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TRAVELING BETWEEN THE LAGRANGIAN POINTS AND THE EARTH

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Abstract—This paper is concerned with minimum energy trajectories to transfer a spacecraft between the five Lagrangian points and Earth. The planar circular restricted three-body problem in two dimensions is used as the model for the Earth–Moon system, and Lemaître regularization is used to avoid singularities.

1. INTRODUCTION

The well-known Lagrangian points that appear in the planar restricted three-body problem[1,2] are very important for astronautical applications[3]. They are five points of equilibrium in the equations of motion. The collinear points (L_1, L_2 and L_3) are always unstable and the triangular points (L_4 and L_5) are stable in the present case (Earth–Moon system). They are all very good points to locate a Space Station, since they require a small amount of fuel for station-keeping. The cislunar point L_1 is an important option as a node to explore the Moon[4], because it can be used as a "parking orbit" to store spacecrafts, fuel and supplies that are needed for the return journey to the Earth, but not in the Moon. The Lagrangian point L_2 is very useful to keep a relay satellite[3], since a satellite in an orbit around this point could see the far side of the Moon and Earth at the same time indefinitely. The triangular points L_4 and L_5 are interesting to keep a Space Station, due to its stability property.

In this paper, the planar restricted three-body problem is regularized (using Lemaître regularization) and combined with numerical integration and gradient methods to solve the two point boundary value problem. This combination is applied to the search of families of transfer orbits between the Lagrangian points and Earth, in the Earth–Moon system, with the minimum possible energy.

2. THE PLANAR CIRCULAR RESTRICTED THREE-BODY PROBLEM

The model used in all phases of this paper is the well-known planar circular restricted three-body problem. This model assumes that two main bodies (M_1 and M_2) are orbiting their common center of mass in circular Keplerian orbits and a third body (M_3) with negligible mass, is orbiting these two primaries. The motion of M_3 is supposed to stay in the plane of the motion of M_1 and M_2[1,2]. The canonical system of units is used, and it implies that: (i) the unit of distance is the distance between M_1 and M_2, (ii) the angular velocity (ω) of the motion of M_1 and M_2 is assumed to be one; (iii) the mass of the smaller primary (M_2) is given by $\mu = m_2/(m_1+m_2)$ (where m_1 and m_2 are the real masses of M_1 and M_2, respectively) and the mass of M_2 is $(1-\mu)$; (iv) the unit of time is defined such that the period of the motion of the primaries is 2π; (v) the gravitational constant is one. Then, the equations of motion in the rotating figure are:

$$\ddot{x} = 2\dot{y} = x - \frac{\partial V}{\partial x} = \frac{\partial \Omega}{\partial x}, \quad \ddot{y} + 2\dot{x} = y - \frac{\partial V}{\partial y} = \frac{\partial \Omega}{\partial y}, \quad (1)$$

where Ω is the pseudo-potential given by:

$$\Omega = \frac{1}{2}(x^2 + y^2) + \frac{(1-\mu)}{r_1} + \frac{\mu}{r_2}. \quad (2)$$

One of the most important reasons why the rotating system is a good choice to describe the motion of M_3 is the existence of an invariant that is called the Jacobi integral. There are many ways to define the Jacobi integral (see Ref.[1], p. 449). In this paper the definition used by Broucke[5] is used. Under this version, the Jacobi integral is given by:

$$E = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) - \Omega(x, y) = \text{Const.} \quad (3)$$

3. LEMAÎTRE REGULARIZATION

The equations of motion given by (1) are not suitable for numerical integration in trajectories passing near one of the primaries. The reason is that the positions of both primaries are singularities in the potential V (since r_1 or r_2 goes to zero, or near zero) and the accuracy of the numerical integration is affected every time this situation occurs.

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The solution for this problem is the use of regularization, that consists of a substitution of the variables for position (x–y) and time (t) by another set of variables (ω_1, ω_2, τ), such that the singularities are eliminated in these new variables. Several transformations with this goal are available in the literature (see Ref.[1]; chap. 3), like Thiele-Burrau, Lemaître and Birkhoff. For the present research the Lemaître regularization[6] is used.

4. THE TWO POINT BOUNDARY VALUE PROBLEM

In the rotating coordinate system all the primaries and the Lagrangian points are in fixed positions. So, the problem considered in this paper can be formulated as:

"Find an orbit (in the three-body problem context) that makes a spacecraft leave a given point A and go to another given point B".

That is the famous TPBVP (two point boundary value problem). There are many orbits that satisfy this requirement, and the way used in this paper to find families of solutions is to specify a time of flight for the transfer. Then, by varying the specified time of flight it is possible to find a whole family of transfer orbits and study them in terms of the ΔV required, energy, initial flight path angle, etc. To solve the TPBVP in the regularized variables the following steps are used: (i) guess an initial velocity $\vec{V}_i$, so, together with the initial prescribed position $\vec{r}_i$, the initial state is known; (ii) guess a final regularized time τ_f and integrate the regularized equations of motion from $\tau_0 = 0$ until τ_f; (iii) check the final position $\vec{r}_f$ obtained from the numerical integration with the prescribed final position and the final real time with the specified time of flight. If there is an agreement (difference less than a specified error allowed) the solution is found and the process can stop here. If there is no agreement, an increment in the initial guessed velocity $\vec{V}_i$ and in the guessed final regularized time is made and the process goes back to step (i). The method used to find the increment in the guessed variables is the standard gradient method, as described in Ref. [7]. The routines in this reference are also used in this research with minor modifications.

5. NUMERICAL RESULTS

The problem is solved for several values of the time of flight. Since the regularized system is used to solve this problem, the Earth's center can be used as the final position for M_3. This choice makes it possible to compare our results with the ones obtained by Broucke[5], which used the Moon's center as the final position in his trajectories. The results are organized in plots of the energy [Jacobi integral, as given by eqn (3)] against the initial flight path angle (in degrees) in the rotating frame. The definition of the angle is such that the zero is in the "x" axis (pointing to the positive direction) and it increases in the counter-clockwise sense. The time of flight for some points are shown in the plots, in days. Emphasis is given in finding the families with the smallest possible energy, although many other families do exist.

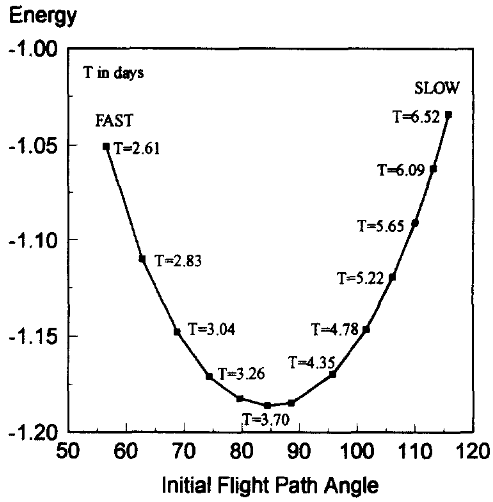

Fig. 1. Initial conditions for trajectories from L_1 to Earth.

Trajectories from L_1: L_1 is the collinear Lagrangian point that exists between the Earth and the Moon. Figure 1 shows the results. The family has a branch of fast transfers (2 to 3 days) and a branch of slow transfers (4 to 6 days). The velocity increment (ΔV) applied at L_1 for the minimum energy transfer is 924 m/s.

Trajectories from L_2: L_2 is the collinear Lagrangian point that exists behind the Moon. Figure 2 shows the results. This family also has a branch of fast transfers (4 to 6 days) and a branch of slow transfers (7 to 13 days). The velocity increment (ΔV) applied at L_2 for the minimum energy transfer is 1248 m/s.

Trajectories from L_3: L_3 is the collinear Lagrangian

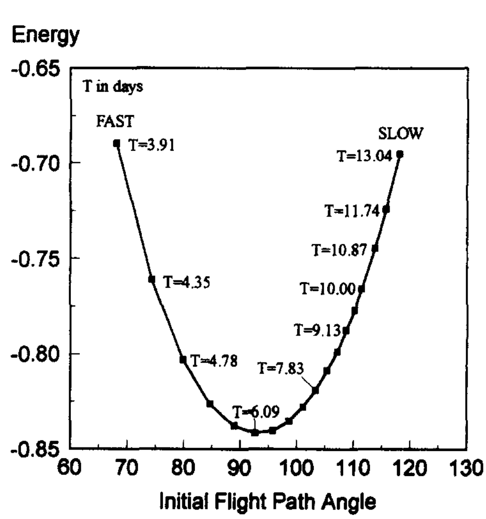

Fig. 2. Initial conditions for trajectories from L_2 to Earth.

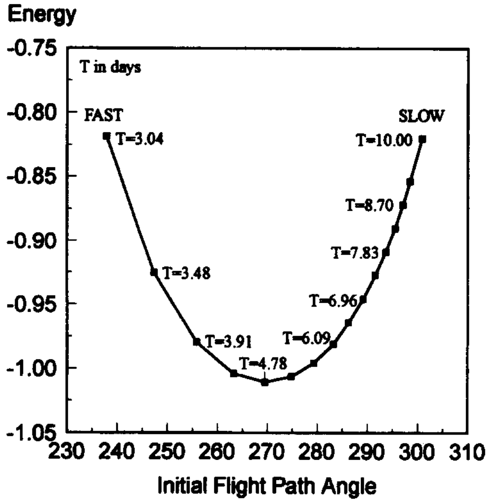

Fig. 3. Initial conditions for trajectories from L_3 to Earth.

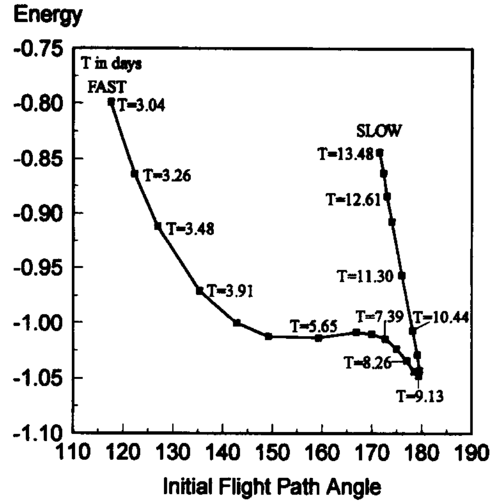

Fig. 5. Initial conditions for trajectories from L_5 to Earth.

point that exists on the opposite side of the Earth. Figure 3 shows the results. The existence of a branch of slow transfers (5 to 10 days) and a branch of fast transfers (3 to 4 days) is shown. The velocity increment (ΔV) applied at L_3 for the minimum energy transfer is 1018 m/s.

Trajectories from L_4: L_4 is one of the triangular Lagrangian points. Figure 4 shows the results. As occurred before, there are two branches: slow transfers (5 to 11 days) and fast transfers (3 to 4 days). The velocity increment (ΔV) applied at L_4 for the minimum energy transfer is 1018 m/s.

Trajectories from L_5: L_5 is the other triangular Lagrangian point. Figure 5 shows the results. The same branches appear: the fast one (3 to 5 days) and the slow one (9 to 13 days). The velocity increment (ΔV) applied at L_1 for the minimum energy transfer is 967 m/s.

One interesting application for the present research is related to missions to the Moon using L_1 as a node of transportation. Details of these missions are available in Ref. [4]. In this paper we only compare the results given by the two and the three-body problem models for this mission analysis. The calculation involved is to find how much more fuel (or ΔV) is required to go to L_1 and then leaving from there to the Moon, instead of going directly to the Moon. In the present paper, the ΔVs at L_1 are calculated to stop a spacecraft coming from the Earth. The ΔVs required are plotted against trip time, to find the minimum ΔV transfer to compare with the same calculations shown in Ref. [4]. The difference between the results obtained in this paper and the ones found in the above reference comes from the fact that the three-body regularized problem is used as the model in this paper and the two-body problem is used

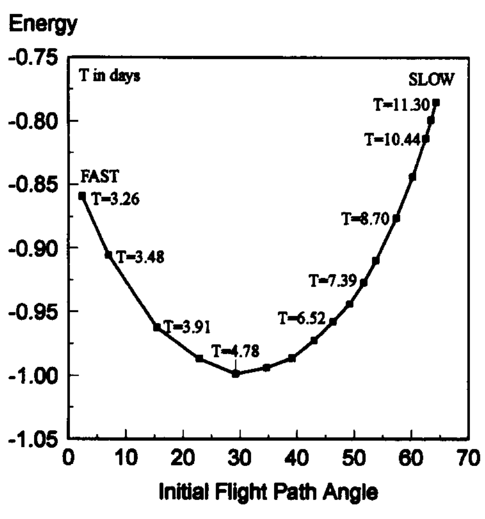

Fig. 4. Initial conditions for trajectories from L_4 to Earth.

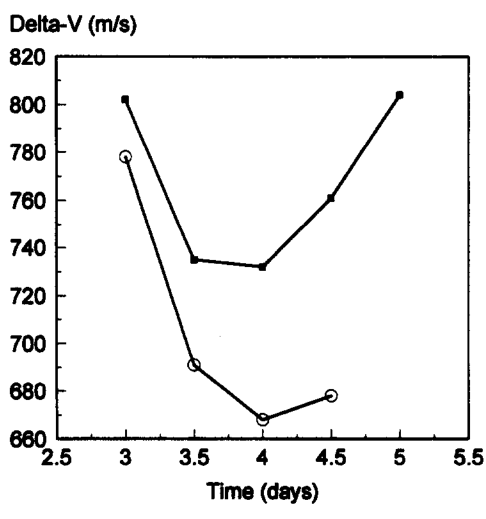

Fig. 6. Two- vs three-body models for a CLP mission.

in Ref.[4]. Figure 6 shows the results that came from both models. The importance of using the three-body problem model can be evaluated, since the conic approximation does not give accurate results in this case. The new difference (in terms of total ΔV required) between the CLP mission and a standard mission increased by 64 m/s each way, that is an increase of about 10% in this part of the mission.

6. CONCLUSIONS

In this paper, the Lemaître regularization is applied to the planar restricted three-body problem to find families of transfer orbits between Earth and all five Lagrangian points that exist in the Earth–Moon system.

Families of transfer orbits with small residual velocities at the Lagrangian points were not found, although they do exist in the case of transfers to the Moon[5]. In particular, it means that the transfer with the absolute minimum ΔV between Earth and the Moon passing by L_1 (as suggested in Ref.[8]) was not found. The minimum energy transfers found in this paper require about 1000 m/s.

Finally, we compared the results given by the two-body model with the results given by the three-body problem model in the particular task of calculating the ΔV for transfers between the important Lagrangian point L_1 and Earth. Differences of about 10% were found between the two models.

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