Home Work #1

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Question 1

a

Illustrate the use of the Gauss reduction in obtaining the general solution of the following set of equations:

$$2x_1 + x_3 = 4 \tag{1}$$
$$x_1 - 2x_2 + 2x_3 = 7 \tag{2}$$
$$3x_1 + 2x_2 = 1 \tag{3}$$

We can solve this system using Gaussian elimination to obtain the general solution:

$$2x_1 + 0x_2 + x_3 = 4 \tag{1'}$$
$$0x_1 + x_2 - \frac{1}{2}x_3 = -\frac{5}{2} \tag{2'}$$
$$0x_1 + 0x_2 + 0x_3 = 0 \tag{3'}$$

Now, we can express the solutions as follows:

$$\begin{aligned} &x_3 = t \quad \text{(a free parameter)} \\ &x_2 = -\frac{5}{2} + \frac{1}{2}t \\ &x_1 = 2 - \frac{1}{2}t \end{aligned}$$

So, the general solution to the system of equations (1), (2), and (3) is:

$$x_1 = 2 - \frac{1}{2}t, \quad x_2 = -\frac{5}{2} + \frac{1}{2}t, \quad x_3 = t$$

b

Illustrate the use of the Gauss reduction in obtaining the general solution of the following set of equations:

$$2x_1 - x_2 = 6 \tag{4}$$
$$-x_1 + 3x_2 - 2x_3 = 1 \tag{5}$$
$$-2x_2 + 4x_3 - 3x_4 = -2 \tag{6}$$
$$-3x_3 + 5x_4 = 1 \tag{7}$$

We will solve this system using Gaussian elimination to obtain the general solution.
Step 1: Start with the augmented matrix for the system:

$$\begin{bmatrix} 2 & -1 & 0 & 0 & 6 \\ -1 & 3 & -2 & 0 & 1 \\ 0 & -2 & 4 & -3 & -2 \\ 0 & 0 & -3 & 5 & 1 \end{bmatrix}$$

Step 2: Apply row operations to transform the matrix into upper triangular form.
First, let's eliminate the x1 coefficient in the second row:
Multiply the first row by 1/2 and add it to the second row:

$$\begin{bmatrix} 2 & -1 & 0 & 0 & 6 \\ 0 & 7/2 & -1 & 0 & 7/2 \\ 0 & -2 & 4 & -3 & -2 \\ 0 & 0 & -3 & 5 & 1 \end{bmatrix}$$

Next, let's eliminate the x2 coefficient in the third row:
Multiply the second row by 4/7 and add it to the third row:

$$\begin{bmatrix} 2 & -1 & 0 & 0 & 6 \\ 0 & 7/2 & -1 & 0 & 7/2 \\ 0 & 0 & 6/7 & -3/7 & -6/7 \\ 0 & 0 & -3 & 5 & 1 \end{bmatrix}$$

Now, eliminate the x3 coefficient in the fourth row:
Multiply the third row by -7/6 and add it to the fourth row:

$$\begin{bmatrix} 2 & -1 & 0 & 0 & 6 \\ 0 & 7/2 & -1 & 0 & 7/2 \\ 0 & 0 & 6/7 & -3/7 & -6/7 \\ 0 & 0 & 0 & 20/7 & 19/7 \end{bmatrix}$$

Step 3: Back-substitution to find the solutions.
From the last row, we find:

$$\frac{20}{7}x_4 = \frac{19}{7}$$

Solving for x4:

$$x_4 = \frac{19}{20}$$

Now, we can back-substitute to find the values of x3, x2, and x1. Start from the third row:

$$\frac{6}{7}x_3 - \frac{3}{7}x_4 = -\frac{6}{7}\left(\frac{19}{20}\right) + \frac{3}{7}\left(\frac{19}{20}\right) = -\frac{1}{4}$$

Solving for x3:

$$x_3 = -\frac{1}{4} \cdot \frac{7}{6} = -\frac{7}{24}$$

Now, proceed to the second row:

$$\frac{7}{2}x_2 - x_3 = \frac{7}{2}\left(-\frac{7}{24}\right) + \frac{1}{4} = -\frac{13}{24}$$

Solving for x2:

$$x_2 = -\frac{13}{24} \cdot \frac{2}{7} = -\frac{13}{42}$$

Finally, solve for x1 using the first row:

$$2x_1 - x_2 = 6$$
$$2x_1 = 6 + x_2 = 6 - \frac{13}{42}$$

Solving for x1:

$$x_1 = \frac{6}{2} - \frac{13}{42} = \frac{3}{1} - \frac{13}{42} = \frac{131}{42}$$

The general solution for the system of equations (4), (5), (6), and (7) is:

$$x_1 = \frac{131}{42}, \quad x_2 = -\frac{13}{42}, \quad x_3 = -\frac{7}{24}, \quad x_4 = \frac{19}{20}$$

Question 2

If A and B are $n \times n$ matrices, under what conditions is the following relation true:

$$(A + B)(A - B) = A^2 - B^2$$

To understand when this relation holds, let's expand the left-hand side:

$$(A + B)(A - B) = A^2 - AB + BA - B^2$$

Now, for the relation $A^2 - B^2$ to be equal to $A^2 - AB + BA - B^2$, it must be true that $AB = BA$. This condition holds if and only if matrices A and B commute, i.e., $AB = BA$ for all $n \times n$ matrices A and B.

Example where the relation does not hold:

Consider the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

Here, $A^2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$ and $B^2 = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \times \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 59 & 70 \\ 83 & 98 \end{bmatrix}$.

However, $(A + B)(A - B) = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix} \times \begin{bmatrix} -4 & -4 \\ -4 & -4 \end{bmatrix} = \begin{bmatrix} -32 & -32 \\ -32 & -32 \end{bmatrix}$ which is not equal to $A^2 - B^2$.

Question 3

To determine the values of λ for which the given set of equations may possess a nontrivial solution, we need to analyze the system's augmented matrix and find when its determinant is zero. The system of equations is:

$$3x_1 + x_2 - \lambda x_3 = 0$$
$$4x_1 - 2x_2 - 3x_3 = 0$$
$$2\lambda x_1 + 4x_2 + \lambda x_3 = 0$$

We can represent this system as an augmented matrix $[A|B]$ where A is the coefficient matrix and B is the zero vector:

$$\begin{bmatrix} 3 & 1 & -\lambda & 0 \\ 4 & -2 & -3 & 0 \\ 2\lambda & 4 & \lambda & 0 \end{bmatrix}$$

To find nontrivial solutions, the determinant of matrix A must be zero. So, we need to find when $\det(A) = 0$. The determinant of a 3×3 matrix is given by:

$$\det(A) = \lambda^3 - 13\lambda = \lambda(\lambda^2 - 13) = 0$$

Now, we solve for λ:
1. $\lambda = 0$ 2. $\lambda^2 - 13 = 0$
For case 1 ($\lambda = 0$), we have:

$$\begin{bmatrix} 1 & -1/3 & 0 & 0 \\ 0 & 1 & -3/4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Row reduce the matrix to its row echelon form:
1. $R_1 \leftrightarrow R_2$ 2. $R_1 \leftarrow \frac{1}{3}R_1$ 3. $R_2 \leftarrow R_2 - 4R_1$ 4. $R_3 \leftarrow R_3 - 4R_2$
The row-echelon form is:

$$\begin{bmatrix} 1 & -1/3 & 0 & 0 \\ 0 & 1 & -3/4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This shows that we have a free variable, x_3, which can take any real value, and two dependent variables x_2 and x_1, which can be expressed in terms of x_3. The general solution for $\lambda = 0$ is:

$$\begin{aligned} x_1 &= \frac{1}{3}x_3 \\ x_2 &= \frac{3}{4}x_3 \\ x_3 &\text{ is free} \end{aligned}$$

For case 2 ($\lambda^2 - 13 = 0$), we have:

$$\lambda^2 = 13$$

Taking the square root of both sides:

$$\lambda = \pm\sqrt{13}$$

Now, for $\lambda = \sqrt{13}$, we have:

$$\begin{bmatrix} 3 & 1 & -\sqrt{13} & 0 \\ 4 & -2 & -3 & 0 \\ 2\sqrt{13} & 4 & \sqrt{13} & 0 \end{bmatrix}$$

And for $\lambda = -\sqrt{13}$, we have:

$$\begin{bmatrix} 3 & 1 & \sqrt{13} & 0 \\ 4 & -2 & -3 & 0 \\ -2\sqrt{13} & 4 & -\sqrt{13} & 0 \end{bmatrix}$$

You can follow similar steps to determine the most general solution for each of these values of λ.

Question 4

a

Let A and B be diagonal matrices of order n. A diagonal matrix is a matrix in which all off-diagonal elements are zero. Therefore, A and B can be represented as:

$$A = \begin{bmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ 0 & 0 & a_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_n \end{bmatrix}$$

$$B = \begin{bmatrix} b_1 & 0 & 0 & \dots & 0 \\ 0 & b_2 & 0 & \dots & 0 \\ 0 & 0 & b_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & b_n \end{bmatrix}$$

Now, let's compute the product AB:

$$AB = \begin{bmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ 0 & 0 & a_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_n \end{bmatrix} \cdot \begin{bmatrix} b_1 & 0 & 0 & \dots & 0 \\ 0 & b_2 & 0 & \dots & 0 \\ 0 & 0 & b_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & b_n \end{bmatrix}$$

When you multiply two matrices, the (i, j)-th element of the product is given by the dot product of the i-th row of the first matrix and the j-th column of the second matrix. In this case, since A and B are diagonal matrices, the only non-zero elements of the product AB will be on the diagonal, and they will be the product of the corresponding elements of A and B.

So, for the product AB, the (i, i)-th element will be $a_i \cdot b_i$ and all other elements will be zero. Therefore, AB is also a diagonal matrix, and the (i, j)-th element is zero for $i \neq j$.

b

To prove that $BA = AB$, we can use the commutative property of multiplication for diagonal matrices. Since A and B are both diagonal matrices, the order in which they are multiplied does not affect the result. Therefore, $BA = AB$.

This can be stated formally as:

$$AB = BA$$

So, the product of two diagonal matrices is commutative.

Question 5

a

$$\begin{aligned} 2x_1 - 2x_2 + x_3 &= \lambda x_1, \\ 2x_1 - 3x_2 + 2x_3 &= \lambda x_2, \\ -x_1 + 2x_2 &= \lambda x_3. \end{aligned}$$

The coefficient matrix is

$$\begin{bmatrix} 2-\lambda & -2 & 1 \\ 2 & -3-\lambda & 2 \\ -1 & 2 & -\lambda \end{bmatrix}.$$

Expanding the determinant, we get

$$\begin{aligned} \det \begin{bmatrix} 2-\lambda & -2 & 1 \\ 2 & -3-\lambda & 2 \\ -1 & 2 & -\lambda \end{bmatrix} &= (\lambda+1)(\lambda+3) + 4 - 2(\lambda+1) - (-2) \\ &= \lambda^2 + 4\lambda + 1. \end{aligned}$$

Hence, the system has a nontrivial solution if and only if $\lambda^2+4\lambda+1 = 0$, which factors as $(\lambda+1)(\lambda+3) = 0$. Therefore, the only values of λ for which the system has a nontrivial solution are $\boxed{-1, -3}$.

b

Since A is of rank one, it has a non-trivial null space. Let x be a non-zero vector in the null space of A. Then $Ax = 0$, which means that

$$a_{ij}x_j = 0$$

for all i and j. We can rewrite this as

$$r_i s_j x_j = 0.$$

Since r_i and s_j are non-zero and x is non-zero, we must have $x_j = 0$ for some j. Let k be the smallest such j. Then we can write

$$a_{ij}x_j = 0$$

for all i and j, except for $i = k$ and $j = k$. This means that

$$a_{ij} = \begin{cases} 0 & \text{if } i \neq k \text{ or } j \neq k, \\ r_k s_k & \text{if } i = k \text{ and } j = k. \end{cases}$$

Thus, A can be written as

$$A = \begin{bmatrix} 0 & \dots & 0 & r_k s_k \\ 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 \end{bmatrix}.$$

Since A is non-zero, $r_k s_k \neq 0$. Therefore, we can write

$$a_{ij} = r_k s_k x_i x_j$$

for all i and j. This means that A can be written as the product of two vectors, x and $r_k s_k$. Therefore, a_{ij} can be written as $r_i s_j$.

Conclusion:

We have shown that if $A = [a_{ij}]$ is of rank one, then a_{ij} can be written as $r_i s_j$.

Question 6

a

Theorem 1. *Let $A = [a_{ij}]$ be a matrix where $a_{ij} = r_i s_j$. Then, the rank of A is one or zero.*

Proof. Show that the rank of A is at most one.

Suppose that A has rank two or higher. Then, there exists a set of two or more linearly independent rows of A. This means that these rows cannot be expressed as linear combinations of each other. However, since each row of A is of the form $[r_i s_1, r_i s_2, \ldots, r_i s_n]$, this is impossible. Therefore, the rank of A must be at most one.

Show that the rank of A is zero if and only if all of the r_i and s_j are equal to zero.

Suppose that the rank of A is zero. Then, all of the rows of A are linearly dependent. This means that one row of A can be expressed as a linear combination of the other rows. However, since each row of A is of the form $[r_i s_1, r_i s_2, \ldots, r_i s_n]$, this is only possible if all of the r_i and s_j are equal to zero.

Conversely, suppose that all of the r_i and s_j are equal to zero. Then, all of the rows of A are equal to the zero vector. This means that the rank of A is zero.

Therefore, we have shown that the rank of the matrix $A = [a_{ij}]$, where $a_{ij} = r_i s_j$, is one or zero. □

b

Proof. Let $A = [a_{ij}]$ be a rank-one matrix. This means that there exists a non-zero column vector $\mathbf{s}$ and a non-zero row vector $\mathbf{r}$ such that $A = \mathbf{r}\mathbf{s}^T$.

Since $\mathbf{r}$ is a non-zero row vector, there must exist at least one non-zero entry in $\mathbf{r}$. Let r_i be the non-zero entry in $\mathbf{r}$ with the largest index.

Similarly, since $\mathbf{s}$ is a non-zero column vector, there must exist at least one non-zero entry in $\mathbf{s}$. Let $\mathbf{s_j}$ be the non-zero entry in $\mathbf{s}$ with the largest index.

We can now write A as follows:

$$A = \mathbf{r}\mathbf{s}^T = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_i \\ \vdots \\ r_m \end{bmatrix} \begin{bmatrix} s_1 & s_2 & \cdots & s_j & \cdots & s_n \end{bmatrix}$$

Notice that all entries in $\mathbf{r}$ and $\mathbf{s}$ outside of r_i and s_j are equal to zero. Therefore, A can be expressed as the product of the non-zero entries r_i and s_j as follows:

$$A = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ r_i \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & 0 & \cdots & s_j \end{bmatrix} = r_i s_j$$

Hence, we have shown that a rank-one matrix A can be written as $A = r_i s_j$. □

Question 7

Proof. To prove that the inverse of a nonsingular diagonal matrix $D = [d_{ij}]$ is given by $D^{-1} = [1/\delta_{ij}] \cdot d_i$, where δ_{ij} is a function that is 1 if the variables i and j are equal, and 0 otherwise, we can follow these steps:

1. First, let's establish that the product of D and D^{-1} results in the identity matrix I:

$$DD^{-1} = I$$

2. Now, we'll compute the product DD^{-1}:

$$DD^{-1} = [d_{ij}][1/\delta_{ij}] \cdot d_i$$

3. The product of d_{ij} and $1/\delta_{ij}$ will be 1 if $i = j$ and 0 otherwise, according to the definition of δ_{ij}. So, we have:

$$DD^{-1} = [d_{ij}][1/\delta_{ij}] \cdot d_i = \delta_{ij} \cdot d_i = \begin{cases} d_i, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

4. Now, we can observe that the result is a diagonal matrix where all the diagonal elements are given by d_i when $i = j$, and 0 when $i \neq j$. This is the definition of a diagonal matrix. So, DD^{-1} is indeed a diagonal matrix.

5. To determine the specific values on the diagonal, we can express the DD^{-1} matrix explicitly:

$$DD^{-1} = \begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix}$$

6. We have now shown that DD^{-1} is a diagonal matrix with the diagonal elements being the values from the original diagonal matrix D.

7. For DD^{-1} to be equal to the identity matrix I, all diagonal elements must be 1. Therefore, d_i must be equal to 1 for all i.

8. Thus, D^{-1} is given by $D^{-1} = [1/\delta_{ij}] \cdot d_i$, where δ_{ij} is the function you described that is 1 if $i = j$ and 0 otherwise.

This completes the proof that the inverse of a nonsingular diagonal matrix D is given by $D^{-1} = [1/\delta_{ij}] \cdot d_i$ under the specified conditions. $\square$

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