Phase_II

November 16, 2023

[1]:
```python
# Reinforcement learning dynamic programin lake of ice

import numpy as np
import matplotlib.pyplot as plt
import seaborn as sns
import random
import time
import pandas as pd
import math
from tqdm import tqdm
from collections import defaultdict
from IPython.display import clear_output
```

[2]:
```python
# FrozenLakeEnv
class FrozenLakeEnv():
    def __init__(self, map_name="5x4", is_slippery=False):
        self.desc = [
            "SFFF",
            "FHFH",
            "FFFH",
            "FHFF",
            "FFFG"
        ]
        self.value_map = np.zeros([5, 4])
        self.nS = 20
        self.nA = 4


    def step(self, index, action):
        if action == "left":
            if index[1] == 0:
                return index
            else:
                return [index[0], index[1]-1]
        elif action == "right":
            if index[1] == 3:
                return index
            else:
```

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                return [index[0], index[1]+1]
        elif action == "up":
            if index[0] == 0:
                return index
            else:
                return [index[0]-1, index[1]]
        elif action == "down":
            if index[0] == 4:
                return index
            else:
                return [index[0]+1, index[1]]
        else:
            return index
    def reward(self, index):
        if self.desc[index[0]][index[1]] == "H":
            return -1
        elif self.desc[index[0]][index[1]] == "G":
            return 1
        else:
            return 0
```

[3]:

```
# plot policy and value function
def plot_policy(policy):
    arr = np.zeros([5, 4])
    for i in range(5):
        for j in range(4):
            if policy[(i, j)] == "left":
                arr[i, j] = 0
            elif policy[(i, j)] == "right":
                arr[i, j] = 1
            elif policy[(i, j)] == "up":
                arr[i, j] = 2
            elif policy[(i, j)] == "down":
                arr[i, j] = 3
    plt.figure(figsize=(5,4))
    sns.heatmap(arr, cmap="YlGnBu", annot=True, cbar=False, square=True)
    # plot arrow in cells
    for i in range(5):
        for j in range(4):
            if policy[(i, j)] == "left":
                plt.annotate("←", (j+0.3, i+0.5), size=20)
            elif policy[(i, j)] == "right":
                plt.annotate("→", (j+0.7, i+0.5), size=20)
            elif policy[(i, j)] == "up":
                plt.annotate("↑", (j+0.5, i+0.3), size=20)
            elif policy[(i, j)] == "down":
                plt.annotate("↓", (j+0.5, i+0.7), size=20)
```

```
    # title
    plt.title("Policy")
    plt.show()
def plot_value_function(V):
    arr = np.zeros([5, 4])
    for i in range(5):
        for j in range(4):
            arr[i, j] = V[(i, j)]
    plt.figure(figsize=(4,4))
    sns.heatmap(arr, cmap="YlGnBu", annot=True, cbar=False, square=True)
    # title
    plt.title("Value function")
    plt.show()
```

0.1 Policy Iteration Algorithm

The Policy Iteration algorithm can be summarized as follows:

$$\begin{aligned}
\text{Initialize:} \quad & \text{Choose an initial policy } \pi_0. \\
\text{Iterate until convergence:} \quad & \\
& \text{Policy Evaluation:} \\
& V^{\pi_k}(s) = \sum_{a \in \mathcal{A}} \pi_k(a|s) \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a V^{\pi_k}(s') \right) \quad \text{for all } s \in \mathcal{S}. \\
& \text{Policy Improvement:} \\
& \pi_{k+1}(a|s) = \begin{cases} 1 & \text{if } a = \arg\max_{a'} \left(\mathcal{R}_s^{a'} + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^{a'} V^{\pi_k}(s') \right) \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

[4]:
```
## make random policy 4x4
def make_random_policy():
    policy = {}
    for i in range(5):
        for j in range(4):
            policy[(i, j)] = random.choice(["left", "right", "up", "down"])
    return policy
policy = make_random_policy()

# policy evaluation
def policy_evaluation(policy, env, discount_factor=1.0, theta=0.01):
    V = defaultdict(float)
    while True:
        delta = 0
        for i in range(5):
            for j in range(4):
                action = policy[(i, j)]
```

```python
                next_state = env.step([i, j], action)
                reward = env.reward(next_state)
                v = (reward + discount_factor * V[tuple(next_state)])
                if env.desc[i][j] == "G":
                    v = 0
                delta = max(delta, np.abs(v - V[(i, j)]))
                V[(i, j)] = v
        plot_value_function(V)
        if delta < theta:
            break
    return V

# policy iteration
def policy_iteration(env, policy_eval_fn=policy_evaluation, discount_factor=0.9):
    policy = make_random_policy()
    plot_policy(policy)
    while True:
        plot_policy(policy)
        V = policy_eval_fn(policy, env, discount_factor)
        policy_stable = True
        for i in range(5):
            for j in range(4):
                old_action = policy[(i, j)]
                new_action = None
                best_value = float('-inf')
                for action in ["left", "right", "up", "down"]:
                    next_state = env.step([i, j], action)
                    reward = env.reward(next_state)
                    v = reward + discount_factor * V[tuple(next_state)]
                    if v > best_value:
                        best_value = v
                        new_action = action
                policy[(i, j)] = new_action
                if new_action != old_action:
                    policy_stable = False
        if policy_stable:
            return policy, V
```

0.2 Value Iteration Algorithm

Initialize: Choose an initial value function V_0.

Iterate until convergence:

Update the value function:

$$V_{k+1}(s) = \max_{a \in \mathcal{A}} \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a V_k(s') \right) \quad \text{for all } s \in \mathcal{S}.$$

Extract the policy:

$$\pi(s) = \arg\max_{a \in \mathcal{A}} \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a V_{\text{final}}(s') \right)$$

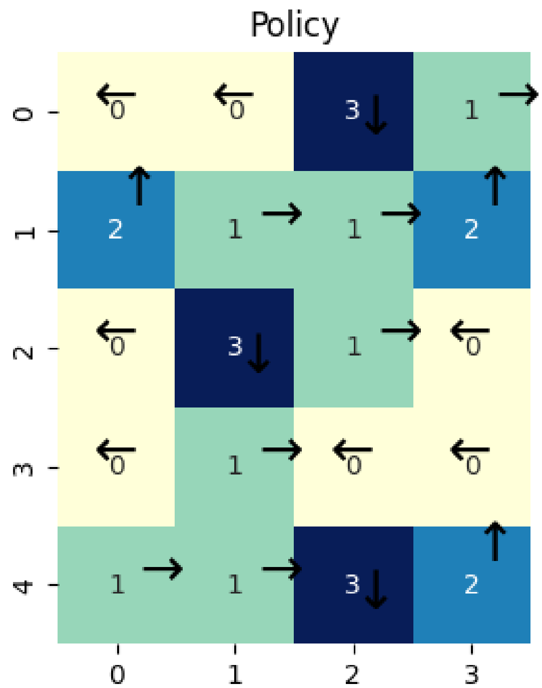
[5]:

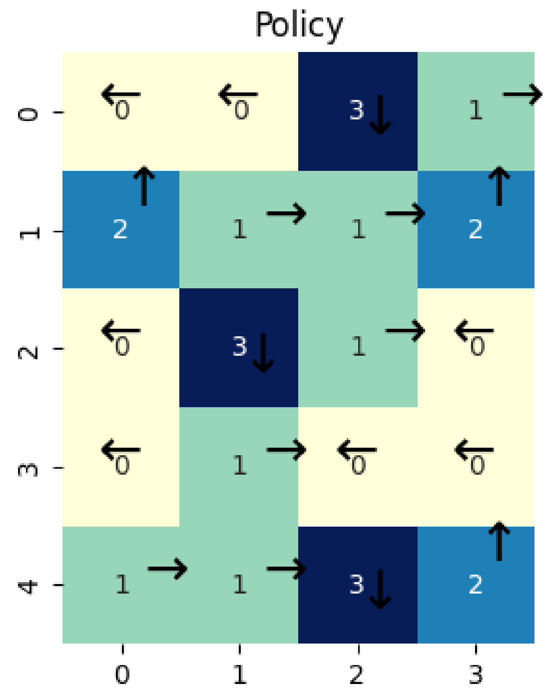
```python
# value iteration
def value_iteration(env, theta=0.01, discount_factor=0.9):
    V = defaultdict(float)
    while True:
        delta = 0
        for i in range(5):
            for j in range(4):
                best_value = float('-inf')
                for action in ["left", "right", "up", "down"]:
                    next_state = env.step([i, j], action)
                    reward = env.reward(next_state)
                    v = reward + discount_factor * V[tuple(next_state)]
                    if env.desc[i][j] == "G":
                        v = 0
                    if v > best_value:
                        best_value = v
                delta = max(delta, np.abs(best_value - V[(i, j)]))
                V[(i, j)] = best_value
        plot_value_function(V)
        if delta < theta:
            break
    policy = {}
    for i in range(5):
        for j in range(4):
            best_value = float('-inf')
            best_action = None
            for action in ["left", "right", "up", "down"]:
                next_state = env.step([i, j], action)
                reward = env.reward(next_state)
                v = reward + discount_factor * V[tuple(next_state)]
                if v > best_value:
                    best_value = v
                    best_action = action
            policy[(i, j)] = best_action
```

```
    return policy, V
```

[6]:

```
# main
env = FrozenLakeEnv()
policy, v = policy_iteration(env)
plot_policy(policy)
```

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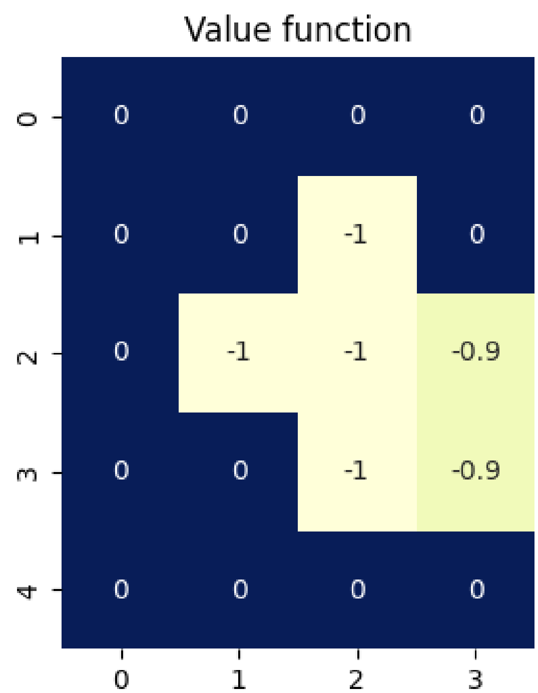
3
4
0 1 2 3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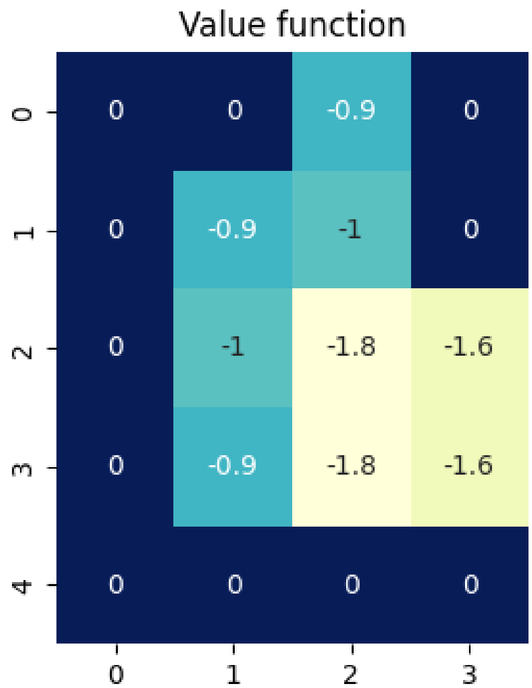
0 0 -1 -0.9
0 0 0 0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-0.9
-1
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0
-1
-1.8
-1.6
0
-0.9
-1.8
-1.6
0
0
0
0
0
1
2
3
4
0
1
2
3


Value function
0
0
-0.9
0
0
-0.9
-1
0
0
-1.8
-2.5
-2.2
0
-1.6
-2.5
-2.2
0
0
0
0
0
1
2
3
4
0
1
2
3


Value function
0
0
-0.9
0
0
-0.9
-1
0
0
-2.5
-3
-2.7
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-3
-2.7
0
0
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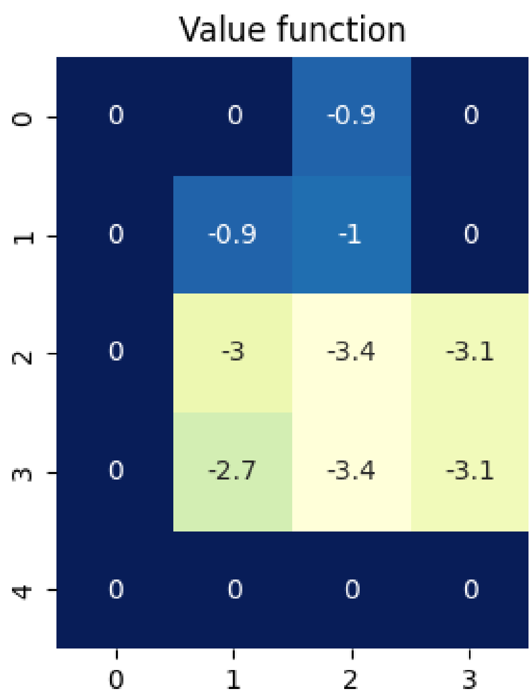
0
0
1
2
3
4
0
1
2
3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-3.4
-3.1
0
0
0
0
0
1
2
3
4
0
1
2
3


Value function
0
0
-0.9
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-0.9
-1
0
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-3.4
-3.8
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Value function
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-4.1
-3.7
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Value function
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-0.9
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0
-0.9
-1
0
0
-4.1
-4.3
-3.9
0
-3.7
-4.3
-3.9
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0
1
2
3
4
0
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2
3


Value function
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-0.9
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-0.9
-1
0
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-4.3
-4.5
-4
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-3.9
-4.5
-4
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0
1
2
3
4
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1
2
3


Value function
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-0.9
-1
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-4.5
-4.6
-4.2
0
-4
-4.6
-4.2
0
0
0
0
0 1 2 3
0 1 2 3 4


Value function
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0
-0.9
0
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-0.9
-1
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-4.6
-4.7
-4.3
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-4.2
-4.7
-4.3
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0 1 2 3
0 1 2 3 4


Value function
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0 1 2 3 4


Value function
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-4.4
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-4.4
-4.9
-4.4
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0 1 2 3
0 1 2 3 4


Value function
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-1
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-4.9
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0 1 2 3
0 1 2 3 4


Value function
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-0.9
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-4.5
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-4.5
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-4.5
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0 1 2 3
0 1 2 3 4


Value function
0 0 -0.9 0
0 -0.9 -1 0
0 -5 -5.1 -4.6
0 -4.5 -5.1 -4.6
0 0 0 0


Value function
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Value function
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-5.1
-4.6
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0
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Value function
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-0.9
-1
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-5.1
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-4.7
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-4.6
-5.2
-4.7
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1
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3
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0
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Value function
0
1
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4
0 0 -0.9 0
0 -0.9 -1 0
0 -5.2 -5.2 -4.7
0 -4.7 -5.2 -4.7
0 0 0 0
0 1 2 3


Value function
0
1
2
3
4
0 0 -0.9 0
0 -0.9 -1 0
0 -5.2 -5.2 -4.7
0 -4.7 -5.2 -4.7
0 0 0 0
0 1 2 3


Value function
0 0 -0.9 0
0 -0.9 -1 0
0 -5.2 -5.2 -4.7
0 -4.7 -5.2 -4.7
0 0 0 0


Value function
0 0 -0.9 0
0 -0.9 -1 0
0 -5.2 -5.2 -4.7
0 -4.7 -5.2 -4.7
0 0 0 0


Value function
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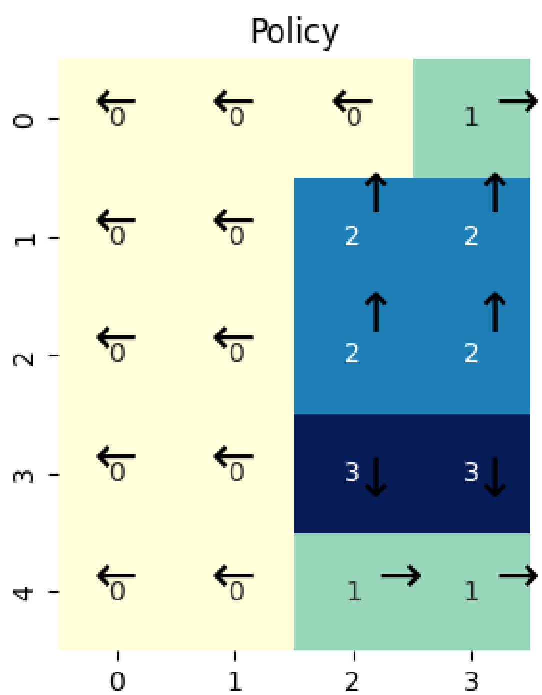
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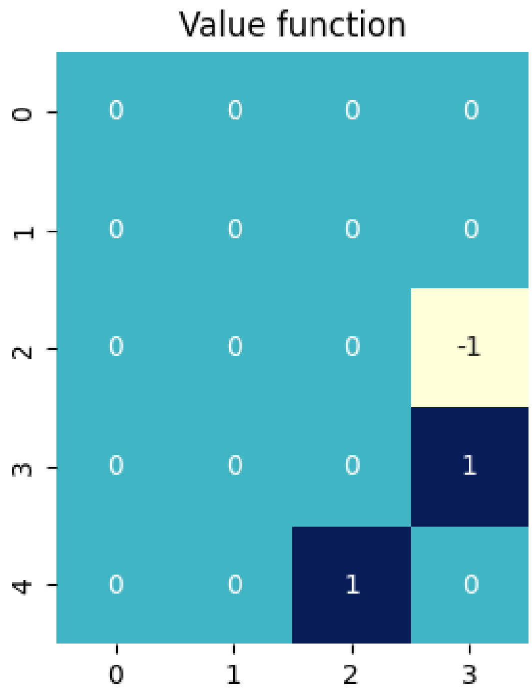
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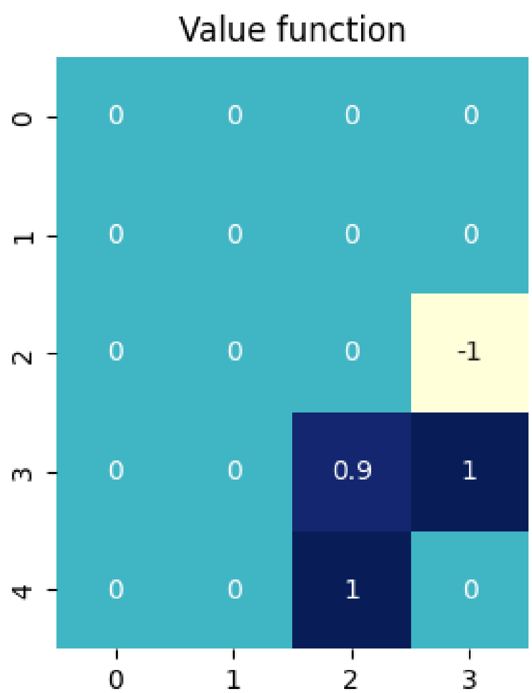
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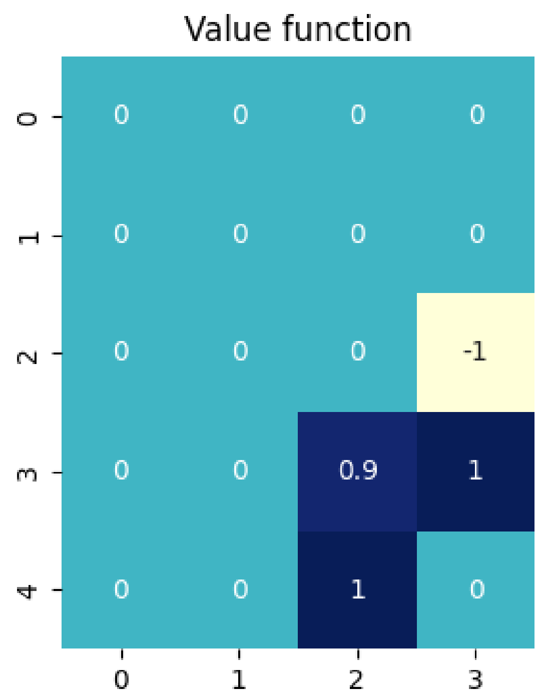
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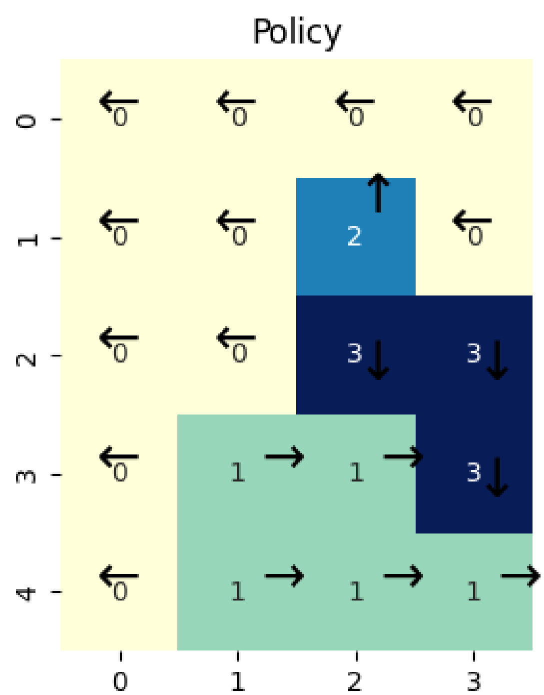
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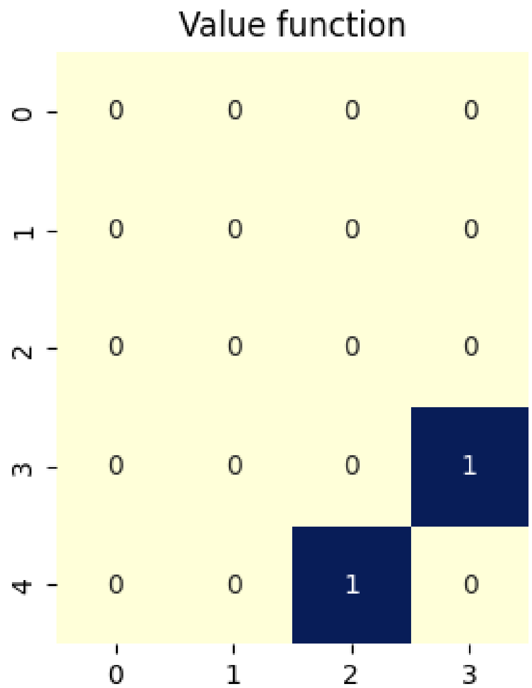
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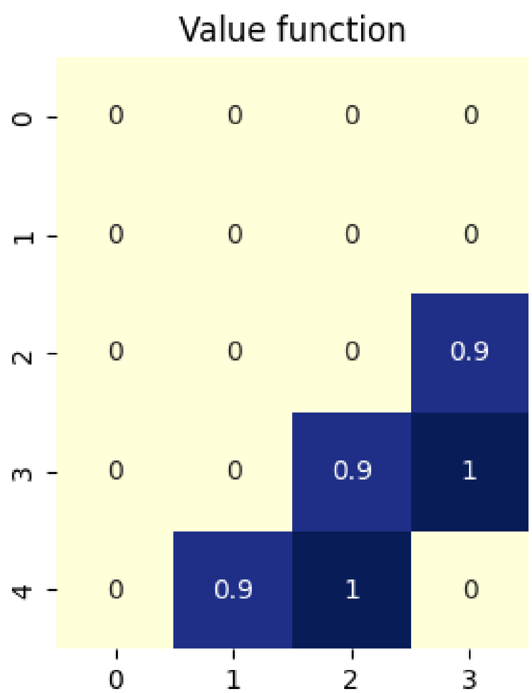
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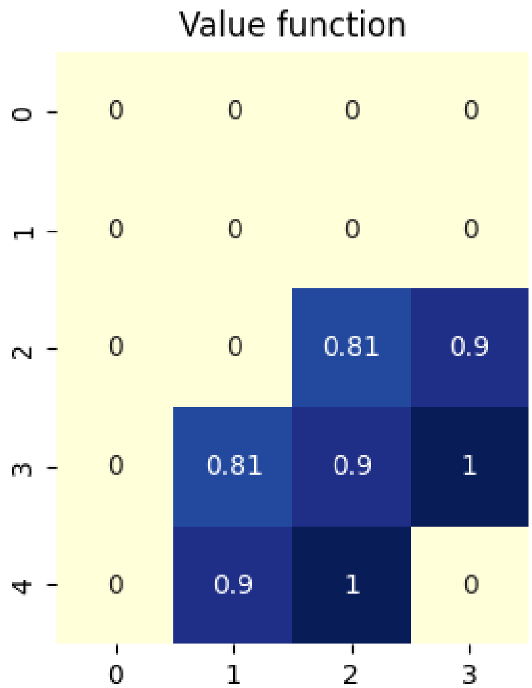
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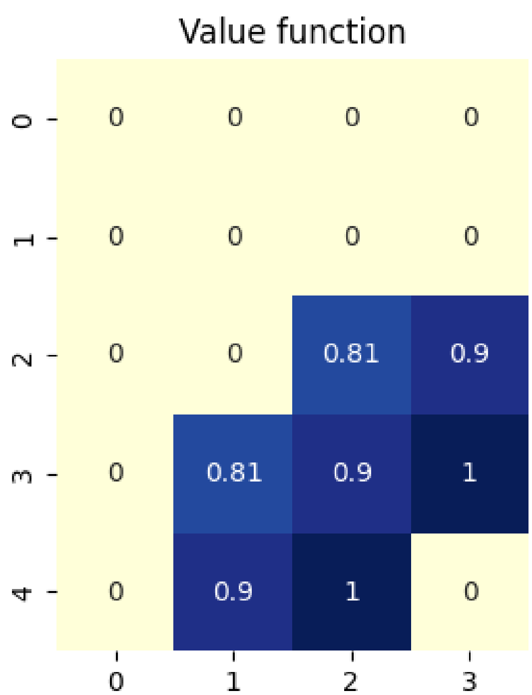
Value function
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2
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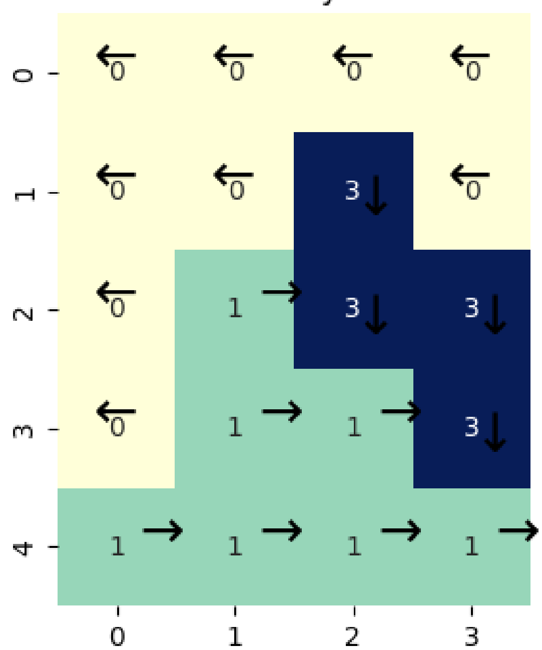
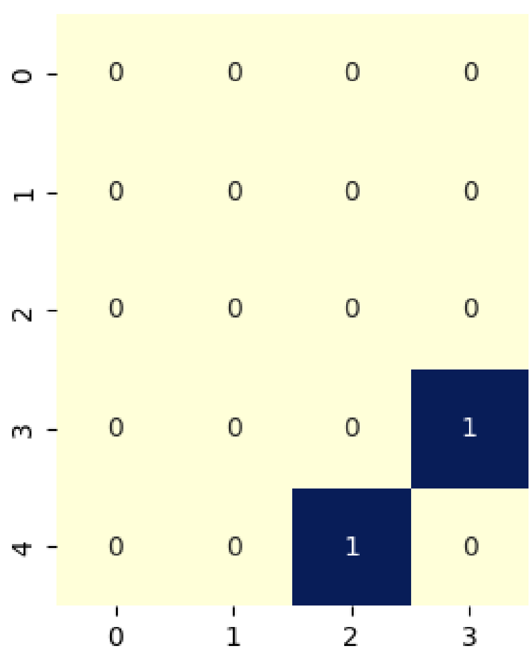
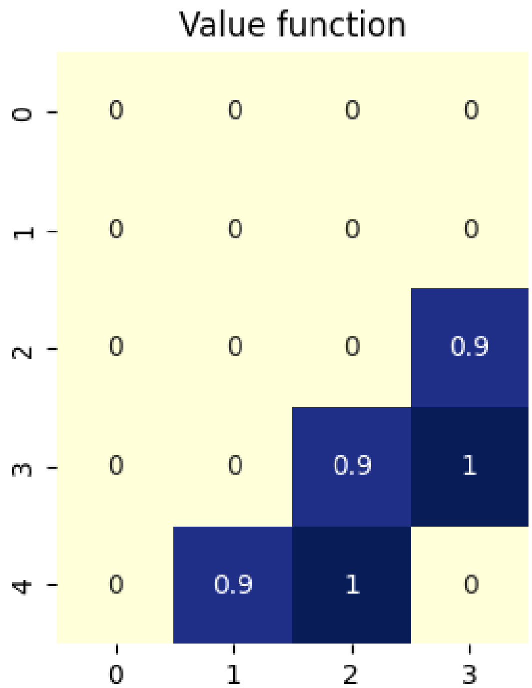
0 0.9 1 0

Policy



Value function

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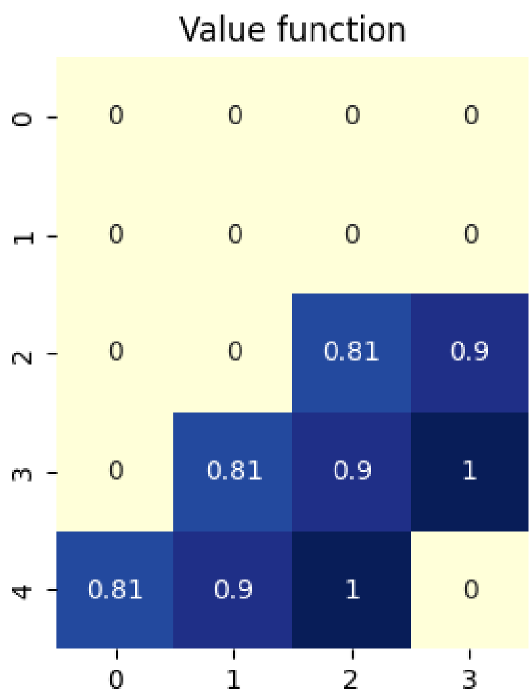
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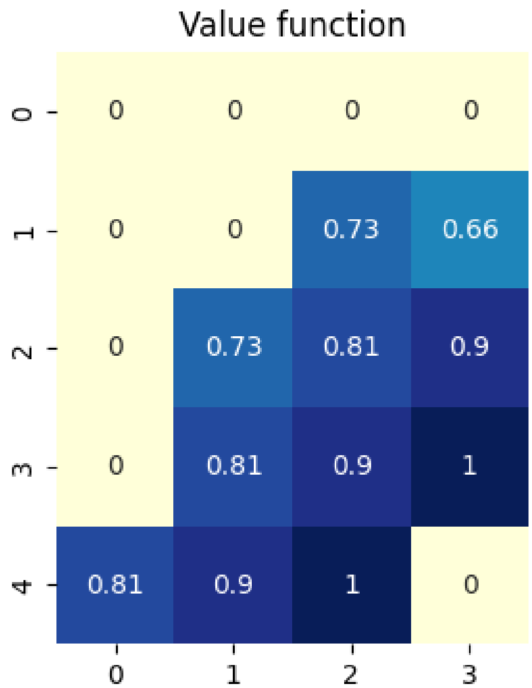
Value function
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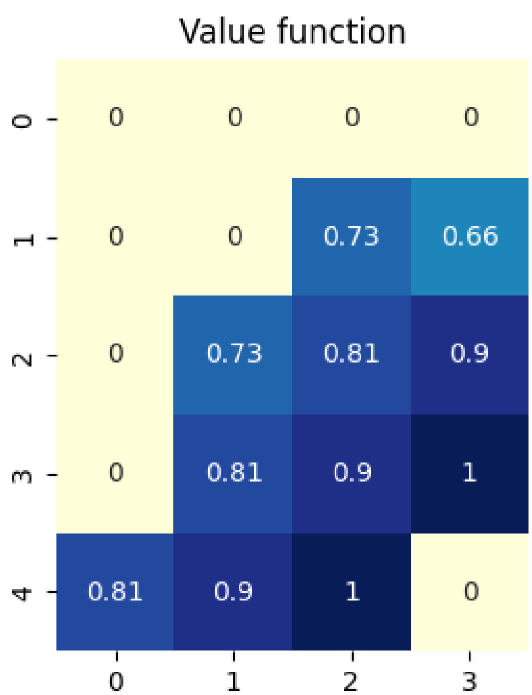
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0.81 0.9 1 0
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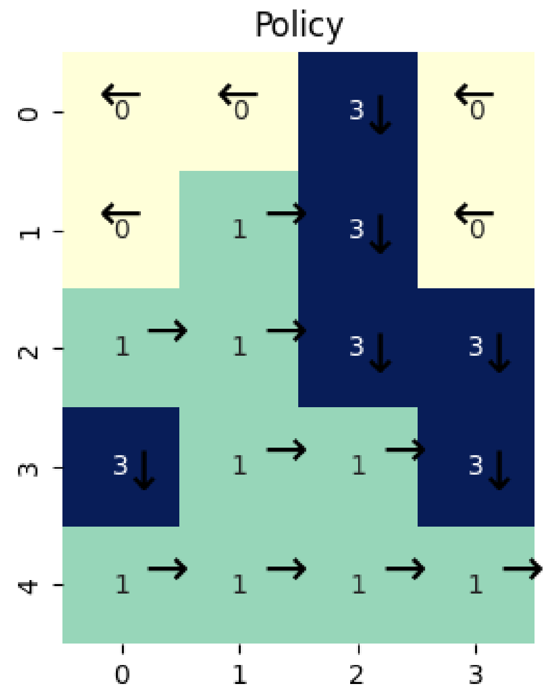
0 1 2 3 4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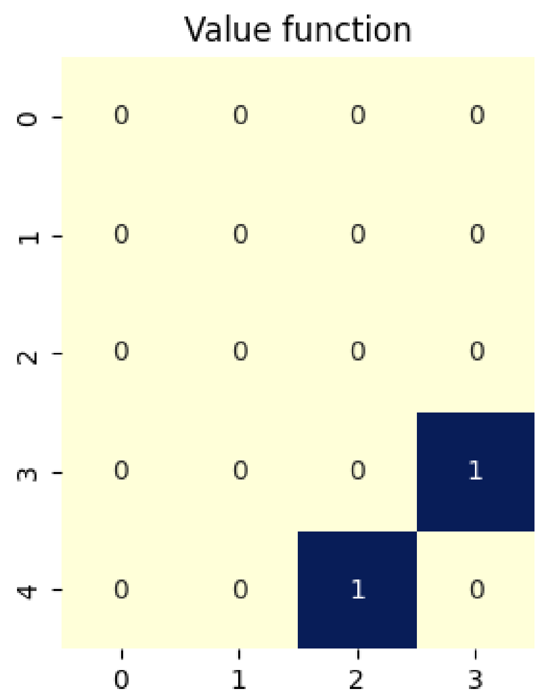
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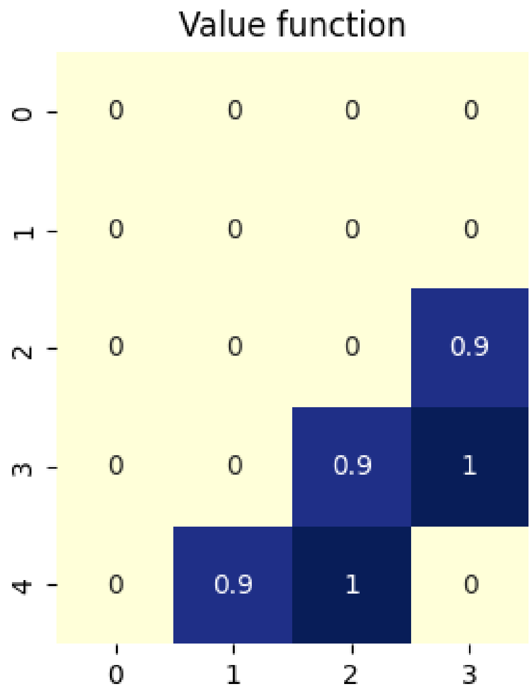
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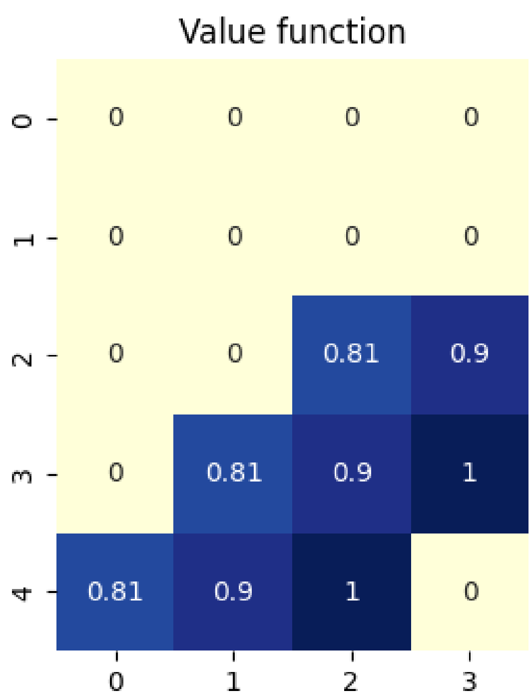
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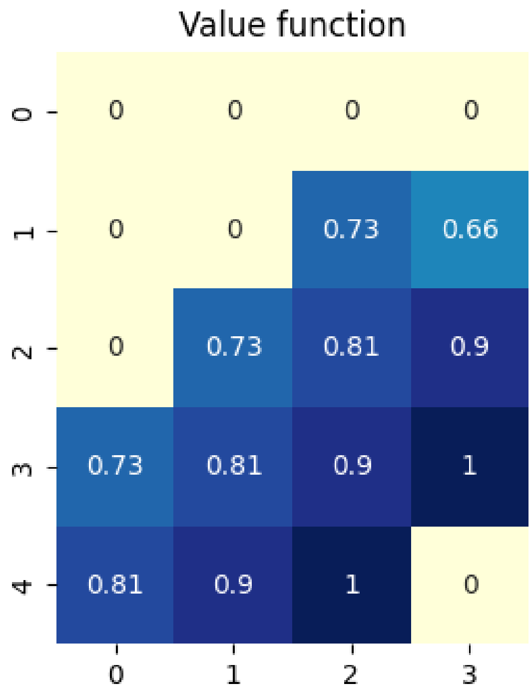
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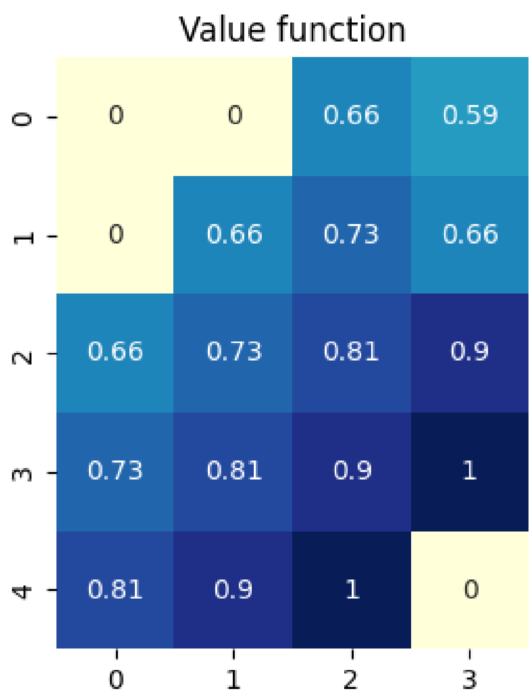
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0 1 2 3
0 1 2 3 4


Value function
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0
0.66
0.59
0
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0.73
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0.81
0.9
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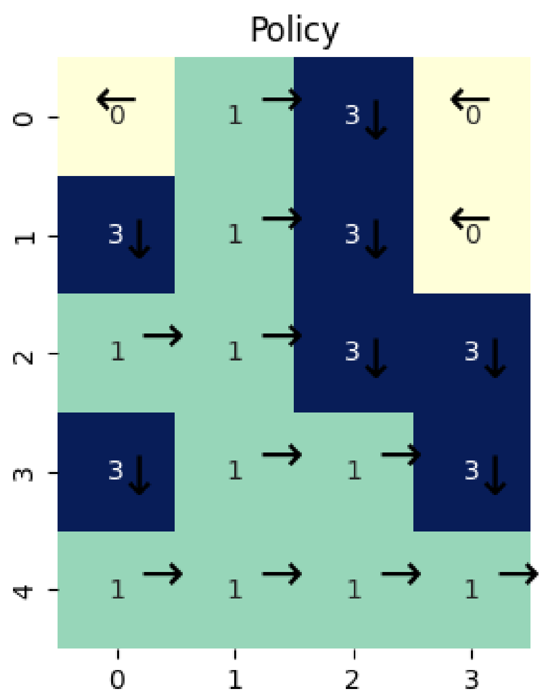
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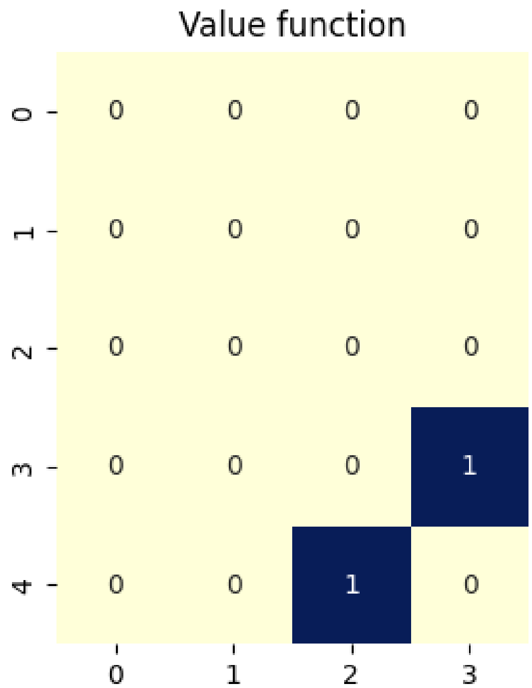
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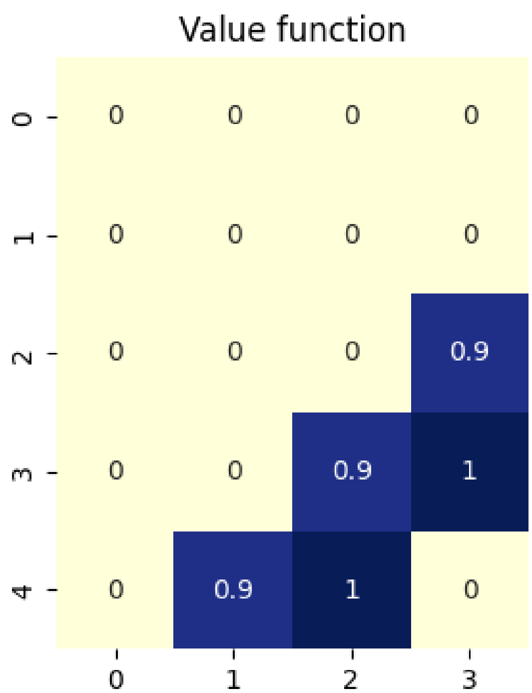
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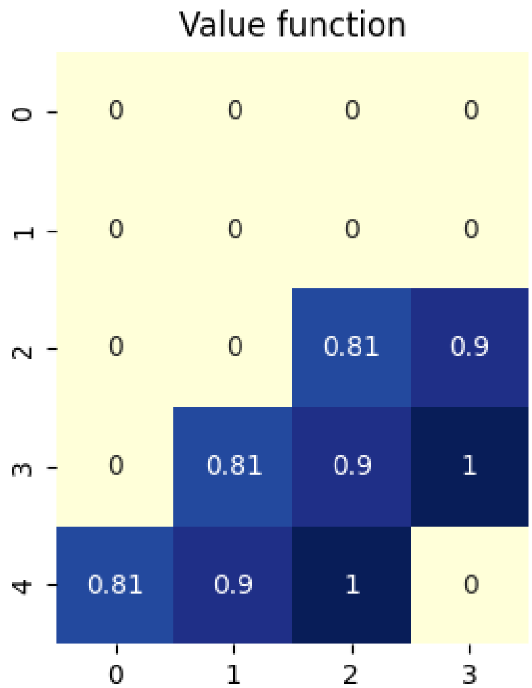
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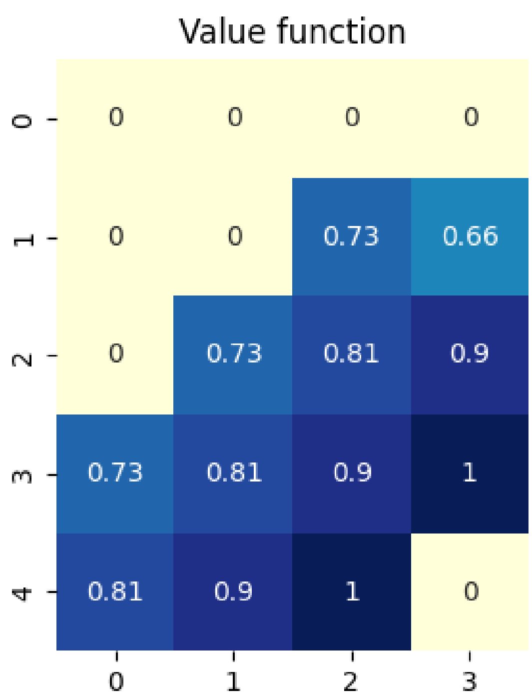
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0.81 0.9 1 0
0 1 2 3
0 1 2 3 4


Value function
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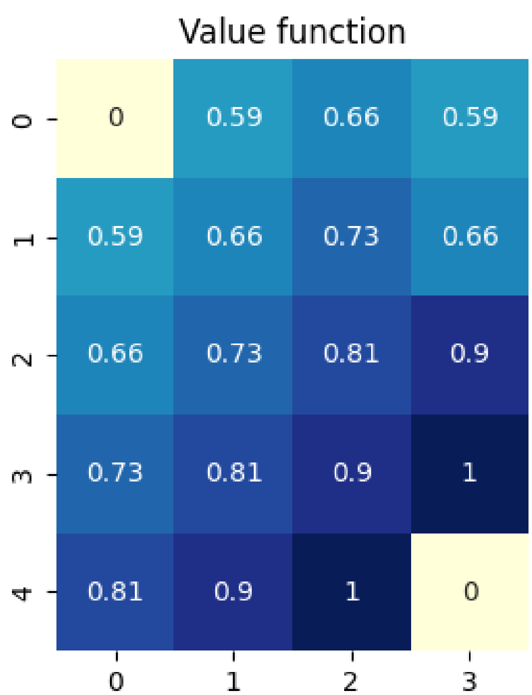
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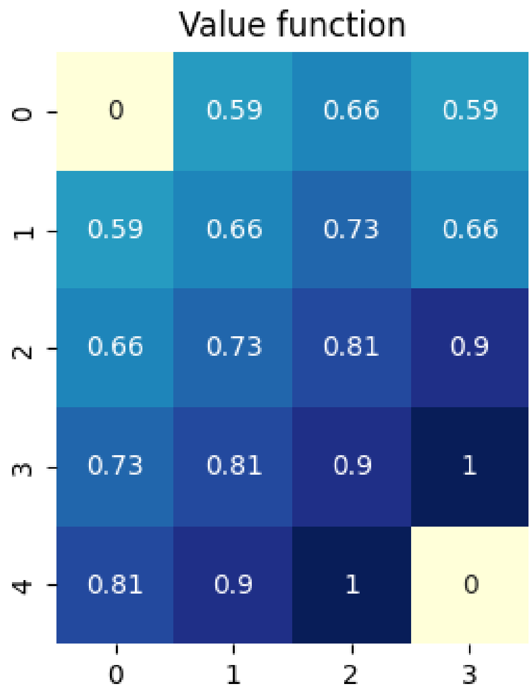
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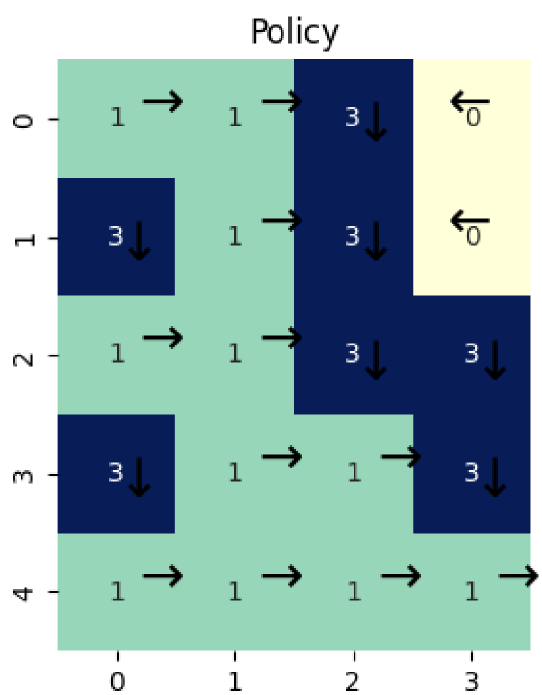
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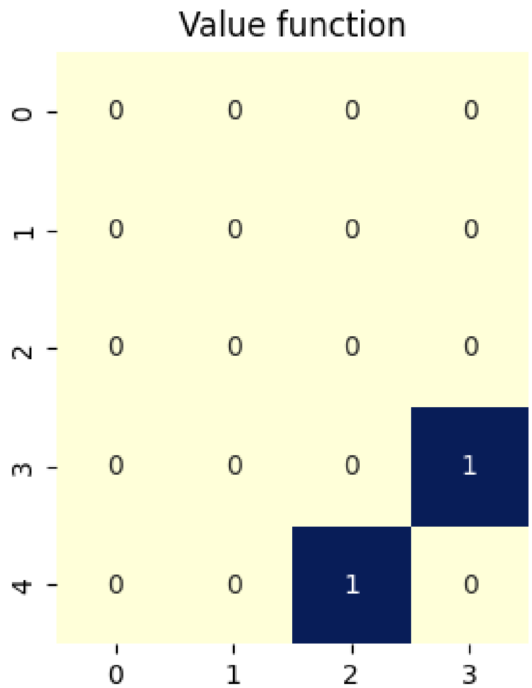
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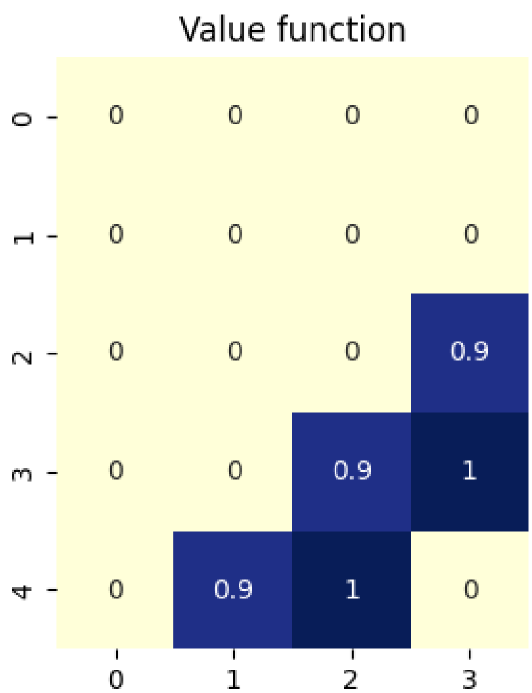
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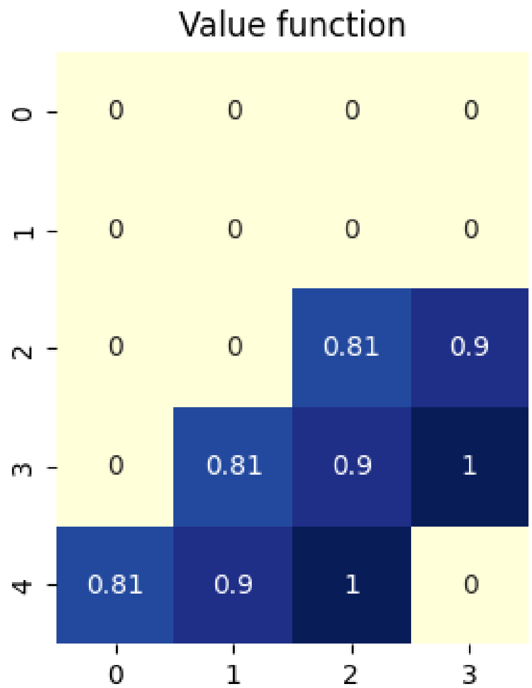
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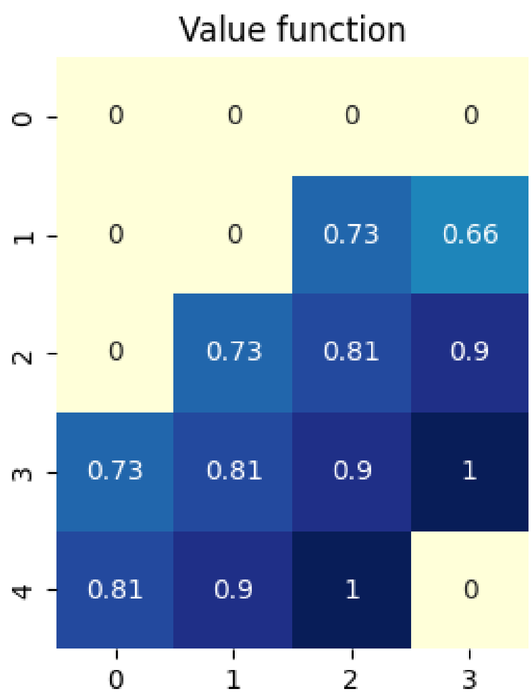
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Value function
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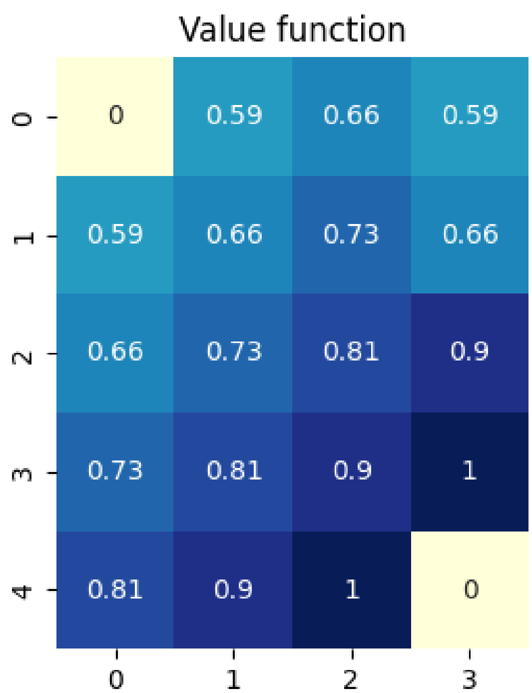
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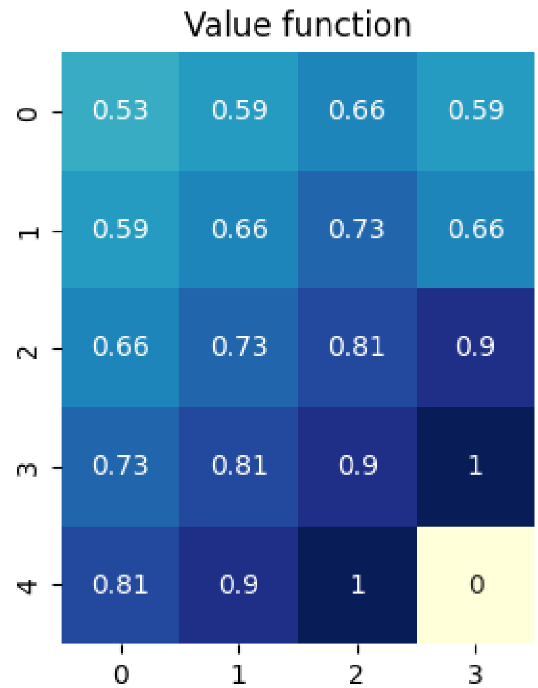
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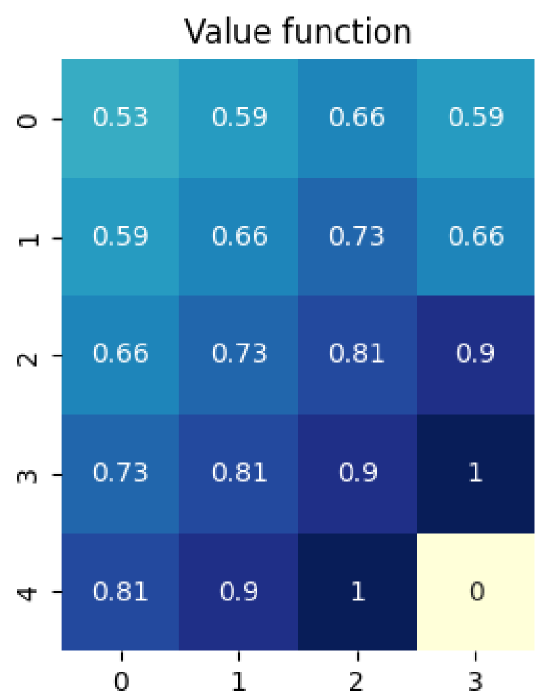
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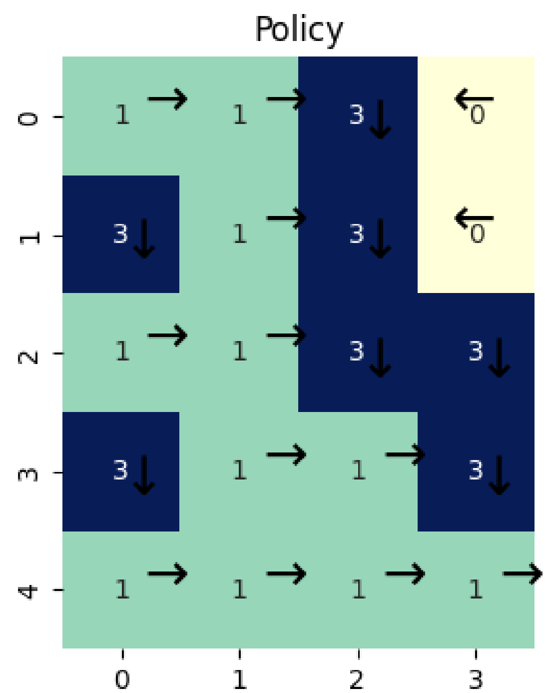

[7]:
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policy, v = value_iteration(env)
plot_policy(policy)
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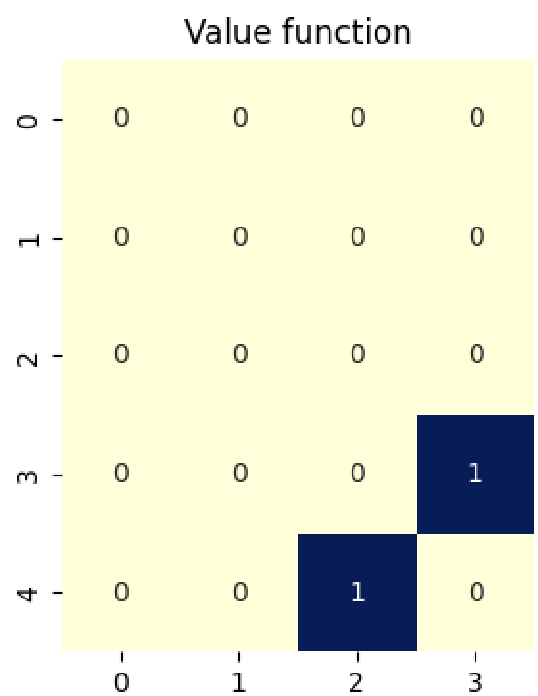

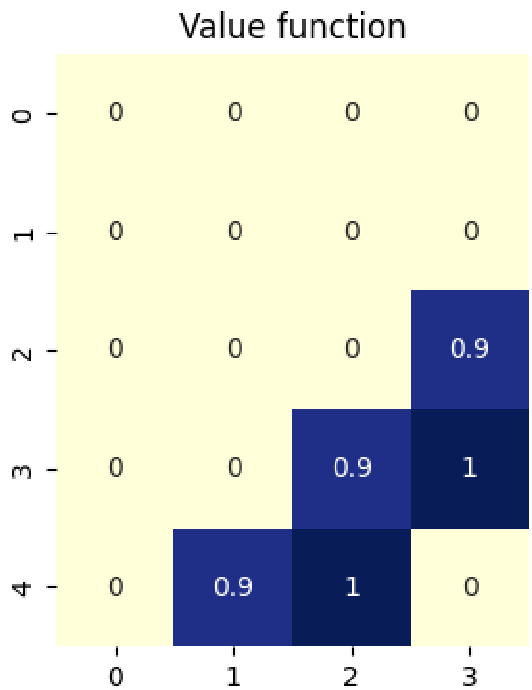
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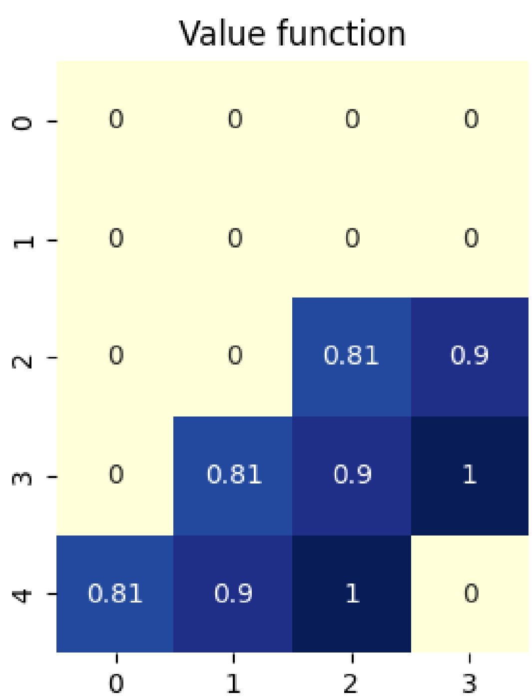
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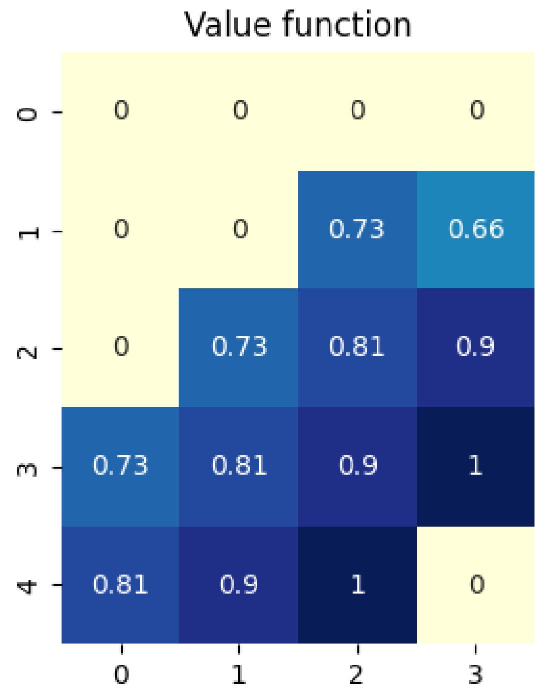
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Value function
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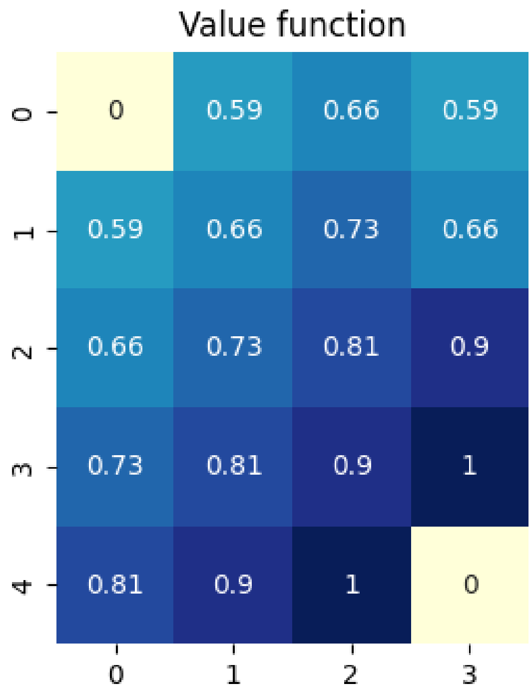
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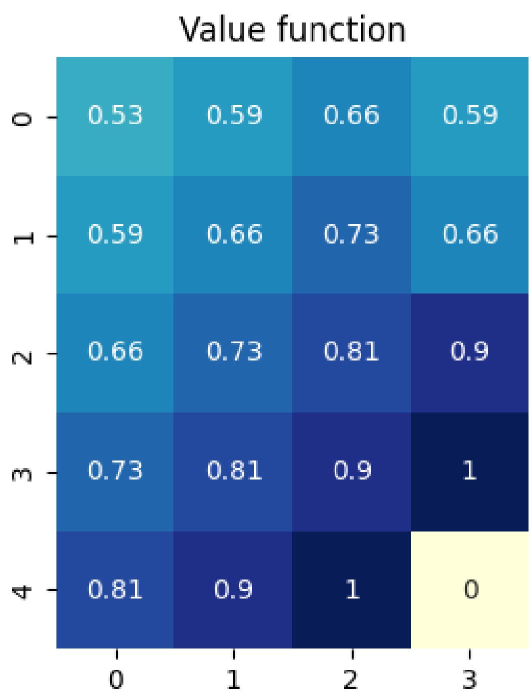
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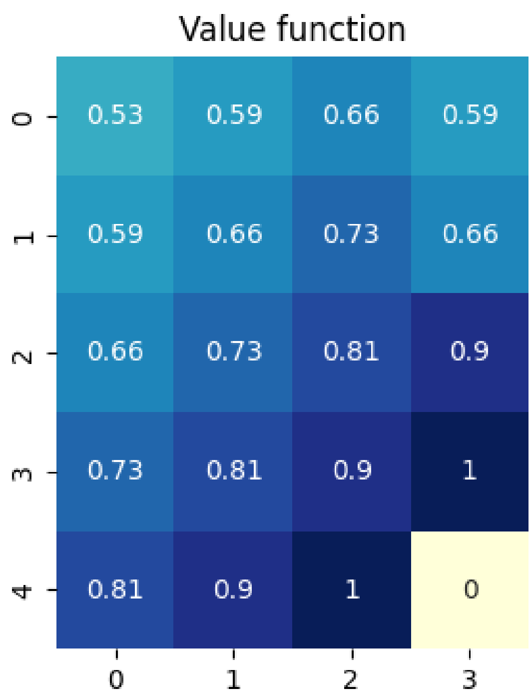
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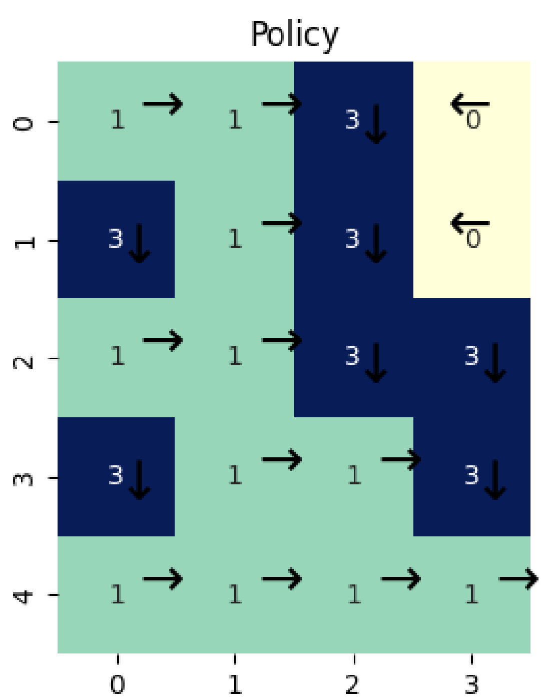
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