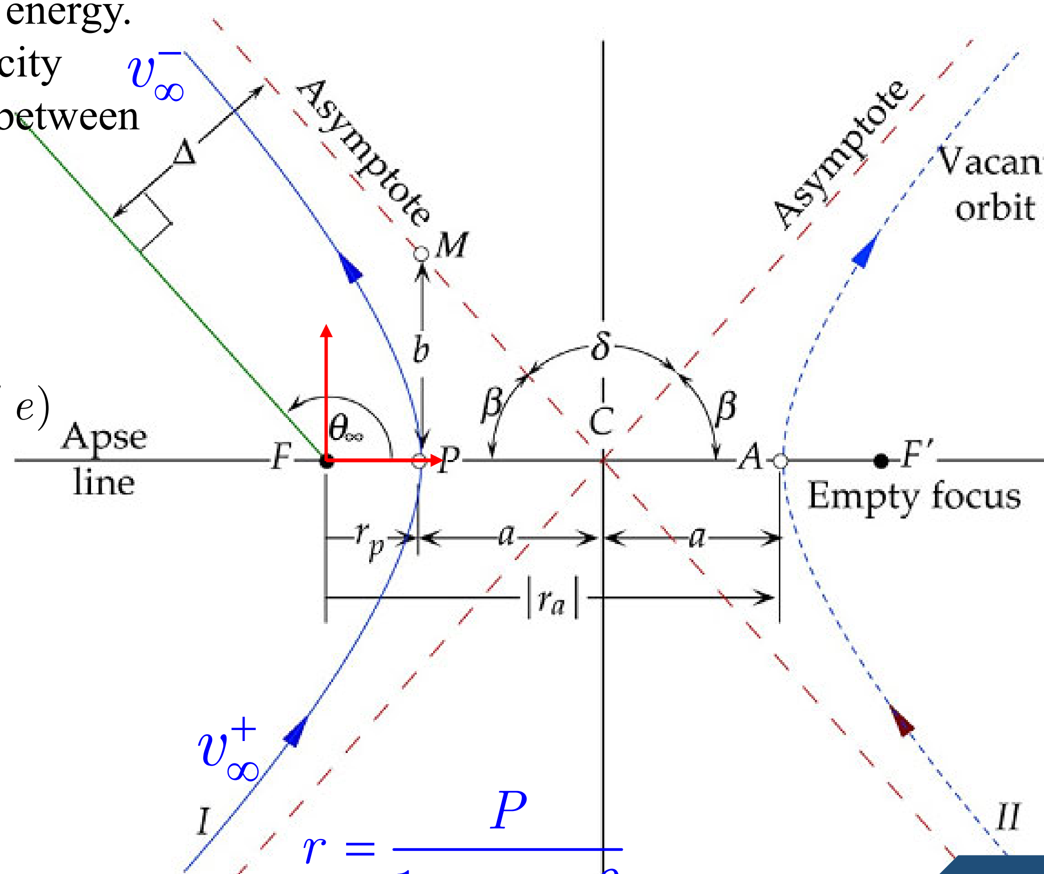
$$1 + e\cos\theta = 0 \to \theta_\infty = \cos^{-1}(-1/e)$$

$$\cos(\pi - \theta_\infty) = -\cos\theta_\infty = \cos\beta \Rightarrow \beta = \cos^{-1}(1/e)$$

2. *Excess Hyperbolic velocity*

$$\varepsilon = \frac{v^2}{2} - \frac{\mu}{r} = constant = \frac{v_\infty^2}{2} - \cancel{\frac{\mu}{r_\infty}} = -\frac{\mu}{2a}$$

$$\to v_\infty^+ = v_\infty^- = \sqrt{-\frac{\mu}{2a}}$$



$$r = \frac{P}{1 + e\cos\theta}$$

Hyperbolic Trajectory $e > 1;\ \varepsilon > 0$

3. *Turn angle δ, and Aiming Distance Δ*

$$\frac{\pi}{2} - \frac{\delta}{2} = \pi - \theta_\infty \Rightarrow \theta_\infty = \frac{\pi}{2} + \frac{\delta}{2}$$

$$\cos\theta_\infty = -\sin(\frac{\delta}{2}) \Rightarrow \sin(\frac{\delta}{2}) = 1/e$$

$$\rightarrow \delta = 2\sin^{-1}(1/e)$$

Aiming Distance Δ : a design/control parameter for interplanetory missions

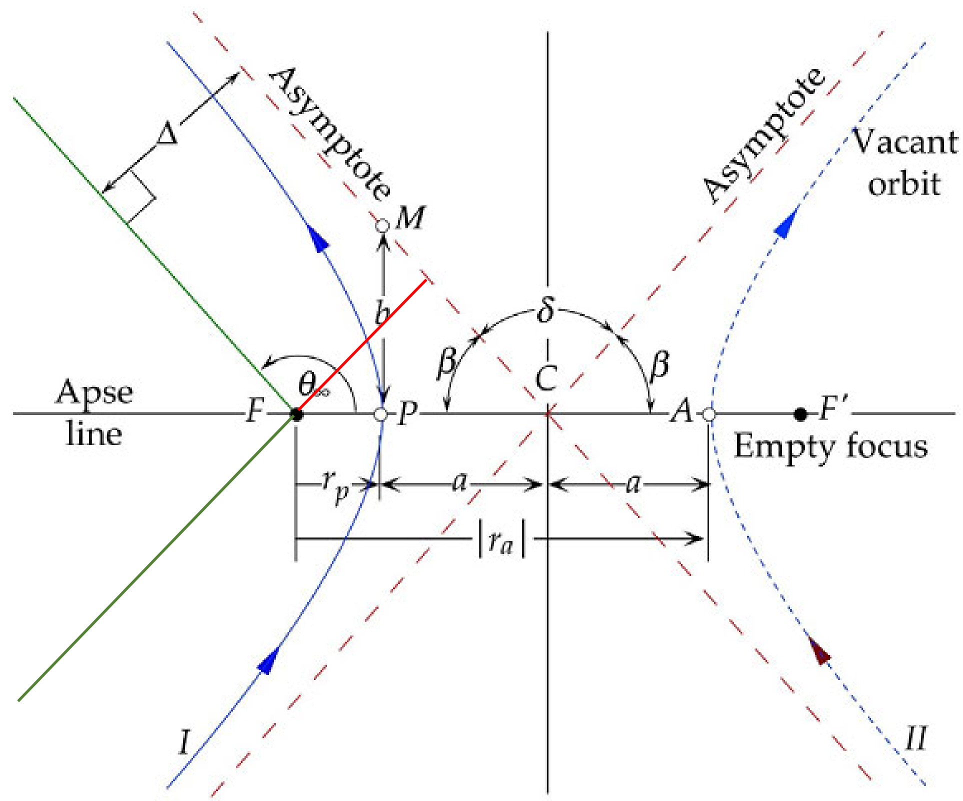
$$\left|\vec{v}_\infty^{\,+}\right| = \left|\vec{v}_\infty^{\,-}\right| = v_\infty$$

$$h = v_\infty \Delta \Rightarrow \Delta = h/v_\infty$$

Or

$$\Delta = (r_p + a)\sin\beta = CF\sin\beta = C\sin\beta = ae\sin\beta$$

$$\Delta = ae\frac{\sqrt{e^2-1}}{e} = a\sqrt{e^2-1} = b$$



$$\varepsilon = -\frac{\mu}{2a} > 0 \Rightarrow a < 0$$

Example 3

Determine the true anomaly of the metorite at the instant of observation.

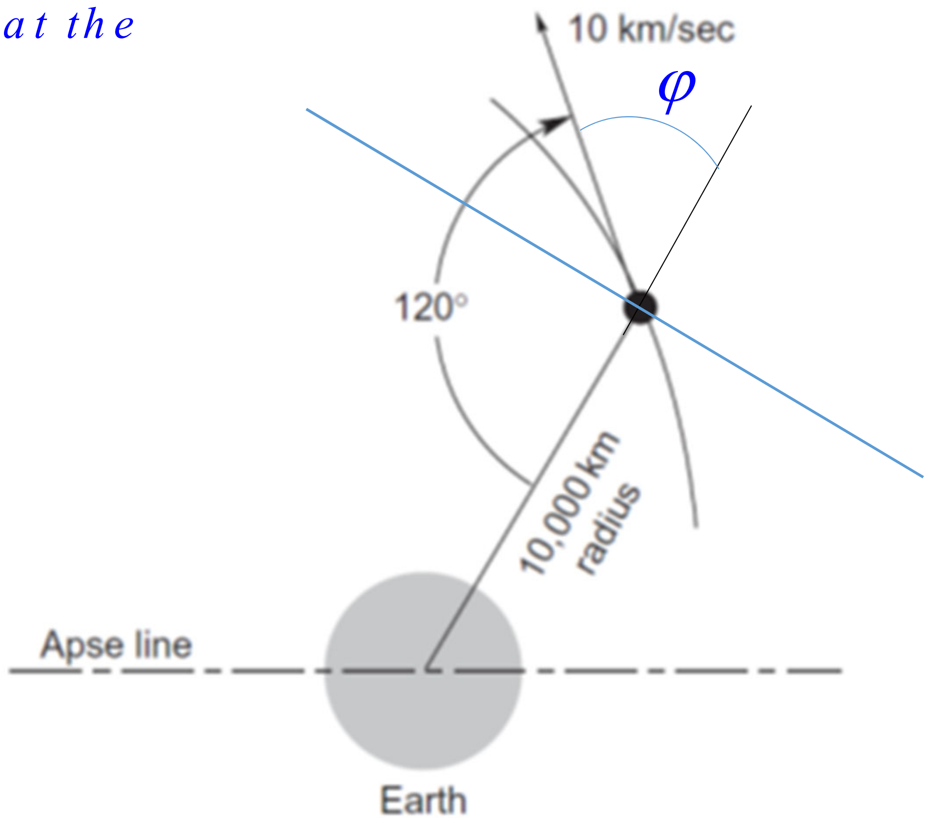
$$\frac{v^2}{2} - \frac{\mu}{r} = 10.14 > 0 \rightarrow Hyperbolic\ \ Orbit$$

$$\varepsilon = -\frac{\mu}{2a} \rightarrow a = -19655\ km$$

$$h = rv\sin\varphi = rv\sin(\pi - 120) = 86603\ km^2 / s$$

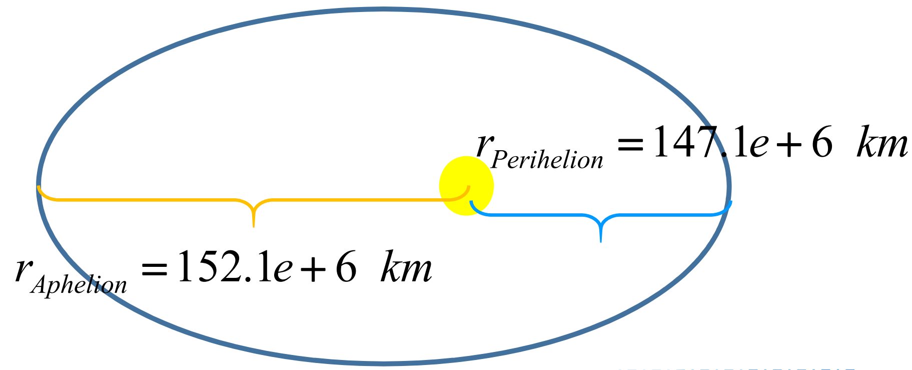
$$\frac{h^2}{\mu} = a(1-e^2) \rightarrow e = 1.39$$

$$r = \frac{a(1-e^2)}{1+e\cos\theta} \rightarrow \theta \approx 51\ deg$$



Astronomical/Canonical Units

As the mass and distances of celestial bodies are not exact and dealing with large numbers are difficult, Astronomical or canonical units are sometimes utilized to describe interplanetary motion:



A : for Planets around the Sun

1. Distance Unit : $1AU = 149\ 597\ 870.691 \; km \approx 150e+6 \; km$

- Examples:

$$Sun - Venus \text{ distance} = 108.2e+6 \; km = 0.72 \; AU$$

$$Sun - Mars \text{ distance} = 227.9e+6 \; km = 1.52 \; AU$$

2. Time unit *(TU)* is defined based on one orbital revolution (2π) of the Earth around the Sun.

$$1 \; TU = 2\pi \sqrt{\frac{a^3_{Sun-Earth}}{\mu_{Sun}}} = 31.685e+6 \; Sec$$

Standard gravitational parameter $\mu = G\,M$ (mass of the celestial body)

Celestial Body	μ [km^3 s^{-2}]
Sun	$1.32712440018 \times 10^{11}$
Mercury	2.2032×10^{4}
Venus	3.24859×10^{5}
Earth	$3.986004418 \times 10^{5}$
Moon	4.9048695×10^{3}
Mars	4.282837×10^{4}
Ceres	6.26325×10
Jupiter	1.26686534×10^{8}
Saturn	3.7931187×10^{7}
Uranus	5.793939×10^{6}
Neptune	6.836529×10^{6}
Pluto	8.71×10^{2}
Eris	1.108×10^{3}

Astronomical/Canonical Units

B : for Orbits around the Earth or Planets

1. 1 Distance Unit is defined as the radius of the Celestial body under study.

2. Time unit (TU) is defined such that circular velocity of the satellite around the celestial body will be equal to 1 DU/TU . In this sense, the planet gravity constant will be, $\mu = 1$ since the circular velocity is : $v_c = \sqrt{\mu / r}$

- Example for an Earth orbiting Satellites:

$$1\ DU = 6378.137\ km;\ \mu = 1$$

$$\tau = 2\pi\sqrt{\frac{r^3}{\mu}} \Rightarrow \tau = 2\pi\ \ ;\ 1TU = 2\pi\sqrt{\frac{1\,DU^3}{\mu}}$$

$$So,\ 1TU = 2\pi\sqrt{\frac{6378.137^3}{398600}}\ sec \rightarrow 1\ TU \approx 806.8\ sec \approx 13.44\ min$$

Example 4

The instantaneous position and velocity of and Earth orbiting satellite with respect to the ECI is provided below in Astronomical units.

1- Find the satellite period.

2- The satellite position after 30 degrees increase in its true anomaly.

1.

$$\vec{r}(t_0) = [1.1,\ 0,\ 0]^T\ DU \Rightarrow r = 1.1\ DU$$

$$\vec{v}(t_0) = [0.9,\ 0.9,\ 0]^T\ DU/TU \Rightarrow v = 1.2728\ \ DU/TU$$

$$\frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a} \to \frac{1.2728^2}{2} - \frac{1}{1.1} = -0.09909 = -\frac{\mu}{2a} \Rightarrow a = 5.0459\ DU$$

$$\tau = 2\pi\sqrt{\frac{a^3}{\mu}} = 2\pi\sqrt{\frac{5.0459^3}{\mu}} = 71.2171\ \ TU = 5.7459e{+}04\ sec$$

Example 4 Continued

$2.$

$$h = norm(\mathbf{r}_0 \times \mathbf{v}_0) = 0.99\ DU^2/TU \Rightarrow P\mu = h^2 \Rightarrow P = 0.9801\ DU$$

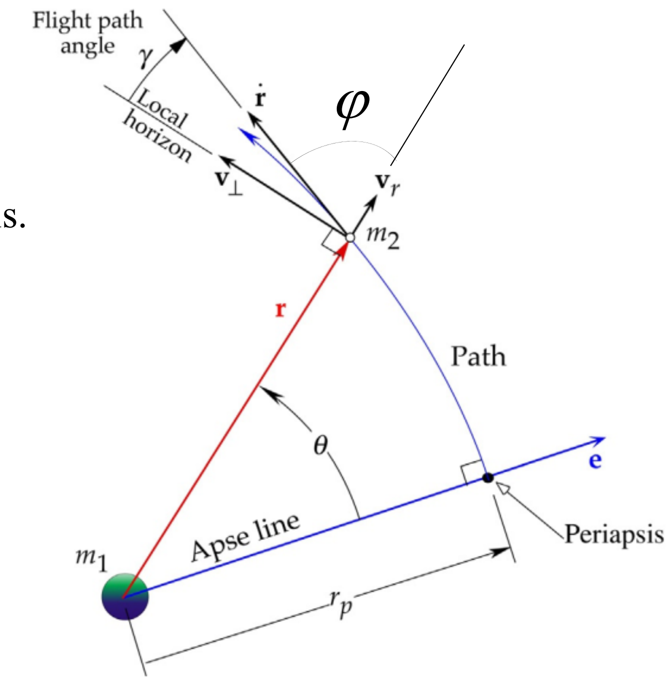
$$P = a(1-e^2) \Rightarrow e = 0.8976$$

$$r = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta} = 1.1 \Rightarrow \theta = 96.975\ at\ current\ position$$

$$r = \frac{h^2}{\mu}\frac{1}{1+e\cos(\theta+30)} = 2.13DU$$

$$Altitude = r - R_E = 1.1302\ \ DU = 7,208.4\ \ km$$

General Orbit Time Relation (GOTR)



Orbital time is related to SC position in orbit :

1- The direct problem : Given an orbital position, find the time elapsed since Periapsis.

2- The indirect problem : Given time elapsed since Periapsis, find SC position.

Recall: $h = rv\sin\phi = rv\cos\gamma = rv_\perp$, *But* $v_\perp = r\dot{\theta} \rightarrow h = r^2\dot{\theta}$

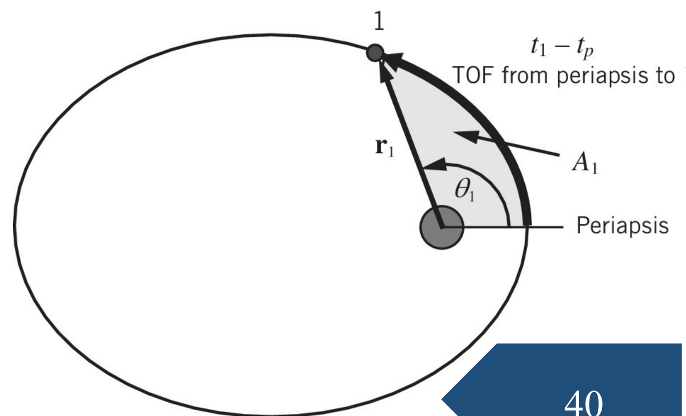
$$\frac{d\theta}{dt} = \frac{h}{r^2} \Rightarrow dt = \frac{r^2}{h}d\theta;\ \ and\ since\ \ r = \frac{h^2/\mu}{1+e\cos\theta}$$

$$dt = \frac{h^3/\mu^2}{(1+e\cos\theta)^2}d\theta \Rightarrow \frac{\mu^2}{h^3}\int_{t_0}^{t} dt = \int_0^{\theta}\frac{d\theta}{(1+e\cos\theta)^2}$$

$$\frac{\mu^2}{h^3}(t-t_p) = \int_0^{\theta}\frac{d\theta}{(1+e\cos\theta)^2}$$

t_p *= time at periapsis passage (θ=0), is usually assumed 0. So:*

t is the time from perapsis passage using the following GOTR,



$$General\ Orbit\ Time\ Relation:\ \ \frac{\mu^2}{h^3}t = \int_0^{\theta}\frac{d\theta}{(1+e\cos\theta)^2}$$

Some Mathematical Relations for Integration of GOTR

$$\frac{\mu^2}{h^3} t = \int_0^\theta \frac{d\theta}{(1+e\cos\theta)^2} = ?$$

$$\int \frac{dx}{(a+b\cos x)^2} = \frac{1}{(a^2-b^2)^{3/2}} \left(2a \tan^{-1} \sqrt{\frac{a-b}{a+b}} \tan\frac{x}{2} - \frac{b\sqrt{a^2-b^2}\sin x}{a+b\cos x} \right) \qquad (b<a)$$

$$\int \frac{dx}{(a+b\cos x)^2} = \frac{1}{a^2} \left(\frac{1}{2}\tan\frac{x}{2} + \frac{1}{6}\tan^3\frac{x}{2} \right) \qquad (b=a)$$

$$\int \frac{dx}{(a+b\cos x)^2} = \frac{1}{(b^2-a^2)^{3/2}} \left(\frac{b\sqrt{b^2-a^2}\sin x}{a+b\cos x} - a\ln\left(\frac{\sqrt{b+a}+\sqrt{b-a}\tan(x/2)}{\sqrt{b+a}-\sqrt{b-a}\tan(x/2)} \right) \right) \qquad (b>a)$$

Application of GOTR to Circular and Elliptic Orbits

1- Circular Orbits

$$\frac{\mu^2}{h^3}t = \int_0^\theta \frac{d\theta}{(1+e\cos\theta)^2}, \quad e = 0 \rightarrow \frac{\mu^2}{h^3}t = \int_0^\theta d\theta$$

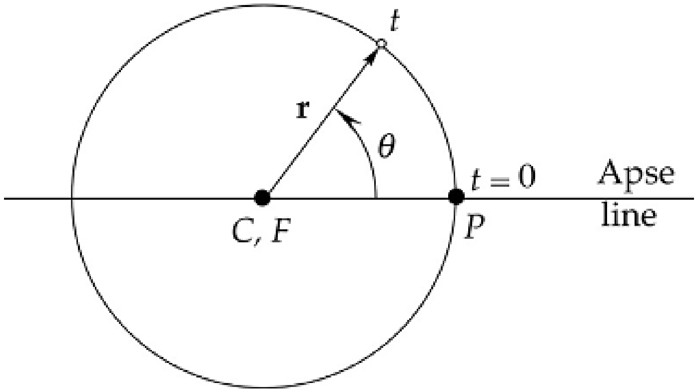
$$t = \frac{h^3}{\mu^2}\theta, \quad But:\ r = h^2/\mu \rightarrow h^3 = r^{3/2}\mu^{3/2}$$

$$\rightarrow t = \frac{r^{3/2}}{\sqrt{\mu}}\theta$$

$$Note:\ \tau = 2\pi\sqrt{\frac{a^3}{\mu}} \Rightarrow \frac{r^{3/2}}{\sqrt{\mu}} = \frac{\tau}{2\pi}$$

$$\rightarrow t = \frac{\tau}{2\pi}\theta \Rightarrow inverse\ relation:\ \theta = \frac{2\pi}{\tau}t$$

$$n = Mean\ Motion = \frac{2\pi}{\tau} \Rightarrow \theta = nt$$



Application of GOTR to Circular and Elliptic Orbits

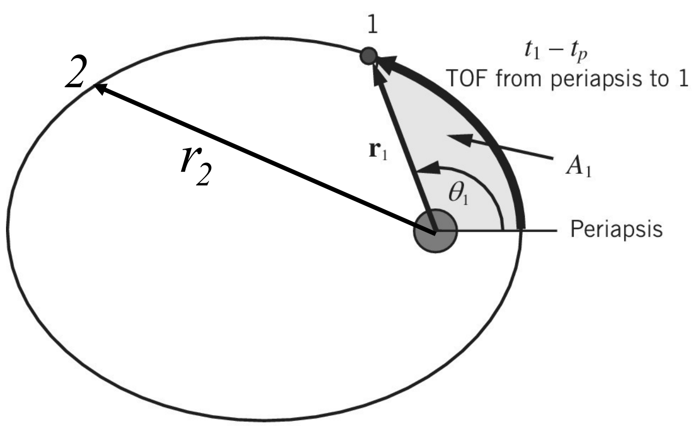
2(a) - Elliptical Orbits time via Elliptical Mean Anomaly(EMA)

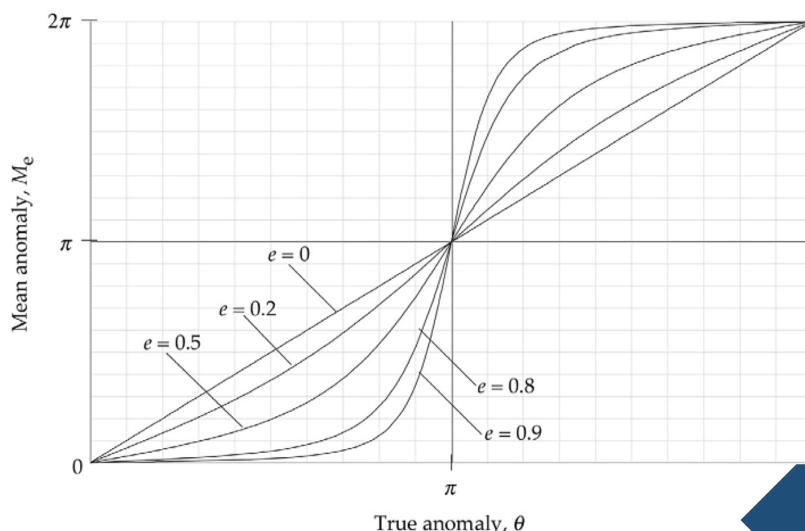
$$\frac{\mu^2}{h^3}t = \int_0^\theta \frac{d\theta}{(1+e\cos\theta)^2} = \frac{1}{(1-e^2)^{3/2}}\left(2\tan^{-1}\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2} - \frac{e\sqrt{1-e^2}\sin\theta}{1+e\cos\theta}\right)$$

$$Let\ M_e = \left(2\tan^{-1}\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2} - \frac{e\sqrt{1-e^2}\sin\theta}{1+e\cos\theta}\right) \equiv f(\theta, e) = Elliptical\ Mean\ Anomaly(EMA)$$

$$\Rightarrow M_e = \frac{\mu^2}{h^3}(1-e^2)^{3/2}t_P$$

$$Or: \Delta M_e = \frac{\mu^2}{h^3}(1-e^2)^{3/2}\Delta t$$





Elliptical Orbits $(0 < e < 1)$

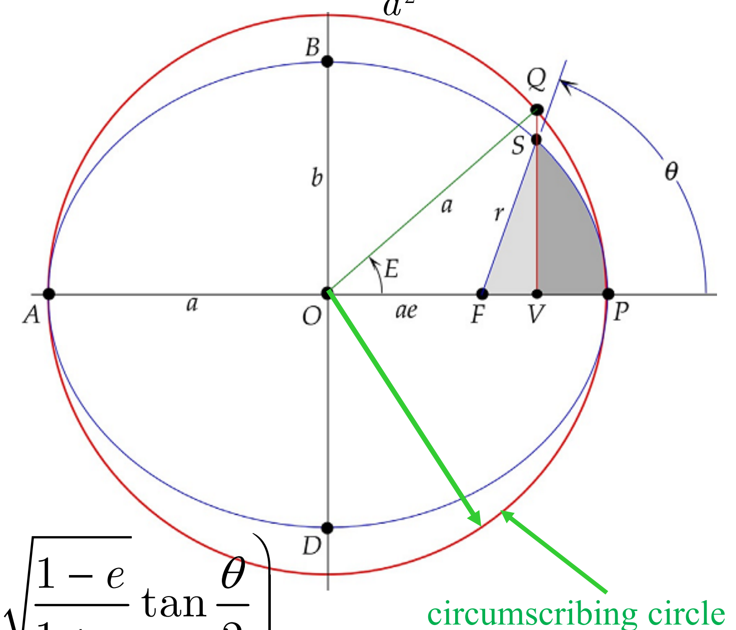
2(b)- Elliptical Orbits time via utility of mean motion

$$\frac{h^2}{\mu} = a(1-e^2) \rightarrow \left[\frac{h^2}{\mu}\right]^{\frac{3}{2}} = \left[a(1-e^2)\right]^{\frac{3}{2}} \rightarrow h^3 = \left[\mu a(1-e^2)\right]^{\frac{3}{2}}$$

$$M_e = \frac{\mu^2}{h^3}(1-e^2)^{3/2} t_P = \frac{\mu^2}{\left[\mu a(1-e^2)\right]^{\frac{3}{2}}}(1-e^2)^{3/2} t_P = \frac{\mu^{\frac{1}{2}}}{a^{\frac{3}{2}}} t = \frac{2\pi}{\tau} t = n t_P$$

$$\Delta M_e = n \Delta t$$

Note: $\tau = 2\pi\sqrt{\frac{a^3}{\mu}} \Rightarrow \frac{\mu^{\frac{1}{2}}}{a^{\frac{3}{2}}} = \frac{2\pi}{\tau} = n$



2(c)- Elliptical Orbits time (c) via utility of Eccentric anomaly (E)

$$OV = a\cos E = ae + r\cos\theta$$

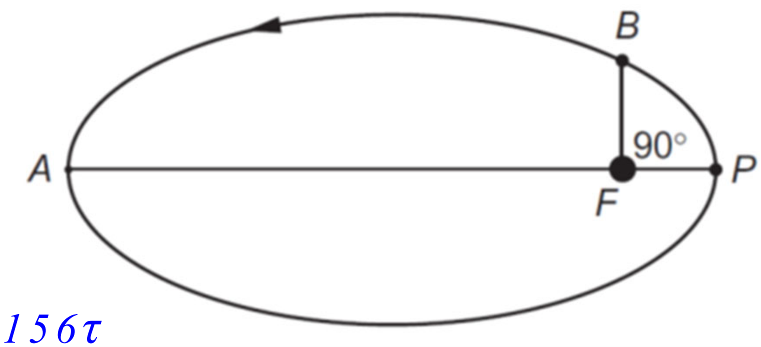
$$a\cos E = ae + \frac{a(1-e^2)\cos\theta}{1+e\cos\theta} \Rightarrow \cos E = \frac{e+\cos\theta}{1+e\cos\theta} \Rightarrow E = 2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right)$$

$$\Rightarrow M_e = \left(2\tan^{-1}\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2} - \frac{e\sqrt{1-e^2}\sin\theta}{1+e\cos\theta}\right) = E - e\sin E = nt$$

$$\theta \rightarrow E \xrightarrow{n} t$$

Example 5

The eccentricity of an elliptic orbit is equal to *0.3*, determine the time needed to reach point B as a fraction of the orbital period.



$$E - e\sin E = nt;\ \tau = 2\pi\sqrt{\frac{a^3}{\mu}};\ n = \frac{2\pi}{\tau}$$

$$\cos E = \frac{e + \cos\theta}{1 + e\cos\theta} = e \Rightarrow E = 72.54\ deg$$

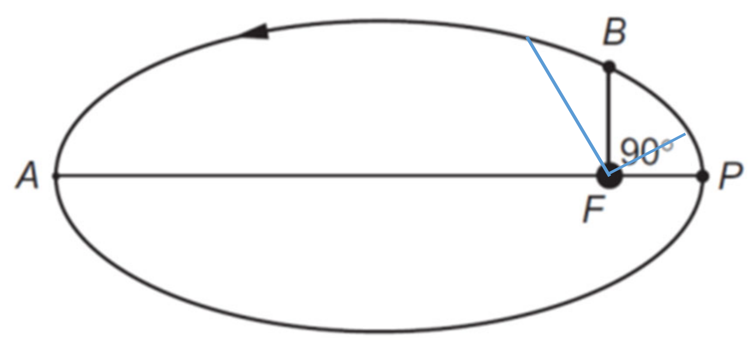
$$\frac{2\pi}{\tau}t = 72.54 \times \frac{\pi}{180} - 0.3\sin(72.54) = 0.979 \Rightarrow t = 0.156\tau$$

Example 6

Compute the flight time between the true anomalies of *30* to *120* degrees , for an Earth orbiting satellite where $e=0.6$ and $P=4000$ *km*.

$$E - e\sin E = nt;\ \tau = 2\pi\sqrt{\frac{a^3}{\mu}};\ n = \frac{2\pi}{\tau};\ P = a(1 - e^2);\ \cos E = \frac{e+\cos\theta}{1+e\cos\theta}$$

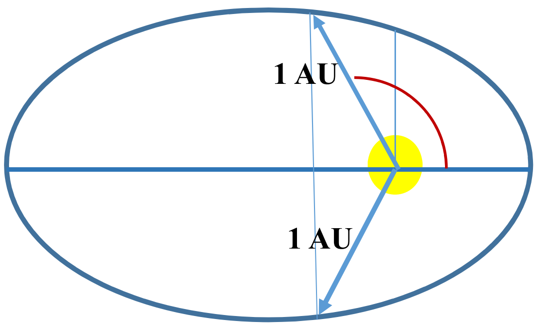
$$a = 6250Km, E_{30} = 15.26^o, E_{120} = 81.78^o, \tau = 4917.2\ sec$$
$$\Rightarrow t_{30} = 84.86sec, t_{120} = 652sec$$
$$\Rightarrow \Delta t = 576.14sec$$



Example 7

A space probe is in an elliptical orbit around the Sun. Perihelion distance is .5 AU and the Apohelion is 2.5 AU. How many days during each orbit is the probe closer than 1 AU to the Sun?

$$\begin{cases} r_p = 0.5AU, r_a = 2.5AU \\ r_1 = 1AU, \quad r_2 = 1AU \end{cases}$$



$$e = \frac{r_a - r_p}{r_a + r_p} = \frac{2}{3} \to a = \frac{r_p + r_a}{2} = \frac{3}{2}$$

$$p = a(1 - e^2) = 0.83333$$

$$r = \frac{a\left(1 - e^2\right)}{1 + e\cos\theta} = 1, \cos\theta = -\frac{1}{4}$$

$$\Rightarrow \theta_{1,2} = 104.47 \text{ and } 255.53$$

$$\cos E = \frac{e + \cos\theta}{1 + e\cos\theta} = 0.5$$

$$E = 60^\circ = 1.048 \text{ radians}, \sin E = 0.866$$

$$\to$$

$$\tau = 2\pi\sqrt{\frac{a^3}{\mu}} = 4.78TU$$

$$n = \frac{2\pi}{\tau} = \sqrt{\frac{\mu}{a^3}}$$

$$E - e\sin E = nt$$

$$\left[1.048 - \frac{2}{3}(.866)\right] = \sqrt{\frac{\mu}{a^3}}t \Rightarrow t = 0.862TU_\odot$$

$$TOF = 2t = 1.724TU_\odot \approx (1.724 / 2\pi)365.24 = 100 Days$$

Elliptical Orbits $(0 < e < 1)$

2(d)- Elliptical Orbits: indirect problem via utility of Eccentric anomaly (E)

$$\cos E = \frac{e + \cos\theta}{1 + e\cos\theta} \Rightarrow E = 2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right); n = \frac{2\pi}{\tau} = \sqrt{\frac{\mu}{a^3}}$$

Direct problem : $\theta \to E \xrightarrow{n} t_P : M_e = E - e\sin E = nt_P,$

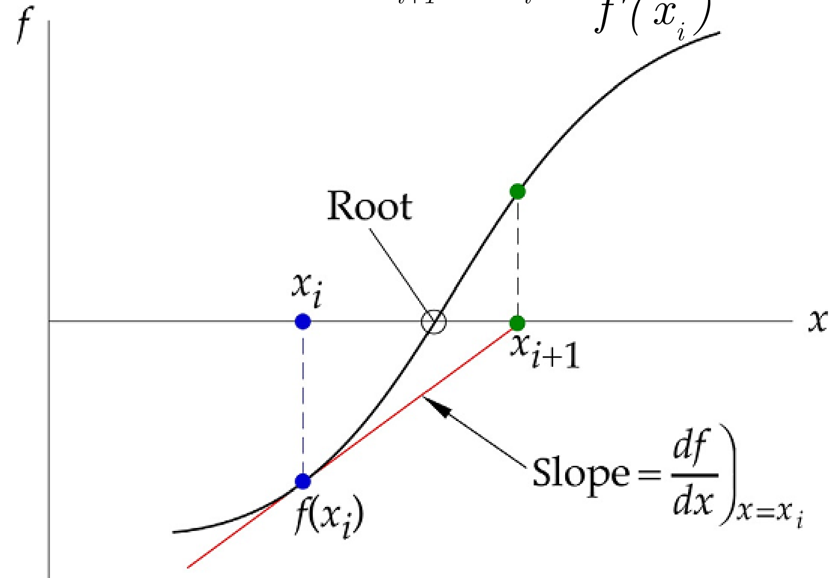
Indirect problem : $t_P \xrightarrow{n} M_e = E - e\sin E \longrightarrow \theta$

$$f(E) = E - e\sin E - M_e \Rightarrow f'(E) = 1 - e\cos E$$

Utility of the iterative Newton's Root finding Method:

$$f(x) = 0$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$



$$E_{i+1} = E_i - \frac{E_i - e\sin E_i - M_e}{1 - e\cos E_i}$$

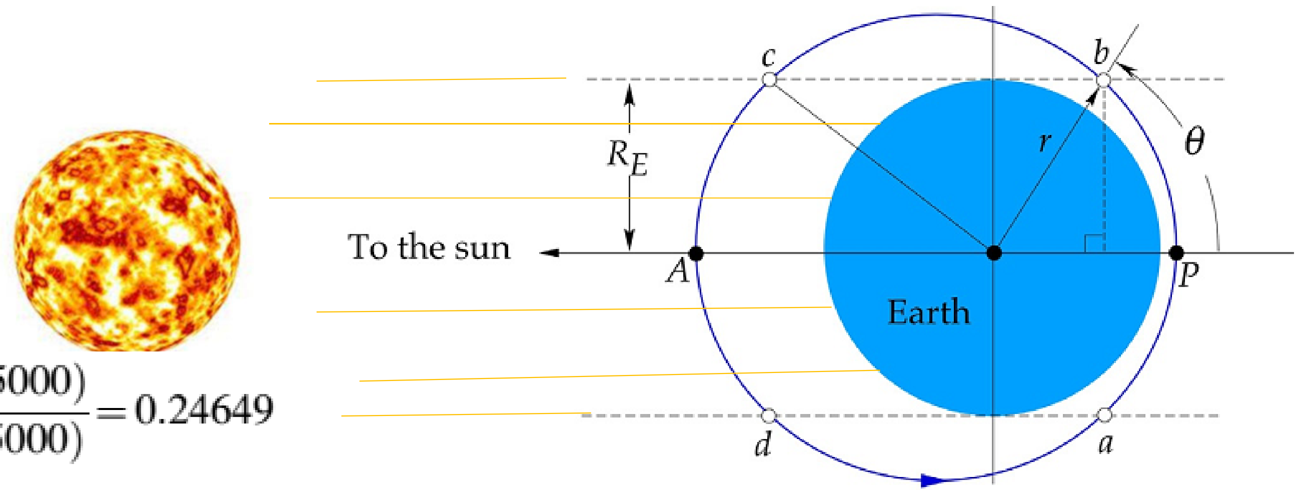
$$E_0 = \begin{cases} M_e + e/2, & M_e < \pi \\ M_e - e/2, & M_e > \pi \end{cases}$$

Stopping Criterion : $|E_{i+1} - E_i| < \varepsilon$ or $|f(E_{i+1})| < \varepsilon$

Example 8

EXAMPLE 3.3

Let a satellite be in a 500 km by 5000 km orbit with its apse line parallel to the line from the earth to the sun, as shown in Fig. 3.9. Find the time that the satellite is in the earth's shadow if
(a) the apogee is toward the sun
(b) the perigee is toward the sun.



$$e = \frac{r_a - r_p}{r_a + r_p} = \frac{(6378+5000)-(6378+5000)}{(6378+5000)+(6378+5000)} = 0.24649$$

$$r_p = \frac{h^2}{\mu}\frac{1}{1+e\cos(0)} \Rightarrow 6878 = \frac{h^2}{398{,}600}\frac{1}{1+0.24649} \Rightarrow h = 58{,}458\,\mathrm{km}^2/\mathrm{s}$$

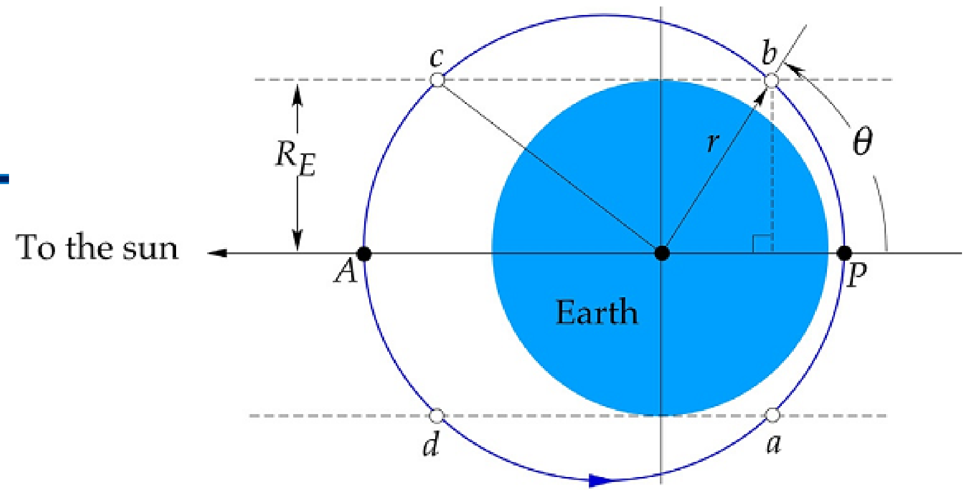
Example 8

$$a = \frac{h^2}{\mu}\frac{1}{1-e^2} = \frac{58,458^2}{398,600}\frac{1}{1-0.24649^2} = 9128\,\text{km}$$

$$T = \frac{2\pi}{\sqrt{\mu}}a^{3/2} = \frac{2\pi}{\sqrt{398,600}}9128^{3/2} = 86791.1\,\text{s}\,(2.4109\text{h})$$

$$\sin\theta = R_E/r, \quad r = \frac{a(1-e^2)}{1+e\cos\theta} \longrightarrow e\cos\theta - (1-e^2)\frac{a}{R_E}\sin\theta + 1 = 0$$

$$0.24649\cos\theta - 1.3442\sin\theta = -1$$

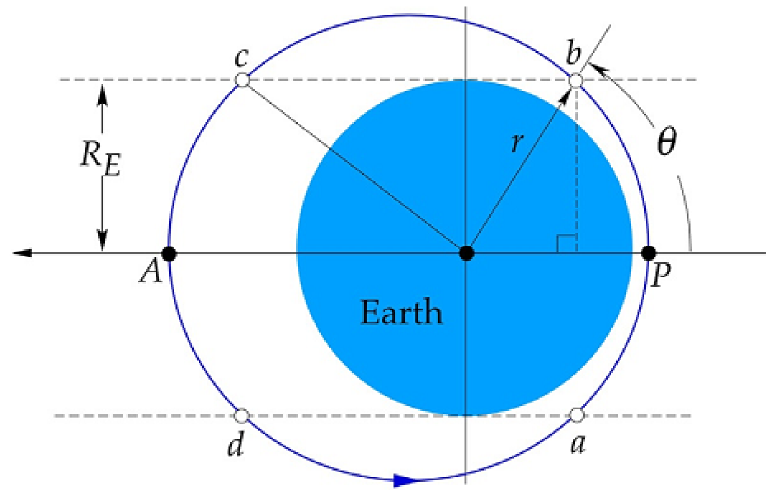


Note:

$$A\cos\theta + B\sin\theta = C$$

$$\theta = \tan^{-1}\frac{B}{A} \pm \cos^{-1}\left[\frac{C}{A}\cos\left(\tan^{-1}\frac{B}{A}\right)\right]$$

Example 8



$$\theta = \tan^{-1}\frac{-1.3442}{0.24649} \pm \cos^{-1}\left[\frac{-1}{0.24649}\cos\left(\tan^{-1}\frac{-1.3442}{0.24649}\right)\right] = -79.607° \pm 137.03°$$

$$\theta_b = 57.423°$$
$$\theta_c = -216.64°\,(+143.36°)$$

$$E_b = 2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta_b}{2}\right) = 2\tan^{-1}\left(\sqrt{\frac{1-0.24649}{1+0.24649}}\tan\frac{57.423°}{2}\right) = 0.80521\,\text{rad}$$

$$M_e = E - e\sin E = 0.80521 - 0.24649\sin 0.80521 = 0.62749\,\text{rad}$$

$$t_b = \frac{M_e}{2\pi}T = \frac{0.62749}{2\pi}8679.1 = 866.77\,\text{s}$$

$$\boxed{t = 2t_b = 1734\,\text{s}\ (28.98\,\text{min})}$$

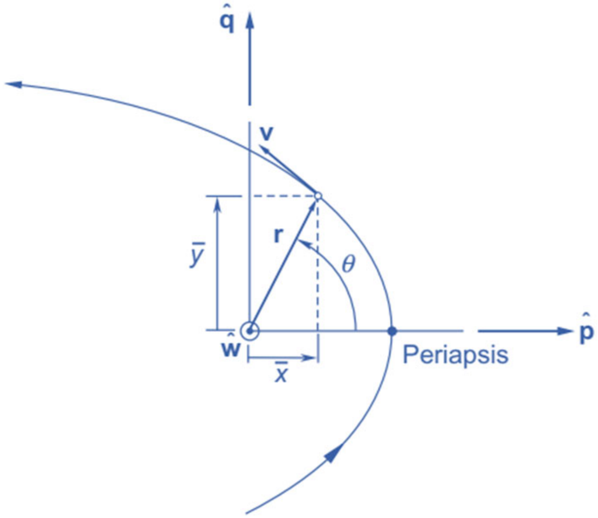
Parabolic Trajectories $(e = 1)$

3(a) -Direct Problem:

Parabolic Orbits time via Parabolic Mean Anomaly (PMA)

$$\frac{\mu^2}{h^3}t = \int_0^{\theta} \frac{d\theta}{(1+\cos\theta)^2} = \frac{1}{2}\tan\frac{\theta}{2} + \frac{1}{6}\tan^3\frac{\theta}{2} = M_P$$

$$M_P = \frac{\mu^2 t}{h^3}$$



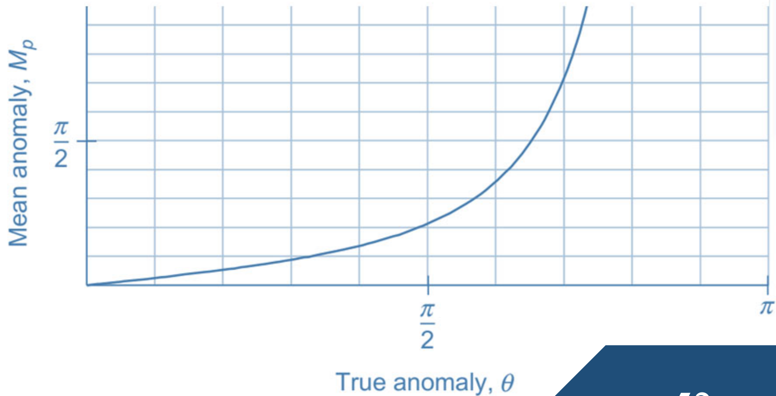
3(b) -Indirect Problem:

Parabolic Orbits position via Parabolic Mean Anomaly (PMA)

$$M_P = \frac{\mu^2 t}{h^3}$$

$$Let : b = 3M_P + \sqrt{(3M_P)^2 + 1}$$

$$\tan\frac{\theta}{2} = (b)^{\frac{1}{3}} - (b)^{-\frac{1}{3}}$$



Example 9

An Earth bound space probe on a parabolic path has a perigee velocity of 10 Km/s. What will be its outbound altitude after 6 hours passage from the perigee ?

$$\mu_{earth} = 398{,}600 km^3 / s^2 ; v_{esc} = \sqrt{\frac{2\mu}{r}}$$

$$v_p = 10 \Rightarrow r_p = \frac{2\mu}{v_p^2} = 7972\ km$$

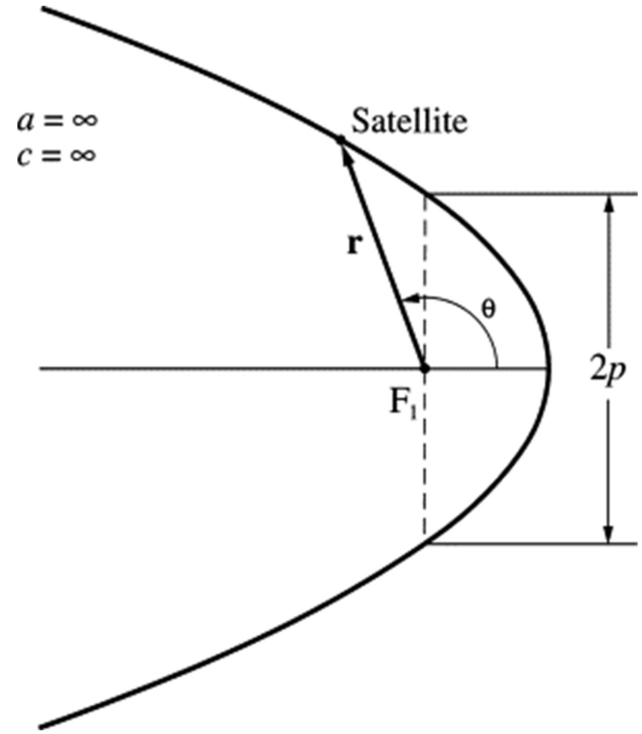
$$h = r_p v_p = 7972\ km^2 / s$$

$$M_P = \frac{\mu^2 t}{h^3} = \frac{\mu^2}{h^3}(6 \times 3600) = 6.7737\ Rad$$

$$b = 3M_P + \sqrt{(3M_P)^2 + 1} = 3 \times 6.7737 + \sqrt{(3 \times 6.7737)^2 + 1} = 40.66$$

$$tan\frac{\theta}{2} = (b)^{\frac{1}{3}} - (b)^{-\frac{1}{3}} = 3.148 \Rightarrow \theta = 144.75^\circ$$

$$r = \frac{h^2 / \mu}{1 + cos\,\theta} = 86899km \Rightarrow H = r - R_E = 80{,}521km$$



Hyperbolic Trajectories $(e > 1)$

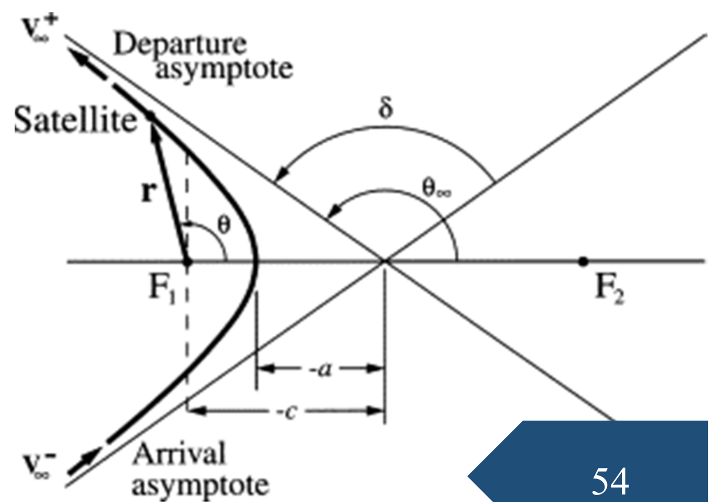
4(a) - Hyperbolic Orbits time via Hyperbolic Mean Anomaly (HMA)

$$\frac{\mu^2}{h^3}t = \int_0^\theta \frac{d\theta}{(1+e\cos\theta)^2} = \frac{1}{e^2-1}\left[\frac{e\sin\theta}{1+e\cos\theta} - \frac{1}{\sqrt{e^2-1}}\ln\left(\frac{\sqrt{e+1}+\sqrt{e-1}\tan(\theta/2)}{\sqrt{e+1}-\sqrt{e-1}\tan(\theta/2)}\right)\right]$$

$$\frac{\mu^2}{h^3}t = \frac{1}{e^2-1}\;\frac{e\sin\theta}{1+e\cos\theta} - \frac{1}{(e^2-1)^{\frac{3}{2}}}\ln\left(\frac{\sqrt{e+1}+\sqrt{e-1}\tan(\theta/2)}{\sqrt{e+1}-\sqrt{e-1}\tan(\theta/2)}\right) \xrightarrow{\times(e^2-1)^{\frac{3}{2}}}$$

$$\frac{\mu^2}{h^3}(e^2-1)^{\frac{3}{2}}t = \underbrace{\frac{e\sqrt{e^2-1}\sin\theta}{1+e\cos\theta} - \ln\left(\frac{\sqrt{e+1}+\sqrt{e-1}\tan(\theta/2)}{\sqrt{e+1}-\sqrt{e-1}\tan(\theta/2)}\right)}_{M_h \,=\, \text{Hyperbolic Mean Anomaly}}$$

$$\frac{\mu^2}{h^3}(e^2-1)^{\frac{3}{2}}t_p = M_h$$



Hyperbolic Trajectories $(e > 1)$

4(b) - Hyperbolic Orbits time via utility of Hyperbolic Eccentric Anomaly (HEA),F

$$M_h = e\sinh F - F = \frac{\mu^2}{h^3}(e^2 - 1)^{\frac{3}{2}} t$$

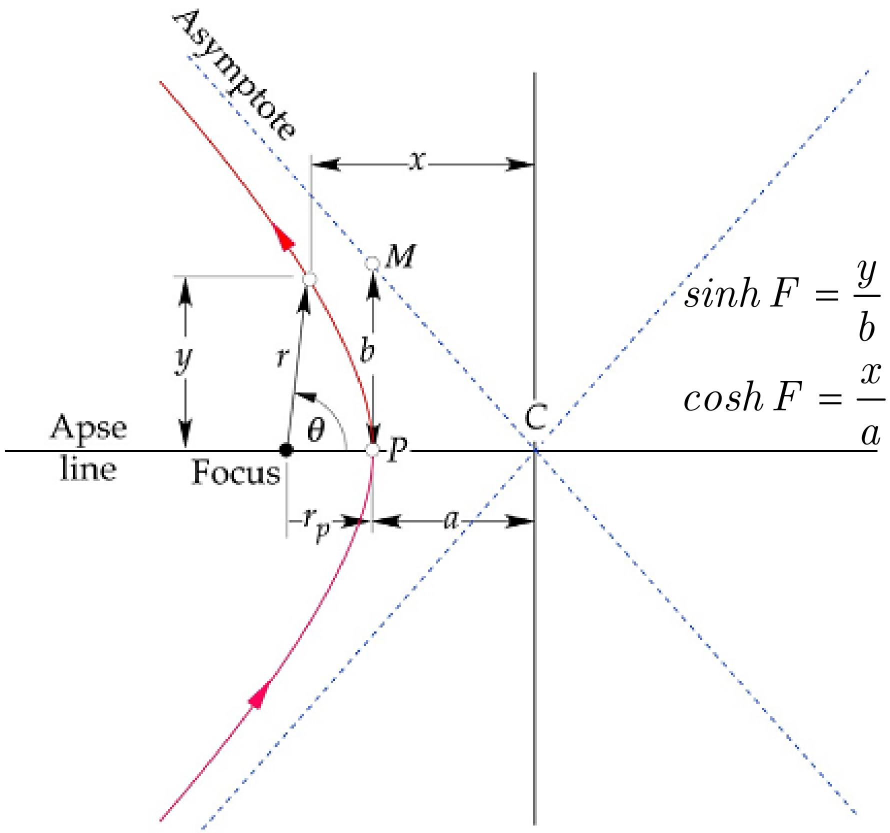
$$\tanh\frac{F}{2} = \sqrt{\frac{e-1}{e+1}}\tanh\frac{\theta}{2}$$

4(c) -Indirect Problem

Hyperbolic Orbits position from time via Newton's Root Finding Method

$$f(F) = e\sinh F - F - M_h = 0 \Rightarrow F \Rightarrow \theta$$

$$\tanh\frac{F}{2} = \sqrt{\frac{e-1}{e+1}}\tanh\frac{\theta}{2}$$



$$\sinh F = \frac{y}{b}$$

$$\cosh F = \frac{x}{a}$$

Example 10

A meteorite has been spotted to approach the Earth at an **altitude** of 37000 km with a speed of 8 km/s. if at the time of initial observation its flight path is -65 degrees,

1- investigate the possibility of its encounter with the Earth or find the closest approach distance.

2- Time to closest approach, if there is no impact.

1-Impact Check

$$\varepsilon=\frac{v^2}{2}-\frac{\mu}{r}=22.81>0 \Rightarrow \text{Hyperbolic Trajectory}$$

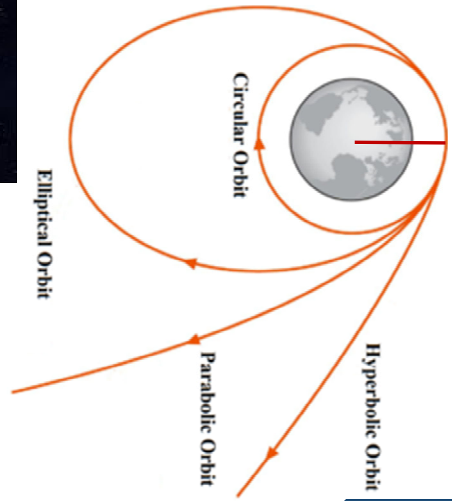
$$h=rv\cos\gamma=1.467e+5\ km^2/s$$

$$\varepsilon=-\frac{1}{2}\frac{\mu^2}{h^2}(1-e^2) \Rightarrow e=2.678$$

$$P=\frac{h^2}{\mu}=53961$$

$$r_p=P/(1+e)=14671.2>R_E \Rightarrow \text{No Encounter!}$$

$$\text{Closest Approach}=r_p-R_E=5605\ km$$



$$r=\frac{53961}{1+2.678\cos\theta}$$

Example 10

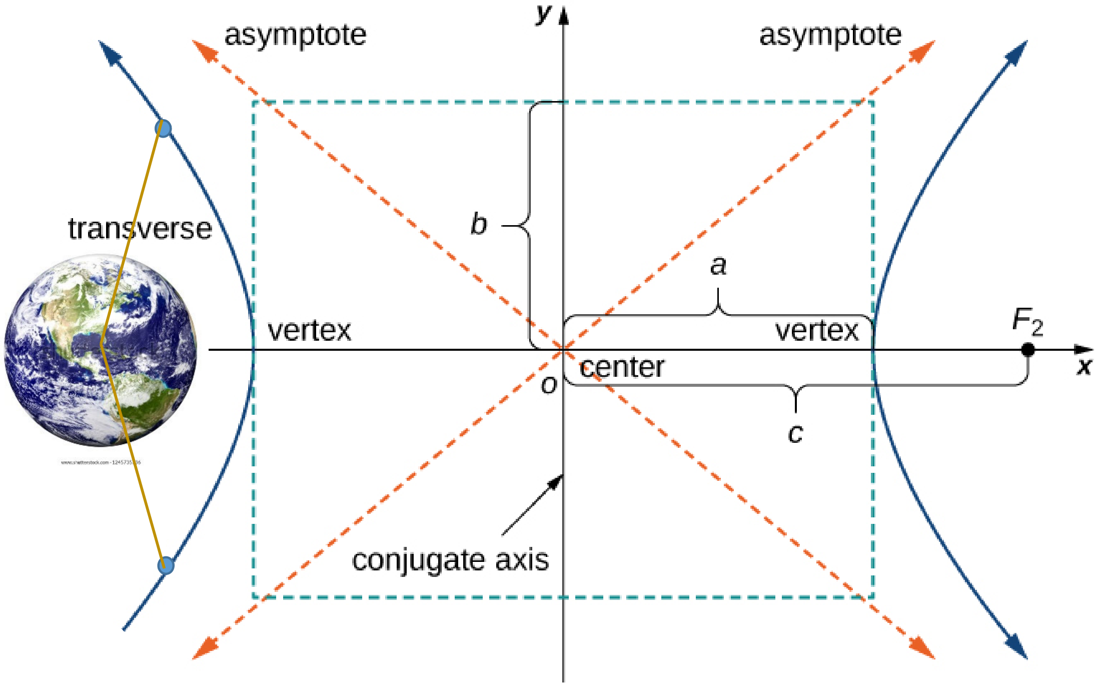
2-Time to closest approach

$$r=\frac{53961}{1+2.678\cos\theta}=37000+6378.145 \Rightarrow \theta=84.77 \text{ or } -84.77$$

$$\tanh\frac{F}{2}=\sqrt{\frac{e-1}{e+1}}\tanh\frac{\theta}{2} \Rightarrow \tanh\frac{F}{2}\sqrt{\frac{e+1}{e-1}}=\tanh\frac{\theta}{2} \Rightarrow F=1.44Rad$$

$$M_h=e\sinh F-F=3.886$$

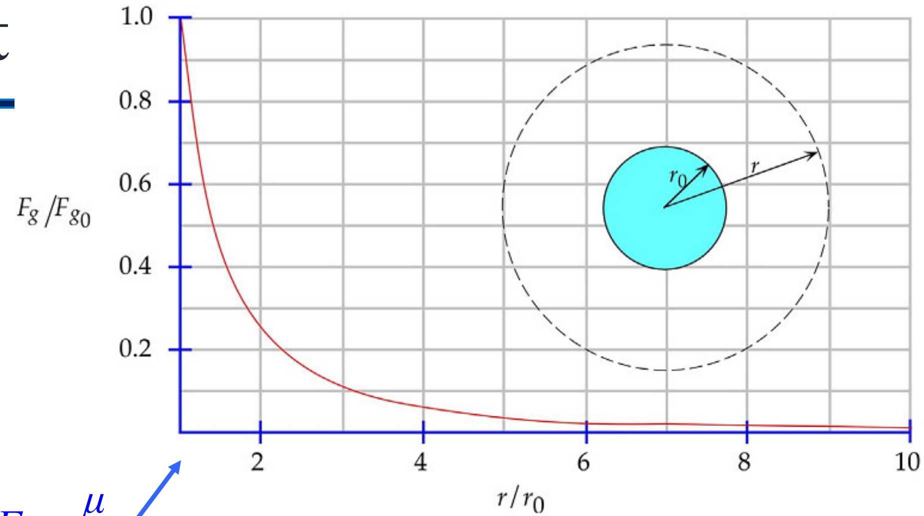
$$M_h=\frac{\mu^2}{h^3}(e^2-1)^{\frac{3}{2}}t \Rightarrow t=5032.8=1hr+24min$$



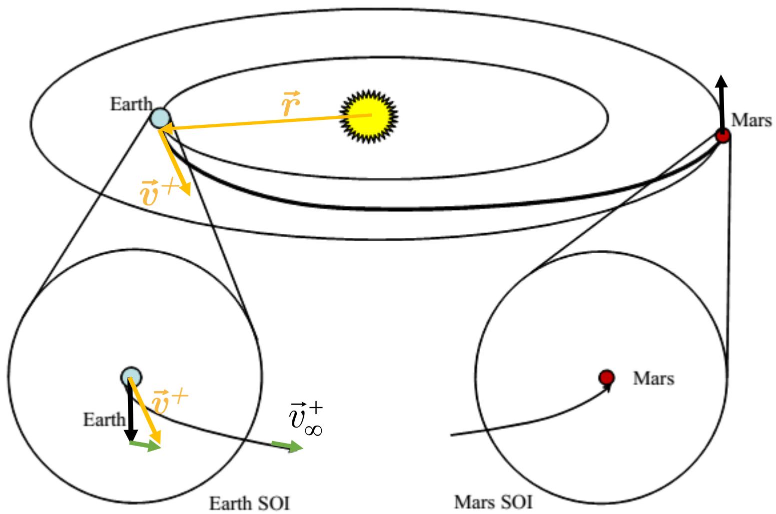
Sphere of Influence (SOI) Concept

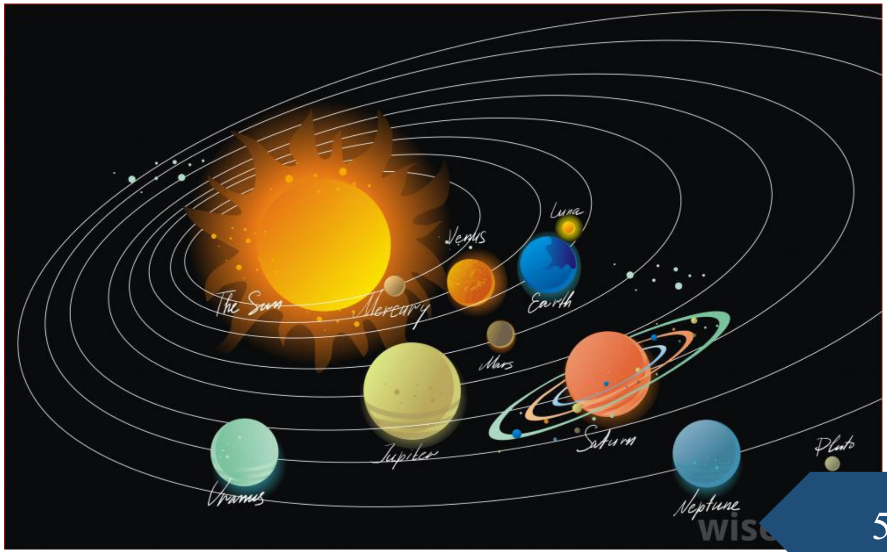
The Sun is the dominant mass within the solar system. It is more than 300,000 times heavier than the Earth.

However, to determine the domain of attraction of each planet within the solar system, an sphere of influence (SOI) is defined as a region around each celestial body where the primary gravitational influence on any orbiting object is that of the celestial body.



$$F_g = \frac{\mu}{r^2}$$



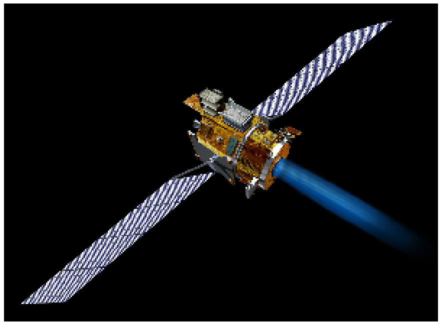


Method of Patched Conics for Interplanetary Transfer

Celestial Body	SOI radius (10^6 km)	SOI radius /(Celestial body radii)
Mercury	0.112	46
Venus	0.616	102
Earth	0.925	145
Moon	0.0661	38
Mars	0.578	170
Jupiter	48.2	687
Saturn	54.5	1025
Uranus	51.9	2040
Neptune	86.8	3525

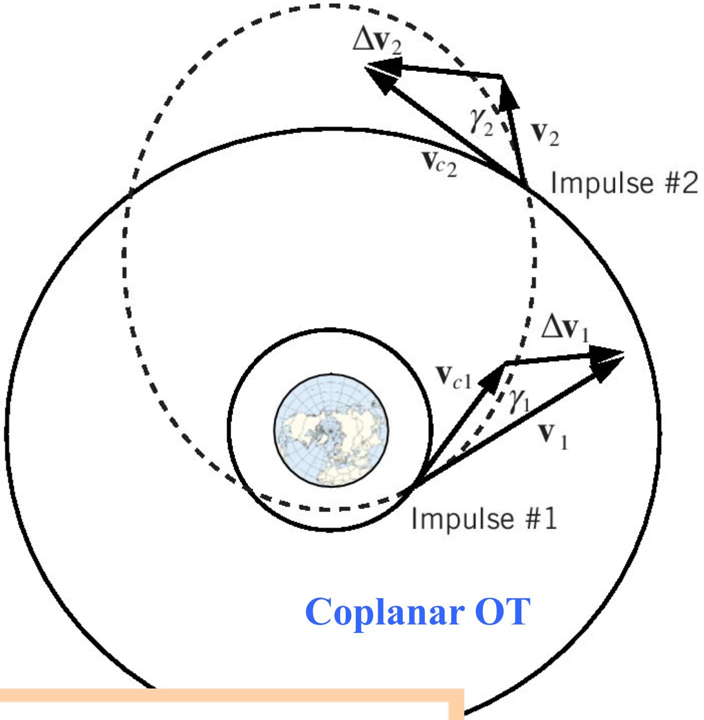
Orbital transfer (OT) or Maneuvers

Orbital maneuvers are performed to transfer a spacecraft form one orbit to another using a velocity change (ΔV). Impulsive Orbital transfer can be justified since the thrust duration is short relative to the orbital period. A simple classification of OT methods are as follows:



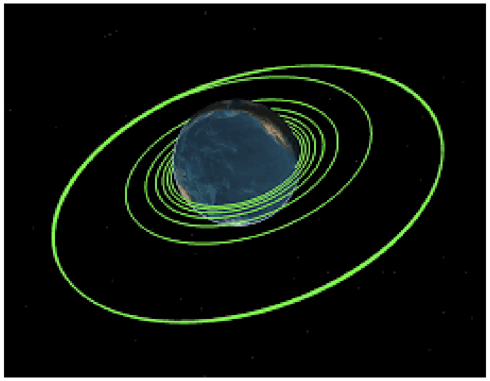
Impulsive Maneuvers

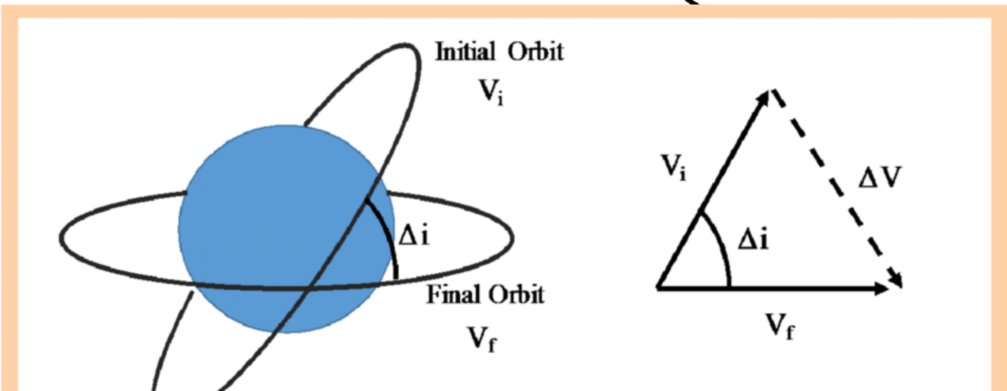
- Hohmann Transfer
- Non-Hohmann Transfers
- Phasing Maneuvers
- Plane Change Maneuvers



Coplanar OT

Non-impulsive Orbital Maneuvers





Plane Change OT

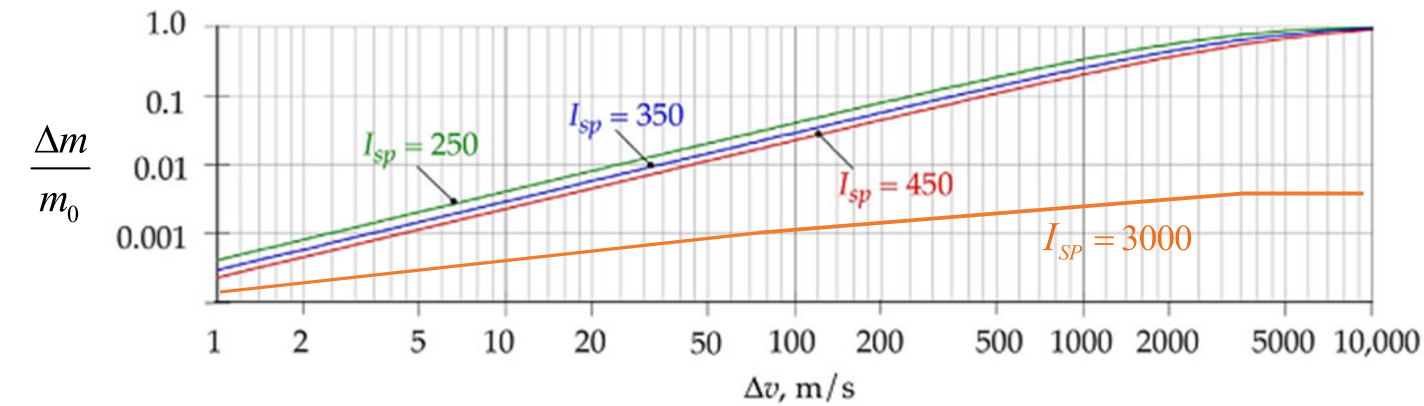
Impulsive Maneuvers

Impulsive maneuvers are those in which brief firings of onboard rocket motors change the magnitude and direction of the velocity vector instantaneously. During an impulsive maneuver, the position of the spacecraft is considered to be fixed; only the velocity changes. The magnitude ΔV of the velocity increment is related to Δm that is equal to Initial mass minus the final mass (i.e. the mass of propellant consumed), via the ideal rocket equation (Tsiolkovsky rocket equation). Where Isp is the specific impulse of the propellants as defined below.

$$\frac{\Delta m}{m_0} = 1 - e^{-\frac{\Delta V}{I_{sp} g_0}} \Rightarrow \Delta V = I_{SP} g_0 \ln\left(\frac{m_0}{m_f}\right) \cancel{-\Delta V_D - \Delta V_g}$$

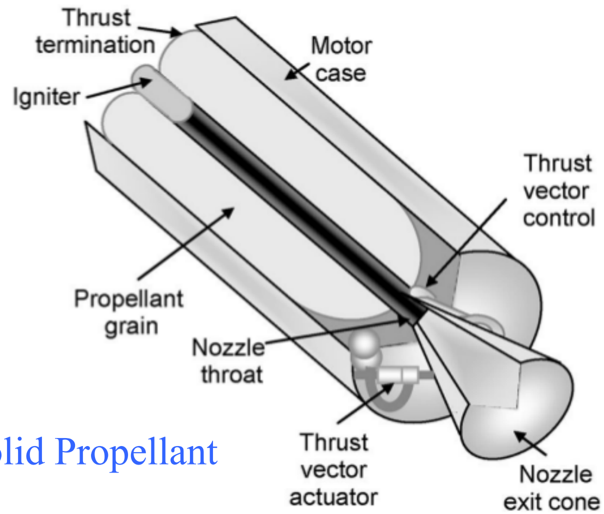
$$I_{SP}(\textit{Specific impulse}) = \frac{\textit{Thrust}}{\textit{Sea level weight rate of fuel consumption}} \textit{ in Sec.} = \frac{T}{\dot{m}_f g_0}$$

g_o: *the sea-level standard acceleration of gravity;* $\Delta m = m_0 - m_f$



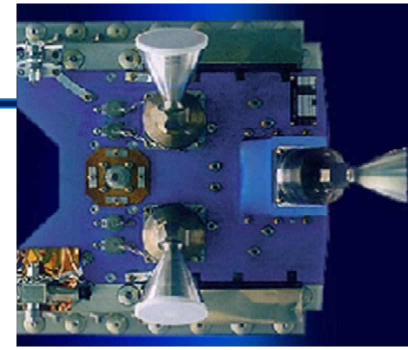
Propellant	I_{sp} (s)
Cold gas	50
Monopropellant hydrazine	230
Solid propellant	290
Nitric acid/monomethylhydrazine	310
Liquid oxygen/liquid hydrogen	455
Ion propulsion	>3000

Impulsive Maneuvers: Typical Thrusters

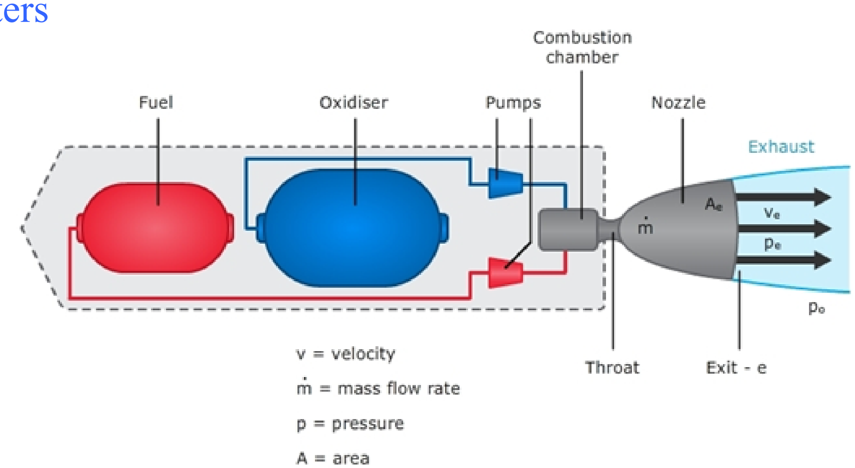


Solid Propellant

Cold Gas Thrusters

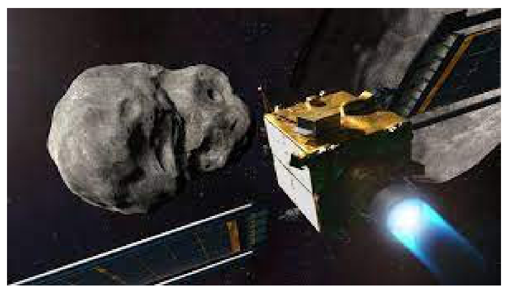


1-400N Hydrazine Thrusters

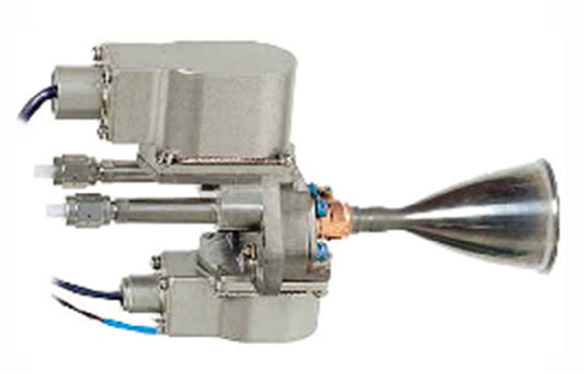


Thrust = F = $\dot{m}\, v_e + (p_e - p_o)\, A_e$

Liquid Propellant



Electric or Ion Thrusters



10N Bipropellant Thrusters

https://www.space-propulsion.com/spacecraft-propulsion/apogee-motors/index.html

Impulsive Maneuvers: Typical Thrusters

Propulsion methods

Method	Effective exhaust velocity (km/s)	Thrust (N)	Firing duration	Maximum delta-v (km/s)	Technology readiness level
Solid-fuel rocket	<2.5	<10^7	Minutes	7	9: Flight proven
Monopropellant rocket	1 – 3[35]	0.1 – 400[35]	Milliseconds – minutes	3	9: Flight proven
Liquid-fuel rocket	<4.4	<10^7	Minutes	9	9: Flight proven
Hybrid rocket	<4		Minutes	>3	9: Flight proven

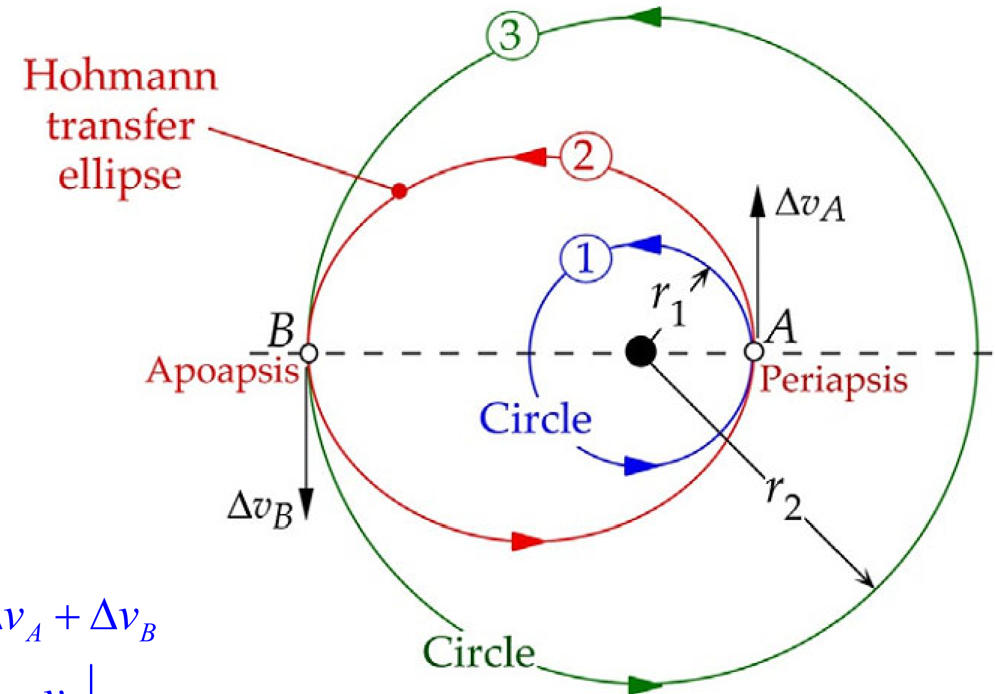
Technology Readiness Levels (TRL) are a method used to measure and assess the maturity of a particular technology.

TRL	Current NASA usage
1	Basic principles observed and reported
2	Technology concept and/or application formulated
3	Analytical and experimental critical function and/or characteristic proof-of concept
4	Component and/or breadboard validation in laboratory environment
5	Component and/or breadboard validation in relevant environment
6	System/subsystem model or prototype demonstration in a relevant environment (ground or space)
7	System prototype demonstration in a space environment
8	Actual system completed and "flight qualified" through test and demonstration (ground or space)
9	Actual system "flight proven" through successful mission operations

https://en.wikipedia.org/wiki/Spacecraft_propulsion

Hohmann Transfer (HT)

Hohmann transfer is the most energy-efficient two-impulse maneuver for transferring between two coplanar circular orbits sharing a common focus. The Hohmann transfer elliptic properties can be computed as follows:



Hohmann transfer can in general be used for transfer between any two coplanar closed orbits sharing a common focus.

$$r_a = \frac{P}{1-e};\ r_p = \frac{P}{1+e} \Rightarrow e = \frac{r_a - r_p}{r_a + r_p}$$

$$r_p = \frac{h^2}{\mu}\frac{1}{1+e} = \frac{h^2}{\mu}\frac{1}{1+\frac{r_a - r_p}{r_a + r_p}} = \frac{h^2}{\mu}\frac{r_a + r_p}{2r_a}$$

$$\Rightarrow h = \sqrt{2\mu}\sqrt{\frac{r_a r_p}{r_a + r_p}} \Rightarrow v_a; v_P$$

$$\Delta v_{total} = \Delta v_A + \Delta v_B$$

$$\Delta v_A = \left| v_{C_1} - v_P \right|$$

$$\Delta v_B = \left| v_a - v_{C_3} \right|$$

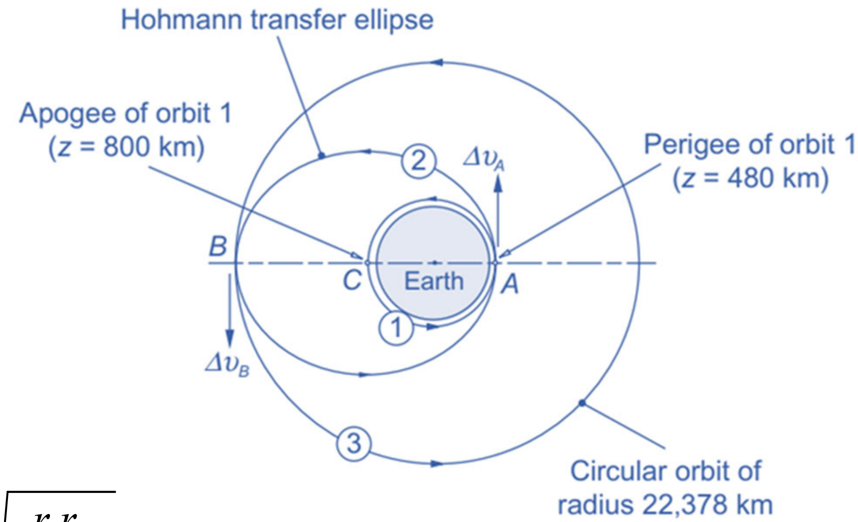
Hohmann Transfer (HT) Example 11

Consider a 2000 kg spacecraft in a 480x800 km (altitude) Earth orbit, Find:

1- The Δv needed at Perigee point A of orbit 1, to place the spacecraft in a 480x16,000 km HT transfer ellipse.

2- The Δv (Apogee kick) required at B of the HT transfer orbit to establish a circular orbit of 16,000 km altitude (orbit 3).

3) The total required propellant mass needed for this transfer, if the specific impulse is 300 s.



$$e_T = \frac{r_a - r_p}{r_a + r_p}; \quad h_T = \sqrt{2\mu}\sqrt{\frac{r_a r_p}{r_a + r_p}} \Rightarrow v_a; v_P$$

$$\Delta v_A = |v_1 - v_P|; \; \Delta v_B = |v_a - v_{C_3}|$$

$$\Delta v_{total} = \Delta v_A + \Delta v_B$$

$$\frac{\Delta m}{m_0} = 1 - e^{-\frac{\Delta V}{I_{sp} g_0}}$$

$$\Delta m = m_0 - m_f$$

Hohmann Transfer (HT) Example 11

$Orbit\ 1:$

$r_p = 6378 + 480 = 6858km \quad r_a = 6378 + 800 = 7178km$

$\Rightarrow h_1 = \sqrt{2 \times 398,600}\sqrt{\dfrac{7178 \times 6858}{7178 + 6858}} = 52,876.5km/s^2$

$Orbit\ 2:$

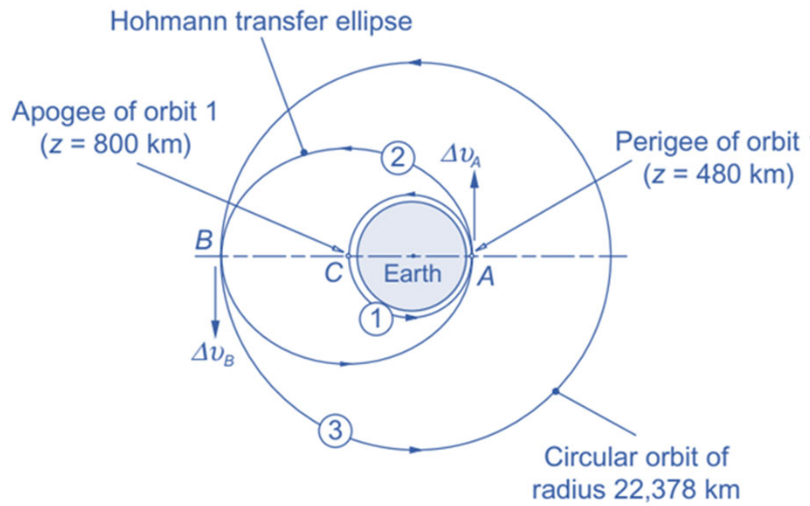
$r_p = 6378 + 480 = 6858km \quad r_a = 6378 + 16,000 = 22,378km$

$\Rightarrow h_2 = \sqrt{2 \times 398,600}\sqrt{\dfrac{22,378 \times 6858}{22,378 + 6858}} = 64,689.5km/s^2$

$v_A)_1 = v_1 = \dfrac{h_1}{r_A} = \dfrac{52,876}{6858} = 7.71019km/s$

$v_A)_2 = v_P = \dfrac{h_2}{r_A} = \dfrac{64,689.5}{6858} = 9.43271km/s$

$\Delta v_A = v_P - v_1 = 1.7225km/s$



$h = \sqrt{2\mu}\sqrt{\dfrac{r_a r_p}{r_a + r_p}} \Rightarrow v_a; v_P$

Hohmann Transfer (HT) Example 11

$Orbit\ 3:$

$r_a = r_p = P = 22,378km$

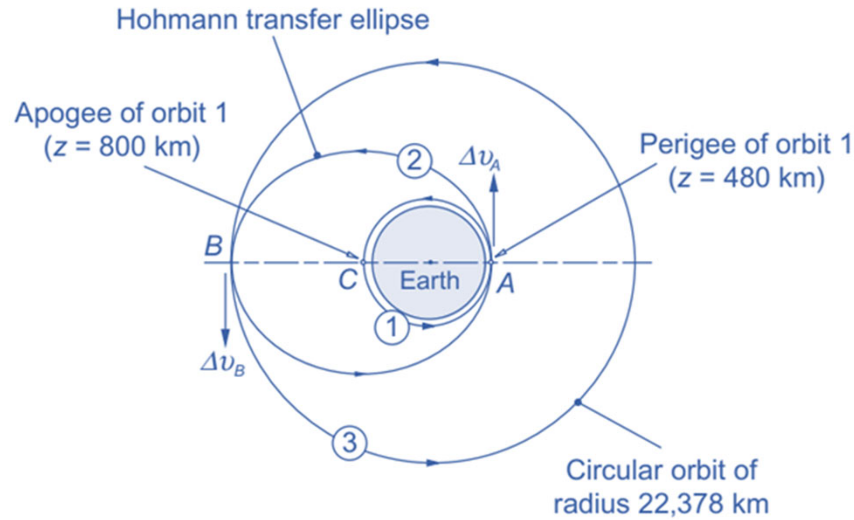
$\Rightarrow h_3 = \sqrt{398,600 \times 22,378} = 94,445.1km/s^2$

$v_B)_2 = v_a = \dfrac{h_2}{r_B} = \dfrac{64,689.5}{22,378} = 2.89076km/s$

$v_B)_3 = v_{C_3} = \dfrac{h_3}{r_B} = \dfrac{94,445.1}{22,378} = 4.22044km/s$

$\Delta v_B = v_B)_3 - v_B)_2 = 1.3297km/s$

$\Delta v_{total} = |\Delta v_A| + |\Delta v_B| = 1.7225 + 1.3297 = 3.0522km/s$



$\Delta v_B = |v_a - v_{C_3}|$

$\Delta v_{total} = \Delta v_A + \Delta v_B$

$\dfrac{\Delta m}{m_0} = 1 - e^{-\frac{\Delta V}{I_{sp} g_0}}$

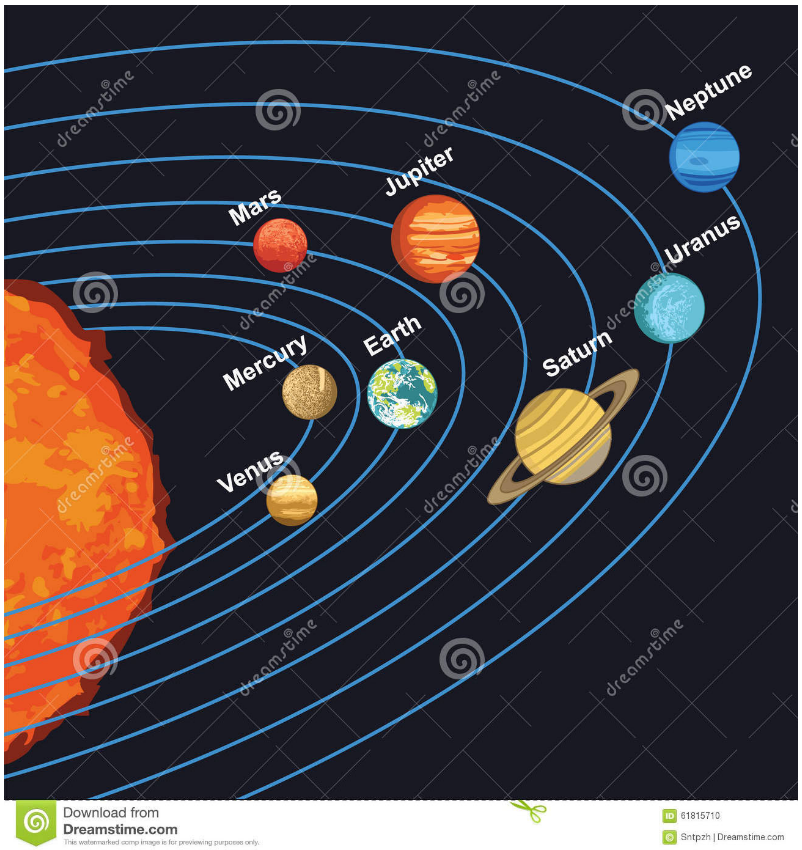
$\Delta m = m_0 - m_f$

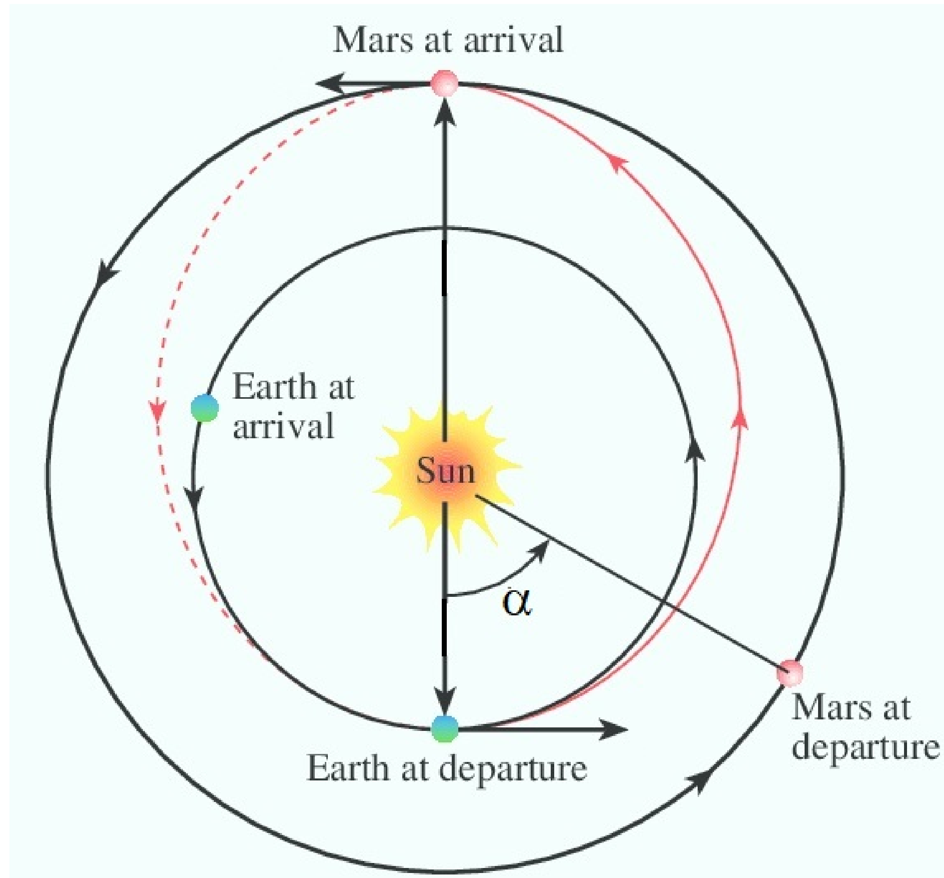
$\dfrac{\Delta m}{m_0} = 1 - e^{-\frac{3052.2}{300 \times 9.807}} = 0.64563$

$\Delta m = 0.64563 \times 2000 = 1291.3kg$

Hohmann Transfer for interplanetary missions

Hohmann transfer can also be used by proper planning (launch window) to send a spacecraft from Low Earth (LEO) orbit to Mars or other planets .





Bi-elliptic Hohmann Transfer (BE)

The **HT** between two circular orbits is efficient as long as the ratio of the bigger to smaller radius is smaller than 11.94 . In contrast Bi-elliptic Hohmann Transfer (**BE)**, as a three impulse transfer is proposed for cases that contradict the above assumption. The idea behind **BE** is to place point B sufficiently far from the focus such that the required Δv_B to put the spacecraft on the target orbit (4) will be very small. It can be shown:

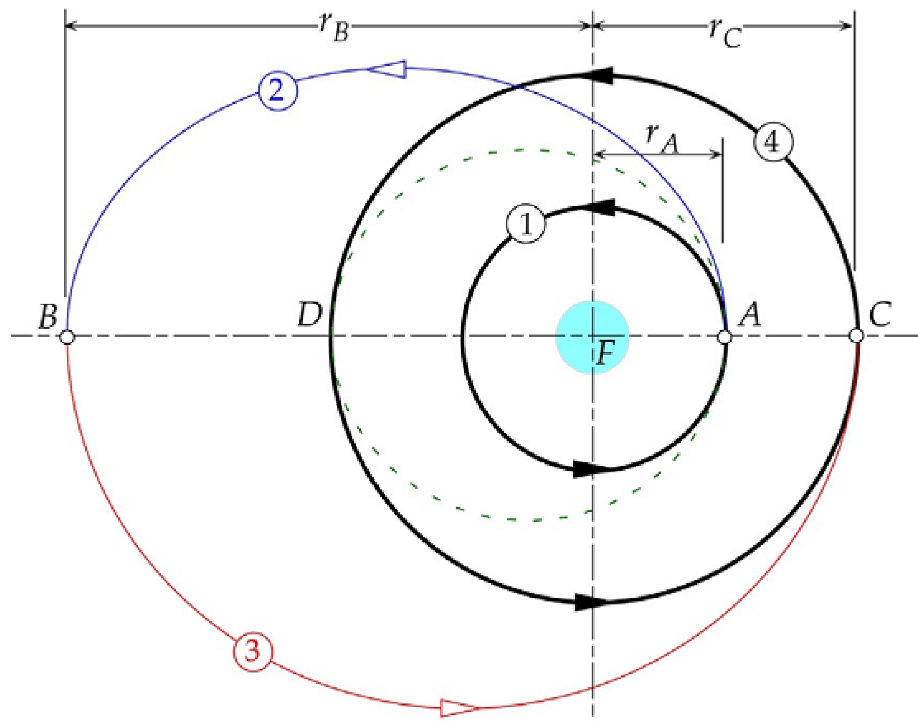
$$v_0 = \sqrt{\frac{\mu}{r_A}} \ ; \ speed\ \ in\ \ circular\ \ inner\ \ orbit\ 1$$

$$\Delta \bar{v}_H = \frac{1}{\sqrt{\alpha}} - \frac{\sqrt{2}(1-\alpha)}{\sqrt{\alpha(1+\alpha)}} - 1$$

$$\Delta \bar{v}_{BE} = \sqrt{\frac{2(\alpha+\beta)}{\alpha\beta}} - \frac{1+\sqrt{\alpha}}{\sqrt{\alpha}} - \sqrt{\frac{2}{\beta(1+\beta)}}(1-\beta)$$

$where:$

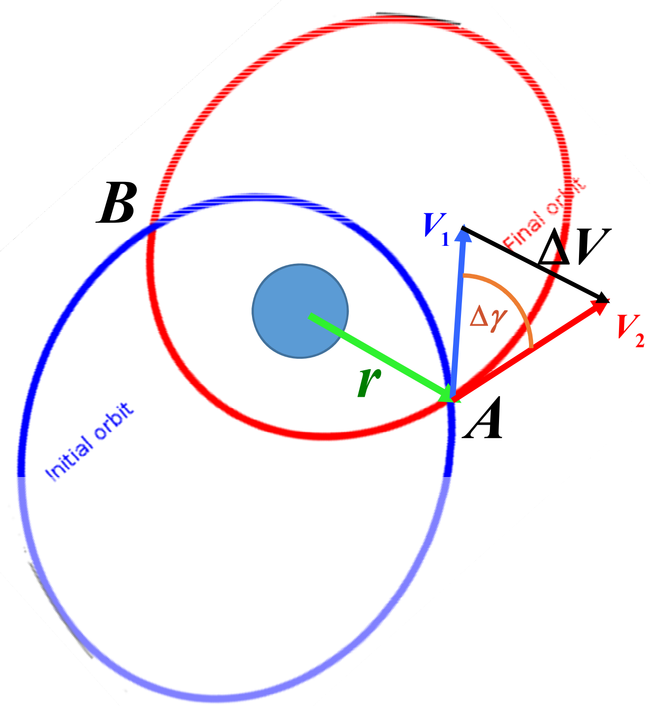
$$\alpha = \frac{r_C}{r_A} \qquad \beta = \frac{r_B}{r_A}$$



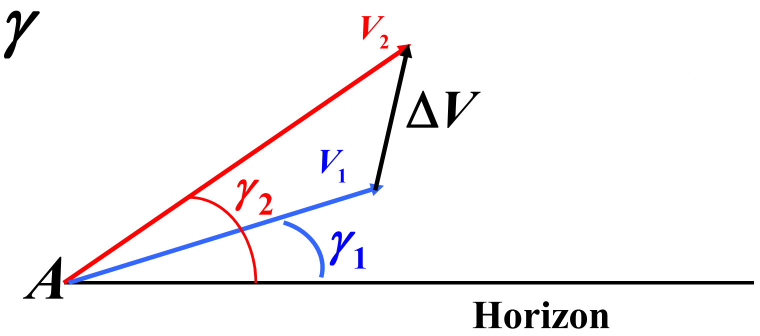
1: Initial Orbit
4: Target orbit
2:Auxililary orbit
3:Auxiliary orbit

Non-Hohmann Transfers

Single Impulsive transfer between two intersecting coplanar orbits is possible by computing the required ΔV to switch from the initial orbit to final orbit at one of the intersection points. Computation of ΔV is possible if the two orbit polar equations or properties are provided .



$$\Delta V^2 = V_1^2 + V_2^2 - 2V_1V_2\cos\Delta\gamma$$
$$where: \Delta\gamma = \gamma_2 - \gamma_1$$



Example 11

Consider two coplanar Earth orbits with the given properties. It is desired to transfer from orbit 1 at the true anomaly of 150 deg. to orbit 2 , find the required ΔV for this transfer.

Orbit 1 : $P_1 = 7200 Km; \quad e_1 = 0.5$

Orbit 2 : $a_2 = 1.5 a_1; \quad e_2 = 0.5$

$$P_1 = a_1(1-e_1^2) \Rightarrow a_1 = 9600 Km \Rightarrow a_2 = 14400 Km$$

$$P_2 = a_2(1-e_2^2) = 10800, \text{ at the intersection point } r_1 = r_2 = \frac{7200}{1+0.5\cos 150} = 12698.7 Km$$

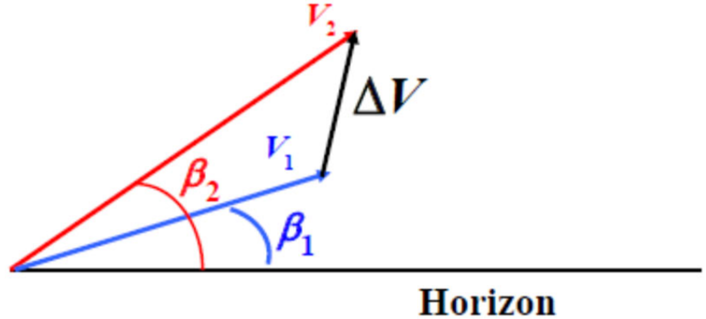
$$r_2 = 12698.7 = \frac{10800}{1+0.5\cos\theta_2} \Rightarrow \theta_2 = 107.4 \text{ deg}$$

$$h_1 = r_1 v_1 \cos\beta_1 = (P_1\mu)^{1/2} \Rightarrow \beta_1 = 23.76^\circ$$

$$h_2 = r_2 v_2 \cos\beta_2 = (P_2\mu)^{1/2} \Rightarrow \beta_2 = 29.35^\circ$$

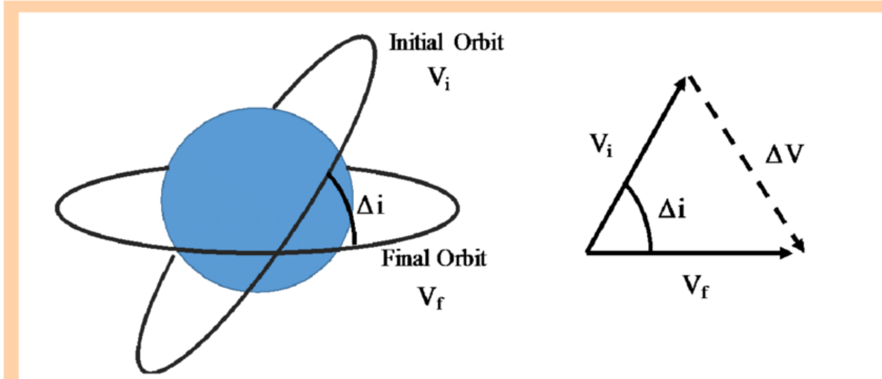
$$\varepsilon = \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a} \Rightarrow v_1 = 4.61 km/s \text{ and } v_2 = 5.92 km/s$$

$$\Delta V^2 = V_1^2 + V_2^2 - 2V_1V_2\cos(\beta_2 - \beta_1) \Rightarrow \Delta V = 1.41 km/s$$



Plane Change Maneuvers (PCM) between intersecting orbits

Single Impulsive transfer between two intersecting Non-coplanar circular orbits is possible by computing the required ΔV to switch from the initial orbit to final orbit at one of the intersection points. If the initial and final orbits are not intersecting two impulsive ΔV will be needed.



The thruster firing direction is important.

$$\Delta V^2 = V_i^2 + V_f^2 - 2V_iV_f \cos \Delta i$$

Phasing Maneuvers for Rendezvous

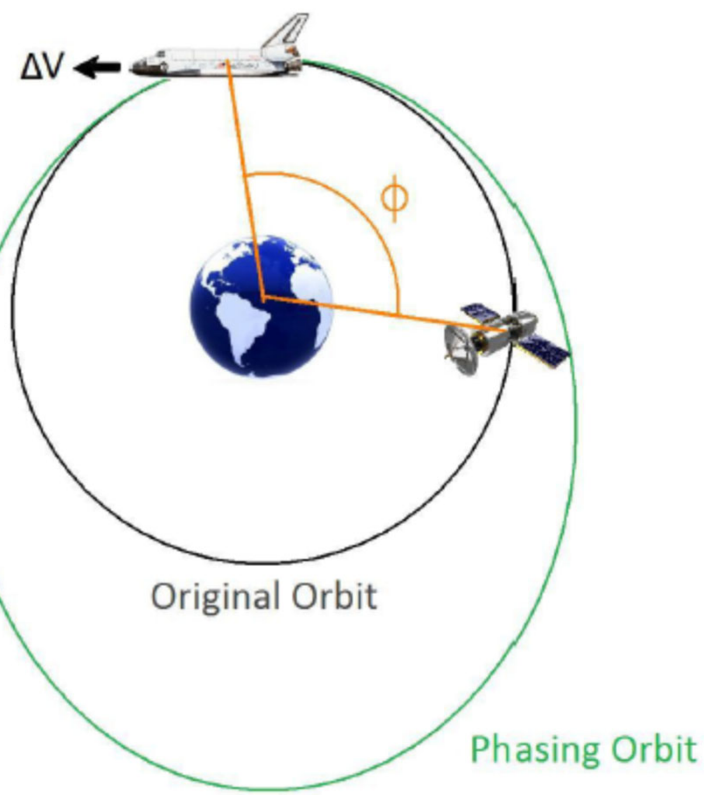
Phasing maneuvers are performed to change the size of the original orbit for a few periods in order to meet back the original orbit at a different point in time.

In this sense, the phasing maneuver has two burns and thus can be categorized as a two-impulse Hohmann transfer from and back to the same orbit. The first burn occurs at the original orbit. This shrinks or expands the orbit from its original state. Once the spacecraft completes the **transfer orbit** and returns to its original burn point, it then performs its second maneuver to coincide with its original orbit.

There are several utilities for phasing maneuvers.

1- If the main spacecraft needs to rendezvous with another spacecraft behind it, it would increase its period just enough to intercept the other spacecraft.

For example as shown in the picture, the space shuttle uses a phasing maneuver to rendezvous with a satellite that is Φ degrees behind it.



Phasing Maneuver Diagram

Phasing Maneuvers for Rendezvous

2- If it is needed to catch up with another spacecraft ahead of it, the main spacecraft would slows down to decrease its period.

3- the Phasing maneuvers is also used for station keeping of GEO satellites (East-West correction) or to change their observation area . There are basically only a few steps in computation of the phasing maneuver:

I. Determination of the phasing orbit Period,(τ or T).

This is determined by computing how far ahead or behind the target is in time.

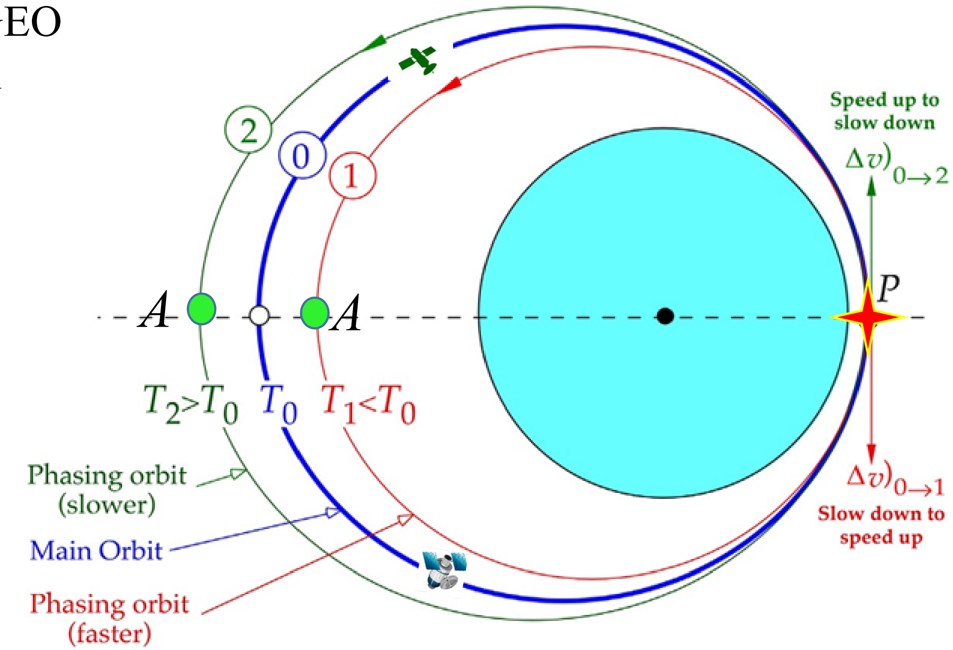
II. Determined the SMA of the Phasing orbit, knowing τ.

$$a = \left(\frac{\tau\sqrt{\mu}}{2\pi}\right)^{2/3}$$

III. Calculate the Δv needed to perform the maneuver.

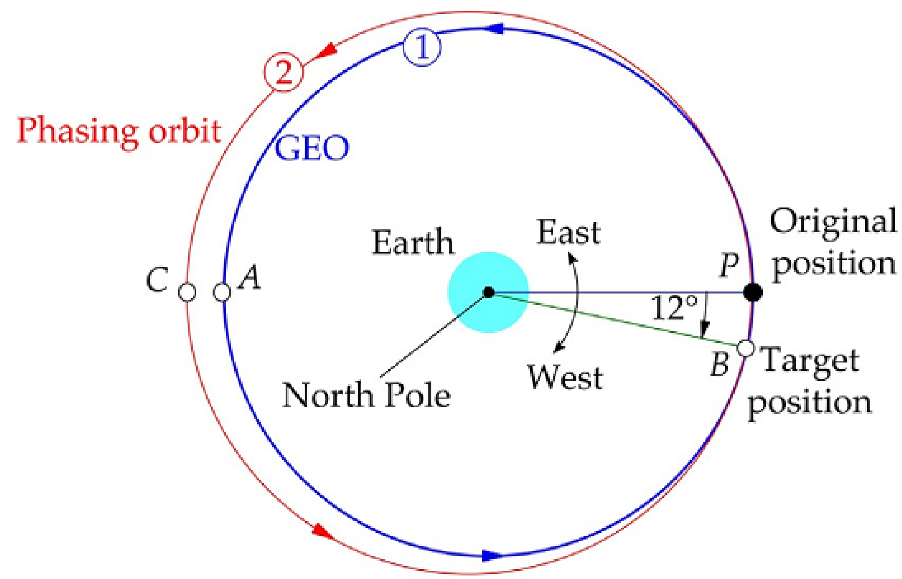
$$2a = r_P + r_A$$

$$h = \sqrt{2\mu}\sqrt{\frac{r_A r_P}{r_A + r_P}} \Rightarrow v_A; v_P$$



Example 12

It is desired to shift the longitude of a GEO satellite 12° westward in three revolutions of its phasing orbit. Calculate the delta-v requirement.



$$\omega_E = \omega_{\text{GEO}} = 72.922\left(10^{-6}\right)\text{rad/s}$$
$$r_{\text{GEO}} = 42,164\,\text{km}$$
$$V_{\text{GEO}} = 3.0747\,\text{km/s}$$

$$\omega_E(3T_2) = 3 \cdot 2\pi + \Delta\Lambda$$

$$T_2 = \frac{1}{3}\frac{\Delta\Lambda + 6\pi}{\omega_E} = \frac{1}{3}\frac{12^\circ \cdot \dfrac{\pi}{180^\circ} + 6\pi}{72.922 \times 10^{-6}} = 87,121\,\text{s}$$

Example 12

$$T_{\text{GEO}} = \frac{2\pi}{\omega_{\text{GEO}}} = 86,163\text{s}$$

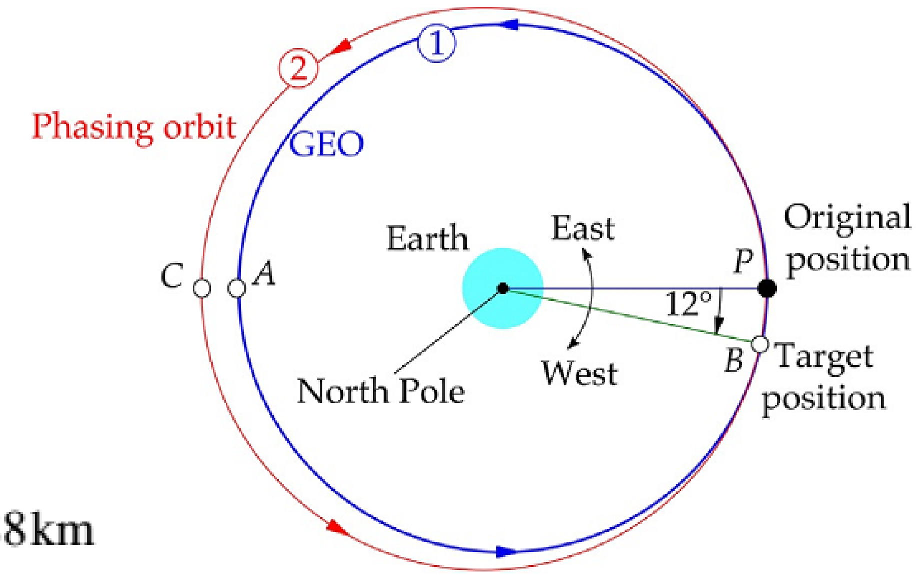
$$\dot{\Lambda} = \frac{\Delta\Lambda}{3T_2} = 8.0133 \times 10^{-7}\,\text{rad/s} = 3.9669\text{degrees/day}$$

$$a_2 = \left(\frac{T_2\sqrt{\mu}}{2\pi}\right)^{2/3} = \left(\frac{87,121\sqrt{398,600}}{2\pi}\right)^{2/3} = 42,476\text{km}$$

$$2a_2 = r_p + r_C \quad \Rightarrow \quad r_C = 2 \cdot 42,476 - 42,164 = 42,788\text{km}$$

$$h_2 = \sqrt{2\mu}\sqrt{\frac{r_B r_C}{r_B + r_C}} = \sqrt{2 \cdot 398,600}\sqrt{\frac{42,164 \cdot 42,748}{42,164 + 42,788}} = 130,120\text{km}^2/\text{s}$$

$$v_P)_2 = \frac{130,120}{42,164} = 3.0859\text{km/s}$$

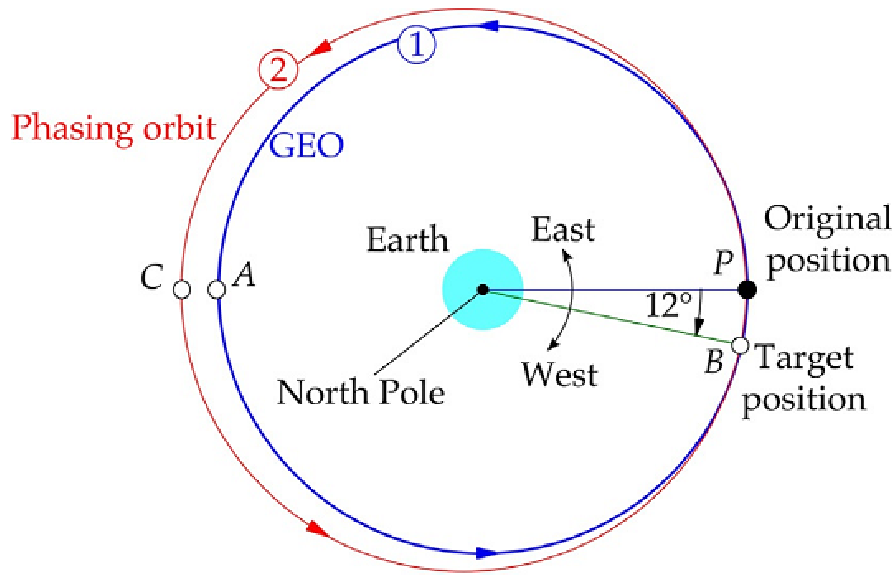


Example 12

$$\Delta v = v_P)_2 - v_{GEO} = 3.0859 - 3.0747 = 0.01126 \text{km/s}$$

$$\Delta v = v_{GEO} - v_P)_2 = 3.0747 - 3.08597 = -0.01126 \text{km/s}$$

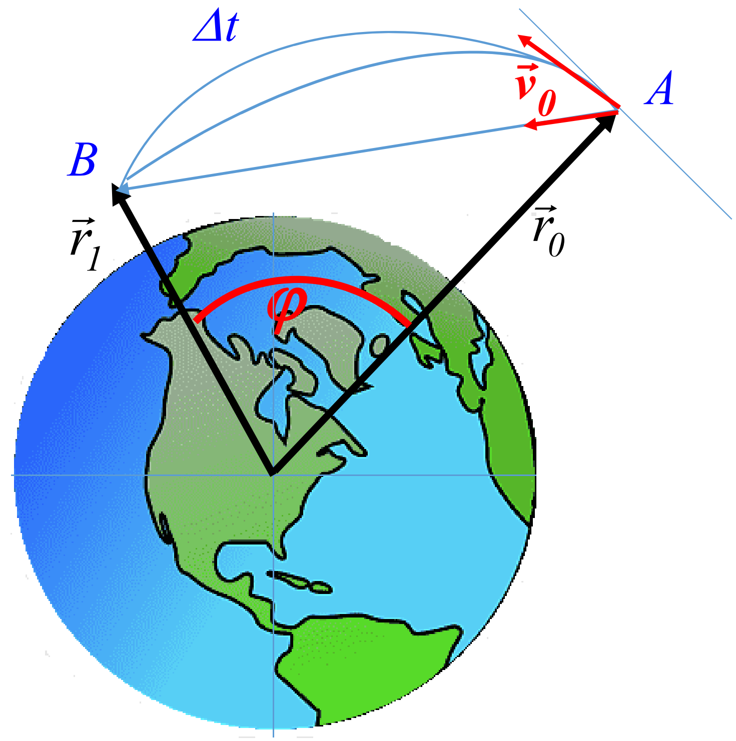
$$\Delta v_{total} = |0.01126| + |-0.01126| = \boxed{0.02252 \text{km/s}}$$



Chasing Maneuver and the Lambert Problem

Occasionally we are interested to know how we can have a **ballistic** flight from point A to point B in space in a given or specified time. The Lambert's problem addresses this issue.

Problem Statement: Determine the initial **required** velocity, $\vec{v}_0$ at $\vec{r}_0$, such that the space vehicle can have a ballistic trajectory to $\vec{r}_1$ in a specified time interval Δt over a specific range parameter denoted by φ.



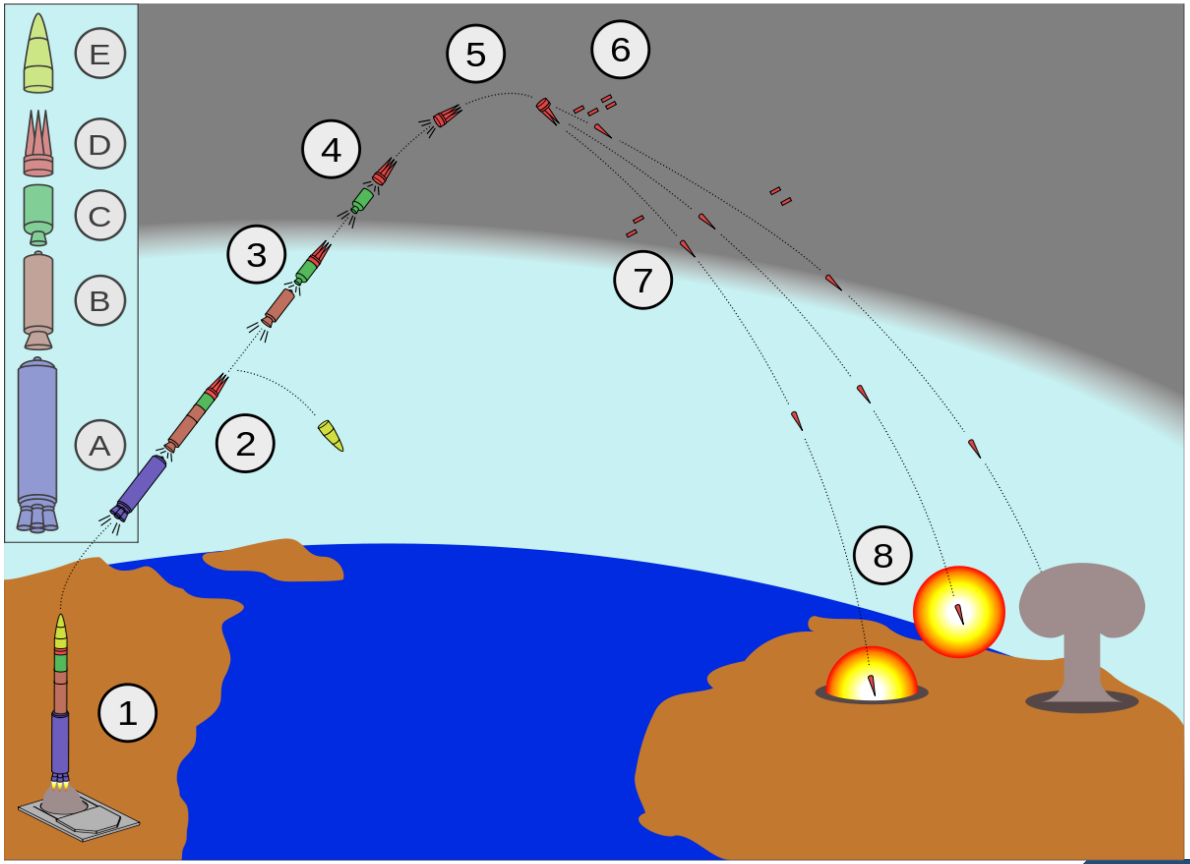
$$v_0 = \sqrt{\frac{\mu(1-\cos\varphi)}{r_0\cos\gamma_0\left[\dfrac{r_0\cos\gamma_0}{r_1} - \cos(\varphi+\gamma_0)\right]}}$$

$$\Delta t = g(r_0, r_1, \varphi, \gamma_0)$$

Note: In fact if the time of flight was not an issue, we could have **various ballistic trajectories** to reach the target vector via different initial speed and initial flight path angle. In other words :

$$\boldsymbol{v_0 = f(r_0, r_1, \varphi, \gamma_0)}$$

Various applications of ballistic Trajectories



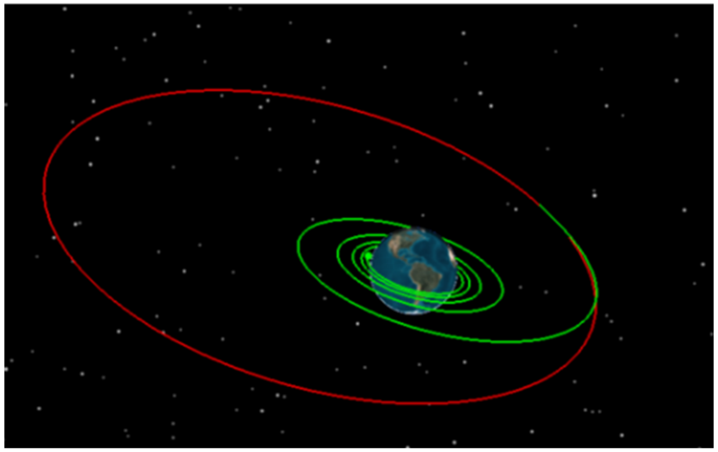
Various applications of ballistic Trajectories

Governing EOM for Continuous Thrust Non-Impulsive Orbital Maneuvers

$$\ddot{\mathbf{r}} = -\mu \frac{\mathbf{r}}{r^3} + \frac{\mathbf{F}}{m}$$

$$\ddot{\mathbf{r}} = -\mu \frac{\mathbf{r}}{r^3} + \frac{T}{m}\frac{\mathbf{v}}{v} \qquad (\mathbf{v} = \dot{\mathbf{r}})$$

$$\frac{dm}{dt} = -\frac{T}{I_{sp} g_0}$$



Accelerated Trajectories need a guidance law

$$\vec{y} = \begin{Bmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \\ m \end{Bmatrix} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{Bmatrix} \Rightarrow \dot{y} = \begin{Bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \ddot{x} \\ \ddot{y} \\ \ddot{z} \\ \dot{m} \end{Bmatrix} = \begin{Bmatrix} y_4 \\ y_5 \\ y_6 \\ -\mu \frac{y_1}{r^3} + \frac{T}{m}\frac{y_4}{v} \\ -\mu \frac{y_2}{r^3} + \frac{T}{m}\frac{y_5}{v} \\ -\mu \frac{y_3}{r^3} + \frac{T}{m}\frac{y_6}{v} \\ -\frac{T}{I_{sp} g_0} \end{Bmatrix}$$

Q-Guidance: A simple guidance strategy is to apply acceleration (i.e. engine thrust) in the direction of V_{TBG} that is the **velocity to be gained.**

$$V_{\mathrm{TBG}} = V_{\mathrm{Lambert}} - V$$

Crew exploration vehicle docking with ISS

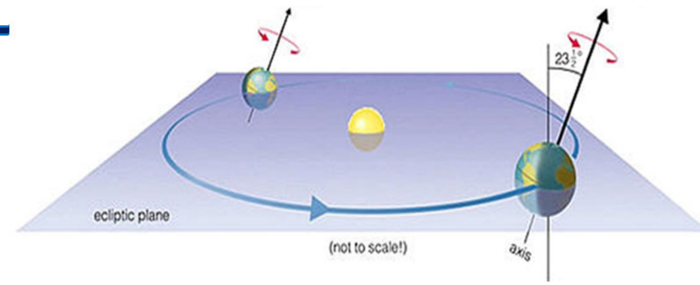
Coordinate Systems

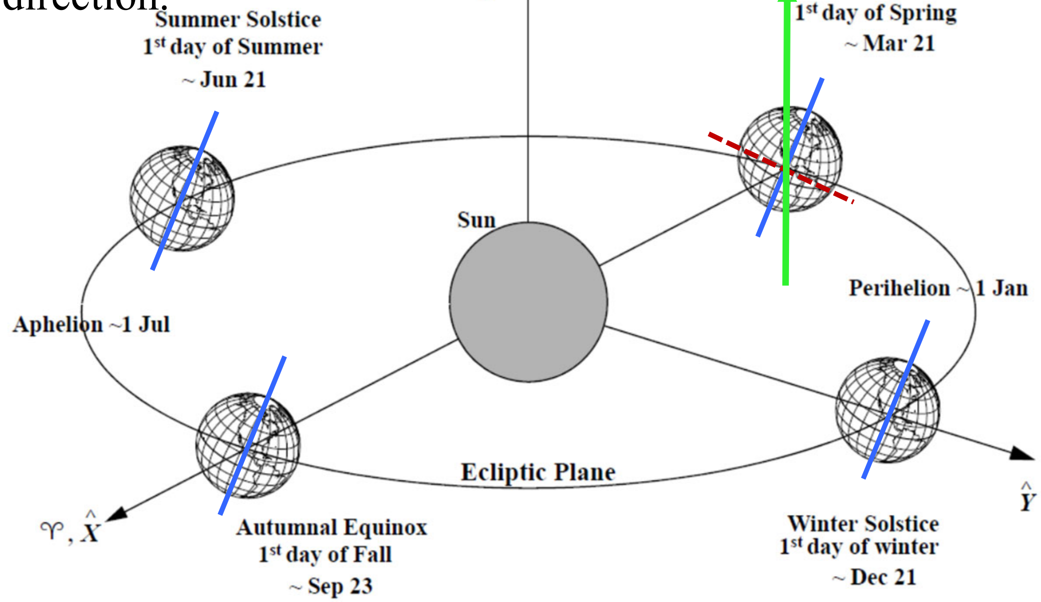
We need coordinate systems to describe and quantify motions.

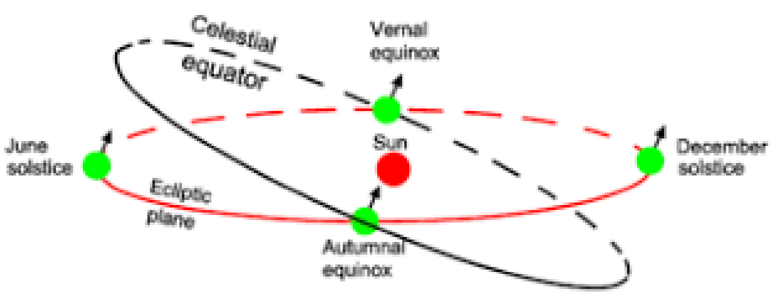
Basic laws of motion are valid in an Inertial Coordinate System.

Earth revolves around the Sun:

- In the Ecliptic Plane,
- Earth axis of rotaion has an Obliquity angle of almost ~23.45º
- Vernal Equinox (V.E.) is a reference direction.







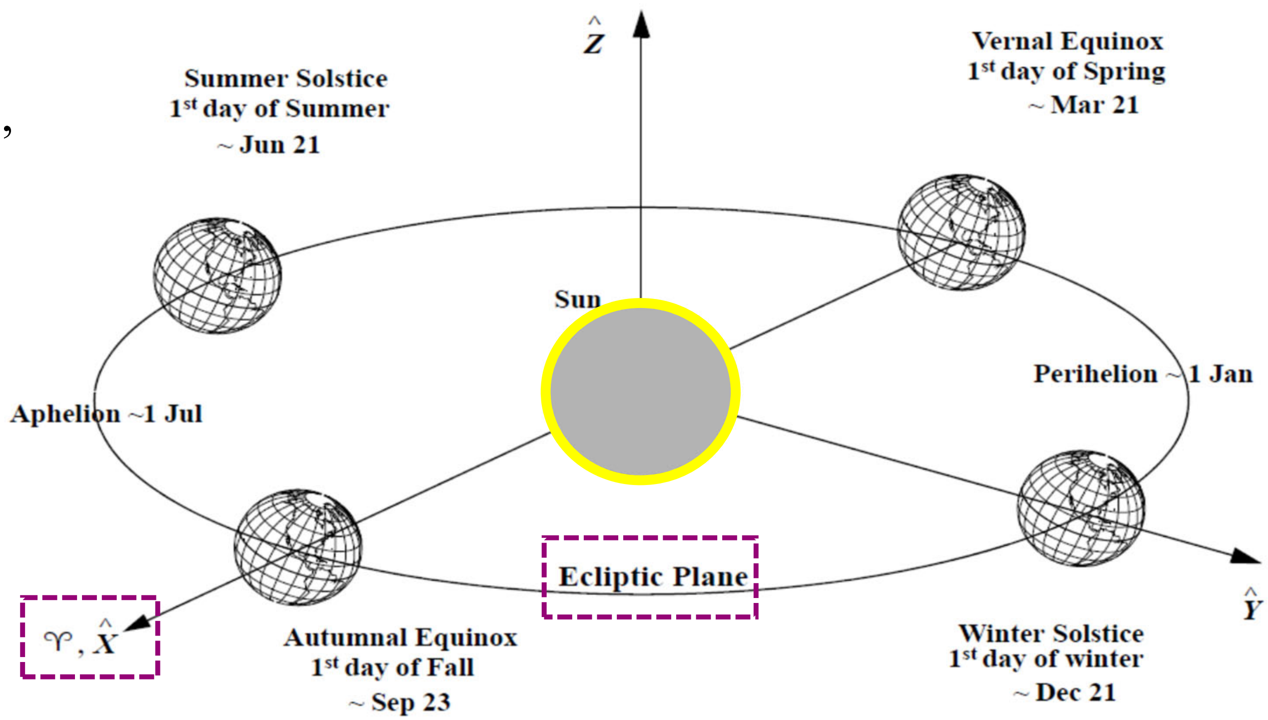
Key Reference Coordinate Systems

I. HRF (Heliocentric Reference Frame), Inertial

Origin: Sun center,

Fundamental plane: The Ecliptic plane,

Preferred direction: V.E.



Aries constellation
symbol for ram horn

Vernal equinox (V.E), there are two moments in the year when the Sun is exactly above the Equator and day and night are of equal length. On one of these moments, the line connecting the Earth (intersection of the ecliptic with the Earth equator) to Sun points toward a Celestial direction that was initially towards the Aries constellation.

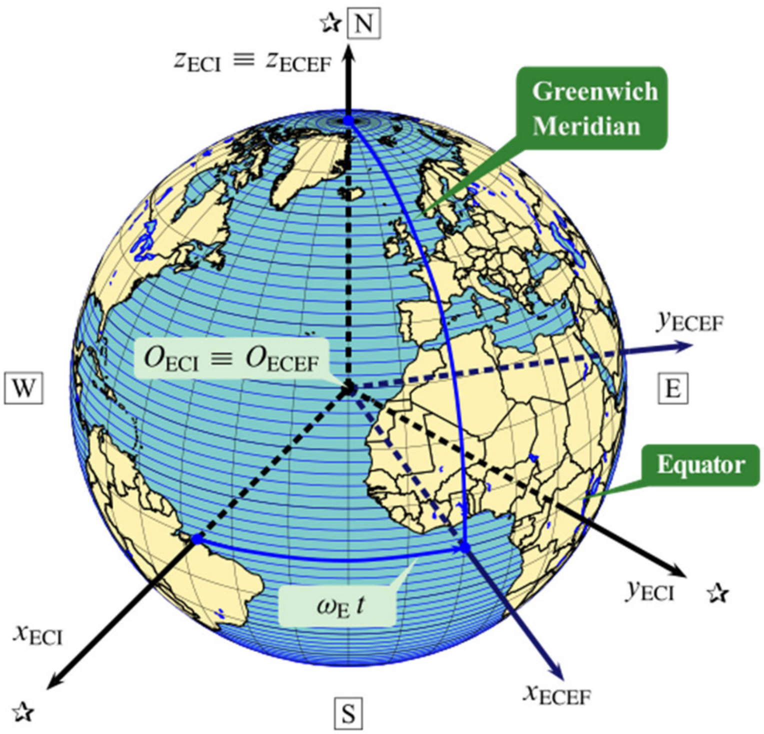
Key Reference Coordinate Systems

II. ECI (Earth Centered Inertial)

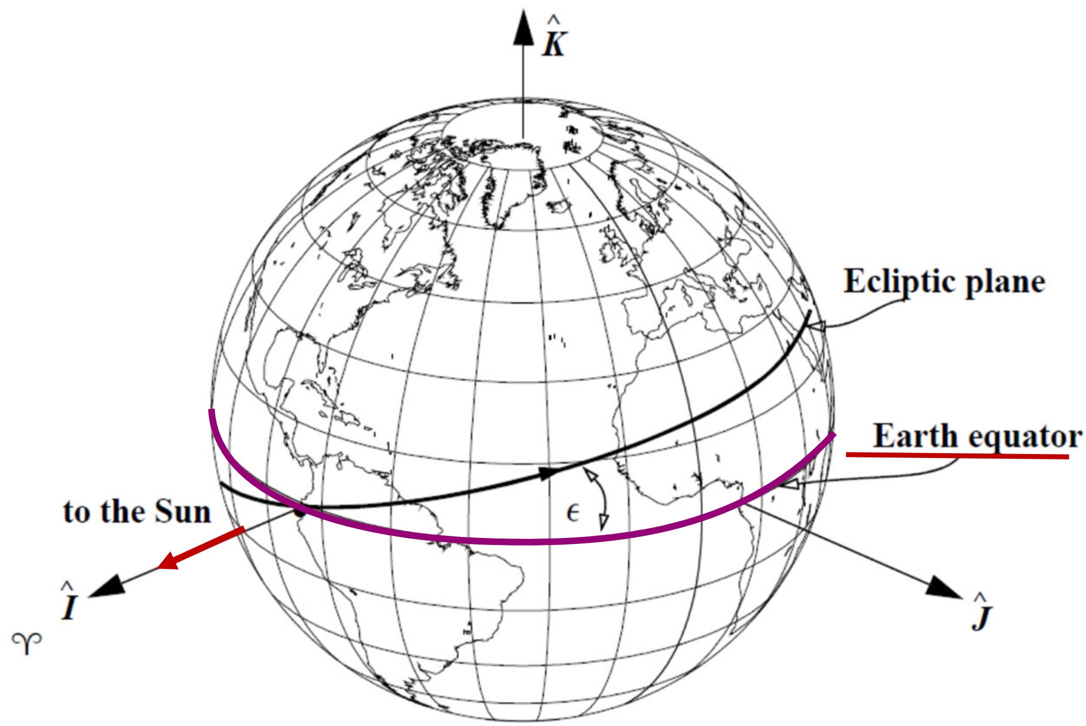
Origin: Earth center

Fundamental plane: Equatorial plane,

Preferred direction: V.E.



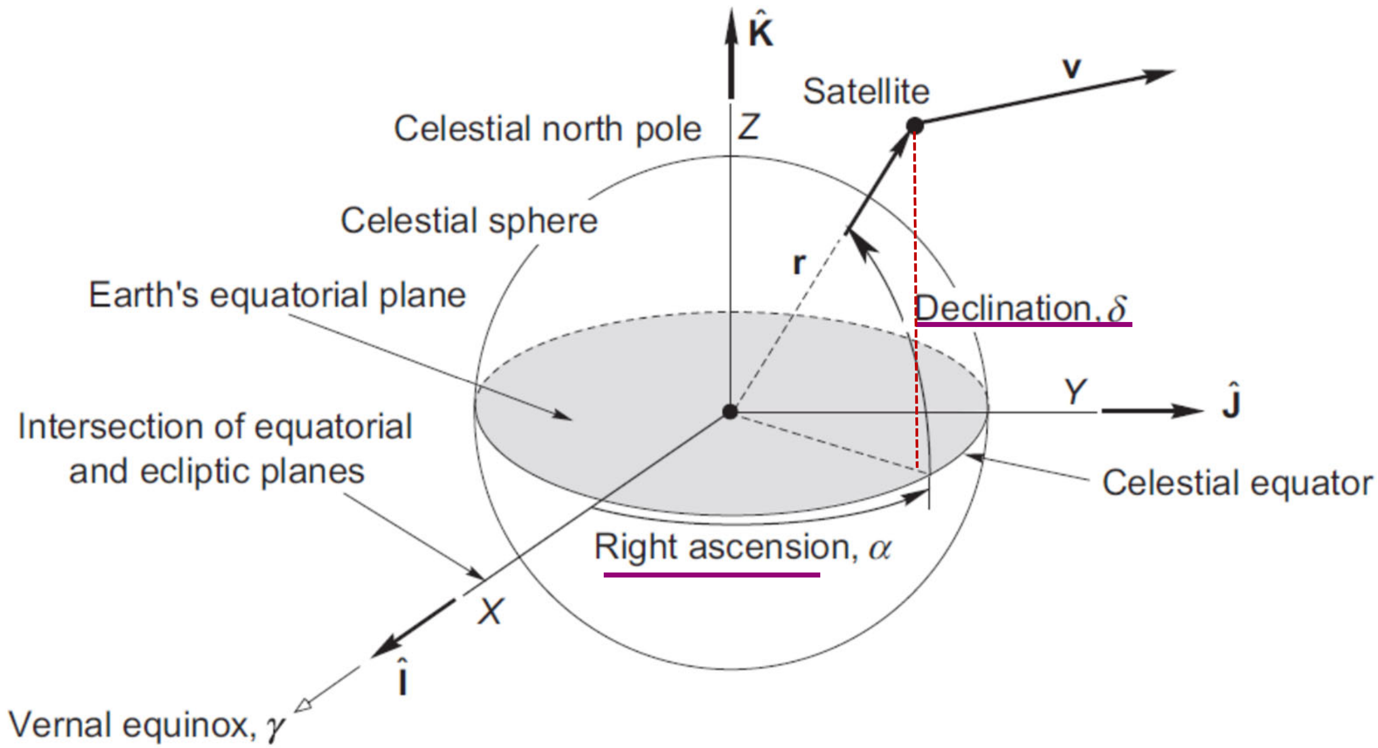
[ECI and ECEF Frames]

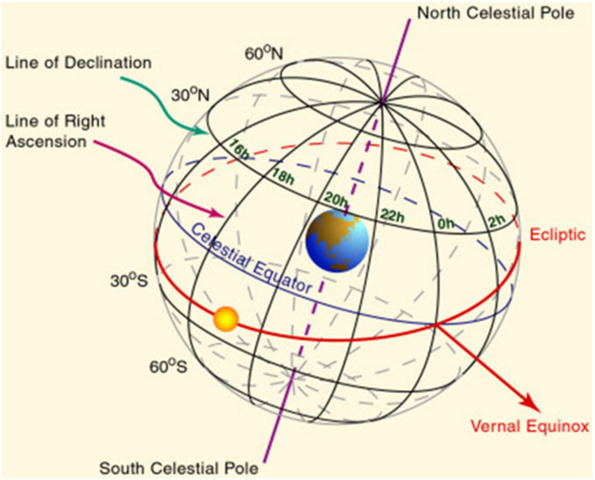


Key Reference Coordinate Systems

III-Celestial Sphere or Geocentric Equatorial CS (Inertial)

Points in this system are coordinated via right ascension and declination.





$$\boldsymbol{r}_{ECI} = X\hat{\boldsymbol{I}} + Y\hat{\boldsymbol{J}} + Z\hat{\boldsymbol{K}}$$

$$\boldsymbol{r}_{ECI} = r\hat{\boldsymbol{u}}_r$$

$$\begin{aligned}\hat{\mathbf{u}}_r &= l\hat{\mathbf{I}} + m\hat{\mathbf{J}} + n\hat{\mathbf{K}} \\ &= \cos\delta\cos\alpha\hat{\mathbf{I}} \\ &+ \cos\delta\sin\alpha\hat{\mathbf{J}} \\ &+ \sin\delta\hat{\mathbf{K}}\end{aligned}$$

In astronomy the **celestial sphere** is a fictitious **sphere** of infinite radius centered at the Earth. All objects in the sky can be conceived as being projected upon the inner surface of the **celestial sphere.** In astronomy and celestial navigation, ephemeris is used to indicate the trajectory (coordinates) of objects in CS with time**.**

Key Reference Coordinate Systems

Example 13

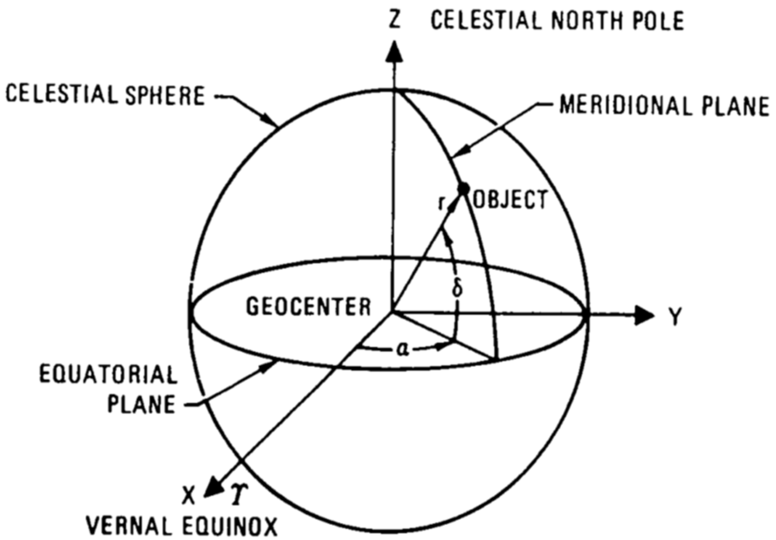
$$Given\ \vec{r}_{ECI} = X\hat{I} + Y\hat{J} + Z\hat{K}\ \ ,compute\ \alpha,\delta\ ?$$

$$1.\ \ r = \sqrt{X^2 + Y^2 + Z^2};\ \ 2.\ \ l = \frac{X}{r}\quad m = \frac{Y}{r}\quad n = \frac{Z}{r}$$

$$3.\ \ \hat{u}_r = l\hat{I} + m\hat{J} + n\hat{K}\quad = \cos\delta\cos\alpha\hat{I}\quad + \cos\delta\sin\alpha\hat{J}\quad + \sin\delta\hat{K}$$

$$4.\ \ \delta = \sin^{-1} n;\quad -90° \le \delta \le 90°$$

$$5.\ \ \alpha = \begin{cases} \cos^{-1}\left(\dfrac{l}{\cos\delta}\right) & m > 0 \\ 360° - \cos^{-1}\left(\dfrac{l}{\cos\delta}\right) & m \le 0 \end{cases}\ ;\ \ 0° \le \alpha \le 360°$$



$$Example: \vec{r} = -5368\hat{I} - 1784\hat{J} + 3941\hat{K}; Km \Rightarrow r = 6574Km$$

$$\hat{u}_r = -0.7947\hat{I} - 0.2642\hat{J} + 0.5464\hat{K}$$

$$\delta = \sin^{-1} n = \sin^{-1} 0.5464 \Rightarrow \delta = 33.12°$$

$$\alpha = 360° - \cos^{-1}\left(\frac{l}{\cos\delta}\right) = 198.4°$$

Key Reference Coordinate Systems

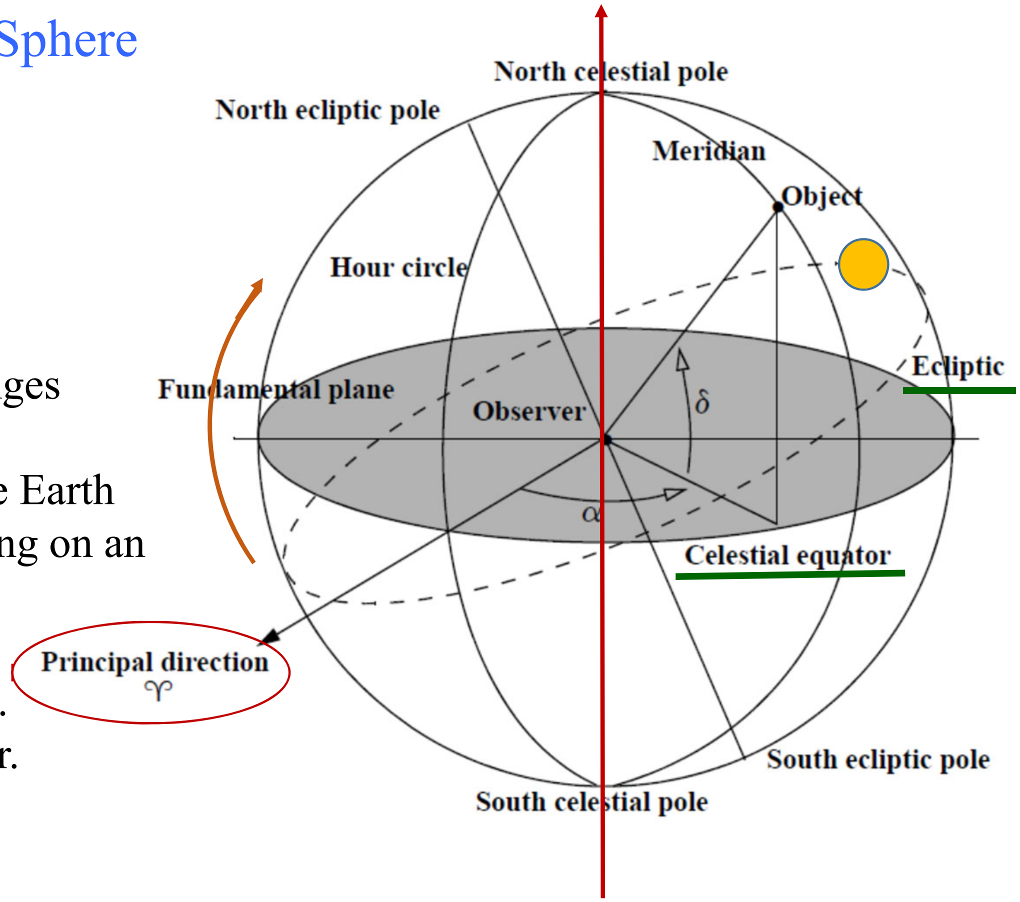
More About the Sun orbit within the Celestial Sphere

Vernal Equinox (V.E.): Ascending node: 21 March

Autumnal Equinox: Descending Node: 23 Sep.

Another definition for V.E. as a reference axis:

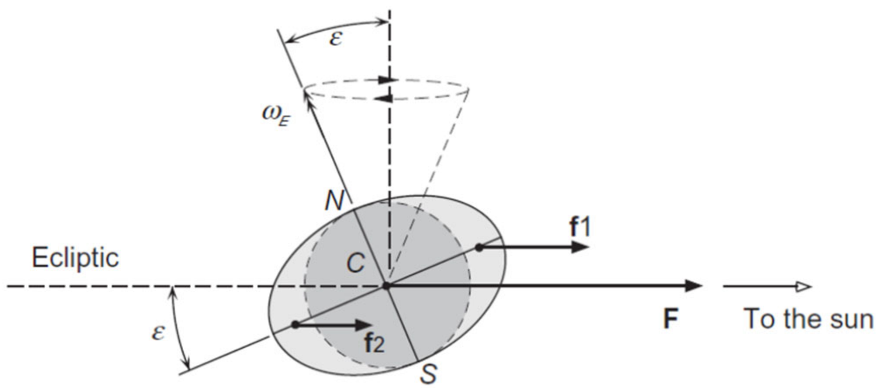
1-It occurs when the Sun’s declination is zero as it changes from negative to positive values.
2-Thus the Sun position can be approximated around the Earth for simulation purposes or is effect on satellites depending on an special calendar called the Julian date.
3-From the point of view of the celestial sphere, the sun follows on orbit over the course of a year on the ecliptic. Its orbit is tilted 23.5 with respect to the celestial equator.

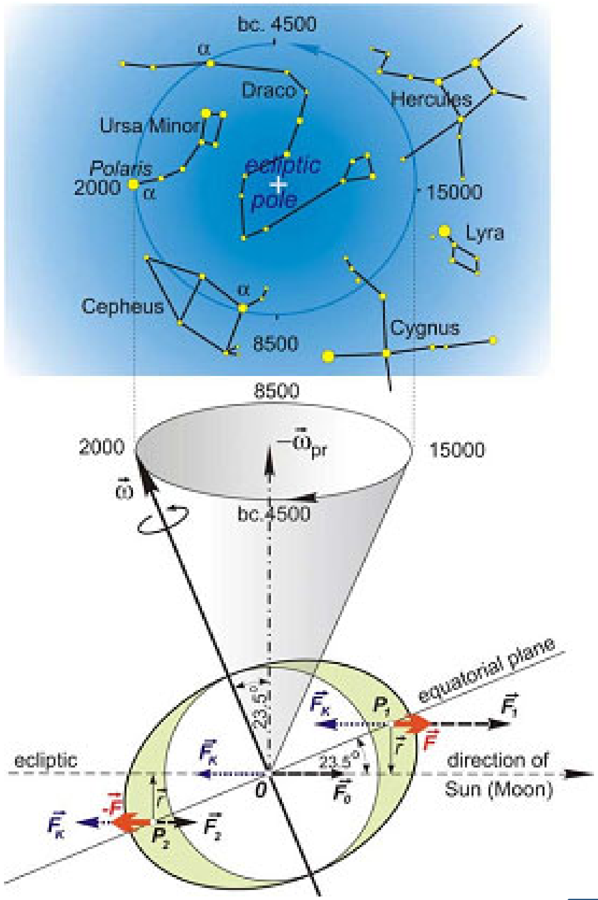


Precession and Nutation

Unfortunately due to a number of factors the V.E in not really a fixed celestial direction. These include : the effect of the Sun and the Moons pull as well as the non-spherical not homogeneous natural shape of the Earth. Equinox location is function of time (Julian date).

- Sun and Moon interact with Earth oblateness to produce
  - ☑ Precession of equinox
  - ☑ Nutation

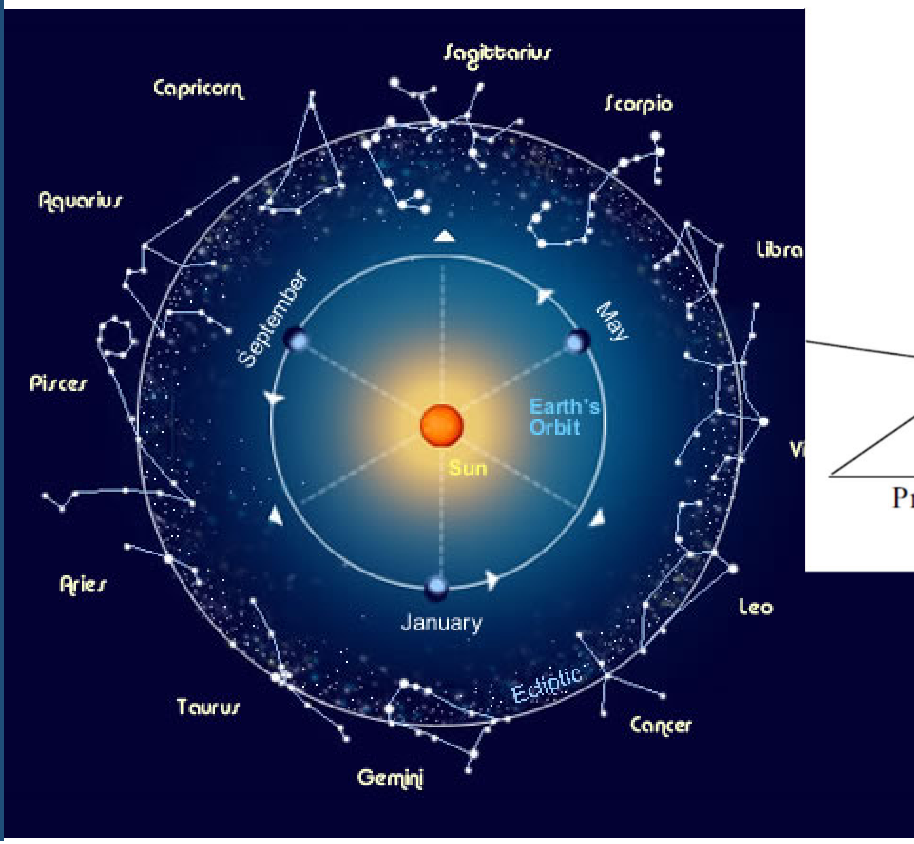


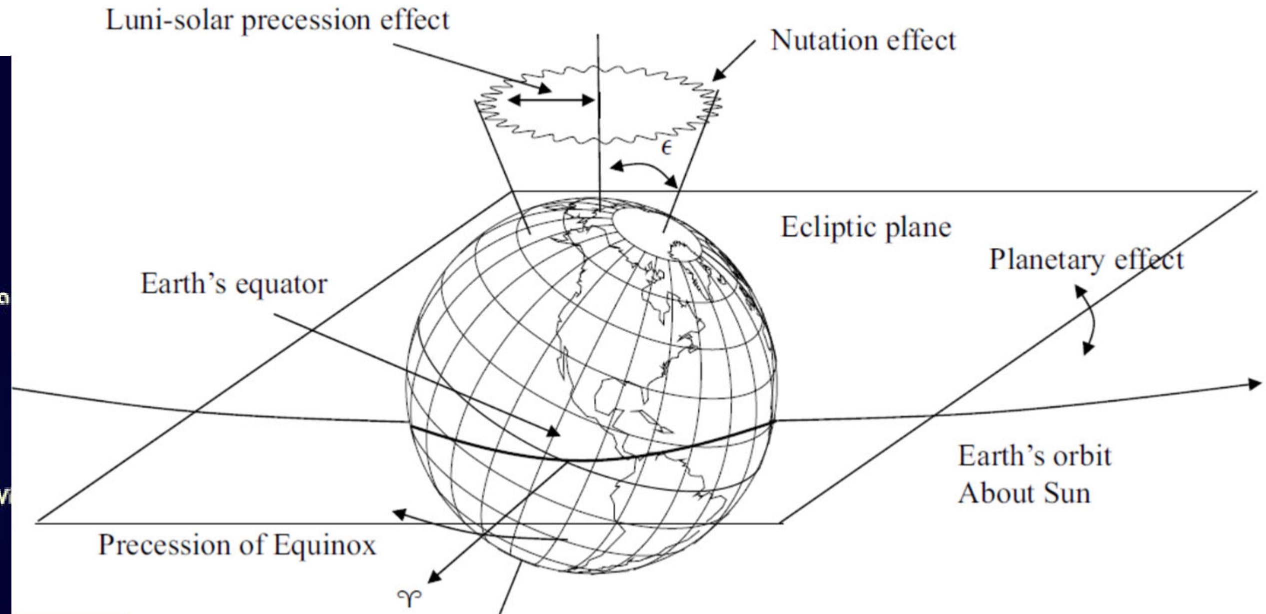


- Gyroscopic Precession
- Earth Precession

Precession and Nutation

Therefor there are two motions (Precession and Nutation of different amplitude and frequency) imposed on the direction of the Earth rotation axis , that in fact cause the Equinox Motion to be a function of time.





Precession:
Amplitude : 23.5
Period: 26000 Years
Approximately 1.4Deg per century.

Nutation:
Amplitude :0.0025
Period: 18.6 Years

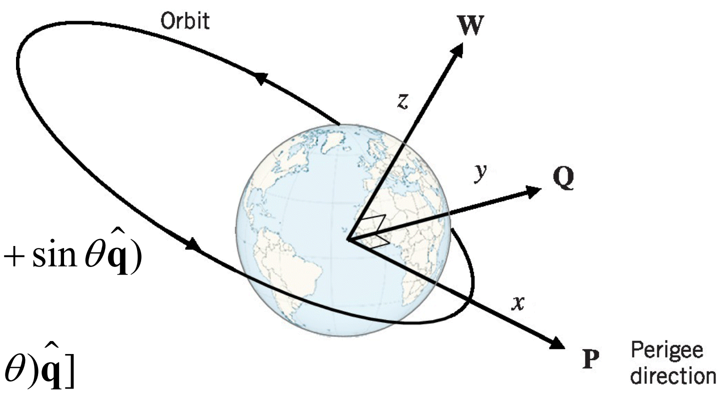
Key Reference Coordinate Systems

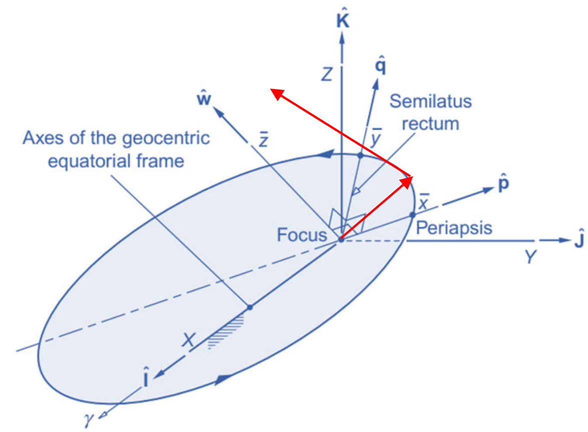
IV- Perifocal Coordinate Frame , Inertial

Origin: Earth Center

Fundamental plane: The Orbital Plane,

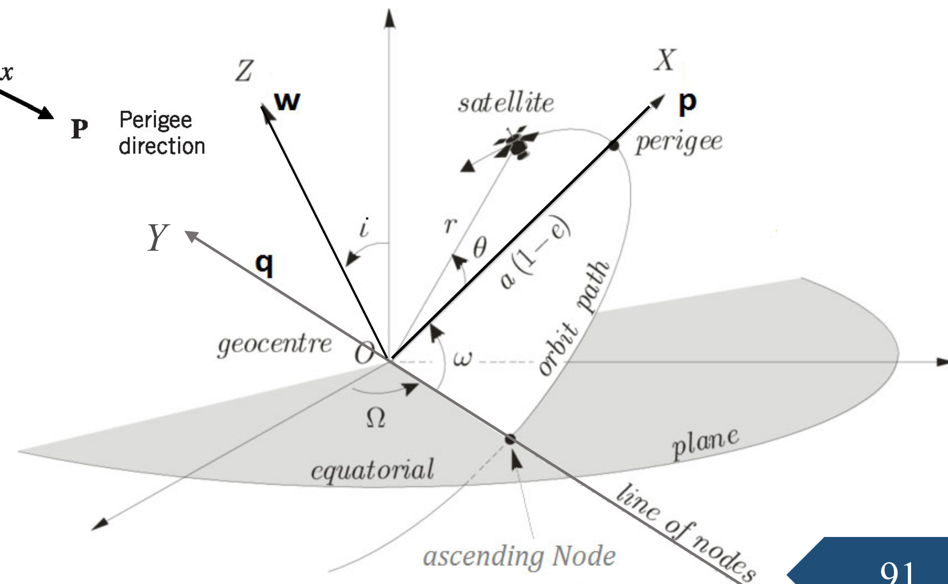
preferred direction: Perigee (Eccentricity Vector)





$Coordination:$

$$
\begin{cases}
\mathbf{r} = \bar{x}\hat{\mathbf{p}} + \bar{y}\hat{\mathbf{q}} = \dfrac{h^2}{\mu}\dfrac{1}{1+e\cos\theta}(\cos\theta\hat{\mathbf{p}} + \sin\theta\hat{\mathbf{q}}) \\
\mathbf{v} = \dot{\bar{x}}\hat{\mathbf{p}} + \dot{\bar{y}}\hat{\mathbf{q}} = \dfrac{\mu}{h}[-\sin\theta\hat{\mathbf{p}} + (e+\cos\theta)\hat{\mathbf{q}}] \\
\to \begin{cases}
\{\mathbf{r}\}_{\bar{x}} = \dfrac{h^2}{\mu}\dfrac{1}{1+e\cos\theta}\begin{Bmatrix}\cos\theta \\ \sin\theta \\ 0\end{Bmatrix} \\
\{\mathbf{v}\}_{\bar{x}} = \dfrac{\mu}{h}\begin{Bmatrix}-\sin\theta \\ e+\cos\theta \\ 0\end{Bmatrix}
\end{cases}
\end{cases}
$$

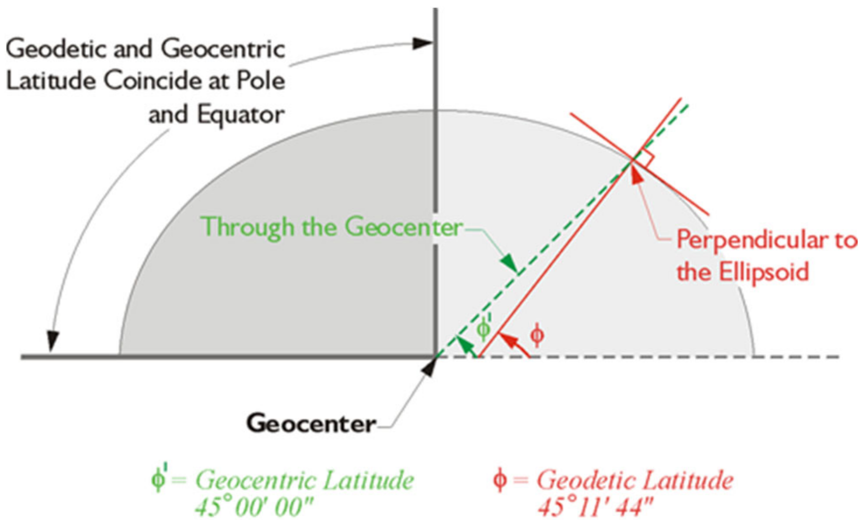


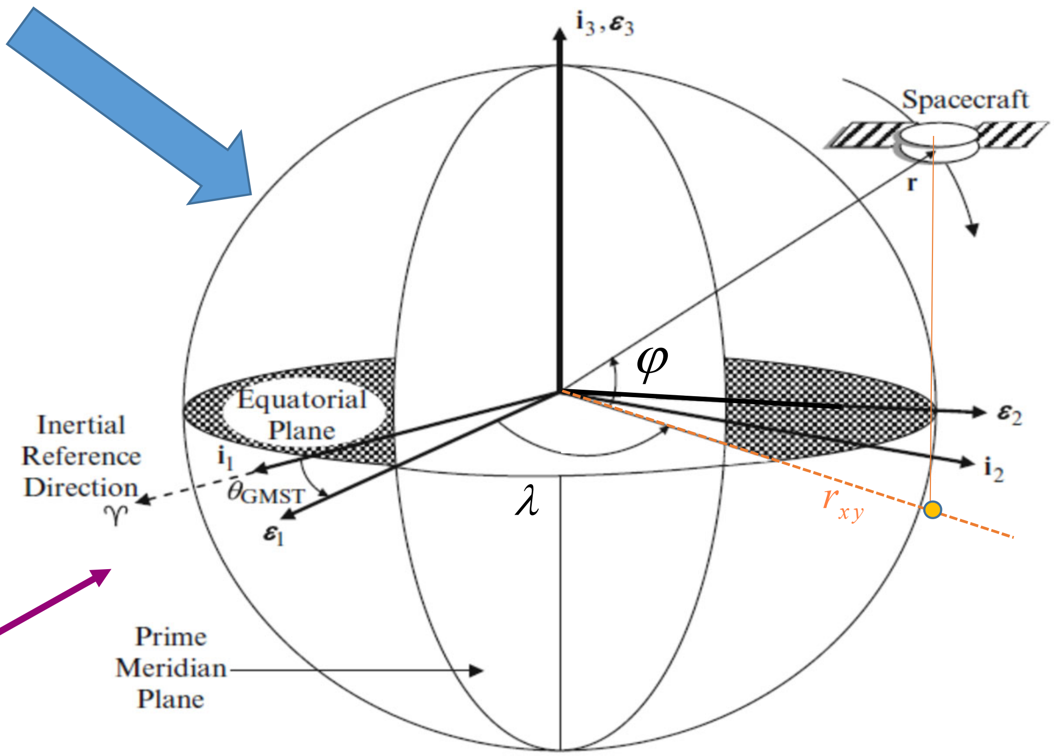
Key Reference Coordinate Systems

V- Earth-Centered Earth-Fixed Frame (ECEF), Non-Inertial

Origin: Earth center
Fundamental plane: Equatorial plane,
Preferred direction: Intersection of Greenwich meridian
With the Equator
Coordinates:

- Longitude (λ),
- Latitude (φ_{gc} *or* φ_{gd})





Greenwich Mean Sidereal Time (θ_{GMST}) is the hour angle of the average position of the vernal equinox, neglecting short term motions of the equinox due to nutation.

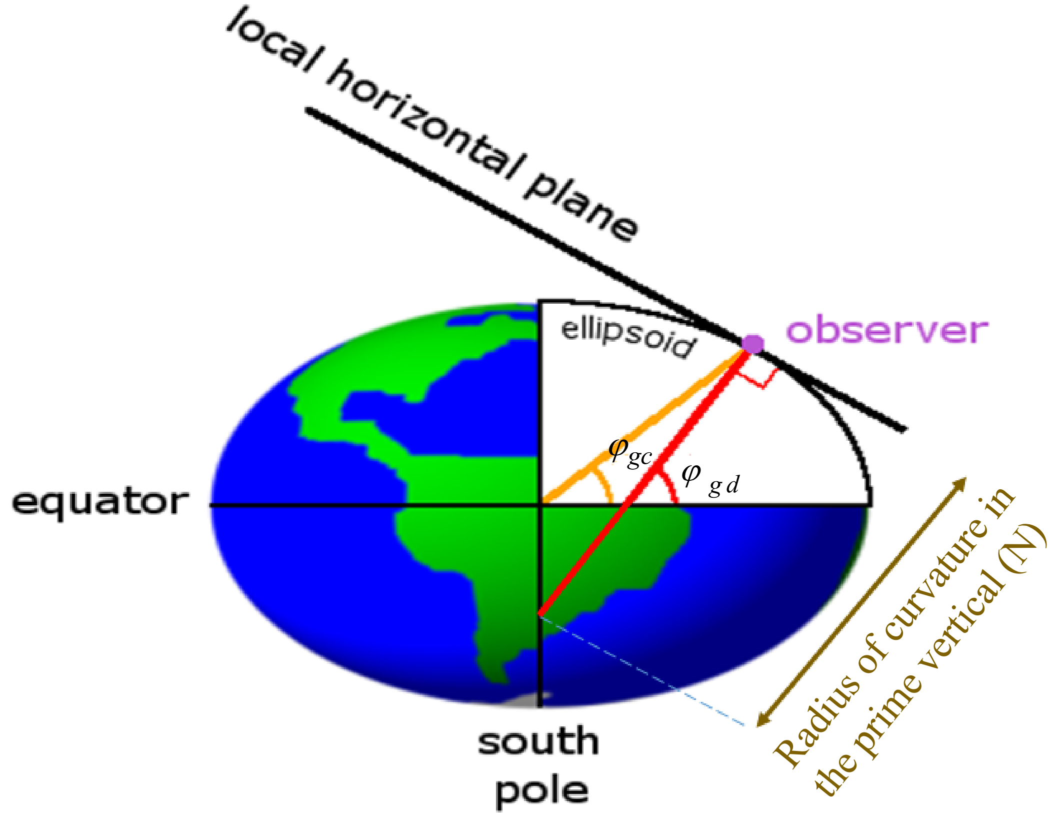
Key Reference Coordinate Systems

- Shape of the Earth
  - Spherical

$$r = \begin{bmatrix} r\cos\varphi_{gc}\cos\lambda \\ r\cos\varphi_{gc}\sin\lambda \\ r\sin\varphi_{gc} \end{bmatrix}$$

  - Ellipsoid

$$r = \begin{bmatrix} (N + h_{ellp.})\cos\varphi_{gd}\cos\lambda \\ (N + h_{ellp.})\cos\varphi_{gd}\sin\lambda \\ (N(1-e_E^2) + h_{ellp.})\sin\varphi_{gd} \end{bmatrix}$$

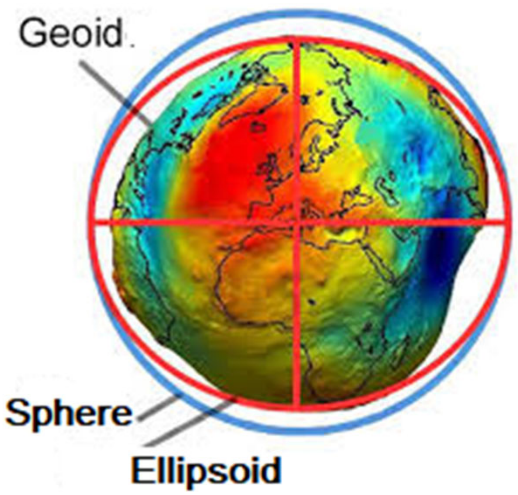


$$N = \frac{R_E}{\sqrt{1 - e_E^2 \sin^2\varphi_{gd}}}; \quad h_{ellp} = \frac{r_{xy}}{\cos(\varphi_i)} - N; \quad e_E \approx 0.08081$$

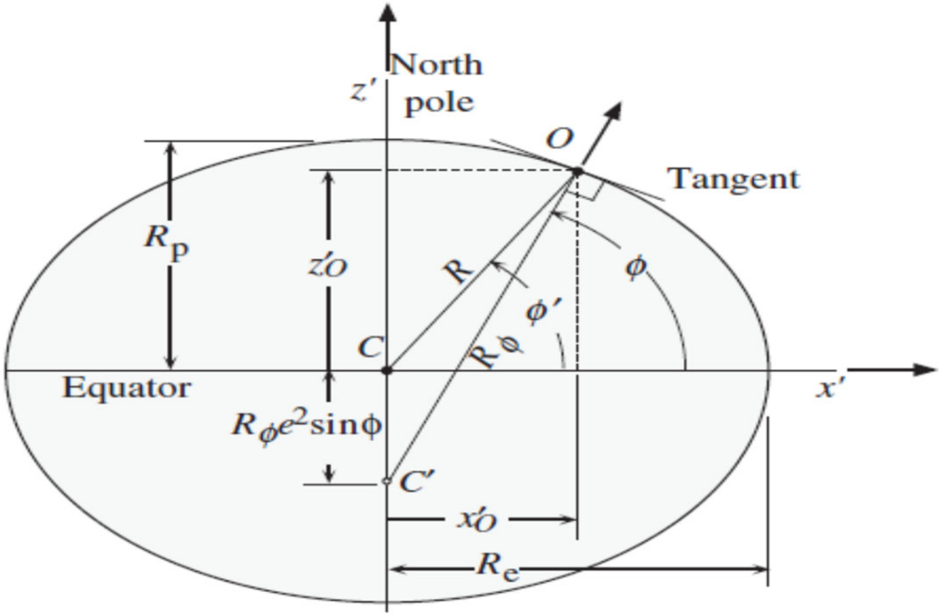
Key Reference Coordinate Systems

- Shape of the Earth

$$\tan\varphi_{gd} = \frac{N + h_{ellp}}{N(1-e_E^2) + h_{ellp}} \tan\varphi_{gc}$$



Oblate Spheroid



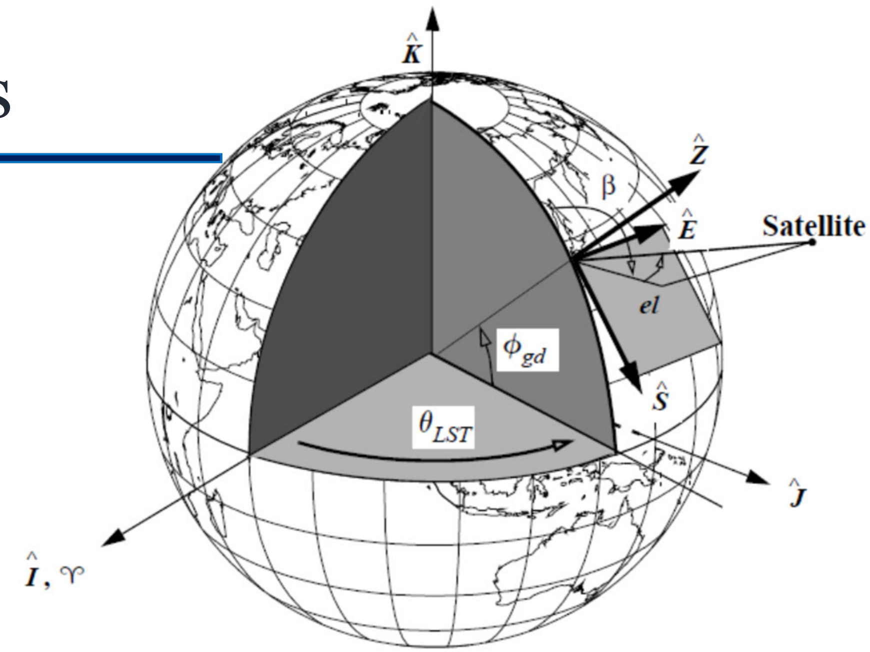
Key Reference Coordinate Systems

IV- Topocentric Coordinate Systems, Non-Inertial

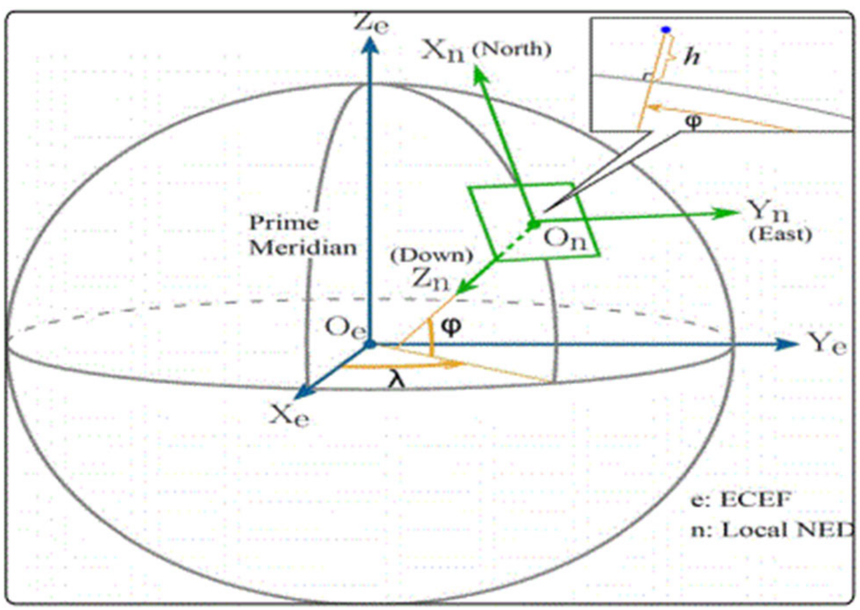
Origin: Projection Point on the Earth,

Fundamental plane: The Local Tangent Plane

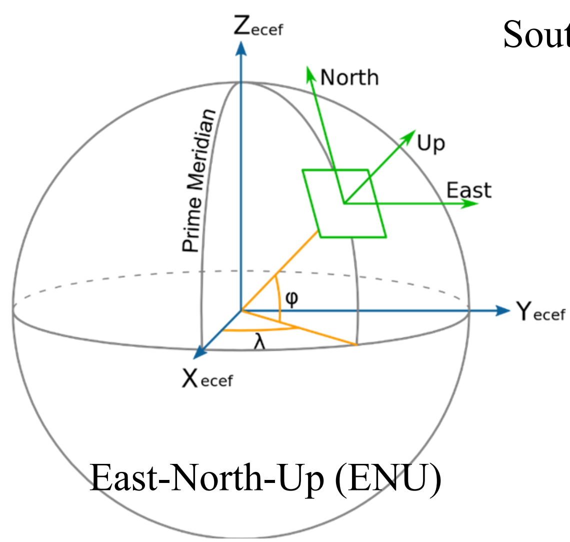
Utility: Different



South-East-Zenith (**SEZ**) or (SEU)



North-East-Down (NED)



East-North-Up (ENU)

Key Reference Coordinate Systems

V- RSW orbit reference Coordinate Systems, Non-Inertial

Origin: The satellite center of mass,

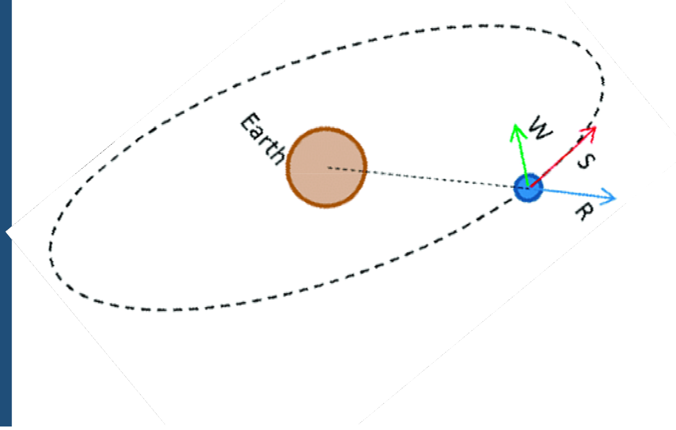
Fundamental plane: The orbital Plane

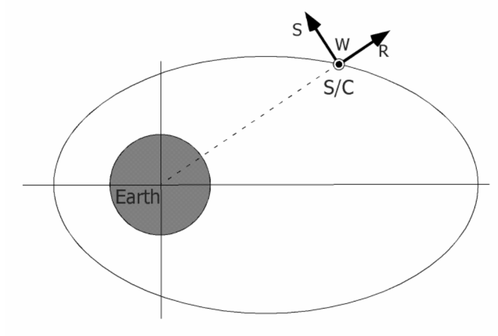
The axis are defined as follows:

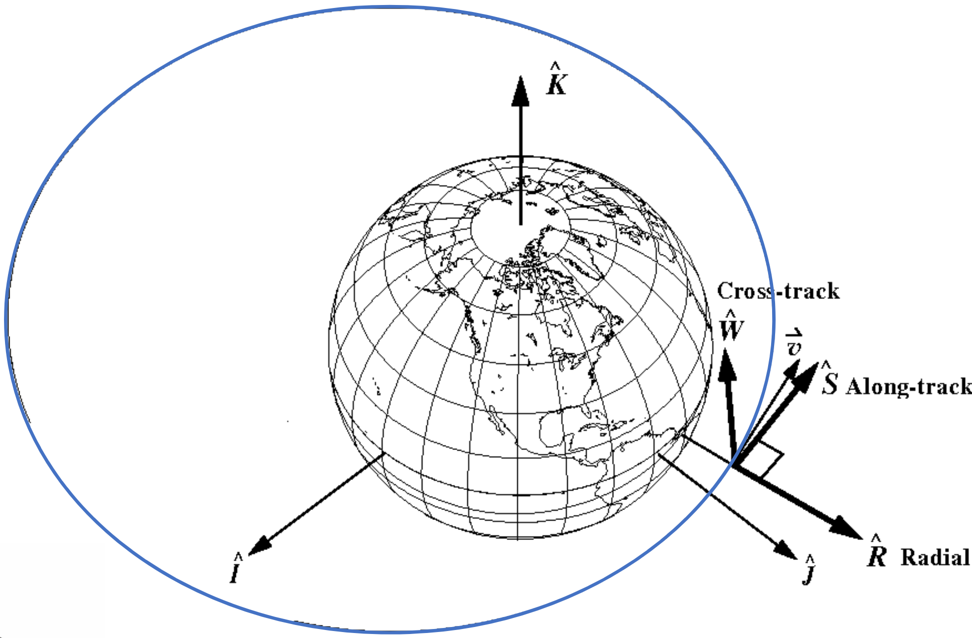
1-The R axis is defined as pointing from the Earth's center along the radius vector toward the satellite,

2- The S axis points in the flight direction (along or in track) and is perpendicular to the radius vector.

3- The W axis is fixed along the (cross track) direction normal to the orbital plane.



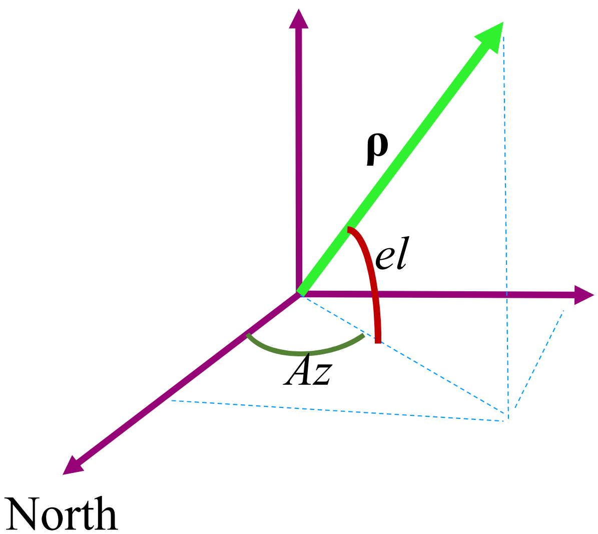




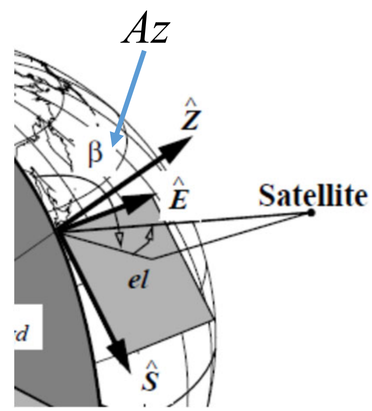
RSW Coordinate system

Key Reference Coordinate Systems for Satellites

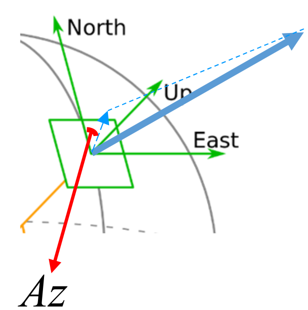
- Azimuth , Elevation



$$[\boldsymbol{\rho}]^{SEZ} = \rho \begin{bmatrix} \cos(el)\cos(\pi - Az) \\ \cos(el)\sin(\pi - Az) \\ \sin(el) \end{bmatrix}$$



$$[\boldsymbol{\rho}]^{ENU} = \rho \begin{bmatrix} \cos(el)\cos(\pi/2 - Az) \\ \cos(el)\sin(\pi/2 - Az) \\ \sin(el) \end{bmatrix}$$



Orbital Elements and the State Vector Relation

Orbital elements (OE) are a collection of six parameters that can uniquely identify a specific orbit in terms of shape and orientation within the 3D space. Though , there are a different ways to mathematically describe the same orbit, but all schemes consist of a set of six parameters commonly used in astronomy and orbital mechanics. In reality due to gravitational perturbations by other celestial objects OE may change with time.

Orbit Representation Methods:

1-Cartesian (Inertial) Coordinates, State vector notation

$$\boldsymbol{X} = [\boldsymbol{r} \quad \boldsymbol{v}]^T$$

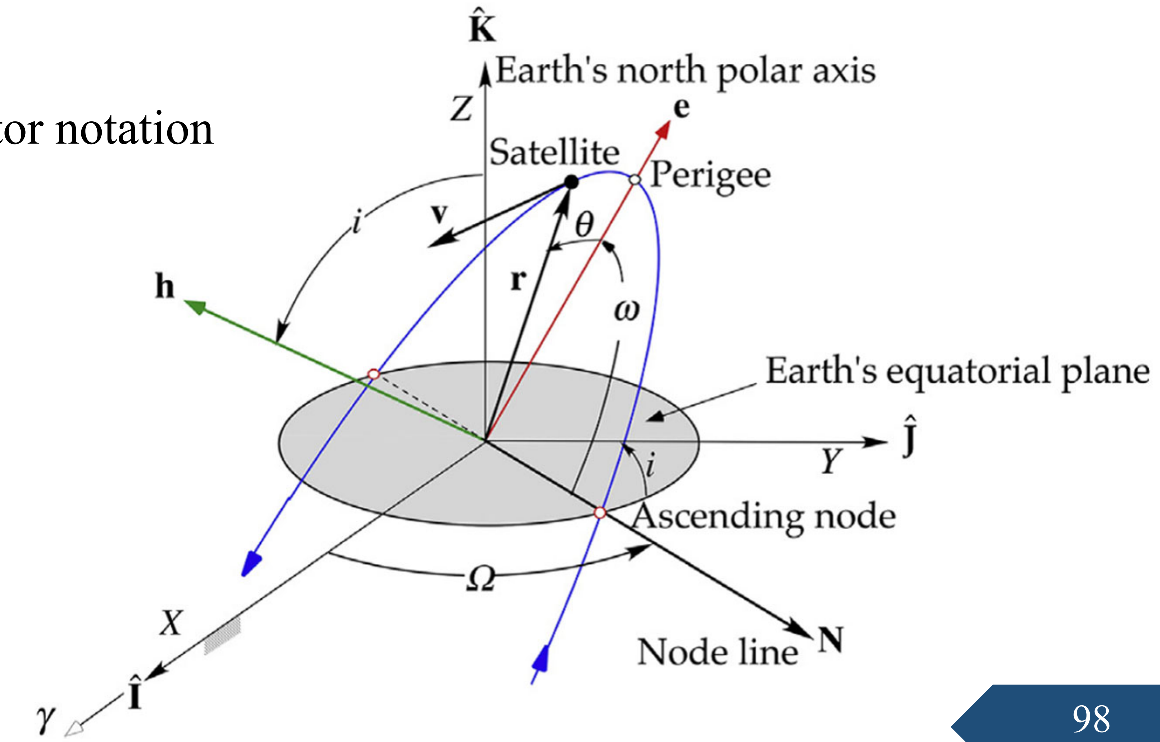
2-True Classical Orbital Elements

$$\alpha = [\, a \quad e \quad i \quad \Omega \quad \omega \quad t \,]^T$$

3-Modified Classical Element Set

$$\alpha = [\, a \quad e \quad i \quad \Omega \quad \omega \quad \theta \,]^T \quad or$$

$$\alpha = [\, h \quad e \quad i \quad \Omega \quad \omega \quad M \,]^T$$



Orbital Elements and the State Vector Relation

OE and the satellite state vector are defined below, where we can do conversion between them as well.

$$\mathbf{r} = [X \quad Y \quad Z]^T, \mathbf{v} = [v_X \quad v_Y \quad v_Z]^T$$

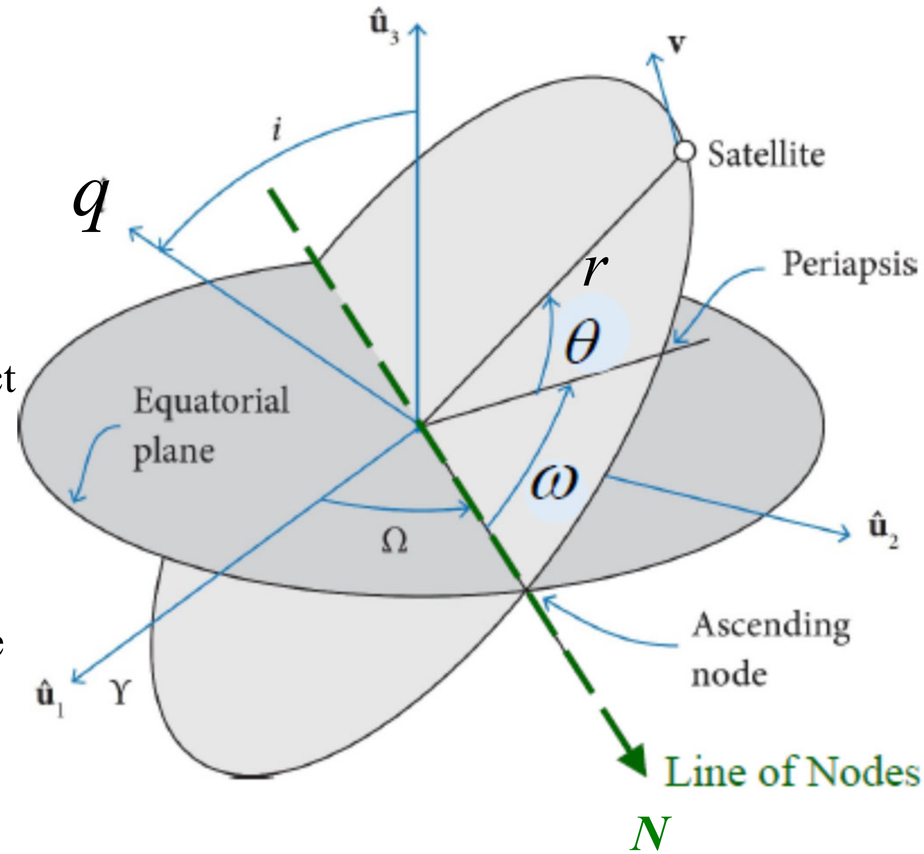
$$\alpha = [a \quad e \quad \underbrace{i \quad \Omega \quad \omega}_{\text{3D Orientation}} \quad \theta]^T$$

h or p
2D Size

Shape

3D Orientation

Position, Reference time
$\theta \equiv M \equiv t$



Inclination (i) - vertical tilt of the orbital plane with respect to ellipse with respect to the reference (equatorial) plane.
Longitude or Right ascension of the ascending node (Ω) - This angle is measured in the reference plane, and is the angle between the V.E. direction and the lines of nodes (LN). In turn the LN is the intersection of the orbital plane with the equatorial reference plane.
Argument of periapsis or perigee (ω) – This angle defines the orientation of the ellipse in the orbital plane and is measure from the ascending node to the periapsis point of the orbit.

State Vector to OE transformation Algorithm

State vector (SV) to Orbital elements (OE) transformation can be achieved via the following algorithms. Inverse transformation from OE to SV is also possible. The problem is defined as : Given the satellite SV, find its corresponding OE. The procedure is stated below.

The satellite SV is usually given in ECI, as :

$$\boldsymbol{r} = [\,X \quad Y \quad Z\,]^T, \boldsymbol{v} = [\,v_X \quad v_Y \quad v_Z\,]^T$$

Find its corresponding OE,

$$\boldsymbol{\alpha} = [\,a \quad e \quad i \quad \Omega \quad \omega \quad \theta\,]^T$$

1. $r = \sqrt{\boldsymbol{r}.\boldsymbol{r}} = \sqrt{X^2 + Y^2 + Z^2}$

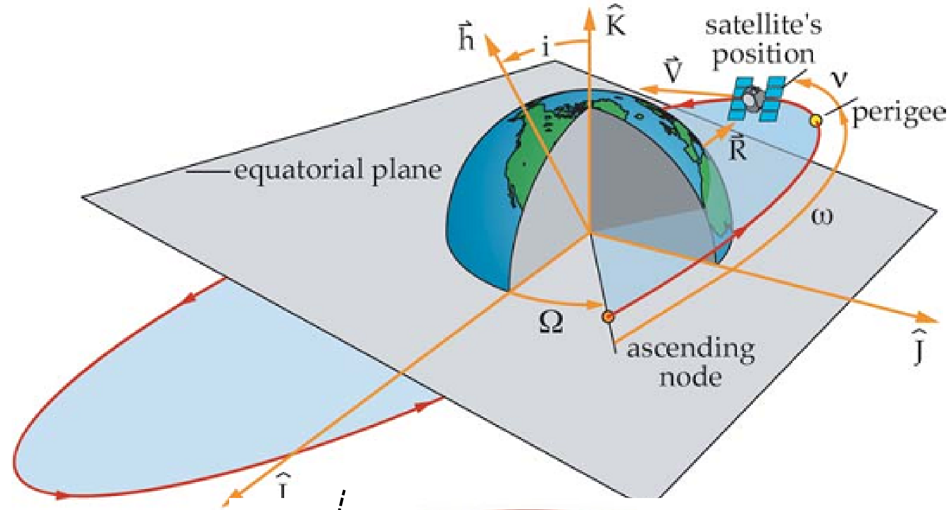
$v = \sqrt{\boldsymbol{v}.\boldsymbol{v}} = \sqrt{v_X{}^2 + v_Y{}^2 + v_Z{}^2}$

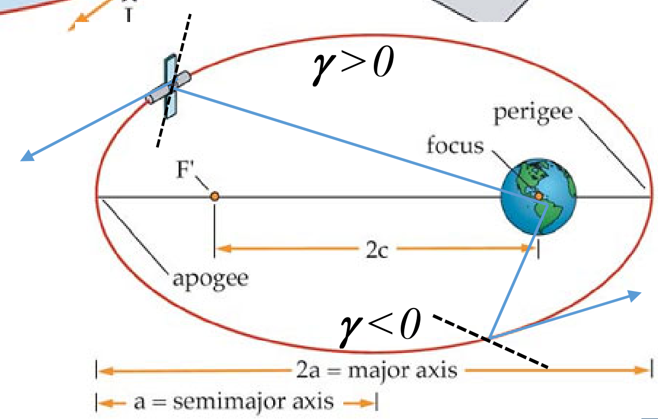
$v_r = \boldsymbol{r}.\boldsymbol{v} / r = \left(Xv_X + Yv_Y + Zv_Z\right) / r = v \sin\gamma$

Note that $: \boldsymbol{r}.\boldsymbol{v} = rv\cos\boldsymbol{\beta} = rv\sin\boldsymbol{\gamma}$

$\Rightarrow v_r > \boldsymbol{0}$ *The satellite is moving away from the perigee*

Or if $v_r < \boldsymbol{0}$ *The satellite is moving towards the perigee*





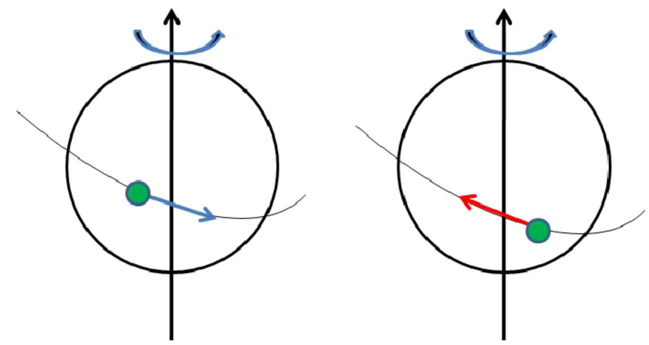
State Vector to OE transformation

$$\textbf{2.}\quad \boldsymbol{h} = \boldsymbol{r}\times\boldsymbol{v}\quad ;\ h=\sqrt{\boldsymbol{h}.\boldsymbol{h}} \rightarrow p = \frac{h^2}{\mu}$$

$$\textbf{3.}\quad \boldsymbol{h}.\hat{\boldsymbol{K}} = h\ cos i = h_Z \Rightarrow\quad i = cos^{-1}\left(\frac{h_Z}{h}\right)$$

Note that :

$0<i<90^\circ \Rightarrow$ *Prograde Orbit* *Or* $90^\circ<i<180^\circ \Rightarrow$ *Retograde Orbit*



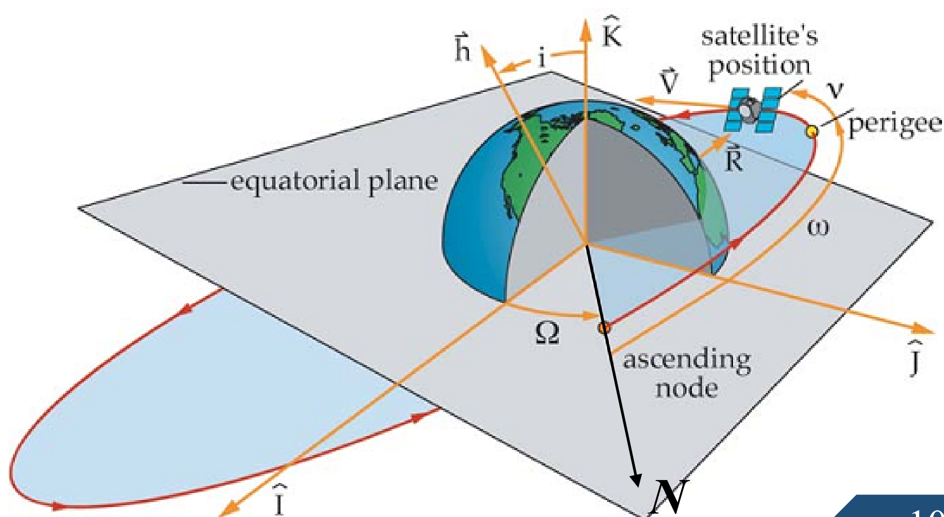
a) Prograde orbit

b) Retrograde orbit

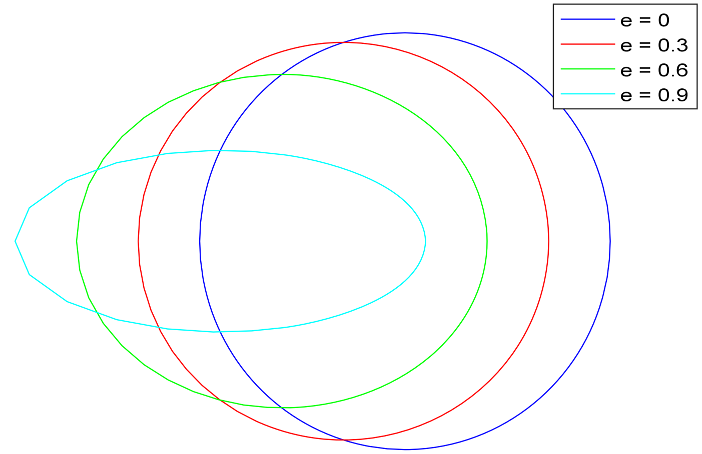
$$\textbf{4.}\quad \boldsymbol{N} = \hat{\boldsymbol{K}}\times\boldsymbol{h} = \begin{vmatrix} \hat{\boldsymbol{I}} & \hat{\boldsymbol{J}} & \hat{\boldsymbol{K}} \\ 0 & 0 & 1 \\ h_X & h_Y & h_Z \end{vmatrix}\quad ;\ N=\sqrt{\boldsymbol{N}.\boldsymbol{N}}$$

$$\boldsymbol{N}.\hat{\boldsymbol{I}} = N cos\Omega$$

$$\textbf{5.}\quad \Omega=\begin{cases} cos^{-1}\left(\dfrac{N_X}{N}\right) & (N_Y \geq 0) \\ 360^\circ - cos^{-1}\left(\dfrac{N_X}{N}\right) & (N_Y<0) \end{cases}$$



State Vector to OE transformation

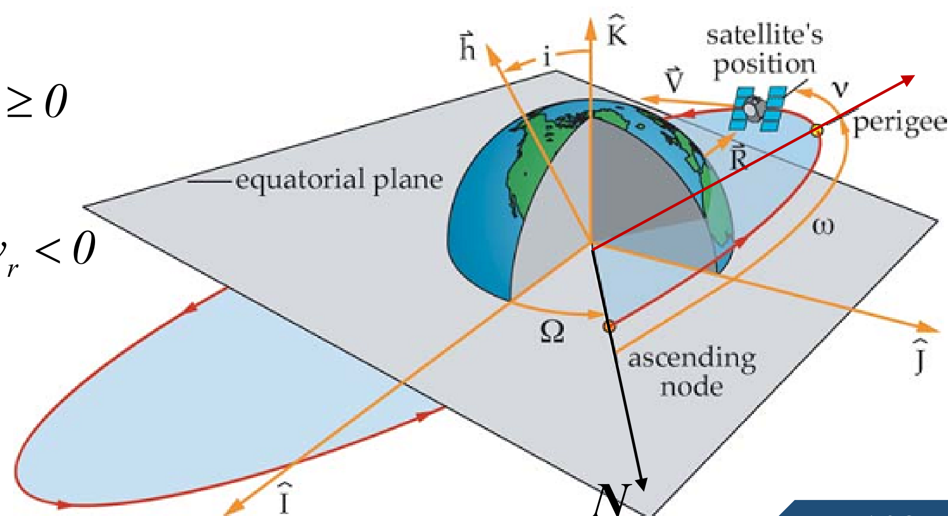


6. $$\boldsymbol{e} = \frac{\boldsymbol{v}\times\boldsymbol{h}}{\mu} - \frac{\boldsymbol{r}}{r} = \frac{1}{\mu}\left[\left(v^2 - \frac{\mu}{r}\right)\boldsymbol{r} - r v_r \boldsymbol{v}\right] \quad ;e = \sqrt{\boldsymbol{e}.\boldsymbol{e}}$$

7. $$\boldsymbol{N}.\boldsymbol{e} = Necos\omega \quad \rightarrow \boldsymbol{\omega} = \begin{cases} cos^{-1}\left(\dfrac{\boldsymbol{N}.\boldsymbol{e}}{Ne}\right) & for \quad e_Z \geq 0 \\ 360°\text{-}cos^{-1}\left(\dfrac{\boldsymbol{N}.\boldsymbol{e}}{Ne}\right) & for \quad e_Z < 0 \end{cases}$$

8. $$\boldsymbol{r}.\boldsymbol{e} = r\,e\,cos\theta \quad \rightarrow \boldsymbol{\theta} = \begin{cases} cos^{-1}\left(\dfrac{\boldsymbol{e}.\boldsymbol{r}}{er}\right) & for \quad v_r \geq 0 \\ 360°\text{-}cos^{-1}\left(\dfrac{\boldsymbol{e}.\boldsymbol{r}}{er}\right) & for \quad v_r < 0 \end{cases}$$

$$\frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a} \Rightarrow a = \frac{\mu}{\dfrac{2\mu}{r} - v^2} \Rightarrow r = \frac{p}{1 + ecos\boldsymbol{\theta}}$$



Example 14

A satellite position and velocity vector (SV) at some instant if time is given in the ECI. Determine its corresponding Orbital elements (OE).

$$\mathbf{r} = -6045\hat{\mathbf{I}} - 3490\hat{\mathbf{J}} + 2500\hat{\mathbf{K}} \text{ (km)}$$

$$\mathbf{v} = -3.457\hat{\mathbf{I}} + 6.618\hat{\mathbf{J}} + 2.533\hat{\mathbf{K}} \text{ (km/s)}$$

Solution

Step 1:

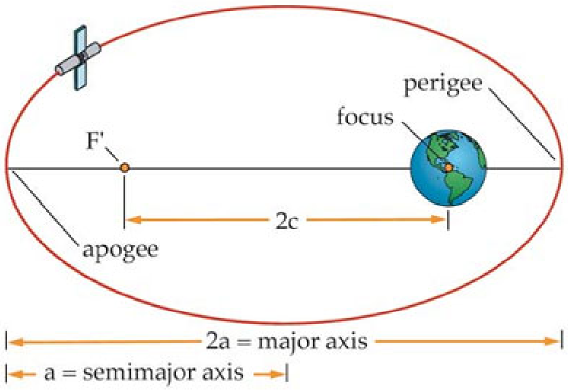
$$r = \sqrt{\mathbf{r}\cdot\mathbf{r}} = \sqrt{(-6045)^2 + (-3490)^2 + 2500^2} = 7414 \text{ km}$$

Step 2:

$$v = \sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{(-3.457)^2 + 6.618^2 + 2.533^2} = 7.884 \text{ km/s}$$

Step 3:

$$v_r = \frac{\mathbf{v}\cdot\mathbf{r}}{r} = \frac{(-3.457)\cdot(-6045) + 6.618\cdot(-3490) + 2.533\cdot 2500}{7414} = 0.5575 \text{ km/s}$$



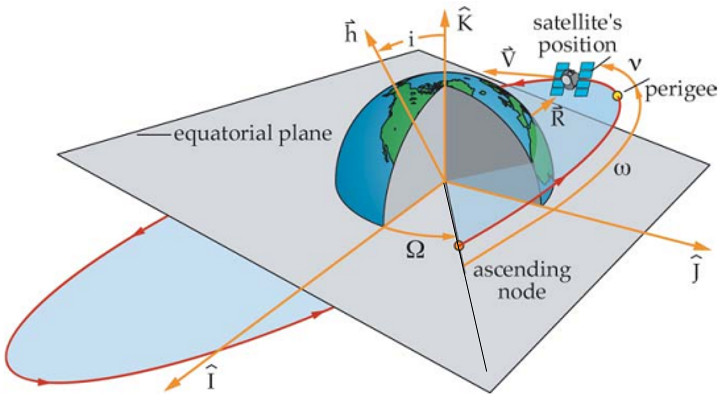
Example 14

Step 4:

$$\mathbf{h} = \mathbf{r} \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ -6045 & -3490 & 2500 \\ -3.457 & 6.618 & 2.533 \end{vmatrix} = -25{,}380\hat{\mathbf{I}} + 6670\hat{\mathbf{J}} - 52{,}070\hat{\mathbf{K}} \left(\text{km}^2/\text{s}\right)$$

Step 5:

$$h = \sqrt{\mathbf{h}\cdot\mathbf{h}} = \sqrt{(-25{,}380)^2 + 6670^2 + (-52{,}070)^2} = \boxed{58{,}310\ \text{km}^2/\text{s}}$$



Step 6:

$$i = \cos^{-1}\frac{h_Z}{h} = \cos^{-1}\left(\frac{-52{,}070}{58{,}310}\right) = \boxed{153.2°}$$

Step 7:

$$\mathbf{N} = \hat{\mathbf{K}} \times \mathbf{h} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ 0 & 0 & 1 \\ -25{,}380 & 6670 & -52{,}070 \end{vmatrix} = -6670\hat{\mathbf{I}} - 25{,}380\hat{\mathbf{J}} \left(\text{km}^2/\text{s}\right)$$

Step 8:

$$N = \sqrt{\mathbf{N}\cdot\mathbf{N}} = \sqrt{(-6670)^2 + (-25{,}380)^2} = 26{,}250\ \text{km}^2/\text{s}$$

Example 14

Step 9:

$$\Omega = \cos^{-1}\frac{N_X}{N} = \cos^{-1}\left(\frac{-6670}{26{,}250}\right) = 104.7^\circ \text{ or } \boxed{255.3^\circ}$$

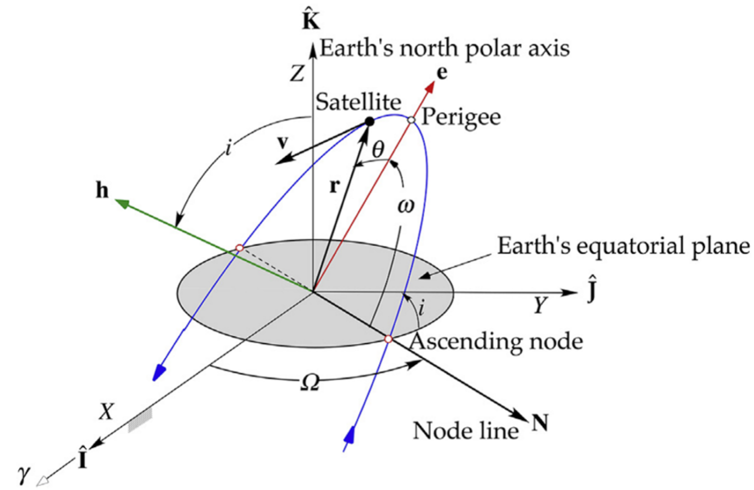
Step 10:

$$\mathbf{e} = \frac{1}{\mu}\left[\left(v^2 - \frac{\mu}{r}\right)\mathbf{r} - r v_r \mathbf{v}\right]$$

$$= \frac{1}{398{,}600}\left[\left(7.884^2 - \frac{398{,}600}{7414}\right)\left(-6045\hat{\mathbf{I}} - 3490\hat{\mathbf{J}} + 2500\hat{\mathbf{K}}\right)\right.$$

$$\left. -(7414)(0.5575)\left(-3.457\hat{\mathbf{I}} + 6.618\hat{\mathbf{J}} + 2.533\hat{\mathbf{K}}\right)\right]$$

$$\mathbf{e} = -0.09160\hat{\mathbf{I}} - 0.1422\hat{\mathbf{J}} + 0.02644\hat{\mathbf{K}}$$



Step 11:

$$e = \sqrt{\mathbf{e}\cdot\mathbf{e}} = \sqrt{(-0.09160)^2 + (-0.1422)^2 + (0.02644)^2} = \boxed{0.1712}$$

Example 14

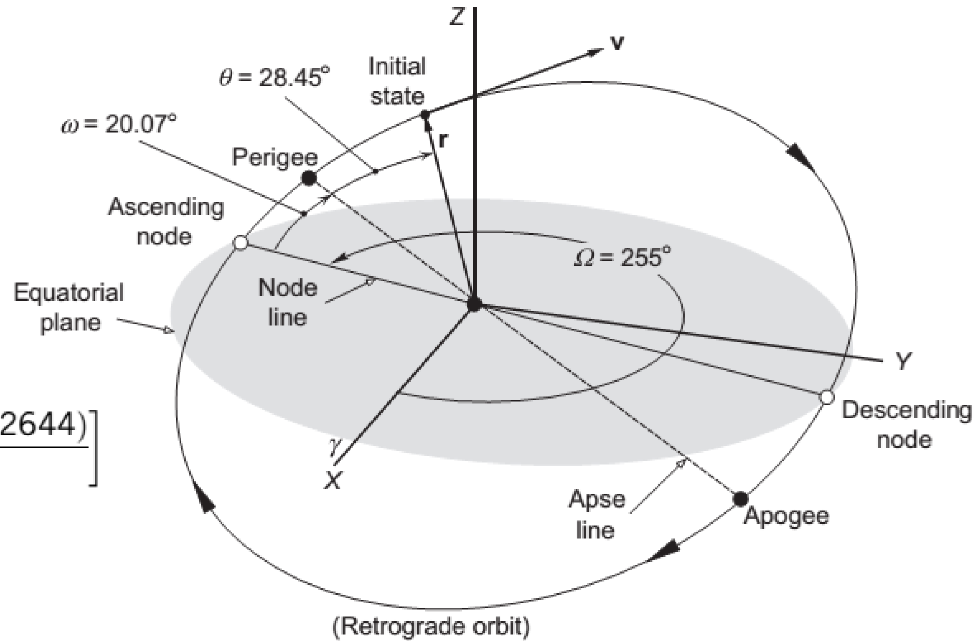
Step 12:

$$\omega = \cos^{-1}\frac{\mathbf{N}\cdot\mathbf{e}}{Ne}$$

$$= \cos^{-1}\left[\frac{(-6670)(-0.09160)+(-25{,}380)(-0.1422)+(0)(0.02644)}{(26{,}250)(0.1712)}\right]$$

$$= \boxed{20.07^\circ} \text{ or } 339.9^\circ$$

$e_Z > 0$



Step 13:

$$\theta = \cos^{-1}\left(\frac{\mathbf{e}\cdot\mathbf{r}}{er}\right) = \cos^{-1}\left[\frac{(-0.09160)(-6045)+(-0.1422)\cdot(-3490)+(0.02644)(2500)}{(0.1712)(7414)}\right]$$

$$= \boxed{28.45^\circ} \text{ or } 331.6^\circ$$

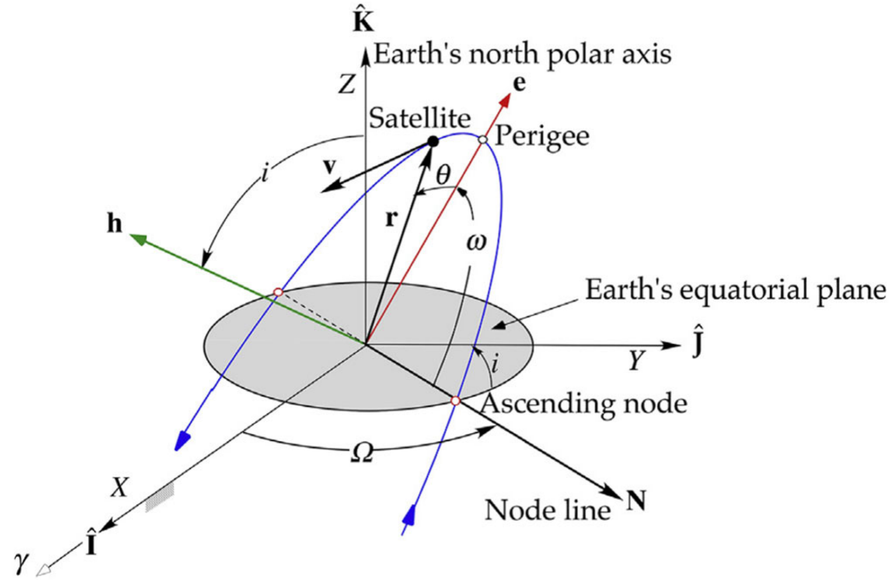
$v_r > 0$

Example 15

A satellite position and eccentricity vectors is given in the ECI at some instant.
If $\gamma < 0$, determine the orbital plane dihedral angles, i.e $(i,\ \Omega\ ,\ \omega)$.

$$[\mathbf{r}]^{ECI} = [-6634.2 \quad -1261.8 \quad -5230.9]^T \ km$$

$$[\mathbf{e}]^{ECI} = [-0.4097 \quad -0.4875 \quad -0.6364]^T$$



Solution:

$$\cos\ \theta = \mathbf{e}.\mathbf{r}/(er) \rightarrow \theta = -30\ \text{ deg}$$

$$\mathbf{u}_h = \frac{\mathbf{r}\times\mathbf{e}}{|\mathbf{r}\times\mathbf{e}|} = [-0.4545 \quad -0.5417 \quad 0.7071]^T$$

$$\cos(i) = \mathbf{u}_h(3) \rightarrow i = 45\ \text{ deg}$$

$$\mathbf{n} = \mathbf{K}\times\mathbf{u}_h = [0.5417 \quad -0.4545 \quad 0]^T$$

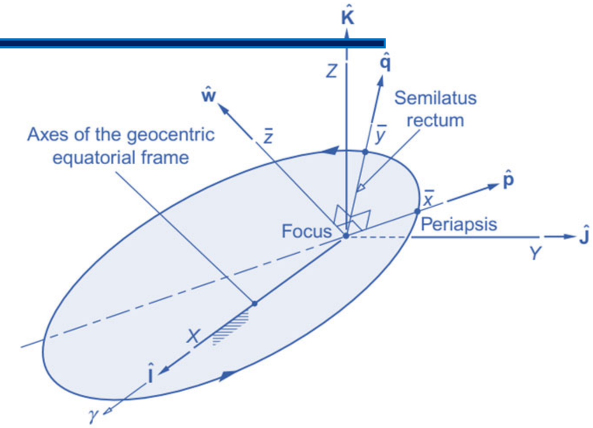
$$\cos(\Omega) = \mathbf{n}(1) \xrightarrow{n(2)<0} \Omega = 360 - 57.2 = 302.8\ \text{ deg}$$

$$\cos\ \omega = \mathbf{n}.\mathbf{e}/e \xrightarrow{e(3)<0} \omega = 360 - 90\ \text{ deg} = 270\ \text{ deg}$$

OE to State Vector to transformation

In some instances, it is desired to convert a satellite Orbital elements (OE) to its corresponding State vector (SV) . Inverse transformation from OE to SV is possible using a transformation matrix between the Perifocl CS and ECI. The problem is defined as :

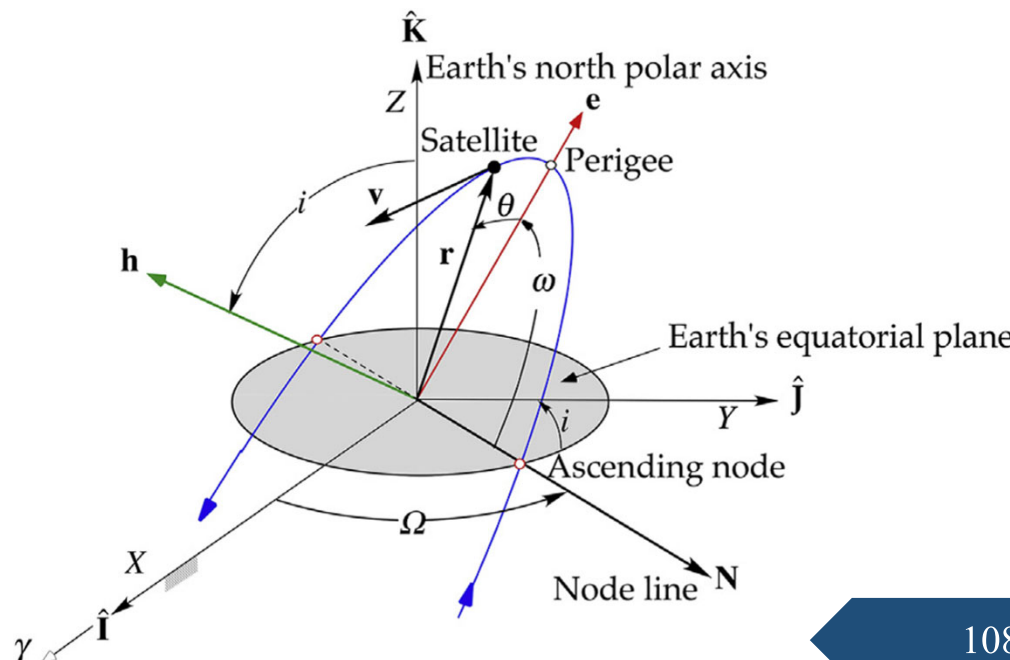
Given the satellite OE, find its corresponding SV.



$$\alpha = [\,a \quad e \quad i \quad \Omega \quad \omega \quad \theta\,]^T \Rightarrow X = [\vec{r} \quad \vec{v}]^T$$

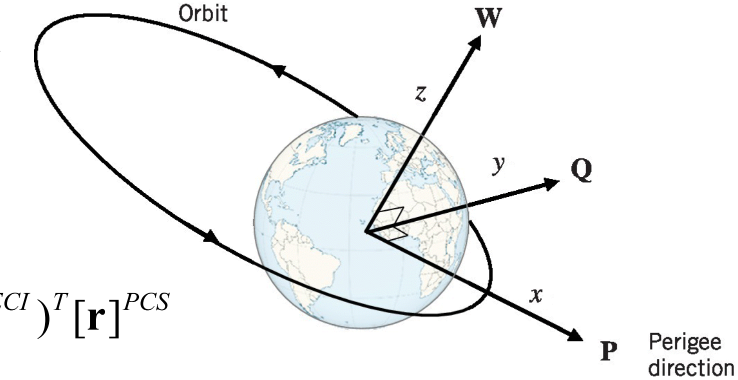
$$Coordination\ in\ the\ PCS: \bar{x} \equiv PCS; X \equiv ECI$$

$$\begin{cases} \mathbf{r} = \bar{x}\hat{\mathbf{p}} + \bar{y}\hat{\mathbf{q}} = \dfrac{h^2}{\mu}\dfrac{1}{1+e\cos\theta}(\cos\theta\hat{\mathbf{p}} + \sin\theta\hat{\mathbf{q}}) \\ \mathbf{v} = \dot{\bar{x}}\hat{\mathbf{p}} + \dot{\bar{y}}\hat{\mathbf{q}} = \dfrac{\mu}{h}[-\sin\theta\hat{\mathbf{p}} + (e+\cos\theta)\hat{\mathbf{q}}] \\ \rightarrow \begin{cases} \{\mathbf{r}\}_{\bar{x}} = \dfrac{h^2}{\mu}\dfrac{1}{1+e\cos\theta}\begin{Bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{Bmatrix} \\ \{\mathbf{v}\}_{\bar{x}} = \dfrac{\mu}{h}\begin{Bmatrix} -\sin\theta \\ e+\cos\theta \\ 0 \end{Bmatrix} \end{cases} \end{cases}$$



Transformation between ECI and PCS

In other words given the OE, one can simply generate the Position and Velocity vector of the satellite in the PCS. Next via a Transformation matrix presented in the next slide, this data can be converted into the ECI coordinate system.



$$[\mathbf{r}]^{PCS} = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta}\begin{bmatrix}\cos\theta\\ \sin\theta\\ 0\end{bmatrix} \Rightarrow$$

$$[\mathbf{r}]^{ECI} = \mathbf{T}^{ECI-PCS}[\mathbf{r}]^{PCS} = (\mathbf{T}^{PCS-ECI})^T[\mathbf{r}]^{PCS}$$

$$[\mathbf{r}]^{PCS} = \begin{Bmatrix}\bar{x}\\ \bar{y}\\ 0\end{Bmatrix} = \mathbf{T}^{PCS-ECI}[\mathbf{r}]^{ECI}$$

$$[\mathbf{v}]^{PCS} = \frac{\mu}{h}\begin{bmatrix}-\sin\theta\\ e+\cos\theta\\ 0\end{bmatrix} \Rightarrow$$

$$[\mathbf{v}]^{ECI} = \mathbf{T}^{ECI-PCS}[\mathbf{v}]^{PCS} = (\mathbf{T}^{PCS-ECI})^T[\mathbf{v}]^{PCS}$$

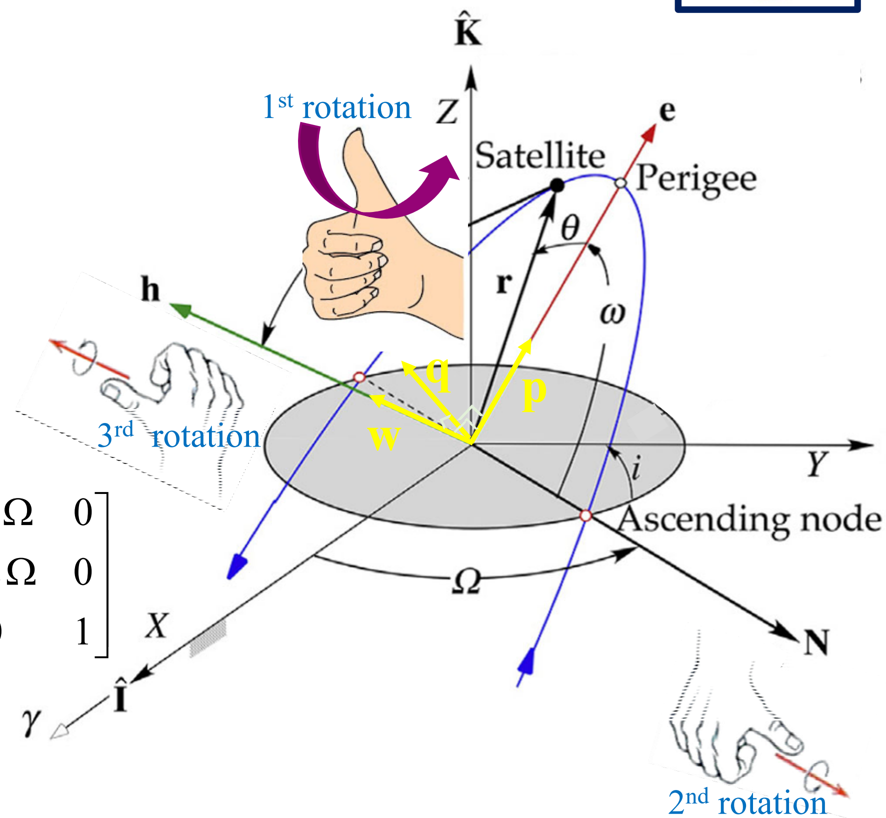
$$[\mathbf{v}]^{PCS} = \begin{Bmatrix}\dot{\bar{x}}\\ \dot{\bar{y}}\\ 0\end{Bmatrix} = \mathbf{T}^{PCS-ECI}[\mathbf{v}]^{ECI}$$

Transformation Matrix between ECI and Perifocal CS

ECI to PCS Transformation

$$[\mathbf{Q}]_{\overrightarrow{X\bar{x}}} = \mathbf{T}^{\overleftarrow{PCS-ECI}} = [\mathbf{R}_3(\omega)][\mathbf{R}_1(i)][\mathbf{R}_3(\Omega)]$$

3rd rotation 2nd rotation 1st rotation



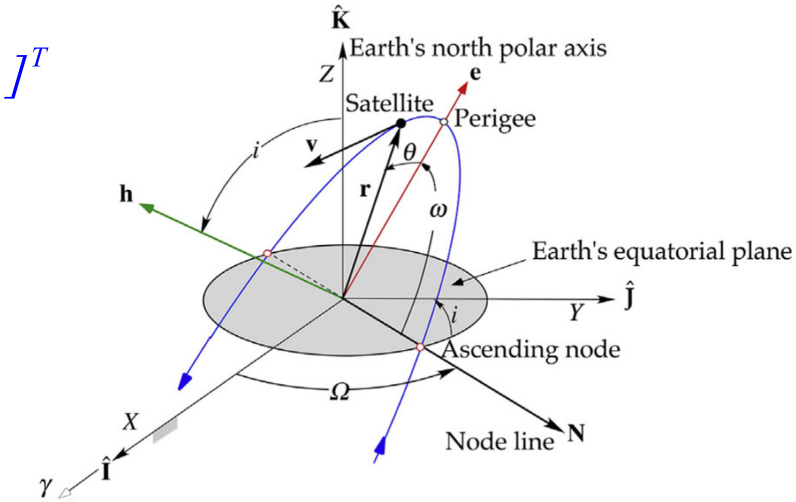
$$[\mathbf{Q}]_{X\bar{x}} = \mathbf{T}^{PCS-ECI} = \begin{bmatrix} \cos\omega & \sin\omega & 0 \\ -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & \sin i \\ 0 & -\sin i & \cos i \end{bmatrix} \begin{bmatrix} \cos\Omega & \sin\Omega & 0 \\ -\sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Note: [\mathbf{Q}]_{X\bar{x}}^T = [\mathbf{Q}]_{\bar{x}X}$$

Example 16

Consider an Earth orbiting satellite whose OE at one instant is provided below.
Determine its corresponding SV in the ECI coordinate system.

$\alpha=[\ h=8\times10^4 Km^2/s \quad e=1.4 \quad i=30^\circ \quad \Omega=40^\circ \quad \omega=60^\circ \quad \theta=30^\circ\ ]^T$



$Coordination\ in\ the\ PCS: \bar{x}\equiv PCS; X\equiv ECI$

$$\begin{cases} \mathbf{r}=\bar{x}\hat{\mathbf{p}}+\bar{y}\hat{\mathbf{q}}=\dfrac{h^2}{\mu}\dfrac{1}{1+e\cos\theta}(\cos\theta\hat{\mathbf{p}}+\sin\theta\hat{\mathbf{q}}) \\ \mathbf{v}=\dot{\bar{x}}\hat{\mathbf{p}}+\dot{\bar{y}}\hat{\mathbf{q}}=\dfrac{\mu}{h}[-\sin\theta\hat{\mathbf{p}}+(e+\cos\theta)\hat{\mathbf{q}}] \\ \rightarrow\begin{cases}\{\mathbf{r}\}_{\bar{x}}=\{6285 \quad 3628.6 \quad 0\}Km \\ \{\mathbf{v}\}_{\bar{x}}=\{-2.4913 \quad 11.29 \quad 0\}Km/s\end{cases} \end{cases}$$

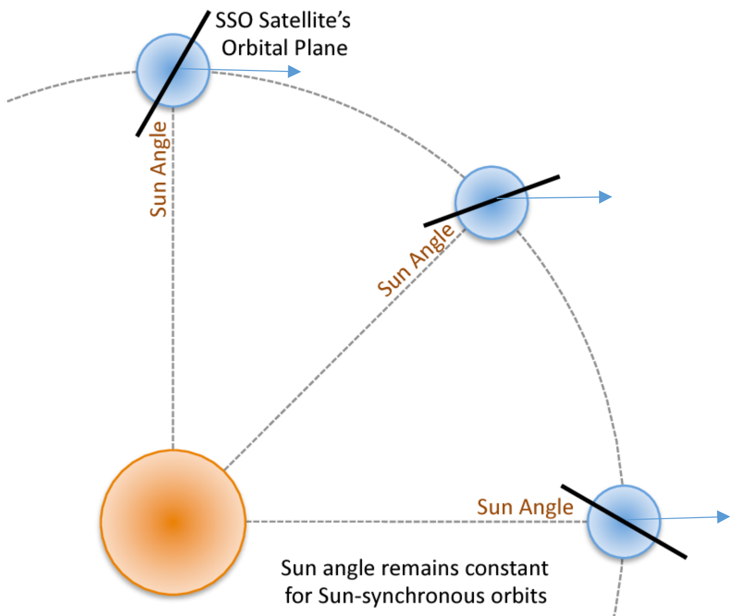
$$[\mathbf{Q}]_{X\bar{x}}=\mathbf{T}^{PCS-ECI}=\begin{bmatrix}\cos\omega & \sin\omega & 0 \\ -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 \\ 0 & \cos i & \sin i \\ 0 & -\sin i & \cos i\end{bmatrix}\begin{bmatrix}\cos\Omega & \sin\Omega & 0 \\ -\sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}-0.09907 & -0.94175 & 0.3214 \\ 0.8959 & 0.2249 & -0.383 \\ 0.433 & 0.25 & 0.866\end{bmatrix}$$

$$\rightarrow\begin{cases}\{\mathbf{r}\}_X=[\mathbf{Q}]_{\bar{x}X}\{6285 \quad 3628.6 \quad 0\}^T Km=\{-4040 \quad 4815 \quad 3629\}^T Km \\ \{\mathbf{v}\}_X=[\mathbf{Q}]_{\bar{x}X}\{-2.4913 \quad 11.29 \quad 0\}^T=\{-10.39 \quad -4.772 \quad 1.744\}^T Km/s\end{cases}$$

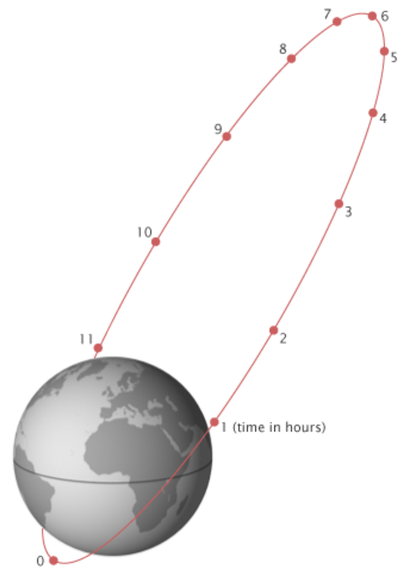
Orbital Classifications

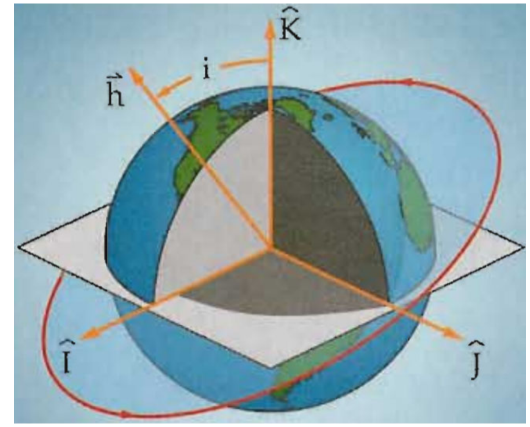
Orbit Classification in terms of Inclination

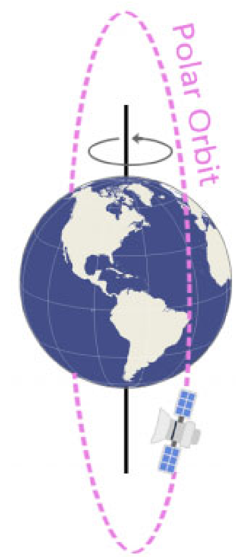
- ❑ Equatorial and Polar Orbits
- ❑ Prograde/ Retrograde Orbits
- ❑ Sun Synchronous Orbits



$$\dot{\Omega} = \omega_{Earth\_Sun}$$



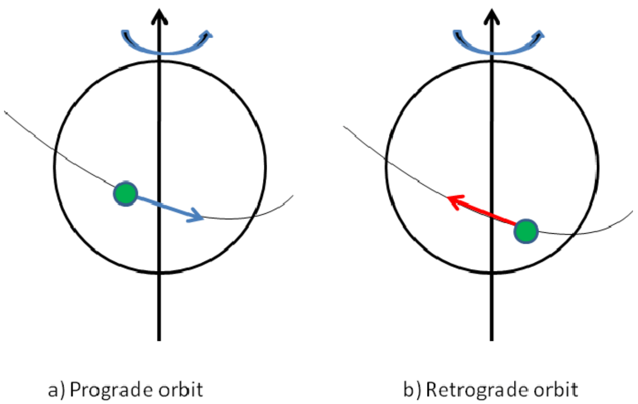




Orbit Classification in terms of Inclination

Prograde/Direct Orbits:

- Natural bodies within the Solar system : All of the planets and most of the other objects that orbit the Sun, with the exception of many comets.
- Artificial Satellites: Almost all communication satellites .



Retrograde Orbits:

Most commercial Earth observing satellites,

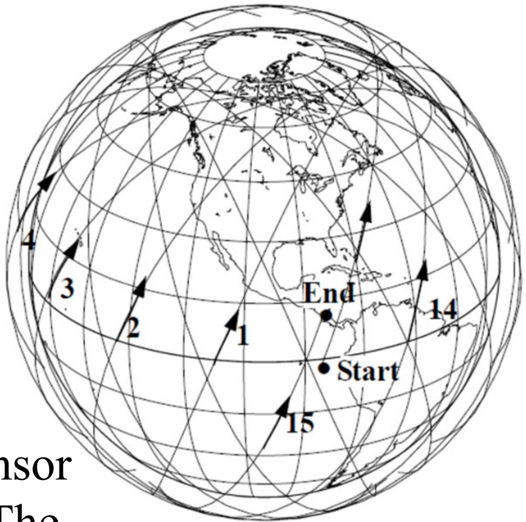
Sun-synchronous Orbits (useful for imaging, spy, and weather satellites.)

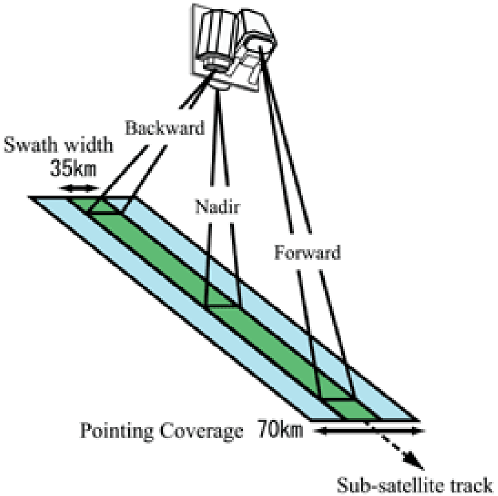
$$0<i<90^\circ \Rightarrow \text{Prograde Orbit Or } 90^\circ<i<180^\circ \Rightarrow \text{Retograde Orbit}$$

Ground Track

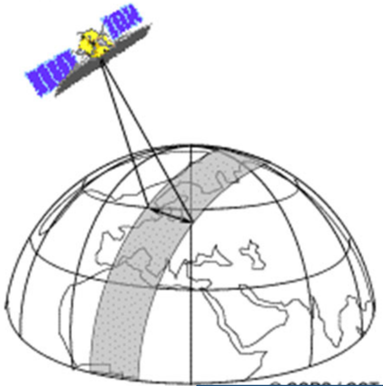
Ground track (GT) or ground trace is the projection of the satellite's orbit onto the surface of the Earth. Some of its properties and advantages are:

- GT is a powerful tool for determining orbit position and location relative to a ground site.
- Typically, satellites have a roughly sinusoidal ground track.
- Ground track of a SC can take a number of different forms, depending on the values of the orbital elements.



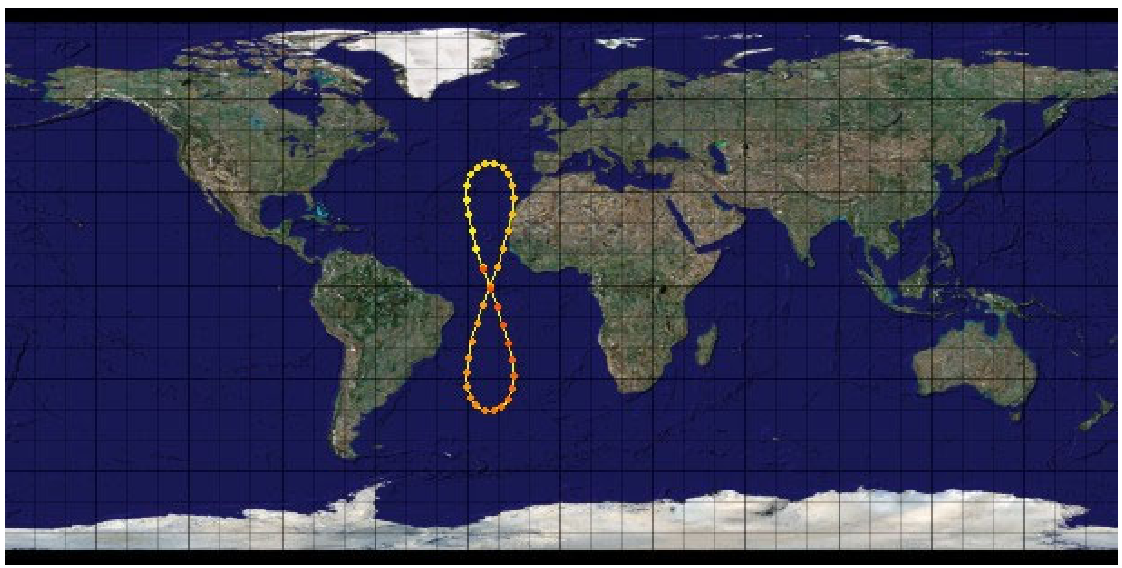


As a satellite revolves around the Earth, the sensor "sees" a certain portion of the Earth's surface. The area imaged on the surface, is referred to as the swath. Imaging swaths for space borne sensors generally vary between tens and hundreds of kilometers wide. Due to Earth rotation, the satellite swath covers a new area with each consecutive pass or orbit around the Earth.



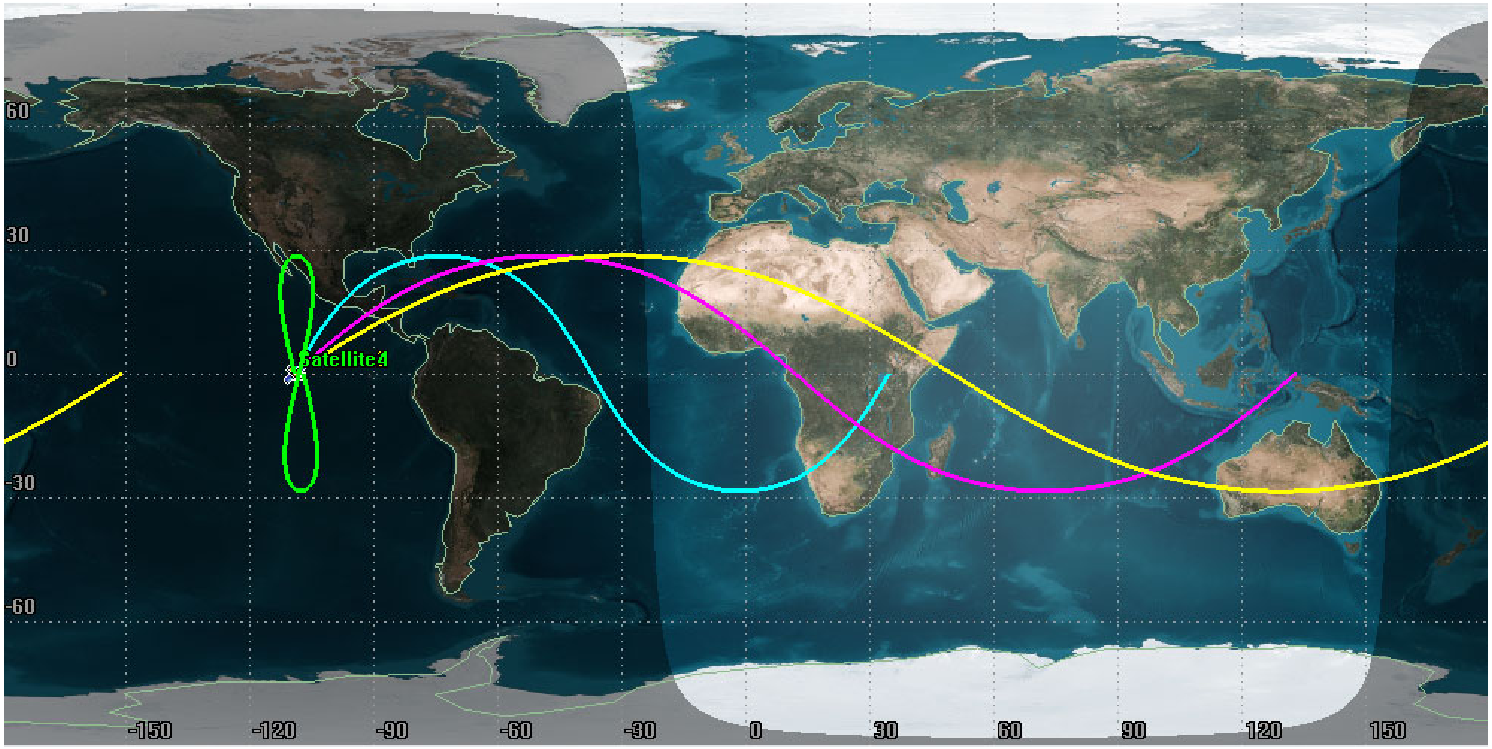
Geosynchronous Ground Track

In general a geosynchronous orbit (GSO) is an Earth-centered circular or elliptical orbit with an orbital period that matches the Earth's rotation rate (23 hours, 56 minutes, and 4 seconds denotes by one sidereal day). This synchronization means that a satellite in GSO returns to the same position after a period of one sidereal day. During the course of a day, the satellite position in the sky may remain still or trace out a path typically in a figure-8 form whose precise characteristics depend on the orbit's inclination and eccentricity. A circular geosynchronous orbit has a constant altitude of 35,786 km (22,236 mi), and all geosynchronous orbits share that semi-major axis. A special case of geosynchronous orbit is the geostationary orbit, which is a circular geosynchronous orbit in the Earth's equatorial plane. A satellite in a geostationary orbit remains in the same position in the sky to observers on the surface that is a single point on the Earth's equator.



Ground Track

- Effect of the semi-major axis



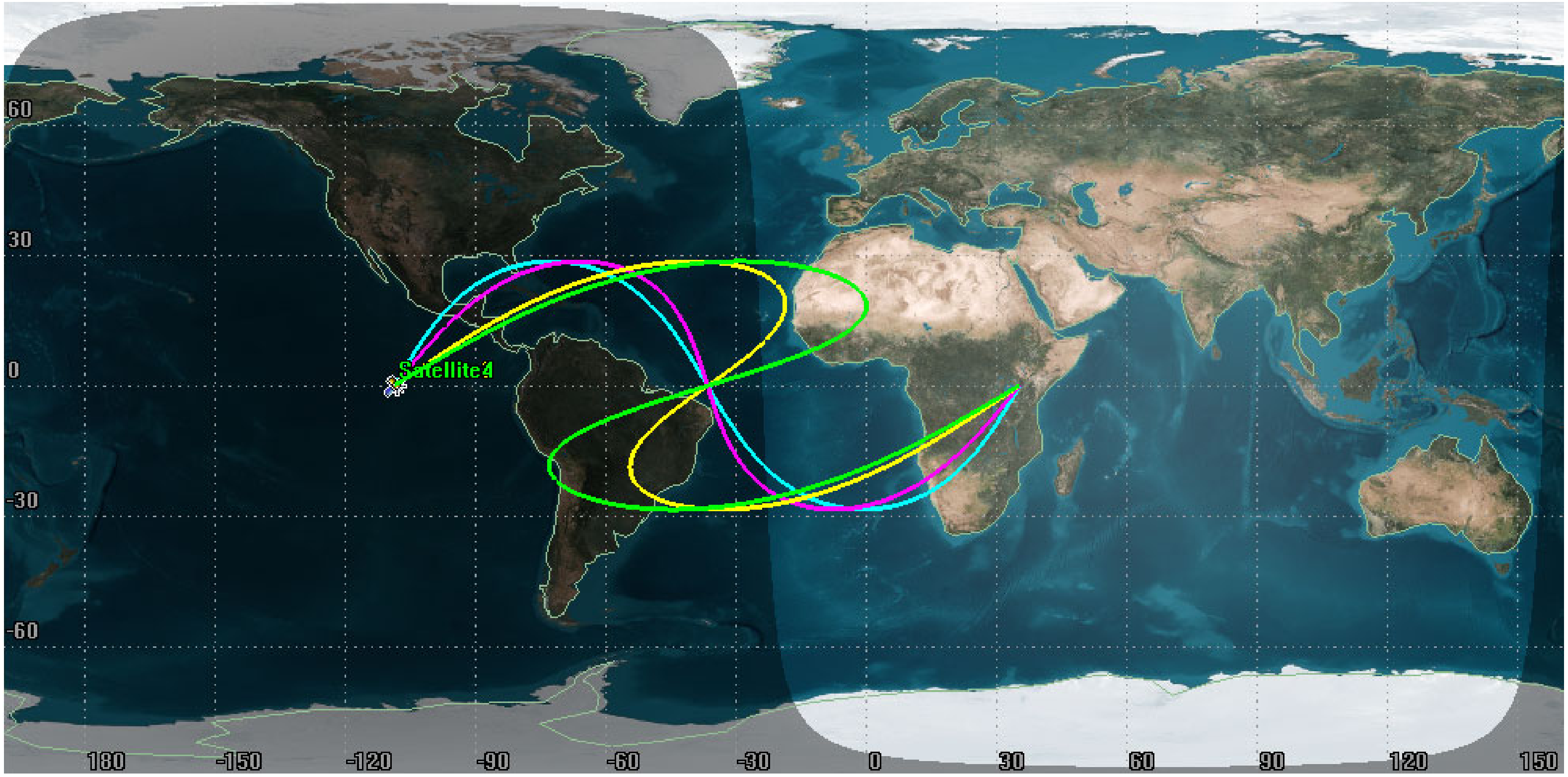
a = 10000 *km*

a = 20000 *km*

a = 30000 *km*

a = 42000 *km*

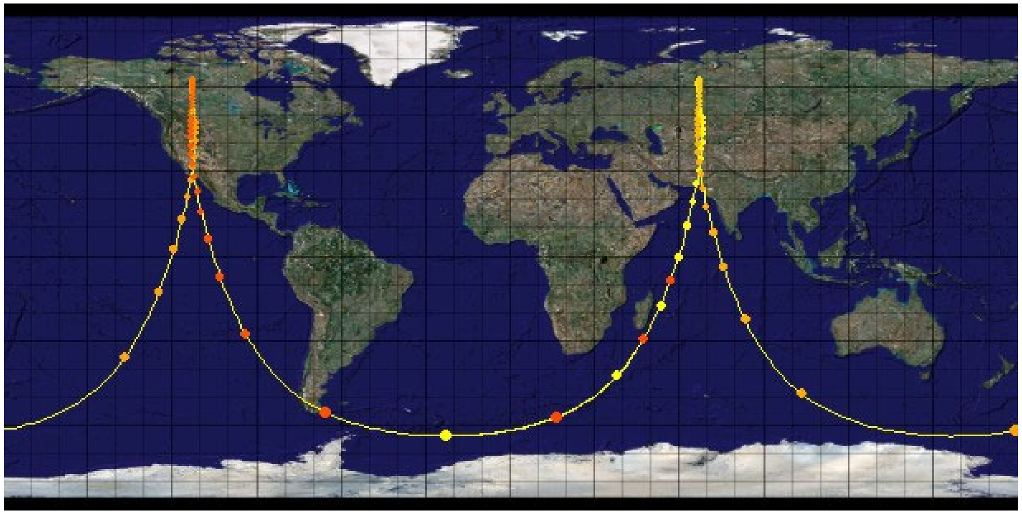
Ground Track : Effect of Eccentricity

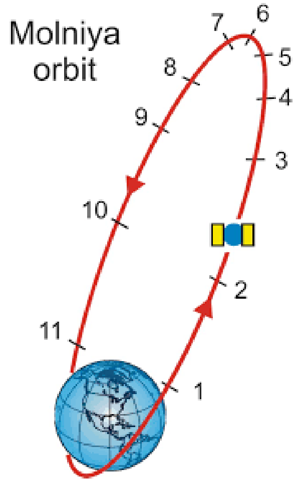


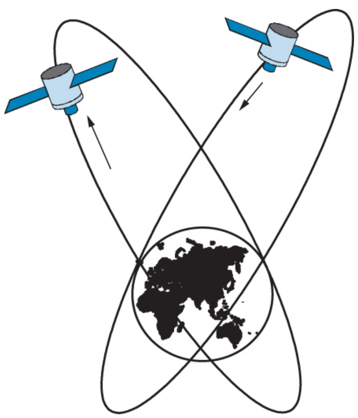
e = 0 e = 0.1 e = 0.5 e = 0.7

Molniya Ground Track

The Molniya is a type of satellite orbit designed to provide communications and remote sensing coverage over high latitudes. Its orbit is designed such that its argument of perigee is fixed. An orbit designed in this manner is called a frozen orbit.
of It is a highly elliptical orbit with the following properties:
Argument of perigee: 270°
Inclination: 63.4°
Period: 718 minutes
Eccentricity: 0.74
Semi-major axis: 26,600 km (16,500 mi)





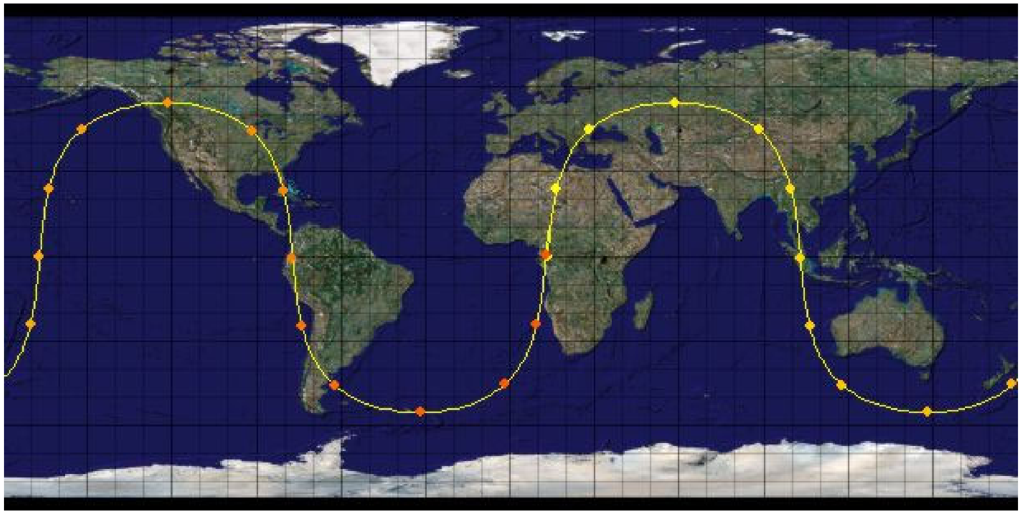


Molniya orbit.

GPS Orbit Ground Track

The satellites in the GPS constellation are arranged into six equally-spaced orbital planes surrounding the Earth. Each plane contains four "slots" occupied by baseline satellites. This 24-slot arrangement ensures users can view at least four satellites from virtually any point on the planet.

- The altitude of each of the 24 space vehicles providing the GPS signal is 20200 km.
- The satellites are distributed equally among six circular orbital planes, which have an inclination of approximately 55° and are separated by 60°.



Ground Track Computations

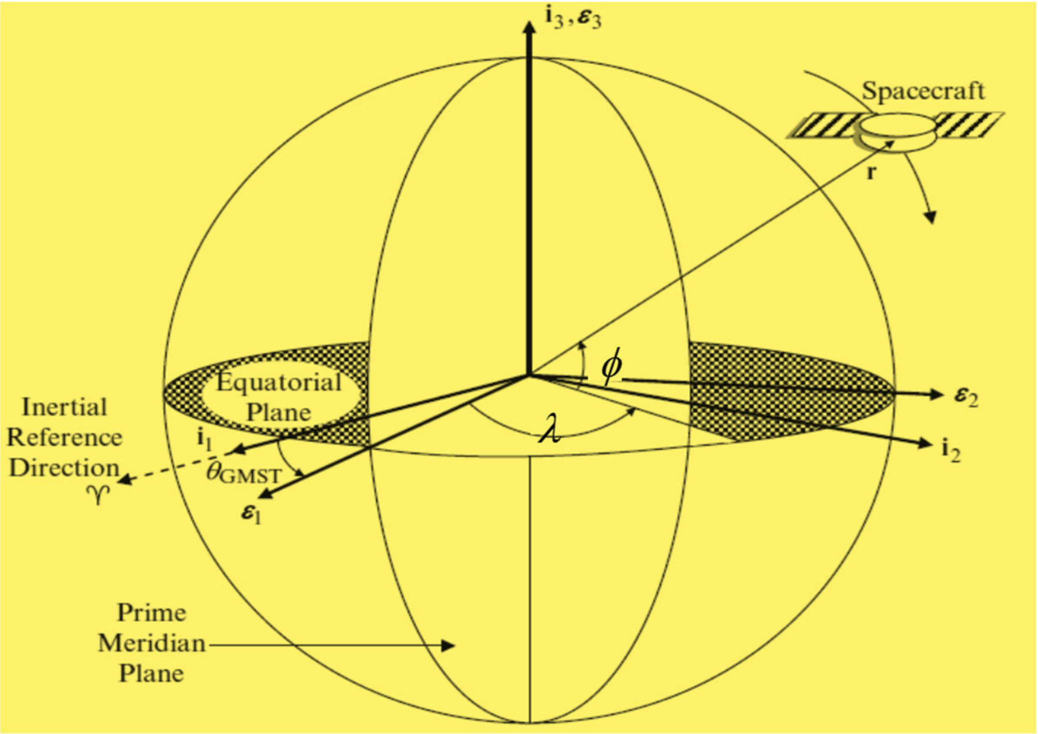
Ground track (GT) or ground trace is the projection of the satellite's orbit onto the surface of the Earth.

$$[\mathbf{R}_3(\theta)] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Where :

$$\theta = \theta_{GMST} + \omega_E (t - t_0)$$

$$[\mathbf{r}]^{ECEF} = [\mathbf{R}_3(\theta)][\mathbf{r}]^{ECI}$$

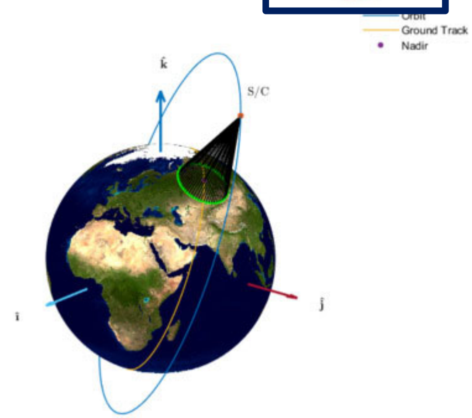


Ground Track

Longitude and Latitude

- Spherical Earth

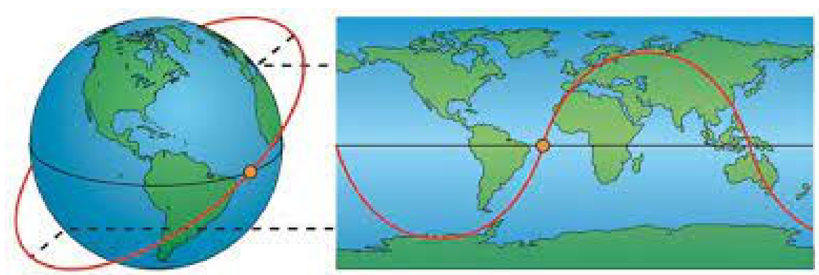
$$\color{blue}{input : [\mathbf{r}]^{ECEF} = [r_x \quad r_y \quad r_z]^T} = [r\cos\varphi_{gc}\cos\lambda \quad r\cos\varphi_{gc}\sin\lambda \quad r\sin\varphi_{gc}]^T$$



$$\hat{\mathbf{u}}_r = \frac{\mathbf{r}}{\|\mathbf{r}\|} = [l \quad m \quad n]^T$$

$$\varphi_{gc} = \sin^{-1} n$$

$$\lambda = \begin{cases} \cos^{-1}\left(\dfrac{l}{\cos\varphi_{gc}}\right) & m > 0 \\ 360^\circ - \cos^{-1}\left(\dfrac{l}{\cos\varphi_{gc}}\right) & m \le 0 \end{cases}$$



Ground Track

- Ellipsoidal Earth

$$\mathbf{r} = \begin{bmatrix} (N+h_{ellp.})\cos\varphi_{gd}\cos\lambda \\ (N+h_{ellp.})\cos\varphi_{gd}\sin\lambda \\ (N(1-e_E^2)+h_{ellp.})\sin\varphi_{gd} \end{bmatrix}$$

$$N = \frac{R_E}{\sqrt{1-e_E^2\sin^2\varphi_{gd}}}$$

e_E=0.08081

$$input: [\mathbf{r}]^{ECEF} = [r_x \quad r_y \quad r_z]^T$$

$$Longitude: \lambda = ATAN2(r_y, r_x)$$

$$Equatorial\ Range: r_{xy} = SQRT(r_x^2 + r_y^2)$$

$$Initial\ Latitude: \varphi_0 = ATAN2(r_z, r_{xy})$$

$$i = 0;$$

$$Error = K >> \varepsilon$$

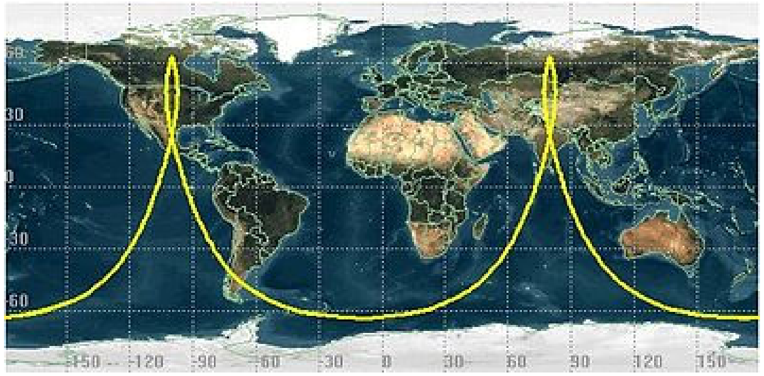
$$while\ Error > \varepsilon$$

$$N = \frac{R_E}{\sqrt{1-e_E^2\sin^2\varphi_i}}$$

$$\varphi_{i+1} = ATAN2(r_z + Ne_E^2\sin(\varphi_i), r_{xy})$$

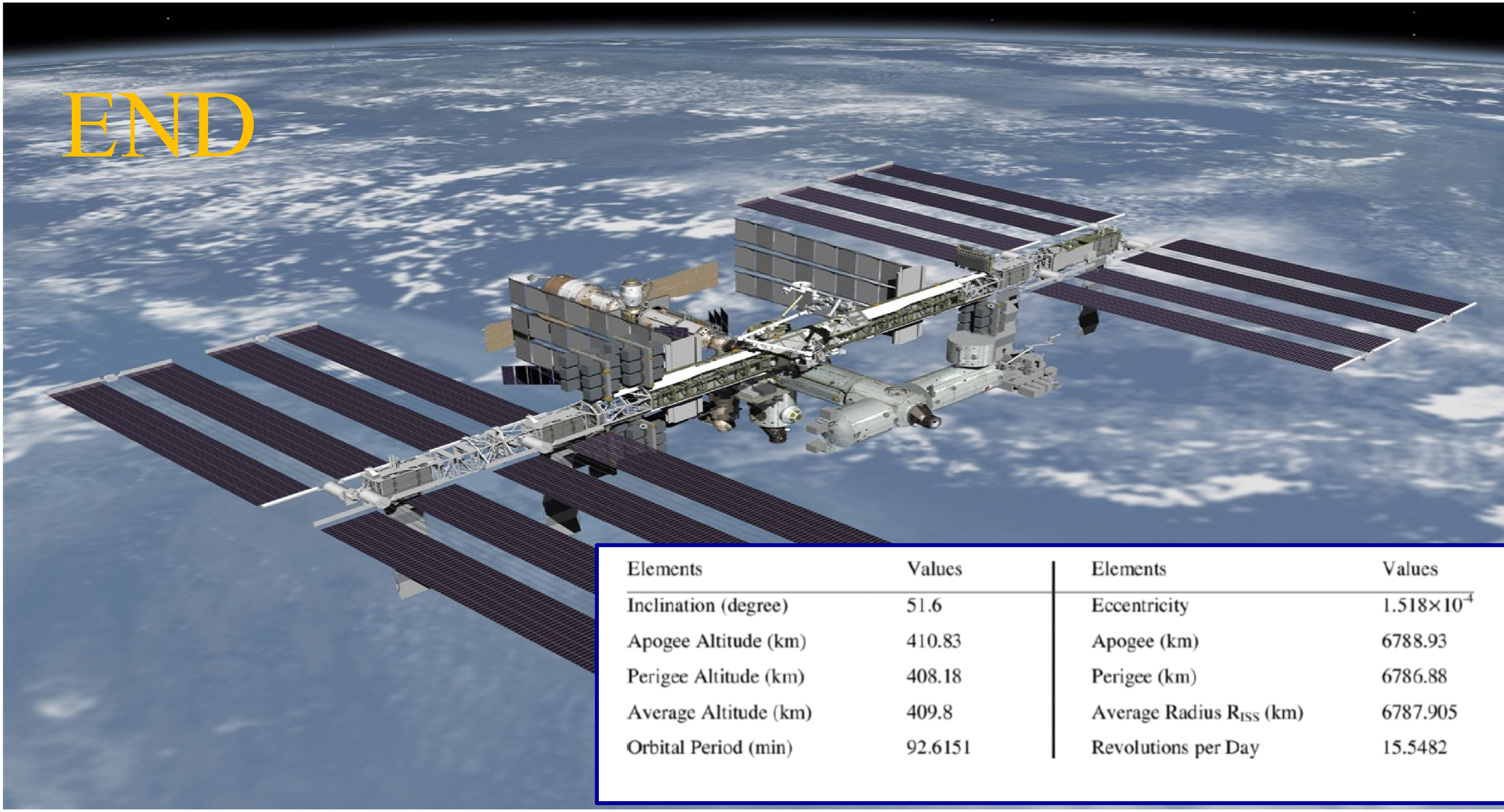
$$error = |\varphi_{i+1} - \varphi_i|$$

$$end$$



$$h_{ellp} = \frac{r_{xy}}{\cos(\varphi_i)} - N, \qquad if\ near\ poles(\sim 1°): h_{ellp} = \frac{r_z}{\sin(\varphi_i)} - N\sqrt{1-e_E^2}$$

END



Elements	Values	Elements	Values
Inclination (degree)	51.6	Eccentricity	1.518×10^{-4}
Apogee Altitude (km)	410.83	Apogee (km)	6788.93
Perigee Altitude (km)	408.18	Perigee (km)	6786.88
Average Altitude (km)	409.8	Average Radius R_{ISS} (km)	6787.905
Orbital Period (min)	92.6151	Revolutions per Day	15.5482