Home Work #4

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January 25, 2023

1 Question 1

We know inertia matrix is symetric so we have:

$$I_{xy} = I_{yx} = 10, \quad I_{xz} = I_{zx} = 0, \quad I_{yz} = I_{zy}$$

$$\mathbf{I} = \begin{bmatrix} 30 & -10 & 0 \\ -10 & 20 & -I_{yz} \\ 0 & -I_{yz} & 30 \end{bmatrix}$$

1.1 part a

We know that:

$$\mathbf{I} \times \boldsymbol{\omega} = \mathbf{h} \tag{1}$$

$$\boldsymbol{\omega} = \begin{bmatrix} 10 & 10 & 10 \end{bmatrix}^T_{RPS} = \begin{bmatrix} 10 \times 2\pi & 10 \times 2\pi & 10 \times 2\pi \end{bmatrix}^T_{rad/\sec}, \quad \mathbf{h} = \begin{bmatrix} 200 & 200 & 400 \end{bmatrix}^T_{kg.m^2/s}$$

If we use radian per second instead of revelotion per second $\mathbf{I} \times \boldsymbol{\omega} \neq \mathbf{h}$ would happen.

$$\mathbf{I} \times \boldsymbol{\omega} = \begin{bmatrix} 200 \\ 100 - 10I_{yz} \\ 300 - 10I_{yz} \end{bmatrix} = \begin{bmatrix} 200 \\ 200 \\ 400 \end{bmatrix} \to I_{yz} = -10 \to \mathbf{I} = \begin{bmatrix} 30 & -10 & 0 \\ -10 & 20 & 10 \\ 0 & 10 & 30 \end{bmatrix}$$

$$T_{Rotational} = \frac{1}{2}\boldsymbol{\omega}^T \times \mathbf{I} \times \boldsymbol{\omega} = \frac{1}{2}\begin{bmatrix} 10 & 10 & 10 \end{bmatrix} \times \begin{bmatrix} 30 & -10 & 0 \\ -10 & 20 & 10 \\ 0 & 10 & 30 \end{bmatrix} \times \begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix} = 4000 \tag{2}$$

1.2 part b

$$T_{Rotational} = \frac{1}{2}\mathrm{I}_\xi \omega^2 \to \mathrm{I}_\xi = \frac{2T_{Rotational}}{\omega^2} = \frac{2 \times 4000}{300} = 26.67 \tag{3}$$

Rotation matrix calculated via eigen vector of inertia matrix (used MATLAB to calculate).

$$\mathbf{A} = \mathrm{eig}(\mathbf{I}) = \begin{bmatrix} -0.4082 & -0.7071 & -0.5774 \\ -0.8165 & -0.0000 & 0.5774 \\ 0.4082 & -0.7071 & 0.5774 \end{bmatrix} \tag{4}$$

$$\mathbf{I}' = \mathbf{A}^T \times \mathbf{I} \times \mathbf{A} = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 40 \end{bmatrix}, \quad \boldsymbol{\omega}' = \mathbf{A}^T \boldsymbol{\omega} = \begin{bmatrix} -8.1650 \\ -14.1421 \\ 5.7735 \end{bmatrix}$$

$$h = |\mathbf{I} \times \boldsymbol{\omega}| = 489.8979$$

We know that above matrix shows the direction cosines of the principal axes with the primary body axes. Used MATLAB function (dcm2angle) to calculate euler angles between two corinate system.

$$\begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} = \begin{bmatrix} 45.00° \\ 35.26° \\ -120.00° \end{bmatrix}$$

1.3 part c

Ellipsoid of inertia calculated as folow:

$$\frac{X^2}{\left(\sqrt{\frac{1}{\mathrm{I}_x}}\right)^2} + \frac{Y^2}{\left(\sqrt{\frac{1}{\mathrm{I}_y}}\right)^2} + \frac{Z^2}{\left(\sqrt{\frac{1}{\mathrm{I}_z}}\right)^2} = \frac{X^2}{\left(\sqrt{\frac{1}{10}}\right)^2} + \frac{Y^2}{\left(\sqrt{\frac{1}{30}}\right)^2} + \frac{Z^2}{\left(\sqrt{\frac{1}{40}}\right)^2} = 1 \tag{5}$$

Figure 1: ellipsoid of inertia

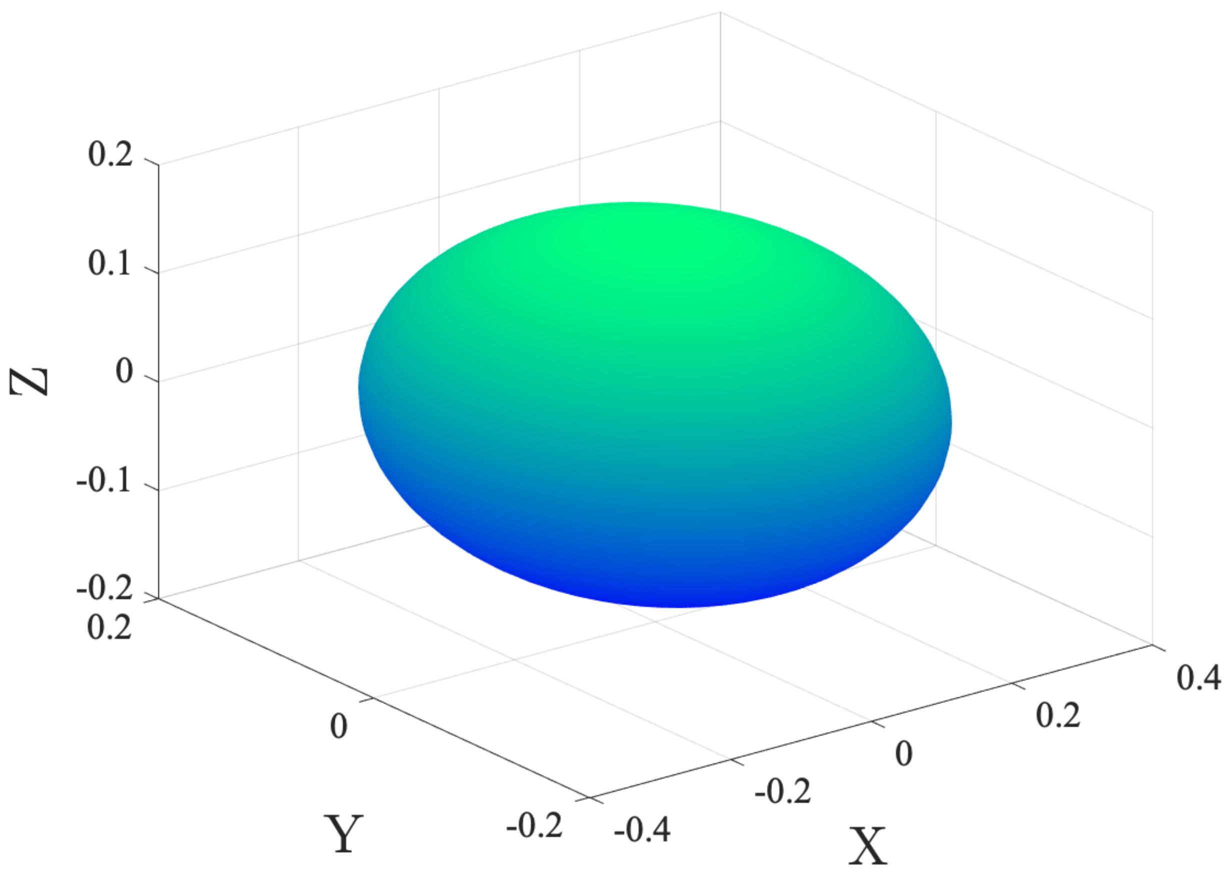


Figure 2: ellipsoid of inertia in zx plane

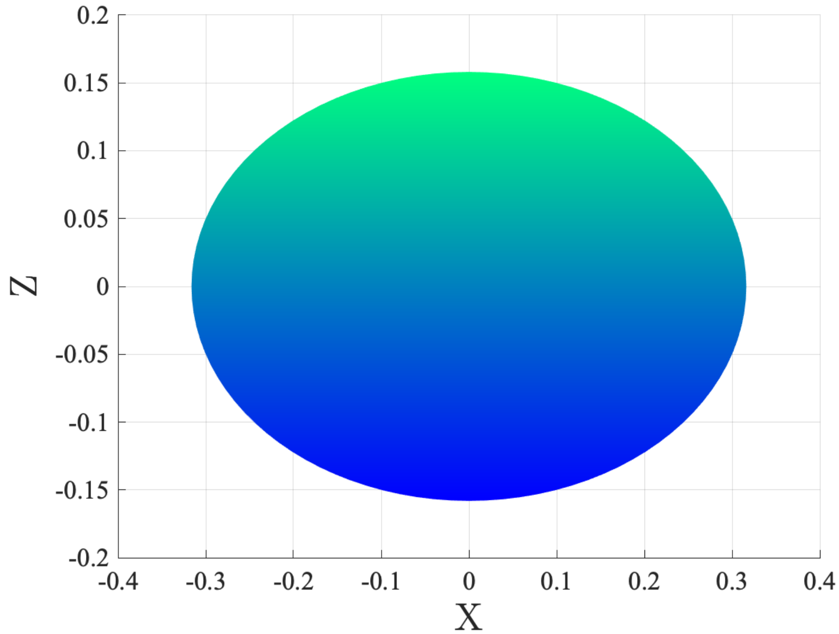


Figure 3: ellipsoid of inertia zy plane

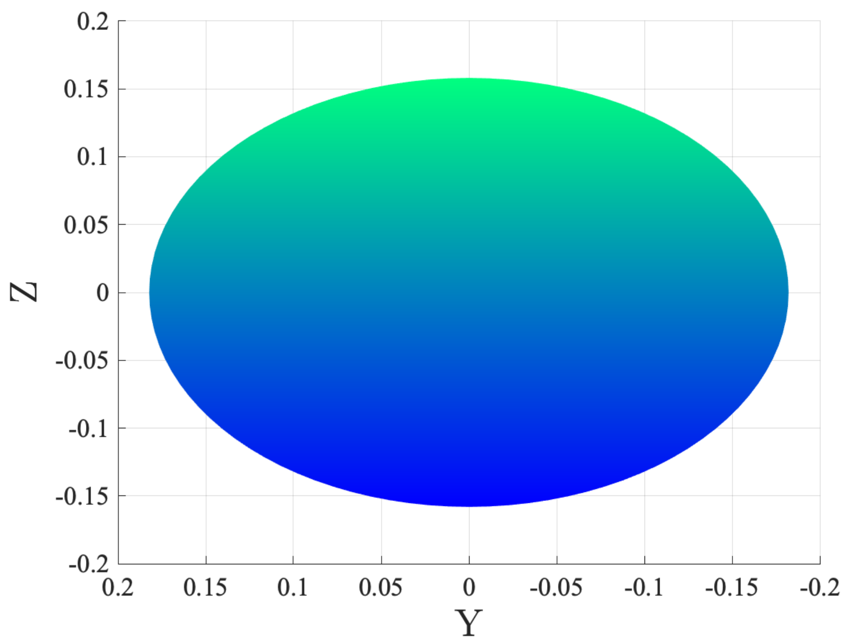
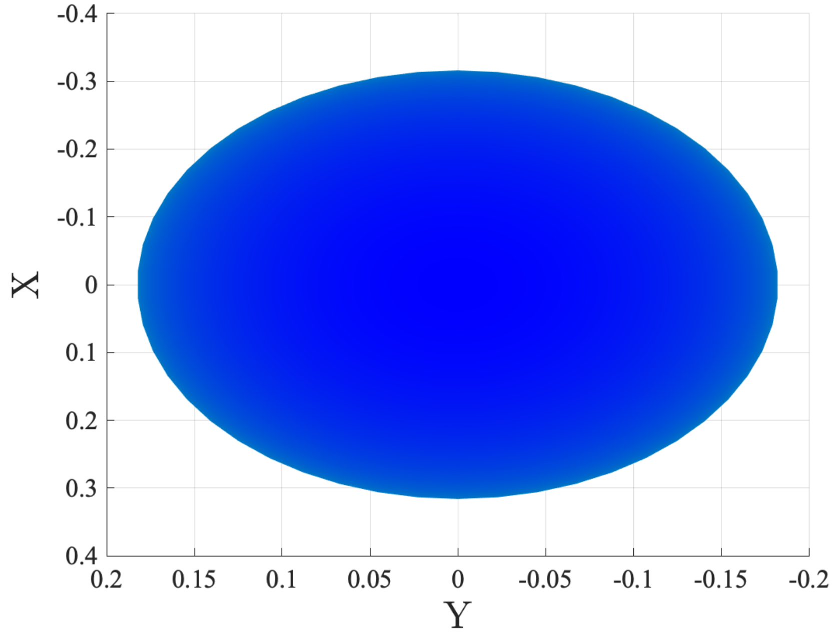


Figure 4: ellipsoid of inertia in xy plane



1.4 part d

Angular momentum and rotational kinetic energy ellipsoid calculated as folow:

$$\frac{\omega_x^2}{\left(\dfrac{h}{\mathrm{I}_x}\right)^2} + \frac{\omega_y^2}{\left(\dfrac{h}{\mathrm{I}_y}\right)^2} + \frac{\omega_z^2}{\left(\dfrac{h}{\mathrm{I}_z}\right)^2} = \frac{\omega_x^2}{\left(\dfrac{490}{10}\right)^2} + \frac{\omega_y^2}{\left(\dfrac{490}{30}\right)^2} + \frac{\omega_z^2}{\left(\dfrac{490}{40}\right)^2} = 1 \tag{6}$$

$$\frac{\omega_x^2}{\left(\sqrt{\dfrac{2T}{\mathrm{I}_x}}\right)^2} + \frac{\omega_y^2}{\left(\sqrt{\dfrac{2T}{\mathrm{I}_y}}\right)^2} + \frac{\omega_z^2}{\left(\sqrt{\dfrac{2T}{\mathrm{I}_z}}\right)^2} = \frac{\omega_x^2}{\left(\sqrt{\dfrac{2\times 4000}{10}}\right)^2} + \frac{\omega_y^2}{\left(\sqrt{\dfrac{2\times 4000}{30}}\right)^2} + \frac{\omega_z^2}{\left(\sqrt{\dfrac{2\times 4000}{40}}\right)^2} = 1 \tag{7}$$

Ellipsoid parameter calculated before, now, use them to draw the ellipsoid.

Figure 5: Angular momentum and rotational kinetic energy ellipsoid of inertia

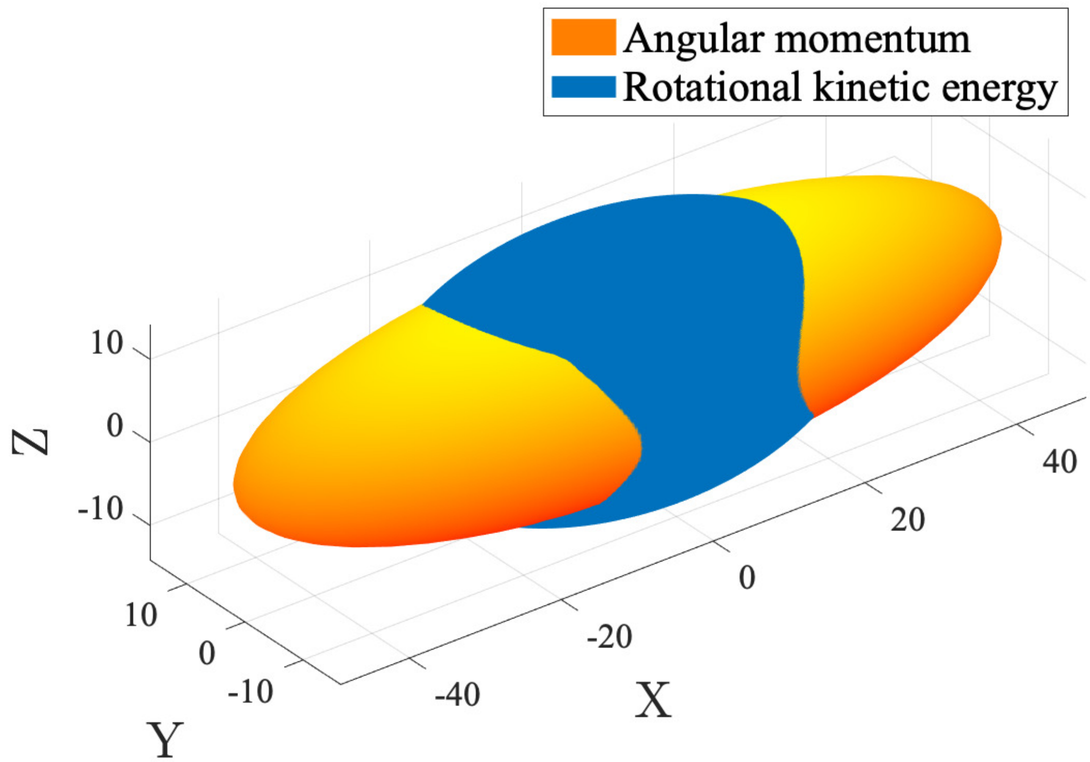


Figure 6: Angular momentum and rotational kinetic energy ellipsoid of inertia in zx plane

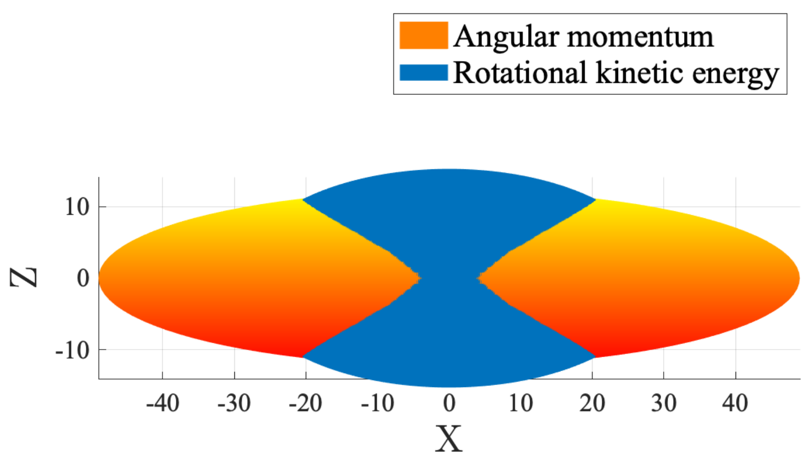


Figure 7: Angular momentum and rotational kinetic energy ellipsoid of inertia zy plane

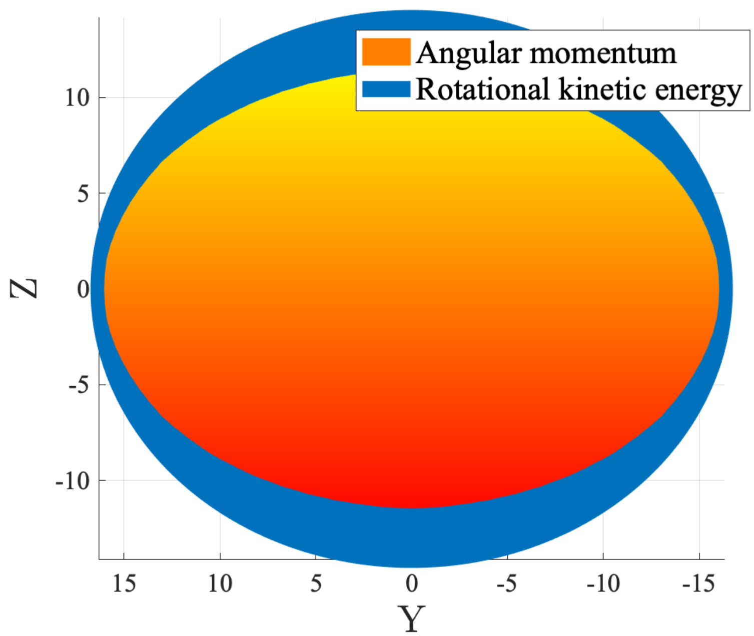
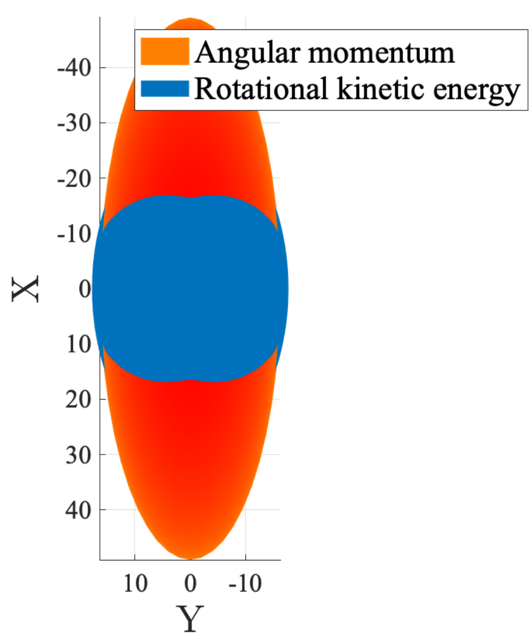


Figure 8: Angular momentum and rotational kinetic energy ellipsoid of inertia in xy plane



1.5 Bonus

$$\frac{\omega_x^2}{\left(\frac{h}{\mathrm{I}_x}\right)^2}+\frac{\omega_y^2}{\left(\frac{h}{\mathrm{I}_y}\right)^2}+\frac{\omega_z^2}{\left(\frac{h}{\mathrm{I}_z}\right)^2}=1\rightarrow\omega_x^2=-\frac{\left(\frac{h}{\mathrm{I}_x}\right)^2\omega_y^2}{\left(\frac{h}{\mathrm{I}_y}\right)^2}-\frac{\left(\frac{h}{\mathrm{I}_x}\right)^2\omega_z^2}{\left(\frac{h}{\mathrm{I}_z}\right)^2}+\left(\frac{h}{\mathrm{I}_x}\right)^2 \tag{8}$$

$$\frac{\omega_x^2}{\left(\sqrt{\frac{2T}{\mathrm{I}_x}}\right)^2}+\frac{\omega_y^2}{\left(\sqrt{\frac{2T}{\mathrm{I}_y}}\right)^2}+\frac{\omega_z^2}{\left(\sqrt{\frac{2T}{\mathrm{I}_z}}\right)^2}=1\rightarrow\frac{-\frac{\left(\frac{h}{\mathrm{I}_x}\right)^2\omega_y^2}{\left(\frac{h}{\mathrm{I}_y}\right)^2}-\frac{\left(\frac{h}{\mathrm{I}_x}\right)^2\omega_z^2}{\left(\frac{h}{\mathrm{I}_z}\right)^2}+\left(\frac{h}{\mathrm{I}_x}\right)^2}{\left(\sqrt{\frac{2T}{\mathrm{I}_x}}\right)^2}+\frac{\omega_y^2}{\left(\sqrt{\frac{2T}{\mathrm{I}_y}}\right)^2}+\frac{\omega_z^2}{\left(\sqrt{\frac{2T}{\mathrm{I}_z}}\right)^2}=1 \tag{9}$$

$$\rightarrow-\frac{\left(\frac{h}{\mathrm{I}_x}\right)^2\omega_y^2}{\left(\frac{h}{\mathrm{I}_y}\right)^2}-\frac{\left(\frac{h}{\mathrm{I}_x}\right)^2\omega_z^2}{\left(\frac{h}{\mathrm{I}_z}\right)^2}+\left(\frac{h}{\mathrm{I}_x}\right)^2+\frac{\left(\sqrt{\frac{2T}{\mathrm{I}_x}}\right)^2\omega_y^2}{\left(\sqrt{\frac{2T}{\mathrm{I}_y}}\right)^2}+\frac{\left(\sqrt{\frac{2T}{\mathrm{I}_x}}\right)^2\omega_z^2}{\left(\sqrt{\frac{2T}{\mathrm{I}_z}}\right)^2}=\left(\sqrt{\frac{2T}{\mathrm{I}_x}}\right)^2 \tag{10}$$

$$\omega_y^2\left(-\frac{\left(\frac{h}{\mathrm{I}_x}\right)^2}{\left(\frac{h}{\mathrm{I}_y}\right)^2}+\frac{\left(\sqrt{\frac{2T}{\mathrm{I}_x}}\right)^2}{\left(\sqrt{\frac{2T}{\mathrm{I}_y}}\right)^2}\right)+\omega_z^2\left(-\frac{\left(\frac{h}{\mathrm{I}_x}\right)^2}{\left(\frac{h}{\mathrm{I}_z}\right)^2}+\frac{\left(\sqrt{\frac{2T}{\mathrm{I}_x}}\right)^2}{\left(\sqrt{\frac{2T}{\mathrm{I}_z}}\right)^2}\right)=\left(\sqrt{\frac{2T}{\mathrm{I}_x}}\right)^2-\left(\frac{h}{\mathrm{I}_x}\right)^2 \tag{11}$$

$$\omega_y^2\left(\frac{\mathrm{I}_y}{\mathrm{I}_x}-\frac{\mathrm{I}_y^2}{\mathrm{I}_x^2}\right)+\omega_z^2\left(\frac{\mathrm{I}_z}{\mathrm{I}_x}-\frac{\mathrm{I}_z^2}{\mathrm{I}_x^2}\right)=\left(\sqrt{\frac{2T}{\mathrm{I}_x}}\right)^2-\left(\frac{h}{\mathrm{I}_x}\right)^2 \tag{12}$$

Mathematical formulas for the Polhode curves is a ellipsoid in yz plane.

2 Question 2

Data of the question:

$$\mathrm{I}_x=8_{kg.m^2},\quad \mathrm{I}_y=10_{kg.m^2},\quad \mathrm{I}_z=14_{kg.m^2}$$

$$\mathbf{q}(0)=\begin{bmatrix}0.3 & 0.2 & 0.5 & 0.7874\end{bmatrix}^T$$

$$\boldsymbol{\omega}(0)=\begin{bmatrix}2 & 3 & -5\end{bmatrix}^T\times 10^{-3}_{rps}=\begin{bmatrix}2\times 2\pi & 3\times 2\pi & -5\times 2\pi\end{bmatrix}^T\times 10^{-3}_{rad/\sec}$$

2.1 part a

Used MATLAB function(quat2eul) to calculate initial euler angles.

$$\begin{bmatrix}\phi\\ \theta\\ \psi\end{bmatrix}=\begin{bmatrix}137.74^\circ\\ -0.86^\circ\\ 65.16^\circ\end{bmatrix}$$

2.2 part b

Equation of motion with gravity gradient:

$$I_x\dot{\omega}_x + \omega_y\omega_z(I_z - I_y) = G_x \tag{13}$$
$$I_y\dot{\omega}_y + \omega_x\omega_z(I_x - I_z) = G_y \tag{14}$$
$$I_z\dot{\omega}_z + \omega_x\omega_y(I_y - I_x) = G_z \tag{15}$$
$$\tag{16}$$

$$\dot{\omega}_x = (G_x + \omega_y\omega_z(I_y - I_z))/I_x$$

$$\dot{\omega}_y = (G_y + \omega_x\omega_z(I_z - I_x))/I_y$$

$$\dot{\omega}_z = (G_z + \omega_y\omega_x(I_x - I_y))/I_z$$

$$\begin{bmatrix}\dot{\omega}_x\\ \dot{\omega}_y\\ \dot{\omega}_z\end{bmatrix} = \begin{bmatrix}p\\ q\\ r\end{bmatrix} + \mathbf{C}_R^b\begin{bmatrix}0\\ -\omega_0\\ 0\end{bmatrix} \rightarrow \begin{bmatrix}p\\ q\\ r\end{bmatrix} = \begin{bmatrix}\dot{\omega}_x\\ \dot{\omega}_y\\ \dot{\omega}_z\end{bmatrix} + \mathbf{C}_R^b\begin{bmatrix}0\\ \omega_0\\ 0\end{bmatrix} \tag{17}$$

where $\mathbf{C}_R^b$ is transformation matrix aand $\omega_0 = \sqrt{\dfrac{\mu}{R_0^3}}$.

$$\mathbf{C}_b^R = \begin{bmatrix}\cos(\theta)\cos(\psi) & -\cos(\phi)\sin(\psi) + \sin(\phi)\sin(\theta)\cos(\psi) & \sin(\phi)\sin(\psi) + \cos(\phi)\sin(\theta)\cos(\psi)\\ \cos(\theta)\sin(\psi) & \cos(\phi)\cos(\psi) + \sin(\phi)\sin(\theta)\sin(\psi) & -\sin(\phi)\cos(\psi) + \cos(\phi)\sin(\theta)\sin(\psi)\\ -\sin(\theta) & \sin(\phi)\cos(\theta) & \cos(\phi)\cos(\theta)\end{bmatrix} \tag{18}$$

Using euler propagation:

$$\begin{bmatrix}\dot{\phi}\\ \dot{\theta}\\ \dot{\psi}\end{bmatrix} = \begin{bmatrix}1 & \sin(\phi)\tan(\theta) & \cos(\phi)\tan(\theta)\\ 0 & \cos(\phi) & -\sin(\phi)\\ 0 & \dfrac{\sin(\phi)}{\cos(\theta)} & \dfrac{\cos(\phi)}{\cos(\theta)}\end{bmatrix}\begin{bmatrix}p\\ q\\ r\end{bmatrix} \tag{19}$$

In linearized equation of motion we assume:

$$\mathbf{C}_b^R = \begin{bmatrix}1 & -\psi & \theta\\ \psi & 1 & -\phi\\ -\theta & \phi & 1\end{bmatrix}, \quad \begin{bmatrix}p\\ q\\ r\end{bmatrix} = \begin{bmatrix}\dot{\phi}\\ \dot{\theta}\\ \dot{\psi}\end{bmatrix} \tag{20}$$

For the transformation from reference to body use the transpose of the desciberd matrix above.

The equations of motions solved in MATLAB with ode45 function. Below is the figure of euler angles simulation in 50 and 1000 seconds.

Figure 9: simulation of ϕ for 50 seconds

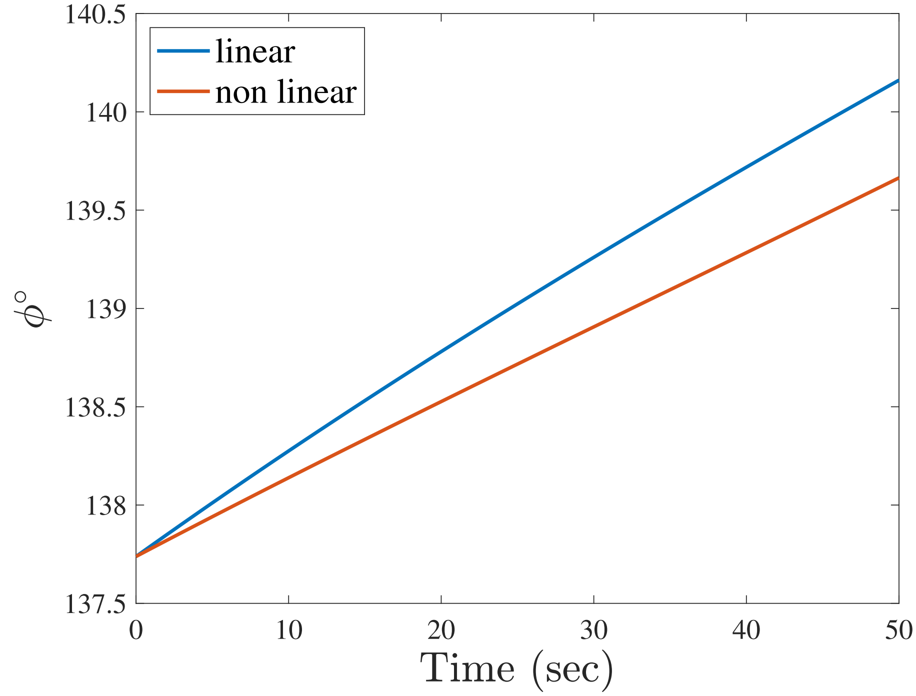


Figure 10: simulation of θ for 50 seconds

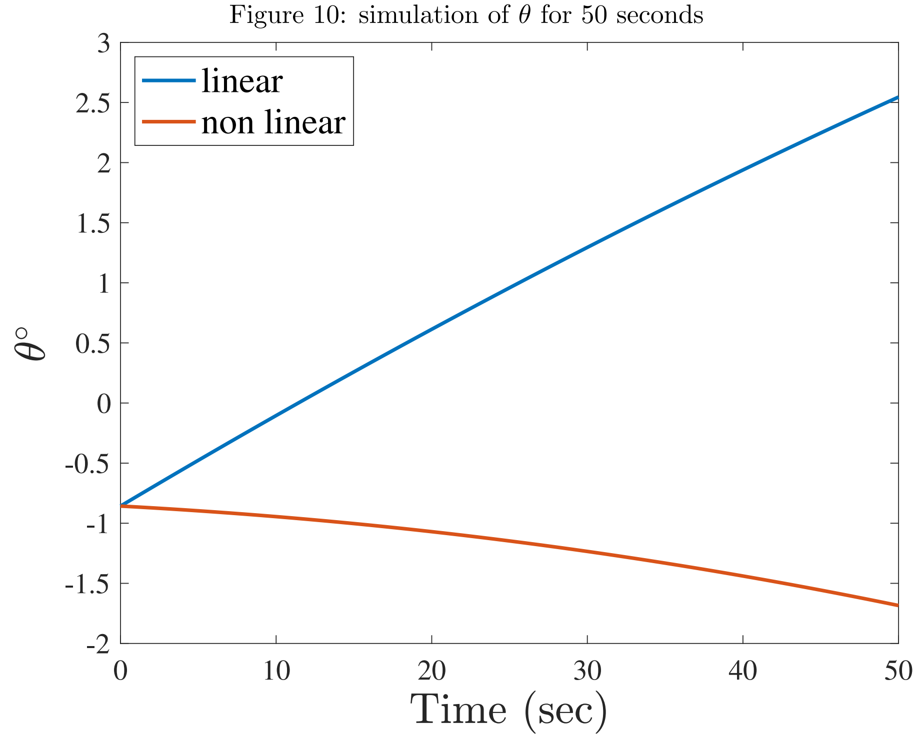


Figure 11: simulation of ψ for 50 seconds

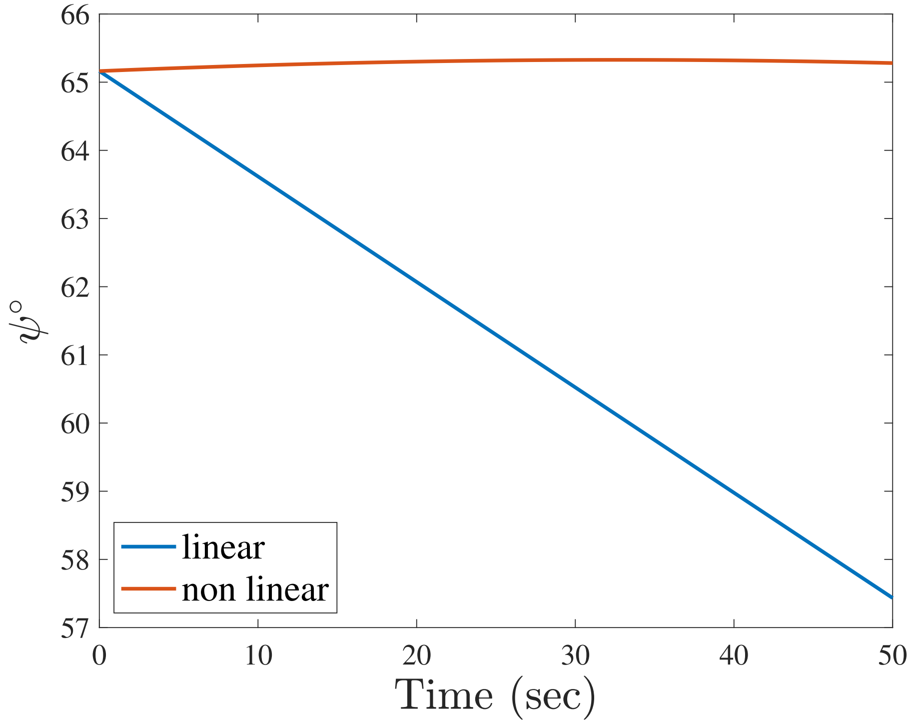


Figure 12: simulation of ϕ for 5000 seconds

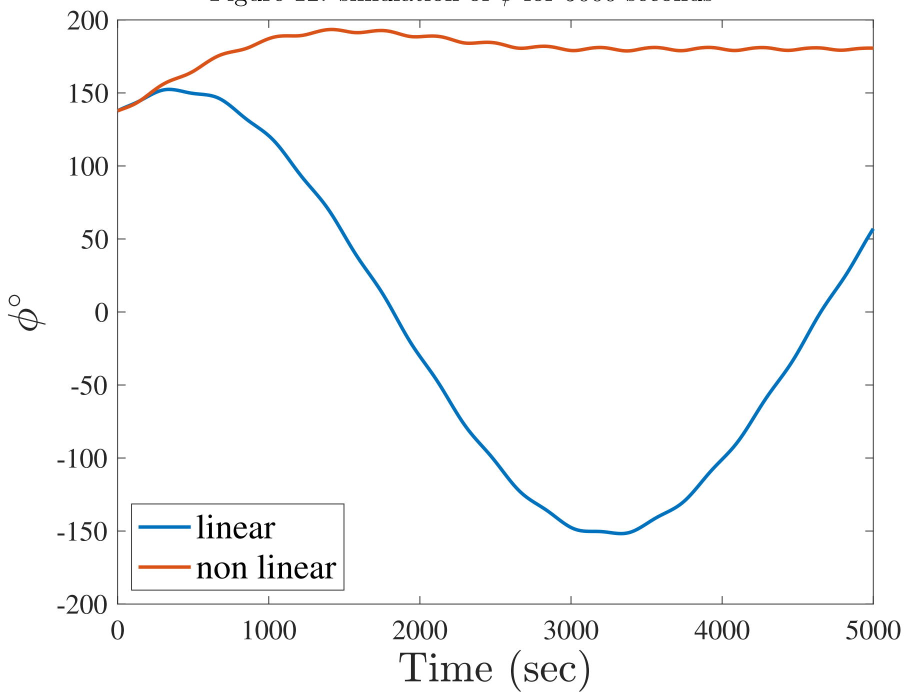


Figure 13: simulation of θ for 5000 seconds

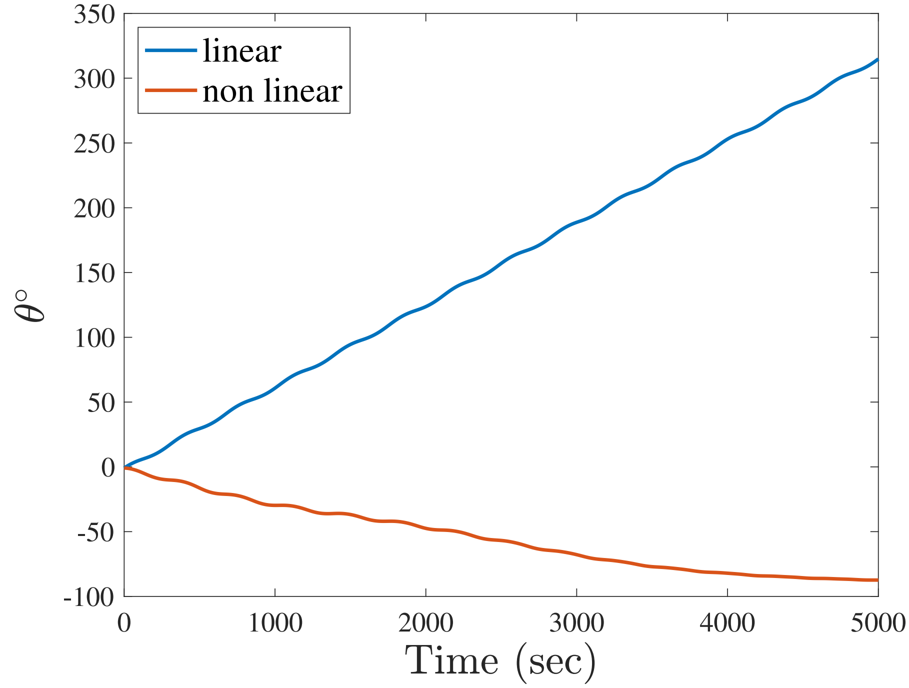
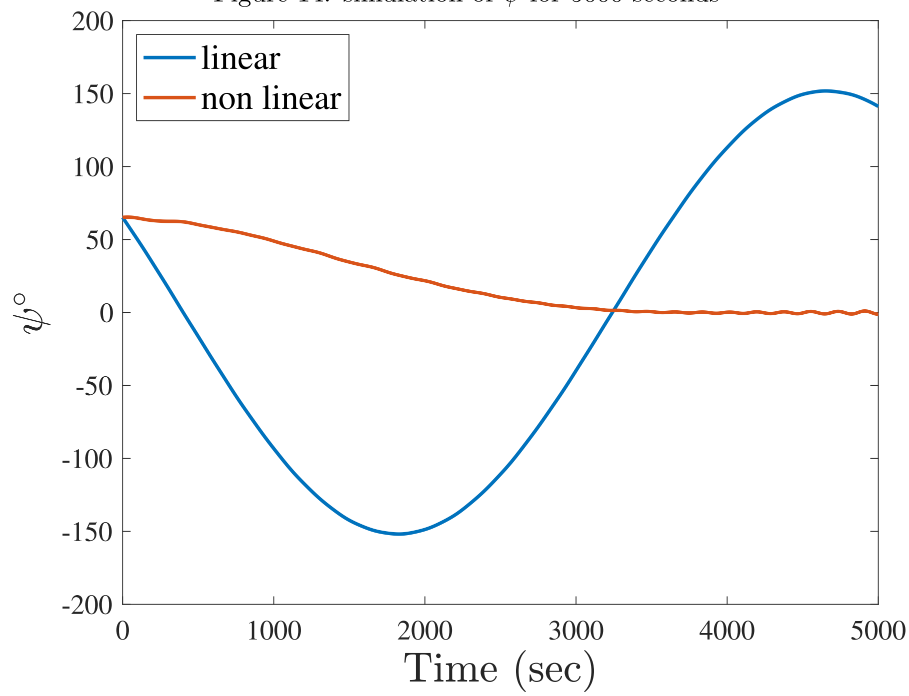


Figure 14: simulation of ψ for 5000 seconds



2.3 part d

Used above equation with different initial conditions and add initial torque disturbance.

Figure 15: simulation of ϕ with torque disturbance

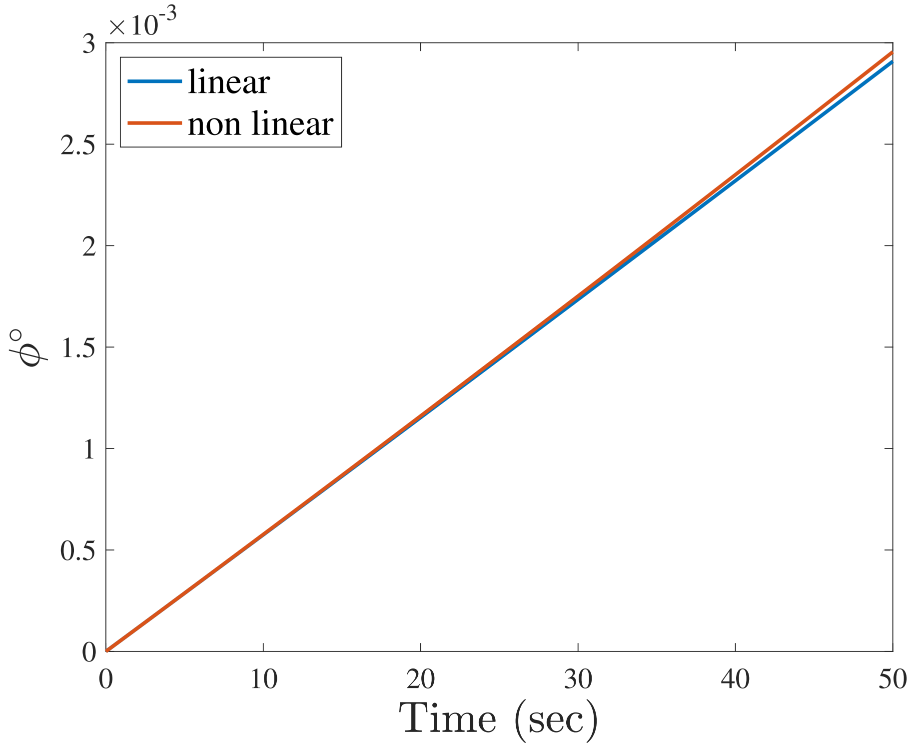


Figure 16: simulation of θ with torque disturbance

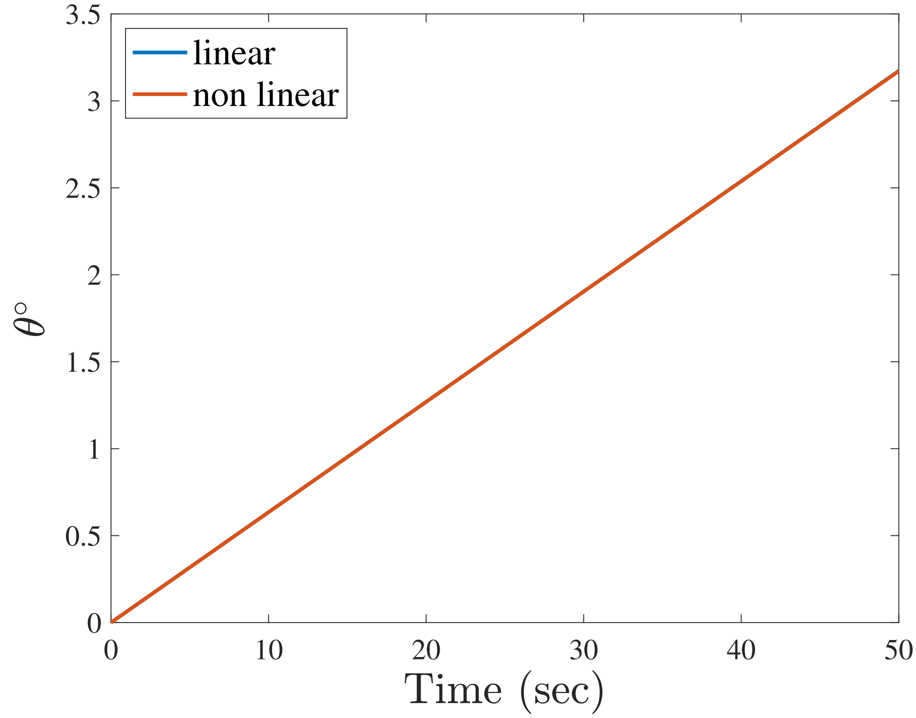
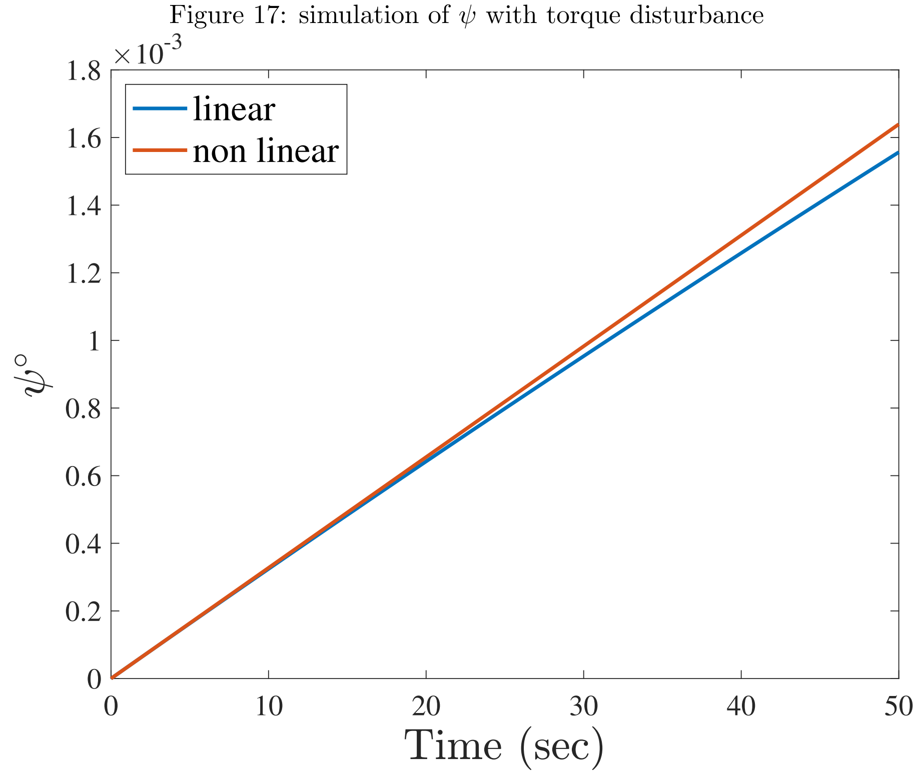


Figure 17: simulation of ψ with torque disturbance



3 Question 3

Equation of motion:

$$
\begin{aligned}
T_{Dx} = I_x\ddot{\phi} + 4\omega_0^2\,(I_y - I_z)\,\phi - \omega_0\,(I_x + I_y - I_z)\,\dot{\psi} &\xrightarrow{\mathscr{L}} 4\,\phi\,(I_y - I_z)\,{\omega_0}^2 - \psi\,(I_x + I_y - I_z)\,\omega_0\,s + I_x\,\phi\,s^2 = 0\\
T_{Dz} = I_z\ddot{\psi} + \omega_0^2\,(I_z + I_x - I_y)\,\dot{\phi} + \omega_0^2\,(I_y - I_x)\,\psi &\xrightarrow{\mathscr{L}} \phi\,(I_x - I_y + I_z)\,{\omega_0}^2\,s - \psi\,(I_x - I_y)\,{\omega_0}^2 + I_z\,\psi\,s^2 = 0
\end{aligned}
$$

Using below terms to simplify the equations.

$$
\sigma_x = \frac{I_y - I_z}{I_x}, \quad \sigma_z = \frac{I_y - I_x}{I_z}
$$

$$
\begin{aligned}
4\,\phi\,\sigma_x\,{\omega_0}^2 - \psi\,(1 - \sigma_x)\,\omega_0\,s + \phi\,s^2 = 0 &\rightarrow \phi[s^2 + 4\sigma_x\omega_0^2] + \psi[-\omega_0 s(1 - \sigma_x)] = 0\\
-\phi\,(\sigma_z - 1)\,{\omega_0}^2\,s - \psi\,\sigma_z\,{\omega_0}^2 + \psi\,s^2 = 0 &\rightarrow \phi[\omega_0 s(1 - \sigma_z)] + \psi[s^2 + \omega_0^2\sigma_z] = 0
\end{aligned}
\tag{21}
$$

$$
\begin{bmatrix} s^2 + 4\sigma_x\omega_0^2 & -\omega_0 s(1 - \sigma_x) \\ \omega_0 s(1 - \sigma_z) & s^2 + \omega_0^2\sigma_z \end{bmatrix} \begin{bmatrix} \phi \\ \psi \end{bmatrix} = \mathbf{A} \begin{bmatrix} \phi \\ \psi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

To avoid the trivial solution ($\phi = \psi = 0$), the determinant of the coefficient matrix must be zero.

$$
\begin{aligned}
\det \mathbf{A} &= {\omega_0}^2\,s^2 + s^4 + 4\,{\omega_0}^4\,\sigma_x\,\sigma_z + 3\,{\omega_0}^2\,s^2\,\sigma_x + {\omega_0}^2\,s^2\,\sigma_x\,\sigma_z\\
&= s^4 + \omega_0^2(3s^2\sigma_x + \sigma_x\sigma_y + 1) + 4\,{\omega_0}^4\,\sigma_x\,\sigma_z = 0
\end{aligned}
$$

4 Question 4

Inertia:

$$
I_x = 8, \quad I_y = 10, \quad I_z = 14
$$

$$
\sigma_x = \frac{I_y - I_z}{I_x} = -0.5, \quad \sigma_z = \frac{I_y - I_x}{I_z} = 0.1429
$$

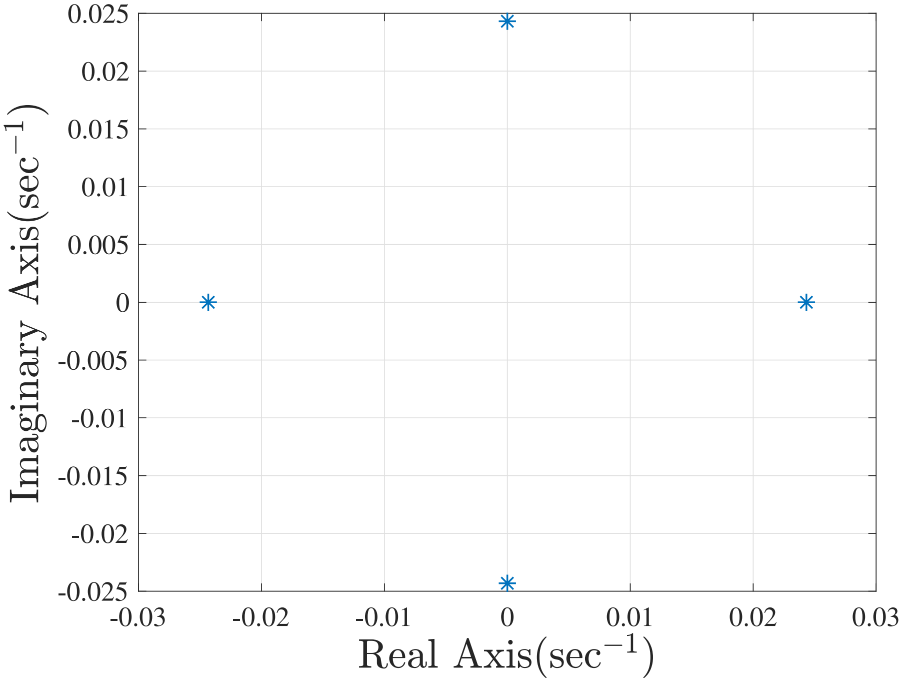
$$
\omega_0 = \sqrt{\frac{\mu}{R_0^3}} = 0.0011
$$

The characteristic equation of system is:

$$
s^4 + \omega_0^2(3s^2\sigma_x + \sigma_x\sigma_y + 1) + 4\,{\omega_0}^4\,\sigma_x\,\sigma_z = s^4 - 6.998 \times 10^{-07}s^2 - 3.499 \times 10^{-7}
\tag{22}
$$

poles of the system ploted if figure 4.

Figure 18: poles of system



Beacuse system has poles in right side of real axis, system is unstable and we have seen it in question 2.

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