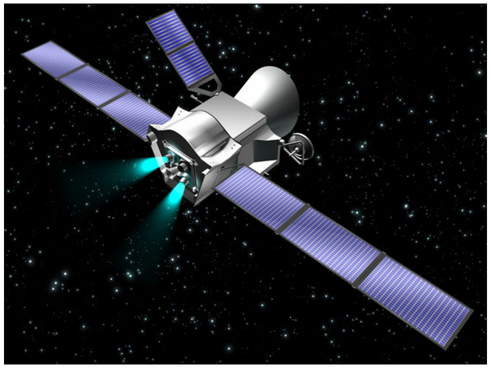
In The Name of God



AE 45780: Spacecraft Dynamics and Control

Fall 1401

3- Attitude Dynamics

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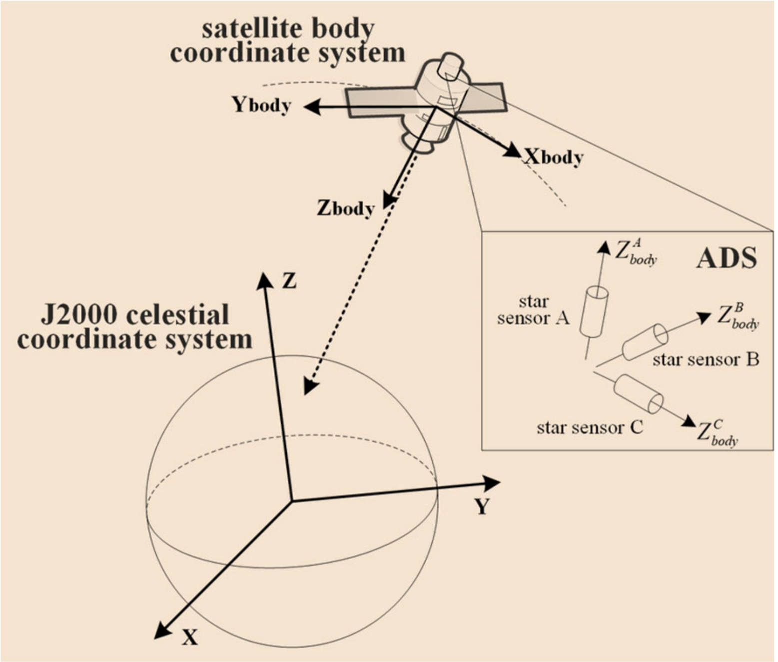
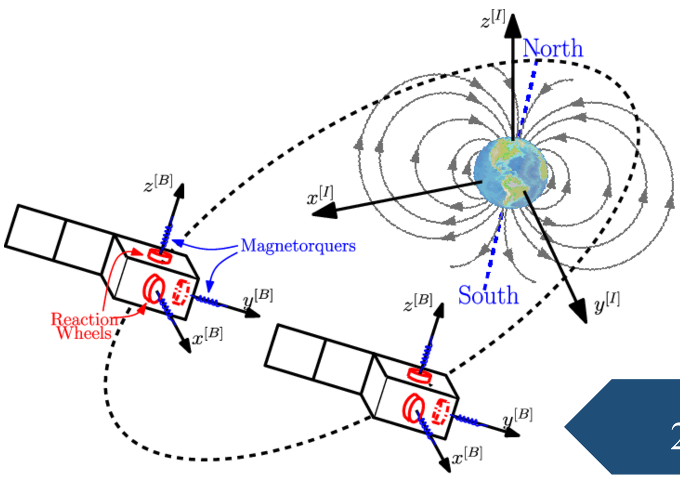
Attitude Dynamics and Control (ADC)

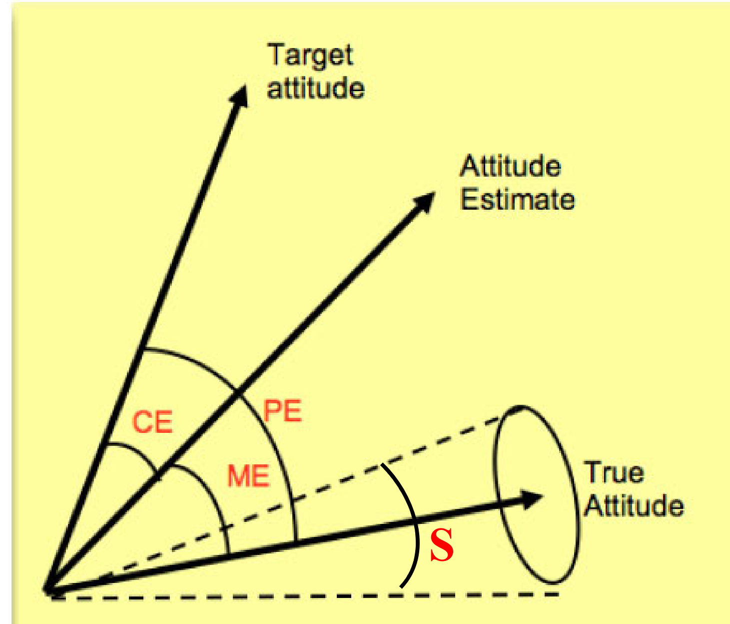
Attitude Dynamics (AD) pertains to the rotational motion of satellites and space vehicles around their center of mass. **Attitude control (AC)** is the process of controlling the orientation of a spacecraft with respect to a reference frame. Initial understanding of feasible motion dynamics of orbiting spacecraft (2BP, Perturbation effect, 3PB...) was a prerequisite to orbit transfer, maneuver and control. A similar approach is followed in the study the attitude dynamics, its propagation and kinematics, for subsequent control of SC attitude motion. In this sense, basic laws and concepts regarding angular kinetic energy and momentum will be introduced and used in the derivation of the fundamental laws of angular motion based on the Euler's equations. But, first some related terminologies are should be reviewed.





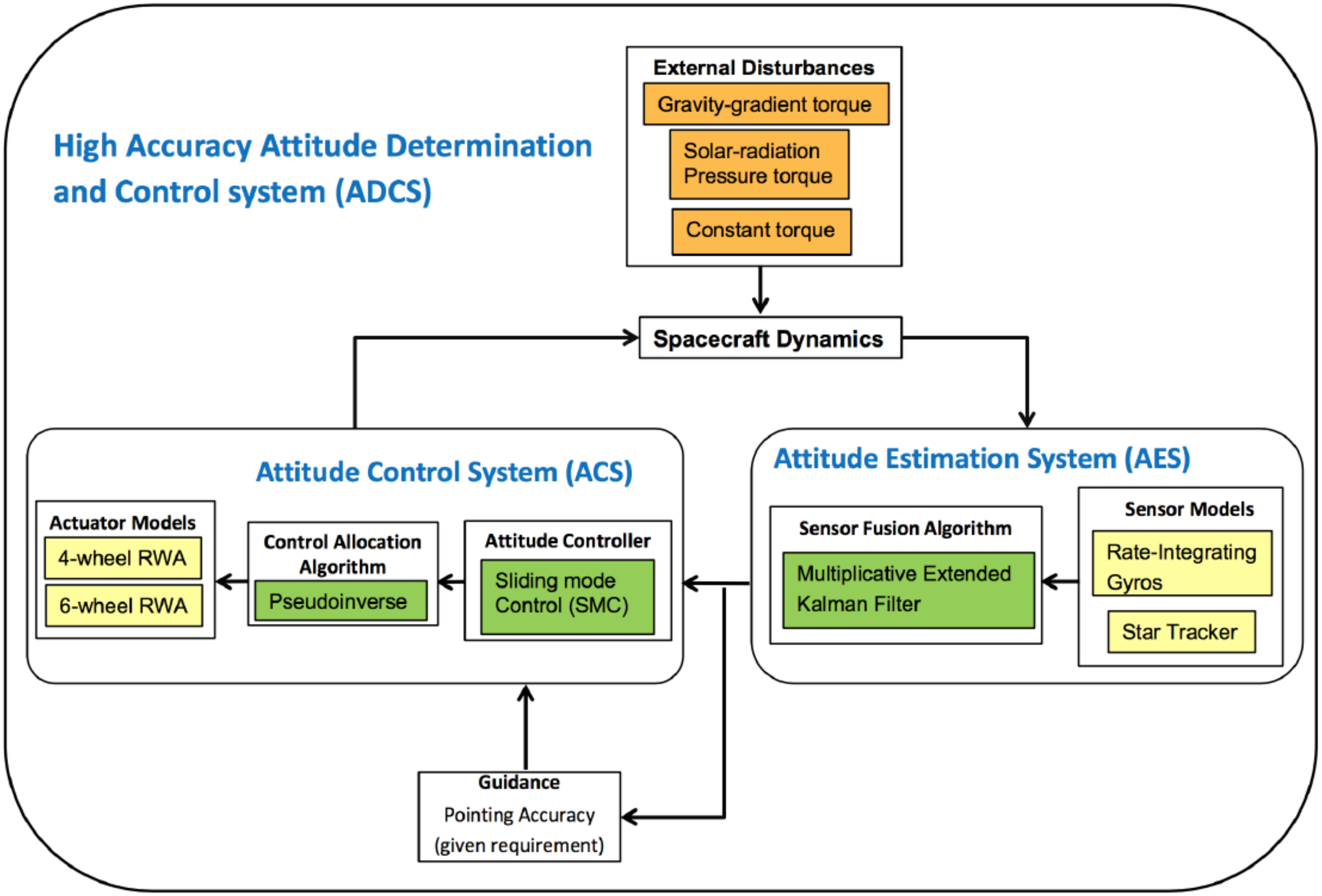
AD Related Terminologies

Attitude: Orientation of a SC body coordinate system WRT an external reference frame.
Attitude Determination: Real time knowledge or determination of SC attitude (within a given tolerance).
Attitude Control: Maintenance (keeping) of a desired specified attitude within a given tolerance.
Attitude or Pointing Error (PE) : "Low frequency" SC misalignment WRT desired attitude. Or, the amount of angular separation between the desired or commanded direction and the actual instantaneous (true) direction.
Attitude Jitter (S) : "High frequency" SC misalignment WRT desired attitude. This is usually ignored by SC attitude determination and control system (ADCS), but is important in good design for fine pointing.
Target Attitude: Desired pointing direction.
True Attitude : Actual pointing direction.
Attitude Estimate : An instantaneous estimate of the true attitude by (AES).
Measurement Error (ME) : Pointing accuracy from the attitude estimation system (AES).
Control Error (CE) : Pointing accuracy from the attitude control system(ACS).



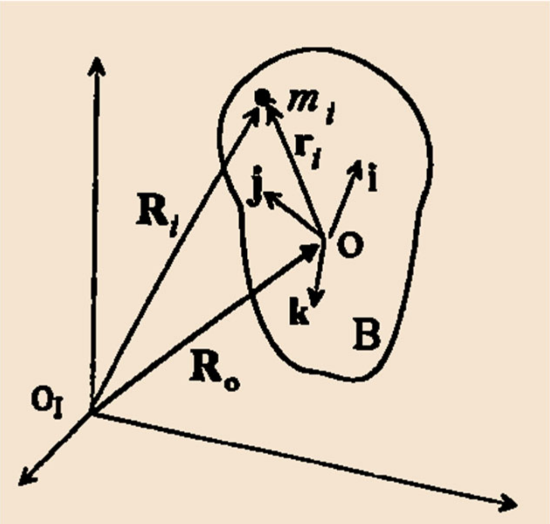
Pointing definitions

Attitude Determination and Control System (ADCS)



Components of the Closed-Loop Attitude Determination and Control System of IRASSI Spacecraft.

Angular Momentum of a Rigid Body (SC)



Suppose the origin of the SC body coordinate (moving) is located at its center of mass ,O .The angular momentum of a particle within this body about point O is: $\vec{h}_{O_i} = \vec{r}_i \times m_i \vec{V}_i$

where $\vec{V}_i$ is the inertial velocity of the particle,

which via Coriolis can be wrriten in body ccordinates as:

$$\vec{V}_i = \dot{\vec{R}}_i = \dot{\vec{R}}_O + \vec{V}_{i_b} + \vec{\omega} \times \vec{r}_i = \vec{V}_O + \dot{\vec{r}}_i + \vec{\omega} \times \vec{r}_i$$

Substituting in the relation for particle angular momentum and assuming a rigid SC :

$\vec{h}_{O_i} = \vec{r}_i \times m_i \left(\vec{V}_O + \vec{\omega} \times \vec{r}_i \right)$, where in body coordinate: $\vec{V}_{i_b} = \dot{\vec{r}}_i = 0$ is used.

If the SC body is considered as a set of particles or small masses, the total SC angular momentum will be the sum of the particle angular momentums around the point O. In other words:

$$\vec{H}_O = \sum_i h_{O_i} = \sum_i \vec{r}_i \times m_i \left(\vec{V}_O + \vec{\omega} \times \vec{r}_i \right) = \sum_i \left[\vec{r}_i \times m_i \vec{V}_O + \vec{r}_i \times m_i \vec{\omega} \times \vec{r}_i \right]$$

$$= \sum_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i) m_i - \underbrace{\vec{V}_O \times \sum_i m_i \vec{r}_i}_{=0} = \sum_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i) m_i \text{ or } \int_B \vec{r} \times (\vec{\omega} \times \vec{r}) \mathrm{d}m$$

Angular Momentum of a Rigid SC

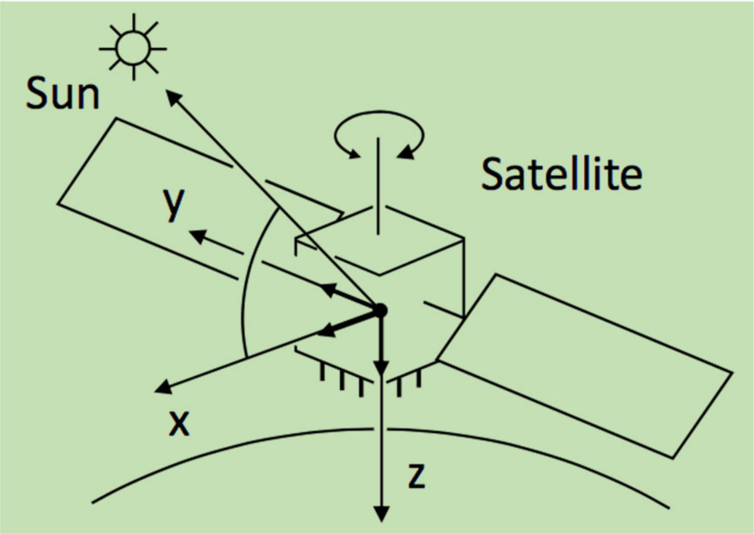
The previous relation can be presented based on the expression of the $\vec{r}_i$ and $\vec{\omega}_i$ vectors in body coordinate system. In addition using the definition of moments of inertia of the SC about the body axes will further simplify the equation:

$$\vec{h}_O = \sum_i \vec{r}_i \times \left(\vec{\omega} \times \vec{r}_i\right) m_i \text{ or } \int_B \vec{r} \times \left(\vec{\omega} \times \vec{r}\right) \mathrm{d}m$$

$$\vec{r}_i = \left[x_i \quad y_i \quad z_i\right]^\top \quad , \quad \vec{\omega} = \left[\omega_x \quad \omega_y \quad \omega_z\right]^\top$$

$$\vec{\omega} \times \vec{r}_i = \left(\omega_y z_i - \omega_z y_i\right)\vec{i} + \left(\omega_z x_i - \omega_x z_i\right)\vec{j} + \left(\omega_x y_i - \omega_y x_i\right)\vec{k}$$

$$\vec{r}_i \times \omega \times \vec{r}_i = \begin{Bmatrix} \left[\omega_x\left(y_i^2 + z_i^2\right) - \omega_y\left(x_i y_i\right) - \omega_z\left(x_i z_i\right)\right]\vec{i} \\ \left[-\omega_x\left(x_i y_i\right) + \omega_y\left(x_i^2 + z_i^2\right) - \omega_z\left(y_i z_i\right)\right]\vec{j} \\ \left[-\omega_x\left(x_i z_i\right) - \omega_y\left(y_i z_i\right) + \omega_z\left(x_i^2 + y_i^2\right)\right]\vec{k} \end{Bmatrix}$$

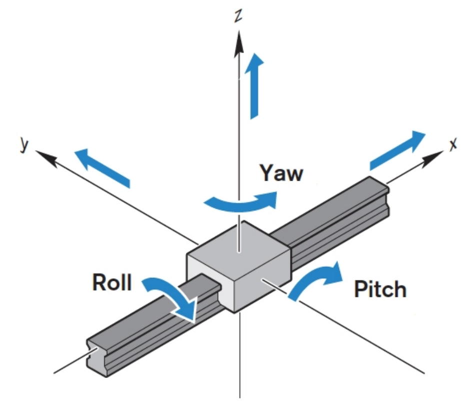


In this way, using equation of angular momentum **per unit mass** and the definition of moments of inertia of the body around its axes, yields the SC total angular momentum. Subsequently, the SC Kinematic EOM will be later derived from the basic law governing rotational dynamics via Euler EOM.

Angular Momentum of a Rigid SC

$$\begin{cases} I_{xx} = \sum_i \left(y_i^2 + z_i^2\right) m_i = \int_B \left(y^2 + z^2\right) \mathrm{d}m = I_x \\ I_{yy} = \sum_i \left(x_i^2 + z_i^2\right) m_i = \int_B \left(x^2 + z^2\right) \mathrm{d}m = I_y \\ I_{zz} = \sum_i \left(x_i^2 + y_i^2\right) m_i = \int_B \left(x^2 + y^2\right) \mathrm{d}m = I_z \end{cases} , \quad \begin{cases} I_{xy} = \sum_i \left(x_i y_i\right) m_i = \int_B \left(xy\right) \mathrm{d}m \\ I_{xz} = \sum_i \left(x_i z_i\right) m_i = \int_B \left(xz\right) \mathrm{d}m \\ I_{yz} = \sum_i \left(y_i z_i\right) m_i = \int_B \left(yz\right) \mathrm{d}m \end{cases}$$

Note that the SC moments of inertias (MOI) can be written in Tensor form and is usually **symmetric**. In addition the product moments of inertia may be negative. Using MOI tensor, and the last equation for the SC angular momentum ,one can show that:



$$I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \Rightarrow \vec{h} = I\vec{\omega} \Rightarrow \vec{h} = h_x\vec{i} + h_y\vec{j} + h_z\vec{k} \quad , \quad \begin{cases} h_x = \omega_x I_{xx} - \omega_y I_{xy} - \omega_z I_{xz} \\ h_y = -\omega_x I_{yx} + \omega_y I_{yy} - \omega_z I_{yz} \\ h_z = -\omega_x I_{zx} - \omega_y I_{zy} + \omega_z I_{zz} \end{cases}$$

Rotational Kinetic Energy of a Rigid Body

A 50 Kg Demonstration satellite with four movable solar cell paddles for high-speed satellite attitude and orbit control via paddle movement.
Source: https://www.titech.ac.jp/english/news/2021/062360

Rotational Kinetic Energy of a Rigid Body

The kinetic energy of a rotating rigid body around its center of mass is known as **rotational kinetic energy**, which can be shown to be related to its angular momentum. In this regard, let us first consider the kinetic energy of an element with a differential mass dm:

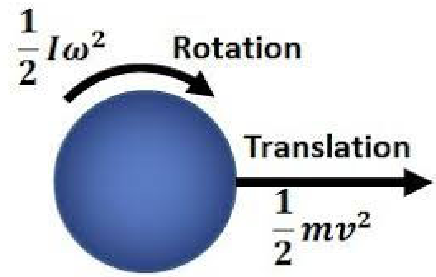
$$dT = \frac{1}{2}(dm)V^2$$

Where V is the magnitude of the absolute (inertial) velocity of dm, that can be written as $\vec{V}_i = \vec{V}_0 + \vec{\omega} \times \vec{r}_i$. Therefore: $V^2 = \vec{V}_i \cdot \vec{V}_i = V_0^2 + 2\vec{V}_O \cdot (\vec{\omega} \times \vec{r}) + (\vec{\omega} \times \vec{r}_i) \cdot (\vec{\omega} \times \vec{r}_i)$

Thus, putting back for dT, gives :

$$T = \int dT = \frac{1}{2}\int_B V^2 dm = \frac{1}{2}\int V_O^2 dm + \int \vec{V}_O \cdot (\vec{\omega} \times \vec{r})\, dm + \frac{1}{2}\int (\vec{\omega} \times \vec{r}) \cdot (\vec{\omega} \times \vec{r})\, dm$$

$$= \frac{1}{2} M V_O^2 + \vec{V}_O \cdot \underbrace{\vec{\omega} \times \int \vec{r} dm}_{=0} + \frac{1}{2}\int (\vec{\omega} \times \vec{r}) \cdot (\vec{\omega} \times \vec{r})\, dm = T_{Translational} + T_{Rotational}$$



In other words:

$$T_{Rotational} = \frac{1}{2}\int_B (\omega \times r) \cdot (\omega \times r)\, \mathrm{d}m$$

Rotational Kinetic Energy of Rigid SC

$$\vec{\omega}\times\vec{r}_i=\left(\omega_y z_i-\omega_z y_i\right)\vec{i}+\left(\omega_z x_i-\omega_x z_i\right)\vec{j}+\left(\omega_x y_i-\omega_y x_i\right)\vec{k}$$

$$h_x=\omega_x I_{xx}-\omega_y I_{xy}-\omega_z I_{xz}$$

$$h_y=-\omega_x I_{yx}+\omega_y I_{yy}-\omega_z I_{yz}$$

$$h_z=-\omega_x I_{zx}-\omega_y I_{zy}+\omega_z I_{zz}$$

Utilizing the previous relation we had :

$$T_{Rotational}=\frac{1}{2}\int_B(\omega\times r)\cdot(\omega\times r)\,dm$$

$$T_{Rotational}=\frac{1}{2}\int_B\left[\left(\omega_y z-\omega_z y\right)^2+\left(\omega_z x-\omega_x z\right)^2+\left(\omega_x y-\omega_y x\right)^2\right]dm$$

Subsequently, via expanding, integrating and utility of the MOI definitions, we will arrive at the following relation:

$$\begin{aligned}2T_{Rotational}&=\omega_x^2 I_{xx}+\omega_y^2 I_{yy}+\omega_z^2 I_{zz}-2\omega_x\omega_z I_{xz}-2\omega_y\omega_z I_{yz}-2\omega_x\omega_y I_{xy}\\&=\omega_x\left[\omega_x I_x-\omega_y I_{xy}-\omega_z I_{xz}\right]+\omega_y\left[-\omega_x I_{yx}+\omega_y I_y-\omega_z I_{yz}\right]+\omega_z\left[-\omega_x I_{zx}-\omega_y I_{zy}+\omega_z I_z\right]\\&=\vec{\omega}\cdot\vec{h}=\omega_x h_x+\omega_y h_y+\omega_z h_z\\&=\vec{\omega}\cdot I\vec{\omega}\Rightarrow or\quad T_{Rotational}=\frac{1}{2}\vec{\omega}^T I\vec{\omega}\end{aligned}$$

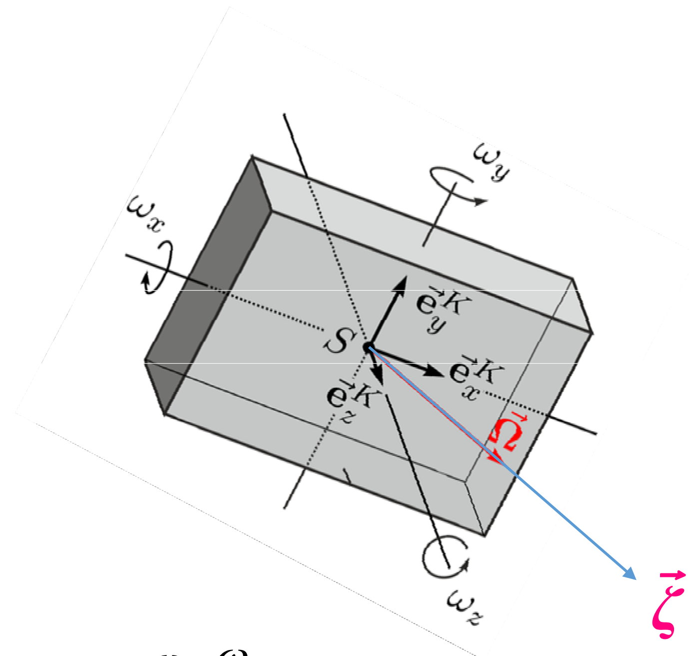
$$Note:\vec{h}_i=\frac{1}{2}\frac{\partial}{\partial\omega_i}(2T_{ROT});\ i=x,y,y$$

Moment of Inertia in Special Coordinates

The moment of inertia of a body can be defined around any axis that passes through its center of mass. In this regard, suppose an **axis is taken** parallel to the $\vec{\omega}$ vector, which passes through the SC center of mass. In this case, for $\vec{\omega}$, would only have one nonzero component . Thus it can be shown that $\vec{\omega} I \vec{\omega} = I_\xi \omega^2$. That is, if ξ is an axis along $\vec{\omega}$ and passing through the SC the center of mass, then we have:

$$T_{Rotational} = \frac{1}{2} I_\xi \omega^2$$

$$\vec{\omega} = \omega \vec{1}_\xi \;\; or \;\; \vec{\omega} = \begin{bmatrix} \omega_x & \omega_y & \omega_z \end{bmatrix}^\top$$



In the latter situation, notice that T_{ROT} can be written independent of the body coordinate system. Then using the definition of T_{ROT} . Also, let , $I_{ii} = I_i$ for $i = x, y, z$.

$$I_\zeta \omega^2 = \omega_x^2 I_x + \omega_y^2 I_y + \omega_z^2 I_z - 2\omega_y \omega_x I_{xy} - 2\omega_y \omega_z I_{yz} - 2\omega_z \omega_x I_{xz}$$

$$\text{or } I_\xi = \left(\frac{\omega_x}{\omega}\right)^2 I_x + \left(\frac{\omega_y}{\omega}\right)^2 I_y + \left(\frac{\omega_z}{\omega}\right)^2 I_z - 2\frac{\omega_x}{\omega}\frac{\omega_z}{\omega} I_{xz} - 2\frac{\omega_y}{\omega}\frac{\omega_z}{\omega} I_{yz} - 2\frac{\omega_x}{\omega}\frac{\omega_y}{\omega} I_{xy}$$

Moment of Inertia in Special Coordinates

$$T_{Rotational} = \frac{1}{2} I_\xi \omega^2$$

$$\vec{\omega} = \omega \vec{1}_\xi \ or\ \vec{\omega} = \begin{bmatrix} \omega_x & \omega_y & \omega_z \end{bmatrix}^T$$

Since $\vec{\omega} = \omega \vec{1}_\xi$, the components of this vector in body coordinate are obtained using inner product with $\vec{i}, \vec{j}, \vec{k}$, or in other words:

$$\omega_x = \vec{\omega} \cdot \vec{i} = \omega \vec{1}_\xi \cdot \vec{i}$$

$$\omega_y = \vec{\omega} \cdot \vec{j} = \omega \vec{1}_\xi \cdot \vec{j}$$

$$\omega_z = \vec{\omega} \cdot \vec{k} = \omega \vec{1}_\xi \cdot \vec{k}$$

In addition by introducing: $a_x = \vec{1} \cdot \vec{i}$, $a_y = \vec{1} \cdot \vec{j}$, and $a_z = \vec{1} \cdot \vec{k}$ (direction cosine angles between the $\vec{\omega}$ and each of the body coordinate axes) , that indicate the position of the axis ξ relative to the *x, y, z* axes,

one can write: $a_x = \dfrac{\omega_x}{\omega}$, $a_y = \dfrac{\omega_y}{\omega}$, $a_z = \dfrac{\omega_z}{\omega}$

That leads to the following relation in relation for I_ξ:

$$I_\xi = \left(\frac{\omega_x}{\omega}\right)^2 I_x + \left(\frac{\omega_y}{\omega}\right)^2 I_y + \left(\frac{\omega_z}{\omega}\right)^2 I_z - 2\frac{\omega_x}{\omega}\frac{\omega_z}{\omega} I_{xz} - 2\frac{\omega_y}{\omega}\frac{\omega_z}{\omega} I_{yz} - 2\frac{\omega_x}{\omega}\frac{\omega_y}{\omega} I_{xy}$$

$$I_\xi = (a_x)^2 I_x + (a_y)^2 I_y + (a_z)^2 I_z - 2a_y a_z I_{zy} - 2a_x a_z I_{xz} - 2a_x a_y I_{yx}$$

Moment of Inertia in Special Coordinates

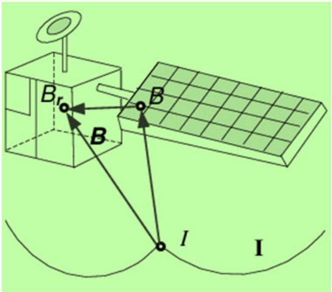
$$T_{Rotational} = \frac{1}{2} I_{\xi} \omega^2 ; \; \vec{\omega} = \omega \vec{1}_{\xi}$$

$$I_{\xi} = \left(a_x\right)^2 I_x + \left(a_y\right)^2 I_y + \left(a_z\right)^2 I_z - 2a_y a_z I_{zy} - 2a_x a_z I_{xz} - 2a_x a_y I_{yx}$$

- The given relation for T_{ROT} is very simple and attractive, but its utility requires the direction cosines as well as MOI properties in the body CS.
- I_{ζ} could be continuously changing if the SC angular velocity vector changes.

However, it would be interesting to choose a body CS in such a way that the product of inertias are zero! that further simplifies the relation. In addition, control engineers usually prefer to deal with SC with zero or negligible product of inertia that subsequently simplifies ACS designs. In these cases, small nonzero product of inertias emanating out of SC assembly and production will be considered as disturbances to check for robust ACS designs.

Inertial properties	Value	Unit
Mass	2501.45	Kg
I_{xx}	1.0225E+05	kg·m^2
I_{yy}	1.7444E+04	kg·m^2
I_{zz}	1.1472E+05	kg·m^2



Principal Axes of Inertia

The problem is to convert the inertia matrix or the MOI tensor into a diagonal matrix. As this is a common routine in linear algebra, its proof is not presented here.

$$T_{Rotational} = \frac{1}{2}\vec{\omega}^T I \vec{\omega}$$

In the primary body CS, the components of $\vec{\omega}$ are ω_x , ω_y , ω_z. In order to obtain a new coordinate frame in which the MOI matrix is diagonal, a transformation or rotation matrix (TM) will be required. Lets assume that this TM is denoted by ***A***. Thus, the relationship between the components of $\vec{\omega}$ in two body CSs will be :

$$\vec{\omega} = A\vec{\omega}'; \ \omega' \triangleq \text{the angular velocity in the new frame}$$

Now we can rewrite the T_{ROT} relation as :

$$2T_{rotational} = \vec{\omega}^T I \vec{\omega} \Rightarrow 2T_{rotational} = \left(A\vec{\omega}'\right)^T I \left(A\vec{\omega}'\right) = \vec{\omega}'^T A^T I A \vec{\omega}' = \vec{\omega}'^T I' \vec{\omega}' \text{ ; where } \left(I' \triangleq diagonal = A^T I A\right)$$

That means that in the new frame, the inertia tensor will be diagonal. $2T_{Rotational} = \vec{\omega}'^T I' \vec{\omega}'$

The eigenvalues (λ_i) of the **inertia matrix *I*** will be the principal moments of inertias or in other words, the diagonal elements of I' that are obtained using $\det\big[I - \lambda[1]\big] = 0$. Likewise, the eigenvectors $I, \vec{e}_1, \vec{e}_2, \vec{e}_3$ will be the columns of the rotation matrix ***A***.

$$A = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{bmatrix}$$

Principal Axes of Inertia

$$2T_{ROT} = \omega_x^2 I_{xx} + \omega_y^2 I_{yy} + \omega_z^2 I_{zz} - 2\omega_x\omega_z I_{xz} - 2\omega_y\omega_z I_{yz} - 2\omega_x\omega_y I_{xy}$$

To generate the eigenvectors, you can use the Matlab or the solution of the following equation:

$$\lambda_i \vec{e}_i = [I]\vec{e}_i \quad , \quad i = 1,2,3$$

Conclusion: The diagonal elements of I' will be the principal moment of inertia and the new axis or frame is called the principal axes. The principal axes will contain the axis of maximum and minimum inertias.

Example: Suppose a SC MOI tensor is given. It is desired to diagonalize the MOI, by finding the corresponding TM or (A) as well as the new MOI in the principal axes frame (I').

$$I = \begin{bmatrix} 20 & -10 & 0 \\ -10 & 30 & 0 \\ 0 & 0 & 40 \end{bmatrix} Nm/s^2 \Rightarrow 2T_{ROT} = 20\omega_x^2 + 30\omega_y^2 + 40\omega_z^2 + 20\omega_x\omega_y$$

The diagonalization process leads to:

$$I' = \begin{bmatrix} 13.82 & 0 & 0 \\ 0 & 36.18 & 0 \\ 0 & 0 & 40 \end{bmatrix} Nm/s^2 ; \quad A = \begin{bmatrix} 0.85066 & -0.527 & 0 \\ 0.527 & 0.85066 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\vec{\omega}' = A^T\vec{\omega} \Rightarrow \omega_1 = 0.85066\omega_x + 0.527\omega_y ; \omega_2 = -0.52571\omega_x + 0.85066\omega_y ; \omega_3 = \omega_z$$

$$2T_{ROT} = 13.82\omega_1^2 + 36.18\omega_2^2 + 40\omega_3^2$$

Principal Axes of Inertia : Example

1- generating eigenvalues using: $\det\left[I-\lambda[1]\right]=0$

$$\lambda^3-65\lambda^2+1025\lambda-750=0 \Rightarrow \begin{cases}\lambda_1=13.82\\ \lambda_2=36.18\\ \lambda_3=40\end{cases}$$

2- for each λ_i, $\vec{e}_i$ is obtained by solving $[I]\vec{e}_i=\lambda_i\vec{e}_i$. For example, to determine $\vec{e}_1$:

$$\begin{array}{ccccccc}(20-13.82)e_{1x} & - & 10e_{1y} & + & 0 & = & 0\\ -10e_{1x} & + & (30-13.82)e_{1y} & + & 0 & = & 0\\ 0 & + & 0 & + & (40-13.82)e_{1z} & = & 0\end{array}$$

$$\Rightarrow\begin{cases}\vec{e}_1=\begin{bmatrix}0.85066 & 0.527 & 0\end{bmatrix}^T\\ \vec{e}_2=\begin{bmatrix}-0.52571 & 0.85066 & 0\end{bmatrix}^T\\ \vec{e}_3=\begin{bmatrix}0 & 0 & 1\end{bmatrix}^T\end{cases}$$

Principal Axes of Inertia: Example

And similarly other columns of the TM can be computed:

$$A = \begin{bmatrix} e_{1x} & e_{2x} & e_{3x} \\ \vdots & \vdots & \vdots \end{bmatrix} = \begin{bmatrix} 0.85066 & -0.527 & 0 \\ 0.527 & 0.85066 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \vec{\omega}' = A^{-1}\vec{\omega} \text{ or } \vec{\omega}' = A^{T}\vec{\omega} \text{ (duo to orthogon property of } A\text{)}$$

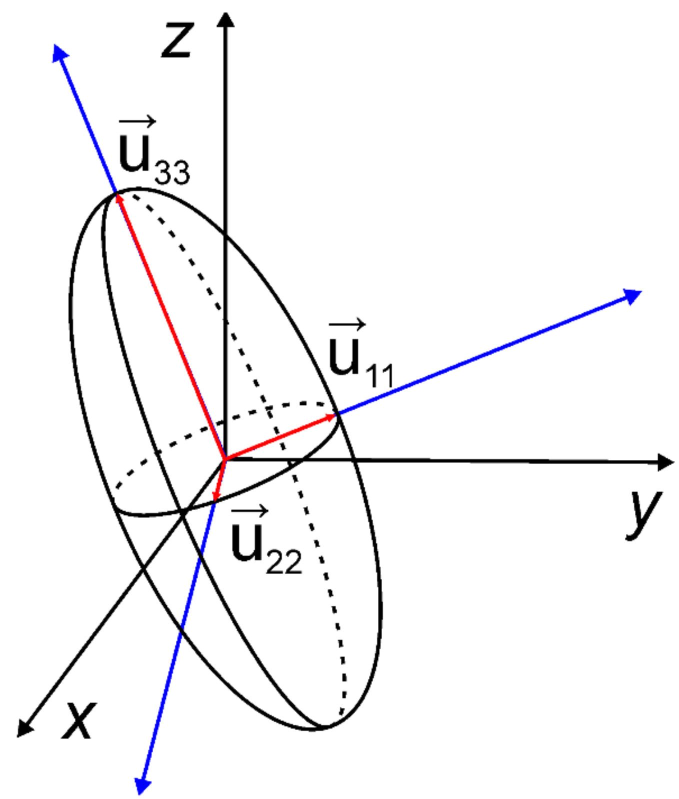
$$I' = A^{T} I A = \begin{bmatrix} 13.82 & 0 & 0 \\ 0 & 36.18 & 0 \\ 0 & 0 & 40 \end{bmatrix}$$

Note and summary: The column of the matrix A shows the direction cosines of the principal axes with the primary body axes. The third column of this matrix indicates that in reaching the principal axes system, the body CS must rotate around z_B and that is why in this process, the moment around z_B has not changed and the third column of A is as found.

Principal Axes of Inertia

Final Notes:

Fortunately for most regular geometries, the principal axes system can be determined by inspection, but the following guidelines can also be used:



- The axis of rotation for a body of revolution is always one of the principal axes. In addition, any transverse axes that passes through the center of mass will the other two principal axes.
- The plane of symmetry of a body will contain two principal axes, where third axis will be perpendicular to the symmetry plane.
- In general, the three principal axes that pass through the center of mass will include the axis of maximum and minimum inertia, " minor & major axes" respectively.

Principal Axes of Inertia

- Using the principal axes system and its associated moments simplify most of the relations. For example, components of the angular momentum vector will be:

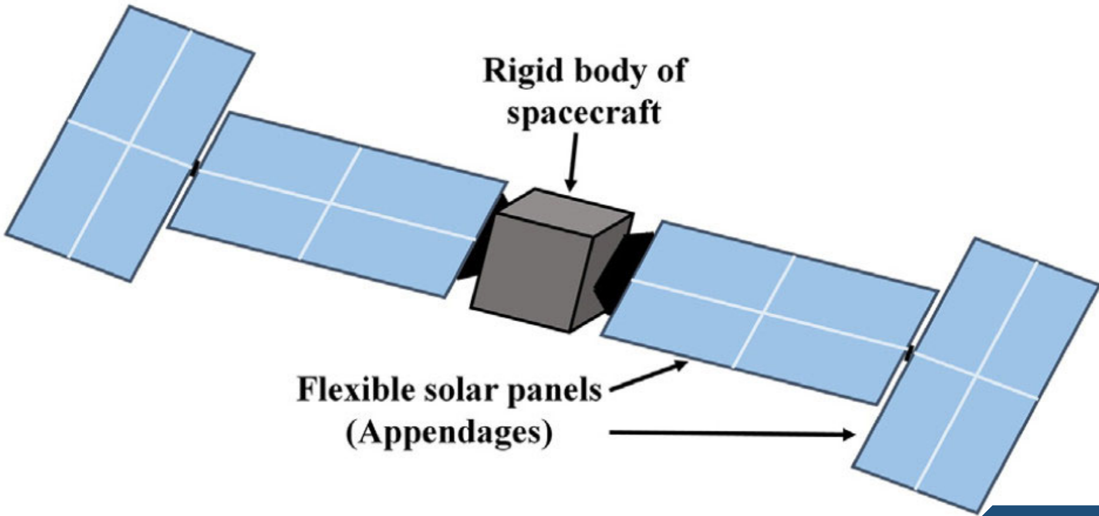
$$\begin{cases} h_1 = I_1\omega_1 \\ h_2 = I_2\omega_2 \Rightarrow \vec{h} = \left(I_1\omega_1\right)\vec{e}_1 + \left(I_2\omega_2\right)\vec{e}_2 + \left(I_3\omega_3\right)\vec{e}_3 \\ h_3 = I_3\omega_3 \end{cases}$$

- Based on previous relations, the moment of inertia around a principal spin axis (body will be:

$$I_\xi = \left(a_x\right)^2 I_1 + \left(a_y\right)^2 I_2 + \left(a_z\right)^2 I_3$$

$where: a_x, a_y, a_z$ are the direction cosines locating $\vec{\omega}$ w.r.t the principal axes

$$\text{i.e.:} \begin{cases} a_x = \vec{1}_\xi \cdot \vec{e}_1 \\ a_y = \vec{1}_\xi \cdot \vec{e}_2 \\ a_z = \vec{1}_\xi \cdot \vec{e}_3 \end{cases}$$



Ellipsoid of Inertia

Suppose that the principal axes of a SC are the same as the body axes frame, so according to previous equation, the moment of inertia around the spin axis or in the direction of $\vec{\omega}$ will be:

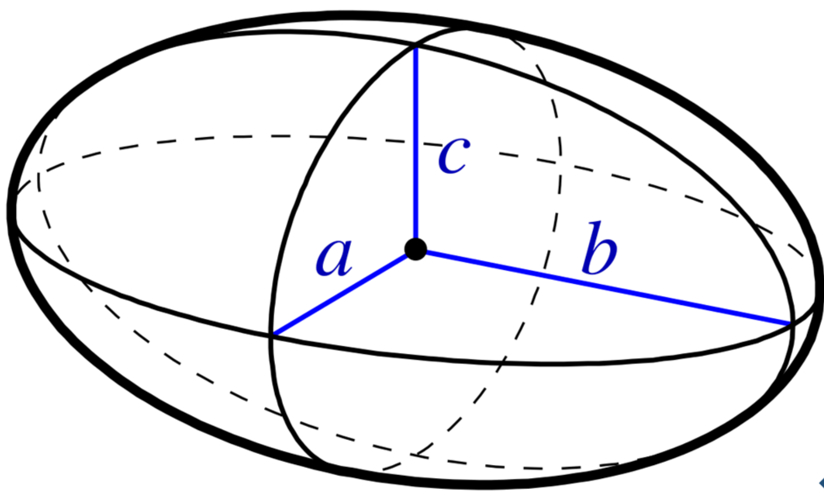
$$I_\xi = (a_x)^2 I_x + (a_y)^2 I_y + (a_z)^2 I_z \; ; \textit{ where } a_i s \textit{ are direction cosines}$$

That is the MOI about any instantaneous rotation axis (in direction of $\vec{1}_\xi$). In addition, this form suggests that a surface can be defined to describe I_ξ changes with respect to the SC instantaneous spin axes relative to the principal axes. For this purpose, let :

$$\rho = \left(\frac{1}{I_\xi}\right)^{\frac{1}{2}} ; and : X = a_x\rho, Y = a_y\rho, Z = a_z\rho$$

That allows one to rewrite I_ξ into an ellipsoidal form with axial dimensions :

$$1 = \frac{1}{\sqrt{I_x}} \quad , \quad 2 = \frac{1}{\sqrt{I_y}} \quad , \quad 3 = \frac{1}{\sqrt{I_z}}$$



Ellipsoid of Inertia

$$\rho = \left(\frac{1}{I_\xi}\right)^{\frac{1}{2}} ; and : X = a_x\rho, Y = a_y\rho, Z = a_z\rho$$

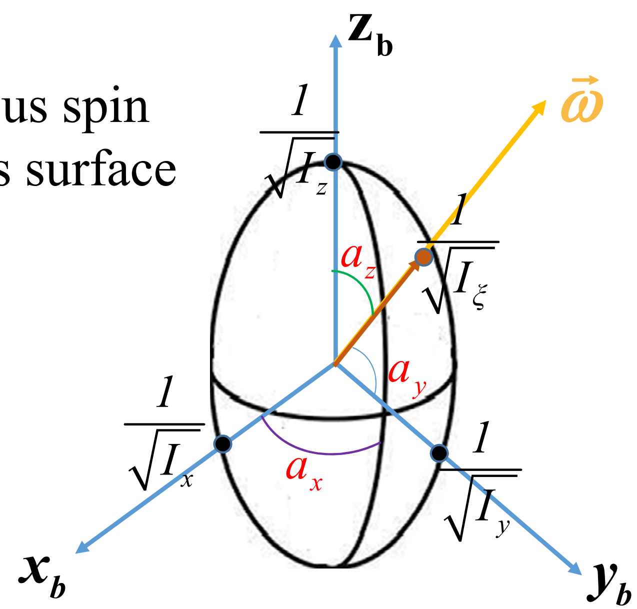
Therefore, we have:

$$I_\xi = (a_x)^2 I_x + (a_y)^2 I_y + (a_z)^2 I_z \Rightarrow 1 = X^2 I_x + Y^2 I_y + Z^2 I_z$$

$$or: \frac{X^2}{\left(\frac{1}{\sqrt{I_x}}\right)^2} + \frac{Y^2}{\left(\frac{1}{\sqrt{I_y}}\right)^2} + \frac{Z^2}{\left(\frac{1}{\sqrt{I_z}}\right)^2} = 1 \Rightarrow \text{ Axes dimensions: } 1 = \frac{1}{\sqrt{I_x}} \quad , \quad 2 = \frac{1}{\sqrt{I_y}} \quad , \quad 3 = \frac{1}{\sqrt{I_z}}$$

The above surface shows all possible values of inertia about various spin axis and is called the **ellipsoid of inertia**. Where, each point on its surface represents the value of the body's moment of inertia about a line (axis) that connects that point to the center.

Note that the angular momentum along with the rotational kinetic energy describes the dynamic state of a rotating body, therefore, using the principal axes, we have:



$$\vec{h} = I\vec{\omega} \quad , \quad h^2 = I_x^2\omega_x^2 + I_y^2\omega_y^2 + I_z^2\omega_z^2$$

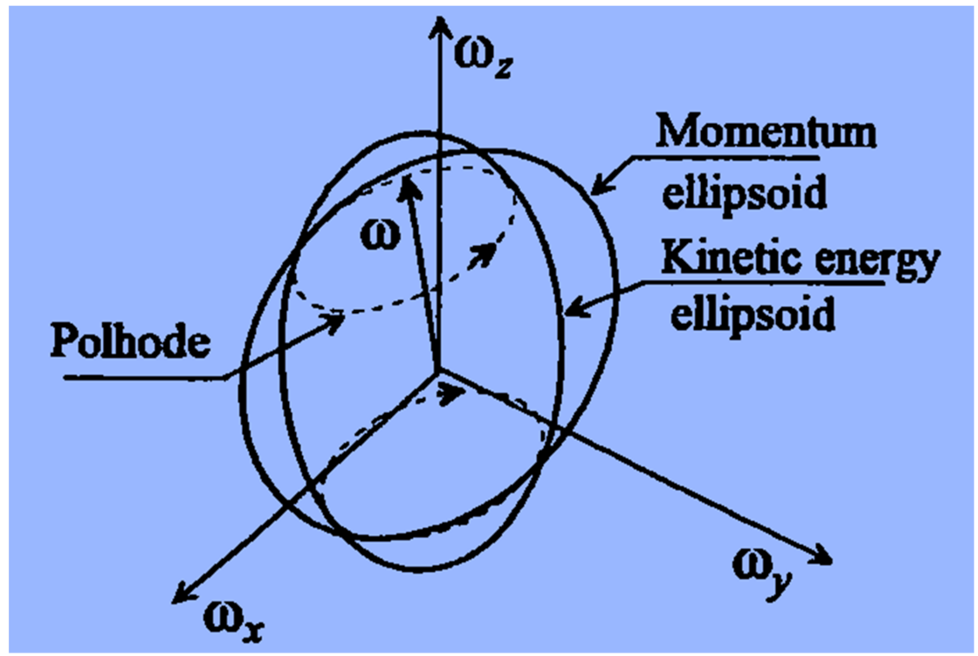
$$2T_{Rotational} = \vec{\omega}^T I\vec{\omega} = I_x^2\omega_x^2 + I_y^2\omega_y^2 + I_z^2\omega_z^2$$

Ellipsoid of Inertia

Using the previous equations, one can similarly define the **angular momentum (AM) and rotational kinetic energy (RKE) ellipsoids**, in which the components of the SC **angular velocity vector** are defined as variables. To simplify the relations, let $T_{ROT} = T$, which will now yield the T and h ellipsoids.

$$\frac{\omega_x^2}{\left(\frac{h}{I_x}\right)^2} + \frac{\omega_y^2}{\left(\frac{h}{I_y}\right)^2} + \frac{\omega_z^2}{\left(\frac{h}{I_z}\right)^2} = 1$$

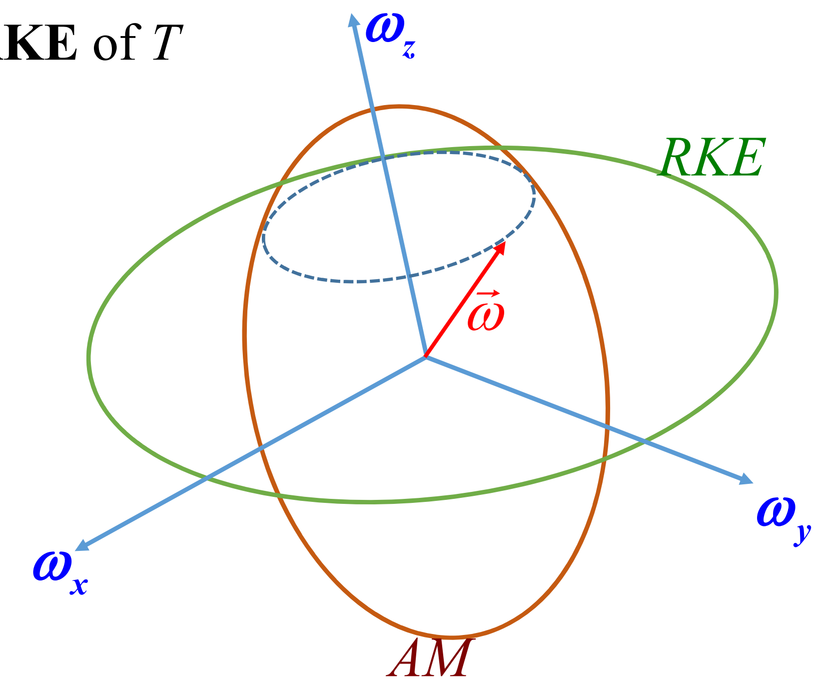
$$\frac{\omega_x^2}{\left(\sqrt{\frac{2T}{I_x}}\right)^2} + \frac{\omega_y^2}{\left(\sqrt{\frac{2T}{I_y}}\right)^2} + \frac{\omega_z^2}{\left(\sqrt{\frac{2T}{I_z}}\right)^2} = 1$$



Ellipsoid of Inertia

Note that the **RKE** ellipsoid has axial dimensions of $\sqrt{2T/I_x}$, $\sqrt{2T/I_y}$, $\sqrt{2T/I_z}$, and its surface represents all possible values of $\vec{\omega}$ that satisfy its relation. Similarly, the axial dimensions of the **AM** ellipsoid will be $\frac{h}{I_x}$; $\frac{h}{I_y}$; $\frac{h}{I_z}$ that is the geometric loci of all possible values of $\vec{\omega}$ that satisfy its relation. Therefore, for any spinning SC with **RKE** of T and **AM of** h, two ellipsoids define all possible values of angular velocities. In particular, the two ellipsoids intersect each other along the curves that satisfying both relations are of interest. These curves are known as **Polhode** and represent simultaneous solutions of both ellipsoids, as the loci of all possible values of $\vec{\omega}$ at that energy level.

Note that for different angular momentum and rotational energy, different polhodes are created.



Euler's Based Attitude Dynamics (AD)

As noted earlier, the rotational dynamic EOM of a rigid SC can be derived via the basic Euler's law of angular momentum. $\left.\frac{d\vec{h}}{dt}\right|_I = \vec{M}^I$

That led to the conservation of $\vec{h}$ in the inertial space in 2BP under pure mutual gravitational attraction and nothing else. Now, the SC can be acted upon by some torques $\vec{M}$ due to external disturbances or attitude control commands, that is : $\vec{M} = \vec{M}_D + \vec{M}_C$

As it is easier to work in the Body frame, again application of the Coriolis law gives:

$\dot{\vec{h}}_I = \dot{\vec{h}}_B + (\vec{\omega} \times \vec{h})_B$, that is called Euler's Moment Equation. By introducing the vector components of $\vec{h}$ and $\vec{\omega}$ in the body frame, we will arrive at basic AD EOM.

$\dot{\vec{h}}_B \triangleq \textit{Derivative in Body Frame; so } \dot{\vec{h}}_B = \begin{bmatrix} \dot{h}_x & \dot{h}_y & \dot{h}_z \end{bmatrix}^T \textit{;and} \quad \vec{\omega}_B = \begin{bmatrix} \omega_x & \omega_y & \omega_z \end{bmatrix}^T$

$$\begin{cases} M_x = \dot{h}_x + \omega_y h_z - \omega_z h_y \\ M_y = \dot{h}_y + \omega_z h_x - \omega_x h_z \\ M_z = \dot{h}_z + \omega_x h_y - \omega_y h_x \end{cases}$$

Euler's Moment Equation (EME)

The EME describes the general attitude behavior of a rigid SC that can either be converted into a set of **first order** DE for $\vec{\omega}$ using the $\vec{h}$ components derived earlier, or to a set of **second order** DE based on the Euler angles using the kinematic EOMs. This issue will be detailed again later. Now, for example assuming the BCS to be the principal axes system:

$$\begin{cases} M_x = \dot{h}_x + \omega_y h_z - \omega_z h_y \\ M_y = \dot{h}_y + \omega_z h_x - \omega_x h_z \\ M_z = \dot{h}_z + \omega_x h_y - \omega_y h_x \end{cases}$$

$$\vec{M} = \begin{Bmatrix} \left[\dot{h}_x + \left(\omega_y h_z - \omega_z h_y\right)\right]\vec{i} \\ \left[\dot{h}_y + \left(\omega_z h_x - \omega_x h_z\right) + \right]\vec{j} \\ \left[\dot{h}_z + \left(\omega_x h_y - \omega_y h_x\right) + \right]\vec{k} \end{Bmatrix} \Rightarrow \vec{M} = \begin{Bmatrix} \left[\dot{\omega}_x I_{xx} + \left(\omega_y \omega_z I_{zz} - \omega_z \omega_y I_{yy}\right)\right]\vec{i} \\ \left[\dot{\omega}_y I_{yy} + \left(\omega_z \omega_x I_{xx} - \omega_x \omega_z I_{zz}\right) + \right]\vec{j} \\ \left[\dot{\omega}_z I_{zz} + \left(\omega_x \omega_y I_{yy} - \omega_y \omega_x I_{xx}\right) + \right]\vec{k} \end{Bmatrix}$$

and using the Euler angles method for propagation, the above AD EOM can be directly related to Euler angles and the SC attitude due to $\vec{M}$ as it orbits.

$$\begin{cases} \omega_x = \dot{\phi} - \dot{\psi}\sin\theta \\ \omega_y = \dot{\theta}\cos\phi + \dot{\psi}\cos\theta\sin\phi \\ \omega_z = \dot{\psi}\cos\theta - \dot{\theta}\sin\psi \end{cases}$$

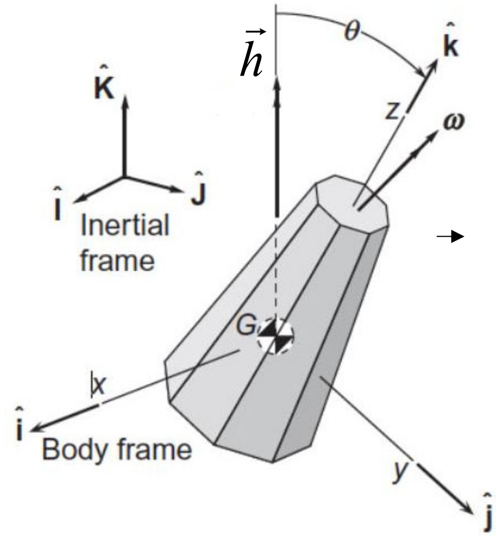
Some Observations with EME

Torque Free Motion of an Axisymmetric SC Body

Using the EME in principal axes while assuming a Torque free motion, $\vec{M} = 0$ yields:

$$\begin{Bmatrix} \left[\dot{\omega}_x I_{xx} + \left(\omega_y \omega_z I_{zz} - \omega_z \omega_y I_{yy}\right)\right]\vec{i} \\ \left[\dot{\omega}_y I_{yy} + \left(\omega_z \omega_x I_{xx} - \omega_x \omega_z I_{zz}\right) + \right]\vec{j} \\ \left[\dot{\omega}_z I_{zz} + \left(\omega_x \omega_y I_{yy} - \omega_y \omega_x I_{xx}\right) + \right]\vec{k} \end{Bmatrix} = \vec{M}$$

$$\begin{cases} I_x \dot{\omega}_x + \omega_y \omega_z \left(I_z - I_y\right) = 0 \\ I_y \dot{\omega}_y + \omega_x \omega_z \left(I_x - I_z\right) = 0 \\ I_z \dot{\omega}_z + \omega_x \omega_y \left(I_y - I_x\right) = 0 \end{cases}$$



As the above equations are nonlinear, there is no analytical solution in closed form. But, some observation can be made. Assuming the axis of symmetry in z the direction ($I_x = I_y$) , further simplifies the EME:

$$\begin{cases} I_x \dot{\omega}_x + \omega_y n \left(I_z - I_y\right) = 0 \\ I_y \dot{\omega}_y + \omega_x n \left(I_x - I_z\right) = 0 \\ I_z \dot{\omega}_z = 0 \end{cases}$$

A body is ***axisymmetric*** if any two of its three principal MOIs are equal. The axis corresponding to the third MOI is called the axis of symmetry. An axisymmetric body having the axis of symmetry as its minor principal axis is called **prolate.** An axisymmetric body is called **oblate** if its axis of symmetry is its major principal axis of inertia.

Torque Free Motion of an Axisymmetric SC

$$\begin{cases} I_x\dot{\omega}_x + \omega_y n\left(I_z - I_y\right) = 0 \\ I_y\dot{\omega}_y + \omega_x n\left(I_x - I_z\right) = 0 \\ I_z\dot{\omega}_z = 0 \end{cases}$$

$$n = \omega_z, spin\ rate$$

Let's define the parameter $\lambda = \frac{n(I_z - I_x)}{I_x}$, and reduce the governing equation further : (Note that $I_x = I_y$):

$$\begin{cases} \dot{\omega}_x + \lambda\omega_y = 0 \\ \dot{\omega}_y - \lambda\omega_x = 0 \end{cases}$$

If we multiply the previous relations by ω_x, ω_y respectively and add them together, we arrive at the following equation:

$$\omega_x\dot{\omega}_x + \omega_y\dot{\omega}_y = 0 \Rightarrow \omega_x \mathrm{d}\omega_x + \omega_y \mathrm{d}\omega_y = 0 \Rightarrow \omega_x^2 + \omega_y^2 = \text{const} = \omega_{xy}^2$$

where ω_{xy} is the component of $\vec{\omega}$ in the body *xy*-plane! Since ω_z (spin rate) is also constant, we conclude that in such conditions the SC will have an angular velocity vector with constant norm!. In other words, ($|\vec{\omega}| = const$), i.e. $\omega^2 = \omega_x^2 + \omega_y^2 + \omega_z^2$ will be constant not $\vec{\omega}$. In addition, via the simplified equations above (with λ), the initial conditions will be :

$$\dot{\omega}_x(0) = -\lambda\omega_y(0) \quad , \quad \dot{\omega}_y(0) = -\lambda\omega_x(0)$$

Torque Free Motion of an Axisymmetric SC

$$\begin{cases} \dot{\omega}_x + \lambda\omega_y = 0 \\ \dot{\omega}_y - \lambda\omega_x = 0 \end{cases}$$

Taking derivative from the $\dot{\omega}_x$ equation gives :

$$\ddot{\omega}_x + \lambda\dot{\omega}_y = \ddot{\omega}_x + \lambda^2\omega_x = 0 \Rightarrow \omega_x(s) = \frac{\dot{\omega}_x(0) + s\omega_x(0)}{s^2 + \lambda^2}$$

$$\omega_x(t) = \omega_x(0)\cos\lambda t + \frac{\dot{\omega}_x(0)}{\lambda}\sin\lambda t$$

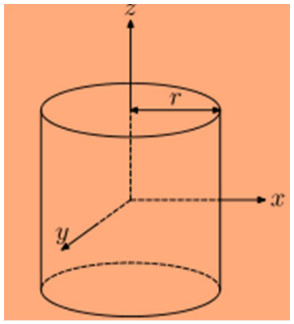
$$\omega_y(t) = -\frac{\dot{\omega}_x(t)}{\lambda} = \omega_x(0)\sin\lambda t - \frac{\dot{\omega}_x(t)}{\lambda}\cos\lambda t$$

Finally, using complex numbers notation a relation can be defined for a **in-plane** or transverse component of $\vec{\omega}$ or ω_{xy}. In other words:

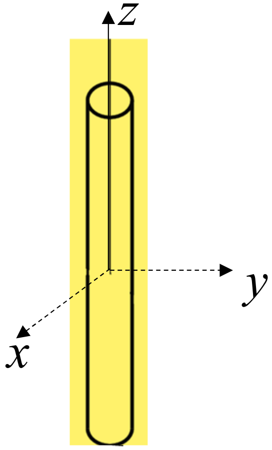
$$\omega_{xy}(t) = \omega_x + i\omega_y = \left[\omega_x(0) + i\omega_y(0)\right]\left(\cos\lambda t + i\sin\lambda t\right)$$

$$= \left[\omega_x(0) + i\omega_y(0)\right]e^{i\lambda t}$$

$$= \omega_{xy}(0)e^{i\lambda t}$$



$I_z > I_x$



$I_z < I_x$

Nutation Motion of a Spinning SC

$$\omega_{xy}(t) = \omega_{xy}(0)e^{i\lambda t}$$

The latter equation implies that ω_{xy} rotates at a rate of λ relative to the principal axes. That is, a body observer on the symmetry axis z, will see $\vec{\omega}$ changing at the rate λ, in the positive direction around z. Of course, note the this motion due to initial conditions (disturbances, $\omega_{xy}(0) \neq 0$) and there are externally applied torques.

Review of results:

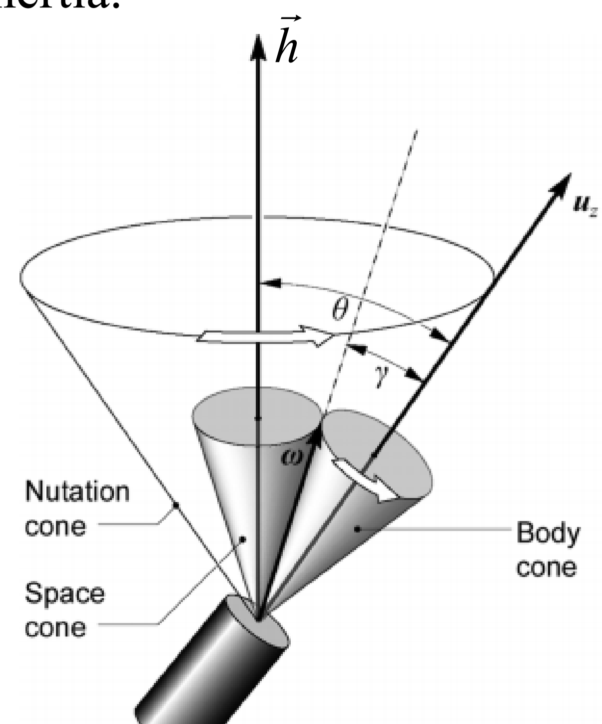
Considering the torque free motion, $\vec{M} = 0$, the angular momentum vector $\vec{h}$ will be constant in the inertial space, $\vec{h} = [I_x\omega_x \quad I_y\omega_y \quad I_z\omega_z]^T$. On the other hand, $\vec{h}$ has a component in the xy plane,

$$\vec{h}_{xy} = I_x\left(\omega_x\vec{i} + \omega_y\vec{j}\right) = I_x\vec{\omega}_{xy} \Rightarrow \vec{h} = I_x\vec{\omega}_{xy} + I_z\omega_z\vec{k}$$

Given that $\vec{\omega} = \vec{\omega}_{xy} + \omega_z\vec{k}$, It can be said that the three vectors $\vec{h}, \vec{\omega}, \vec{k}$ will always be in the same plane (despite the time behavior of ω_{xy}) and that $\vec{h}$ is **constant in the inertial space**. While usually, $\vec{h}$ *and* $\vec{\omega}$ are not collinear (or parallel, that is related to pure spin about the symmetry axis), the SC motion is described by two cones rolling against each other. This motion is graphically described in the next slide.

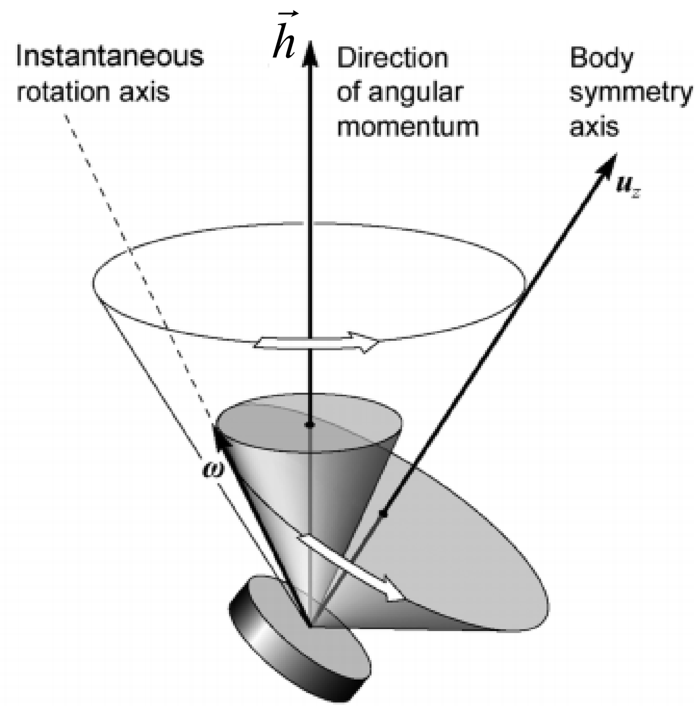
Nutation Motion of a Spinning SC

As shown in both figures, the trajectory of $\vec{\omega}$ creates a **body cone** rolling against what is called a **space cone** where $\vec{\omega}$ represents the tangent (contact) line. The body and space cones will be different based on the SC configuration (rod/disc) or relative values of the principal components of inertia.



Prolate: $I_z < I_x = I_y$

https://www.aero.iitb.ac.in/~bhat/prograde.gif



Oblate: $I_z > I_x = I_y$

https://www.aero.iitb.ac.in/~bhat/retrograde.gif

Nutation of a Spinning S/C

$$\vec{h} = I_x \vec{\omega}_{xy} + I_z \omega_z \vec{k}$$

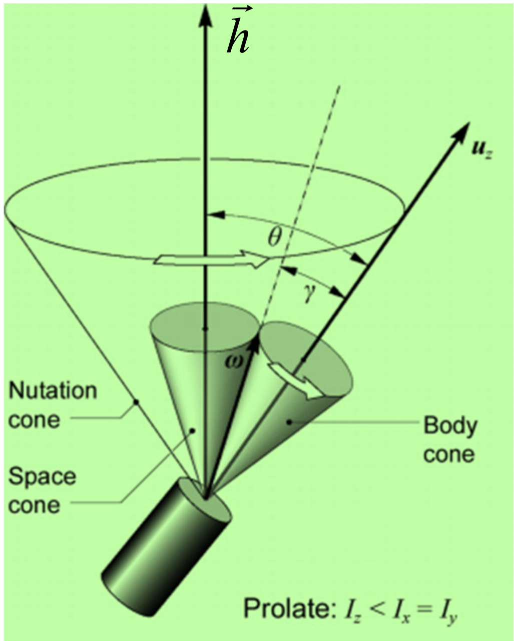
$$\vec{\omega} = \vec{\omega}_{xy} + \omega_z \vec{k}$$

To fully describe the motion, one can easily determine the θ (nutation angle) and γ (the angle between $\vec{\omega}$ and $\vec{z}$). In essence, a perturbed rotating SC (could be due initial condition or injection) with no applied external torques will be **wobbling in space**. This is not a desirable behavior and will degrade the mission performance. In addition, keeping the nutation angle small is one of the important tasks of attitude control system. In addition, the following are true :

1- $\tan\theta = \dfrac{h_{xy}}{h_z} = \dfrac{I_x \omega_{xy}}{I_Z \omega_z}$, $\tan\gamma = \dfrac{\omega_{xy}}{\omega_z}$

2- $\tan\theta = \dfrac{I_x}{I_z}\tan\gamma$

3- $\begin{cases} \text{for a rod or rod type SC}: \ \theta > \gamma \quad \text{if} \quad I_z < I_x \\ \text{for a disk type SC}: \ \theta < \gamma \quad \text{if} \quad I_z > I_x \end{cases}$



Note: If $\omega_{xy} = 0$,then $\theta = \gamma = 0$ and spin motion will be stable.

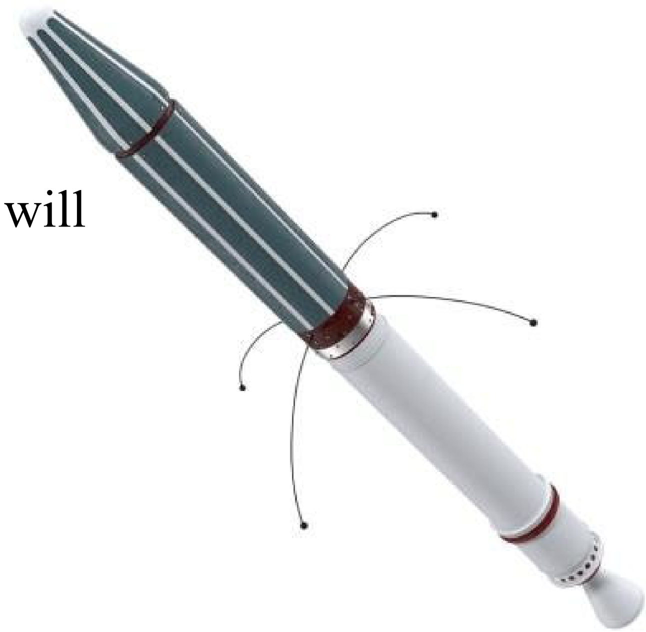
Energy Dissipation & Nutation Destabilization

In the previous parts, it was observed that the KRE around an instantaneous axis of rotation is equal to:

$$T = T_{Rot} = \frac{1}{2} I_{\xi} \omega^2 = \frac{1}{2} \frac{h^2}{I_{\xi}}$$

In the torque free motion, the rotating SC angular momentum, $\vec{h}$ will remain constant. Where I_{ξ} depends on the direction of the axis of rotation in the body coordinate frame. So, since h is constant, the maximum and minimum values of T will occur for the minor and major axes. In other words:

$$\begin{cases} T_{max} = \dfrac{h^2}{2I_{min}} & \textit{at the minor axis} \\ T_{min} = \dfrac{h^2}{2I_{max}} & \textit{at the major axis} \end{cases}$$



Explorer-1, the first U.S. satellite lunched in 1958

Energy Dissipation & Nutation Destabilization

$$\begin{cases} T_{max} = \dfrac{h^2}{2I_{min}} & \textit{at the minor axis} \\ T_{min} = \dfrac{h^2}{2I_{max}} & \textit{at the major axis} \end{cases}$$

According to T_{max} relation with h being constant, if the body spins about the minor axis and there is some internal **energy dissipation** mechanism that tends to decrease the RKE to its minimum, then the body will switch the spin of rotation to the major axis in order to satisfy T_{min} relation. This phenomenon was observed Explorer I, that made it rotationally unstable. For simplification, assume $I_x = I_y$:

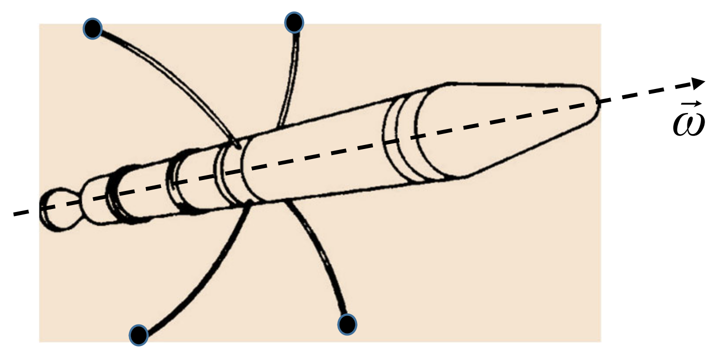
$$h^2 = \left[\omega_x^2 + \omega_y^2\right] I_x^2 + I_z^2 \omega_z^2 \xrightarrow{2T = \vec{\omega}^T I \vec{\omega}} 2T = \left[\omega_x^2 + \omega_y^2\right] I_x + \omega_z^2 I_z$$

Multiply the 2nd equation by I_x and subtract it from the first, to get:

$$h^2 - 2TI_x = \omega_z^2 \left(I_z\right)\left(I_z - I_x\right) = \omega_z^2 I_z^2 \left(\frac{I_z - I_x}{I_z}\right)$$

Noting that, $\cos\theta = \frac{h_z}{h} = \frac{I_z \omega_z}{h}$, yields:

$$2TI_x = h^2 - h^2 \cos^2\theta \left(\frac{I_z - I_x}{I_z}\right) = h^2 \sin^2\theta \left(\frac{I_z - I_x}{I_z}\right)$$



This energy dissipation was attributed to small flexible wire type antennas.

Energy Dissipation & Nutation Destabilization

$$2TI_x = h^2 \sin^2\theta \left(\frac{I_z - I_x}{I_z} \right)$$

Differentiating this equation with time yields:

$$2\dot{T}I_x = 2h^2\dot{\theta}\sin\theta\cos\theta\left(1 - \frac{I_x}{I_z}\right) \Rightarrow \dot{T} = \frac{h^2}{I_x}\sin\theta\cos\theta\left(1 - \frac{I_x}{I_z}\right)\dot{\theta}$$

In conclusion, one can write (simplify):

$$\dot{T} = \frac{h^2}{I_z}\sin\theta\cos\theta\left(\frac{I_z}{I_x} - 1\right)\dot{\theta}$$

1- If $\dot{T} < 0$ and $I_z > I_x$ (disk type), then $\dot{\theta} < 0 \Rightarrow$ Nutation stability

2- If $\dot{T} < 0$ and $I_z < I_x$ (rod type), then $\dot{\theta} > 0 \Rightarrow$ Nutation instability

In other words, in presense of a dissipation mechanism, a spinning SC would have nutation stability if the spin axes is along the major axis.

Inertial stabilization via spinning is a common, cheap and efficient solution that was used for pioneer satellites. Modern SC can achieve stabilization via active 3-axis-stabilization methods.

Stability of Rotation about Principal Axes with no Precondition

In the previous discussion, an axisymmetric body with equal $I_x = I_y$ was considered in terms of stability. In this section, torque free motion will be investigated without this equality requirement. In this context, the stability criteria can be investigated around each spin axis (after disturbance) from an steady spin motion. Hence, the motion will be considered stable if the perturbed quantities show bounded behavior. The existence of stable motion can be assessed by perturbing the steady motion. That is, if $\vec{\omega}$ is initially parallel any of the x, y, and z axes, the torque free equations of motion must show stability criteria for ω_x, ω_y, and ω_z. Let's assume stability condition is sought, if the SC is spinning **about the z axis.**

In other words, initially $\vec{\omega} = n\vec{k}$. $Now,$ let $\omega_z = n + \varepsilon, where\ \varepsilon\ is$ the perturbation. Since, ω_x, ω_y are initially zero, they will refer to pertubation in the following along with ε. Using the previous Torque free EOMs (as perturbed form), we can investigate SC stability without precondition :

$$\begin{cases} I_x\dot{\omega}_x + \omega_y\omega_z\left(I_z - I_y\right) = 0 \\ I_y\dot{\omega}_y + \omega_x\omega_z\left(I_x - I_z\right) = 0 \\ I_z\dot{\omega}_z + \omega_x\omega_y\left(I_y - I_x\right) = 0 \end{cases} \xrightarrow{Linrearized\ \ Form} \begin{cases} I_x\dot{\omega}_x + \omega_y n\left(I_z - I_y\right) = 0 \\ I_y\dot{\omega}_y + \omega_x n\left(I_x - I_z\right) = 0 \\ I_z\dot{\varepsilon} + \omega_x\omega_y\left(I_y - I_x\right) = 0 \end{cases}$$

Stability of Rotation about Principal Axes with no Precondition

$$\begin{cases} I_x\dot{\omega}_x + \omega_y n\left(I_z - I_y\right) = 0 \\ I_y\dot{\omega}_y + \omega_x n\left(I_x - I_z\right) = 0 \\ I_z\dot{\varepsilon} + \omega_x\omega_y\left(I_y - I_x\right) = 0 \end{cases}$$

The first two equations are linear. Differentiating, the first equation While using the second equation in the process yields:

$$I_x\ddot{\omega}_x + n\left(I_z - I_y\right)\left(\frac{-\omega_x n\left(I_x - I_z\right)}{I_y}\right) = 0$$

$$\Rightarrow \ddot{\omega}_x + n^2\frac{I_z - I_y}{I_y}\frac{I_z - I_x}{I_x}\omega_x = 0$$

$$s^2 + \beta^2 = 0 \quad , \quad \beta = n\sqrt{\left[\left(1 - \frac{I_z}{I_x}\right)\left(1 - \frac{I_z}{I_y}\right)\right]}$$

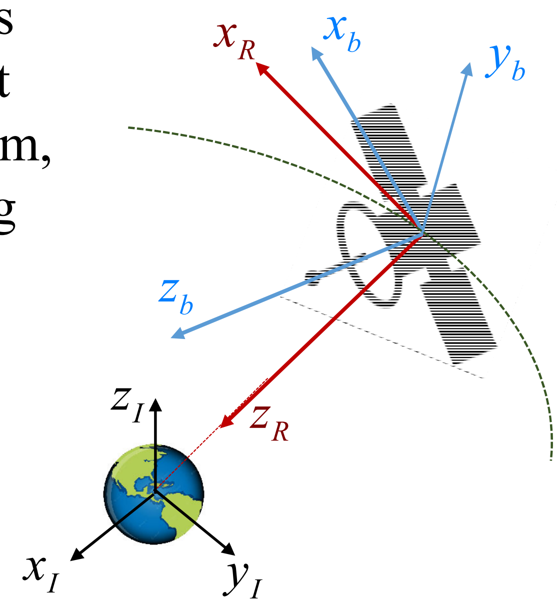
The stability condition can be detected using the characteristic equation for ω_x, the solution of which $s^2 = -\beta^2$. Thus, for stability (neutral), β is required to be real. Similar result will be obtained for ω_y .

In other words, **the stability conditions are**: $I_z > I_x, I_y$ or $I_z < I_x, I_y$

That means , for a rigid spinning SC about its maximum axis (or maximum) of moment of inertia, the angular motion WRT will be stable about other axes (spin about x and y). In addition, spinning about an intermediate axis will be unstable.

Kinematics Equation of Motion (KEOM)

Before introducing the kinematic equations of motion, it is necessary to state that the choice of a coordinate system at any stage of the satellite's operation or life is dependent on the task or conditions of the SC at that stage. For example, for an interplanetary mission within the solar system, the SC starts from a parking orbit around the Earth, for which a moving reference frame will be appropriate SC orientation relative to ECI. Midway, out of the Earth SOI, another inertial reference is utilized for attitude in space. And finally, in the vicinity of the target planet, the reference orbit around the target planet will be the best choice for attitude planning. There are in general various ways of attitude representation and propagation **(ARP)** that include:



- Direction Cosine Matrix
- Rotation Vector
- Euler Angles method
- Quaternions
- Gibbs/Rodrigues Parameters

$$\vec{i}_R = \vec{j}_R \times \vec{k}_R \quad ; \quad \vec{j}_R = \frac{\vec{v} \times \vec{r}}{\left|\vec{v} \times \vec{r}\right|} \quad ; \quad \vec{k}_R = -\frac{\vec{r}}{r}$$

Kinematics Equation of Motion (KEOM)

The attitude of the satellite relative to any reference frame (RF) is possible through utility of any of the previously mentioned ARP methods.

1- **The Euler Angles Method (EA)** ,

This is the most popular method for ARP, though it has some singularities in the propagation process. Via utility of EA, it is possible to establish a continuous relationship between the body frame and the orbital reference frame. EA is defined by three subsequent orderly rotations of the orbit RF to coincide it with the SC BF. The 3,2,1 sequence of rotations with ψ, θ, ϕ develops the following TM and ARP relations:

$$\mathbf{C}_{\mathbf{b}}^{\mathrm{R}} = \begin{bmatrix} \cos\theta\cos\psi & -\cos\varphi\sin\psi + \sin\varphi\sin\theta\cos\psi & \sin\varphi\sin\psi + \cos\varphi\sin\theta\cos\psi \\ \cos\theta\sin\psi & \cos\varphi\cos\psi + \sin\varphi\sin\theta\sin\psi & -\sin\varphi\cos\psi + \cos\varphi\sin\theta\sin\psi \\ -\sin\theta & \sin\varphi\cos\theta & \cos\varphi\cos\theta \end{bmatrix}$$

$$\begin{cases} p = \dot{\varphi} - \dot{\psi}\, sin\,\theta \\ q = \dot{\theta} cos\varphi + \dot{\psi}\, cos\,\theta sin\varphi \\ r = \dot{\psi}\, cos\,\theta cos\varphi - \dot{\theta} sin\varphi \end{cases} \quad or \quad \begin{cases} \dot{\varphi} = p + [q sin\varphi + r cos\varphi]\, tan\,\theta \\ \dot{\theta} = q cos\varphi - r sin\varphi \\ \dot{\psi} = [q sin\varphi + r cos\varphi]\, sec\,\theta \end{cases} \quad or \quad \begin{Bmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{Bmatrix} = \begin{bmatrix} 1 & sin\varphi\, tan\theta & cos\varphi\, tan\theta \\ 0 & cos\varphi & -sin\varphi \\ 0 & \dfrac{sin\varphi}{cos\theta} & \dfrac{cos\varphi}{cos\theta} \end{bmatrix} \begin{Bmatrix} p \\ q \\ r \end{Bmatrix}$$

Kinematics Equation of Motion (KEOM)

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