Home Work #1

Ali BaniAsad 401209244

October 13, 2023

Question 1

a

Illustrate the use of the Gauss reduction in obtaining the general solution of the following set of equations:

$$2x_1 + x_3 = 4 \tag{1}$$
$$x_1 - 2x_2 + 2x_3 = 7 \tag{2}$$
$$3x_1 + 2x_2 = 1 \tag{3}$$

We can solve this system using Gaussian elimination to obtain the general solution:

$$2x_1 + 0x_2 + x_3 = 4 \tag{1'}$$
$$0x_1 + x_2 - \frac{1}{2}x_3 = -\frac{5}{2} \tag{2'}$$
$$0x_1 + 0x_2 + 0x_3 = 0 \tag{3'}$$

Now, we can express the solutions as follows:

$$\begin{aligned} &x_3 = t \quad \text{(a free parameter)} \\ &x_2 = -\frac{5}{2} + \frac{1}{2}t \\ &x_1 = 2 - \frac{1}{2}t \end{aligned}$$

So, the general solution to the system of equations (1), (2), and (3) is:

$$x_1 = 2 - \frac{1}{2}t, \quad x_2 = -\frac{5}{2} + \frac{1}{2}t, \quad x_3 = t$$

b

Illustrate the use of the Gauss reduction in obtaining the general solution of the following set of equations:

$$2x_1 - x_2 = 6 \tag{4}$$
$$-x_1 + 3x_2 - 2x_3 = 1 \tag{5}$$
$$-2x_2 + 4x_3 - 3x_4 = -2 \tag{6}$$
$$-3x_3 + 5x_4 = 1 \tag{7}$$

We will solve this system using Gaussian elimination to obtain the general solution.
Step 1: Start with the augmented matrix for the system:

$$\begin{bmatrix} 2 & -1 & 0 & 0 & 6 \\ -1 & 3 & -2 & 0 & 1 \\ 0 & -2 & 4 & -3 & -2 \\ 0 & 0 & -3 & 5 & 1 \end{bmatrix}$$

Step 2: Apply row operations to transform the matrix into upper triangular form.
First, let's eliminate the x1 coefficient in the second row:
Multiply the first row by 1/2 and add it to the second row:

$$\begin{bmatrix} 2 & -1 & 0 & 0 & 6 \\ 0 & 7/2 & -1 & 0 & 7/2 \\ 0 & -2 & 4 & -3 & -2 \\ 0 & 0 & -3 & 5 & 1 \end{bmatrix}$$

Next, let's eliminate the x2 coefficient in the third row:
Multiply the second row by 4/7 and add it to the third row:

$$\begin{bmatrix} 2 & -1 & 0 & 0 & 6 \\ 0 & 7/2 & -1 & 0 & 7/2 \\ 0 & 0 & 6/7 & -3/7 & -6/7 \\ 0 & 0 & -3 & 5 & 1 \end{bmatrix}$$

Now, eliminate the x3 coefficient in the fourth row:
Multiply the third row by -7/6 and add it to the fourth row:

$$\begin{bmatrix} 2 & -1 & 0 & 0 & 6 \\ 0 & 7/2 & -1 & 0 & 7/2 \\ 0 & 0 & 6/7 & -3/7 & -6/7 \\ 0 & 0 & 0 & 20/7 & 19/7 \end{bmatrix}$$

Step 3: Back-substitution to find the solutions.
From the last row, we find:

$$\frac{20}{7}x_4 = \frac{19}{7}$$

Solving for x4:

$$x_4 = \frac{19}{20}$$

Now, we can back-substitute to find the values of x3, x2, and x1. Start from the third row:

$$\frac{6}{7}x_3 - \frac{3}{7}x_4 = -\frac{6}{7}\left(\frac{19}{20}\right) + \frac{3}{7}\left(\frac{19}{20}\right) = -\frac{1}{4}$$

Solving for x3:

$$x_3 = -\frac{1}{4} \cdot \frac{7}{6} = -\frac{7}{24}$$

Now, proceed to the second row:

$$\frac{7}{2}x_2 - x_3 = \frac{7}{2}\left(-\frac{7}{24}\right) + \frac{1}{4} = -\frac{13}{24}$$

Solving for x2:

$$x_2 = -\frac{13}{24} \cdot \frac{2}{7} = -\frac{13}{42}$$

Finally, solve for x1 using the first row:

$$2x_1 - x_2 = 6$$
$$2x_1 = 6 + x_2 = 6 - \frac{13}{42}$$

Solving for x1:

$$x_1 = \frac{6}{2} - \frac{13}{42} = \frac{3}{1} - \frac{13}{42} = \frac{131}{42}$$

The general solution for the system of equations (4), (5), (6), and (7) is:

$$x_1 = \frac{131}{42}, \quad x_2 = -\frac{13}{42}, \quad x_3 = -\frac{7}{24}, \quad x_4 = \frac{19}{20}$$

Question 2

If A and B are $n \times n$ matrices, under what conditions is the following relation true:

$$(A + B)(A - B) = A^2 - B^2$$

To understand when this relation holds, let's expand the left-hand side:

$$(A + B)(A - B) = A^2 - AB + BA - B^2$$

Now, for the relation $A^2 - B^2$ to be equal to $A^2 - AB + BA - B^2$, it must be true that $AB = BA$. This condition holds if and only if matrices A and B commute, i.e., $AB = BA$ for all $n \times n$ matrices A and B.

Example where the relation does not hold:

Consider the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

Here, $A^2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$ and $B^2 = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \times \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 59 & 70 \\ 83 & 98 \end{bmatrix}$.

However, $(A + B)(A - B) = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix} \times \begin{bmatrix} -4 & -4 \\ -4 & -4 \end{bmatrix} = \begin{bmatrix} -32 & -32 \\ -32 & -32 \end{bmatrix}$ which is not equal to $A^2 - B^2$.

Contents