Stability Theory

Lecture Notes

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Based on handwritten lecture notes

Disclaimer. These are my personal lecture notes that I've transcribed from the handwritten notes of Dr. Tingxiang Zou that can be found on her website. For accuracy I have not changed or edited the content in there, there is another version of these notes where I write more explanations (that I did personally for my own understanding) and provide more examples or sometimes prove something that was left to the reader explicitly with details. You can find that version on my personal website as well
Any questions, recommandations or if you find any mistake feel free to reach me at: `berhanpalabiyik@gmail.com`.
Have fun!

Contents

Introduction

Q: When does a theory T characterise models up to isomorphism?

Theorem 0.1 *(Löwenheim–Skolem)*.
Let T be a first-order theory in a language $\mathcal{L}$ and with infinite models. Then for any $\kappa \geq \max\{|\mathcal{L}|, \aleph_0\}$, T has a model of size κ.

A first-order theory T can never characterise infinite models.

Q: Fix a cardinality $\kappa \geq \max\{|\mathcal{L}|, \aleph_0\} = |\mathcal{L}| + \aleph_0$. When does T have a unique model (up to isomorphism) of cardinality κ?

Theorem 0.2 *(Morley's Theorem)*.
Let T be a countable theory. If T has a unique model of cardinality κ for some $\kappa > \aleph_0$, then T has a unique model of cardinality λ for any uncountable λ.

These theories are called **uncountably categorical** theories.

Example 0.3.
ACF_p, ACF_0 (algebraically closed fields of a fixed characteristic) are uncountably categorical.

Let K and F be algebraically closed fields of the same characteristic. Let k be their prime field. Then $K \cong F$ iff $\mathrm{trd}(K/k) = \mathrm{trd}(F/k)$.

If $\mathrm{trd}(K/k) = \mathrm{trd}(F/k)$ and $|K| = |F| > \aleph_0$, then $\mathrm{trd}(K/k) = \mathrm{trd}(F/k) = |K|$, so $K \cong F$.

If $|K| = |F| = \aleph_0$, then $\mathrm{trd}(K/k) \in \{1, 2, 3, \dots, \aleph_0\}$, so there are countably many non-isomorphic algebraically closed fields of cardinality $\aleph_0$ of a fixed characteristic.

Definition 0.4 *(ω-categorical)*.
A theory T is called **ω-categorical** if T has a unique countable model up to isomorphism.

Non-example: ACF.

Example 0.5.
DLO (dense linear order without end points) is ω-categorical. But DLO is not uncountably categorical: $(\mathbb{R}, <) \ncong (\mathbb{R} \cup \mathbb{Q}, <)$ where $\mathbb{R} \prec \mathbb{Q}$.

Spectrum of a theory

Definition 0.6 *(Spectrum)*.
Let T be a complete theory with infinite models. Let κ be a cardinal. Write $I(T, \kappa)$ for the number of models of T (up to isomorphism) of cardinality κ.

Q: What can $I(T, \kappa)$ be?

Observation: $1 \leq I(T, \kappa) \leq 2^\kappa$ for $\kappa \geq |\mathcal{L}| + \aleph_0$.

Q: When is $I(T,\kappa) < 2^\kappa$?

Shelah's classification project (stable, simple, NIP, NSOP, ...):

Theorem 0.7 *(Shelah)*.
If T is not stable, then T has 2^κ-many models for all $\kappa > |T| + \aleph_0$.

Unstable theories have as many models as possible (some stable theories as well).

Back to spectrum: for countable theories, the function $I(T,\kappa)$ has been almost completely solved by Hart–Hrushovski–Laskowski (2000).

Vaught's conjecture: Let T be a countable theory. Then $I(T,\aleph_0)$ is either $\leq \aleph_0$ or $= 2^{\aleph_0}$.

Remark 0.8.
Assuming CH (continuum hypothesis), the Vaught conjecture is trivially true.

Theorem 0.9 *(Shelah, 1983)*.
The Vaught conjecture is true for ω-stable theories.

Definition 0.10 *(Stability)*.
Let T be a complete theory with infinite models. Let $\mathcal{M} \models T$.

(1) A formula $\varphi(x,y)$ with $|x| = n$, $|y| = m$, is said to have the **k-order property** for some $k \in \omega$, if there are $a_i \in M^n$, $b_j \in M^m$ for $i < k$ such that $\mathcal{M} \models \varphi(a_i, b_j) \Leftrightarrow i < j$.

(2) $\varphi(x,y)$ has the **order property** if it has the k-order property for all $k \in \omega$.

(3) We say a formula $\varphi(x,y)$ is **stable** if $\varphi(x,y)$ does not have the k-order property, i.e. there exists $k \in \omega$ such that $\varphi(x,y)$ does not have the k-order property.

(4) The theory T is called **stable** if all formulas are stable.

Remark 0.11.
The definition does not depend on the model $\mathcal{M}$, since if $\mathcal{M} \equiv N$, then $\varphi(x,y)$ has the k-order property in $\mathcal{M}$ iff it has the k-order property in N. (As k is finite.)

Examples of stable theories: ACF, uncountably categorical theories.

Non-examples: DLO, RCF (real closed fields) — the formula $x < y$ witnesses the k-order property for all k.

Overview of the course

(1) **Preliminaries.** Type spaces, acl, dcl, elimination of imaginaries, saturated models, ...

(2) **General stability.**

2.1. Counting types in stable theories.

2.2. Local ranks, definability of types.

2.3. Forking calculus, ranks in stable theories — notion of independence: $a \mathop{\not\perp}_A b$.

2.4. Canonical base.

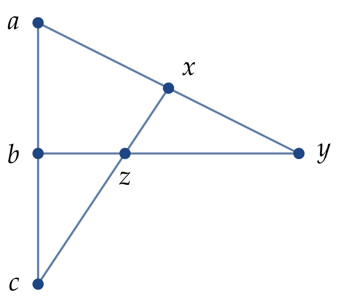
Also: other tame theories — simple, NIP, ...

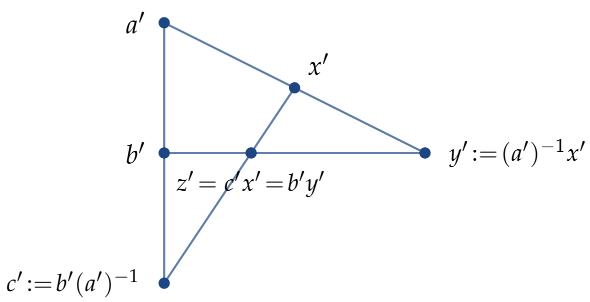
(3) **Stable groups / fields.** Chain condition, generic types, indecomposability theorem.

Application: Infinite superstable fields are algebraically closed (Cherlin–Shelah '79).

(4) **Geometric stability.** Group configuration theorem. Zilber's trichotomy.

Group configuration theorem: If there are tuples (a, b, c, x, y, z) in a stable theory T satisfying: any non-collinear triple are independent, and in any collinear triple (e.g. a, b, c) any two are interalgebraic over the third one (e.g. $a \in \mathrm{acl}(b, c)$, $b \in \mathrm{acl}(a, c)$, $c \in \mathrm{acl}(a, b)$), plus some condition on the canonicity of interalgebraicity, then there is a type-definable group G in T such that the above configuration basically comes from: take $(a', b', x') \in G^3$ generic (so a', b', x' independent), with $c' := b' \cdot (a')^{-1}$, $y' := (a')^{-1} \cdot x'$.





Zilber's trichotomy: Let D be strongly minimal. The pregeometry from acl is either:

- trivial: $\mathrm{acl}(A) = \bigcup_{a \in A} \mathrm{acl}(a)$,
- "vector space like" — locally modular,
- or "field like" — interpret an algebraically closed field.

Remark 0.12.
The above trichotomy is not true but is true in a lot of settings.

1 Preliminaries

1.1 Notation

A **structure** $\mathcal{M} = (M, R_1, R_2, \ldots, f_1, f_2, \ldots, c_1, c_2, \ldots)$ consists of an underlying set M, relations R_i, functions f_j, and constants c_k.

Example 1.1.
Group: $(G, \cdot, ()^{-1}, 1)$. Ring/field: $(R, +, \cdot, 0, 1)$.

Remark 1.2.
We often talk about fields in $\mathcal{L}_{\text{ring}}$.

A **definable set** $\varphi(x)$ with $|x| = n$: we write $X = \varphi(M^n)$ for $\mathcal{M} = (M, \ldots)$.

Interpretation

Definition 1.3 *((Bi-)interpretability)***.**
Let $\mathcal{M}$ be an $\mathcal{L}$-structure and N be an $\mathcal{L}'$-structure.

- An **interpretation** of $\mathcal{M}$ in N is given by a surjective map $f \colon N^n \to M$ such that for any definable set $X \subseteq M^k$, the preimage $f^{-1}(X) \subseteq N^{n \cdot k}$ is definable in N.
- Two structures $\mathcal{M}$ and N are **bi-interpretable** if there exists an interpretation of $\mathcal{M}$ in N and an interpretation of N in $\mathcal{M}$ such that the composite interpretations of $\mathcal{M}$ in itself and of N in itself are **definable** in $\mathcal{M}$ and N respectively.

Example 1.4.
$(\mathbb{C}, +, \cdot)$ is interpretable in $(\mathbb{R}, +, \cdot)$ but not the other way around.

1.2 Saturation, monster models, definable and algebraic closure

Let $\mathcal{M}$ be a model of some complete theory T.

Definition 1.5 *(Partial and complete types)***.**
Let $A \subseteq M$. By a **partial type** $\Phi(x)$ over A we mean a collection of formulas $\varphi(x)$ with parameters from A, such that every finite subcollection is consistent, i.e. has some element $a \in M^{|x|}$ satisfying this finite subcollection: $\mathcal{M} \models \varphi(a)$ for $\varphi(x) \in \Delta$ where Δ is the finite subcollection.

By a **complete type** $p(x)$ over A we mean a partial type such that for all $\varphi(x) \in \mathcal{L}(A)$, either $\varphi(x) \in p$ or $\neg\varphi(x) \in p$.

For $b \in M^{|x|}$, denote $\operatorname{tp}(b/A) := \{\varphi(x) \in \mathcal{L}(A) : \mathcal{M} \models \varphi(b)\}$, the complete type of b over A.

Definition 1.6 *(κ-saturated, κ-homogeneous)***.**
Let κ be an infinite cardinal.

(1) We say that $\mathcal{M}$ is **κ-saturated** if for any set of parameters $A \subseteq M$ with $|A| < \kappa$, every partial type $\Phi(x)$ over A with $|x| < \kappa$ can be realised in $\mathcal{M}$.

(2) We say $\mathcal{M}$ is **κ-homogeneous** if any partial elementary map from M to itself with a domain of size $< \kappa$ can be extended to an automorphism of $\mathcal{M}$.

Fact 1.7.
For any T and κ, there is a κ-saturated and κ-homogeneous model $\mathcal{M}$ of T. (Exercise.)

We say $\mathcal{M}$ is **saturated** if it is $|M|$-saturated.

Remark 1.8.
Under set-theoretic assumptions, e.g. GCH, a saturated model always exists.

Monster Model (Universal Domain)

Given a complete theory T, we fix some large cardinal κ and work in a κ-saturated, κ-homogeneous model $\mathfrak{U} = (U, \dots)$ and we call this model a **monster model**.

And all models we consider will be of size $\leq \kappa$, hence are all submodels of $\mathfrak{U}$. Same for parameters: we only consider sets of parameters of "small" size, and small means size $< \kappa$.

Fact 1.9 *(Compactness applies to small sets of formulas)***.**
Let A be a small set. Suppose $(X_i)_{i\in I}$, $(Y_j)_{j\in J}$ are A-definable subsets of M^n and $\bigcap_{i\in I} X_i \subseteq \bigcup_{j\in J} Y_j$. Then there exist $I_0 \subseteq I$, $J_0 \subseteq J$ finite subsets, such that $\bigcap_{i\in I_0} X_i \subseteq \bigcup_{j\in J_0} Y_j$.

Let $\Phi(x)$ be a small set of $\mathcal{L}(\mathfrak{U})$-formulas. If $\Phi(x) \models \varphi(x)$, then $\Phi_0(x) \models \varphi(x)$ for some finite $\Phi_0 \subseteq \Phi$.

Lemma 1.10.
Let X be a definable subset of $\mathfrak{U}^n$. Then X is A-definable iff $\sigma(X) = X$ (as a set) for all $\sigma \in \mathrm{Aut}(\mathfrak{U}/A)$ (the set of automorphisms of $\mathfrak{U}$ fixing A point-wise).

Proof. ($\Rightarrow$) Easy. $X = \varphi(\mathfrak{U}^n, b)$ for some $b \in A$. Then $a \in X \Leftrightarrow \mathfrak{U} \models \varphi(a,b) \Leftrightarrow \mathfrak{U} \models \varphi(\sigma(a), \sigma(b)) \Leftrightarrow \mathfrak{U} \models \varphi(\sigma(a), b) \Leftrightarrow \sigma(a) \in X$.

($\Leftarrow$) Assume $X = \varphi(\mathfrak{U}^n, b)$, $b \in U^m$. Let $p(y) := \mathrm{tp}(b/A)$.

Claim 1: $p(y) \models \forall x\,(\varphi(x,y) \leftrightarrow \varphi(x,b))$.

Suppose $b' \models p(y)$. Then there exists $\sigma \in \mathrm{Aut}(\mathfrak{U}/A)$ with $\sigma(b) = b'$. So $X = \varphi(\mathfrak{U}^n, b) \implies \sigma(X) = \varphi(\mathfrak{U}^n, \sigma(b)) = \varphi(\mathfrak{U}^n, b')$. But $\sigma(X) = X$ by assumption. $\triangle$ Claim 1

By Fact 1.9, there exists $\psi(y) \in p(y)$ such that $\psi(y) \models \forall x\,(\varphi(x,y) \leftrightarrow \varphi(x,b))$.

Let $\theta(x) := \exists y\,(\psi(y) \wedge \varphi(x,y))$. Then θ is over A.

Claim 2: $X = \theta(\mathfrak{U}^n)$.

If $a \in X$, then $\varphi(a,b)$ holds. Hence we can take $b' := b$ which satisfies $\psi(b')$ (since $\mathfrak{U} \models \psi(b)$ as $\psi \in p = \mathrm{tp}(b/A)$) and $\varphi(a,b')$, so $\mathfrak{U} \models \theta(a)$.

Conversely, if $\mathfrak{U} \models \psi(b') \wedge \varphi(a,b')$, then by Claim 1, $\mathfrak{U} \models \varphi(a,b)$ and $a \in X$. $\triangle$ Claim 2 $\square$

Lemma 1.11.
Let $X \subseteq \mathfrak{U}^n$ be definable. TFAE:

(1) X is **almost A-definable**, i.e. there is an A-definable equivalence relation E on $\mathfrak{U}^n$ with finitely many classes, such that X is a union of E-classes.

(2) The set $\{\sigma(X) : \sigma \in \mathrm{Aut}(\mathfrak{U}/A)\}$ is finite.

(3) The set $\{\sigma(X) : \sigma \in \mathrm{Aut}(\mathfrak{U}/A)\}$ is small.

Proof. (1) $\Rightarrow$ (2): Easy. $\sigma \in \mathrm{Aut}(\mathfrak{U}/A)$ permutes the E-classes, and there are only finitely many of them.

(2) $\Rightarrow$ (3): Obvious.

(3) $\Rightarrow$ (1): Let $X := \varphi(\mathfrak{U}, b)$. Let $p(y) := \mathrm{tp}(b/A)$. By assumption, there exist $(b_i)_{i\in I}$ with I small, such that for all $\sigma \in \mathrm{Aut}(\mathfrak{U}/A)$, $\sigma(X) = \varphi(\mathfrak{U}, b_i)$ for some i. Thus $p(y) \models \bigvee_{i\in I} \forall x\,(\varphi(x,y) \leftrightarrow \varphi(x,b_i))$.

Since I is small, by compactness, there exists $\psi(y) \in p$ and $I_0 \subseteq I$ such that $\psi(y) \models \bigvee_{i\in I_0} \forall x\,(\varphi(x,y) \leftrightarrow \varphi(x,b_i))$.

Define $E(x_1,x_2)$ as:

$$\forall y\,(\psi(y) \to (\varphi(x_1,y) \leftrightarrow \varphi(x_2,y))).$$

Now $E(a_1,a_2)$ iff a_1 and a_2 agree on $\varphi(x,b_i)$ for all $i \in I_0$. The number of E-classes is finite, and X is a finite union of E-classes (since $X = \varphi(\mathfrak{U}, b)$ and $\psi(b)$ holds, hence there exists b_i with $i \in I_0$ such that $\varphi(\mathfrak{U}, b_i) = X$). $\square$

Definition 1.12 *(Definable closure and algebraic closure)***.**
Let A be a set of parameters and b a tuple.

(1) We say b is **definable over** A if there is $\varphi(x) \in \mathcal{L}(A)$ such that $\{b\} = \varphi(\mathfrak{U})$, i.e. $\mathrm{dcl}(A) = \{b \in \mathfrak{U} : b \text{ is definable over } A\}$.

(2) We say b is **algebraic over** A if there is $\varphi(x) \in \mathcal{L}(A)$ such that $b \in \varphi(\mathfrak{U})$ and $|\varphi(\mathfrak{U})| < \infty$, i.e. $\mathrm{acl}(A) = \{b \in \mathfrak{U} : b \text{ is algebraic over } A\}$.

Remark 1.13.
$\mathrm{dcl}(A) \subseteq \mathrm{acl}(A)$ and they are $\mathrm{Aut}(\mathfrak{U}/A)$-invariant.

Corollary 1.14.

(1) $b \in \operatorname{dcl}(A)$ iff $\sigma(b) = b$ for all $\sigma \in \operatorname{Aut}(\mathfrak{U}/A)$.

(2) $b \in \operatorname{acl}(A)$ iff the $\operatorname{Aut}(\mathfrak{U}/A)$-orbit of b is finite, iff the $\operatorname{Aut}(\mathfrak{U}/A)$-orbit of b is small.

1.3 $\mathcal{M}^{\text{eq}}$ and strong types

Definition 1.15 *(Imaginary element)***.**
Let T be a theory and $\mathfrak{U}$ a monster model. An **imaginary element** in $\mathfrak{U}$ is an equivalence class d/E, where E is a $\emptyset$-definable equivalence relation on a definable set $D \subseteq U^n$ and $d \in D$.

Definition 1.16 *(Elimination of imaginaries)***.**
We say T has **elimination of imaginaries** (EI) if for any $\emptyset$-definable equivalence relation E on $\mathfrak{U}^n$ and an E-class X, there is some tuple a in $\mathfrak{U}$ such that for some $\mathcal{L}$-formula $\varphi(x, y)$:

(1) X is defined by $\varphi(x, a)$, and

(2) if $\varphi(x, a')$ also defines X, then $a' = a$.

Remark 1.17.
We think of a as the "code" for the class X, and hence can treat the equivalence class X (the imaginary element) as the element tuple a which lives in $\mathfrak{U}$ (the real element).

Some theories have EI, e.g. ACF, but some do not. There is a canonical way of enlarging the language and theory so that it has EI.

Construction of $\mathcal{L}^{\text{eq}}$ and $\mathcal{M}^{\text{eq}}$

Let $\mathcal{M}$ be an $\mathcal{L}$-structure and let $T = \operatorname{Th}(\mathcal{M})$. We will construct a multi-sorted language $\mathcal{L}^{\text{eq}}$ and expand $\mathcal{M}$ to a multi-sorted structure $\mathcal{M}^{\text{eq}}$ and get $T^{\text{eq}} = \operatorname{Th}_{\text{eq}}(\mathcal{M}^{\text{eq}})$.

Construction: Let $\operatorname{ER}(T)$ be the collection of all $\mathcal{L}$-formulas $E(x, y)$ over $\emptyset$, such that $E(x, y)$ defines an equivalence relation on $M^{|x|}$ for some n.

Let $\mathcal{L}^{\text{eq}} := \mathcal{L} \cup \{S_E : E \in \operatorname{ER}(T)\} \cup \{f_E : E \in \operatorname{ER}(T)\}$, where $\mathcal{L} = (S, (R_i)_i, (f_j)_j, (c_\ell)_\ell)$ has original sort S, relations, functions, and constants on sort S. Each S_E is a new sort, and $f_E \colon S^{|x|} \to S_E$ where $E = E(x, y) \in \operatorname{ER}(T)$.

Note: $|\mathcal{L}| = |\mathcal{L}^{\text{eq}}|$.

$\mathcal{M}^{\text{eq}}$: the original structure $\mathcal{M}$ is preserved in sort S. For S_E, it is given by $\{a/E : a \in M^{|x|}\}$. For f_E, it is interpreted as $a \mapsto a/E$ (the quotient function).

$$T^{\text{eq}} := T \cup \{\forall y \in S_E\, \exists x \in S^{|x|}\, (f_E(x) = y) : E \in \operatorname{ER}(T)\} \cup \{\forall x_1, x_2 \in S^{|x|}\, (f_E(x_1) = f_E(x_2) \leftrightarrow E(x_1, x_2)) : E$$

Clearly $\mathcal{M}^{\mathrm{eq}} \models T^{\mathrm{eq}}$.

Lemma 1.18.
(1) Every $\mathcal{M}^* \models T^{\mathrm{eq}}$ is of the form $\mathcal{M}^{\mathrm{eq}}$ for some $\mathcal{M} \models T$.

(2) Given $E_1, \ldots, E_k \in \mathrm{ER}(T)$ and $\varphi(x_1, \ldots, x_k) \in \mathcal{L}^{\mathrm{eq}}$, there exists $\psi(y_1, \ldots, y_k) \in \mathcal{L}$ such that

$$T^{\mathrm{eq}} \models \forall y_1 \cdots y_k \in S\,(\psi(y_1, \ldots, y_k) \leftrightarrow \varphi(f_{E_1}(y_1), \ldots, f_{E_k}(y_k))).$$

(3) T^{eq} is complete.

(4) $\mathcal{M}^{\mathrm{eq}} = \mathrm{dcl}^{\mathrm{eq}}(\mathcal{M})$.

(5) Every $X \subseteq M^n$ definable in $\mathcal{M}^{\mathrm{eq}}$ is already definable in $\mathcal{M}$.

(6) If $\mathcal{M}$ is κ-saturated (κ-homogeneous), then $\mathcal{M}^{\mathrm{eq}}$ is κ-saturated (κ-homogeneous, resp.).

(7) Every automorphism of $\mathcal{M}$ extends in a unique way to an automorphism of $\mathcal{M}^{\mathrm{eq}}$.

Proof. (2). Exercise.

(5). Induction on the complexity of $\varphi(x)$ with $x \in S^{|x|}$.

Let X be defined by $\mathcal{L}^{\mathrm{eq}}$-formula $\varphi(x)$ with x in S.

Claim: there is an $\mathcal{L}$-formula $\varphi^*(x)$ such that $\varphi^*(x)$ defines X.

Proof of claim: Induction on the complexity of φ.

Atomic case: either φ is $\mathcal{L}$-atomic, in which case set $\varphi^*(x) := \varphi(x)$, or φ is of the form $f_E(x_1) = f_E(x_2)$, and we replace with $\varphi^*(x_1, x_2) := E(x_1, x_2)$.

The operation * commutes with $\neg$, $\wedge$, $\exists x \in S$, namely: $(\neg\varphi)^* = \neg\varphi^*$, $(\varphi \wedge \psi)^* = \varphi^* \wedge \psi^*$, $(\exists y \in S\, \varphi(x, y))^* = \exists y \in S\, (\varphi(x, y))^*$.

Finally, set $(\exists y \in S_E\, \varphi(x, y))^* = \exists y' \in S\, (\varphi(x, f_E(y')))^*$. □

Canonical parameters

Definition 1.19 *(Canonical parameter)*.
Let X be an $\mathcal{L}$-definable set in a monster model $\mathfrak{U}$ of some theory T. A finite tuple $d \subseteq \mathfrak{U}$ is called a **canonical parameter** (code) for X if d is fixed by the same set of automorphisms which leaves X invariant, namely: for all $\sigma \in \mathrm{Aut}(\mathfrak{U})$, $\sigma(d) = d$ iff $\sigma(X) = X$.

Remark 1.20.
Suppose d is a canonical parameter for X. Then there is an $\mathcal{L}$-definable formula $\varphi(x, y)$ over $\emptyset$ such that $X = \varphi(\mathfrak{U}, d)$ and for all $z \in \mathfrak{U}^{|d|}$, if $X = \varphi(\mathfrak{U}, z)$ then $z = d$.

Proof. ($\Leftarrow$): Only need to show $\sigma(X) = X$ implies $\sigma(d) = d$. But $\sigma(X) = \varphi(\mathfrak{U}, \sigma(d))$; if $\sigma(X) = X$, then by assumption $\sigma(d) = d$.

($\Rightarrow$): Since X is definable and invariant under all $\sigma \in \text{Aut}(\mathfrak{U}/d)$, by Lemma 1.10, X is definable by $\varphi(x, d)$ where $\varphi(x, y)$ is over $\emptyset$. Suppose $\text{tp}(z) = \text{tp}(d)$; then there is an automorphism σ with $\sigma(d) = z$. Suppose further $\varphi(\mathfrak{U}, d) = \varphi(\mathfrak{U}, z) = X$; then $\sigma(X) = X$. By assumption, $z = \sigma(d) = d$.

Namely $\text{tp}(z) = \text{tp}(d) \models \forall x\, (\varphi(x, d) \leftrightarrow \varphi(x, z)) \to d = z$.

Hence, there exists $\theta(y)$ such that

$$\bigwedge_{\theta(y) \in \text{tp}(d)} \theta(y) \models \forall x\, (\varphi(x, d) \leftrightarrow \varphi(x, y)) \to d = y.$$

By compactness, there exists $\theta(y) \in \text{tp}(d)$ such that $\theta(y) \models \forall x\, (\varphi(x, d) \leftrightarrow \varphi(x, y)) \to d = y$.

Let $\phi(x, y) := \theta(y) \wedge \varphi(x, y)$. Clearly $\phi(\mathfrak{U}, d) = X$, and if $\phi(\mathfrak{U}, z) = X$, then $\mathfrak{U} \models \theta(z) \wedge \forall x\, (\varphi(x, d) \leftrightarrow \varphi(x, z))$. Hence $\mathfrak{U} \models d = z$. □

Example 1.21.
In ACF_0, take $X = \{a, b\}$ for some $a, b \in \mathfrak{U}$. The tuple (a, b) might not be a canonical parameter for X. (Consider $X = \{i, -i\}$; the automorphism $\sigma\colon i \mapsto -i$ fixes X but does not fix $(a, b) = (i, -i)$.)

Let $f(x) := (x - a)(x - b) = x^2 - (a + b)x + ab$, the polynomial which has X as the set of roots. Take the tuple of coefficients of f, namely $d = (-(a + b), ab)$. Then d is a canonical parameter for X: since if $\sigma(d) = d$, then $\sigma(f(x)) = f(x)$, and $X = \sigma(X)$. And if $\sigma \in \text{Aut}(\mathfrak{U})$ and $\sigma(X) = X$, then $\prod_{a' \in \sigma(X)}(x - a') = \prod_{a' \in X}(x - a') = f(x)$, so $\sigma(f) = f$, hence $d = \sigma(d)$.

Remark 1.22.
This proof shows that in fields, finite sets always have canonical parameters.

Remark 1.23.
T eliminates imaginaries iff every definable set has a canonical parameter in $\mathfrak{U}$.

The reason is that every definable set $X \subseteq \mathfrak{U}^n$ can be viewed as an imaginary element, defined by the equivalence relation $E(y_1, y_2) := \forall x\, (\varphi(x, y_1) \leftrightarrow \varphi(x, y_2))$ where $X = \varphi(x, b)$ for some $b \in U^m$ and $\varphi(x, y)$ over $\emptyset$.

In $\mathfrak{U}^{\text{eq}}$, consider the element b/E. Then:

(1) For all $\sigma \in \text{Aut}(\mathfrak{U})$ (equivalently in $\mathfrak{U}^{\text{eq}}$), $\sigma(X) = X$ iff $\sigma(b/E) = b/E$.

(2) X is b/E-definable in $\mathfrak{U}^{\text{eq}}$, by $\exists y\, (f_E(y) = b/E \wedge \varphi(x, y)) =: \psi(x, b/E)$.

(3) If $z \in S_E$ such that $X = \psi(\mathfrak{U}^{\text{eq}}, z)$, then $z = b/E$.

Hence b/E is a canonical parameter/code for X in $\mathfrak{U}^{\mathrm{eq}}$.

Lemma 1.24.
Let $\mathfrak{U}$ be a monster model of T. T has EI iff for any definable set X and a code/canonical parameter $e \in \mathfrak{U}^{\mathrm{eq}}$ for X, there is a tuple $c \in U^m$ such that $e \in \mathrm{dcl}(c)$ and $c \in \mathrm{dcl}(e)$ (dcl in $\mathfrak{U}^{\mathrm{eq}}$).

Proof. Exercise. □

Lemma 1.25 *(T^{eq} has EI)***.**
T^{eq} has EI.

Proof. By Lemma 1.18(2), indeed in the induction one needs to show: for $\varphi(x, x_1, \dots, x_k) \in \mathcal{L}^{\mathrm{eq}}$ with $x \in S^n$ and $x_i \in S_{E_i}$, there is $\psi(x, y_1, \dots, y_k) \in \mathcal{L}$ such that

$$T^{\mathrm{eq}} \models \forall x, y_1, \dots, y_k \, (\psi(x, y_1, \dots, y_k) \leftrightarrow \varphi(x, f_{E_1}(y_1), \dots, f_{E_k}(y_k))).$$

Hence any $\emptyset$-definable equivalence relation $\tilde{E}$ in $\mathfrak{U}^{\mathrm{eq}}$ over $\emptyset$, there is an $\mathcal{L}$-definable equivalence relation $\hat{E}(x_1, x_2)$ in $\mathfrak{U}$ over $\emptyset$ such that $\tilde{E}(a_1, a_2) \Leftrightarrow \hat{E}(b_1, b_2)$ where $b_i = f(a_i)$, for an $\mathcal{L}^{\mathrm{eq}}$-definable function f over $\emptyset$.

Let X be an equivalence class of $\tilde{E}$. Let $\hat{f}^{-1}(X)$ be the corresponding equivalence class of $\hat{E}$ in $\mathfrak{U}$. Then $\hat{f}^{-1}(X)$ has a code $d \in \mathfrak{U}^{\mathrm{eq}}$. So $\sigma(d) = d \Leftrightarrow \sigma(\hat{f}^{-1}(X)) = \hat{f}^{-1}(X) \Leftrightarrow \sigma(X) = X$.
□

Lemma 1.26.
Let $X \subseteq U^n$ be definable and $e \in \mathfrak{U}^{\mathrm{eq}}$ a code (canonical parameter) for X.

(1) X is A-definable iff $e \in \mathrm{dcl}(A)$ in $\mathfrak{U}^{\mathrm{eq}}$ (we write $\mathrm{dcl}^{\mathrm{eq}}(A)$). (By Lemma 1.10.)

(2) X is almost A-definable iff $e \in \mathrm{acl}(A)$ in $\mathfrak{U}^{\mathrm{eq}}$ (we write $\mathrm{acl}^{\mathrm{eq}}(A)$). Equivalently, X is $\mathrm{acl}^{\mathrm{eq}}(A)$-definable in $\mathfrak{U}^{\mathrm{eq}}$. (By Lemma 1.11.)

Principle: Strong types

Intuition: we need to work with $\mathrm{acl}^{\mathrm{eq}}$-closed base sets as bases for types.

Definition 1.27 *(Strong type)***.**
Let a, b be n-tuples from $\mathfrak{U}$. We say they have the **same strong type over** C, written $\mathrm{stp}(a/C) = \mathrm{stp}(b/C)$, if for every C-definable equivalence relation E with finitely many classes, $E(a, b)$ holds. Note that $\mathrm{stp}(a/C) = \mathrm{stp}(b/C)$ implies $\mathrm{tp}(a/C) = \mathrm{tp}(b/C)$ (since a C-definable set gives an equivalence relation).

Example 1.28.
T is the theory of an equivalence relation with two classes of infinitely many elements (in each class). Let a, b be two elements of different classes. Then $\mathrm{tp}(a/\emptyset) = \mathrm{tp}(b/\emptyset)$ but $\mathrm{stp}(a/\emptyset) \neq \mathrm{stp}(b/\emptyset)$.

Lemma 1.29.
TFAE:

(1) $\operatorname{stp}(a/A) = \operatorname{stp}(b/A)$.

(2) If $X \subseteq \mathfrak{U}^n$ is almost A-definable, then $a \in X$ iff $b \in X$.

(3) $\operatorname{tp}(a/\operatorname{acl}^{\mathrm{eq}}(A)) = \operatorname{tp}(b/\operatorname{acl}^{\mathrm{eq}}(A))$ (in $\mathfrak{U}^{\mathrm{eq}}$).

Example of a multi-sorted structure: Valued fields

Definition 1.30 *(Valuation)*.
Let K be a field and Γ be an ordered abelian group. A **valuation** on K with the valued group Γ is a surjective map $v\colon K \to \Gamma \cup \{\infty\}$ such that:

(1) $v(x) = \infty$ iff $x = 0$.

(2) $v(xy) = v(x) + v(y)$.

(3) $v(x + y) \geq \min\{v(x), v(y)\}$.

Example 1.31.
$v_p\colon \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$, the p-adic valuation: write $x = p^v \cdot \frac{c}{d}$ where $c, d \in \mathbb{Z}$ and $\gcd(c, d) = 1$, $p \nmid c$, $p \nmid d$. Define $v_p(x) = v$.

Associated structures:

- Valuation ring: $\mathcal{O}_K := \{x \in K : v(x) \geq 0\}$.
- Maximal ideal: $\mathfrak{m}_K := \{x \in K : v(x) > 0\}$.
- Residue field: $\mathcal{O}_K/\mathfrak{m}_K =: Kv$.
- $\operatorname{res}\colon \mathcal{O}_K \to Kv$.

Fact 1.32.
Up to isomorphism (of ordered abelian groups), Γ can be defined from $\mathcal{O}_K$ by $\Gamma := K^\times/\mathcal{O}_K^\times$ and $x \cdot \mathcal{O}_K^\times \leq y \cdot \mathcal{O}_K^\times$ iff $xy^{-1} \in \mathcal{O}_K$. Units: $\{x \in \mathcal{O}_K : x^{-1} \in \mathcal{O}_K\}$.

We may consider the valued field K in a **one-sorted language**: $(K, +, \cdot, 0, 1, \mathcal{O}_K)$ with $\mathcal{O}_K$ a unary predicate, which gives the valuation ring.

We have: $Kv := \mathcal{O}_K/\mathfrak{m}_K$ is an imaginary sort, $\Gamma := K^\times/\mathcal{O}_K^\times$ is an imaginary sort.

We can also consider K in a multi-sorted language:

2-sorted language:

- Field sort with the ring language $(K, +, \cdot, 0, 1)$.
- Ordered abelian group $\Gamma \cup \{\infty\}$ in the ordered group language $(\Gamma \cup \{\infty\}, +, 0, <)$.
- A map $v\colon K \to \Gamma \cup \{\infty\}$ (the valuation map).

3-sorted language:

- Field sort K.
- Ordered abelian group sort $\Gamma \cup \{\infty\}$.
- Residue field sort Kv in $\mathcal{L}_{\text{ring}}$: $(Kv, +, \cdot, 0, 1)$.
- Map $v\colon K \to \Gamma \cup \{\infty\}$.
- $\text{Res}\colon K \times K \to Kv$, defined by $\text{Res}(a,b) = \text{res}(a/b)$ if $b \neq 0$ and $a/b \in \mathcal{O}_K$, and 0 otherwise.

Remark 1.33.
There are obvious imaginaries in the 1- or 2-sorted language. In the 3-sorted language, generally there are more imaginaries (that cannot be eliminated), e.g. $S_1 = \mathcal{O}_K/(v(x) = v(y))$, i.e. $v(x) = v(y)$ iff $xy^{-1} \in \mathcal{O}_K^\times$.

1.4 Type Spaces

For convenience, we work in single-sorted languages. Fix a complete theory T and $\mathfrak{U} \models T$ a monster model.

Let $S_n(A)$ (or sometimes write $S_x(A)$ with $|x| = n$) be the space of n-types over A, i.e.:

$$S_n(A) := \{p(x) : p(x) \text{ is a complete type over } A,\ |x| = n\}.$$

Equip $S_n(A)$ with the topology where the basis of opens (clopens) are of the form:

$$\langle \varphi(x) \rangle := \{p \in S_n(A) : \varphi(x) \in p\},$$

for $\varphi(x) \in \mathcal{L}(A)$. Then $S_n(A)$ is a compact, Hausdorff, totally disconnected space — a **Stone space**.

Definition 1.34 *(Global types)*.
We call elements of $S_n(\mathfrak{U})$ **global types**.

Remark 1.35.
There are continuous surjections:

$$\pi_x\colon S_{x,y}(A) \to S_x(A), \quad \text{tp}(a,b/A) \mapsto \text{tp}(a/A),$$

$$\pi_y\colon S_{x,y}(A) \to S_y(A), \quad \text{tp}(a,b/A) \mapsto \text{tp}(b/A).$$

But $S_{x,y}(A)$ is *not* $S_x(A) \times S_y(A)$ with the product topology.

For each $q \in S_y(A)$, choose some $b_q \in \mathfrak{U}^n$, a realisation of $q(y)$. For any $p(x,y) \in S_{x,y}(A)$, let $(a,b) \models p(x,y)$. Since $\text{tp}(b/A) = \text{tp}(b_q/A)$ where $q = \pi_y(p(x,y))$, there is $\sigma \in \text{Aut}(\mathfrak{U}/A)$ with $\sigma(b) = b_q$. Consider $a' := \sigma(a)$. Then $(a', b_q) \models p(x,y)$.

Define:

$$\tilde{\pi}_y \colon S_{x,y}(A) \to \bigcup_{q \in S_y(A)} S_x(Ab_q), \quad p(x,y) \mapsto \operatorname{tp}(a'/Ab_q).$$

Lemma 1.36 *($\tilde{\pi}_y$ is injective)*.
$\tilde{\pi}_y$ is injective.

Proof. If $p(x,y) \neq p'(x,y) \in S_{x,y}(A)$. Suppose $\pi_y(p) \neq \pi_y(p')$; then clearly $\tilde{\pi}_y(p) \neq \tilde{\pi}_y(p')$. Otherwise, let $b_q \models \pi_y(p)$. We have $(a, b_q) \models p(x,y)$ and $(a', b_q) \models p'(x,y)$.

Suppose towards contradiction $\operatorname{tp}(a/Ab_q) = \operatorname{tp}(a'/Ab_q)$. Then there is $\sigma \in \operatorname{Aut}(\mathfrak{U}/Ab_q)$ such that $\sigma(a) = a'$. Hence $\sigma(a, b_q) = (a', b_q)$ and σ fixes A point-wise. Namely $\operatorname{tp}(a, b_q) = \operatorname{tp}(a', b_q)$ — i.e. $p(x,y) = p'(x,y)$, a contradiction. □

2 Counting Types, Stability, and Forking

2.1 Counting types and stability

Let T be a complete theory with infinite models.

Definition 2.1 *(κ-stable)*.
Let κ be an infinite cardinal. We say T is **κ-stable** if for each model of T, over every set of parameters of size $\leq \kappa$, and for any $n \in \omega$, there are at most κ-many n-types, i.e. $|A| \leq \kappa \implies |S_n(A)| \leq \kappa$.

Lemma 2.2.
T is κ-stable iff T is κ-stable for 1-types, i.e. $|A| \leq \kappa \implies |S(A)| \leq \kappa$.

Remark 2.3.
$\kappa \leq |S(A)| \leq 2^{\kappa + |T|}$ in general for $|A| = \kappa$. For 1-types, we write $S(A)$ instead of $S_1(A)$.

Proof of Lemma. ($\Leftarrow$): Suppose T is κ-stable for 1-types. Work in a monster model. Suppose $|S(A)| \leq \kappa$ for all $|A| \leq \kappa$. WTS: $|S_n(A)| \leq \kappa$ for all such A.

Induction on n. The case $n = 1$ is given by assumption.

For $S_{n+1}(A)$: by Lemma 1.36, $|S_{n+1}(A)| \leq \left|\bigcup_{q \in S(A)} S_n(Ab_q)\right|$, where b_q is a fixed choice of a realisation of q. Now $\left|\bigcup_{q \in S(A)} S_n(Ab_q)\right| \leq \kappa \cdot \kappa = \kappa$ (by $|S(A)| \leq \kappa$). Let $B = \bigcup_{q \in S(A)} Ab_q$. Hence $|S_n(B)| \leq \kappa$ by assumption, and $|S_{n+1}(A)| \leq |S_n(B)| \leq \kappa$ by the induction hypothesis. □

Recall the definition of stable from the Introduction.

Fact 2.4.
For a countable theory T, T is ω-stable iff T is totally transcendental (i.e. Morley rank of T is not ∞).

T is stable iff T is κ-stable for some κ, iff T is κ-stable for all κ satisfying $\kappa^{|T|} = \kappa$.

Definition 2.5 *(Dedekind cuts)***.**
Let $(I, \leq)$ be a linear order. A **cut** of I is a partition (A, B) where A is an initial segment of I and B is an end segment of I. A cut is uniquely determined by its initial segment.

Let κ be an infinite cardinal. The supremum of the number of Dedekind cuts of linear orders of size κ is denoted

$$\operatorname{ded} \kappa := \sup\{\lambda : \text{there is a linear order of size } \kappa \text{ with } \lambda \text{ cuts}\}.$$

It suffices to consider dense linear orders.

Lemma 2.6.
$\kappa < \operatorname{ded} \kappa \leq 2^\kappa$.

Proof. $\operatorname{ded} \kappa \leq 2^\kappa$ is clear, since every cut is uniquely determined by its initial segment, which is a subset of the underlying set.

To see $\kappa < \operatorname{ded} \kappa$: let μ be the minimal cardinal such that $2^\mu > \kappa$. Consider the tree $2^{<\mu}$ with the lexicographic ordering $\leq$ on it. Then $(2^{<\mu}, \leq)$ is a dense linear order and $|2^{<\mu}| = \left|\bigcup_{\lambda<\mu} 2^\lambda\right| \leq \mu \cdot \kappa$. By minimality of μ, and since $\mu \leq \kappa$, we get $|2^{<\mu}| \leq \kappa \cdot \kappa = \kappa$. Now any branch of 2^μ determines a cut on $(2^{<\mu}, \leq)$. Hence the number of cuts is $\geq 2^\mu > \kappa$.
□

Proposition 2.7.
Assume T is unstable. Then T is not κ-stable for any $\kappa \geq |T|$. Namely, if T is λ-stable for some λ, then T is stable.

Proof. Suppose $\varphi(x, y)$ has the n-order property for all $n \in \omega$. Fix $\kappa \geq |T|$. Let I be a dense linear order of size κ. By compactness and κ^+-saturation, there exist $M \models T$ and $(a_i, b_i : i \in I)$ such that $M \models \varphi(a_i, b_j) \Leftrightarrow i < j$ for all $i, j \in I$. By Löwenheim–Skolem (take the model generated by $(a_i, b_i : i \in I)$), we may assume $|M| = |I| + |T| = \kappa$ (as $\kappa \geq |T|$).

Given a cut $C = (A, B)$ in I, consider the $\mathcal{L}(M)$-formulas

$$\Phi_C := \{\varphi(x, b_j) : j \in B\} \cup \{\neg\varphi(x, b_i) : i \in A\}.$$

Then by density, Φ_C is a partial type. Let p_C be a complete type extending Φ_C. Now $p_C \neq p_{C'}$ for two different cuts C and C'.

Hence $\sup\{|S_x(M)| : M \models T, |M| = \kappa\} \geq \operatorname{ded} \kappa > \kappa$. □

Work in a monster model $\mathfrak{U} \models T$.

Lemma 2.8 *(Properties of stable formulas).*
Let $\varphi(x,y)$, $\psi(x,z)$ be stable formulas (where y, z might not be disjoint). Then:

(1) Let $\varphi^*(y,x) := \varphi(x,y)$. Then $\varphi^*(y,x)$ is stable.

(2) $\neg\varphi(x,y)$ is stable.

(3) $\theta(x,yz) := \varphi(x,y) \wedge \psi(x,z)$ and $\theta'(x,yz) := \varphi(x,y) \vee \psi(x,z)$ are stable.

(4) If $y = u^k$ and $c \in \mathfrak{U}^{|u|^{k-1}}$, then $\theta(x,u) := \varphi(x,u,c)$ is stable.

(5) If T is stable, then every $\mathcal{L}^{\mathrm{eq}}$-formula is stable as well.

Proof. (1). Suppose $\varphi(x,y)$ is stable (does not have the n-order property) but $\varphi^*(y,x)$ is not stable, so it has the n-order property. Let $(a_i, b_j : i,j < n)$ be such that $\mathfrak{U} \models \varphi^*(a_i, b_j) \Leftrightarrow i < j$. Then $\mathfrak{U} \models \varphi(b_j, a_i) \Leftrightarrow i < j$. Let $b'_j := b_{n-1-j}$ and $a'_i := a_{n-1-i}$ for all $i,j < n$. Then $\mathfrak{U} \models \varphi(b'_j, a'_i) \Leftrightarrow \mathfrak{U} \models \varphi(b_{n-1-j}, a_{n-1-i}) \Leftrightarrow n-1-j > n-1-i \Leftrightarrow j < i$, contradicting φ being stable.

(3). Suppose $\varphi(x,y) \vee \psi(x,z)$ is unstable. By $\aleph_0$-saturation of $\mathfrak{U}$, there are $(a_i, b_i, b'_i : i \in \omega)$ such that $\mathfrak{U} \models \varphi(a_i, b_j) \vee \psi(a_i, b'_j) \Leftrightarrow i < j$ for all $i,j \in \omega$.

Let $P := \{(i,j) \in \mathbb{N}^2 : i < j \text{ and } \mathfrak{U} \models \varphi(a_i, b_j)\}$ and $Q := \{(i,j) \in \mathbb{N}^2 : i < j \text{ and } \mathfrak{U} \models \psi(a_i, b'_j)\}$. Then $P \cup Q = \{(i,j) \in \mathbb{N}^2 : i < j\}$. Consider P and $Q \setminus P$ as two colours on the 2-element subsets of $\mathbb{N}$. By Ramsey theory, there is an infinite set $I \subseteq \mathbb{N}$ such that all 2-element subsets of I are either in P or all in $Q \setminus P$. The former contradicts stableness of $\varphi(x,y)$ and the latter contradicts $\psi(x,z)$ being stable.

(2), (4), (5): Exercises. $\square$

Local φ-types and counting local φ-types

Definition 2.9 *(φ-type).*
Let $\varphi(x,y)$ be a formula. By a **complete φ-type** over a set of parameters $A \subseteq \mathfrak{U}^{|y|}$, we mean a maximal collection of formulas of the form $\varphi(x,b)$, $\neg\varphi(x,b)$ where $b \in A$. Let $S_\varphi(A)$ be the space of all complete φ-types over A.

Proposition 2.10.
Assume $|S_\varphi(B)| > |B|$ for some infinite set of parameters B. Then $\varphi(x,y)$ is unstable.

Lemma 2.11 *(Erdős–Makkai).*
Let B be an infinite set and $\mathcal{F} \subseteq \mathcal{P}(B)$ a collection of subsets of B with $|B| < |\mathcal{F}|$. Then there are sequences $(b_i, i < \omega)$ of elements of B and $(S_j, j < \omega)$ of elements of $\mathcal{F}$ such that one of the following holds:

(1) $b_i \in S_j \Leftrightarrow j < i$ for all $i,j \in \omega$.

(2) $b_i \in S_j \Leftrightarrow i \le j$ for all $i,j \in \omega$.

Proof. Let $\mathcal{T} := \{(B_0, B_1) : B_0, B_1$ finite subsets of B, $\exists S \in \mathcal{F}$ with $B_0 \subseteq S$, $B_1 \subseteq B \setminus S\}$. Then $|\mathcal{T}| \leq |B|$. For each (B_0, B_1) choose $S \in \mathcal{F}$ with $B_0 \subseteq S$, $B_1 \subseteq B \setminus S$, and let $\mathcal{F}'$ be the collection of all such S. Then $|\mathcal{F}'| \leq |B|$. By assumption $|\mathcal{F}| > |B|$, so there is $S^* \in \mathcal{F}$ that is not a Boolean combination of elements of $\mathcal{F}'$ (since there are only $\leq |B|$-many such Boolean combinations).

We construct inductively: $(S_i : i \in \omega)$ in $\mathcal{F}'$, $(b'_i : i \in \omega)$ in S^*, $(b''_i : i \in \omega)$ in $B \setminus S^*$ such that $\{b'_0, \ldots, b'_n\} \subseteq S_n$ and $\{b''_0, \ldots, b''_n\} \subseteq B \setminus S_n$, and $b'_n \in S_i \Leftrightarrow b''_n \in S_i$ for all $i < n$.

Step 0: Choose any $S_0 \in \mathcal{F}'$ with $S_0 \neq \emptyset$, and $b'_0 \in S_0$, $b''_0 \notin S_0$.

Step n: Assume $(b'_i : i < n)$, $(b''_i : i < n)$ and $(S_i : i < n)$ have been constructed. We seek $b'_n \in S^*$, $b''_n \in B \setminus S^*$ with $b'_n \in S_i \Leftrightarrow b''_n \in S_i$ for all $i < n$. If no such pair exists, then S^* is a Boolean combination of $(S_i : i < n)$, contradicting our choice. By assumption $\{b'_0, \ldots, b'_{n-1}\} \subseteq S^*$, hence $\{b'_0, \ldots, b'_n\} \subseteq S^*$. Similarly $\{b''_0, \ldots, b''_n\} \subseteq B \setminus S^*$. By choice of $\mathcal{F}'$, there is $S_n \in \mathcal{F}'$ with $\{b'_0, \ldots, b'_{n-1}\} \subseteq S_n$ and $\{b''_0, \ldots, b''_{n-1}\} \subseteq B \setminus S_n$.

Now colour the two-element subsets $\{i, n\}$ of $\mathbb{N}$ with $i < n$ by: $Q := \{(i, n) : i < n,\ b'_n \in S_i\}$ and $P := \{(i, n) : i < n,\ b'_n \notin S_i\}$. By Ramsey theory, there is an infinite $I \subseteq \mathbb{N}$ such that either:

① for all (i, n) with $i < n$ and $i, n \in I$, $b'_n \in S_i$; or

② for all (i, n) with $i < n$ and $i, n \in I$, $b'_n \notin S_i$.

If ① happens, let $b_i := b''_i$. Then $b''_n \notin S_i$ for all $i < n$, $i, n \in I$ (since b'_n and b''_n have the same behaviour over $\{S_i : i < n\}$), and by construction $b''_n \notin S_j$ for all $j \geq n$. Restricting the order on I, we get $b_i \in S_j \Leftrightarrow j < i$.

If ② happens, let $b_i := b'_i$. Then $b'_n \notin S_i$ for all $i < n$, $i, n \in I$, and $b_n \in S_i$ for all $i \geq n$. Restricting the order on I, we get $b_i \in S_j \Leftrightarrow i \leq j$. □

Proof of Proposition 2.10. Work in $\mathfrak{U}$. For any $a \in \mathfrak{U}^{|x|}$, $\operatorname{tp}_\varphi(a/B)$ determines uniquely a subset $S_a := \{b \in B : \mathfrak{U} \models \varphi(a, b)\} \subseteq B$. By assumption $\left|\{S_a : a \in \mathfrak{U}^{|x|}\}\right| = |S_\varphi(B)| > |B|$. By Erdős–Makkai, there are $(b_i : i < \omega)$ and $(a_i : i < \omega)$ such that either:

(1) $b_i \in S_{a_j} \Leftrightarrow j < i$, i.e. $\mathfrak{U} \models \varphi(a_j, b_i) \Leftrightarrow j < i$; or

(2) $b_i \in S_{a_j} \Leftrightarrow i \leq j$, i.e. $\mathfrak{U} \models \varphi(a_j, b_i) \Leftrightarrow i \leq j$.

Case (1) directly contradicts $\varphi(x, y)$ being stable. For case (2), let $a'_j := a_{j+1}$ and $\varphi^*(y, x) := \varphi(x, y)$. Then $\mathfrak{U} \models \varphi^*(b_i, a'_j) \Leftrightarrow \mathfrak{U} \models \varphi(a_{j+1}, b_i) \Leftrightarrow i \leq j + 1$, which for $i < j$ gives the order property, contradicting φ^* being stable. □

Corollary 2.12 *(Characterisations of stability)***.**

(1) T is stable iff T is κ-stable for some κ.

(2) Iff T is κ-stable for all κ satisfying $\kappa^{|T|} = \kappa$.

Proof. (2) $\Rightarrow$ (1): Already seen. (3) $\Rightarrow$ (2): Trivial.

(1) $\Rightarrow$ (3): If T is stable, then by Proposition 2.10, $\left|S_\varphi(B)\right| \leq |B|$ for all $\varphi(x,y)$. Let κ be such that $\kappa^{|T|} = \kappa$. Let $|B| \leq \kappa$ and $B \subseteq \mathfrak{U}$. A complete 1-type is determined by the collection of φ-types over B, for all $\varphi(x,y)$ with $|x| = 1$. So $|S(B)| \leq \prod_{\varphi \in \mathcal{L}} \left|S_\varphi(B)\right| \leq \prod_{\varphi \in \mathcal{L}} |B| \leq |B|^{|T|} \leq \kappa^{|T|} = \kappa$. □

Corollary 2.13.

T is stable iff for all $\varphi(x,y)$ with $|x| = 1$, $\varphi(x,y)$ does not have the order property, i.e. $\varphi(x,y)$ is stable.

Proof. Note that in the proof of (1) $\Rightarrow$ (3) above, we only use that $\varphi(x,y)$ is stable for $|x| = 1$. □

2.2 Local ranks and definability of types

In this section, we give more equivalent characterisations of stability: one by local rank and one by definability of types.

Definition 2.14 *(Shelah's local 2-rank)***.**

Takes values in $\{-\infty\} \cup \omega \cup \{\infty\}$, where $-\infty < \omega < \infty$.

Let Δ be a set of $\mathcal{L}$-formulas and $\Theta(x)$ a partial type over $\mathfrak{U}$.

- $R_\Delta(\Theta(x)) \geq 0$ iff $\Theta(x)$ is consistent. (Otherwise $R_\Delta(\Theta(x)) = -\infty$.)
- $R_\Delta(\Theta(x)) \geq n+1$ if for some $\varphi(x,y) \in \Delta$ and $a \in \mathfrak{U}^{|y|}$, we have both $R_\Delta(\Theta(x) \wedge \varphi(x,a)) \geq n$ and $R_\Delta(\Theta(x) \wedge \neg\varphi(x,a)) \geq n$.
- $R_\Delta(\Theta(x)) = n$ if $R_\Delta(\Theta(x)) \geq n$ and $R_\Delta(\Theta(x)) \ngeq n+1$.
- $R_\Delta(\Theta(x)) = \infty$ if $R_\Delta(\Theta(x)) \geq n$ for all $n \in \omega$.

If $\Delta = \{\varphi(x,y)\}$ we write R_φ instead of $R_{\{\varphi\}}$.

Proposition 2.15.

$\varphi(x,y)$ is stable iff $R_\varphi(x = x)$ is finite. (Hence also $R_\varphi(\Theta(x))$ is finite for any partial type $\Theta(x)$.)

Proof. Assume $\varphi(x,y)$ is unstable. By saturation, there are $(a_i, b_i : i \in [0,1])$ such that $\mathfrak{U} \models \varphi(a_i, b_j) \Leftrightarrow i < j$. Now both $\varphi(x, b_{1/2})$ and $\neg\varphi(x, b_{1/2})$ each contain half of the a_i's.

Each can be split again by $\varphi(x, b_{1/4})$ and $\varphi(x, b_{3/4})$, and so on. Inductively, $R_\varphi(x = x) \geq n$ for all n.

Conversely, assume $R_\varphi(x = x) = \infty$. For any n, $R_\varphi(x = x) \geq n$ implies we can find a binary tree $\mathcal{B}_n = (b_\eta : \eta \in 2^{\leq n})$ such that $\{\varphi^{f(i)}(x, b_{f\restriction i}) : i < n\}$ is consistent for every branch $f \in 2^n$, where $\varphi^0(x, b) := \varphi(x, b)$ and $\varphi^1(x, b) := \neg\varphi(x, b)$.

By saturation, we can find an infinite binary tree $\mathcal{B} = (b_\eta : \eta \in 2^{<\omega})$ such that for any $f : \omega \to \{0, 1\}$, $\{\varphi^{f(i)}(x, b_{f\restriction i}) : i < \omega\}$ is consistent. Now $|S_\varphi(\mathcal{B})| \geq |\{f : \omega \to \{0, 1\}\}| > |\mathcal{B}|$. By Proposition 2.10, $\varphi(x, y)$ is unstable. $\square$

Definition 2.16 *(Definable types)***.**
Let $\varphi(x, y)$ be an $\mathcal{L}$-formula.

(1) A type $p(x) \in S_\varphi(A)$ is **definable over** B if there is some $\mathcal{L}(B)$-formula $\psi(y)$ such that $\varphi(x, a) \in p(x) \Leftrightarrow \mathfrak{U} \models \psi(a)$ for all $a \in A$.

(2) A type $p \in S_x(A)$ is **definable over** B if $p|_\varphi$ is definable over B for all $\varphi(x, y) \in \mathcal{L}$.

(3) A type is **definable** if it is definable over its domain, i.e. $p \in S_x(A)$ is definable if it is definable over A.

(4) We say types in T are **uniformly definable** if for any $\varphi(x, y) \in \mathcal{L}$, there is $\psi(y, z) \in \mathcal{L}$, such that for any A and $p \in S_\varphi(A)$, there is $b \in A^{|z|}$ such that $\varphi(x, a) \in p \Leftrightarrow \mathfrak{U} \models \psi(a, b)$ for all $a \in A$.

Example 2.17.
Consider $(\mathbb{Q}, <) \models \mathrm{DLO}$ and let $p = \mathrm{tp}(\pi/\mathbb{Q})$. Then the set $\{a \in \mathbb{Q} : \pi > a\} = \{a \in \mathbb{Q} : (x > a) \in p(x)\} \neq \psi(\mathbb{Q})$ for any ψ. Hence p is not definable.

For any realised type $p = \mathrm{tp}(a/A)$ where $a \in A^n$, $\{b \in A : \psi(x, b) \in p\} = \{b \in A : \mathfrak{U} \models \psi(a, b)\} = \psi(a, A)$, hence it is definable trivially.

Lemma 2.18.
(1) The set $\{e : R_\varphi(\Theta(x, e)) \geq n\}$ is definable, for all $n \in \omega$.

(2) If $R_\varphi(\Theta(x)) = n$, then for any $a \in \mathfrak{U}^{|y|}$, at most one of $\Theta(x) \wedge \varphi(x, a)$ and $\Theta(x) \wedge \neg\varphi(x, a)$ has R_φ-rank n.

Proof. (1). Induction on n. For $n = 0$: $\{e : R_\varphi(\Theta(x, e)) \geq 0\}$ is defined by $\exists x\, \Theta(x, e)$.

Suppose $\{e : R_\varphi(\Theta(x, e)) \geq n - 1\}$ is definable for all $\Theta(x, y)$. By definition, $\{e : R_\varphi(\Theta(x, e)) \geq n\}$ holds iff there is $a \in \mathfrak{U}^{|y|}$ such that $R_\varphi(\Theta(x, e) \wedge \varphi(x, a)) \geq n - 1$ and $R_\varphi(\Theta(x, e) \wedge \neg\varphi(x, a)) \geq n - 1$.

Let $\psi_1(y_1, y_2)$ and $\psi_2(y_1, y_2)$ (with $|y_1| = |e|$, $|y_2| = |a|$, and y_1, y_2 disjoint) be the formulas defining $\{(e, a) : R_\varphi(\Theta(x, e) \wedge \varphi(x, a)) \geq n - 1\}$ and $\{(e, a) : R_\varphi(\Theta(x, e) \wedge \neg\varphi(x, a)) \geq n - 1\}$ respectively. Then $\{e : R_\varphi(\Theta(x, e)) \geq n\}$ is defined by $\exists y_2\, (\psi_1(y_1, y_2) \wedge \psi_2(y_1, y_2))$.

(2) follows from definition. □

Remark 2.19.
Indeed we have uniform definability for (1). Easy to check.

Proposition 2.20.
Let $\varphi(x, y)$ be a stable formula. Then all φ-types are uniformly definable.

Proof. Write $\neg\varphi(x, a)$ as $\varphi^0(x, a)$ and $\varphi(x, a)$ as $\varphi^1(x, a)$. Let $k := R_\varphi(x = x)$.

Claim 1: For any φ-type $p(x)$, there is a collection Δ_p (possibly empty) of $\leq k$ many formulas in $p(x)$ (recall $p(x)$ contains formulas of the form $\varphi^\varepsilon(x, a)$, $\varepsilon \in \{0, 1\}$) such that $R_\varphi(\Delta_p \wedge \varphi^\varepsilon(x, b)) = R_\varphi(\Delta_p)$ for all $\varphi^\varepsilon(x, b) \in p$. (By R_φ(empty set) we mean $R_\varphi(x = x)$.)

Proof of Claim 1: Start with $\Delta_0 = \emptyset$ and choose inductively: given Δ_i, pick any $\varphi^\varepsilon(x, a) \in p(x)$ with $R_\varphi(\Delta_i \wedge \varphi^\varepsilon(x, a)) < R_\varphi(\Delta_i)$ and let $\Delta_{i+1} := \Delta_i \cup \{\varphi^\varepsilon(x, a)\}$. If no such formula exists, let $\Delta_p := \Delta_i$. This procedure must stop in $\leq k$ steps since $R_\varphi(\Delta_0) = R_\varphi(x = x) = k$.
△

Claim 2: $p(x)$ is defined by the condition $\{a : R_\varphi(\Delta_p \wedge \varphi(x, a)) \geq R_\varphi(\Delta_p)\}$.

Proof of Claim 2: This condition is definable by Lemma 2.18(1). By choice of Δ_p, $R_\varphi(\Delta_p \wedge \varphi(x, a)) = R_\varphi(\Delta_p)$ for all $\varphi(x, a) \in p(x)$. And if $\varphi(x, a) \notin p(x)$ (i.e. $\neg\varphi(x, a) \in p(x)$), then $R_\varphi(\Delta_p \wedge \neg\varphi(x, a)) = R_\varphi(\Delta_p)$. By Lemma 2.18(2), $R_\varphi(\Delta_p \wedge \varphi(x, a)) < R_\varphi(\Delta_p)$, hence not satisfying the defining condition, as desired.

Thus any φ-type is definable. △

Claim 3: The φ-types are uniformly definable.

Note that Δ_p is a conjunction of $\leq k$ many instances of $\varphi^\varepsilon(x, a)$ (possibly with $x = x$ as well). Let $\psi(x; y_0, \ldots, y_k, z_1^1, z_2^1, \ldots)$ be the formula:

$$\bigvee_{f : k \to \{0,1\}} \Big(\bigwedge_{i \in k} \varphi^{f(i)}(x, y_i) \wedge z_i^1 = z_i^1 \Big) \vee (x = x \wedge z_i^1 = z_i^1).$$

Then each Δ_p is equivalent to some instance of ψ, by realising one disjunct and eliminating the others by setting $z_i^1 \neq z_i^1$. Now by Lemma 2.18(1), we get uniform definability. □

Theorem 2.21.
TFAE for a formula $\varphi(x, y)$:

(1) $\varphi(x, y)$ is stable.

(2) $R_\varphi(x = x) < \omega$.

(3) All φ-types are uniformly definable.

(4) All φ-types over models are definable.

(5) $|S_\varphi(M)| \leq \kappa$ for all $\kappa \geq |\mathcal{L}|$ and $M \models T$ with $|M| = \kappa$.

(6) There is some $\kappa \geq |\mathcal{L}|$ such that $|S_\varphi(M)| < \operatorname{ded} \kappa$ for all $M \models T$ with $|M| = \kappa$.

Proof. We have seen $(1) \Rightarrow (2)$ and $(2) \Rightarrow (3)$ and $(3) \Rightarrow (4)$ is trivial. $(4) \Rightarrow (5)$: if all φ-types are definable over the model, then any φ-type is determined by one formula $\psi(y)$ over M. Hence $|S_\varphi(M)| \leq |M| + |\mathcal{L}| = \kappa$ for $M \models T$ with $|M| = \kappa$. $(5) \Rightarrow (6)$: trivial. $(6) \Rightarrow (1)$: Proposition 2.7. □

Corollary 2.22.
T is stable iff all types over models are definable.

2.3 Indiscernible sequences and stability

Definition 2.23 *(Indiscernible sequence)*.
Given a linear order I, a sequence of tuples $(a_i, i \in I)$ with $a_i \in \mathfrak{U}^{|x|}$ is called **indiscernible** over a set of parameters A if $a_{i_0} \cdots a_{i_n} \equiv_A a_{j_0} \cdots a_{j_n}$ for all $i_0 < \cdots < i_n$ and $j_0 < \cdots < j_n$ from I and $n \in \omega$ (i.e. $\varphi(a_{i_0}, \ldots, a_{i_n})$ holds iff $\varphi(a_{j_0}, \ldots, a_{j_n})$ holds in $\mathfrak{U}$).

Example 2.24.
(1) A constant sequence is indiscernible over any set of parameters.

(2) Any increasing (or decreasing) sequence of singletons in a dense linear order is indiscernible.

(3) Any basis in a vector space is indiscernible.

Now we describe a standard method of finding indiscernible sequences in an arbitrary theory.

Definition 2.25 *(Ehrenfeucht–Mostowski type)*.
For any sequence $\bar{a} = (a_i, i \in I)$ and a set of parameters B, we define $\mathrm{EM}(\bar{a}/B)$, the **Ehrenfeucht–Mostowski type** of $\bar{a}$ over B, as a partial type over B in countably many variables indexed by ω and given by

$$\{\varphi(x_0, \ldots, x_n) \in \mathcal{L}(B) : \forall\, i_0 < \cdots < i_n, \mathfrak{U} \models \varphi(a_{i_0}, \ldots, a_{i_n}), n \in \omega\}.$$

Remark 2.26.
If $\bar{a}$ is indiscernible over B, then $\mathrm{EM}(\bar{a}/B)$ is complete.

Proposition 2.27 *(Extraction of indiscernibles)*.
Let $\bar{a} = (a_j : j \in J)$ be an arbitrary sequence in $\mathfrak{U}$ with J an infinite linear order. Let A be a small set of parameters. Then for any small linear order I we can find in $\mathfrak{U}$ an A-indiscernible sequence $(b_i, i \in I)$ based on $\bar{a}$, i.e. for any $i_0 < \cdots < i_n$ in I and a finite set of formulas $\Delta \subseteq \mathcal{L}(A)$, there are some $j_0 < \cdots < j_n$ in J such that $\mathfrak{U} \models \varphi(b_{i_0}, \ldots, b_{i_n}) \Leftrightarrow \mathfrak{U} \models \varphi(a_{j_0}, \ldots, a_{j_n})$ for all $\varphi \in \Delta$.

Remark 2.28.
"Based on $\bar{a}$" is the same as realising $\mathrm{EM}(\bar{a}/A)$.

Proof of Proposition 2.27. Let $(C_i, i \in I)$ be a set of new constant symbols. Consider the theory T in $\mathcal{L}' := \mathcal{L} \cup \{C_i : i \in I\}$ with additional axioms:

① $\varphi(C_{i_0}, \dots, C_{i_n})$ for all $i_0 < \cdots < i_n \in I$ and $\varphi \in \mathrm{EM}(\bar{a}/A)$.

② $\varphi(C_{i_0}, \dots, C_{i_n}) \leftrightarrow \varphi(C_{j_0}, \dots, C_{j_n})$ for all $i_0 < \cdots < i_n$ and $j_0 < \cdots < j_n$ in I and $\varphi \in \mathcal{L}(A)$.

Claim: Φ is consistent.

By compactness, it suffices to show any finite $\Phi_0 \subseteq \Phi$ is consistent. Let Δ be the finite set of formulas appearing in Φ_0, and suppose $n+1$ is the length of variables. Write $\Delta = \Delta_0 \cup \Delta_1$ where $\Delta_0 \subseteq \mathrm{EM}(\bar{a}/A)$ and Δ_1 consists of the equivalences from ②. Colour the n-tuples of $\{a_j : j \in J\}$ by their values of $\varphi(a_{j_0}, \dots, a_{j_n})$ for $\varphi \in \Delta_1$ ($j_0 < \cdots < j_n$). There are at most $2^{|\Delta_1|}$-many colours and $|J|$ is infinite. By Ramsey theory, there is an infinite $J' \subseteq J$ such that $\mathfrak{U} \models \varphi(a_{j'_0}, \dots, a_{j'_n}) \Leftrightarrow \varphi(a_{t'_0}, \dots, a_{t'_n})$ for all $j'_0 < \cdots < j'_n$ in J' and $t'_0 < \cdots < t'_n$ in J' and $\varphi \in \Delta_1$. Clearly $\mathfrak{U} \models \varphi(a_{i'_0}, \dots, a_{i'_n})$ for all $i'_0 < \cdots < i'_n$ in J' and $\varphi \in \Delta_0 \subseteq \mathrm{EM}(\bar{a}/A)$. $\triangle$

Now any type $(b_i : i \in I)$ realising Φ will be an A-indiscernible sequence realising $\mathrm{EM}(\bar{a}/A)$. The "based on $\bar{a}$" property follows since if for some finite Δ and $i_0 < \cdots < i_n$ in I, no suitable $j_0 < \cdots < j_n$ in J existed, then $\bigvee_{\varphi \in \Delta} \neg\varphi^{\varepsilon}(a_{j_0}, \dots, a_{j_n}) \in \mathrm{EM}(\bar{a}/A)$, hence the same would hold for $b_{i_0}, \dots, b_{i_n}$, a contradiction. $\square$

Remark 2.29.
We may phrase the definition of indiscernibles as follows: let $(I, <)$ be a linear order and $\mathcal{L}_{\mathrm{ord}}$ be the language of linear order. A sequence $(a_i, i \in I)$ indexed by $(I, <)$ is indiscernible over B iff for any $(i_0, \dots, i_n) \equiv_{\mathcal{L}_{\mathrm{ord}}} (j_0, \dots, j_n)$, we have $a_{i_0} \cdots a_{i_n} \equiv_{\mathcal{L}(B)} a_{j_0} \cdots a_{j_n}$.

We may generalise the index structure from orders to arbitrary structures, e.g. trees, and use the corresponding Ramsey theory to prove e.g. tree-indiscernibles exist.

Definition 2.30 *(Totally indiscernible)***.**
A sequence $(a_i, i \in I)$ is **totally indiscernible** over A if $a_{i_0} \cdots a_{i_n} \equiv_A a_{j_0} \cdots a_{j_n}$ for any pairwise distinct $i_0, \dots, i_n$ and pairwise distinct $j_0, \dots, j_n$ from I. (I.e. indiscernible in the language of "$=$"; the order is not important.)

Theorem 2.31.
T is stable iff every indiscernible sequence is totally indiscernible.

Proof. ($\Leftarrow$): Suppose T is unstable with $\varphi(x, y)$ witnessing the order property. Let $\varphi^*(x'y, xy') := \varphi(x, y)$ with $|y'| = |y|$ and $|x'| = |x|$. Suppose $(a_i, b_j : i, j \in \omega)$ satisfy $\mathfrak{U} \models \varphi(a_i, b_j) \Leftrightarrow i < j$. Let $c_i := a_i b_i$. Then $\mathfrak{U} \models \varphi^*(c_i, c_j) \Leftrightarrow i < j$. Consider an indiscernible sequence $(c'_i : i < \omega)$ based on $(c_i : i < \omega)$, i.e. $\mathrm{EM}((c'_i)_{i\in\omega}) = \mathrm{EM}((c_i)_{i<\omega})$. Then $\mathfrak{U} \models \varphi^*(c'_i, c'_j) \Leftrightarrow i < j$, so it is not totally indiscernible.

($\Rightarrow$): Suppose there is an indiscernible sequence $(a_i, i \in I)$ over A which is not totally indiscernible. By Proposition 2.27, we may assume $I = \mathbb{Q}$. Then there is $\varphi(x_1, \ldots, x_n) \in \mathcal{L}(A)$ and $i_1 < \cdots < i_n$ and $\sigma \in \mathrm{Sym}(n)$ such that $\mathfrak{U} \models \varphi(a_{i_1}, \ldots, a_{i_n}) \wedge \neg\varphi(a_{i_{\sigma(1)}}, \ldots, a_{i_{\sigma(n)}})$.

Fact: σ is a product of adjacent transpositions of the form (i_s, i_{s+1}) for $1 \leq s < n$. Write σ as a sequence $\mathrm{id}, \sigma_1, \ldots, \sigma_K$ where $\sigma_K = \sigma$ and $\sigma_i = \tau_i \circ \sigma_{i-1}$, where τ_i is an adjacent transposition.

Then we have $\mathfrak{U} \models \varphi(a_{i_1}, \ldots, a_{i_n})$ and $\mathfrak{U} \models \neg\varphi(a_{i_{\sigma_K(1)}}, \ldots, a_{i_{\sigma_K(n)}})$.

Let $\sigma_{\ell+1}$ be the first index such that $\mathfrak{U} \models \neg\varphi(a_{i_{\sigma_{\ell+1}(1)}}, \ldots, a_{i_{\sigma_{\ell+1}(n)}})$. Suppose $\sigma_{\ell+1} = (i_s, i_{s+1}) \circ \sigma_\ell$ and $\sigma_\ell(\tilde{\imath}) = i_s$ and $\sigma_\ell(\tilde{\jmath}) = i_{s+1}$.

Let $\psi(x, y) := \varphi(a_{\sigma_\ell(i_1)}, \ldots, x, \ldots, y, \ldots, a_{\sigma_\ell(i_n)}) \wedge x \neq y$ (replacing $a_{\sigma_\ell(\tilde{\imath})}$ with x and $a_{\sigma_\ell(\tilde{\jmath})}$ with y). Since $\sigma_\ell(\tilde{\imath}) = i_s < i_{s+1} = \sigma_\ell(\tilde{\jmath})$, by indiscernibility of $(a_i, i \in I)$, for all $(a_i : i_s < i < i_{s+1})$ and $(a_j : i_s < j < i_{s+1})$, $\mathfrak{U} \models \psi(a_i, a_j) \Leftrightarrow i < j$.

Since $I = \mathbb{Q}$, the set $\{i_s < i < i_{s+1}\}$ is infinite. We get ψ has the order property and T is not stable. $\square$

Proposition 2.32.

For any stable formula $\varphi(x, y)$ in an arbitrary theory, there is $k_\varphi \in \omega$, depending only on φ, such that for any indiscernible sequence $I \subseteq \mathfrak{U}^{|x|}$ and $b \in \mathfrak{U}^{|y|}$, either $|\varphi(I, b)| \leq k_\varphi$ or $|\neg\varphi(I, b)| \leq k_\varphi$.

Proof. Suppose $\varphi(x, y)$ does not have the k-order property. Let $I = (a_i, i \in I)$ be indiscernible and $b \in \mathfrak{U}^{|y|}$. We claim either $|\varphi(I, b)| \leq 2k$ or $|\neg\varphi(I, b)| \leq 2k$.

Suppose not. Then there are $j_0 < \cdots < j_{4k-1}$ such that $E_0 := \{t : \mathfrak{U} \models \varphi(a_{j_t}, b)\}$ and $E_1 := \{t : \mathfrak{U} \models \neg\varphi(a_{j_t}, b)\}$ each have size $\geq 2k$. Hence either $E_0 \cap \{0, \ldots, 2k-1\} \geq k$ and $E_1 \cap \{2k, \ldots, 4k-1\} \geq k$, or $E_1 \cap \{0, \ldots, 2k-1\} \geq k$ and $E_0 \cap \{2k, \ldots, 4k-1\} \geq k$.

We may assume the first case (the second is analogous). Thus $\mathfrak{U} \models \exists y \bigwedge_{0\leq t<k} \varphi(a_{j_t}, y) \wedge \bigwedge_{k\leq t<2k} \neg\varphi(a_{j_t}, y)$.

Since $(a_i, i \in I)$ is indiscernible, for $j_0 < j_1 < \cdots < j_{k-1} < j_k < \cdots < j_{2k-1}$ we get $\mathfrak{U} \models \exists y \bigwedge_{0\leq t<i} \varphi(a_{j_t}, y) \wedge \bigwedge_{i\leq t<2k} \neg\varphi(a_{j_t}, y)$ for all $0 \leq i < k$.

Let $b_i \in \mathfrak{U}^{|y|}$ witness this. Let $c_t := a_{j_t}$ for $t, j < k$. Then $\mathfrak{U} \models \varphi(c_t, b_i) \Leftrightarrow t < i$, contradicting $\varphi(x, y)$ having no k-order property. $\square$

Corollary 2.33 (*Average type*).

In a stable theory, we can define the **average type** of an indiscernible sequence $\bar{b} = (b_i)_{i\in I}$

over any set of parameters A as:

$$\mathrm{Av}(\bar{b}/A) := \{\varphi(x,a) \in \mathcal{L}(A) : \mathfrak{U} \models \varphi(b_i,a) \text{ for all but finitely many } i \in I\}.$$

By the previous proposition, this type is complete and consistent.

2.4 Number of types and definability of types in NIP

Definition 2.34 *(Independence property and NIP)*.

(1) A formula $\varphi(x,y)$ is said to have the **independence property** (IP) if in $\mathfrak{U}$ there are an infinite sequence $(b_i, i \in \omega)$ and $(a_s, s \subseteq \omega)$ such that $\mathfrak{U} \models \varphi(a_s, b_i) \Leftrightarrow i \in s$.

(2) A theory T has IP if some formula does; otherwise T is called **NIP**.

Remark 2.35.
$\varphi(x,y)$ has IP implies $\varphi(x,y)$ has the order property. Hence T stable $\Rightarrow$ T NIP.

Example 2.36.

(1) The theory of the random graph has IP.

(2) O-minimal theories are NIP. (Fact: T is NIP iff $\varphi(x,y)$ is NIP for all φ with $|x| = 1$.)

Proposition 2.37.
A formula $\varphi(x,y)$ is NIP iff for any indiscernible sequence $\bar{b} = (b_i, i \in I)$ and a parameter a, the alternation of $\varphi(a,y)$ on $\bar{b}$ is finite, bounded by some number $n \in \omega$ depending only on φ. I.e. there are at most n indices $i_0 < \cdots < i_{n-1}$ such that $\mathfrak{U} \models \varphi(a, b_{i_s}) \Leftrightarrow \neg\varphi(a, b_{i_{s+1}})$ for all $s < n-1$.

Proof. ($\Rightarrow$): If $\varphi(x,y)$ has IP, let $(a_s, b_i : s \subseteq \omega, i \in \omega)$ be such that $\mathfrak{U} \models \varphi(a_s, b_i) \Leftrightarrow i \in s$. Let $S \subseteq \omega$ be the set of even numbers. Then $\varphi(a_S, y)$ has infinite alternation on $(b_i)_{i\in\omega}$.

($\Leftarrow$): Assume $\varphi(a,y)$ has infinite alternation on some indiscernible sequence $\bar{b} = (b_i)_{i\in\omega}$ (we can find such by saturation). Let $n \in \omega$ and $J \subseteq n$. Choose $i_0 < i_1 < \cdots < i_{n-1}$ such that i_k is even iff $k \in J$. Hence $\mathfrak{U} \models \varphi(a, b_{i_k}) \Leftrightarrow k \in J$. But $b_0 b_1 \cdots b_{n-1} \equiv b_{i_0} b_{i_1} \cdots b_{i_{n-1}}$ by indiscernibility. Let a_J be a witness. Then $\mathfrak{U} \models \varphi(a_J, b_k) \Leftrightarrow k \in J$. By saturation (since $n \in \omega$ was arbitrary), $\varphi(x,y)$ has IP. □

Remark 2.38 *(Average type in NIP)*.
Let T be NIP and $\bar{b} = (b_i, i \in I)$ an indiscernible sequence with I a linear order without endpoints. Let A be an arbitrary set of parameters. Then we can define a complete and consistent type:

$$\mathrm{Av}(\bar{b}/A) := \{\varphi(a,x) \in \mathcal{L}(A) : \text{the set } \{i \in I : \mathfrak{U} \models \varphi(a,b_i)\} \text{ is cofinal}\}.$$

If T is stable, this coincides with the definition of average type in stable theories.

Counting types

Proposition 2.39.

(1) If $\varphi(x,y)$ has IP, then for each cardinal κ, there is a set A of cardinality κ such that $|S_\varphi(A)| = 2^\kappa$.

(2) If $\varphi(x,y)$ is NIP, then for each cardinal κ and a set of parameters A, if $|A| = \kappa$ then $|S_\varphi(A)| \leq \operatorname{ded}\kappa$.

Remark 2.40.
Recall in the stable case:

(1) If $\varphi(x,y)$ has the order property, then for each κ there is A of cardinality κ with $|S_\varphi(A)| > \operatorname{ded}\kappa$.

(2) If $\varphi(x,y)$ is stable, then for each κ and A with $|A| = \kappa$, $|S_\varphi(A)| \leq |A|$.

Additional cardinal arithmetic: $\kappa < \operatorname{ded}\kappa \leq 2^\kappa$; $\operatorname{ded}\aleph_0 = 2^{\aleph_0}$. Under GCH, $\operatorname{ded}\kappa = 2^\kappa$ for all κ. Mitchell showed it is consistent that $\operatorname{ded}\kappa < 2^\kappa$ for κ of uncountable cofinality.

Lemma 2.41.
If $F \subseteq 2^\lambda$ and $|F| > \operatorname{ded}\lambda$, then for any $n < \omega$, there is some $I \subseteq \lambda$ such that $|I| = n$ and $F \restriction I := \{f \restriction I : f \in F\} = 2^I$.

Proof. Assume F, λ are counterexamples, with λ minimal. We regard F as a subset of branches of the tree $\bigcup_{i<\lambda} F \restriction i$. By minimality of λ, $|F \restriction i| \leq \operatorname{ded}\lambda$ for $i < \lambda$.

For each $f \in \bigcup_{i<\lambda} F \restriction i$, let $F(f) := \{g \in F : f \subseteq g\}$, $G_i := \{f \in F \restriction i : |F(f)| > \operatorname{ded}\lambda\}$, and $G := \{f \in F : f \restriction i \in G_i \text{ for all } i < \lambda\}$. Then $G \subseteq F$ is a subset of branches of the tree $\bigcup_{i<\lambda} G_i$. Since $F \setminus G = \bigcup_{i<\lambda} \bigcup_{f \in (F\restriction i)\setminus G_i} F(g)$ and $|F(g)| \leq \operatorname{ded}\lambda$, we get $|F \setminus G| \leq \lambda \times \operatorname{ded}\kappa \times \operatorname{ded}\kappa = \operatorname{ded}\kappa$. So $|G| > \operatorname{ded}\lambda$, and G is also a counterexample. WLOG $F = G$.

Now we prove by induction on $n < \omega$: for each $f \in \bigcup_{i<\lambda} F \restriction i$, there is $I_f \subseteq \lambda$ with $|I_f| = n$ and $F(f) \restriction I_f = 2^{I_f}$.

Consider $n+1$. Since $|F(f)|$ is a subset of branches of the tree $\bigcup_{i<\lambda} F(f) \restriction i$, and the lexicographic order on this tree has each branch determining a cut, we get $|F(f)| \leq \operatorname{ded}(|\bigcup_{i<\lambda} F(f) \restriction i|)$. By assumption $|F(f)| > \operatorname{ded}\lambda$, hence $|\bigcup_{i<\lambda} F(f) \restriction i| > \lambda$, namely there is $i < \lambda$ with $|F(f) \restriction i| > \lambda$.

By the induction hypothesis, for each $g \in F(f) \restriction i$, there is $I_g \subseteq \lambda$ with $|I_g| = n$ and $F(g) \restriction I_g = 2^{I_g}$. The set $\{I \subseteq \lambda : |I| = n\} = \lambda$. Hence there are $h \neq g \in F(f) \restriction i$ with $I_z := I_g = I_h$. Choose $j < i$ with $h(j) \neq g(j)$. Since $I \cap i = \emptyset$, $j \notin I$. Let $J := I \cup \{j\}$; then $F(f) \restriction J = 2^J$. ☐

Proof of Proposition 2.39. (1): If $\varphi(x,y)$ has IP, then by compactness, for any κ, we can find a set A of size κ such that for any $S \subseteq A$, there is some a_S with $\mathfrak{U} \models \varphi(a_S, b) \Leftrightarrow b \in S$ for all $b \in A$. Hence $|S_\varphi(A)| = 2^\kappa$.

(2): Assume $|A| = \kappa$ and $|S_\varphi(A)| > \operatorname{ded}\kappa$. Fix an enumeration $A = (a_i, i < \kappa)$. For each $p \in S_\varphi(A)$, define $f_p \in 2^\kappa$ by $f_p(i) = 0 \Leftrightarrow \varphi(x, a_i) \in p$. Let $F = \{f_p : p \in S_\varphi(A)\}$. Then $|F| > \operatorname{ded}\kappa$. By Lemma 2.41, for any n, there is $I \subseteq \kappa$ with $|I| = n$ and $F \restriction I = 2^I$. Thus for each $X \subseteq I$, $\{\varphi(x, a_i) : i \in X\} \cup \{\neg\varphi(x, a_i) : i \in I \setminus X\}$ is consistent (as it is contained in some φ-type p). Hence $\varphi(x, y)$ has IP. $\square$

Sauer–Shelah lemma and uniform definability in NIP

For φ-types over finite sets, NIP gives a polynomial bound.

Fact 2.42 *(Sauer–Shelah–Vapnik–Chervonenkis)***.**
A formula $\varphi(x, y)$ is NIP iff there exist $d, c \in \omega$ such that for any finite set A with $|A| = n$, we have $|S_\varphi(A)| \leq c \cdot n^d$.

Example 2.43.
An irrational cut of $\mathbb{Q}$ gives a type that is not definable. But in NIP theories, types over finite sets are uniformly definable.

Fact 2.44 *(Chernikov–Simon, 2015)***.**
Let T be NIP. Then types over finite sets are uniformly definable. I.e. for any formula $\varphi(x, y)$, there is $\psi(y, z) \in \mathcal{L}$ such that for any finite set $A \subseteq \mathfrak{U}^{|y|}$ (with $|A| \geq 2$) and any $p(x) \in S_\varphi(A)$, there is some tuple b from A such that $\varphi(x, a) \in p \Leftrightarrow \mathfrak{U} \models \psi(a, b)$ for all $a \in A$.

3 Forking Calculus

We will define "large" and "small" definable sets via ideals (small means belonging to a certain ideal, hence forking means changing from large to small). We will develop an **independence notion** (namely, not changing from large to small) which generalises linear independence in vector spaces and algebraic independence in fields.

3.1 Keisler measures and S_1-ideals

Definition 3.1 *(Keisler measure)***.**

(1) A **Keisler measure** (over a set of parameters A) is a finitely additive probability measure on the Boolean algebra of A-definable subsets of $\mathfrak{U}^n$ (denoted $\operatorname{Def}_n(A)$), for some n. I.e. a map $\mu \colon \operatorname{Def}_n(A) \to [0, 1]$ such that:

(a) $\mu(\mathfrak{U}^n) = 1$;

(b) $\mu(P \cup Q) = \mu(P) + \mu(Q)$ for $P \cap Q = \emptyset$, $P, Q \in \operatorname{Def}_n(A)$.

(2) A Keisler measure μ is **invariant over** B if $a \equiv_B b$ implies $\mu(\varphi(x,a)) = \mu(\varphi(x,b))$.

Remark 3.2.

(1) A type is a $\{0,1\}$-valued Keisler measure. (From now on, by "measure" we mean Keisler measure unless stated otherwise.)

(2) Recall: $\mathcal{I} \subseteq \mathcal{P}(X)$ is called an **ideal** if: (i) $\emptyset \in \mathcal{I}$; (ii) $A \subseteq B$ and $B \in \mathcal{I}$ implies $A \in \mathcal{I}$; (iii) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$.

Let $\mathcal{D}_\mu \subseteq \mathrm{Def}_n(\mathfrak{U})$ be the collection of definable sets of measure 0 with respect to a measure μ over A. Then $\mathcal{D}_\mu$ is an ideal.

Lemma 3.3 *(Extension of a type avoiding an ideal)***.**
Let $\mathcal{I} \subseteq \mathrm{Def}_n(\mathfrak{U})$ be an ideal. If a partial type $\pi(x)$ over a set A does not imply a formula from $\mathcal{I}$, then for any set $B \supseteq A$, there is a complete type $p(x)$ over B extending $\pi(x)$ not containing any formulas from $\mathcal{I}$.

Proof. Let $\tau(x) := \pi(x) \cup \{\neg\varphi(x,b) : b \in B \text{ and } \varphi(x,b) \in \mathcal{I}\}$.

Claim: $\tau(x)$ is consistent. If not, there are $\varphi_i(x,b_i)$, $i \leq N$, with $\varphi_i(x,b_i) \in \mathcal{I}$, such that $\pi(x) \models \bigvee_{0 \leq i \leq N} \varphi_i(x,b_i)$. Since $\mathcal{I}$ is closed under finite unions, $\bigvee_{0\leq i\leq N} \varphi_i(x,b_i) \in \mathcal{I}$, contradicting the assumption on $\pi(x)$. △

Hence any complete type over B extending $\tau(x)$ does not contain formulas from $\mathcal{I}$. □

Definition 3.4 *(S_1-ideal (Hrushovski))***.**
An A-invariant ideal $\mathcal{I} \subseteq \mathrm{Def}_n(\mathfrak{U})$ is S_1 if for any formula $\varphi(x,y) \in \mathcal{L}(A)$ and indiscernible sequence $(a_i : i \in \omega)$ over A: if $\varphi(x,a_i) \wedge \varphi(x,a_j) \in \mathcal{I}$ for $i \neq j$, then some $\varphi(x,a_i) \in \mathcal{I}$.

Remark 3.5.
Measure-0 ideals are S_1-ideals.

Proof. Let μ be a Keisler measure invariant over A and $\mathcal{D}_\mu$ the ideal of measure-0 definable sets. Suppose $\varphi(x,a_i) \wedge \varphi(x,a_j) \in \mathcal{D}_\mu$ for some indiscernible $(a_i : i \in \omega)$ over A. Suppose $\varphi(x,a_i) \notin \mathcal{D}_\mu$; then $\mu(\varphi(x,a_i)) > 0 \geq 1/n$ for some $0 < n \in \omega$. Take $a_0, a_1, \ldots, a_{n+1}$ and consider $\mu\left(\bigcup_{0\leq i\leq n-1} \varphi(x,a_i)\right) = \sum_{0\leq i\leq n-1} \mu(\varphi(x,a_i)) > 1$, a contradiction. □

3.2 Dividing and forking

Definition 3.6 *(Dividing and forking)***.**

(1) A formula $\varphi(x,a)$ **divides over** B if there is a sequence $(a_i)_{i\in\omega}$ and $k \in \omega$ such that $a_i \equiv_B a$ and $\{\varphi(x,a_i)\}_{i\in\omega}$ is k-inconsistent. Equivalently, if there is a B-indiscernible sequence $(a_i)_{i\in\omega}$ starting with a such that $\{\varphi(x,a_i)\}_{i\in\omega}$ is inconsistent.

(2) A formula $\varphi(x,a)$ **forks over** B if it belongs to the ideal generated by the formulas dividing over B, i.e. if there are $\psi_i(x,c_i)$ dividing over B for $i<n$ such that

$$\varphi(x,a) \vdash \bigvee_{i<n} \psi_i(x,c_i).$$

(3) We denote $\mathrm{F}(B)$ the ideal of formulas forking over B. It is invariant over B.

Example 3.7.
Let T be DLO, then $a<x$ doesn't divide, since any indiscernible sequence $(a_i, i\in\omega)$, $\{a_i<x\}_{i\in\omega}$ is consistent. But $a<x<b$ divides. One can take $(a_i,b_i)_{i\in\omega}$ pairwise disjoint and $a_i<b_i$. Then $\{a_i<x<b_i\}_{i\in\omega}$ are 2-inconsistent. (In general, these formulas do not divide but fork.)

Definition 3.8 *(Non-dividing / non-forking type)***.**
A partial type does not divide (fork) over B if it does not imply any formula which divides (resp. forks) over B.

Remark 3.9.
If $a\notin \mathrm{acl}(A)$, then $\mathrm{tp}(a/Aa)$ divides over A (with the formula $x=a$). Also, if $\pi(x)$ is consistent and defined over $\mathrm{acl}(A)$, then it doesn't divide over A (since if $\psi(x,\bar{a})$ with $\bar{a}\in\mathrm{acl}(A)$, then any indiscernible sequence over A starting with $\bar{a}$ is constant).

Proposition 3.10.
$\mathrm{F}(B)$ is contained in any S_1-ideal invariant over B.

Proof. It is enough to show that if $\psi(x,a)$ divides over B and $\mathcal{I}$ is S_1, then $\psi(x,a)\in\mathcal{I}$.

By definition of dividing, there is an indiscernible sequence $(a_i)_{i\in\omega}$ with $a_0=a$ such that $\{\psi(x,a_i)\}_{i\in\omega}$ is inconsistent. We want to show $\psi(x,a_0)\in\mathcal{I}$. Suppose $\psi(x,a_0)\notin\mathcal{I}$. Then $\psi(x,a_i)\notin\mathcal{I}$ by B-invariance. By the definition of S_1-ideal ($\psi(x,a_i)\wedge\psi(x,a_j)\in\mathcal{I}$ for $i\neq j$, then $\psi(x,a_i)\in\mathcal{I}$), we have $\psi(x,a_i)\wedge\psi(x,a_j)\notin\mathcal{I}$ for $i<j$. Now let $\psi_0 := \psi(x,y)\wedge\psi(x,z)$; we have $\psi_0(x,a_0a_1)\notin\mathcal{I}$. Note that $(a_0a_1, a_2a_3, a_4a_5,\dots)$ is a B-indiscernible sequence, hence $\psi_0(x,a_{2n}a_{2n+1})\notin\mathcal{I}$. By the same argument, we have $\psi_0(x,a_0a_1)\wedge\psi_0(x,a_2a_3)\notin\mathcal{I}$. By induction, for any $n\in\omega$, $\bigwedge_{i<2^n}\psi(x,a_i)\notin\mathcal{I}$.

But by assumption $(\psi(x,a_i))_{i\in\omega}$ is inconsistent, i.e. $\bigwedge_{i<n}\psi(x,a_i)$ is inconsistent, hence in $\mathcal{I}$ (as $\mathcal{I}$ contains the empty set), a contradiction. $\square$

Definition 3.11 *($\mathrm{SI}(A)$ and $\mathrm{O}(A)$)***.**

(1) Let $\mathrm{SI}(A)$ be the smallest A-invariant S_1-ideal. (By Remark 3.5, S_1-ideals are closed under intersections among A-invariant ones.)

(2) Let $\mathrm{O}(A)$ be the ideal of formulas which have measure 0 with respect to every A-invariant Keisler measure.

Proposition 3.12.
In any theory, for any set A, we have $\mathrm{F}(A) \subseteq \mathrm{SI}(A) \subseteq \mathrm{O}(A)$.

Proof. $\mathrm{F}(A) \subseteq \mathrm{SI}(A)$ by Proposition 3.10. By Remark 3.5, $\mathrm{SI}(A) \subseteq \mathcal{D}_\mu(A)$ for any A-invariant Keisler measure μ. Hence $\mathrm{SI}(A) \subseteq \mathrm{O}(A)$. $\square$

3.3 Special extensions of types

Goal: Let $A \subseteq B$ and $p(x) \in S_n(A)$. We want to extend $p(x)$ to a $q(x) \in S_n(B)$ such that $q(x)$ is a "generic" extension.

Definition 3.13 *(Invariant type)*.
A global type $p(x) \in S(\mathfrak{U})$ is called **invariant over** C if for every $a \equiv_C b$ and $\varphi(x,y) \in \mathcal{L}$, $\varphi(x,a) \in p \Leftrightarrow \varphi(x,b) \in p$. Equivalently, $p(\mathfrak{U})$ is invariant under all automorphisms fixing C.

Remark 3.14.
Let $p(x)$ be an A-invariant type, then $\mathcal{D}_p := \{\varphi(x,a) : \varphi(x,a) \notin p\}$ is the measure-0 ideal of the Keisler measure given by p. Hence $\mathcal{D}_p \supseteq \mathrm{O}(A) \supseteq \mathrm{F}(A)$. Namely, $p(x)$ does not fork over A.

Definition 3.15 *(Heir and coheir)*.
Let $A \subseteq B$, $p \in S_n(A)$ and $q \in S_n(B)$ extending p (i.e. $q \restriction \mathrm{Def}_n(A) = p$). We denote $q \restriction A$.

(1) We say q is an **heir** of p (or "an heir over A") if for every formula $\varphi(x,y) \in \mathcal{L}(A)$, if $\varphi(x,b) \in q$ for some $b \in B$, then $\varphi(x,b') \in p$ for some $b' \in A$. (Note: if q is an heir of p, then A has to be a model of T, B is a model, and every formula is represented by a formula in p.)

(2) We say that q is a **coheir** of p ("coheir over A", finitely satisfiable in A) if for any $\varphi(x,b) \in q$ there is some $a \in A$ such that $\mathfrak{U} \models \varphi(a,b)$.

Remark 3.16.
(1) $\mathrm{tp}(a/Mc)$ is an heir of $\mathrm{tp}(a/M)$ iff $\mathrm{tp}(c/Ma)$ is a coheir of $\mathrm{tp}(c/M)$.

(2) If $q \in S_n(B)$ is definable over a submodel $M \subseteq B$, then q is an heir over M.

Example 3.17.
Let $M = (\mathbb{Q}, <)$, consider $p \in S(M)$ given by $\{a < x : a \in M\}$, i.e. the type of $+\infty$. Consider 2 extensions to $\mathfrak{U}$:

(1) $q_1(x) := \{a < x : a \in \mathfrak{U}\}$.

(2) $q_2(x) := p(x) \cup \{x < b : M < b \in \mathfrak{U}\}$.

Then $q_1(x)$ is an heir of $p(x)$, but not finitely satisfiable in M, hence not a coheir. $q_2(x)$ is a coheir, but $x < b$ for $b > M$ is not "represented" in $p(x)$ (there's no such $x < a$ in

$p(x)$ for any $a \in M$).

Proposition 3.18 *(Finitely satisfiable types).*

(1) A type p over $B \supseteq A$ is finitely satisfiable in A iff there is an ultrafilter $\mathcal{U}$ on $\mathcal{P}(A)$ such that for any $b \in B^{|y|}$, $\varphi(x,b) \in p \Leftrightarrow \varphi(A,b) \in \mathcal{U}$.

(2) If p is finitely satisfiable in A, then p is A-invariant.

Proof. (1): Suppose p is finitely satisfiable in A. Let $\mathcal{U}' := \{\varphi(A,b) : \varphi(x,b) \in q\}$. By finitely satisfiability, if $\varphi(x,b) \in q$, then $\varphi(A,b) \neq \emptyset$. So $\mathcal{U}'$ has the finite intersection property. Extend $\mathcal{U}'$ to an ultrafilter $\mathcal{U}$. Then by definition, $\varphi(x,b) \in q \Rightarrow \varphi(A,b) \in \mathcal{U}$, and if $\varphi(x,b) \notin q$, then $\neg\varphi(x,b) \in q$ and $\neg\varphi(A,b) \in \mathcal{U}$; hence $\varphi(A,b) \notin \mathcal{U}$ (note $\varphi(A,b) = A \setminus \neg\varphi(A,b)$).

Conversely, if $\mathcal{U}$ is an ultrafilter on $\mathcal{P}(A)$, let $p_{\mathcal{U}}$ be given by $\varphi(x,b) \in p_{\mathcal{U}}$ iff $\varphi(A,b) \in \mathcal{U}$. Then $p_{\mathcal{U}}$ is clearly finitely satisfiable in A. It is a complete type by $\mathcal{U}$ being an ultrafilter. If $\mathcal{U}$ is a principal ultrafilter generated by $\{a\}$ with $a \in A^n$, then $p_{\mathcal{U}} = \mathrm{tp}(a/B)$.

(2): Suppose p finitely satisfiable over A and $b \equiv_A b'$ but $\varphi(x,b) \in p$, $\neg\varphi(x,b') \in p$. Then $\varphi(x,b) \wedge \neg\varphi(x,b') \in p$. Consider $a \in A^n$ such that $\varphi(a,b) \wedge \neg\varphi(a,b')$. By finitely satisfiability of p, then $b' \not\equiv_A b$, contradiction. $\square$

Proposition 3.19 *(Existence of global coheirs and heirs).*

Let $p \in S_n(M)$ be an arbitrary type over a model $M \models T$.

(1) There is a global coheir q of p.

(2) There is a global heir r of p.

Proof. (1): Note that a type over a model is always finitely satisfiable in this model. Indeed, if $\varphi(x,a) \in p$, then $\mathfrak{U} \models \exists x\, \varphi(x,a)$, since $a \in M^{|y|}$; $M \models \exists x\, \varphi(x,a)$ and hence $\mathfrak{U} \models \varphi(b,a)$ for some $b \in M^{|x|}$. By Proposition 3.18, there is an ultrafilter $\mathcal{U}$ on $\mathcal{P}(M)$ such that $\varphi(x,b) \in p \Leftrightarrow \varphi(M,b) \in \mathcal{U}$ for any $b \in M^{|y|}$. Let q be given by $\varphi(x,b') \in q \Leftrightarrow \varphi(M,b') \in \mathcal{U}$ for $q \in \mathfrak{U}^{|y|}$. Then q is finitely satisfiable in M.

(2): We want to find $r \in S_n(\mathfrak{U})$ such that $\varphi(x,b) \in r \Rightarrow \exists b' \in M,\ \varphi(x,b') \in p$. I.e. if for all $b' \in M$, $\varphi(x,b') \notin p$, then $\varphi(x,b) \notin r$; i.e. $\forall b' \in M,\ \neg\varphi(x,b') \in p \Rightarrow \neg\varphi(x,b) \in r$. Let

$$s(x) := p(x) \cup \{\neg\varphi(x,c) : c \in \mathfrak{U},\ \varphi(x,y) \in \mathcal{L}(M),\ \text{for all } m \in M,\ \varphi(x,m) \notin p\}.$$

If $s(x)$ is consistent, then any complete type extending $s(x)$ is an heir of $p(x)$.

Claim: $s(x)$ is consistent.

If not, there is $\varphi(x,b) \in \mathcal{L}(M)$ and $(\varphi_i(x,c_i))_{i \leq N}$ such that for all $m \in M$, $\varphi_i(x,m) \notin p$ and $\varphi(x,b) \vdash \bigvee_{i \leq N} \varphi_i(x,c_i)$. Hence $M \models \varphi(x,b) \to \exists y_0 \ldots y_N \bigvee_{i \leq N} \neg\varphi_i(x,y_i)$. Let $m_0, \ldots, m_N \in M$ such that $M \models \varphi(x,b) \to \bigvee_{i \leq N} \neg\varphi_i(x,m_i)$. Since $\varphi(x,b) \in p$ and p is a complete type over M, we get $\bigvee_{i\leq N} \neg\varphi_i(x, m_i) \in p$, contradicting $\varphi_i(x, m_i) \notin p$ for all $i \leq N$. $\square$

Proposition 3.20 *(Definable type has unique global heir).*
Let $p \in S_n(M)$ be a definable type over some model M. Then it has a unique global heir $q \supseteq p$, which in fact is definable over M.

Proof. First we show the global heir exists and is definable over M.

For $\varphi(x, y) \in \mathcal{L}$, let $d_\varphi(y) \in \mathcal{L}(M)$ be the formula defines p with respect to φ. I.e. $M \models d_\varphi(a) \Leftrightarrow \varphi(x, y) \in p$ for all $a \in M^{|y|}$. Let

$$q(x) := \{\varphi(x, a) : \varphi(x, y) \in \mathcal{L},\ a \in \mathfrak{U}^{|y|},\ \mathfrak{U} \models d_\varphi(a)\}.$$

Claim: $q(x)$ is consistent. If not, then there are φ_i and a_i, $i \leq N$, such that $\mathfrak{U} \models \bigwedge_{i\leq N} d_{\varphi_i}(a_i) \wedge (\neg\exists x \bigwedge_{i\leq N} \varphi_i(x, a_i))$. Hence $M \models \exists y_0 \ldots y_N \bigwedge_{i\leq N} d_{\varphi_i}(y_i) \wedge (\neg\exists x \bigwedge_{i\leq N} \varphi_i(x, y_i))$. Let $b_0, \ldots, b_N \in M^{|y|}$ such that $M \models \bigwedge_{i\leq N} d_{\varphi_i}(b_i) \wedge (\neg\exists x \bigwedge_{i\leq N} \varphi_i(x, b_i))$. By definition of d_{φ_i}, we get $\varphi_i(x, b_i) \in p$ for $i \leq N$. Hence $\bigwedge_{i\leq N} \varphi_i(x, b_i)$ is consistent, in particular realised in M, a contradiction.

Now, if $\varphi(x, b) \in q$ then $\mathfrak{U} \models d_\varphi(b)$ holds and $\exists b' \in M$, $M \models d_\varphi(b')$, thus $\varphi(x, b') \in p$. So q is an heir of p.

Uniqueness. If $r \supseteq p$ is an heir. Suppose $\varphi(x, b) \in q$ and $\neg\varphi(x, b) \in r$, hence $\neg d_\varphi(b) \wedge \neg\varphi$; then $\mathfrak{U} \models \neg d_\varphi(b)$. By $\varphi(x, b) \in q$, $\mathfrak{U} \models d_\varphi(b)$. By heir of p, there is $a \in M$, $d_\varphi(a) \wedge \neg\varphi(x, a) \in r$. So $d_\varphi(a) \wedge \neg\varphi(x, a) \in p$, a contradiction. $\square$

Proposition 3.21 *(Properties of global invariant types).*
Let $p \in S_n(\mathfrak{U})$ be a global A-invariant type.

(1) If p is definable, then it is definable over A.

(2) If p is finitely satisfiable in some small set, then it is finitely satisfiable in any model $M \supseteq A$.

Proof. (1): For any $\varphi(x, y) \in \mathcal{L}$, there is $d_\varphi(y) \in \mathcal{L}(\mathfrak{U})$ such that $\varphi(x, b) \in p \Leftrightarrow \mathfrak{U} \models d_\varphi(b)$ for any $b \in \mathfrak{U}^{|y|}$. By assumption, p is A-invariant. Hence $d_\varphi(\mathfrak{U})$ is an A-invariant definable set, hence A-definable by Lemma 1.2.6.

(2): Suppose p is finitely satisfiable in some small model N. Let M be a small model containing A. Let $\varphi(x, b) \in p$. Consider $\mathrm{tp}(N/M)$ as a type in $|N|$-variables. Then by Proposition 3.19, $\mathrm{tp}(N/M)$ has a global coheir $r(\bar{x})$. Let $N_1 \models r\restriction_{Mb}$. Then there is $\sigma \in \mathrm{Aut}(\mathfrak{U}/M)$ such that $\sigma(N) = N_1$ (pointwise). As $\varphi(x, b) \in p$, by A-invariance and $M \supseteq A$, $\varphi(x, \sigma(b)) \in p$. Since p is finitely satisfiable in N, there is $\tilde{n} \in N$ with $\mathfrak{U} \models \varphi(\tilde{n}, \sigma(b))$. Thus $\mathfrak{U} \models \varphi(\sigma(\tilde{n}), b)$ and $\sigma(\tilde{n}) \in N_1$.

By construction, $\operatorname{tp}(N_1/Mb)$ is finitely satisfiable in M, and $\varphi(x\tilde{n}, b) \in \operatorname{tp}(N_1/Mb)$, hence there is $\hat{m} \in M$ with $\mathfrak{U} \models \varphi(\hat{m}, b)$ as desired. $\square$

Summary.

(1) Let $p \in S_n(\mathfrak{U})$ be a global type. If p is definable or finitely satisfiable in A, then p is A-invariant and does not fork over A.

(2) Conversely, any type $p(x)$ over a model $M \models T$ can be extended to a global type $q(x)$ which is an heir or a coheir of $p(x)$; hence $q(x)$ doesn't fork over M (in the latter case, when $q(x)$ is a coheir).

(3) If $p(x)$ is a type over B such that $p(x)$ does not fork over B, then by Lemma 3.3, $p(x)$ can be extended to a global type $q(x)$ which does not fork. (But in general, a type over B can fork over B.)

3.4 Forking independence in arbitrary theories

Definition 3.22 *(Forking independence)*.
In an arbitrary theory, we define a ternary relation $\mathop{\smile\!\!\!\!\!\mid}$ (independence) on small subsets of the monster by:

$$a \mathop{\smile\!\!\!\!\!\mid}_C b \iff \operatorname{tp}(a/bC) \text{ does not fork over } C,$$

where $a, b, C \subseteq \mathfrak{U}$ are small subsets. (When we write $\operatorname{tp}(a/bC)$, we regard a as a tuple by some fixed enumeration.)

Now we come to the following (left) transitivity property of $\mathop{\smile\!\!\!\!\!\mid}$:

$$a \mathop{\smile\!\!\!\!\!\mid}_C b \text{ and } a' \mathop{\smile\!\!\!\!\!\mid}_{aC} b \implies aa' \mathop{\smile\!\!\!\!\!\mid}_C b.$$

We will first show it for dividing.

Lemma 3.23 *(Characterising dividing via indiscernibility)*.
The following are equivalent:

(1) $\operatorname{tp}(a/Ab)$ does not divide over A.

(2) For every infinite A-indiscernible sequence I such that $b \in I$, there is some $a' \equiv_{Ab} a$ such that I is Aa'-indiscernible.

(3) For every infinite A-indiscernible sequence I such that $b \in I$, there is some $J \equiv_{Ab} I$ such that J is Aa-indiscernible.

Proof. The equivalence of (2) and (3) follows by taking A-automorphisms.

(3) $\Rightarrow$ (1): Recall $\operatorname{tp}(a/Ab)$ divides iff there is $\psi(x, b') \in \mathcal{L}(\mathfrak{U})$ such that $\operatorname{tp}(a/Ab) \vdash \psi(x, b')$ and $\psi(x, b')$ divides over A.

Claim: If $\psi(x, b) \vdash \psi(x, b')$ and $\psi(x, b')$ divides over A, then $\psi(x, b)$ divides over A.

Proof of Claim: By definition, there are $(b'_i)_{i\in\omega}$ such that $b'_i \equiv_A b'$ and $\{\psi(x,b'_i)\}_{i\in\omega}$ is k-inconsistent. Let $(b_i)_{i\in\omega}$ be such that $b_i b'_i \equiv_A bb'$ (by taking automorphisms). Then $\psi(x,b_i) \vdash \psi(x,b'_i)$ and clearly $b_i \equiv_A b$ for all i, so $\{\psi(x,b_i)\}_{i\in\omega}$ is k-inconsistent. $\triangle$

Thus, we may assume some $\psi(x,b) \in \mathrm{tp}(a/Ab)$ divides over A. I.e. there is an A-indiscernible sequence $(b_i)_{i\in\omega}$ with $b = b_0$ such that $\{\psi(x,b_i)\}_{i\in\omega}$ is inconsistent. Let $J = (c_j)_{j\in\omega} \equiv_{Ab} (b_i)_{i\in\omega}$ be given by (3), such that J is Aa-indiscernible. Since $c_0 = b_0 = b$, we have $\mathfrak{U} \models \psi(a,c_0)$. Then $\mathfrak{U} \models \psi(a,c_i)$ for all $i \in \omega$ (by $(c_j)_{j\in\omega}$ being Aa-indiscernible). Thus $\mathfrak{U} \models \exists x \bigwedge_{0\leq i<k} \psi(x,c_i)$, contradicting $(c_j)_{j\in\omega} \equiv_{Ab} (b_i)_{i\in\omega}$.

(1) $\Rightarrow$ (2): Let $p(x,b) := \mathrm{tp}(a/b)$, let $(b_i)_{i\in I}$ be an infinite A-indiscernible sequence. Let $r(x,(b_i)_{i\in I})$ be the union of $p(x,b)$ and the assertion that $(b_i)_{i\in I}$ is indiscernible over xA, i.e. for any formula $\psi(x,y_0,\dots,y_n)$ and $i_0 < \cdots < i_n$ in I, $j_0 < \cdots < j_n$ in I,

$$\psi(x,b_{i_0},\dots,b_{i_n}) \leftrightarrow \psi(x,b_{j_0},\dots,b_{j_n})$$

is in $r(x,(b_i)_{i\in I})$.

It is enough to show $r(x,(b_i)_{i\in I})$ is consistent. Let $\Delta(x,b_{\tilde{i}_0},\dots,b_{\tilde{i}_K})$ be a finite subset of formulas in $r(x,(b_i)_{i\in I})$ where $\tilde{i}_0 < \cdots < \tilde{i}_K$ in I. To show $\Delta(x,b_{\tilde{i}_0},\dots,b_{\tilde{i}_K})$ is consistent, it is enough to find some $\tilde{j}_0 < \cdots < \tilde{j}_K$ in I such that $\Delta(x,b_{\tilde{j}_0},\dots,b_{\tilde{j}_K})$ is consistent, since $(b_i)_{i\in I}$ is indiscernible over A.

By (1), for any $\varphi(x,y)$ with $\varphi(x,b) \in p(x,b)$, the set $\{\varphi(x,b_i)\}_{i\in I}$ is consistent, as $(b_i)_{i\in I}$ is A-indiscernible. Let $q(x) := \bigcup_{i\in I}\{\varphi(x,b_i) : \varphi(x,b) \in p(x,b)\}$. Then $q(x)$ is consistent. Let $c \in \mathfrak{U}^{|x|}$ be a realisation of $q(x)$. Now colour the tuples $\tilde{i}_0 < \cdots < \tilde{i}_K$ in I by their values on $\psi(c,b_{\tilde{i}_0},\dots,b_{\tilde{i}_K})$ where $\psi(x,b_{j_0},\dots,b_{j_n}) \leftrightarrow \psi(x,b_{j'_0},\dots,b_{j'_n})$ is in Δ for some $j'_0 < \cdots < j'_n$ (note $n \leq K$). Then by Ramsey's theorem, there is an infinite $I_0 \subseteq I$ such that $\psi(c,b_{\tilde{i}_0},\dots,b_{\tilde{i}_n}) \Leftrightarrow \psi(c,b_{\tilde{j}_0},\dots,b_{\tilde{j}_n})$ holds in $\mathfrak{U}$ for any $\tilde{i}_0 < \cdots < \tilde{i}_n$ and $\tilde{j}_0 < \cdots < \tilde{j}_n$ in I_0. Thus $\Delta(x,b_{\tilde{i}_0},\dots,b_{\tilde{i}_K})$ is consistent for any $\tilde{i}_0 < \cdots < \tilde{i}_K$ in I_0. $\square$

Corollary 3.24 *(Left transitivity of non-dividing).*
If $\mathrm{tp}(a/B)$ does not divide over $A \subseteq B$ and $\mathrm{tp}(b/Ba)$ does not divide over Aa, then $\mathrm{tp}(ab/B)$ does not divide over A.

Proof. Let I be an A-indiscernible sequence with $B \in I$. By $\mathrm{tp}(a/B)$ does not divide over A and equivalence (1) $\Leftrightarrow$ (3) in Lemma 3.23, there is $I' \equiv_B I$ such that I' is Aa-indiscernible. By $\mathrm{tp}(b/Ba)$ does not divide over Aa, there is $I'' \equiv_B I'$ such that I'' is Aab-indiscernible. Since $I'' \equiv_B I' \equiv_B I$ and $B \in I$, we also have $B \in I''$ and I'' is Aab-indiscernible. We conclude by Lemma 3.23 again, $\mathrm{tp}(ab/B)$ does not divide over A. $\square$

Proposition 3.25 *(Properties of $\mathop{\smile\!\!\!\!\!\!|}$ in arbitrary theories).*

(1) **Invariance under automorphisms:** $a \mathop{\smile\!\!\!\!\!\!|}_C b$ iff $\sigma(a) \mathop{\smile\!\!\!\!\!\!|}_{\sigma(C)} \sigma(b)$ for any $\sigma \in \mathrm{Aut}(\mathfrak{U})$.

(2) **Finite character:** $a \not\mathop{\smile\!\!\!\!\!\!|}_C b$ implies there are finite $a' \subseteq a$, $b' \subseteq b$ such that $a' \not\mathop{\smile\!\!\!\!\!\!|}_C b'$.

(3) **Monotonicity:** $aa' \mathop{\smile\!\!\!\!\!\!\mid}_C bb'$ implies $a \mathop{\smile\!\!\!\!\!\!\mid}_C b$.

(4) **Base monotonicity:** $a \mathop{\smile\!\!\!\!\!\!\mid}_C bb'$ implies $a \mathop{\smile\!\!\!\!\!\!\mid}_{Cb} b'$.

(5) **Left transitivity:** $a \mathop{\smile\!\!\!\!\!\!\mid}_C b$ and $a' \mathop{\smile\!\!\!\!\!\!\mid}_{aC} b$ implies $aa' \mathop{\smile\!\!\!\!\!\!\mid}_C b$.

(6) **Right extension:** If $a \mathop{\smile\!\!\!\!\!\!\mid}_C b$, then for any d, there is $d' \equiv_{bC} d$ such that $a \mathop{\smile\!\!\!\!\!\!\mid}_C bd'$; equivalently, there is $a' \equiv_{bC} a$ such that $a' \mathop{\smile\!\!\!\!\!\!\mid}_C bd$.

Proof. (1) is obvious. (2), (3), (4) are exercises.

(6): If $\mathrm{tp}(a/bC)$ does not fork over C, by Lemma 3.3, $\mathrm{tp}(a/bC)$ can be extended to a complete type $q(x)$ over bCd such that $q(x)$ does not fork over C. Let $a' \models q(x)$, then $a' \mathop{\smile\!\!\!\!\!\!\mid}_C bd$ and $a' \equiv_{bC} a$.

(5): Let $M_1 \supseteq bC$ be $(|C| + |b| + |T|)^+$-saturated. By (6), there is $\tilde{a} \equiv_{bC} a$ such that $\tilde{a} \mathop{\smile\!\!\!\!\!\!\mid}_C M_1$. Let $\sigma\colon \mathfrak{U} \to \mathfrak{U}$ be an automorphism fixing bC such that $\sigma(a) = \tilde{a}$; then $\sigma(a) \mathop{\smile\!\!\!\!\!\!\mid}_{\sigma(C)} \sigma(b)$, i.e. $\sigma(a') \mathop{\smile\!\!\!\!\!\!\mid}_{\tilde{a}C} b$.

By (6), there is $\tilde{a}' \equiv_{\tilde{a}Cb} \sigma(a')$ such that $\tilde{a}' \mathop{\smile\!\!\!\!\!\!\mid}_{\tilde{a}C} M_1$, in particular $\tilde{a}' \mathop{\smile\!\!\!\!\!\!\mid}_{\tilde{a}C} b$. Hence $\mathrm{tp}(\tilde{a}/M_1)$ does not divide over C, and $\mathrm{tp}(\tilde{a}'/M_1\tilde{a})$ does not divide over $\tilde{a}C$. By Corollary 3.24, $\mathrm{tp}(\tilde{a}'\tilde{a}/M_1)$ does not divide over C. Note that $\tilde{a}'\tilde{a} \equiv_{Cb} \sigma(a')\sigma(a) \equiv_{Cb} a'a$.

Claim: $\mathrm{tp}(\tilde{a}'\tilde{a}/M_1)$ does not fork over C (in particular $\tilde{a}'\tilde{a} \mathop{\smile\!\!\!\!\!\!\mid}_C b$ and $a'a \mathop{\smile\!\!\!\!\!\!\mid}_C b$).

Suppose not, then there are $\varphi(x, m) \in \mathrm{tp}(\tilde{a}'\tilde{a}/M_1)$ and $\varphi_i(x, b_i)$, $i \leq N$, with $b_i \in \mathfrak{U}^{|y_i|}$, such that:

(i) $\varphi(x, m) \vdash \bigvee_{i \leq N} \varphi_i(x, b_i)$.

(ii) $\varphi_i(x, b_i)$ divides over C by sequences $(c^i_j)_{j \in \omega}$.

Let $\tau(x, m, b_i, (c^i_j)_{j\in\omega}, i \leq N)$ be the partial type stating (i) and (ii). Then by $(|C| + |b| + |T|)^+$-saturation, we may assume $(b_i, (c^i_j)_{j\in\omega}, i \leq N)$ live in M_1. Thus $\varphi_i(x, b_i) \in \mathrm{tp}(\tilde{a}'\tilde{a}/M_1)$ for some $i \leq N$ (since $\varphi(x, m) \in \mathrm{tp}(\tilde{a}'\tilde{a}/M_1)$ and $\varphi(x, m) \vdash \bigvee_{i \leq N} \varphi_i(x, b_i)$). Then $\varphi_i(x, b_i)$ divides over C, contradicting $\mathrm{tp}(\tilde{a}'\tilde{a}/M_1)$ does not divide over C. $\square$

4 Simple Theories

4.1 Simplicity (Tent–Ziegler)

Definition 4.1 *(Tree property and simplicity)*.

(1) A formula $\psi(x, y)$ has the k**-tree property** if there is a tree of parameters $(a_s : \emptyset \neq s \in \omega^{<\omega})$ such that:

(a) for all $s \in \omega^{<\omega}$, $(\psi(x, a_{s\frown i}) : i < \omega)$ is k-inconsistent;

(b) for all $\sigma \in \omega^\omega$, $\{\psi(x, a_s) : \emptyset \neq s \subseteq \sigma\}$ is consistent.

(2) A formula $\psi(x, y)$ is said to have the **tree property** if it has the k-tree property for some $k \geq 2$.

(3) A theory T is **simple** if there is no formula $\psi(x, y)$ with the tree property.

Examples:

(1) If $\psi(x, y)$ has the tree property, then it has the order property. In particular, stable theories are simple.

(2) The theory of the random graph is simple.

Definition 4.2 *(Δ-k-dividing sequence)***.**
Let Δ be a finite set of formulas $\psi_i(x, y)$ without parameters. A **Δ-k-dividing sequence over** A is a sequence $(\psi_i(x, a_i) : i \in I)$ such that:

(1) $\psi_i(x, y) \in \Delta$;

(2) $\psi_i(x, a_i)$ divides over $A \cup \{a_j : j < i\}$ with respect to k;

(3) $\{\psi_i(x, a_i) : i \in I\}$ is consistent.

Lemma 4.3.
(1) If ψ has the k-tree property, then for every A and λ, there exists a ψ-k-dividing sequence over A indexed by λ.

(2) If no $\psi \in \Delta$ has the k-tree property, there is no infinite Δ-k-dividing sequence over $\emptyset$.

Proof. (1): It is enough to show for λ when λ is a limit ordinal. Take α an ordinal such that $\alpha > 2^{|A|+|T|+\lambda}$. Build a tree $(a_s : \emptyset \neq s \in \alpha^{<\lambda})$ such that:

(i) $(\psi(x, a_{s\frown i}) : i < \alpha)$ is k-inconsistent for all $s \in \alpha^{<\lambda}$;

(ii) $\{\psi(x, a_s) : \emptyset \neq s \subseteq \sigma\}$ is consistent for all $\sigma \in \alpha^{\lambda}$.

It exists by $\psi(x, y)$ having the k-tree property and saturation. Now we will recursively define a path $\sigma \in \alpha^{\lambda}$ such that $(\psi(x, a_{\sigma\restriction_{i+1}}) : i \in \lambda)$ is a ψ-k-dividing sequence over A.

We only need to define $a_{\sigma\restriction_i}$ when i is a successor ordinal, and for i a limit ordinal, choose any extension of $(a_{\sigma\restriction_j} : j < i)$.

Suppose $(\psi(x, a_{\sigma\restriction_j}) : j < i)$ has been defined. Let $B := A \cup \{a_{\sigma\restriction_j} : j \leq i\}$, then $|B| \leq |A| + \lambda$. Let $s' := \sigma \restriction i$. Then by the Pigeonhole principle, there is an infinite set $I \subseteq \alpha$ such that $\{a_{s'\frown j} : j \in I\}$ are of the same type over B. (Since $|\{a_{s'\frown j} : j < \alpha\}| = \alpha$ and there are at most $2^{|A|+\lambda+|T|}$-many types over B.) Let $\sigma(i + 1) = s' \frown j$ for some chosen $j \in I$. Then $\psi(x, a_{\sigma\restriction_{i+1}})$ k-divides over B witnessed by $(a_{s'\frown j'})_{j' \in I}$.

(2): Suppose there is an infinite Δ-k-dividing sequence over $\emptyset$, say $(\psi_i(x, a_i))_{i\in\omega}$. Let $\psi \in \Delta$ be such that there is an infinite $I \subseteq \omega$ with $\psi_i(x, y) = \psi(x, y)$ for all $i \in I$. By Δ

being a finite set of formulas, then $(\psi(x,a_i))_{i\in I}$ is an infinite ψ-k-dividing sequence. We may assume $I=\omega$.

By assumption, for each i, there is a sequence $(a_i^n : n\in\omega)$ such that $a_i^n \equiv_{(a_j)_{j<i}} a_i$ and $(\psi(x,a_i^n) : n\in\omega)$ is k-inconsistent. We inductively define b_s for any $s\in\omega^i$ such that $(b_{s\restriction_1},\dots,b_{s\restriction_i})\equiv(a_0,\dots,a_{i-1})$ and for any $s'\in\omega^{i-1}$, $(b_{s'\frown j} : j\in\omega)$ is k-inconsistent.

Suppose $(b_s, s\in\omega^{\le i})$ has been defined. For $s\in\omega^{i+1}$, let $\bar b=(b_{s\restriction_1},\dots,b_{s\restriction_i})\equiv a_0\cdots a_{i-1}$. Choose $\sigma\in\mathrm{Aut}(\mathfrak{U})$ such that $\sigma(a_0,\dots,a_{i-1})=\bar b$. Let $b_s=\sigma(a_i^{s(i)})$. Then $\bar b\, b_s\equiv a_0\cdots a_{i-1}a_i^{s(i)}\equiv a_0\cdots a_{i-1}a_i$ (by $a_i^{s(i)}\equiv_{a_0\cdots a_{i-1}} a_i$), and let $s'=s\restriction i$, then $(b_{s'\frown j} : j<\omega)\equiv(a_i^j : j<\omega)$. In particular $(\psi(x,b_{s'\frown j}) : j<\omega)$ is k-inconsistent. $\square$

Remark 4.4.
By Lemma 4.3(2), if T is simple, then for any k and any finite set of formulas Δ, there is a finite bound on the possible length of Δ-k-dividing sequences.

Proposition 4.5 *(Characterisation of simplicity)***.**
Let T be a complete theory. The following are equivalent:

(a) T is simple.

(b) (Local character) For all $p\in S_n(B)$, there is some $A\subseteq B$ with $|A|\le|T|$ such that p does not divide over A.

Proof. $(a)\Rightarrow(b)$: If (b) does not hold, then there is a type $p\in S_n(B)$ such that for any $A\subseteq B$ with $|A|\le|T|$, p divides over A. In particular, p divides over $\emptyset$. Let $\psi_0(x,b_0)\in p$ such that $\psi_0(x,b_0)$ k_0-divides over $\emptyset$.

Now choose inductively a sequence $(b_i : i<|T|^+)$ and formulas $\psi_i(x,y_i)$ such that $\psi_i(x,b_i)$ divides over $\{b_j : j<i\}$ with respect to k_i. Since i is indexed by $|T|^+$, there is $\psi(x,y)\in\mathcal{L}$ and $k\in\omega$ and an infinite set $I\subseteq|T|^+$ such that $\psi_i(x,y)=\psi(x,y)$ and $\psi_i(x,b_i)$ k-divides over $(b_j : j<i)$ for all $i\in I$.

Now $(\psi(x,b_i) : i\in I)$ is an infinite ψ-k-dividing sequence. By Lemma 4.3(2), ψ has the k-tree property; contradiction.

$(b)\Rightarrow(a)$: If ψ has the k-tree property, by Lemma 4.3(1), there are ψ-k-dividing sequences $(\psi(x,b_i) : i<|T|^+)$. Let $B:=\{b_i : i<|T|^+\}$. Complete $\{\psi(x,b_i) : i<|T|^+\}$ into a complete type $p(x)$ over B. Then for any $A\subseteq B$ with $|A|\le|T|$, then $A\subseteq\{b_j : j<i\}$ for some $i<|T|^+$. Hence $\psi(x,b_i)$ k-divides over A, namely (b) does not hold. $\square$

Remark 4.6 *(Exercise)***.**
Suppose $\psi(x,b)$ k-divides over A and $A\subseteq B$. Then there is some A-conjugate B', i.e. $B'\equiv_A B$, such that $\psi(x,b)$ k-divides over B'.

Corollary 4.7 *(Types don't fork over A in simple theories)***.**
Let T be a simple theory and $p\in S_n(A)$. Then p does not fork over A.

Proof. Suppose towards contradiction that $p(x)$ forks over A. Then p implies some $\bigvee_{\ell \le d} \psi_\ell(x, b')$ with each $\psi_\ell(x, b')$ k-dividing over A. Let $\Delta := \{\psi_\ell(x, y) : \ell \le d\}$.

We will prove by induction on n that there is a Δ-k-dividing sequence over A of length n, which is consistent with $p(x)$, for any $n \in \omega$. This will contradict Lemma 4.3(2) (the remark after).

So suppose $(\psi_i(x, a_i) : i < n)$ is a Δ-k-dividing sequence over A which is consistent with $p(x)$. By Remark 4.6, there is $b' \equiv_A b$ such that $(\psi_i(x, a_i) : i < n)$ is a Δ-k-dividing sequence over Ab'. Indeed, let $b_0 \equiv_A b$ such that $\psi_0(x, a_0)$ k_0-divides over Ab_0. Now let $b_1 \equiv_{Aa_0} b_0$ such that $\psi_1(x, a_1)$ k-divides over Aa_0b_1. Since $a_0b_1 \equiv_A a_0b_0$, we still have $\psi_0(x, a_0)$ k-divides over Ab_1. Thus, inductively we can find b' such that $\psi_i(x, a_i)$ k-divides over $Aa_0 \cdots a_{i-1}b'$ for all $i < n$.

Since $p(x) \vdash \bigvee_{\ell \le d} \psi_\ell(x, b')$ by $b \equiv_A b'$, $p(x) \cup \{\psi_i(x, a_i) : i < n\}$ is consistent, some ℓ: $\psi_\ell(x, b')$ is consistent with $p(x) \cup \{\psi_i(x, a_i) : i < n\}$. Now $(\psi_\ell(x, b'), \psi_0(x, a_0), \ldots, \psi_{n-1}(x, a_{n-1}))$ is a Δ-k-dividing sequence over A of length $n + 1$. □

Corollary 4.8 *(Existence of non-forking extensions).*
If T is simple, any type over A has a non-forking extension to any B containing A.

Definition 4.9 *(Independent and Morley sequences).*
Let I be a linear order. A sequence $(a_i)_{i \in I}$ is called:

(1) **Independent over** A if $a_i \mathop{\smile\!\!\!\!\!\!\mid}_A \{a_j : j \ne i\}$ for all i.

(2) A **Morley sequence over** A if it is independent and indiscernible over A.

(3) A **Morley sequence in** $p(x)$ **over** A if it is a Morley sequence over A consisting of realisations of $p(x)$.

Example 4.10.
Let $q(x)$ be a global A-invariant type. Then any sequence $(b_i)_{i \in I}$ where each b_i realises $q \restriction A \cup \{b_j : j < i\}$ is a Morley sequence over A.

Proof. Let us call such a sequence *good*. By definition, a subsequence of a good sequence is good. For indiscernibility, it suffices to show that for all finite good sequences $b_0, \ldots, b_n$ and $b'_0, \ldots, b'_n$, they have the same type over A. By induction, we may assume $b_0 \cdots b_{n-1} \equiv_A b'_0 \cdots b'_{n-1}$. Let $\sigma \in \mathrm{Aut}(\mathfrak{U}/A)$ such that $\sigma(b_0 \cdots b_{n-1}) = b'_0 \cdots b'_{n-1}$. Then $\sigma(\mathrm{tp}(b_n/Ab_0 \cdots b_{n-1})) = \sigma(q \restriction Ab_0 \cdots b_{n-1}) = q \restriction Ab'_0 \cdots b'_{n-1} = \mathrm{tp}(b'_n/Ab'_0 \cdots b'_{n-1})$ by invariance. Thus $b_0 \cdots b_n \equiv_A b'_0 \cdots b'_n$.

By exercise, q does not fork over A; in particular $\mathrm{tp}(b_n/b_0 \cdots b_{n-1}A)$ does not fork over A, i.e. $b_n \mathop{\smile\!\!\!\!\!\!\mid}_A b_0 \cdots b_{n-1}$. □

Remark 4.11.
Let q be a global A-invariant type. We call a sequence $(b_i)_{i\in I}$ given by Example 4.10 a **Morley sequence of q over A**. The proof above shows that the type of such a sequence is uniquely determined by its order type I.

Lemma 4.12 *(Shelah; Erdős–Rado)*.
For all A, there is λ such that for any linear order I of cardinality λ and any family $(a_i)_{i\in I}$, there is an A-indiscernible sequence $(b_j)_{j\in\omega}$ such that for all $j_1 < \cdots < j_n$ in ω, there is a sequence $i_1 < \cdots < i_n$ in I with $a_{i_1}\cdots a_{i_n} \equiv_A b_{j_1}\cdots b_{j_n}$.

Lemma 4.13 *(Morley sequences from non-forking types)*.
If $p \in S_n(B)$ does not fork over A, there is an infinite Morley sequence in p over A which is indiscernible over B. In particular, if T is simple, for every $p \in S_n(A)$ there is an infinite Morley sequence in p over A.

Proof. Let $a_0 \models p$. Take p' a non-forking extension of p to Ba_0. Let $a_1 \models p'$. Continuing this way, we obtain a sequence $(a_i)_{i<\lambda}$ with $a_i \underset{A}{\not\!\downarrow} B(a_j)_{j<i}$ for arbitrary λ. By Lemma 4.12, we obtain a sequence $(b_i)_{i\in\omega}$ which is indiscernible over B. Since $b_ib_{i-1}\cdots b_0 \equiv_B a_{j_i}a_{j_{i-1}}\cdots a_{j_0}$ for some $j_0 < \cdots < j_i$, we have $b_i \underset{A}{\not\!\downarrow} B(b_j)_{j<i}$ as $a_{j_i} \underset{A}{\not\!\downarrow} B(a_{j_k})_{k<i}$. Hence $(b_i)_{i\in\omega}$ is a Morley sequence in p over A. $\square$

Proposition 4.14 *(Kim's lemma)*.
Let T be simple and $\pi(x,y)$ be a partial type over A. Let $(b_i)_{i\in\omega}$ be an infinite Morley sequence over A and $\bigcup_{i<\omega}\pi(x,b_i)$ consistent. Then $\pi(x,b_0)$ does not divide over A.

(Note that if $\bigcup_{i<\omega}\pi(x,b_i)$ is inconsistent, then $\pi(x,b_0)$ divides over A, i.e. checking dividing it is enough to check for any Morley sequence in simple theories.)

Proof. Let $(b_i)_{i\in\omega}$ be an infinite Morley sequence such that $\bigcup\pi(x,b_i)$ is consistent. By the standard lemma, there is a Morley sequence $(b'_i)_{i\in I}$ in $\mathrm{tp}(b_0/A)$ such that $r(x) := \bigcup_{i\in I}\pi(x,b'_i)$ is consistent for any small order I. Let I be the inverse order type of $|T|^+$. Let c be a realisation of $r(x)$.

By Proposition 4.5(b) (local character), there is a subset $I_0 \subseteq I$ with $|I_0| \le |T|$, such that $\mathrm{tp}(c/A \cup \{b'_i : i \in I_0\})$ does not divide over $A \cup \{b'_i : i > i_0\}$ for some $i_0 \in I$.

Claim: $(b'_i)_{i>i_0} \underset{A}{\not\!\downarrow} b'_{i_0}$.

(Proof: by induction on finite subtuples of $(b'_i)_{i>i_0}$. By indiscernibility, enough to show $b_nb_{n-1}\cdots b_1 \underset{A}{\not\!\downarrow} b_0$. We have $b_n \underset{A}{\not\!\downarrow} (b_i)_{i<n}$, hence $b_n \underset{Ab_{n-1}}{\not\!\downarrow} (b_i)_{i<n-1}$ by base monotonicity. By assumption, $b_{n-1} \underset{A}{\not\!\downarrow} (b_i)_{i<n-1}$. By left transitivity, $b_nb_{n-1} \underset{A}{\not\!\downarrow} (b_i)_{i<n-1}$.)

In particular, $\mathrm{tp}((b'_i)_{i>i_0}/Ab'_{i_0})$ does not divide over A. By left transitivity of dividing, $\mathrm{tp}(c\,(b'_i)_{i>i_0}/Ab'_{i_0})$ does not divide over A. In particular, $\pi(x,b'_{i_0})$ does not divide over A.
$\square$

Proposition 4.15 *(Forking equals dividing in simple theories).*
Let T be simple. Then $\pi(x,b)$ divides over A iff $\pi(x,b)$ forks over A.

Proof. By definition, dividing implies forking. For the converse, suppose $\pi(x,b)$ does not divide over A. Suppose, towards a contradiction, that $\pi(x,b)$ forks over A. Then there is $\psi(x,b') := \bigvee_{\ell \le d} \psi_\ell(x,b')$ with each $\psi_\ell(x,b')$ dividing over A and $\pi(x,b) \models \psi(x,b')$. Since $\pi(x,b)$ does not divide, $\psi(x,b')$ does not divide.

Let $(b'_i)_{i\in\omega}$ be a Morley sequence of $\mathrm{tp}(b'/A)$ which exists by T being simple. Then $\{\psi(x,b'_i) : i \in \omega\}$ is consistent. By Pigeonhole, there is some infinite $I_0 \subseteq \omega$ and $\ell \le d$ such that $\{\psi_\ell(x,b'_i) : i \in I_0\}$ is consistent. Since $(b'_i)_{i\in I_0}$ is also a Morley sequence over A, by Proposition 4.14, $\psi_\ell(x,b'_i)$ does not divide over A; contradicts $\psi_\ell(x,b')$ divides over A. □

Proposition 4.16 *(Symmetry of independence in simple theories).*
In simple theories, independence is symmetric, i.e. $A \perp_C B$ iff $B \perp_C A$.

Proof. Assume $A \perp_C B$, let $a \subseteq A$, $b \subseteq B$ be finite tuples. Then $a \perp_C b$. By Lemma 4.13, since $\mathrm{tp}(a/bC)$ does not fork, there is an infinite Morley sequence $(a_i)_{i\in\omega}$ in $\mathrm{tp}(a/bC)$ over C which is indiscernible over bC. Let $p(x,y) := \mathrm{tp}(ab/C)$. Then $\bigcup_{i<\omega} p(a_i,y)$ is consistent, as witnessed by b. By Proposition 4.14, $p(a_0,y)$ does not divide over C. By Proposition 4.15, $b \perp_C a$. Therefore, $B \perp_C A$ by finite character. □

Corollary 4.17 *(Independent sequences and disjoint index sets).*
Let $(a_i)_{i\in I}$ be a sequence of tuples, T simple, such that $a_i \perp_A (a_j)_{j\neq i}$, i.e. an independent sequence. Then for any $I_0, I_1 \subseteq I$ with $I_0 \cap I_1 = \emptyset$,

$$(a_i)_{i\in I_0} \underset{A}{\perp} (a_j)_{j\in I_1}.$$

Proof. We first prove $a_i \perp_A (a_j)_{j\neq i}$. Then by transitivity and base monotonicity, we get inductively $(a_i)_{i\in I_0} \perp_A (a_j)_{j\notin \tilde{I}}$ for any finite $\tilde{I}$, which implies $(a_i)_{i\in I_0} \perp_A (a_j)_{j\in I_1}$.

For any $J \subsetneq K$ with J, K finite: by base monotonicity and transitivity and definition, $a_i a_K \perp_A a_J$ and $a_K \perp_A a_i$. Hence, by symmetry, $a_J \perp_A a_K a_i \Rightarrow a_J \perp_{A a_K} a_i$. By left transitivity, $a_J a_K \perp_A a_i$. By symmetry again, $a_i \perp_A a_J a_K$. □

4.2 The independence theorem and Kim–Pillay theorem

Lemma 4.18 *(Coheir extensions do not fork).*
Let T be arbitrary. Let $p \in S_n(A)$ and $q \in S_n(B)$ a coheir extension of p for some $B \supseteq A$. Then q does not fork over A.

Proof. Suppose $\psi(x) \vdash \bigvee_{\ell \leq d} \psi_\ell(x, b'_\ell)$ with $\psi(x) \in q(x)$ and $\psi_\ell(x, b'_\ell) \in \mathcal{L}(\mathfrak{U})$. By q being finitely satisfiable in A, there is some $a \in A$ such that $\mathfrak{U} \models \psi(a)$. Thus $\mathfrak{U} \models \psi_\ell(a, b'_\ell)$ for some $\ell \leq d$. Let $(b_i)_{i\in\omega}$ be an indiscernible sequence over A such that $b_i \equiv_A b'_\ell$. Then $\mathfrak{U} \models \psi_\ell(a, b_i)$ for all i. Thus $\{\psi_\ell(a, b_i) : i \in \omega\}$ is consistent and $\psi_\ell(x, b'_\ell)$ does not divide over A. Namely, $\psi(x)$ does not fork. $\square$

From now on we assume T is simple unless stated otherwise.

Lemma 4.19 *(Indiscernible sequence gives Morley sequence)***.**
(Let T be simple.) Let $(a_i : i < \omega + \omega)$ be an A-indiscernible sequence and put $I = (a_i : i < \omega)$ and $I' = (a_{\omega+i} : i < \omega)$. Then I' is a Morley sequence in $\mathrm{tp}(a_\omega / AI)$ over AI.

Proof. I' is clearly an AI-indiscernible sequence in $\mathrm{tp}(a_\omega / AI)$. We need to check that I' is independent.

Note that $\mathrm{tp}(a_{\omega+i} / AI\, (a_{\omega+j})_{j>i})$ is finitely satisfiable in AI. Indeed, for any $\psi(x, (a_i)_{i\in I_0}, (a_j)_{j\in\omega+J_0}) \in \mathrm{tp}(a_{\omega+i} / AI\, (a_{\omega+j})_{j>i})$ with I_0, J_0 finite, take $i_0 > I_0$; then $\mathfrak{U} \models \psi(a_{i_0}, (a_i)_{i\in I_0}, (a_j)_{j\in\omega+J_0})$. Hence $a_{\omega+i} \not\downarrow_{AI} (a_{\omega+j})_{j>i}$ by Lemma 4.18.

Thus $(a_{\omega+i})_{i\in\omega}$ with the reverse ordering of ω is a Morley sequence over AI. By Corollary 4.17, $a_{\omega+i} \not\downarrow_{AI} (a_{\omega+j})_{j<i}$, and $(a_{\omega+i})_{i<\omega}$ is a Morley sequence over AI. $\square$

Lemma 4.20 *(Morley sequence and non-forking partial type)***.**
(Let T be simple.) Let $\pi(x, \bar{a})$ be a partial type over Aa which does not fork over A. If $(a_i : i < \omega)$ is a Morley sequence over A in $\mathrm{tp}(a/A)$, then $\tau := \bigcup_{i<\omega} \pi(x, a_i)$ is consistent and does not fork over A.

Proof. Consistency follows from definition. Since forking $=$ dividing, it is enough to show $\bigwedge_{i<n} \psi(x, a_i)$ does not divide, where $\psi(x, a_i) \in \pi(x, a_i)$.

Let $b_j := (a_{jn+i} : i < n)$. Then $(b_j : j < \omega)$ is also an infinite Morley sequence over A. Moreover, $\{\psi(x, a_i) : i < \omega\}$ is realised by some $b \models r(x)$. Let $\psi(x, a_0, \ldots, a_{n-1}) := \bigwedge_{i<n} \psi(x, a_i)$. Then $\{\psi(x, b_j) : j < \omega\}$ is realised by b. Thus, by Proposition 4.14 (Kim's lemma), $\psi(x, b_0) = \bigwedge_{i<n} \psi(x, a_i)$ does not divide over A. $\square$

Theorem 4.21 *(Non-forking extension along indiscernible sequences)***.**
(T simple.) Let $\pi(x, a)$ be a partial type over Aa which does not fork over A. If $(a_i : i < \omega)$ is indiscernible over A with $\mathrm{tp}(a_i/A) = \mathrm{tp}(a/A)$, then $\pi' := \bigcup_{i<\omega} \pi(x, a_i)$ is consistent and does not fork over A.

Proof. Let I be a sequence of order type ω such that $I^\frown(a_i : i < \omega)$ is indiscernible over A. (This is possible since we can take an indiscernible realising $\mathrm{EM}((a_i)_{i<\omega}/A)$ of order type $\omega + \omega$ and take an automorphism.)

Let $p(x, a_0)$ be a completion of $\pi(x, a_0)$ to AIa_0 which does not fork over A. By Lemma 4.19, $(a_i : i < \omega)$ is a Morley sequence in $\operatorname{tp}(a_0/AI)$ over AI. Since $p(x, a_0)$ does not fork over AI as well, by Lemma 4.20, $\bigcup_{i<\omega} p(x, a_i)$ is consistent and does not fork over AI.

Let $d \models \bigcup_{i<\omega} p(x, a_i)$, then $d \mathop{\not\perp}_{AI} (a_i)_{i<\omega}$. Since $d \models p(x, a_0)$ which does not fork over A, we get $d \mathop{\not\perp}_A Ia_0$, hence $d \mathop{\not\perp}_A I$. By right transitivity (from symmetry + left transitivity), we get $d \mathop{\not\perp}_A I\,(a_i)_{i<\omega}$, in particular $d \mathop{\not\perp}_A (a_i)_{i<\omega}$.

Now $d \models \bigcup_{i<\omega} \pi(x, a_i)$, so $\bigcup_{i<\omega} \pi(x, a_i)$ does not fork over A. □

Stable relations

Definition 4.22 *(Invariant and stable relations)*.

(1) A relation $R(x, y)$ (which need not be definable) is A-**invariant** if it is invariant under all $\operatorname{Aut}(\mathfrak{U}/A)$, i.e. if $ab \equiv_A a'b'$ then $R(a, b) \Leftrightarrow R(a', b')$.

(2) An A-invariant relation $R(x, y)$ is **stable** if there is no A-indiscernible sequence $(a_i b_i : i < \omega)$ such that $R(a_i, b_j)$ holds iff $i \leq j$.

Remark 4.23.
In the definition of stable relation, one can equivalently ask for $i < j$, $i > j$, or $i \geq j$ instead of $i \leq j$.

Lemma 4.24 *(Non-forking gives stable relation)*.
(T simple.) Let $\Phi(x, y)$ and $\Psi(x, z)$ be partial types. Then the relation "$\Phi(x, a) \wedge \Psi(x, b)$ does not fork over A" is a stable relation.

Proof. Clearly, it is an A-invariant relation. Let $(a_i b_i : i < \omega)$ be an A-indiscernible sequence such that $\Phi(x, a_i) \wedge \Psi(x, b_j)$ does not fork over A iff $i \leq j$, i.e. $\Phi(x, a_i) \wedge \Psi(x, b_j)$ forks over A iff $i > j$. In particular, $\Phi(x, a_0) \wedge \Psi(x, b_0)$ does not fork over A. By Theorem 4.21, the whole set $\bigcup_{i<\omega} \Phi(x, a_i) \wedge \Psi(x, b_i)$ does not fork over A. In particular $\Phi(x, a_1) \wedge \Psi(x, b_0)$ does not fork over A, a contradiction. □

Lemma 4.25 *(Unstable invariant relation)*.
(T arbitrary.) Let R be an A-invariant relation. Suppose there is an A-indiscernible sequence $(b_i : i \in I)$, some $i_0 \in I$, and some a such that:

(1) $\operatorname{tp}(ab_i/A)$ is constant for all $i \leq i_0$ and all $i > i_0$, i.e. $ab_i \equiv_A ab_j$ for $i, j \leq i_0$ and $ab_i \equiv_A ab_j$ for $i, j > i_0$.

(2) $R(a, b_i)$ holds iff $i \leq i_0$.

(3) Both $\{i \in I : i < i_0\}$ and $\{i \in I : i > i_0\}$ are infinite.

Then R is not a stable relation.

Proof. By compactness/saturation, we may assume $I = \mathbb{Z}$ and $i_0 = 0$. By indiscernibility, let $\sigma_i \in \mathrm{Aut}(\mathfrak{U}/A)$ be such that $\sigma_i(b_j) := b_{j+i}$. Let $a_i := \sigma_i(a)$. Then $R(a_i, b_j)$ iff $j \leq i$ and $R(a_i, b_j)$ iff $j \leq i$.

Consider $(a_i b_i)_{i \in \mathbb{N}}$. We want to make it indiscernible over A. Let $\pi_0(x, y) := \mathrm{tp}(ab_0/A) = \mathrm{tp}(a_i b_j/A)$ for $j \leq i$ and $\pi_1(x, y) := \mathrm{tp}(ab_1/A) = \mathrm{tp}(a_i b_j/A)$ for $j > i$.

Let $r((x_i, y_i) : i \in \mathbb{N})$ be the set of formulas consisting of:

(i) $\pi_0(x_i, y_j)$ for $j \leq i$;

(ii) $\pi_1(x_i, y_j)$ for $j > i$;

(iii) $\psi(x_{i_0} y_{i_0}, \dots, x_{i_n} y_{i_n}) \leftrightarrow \psi(x_{j_0} y_{j_0}, \dots, x_{j_n} y_{j_n})$ for all $\psi \in \mathcal{L}_A$ and $i_0 < \cdots < i_n$, $j_0 < \cdots < j_n$ in $\mathbb{N}$.

Claim: r is consistent. Let Δ be a finite subset of $\psi \in \mathcal{L}_A$ from (iii) of length $n+1$. Colour the $n+1$-tuples $(a_{i_0} b_{i_0}, \dots, a_{i_n} b_{i_n})$ for $i_0 < \cdots < i_n$ in $\mathbb{N}$ by the values of $\psi(x_{i_0} y_{i_0}, \dots, x_{i_n} y_{i_n})$ for $\psi \in \Delta$. By Ramsey's theorem, there is an infinite monochromatic set $I \subseteq \mathbb{N}$. Then $(a_i b_i : i \in I)$ witnesses the consistency of (i), (ii), and the indiscernibility conditions from (iii).

Thus, let $(a_i' b_i')_{i \in \mathbb{N}}$ be a realisation of r. Then it is indiscernible over A by (iii), and $R(a_i', b_j')$ holds iff $j \leq i$ (by (i) and (ii)). □

Theorem 4.26 *(Stable relations over models).*
(T arbitrary.) Suppose M is a model of an arbitrary theory T and R is an M-invariant stable relation such that $R(a, b)$ holds for some $a \mathop{\smile\!\!\!\!\!\!|}_M b$. Then $R(a', b')$ holds for all $a' \models \mathrm{tp}(a/M)$ and $b' \models \mathrm{tp}(b/M)$ with $a' \mathop{\smile\!\!\!\!\!\!|}_M b'$.

Proof. Suppose $R(a', b')$ fails for some $a' \models \mathrm{tp}(a/M)$ and $b' \models \mathrm{tp}(b/M)$ with $a' \mathop{\smile\!\!\!\!\!\!|}_M b'$. By invariance of R over M, we may assume $a' = a$. Let p be a coheir of $\mathrm{tp}(b/M)$ to the monster model $\mathfrak{U}$. Let $(b_i : i < \omega)$ and $(b_i' : i < \omega)$ be Morley sequences of p over M, such that $b_0 = b$ and $b_0' = b'$ (i.e. $b_i \models p \restriction \{b_j\}_{j<i}$, same for b_i').

By Lemma 3.23, we may assume $(b_i)_{i<\omega}$ and $(b_i')_{i<\omega}$ are both indiscernible over $M \cup \{a\}$ (since $\mathrm{tp}(a/Ab)$ does not divide over A). So $R(a, b_i)$ holds for all $i < \omega$ and $R(a, b_i')$ fails for all $i < \omega$.

Now, let $(c_i : i < \omega)$ be a Morley sequence of p over $M \cup \{b_i b_i' : i < \omega\}$. Then $(b_i : i < \omega)^\frown(c_i : i < \omega)$ and $(b_i' : i < \omega)^\frown(c_i : i < \omega)$ are both Morley sequences of p over M. By Ramsey's theorem, we may assume $(c_i : i < \omega)$ is indiscernible over $M \cup \{a\}$.

Now either $R(a, c_i)$ holds for all $i < \omega$, in which case $(b_i' : i < \omega)^\frown(c_i : i < \omega)$ contradicts Lemma 4.25. Or $R(a, c_i)$ fails for all $i < \omega$ and $(b_i : i < \omega)^\frown(c_i : i < \omega)$ contradicts Lemma 4.25. □

Corollary 4.27 *(Independence theorem over a model).*

Let T be simple. Let M be a model, $p \in S_n(M)$, and A, B supersets of M with $A \mathop{\smile\!\!\!\!\!\!\mid}_M B$, and $p_A \in S_n(A)$ and $p_B \in S_n(B)$ non-forking extensions of p. Then $p_A \cup p_B$ does not fork over M. (In particular it is consistent.)

Proof. Let $a \models p_A$ and $a' \models p_B$. There is $\sigma \in \mathrm{Aut}(\mathfrak{U}/M)$ such that $\sigma(a') = a$ and $\sigma(B) = B'$ and $a \mathop{\smile\!\!\!\!\!\!\mid}_M A$, $a \mathop{\smile\!\!\!\!\!\!\mid}_M B'$. Take a non-forking extension of $\mathrm{tp}(A/Ma)$ to MaB'. Then there is $B'' \equiv_{Ma} B$ such that $A \mathop{\smile\!\!\!\!\!\!\mid}_{Ma} B''$. Since $A \mathop{\smile\!\!\!\!\!\!\mid}_M a$, we get $A \mathop{\smile\!\!\!\!\!\!\mid}_M aB''$. Hence $A \mathop{\smile\!\!\!\!\!\!\mid}_{B''} a$ by $B'' \supseteq M$. As $B'' \equiv_{Ma} B$ and $a \mathop{\smile\!\!\!\!\!\!\mid}_M B'$, by symmetry and invariance, $B'' \mathop{\smile\!\!\!\!\!\!\mid}_M a$. Thus $AB'' \mathop{\smile\!\!\!\!\!\!\mid}_M a$.

Therefore $p_A \cup p_{B''}$ does not fork over M. By Lemma 4.24 the relation "$p_x \cup p_y$ does not fork over M" is a stable M-invariant relation, and it holds for $A \mathop{\smile\!\!\!\!\!\!\mid}_M B''$, thus by Theorem 4.26, it also holds for $A \mathop{\smile\!\!\!\!\!\!\mid}_M B$. $\square$

Remark 4.28 *(Three-amalgamation).*

Let $C \models p$. We have three 1-types: $\mathrm{tp}(C/M)$, $\mathrm{tp}(A/M)$, $\mathrm{tp}(B/M)$, and three 2-types: $p_A \supseteq \mathrm{tp}(C/M) \cup \mathrm{tp}(A/M)$, $p_B \supseteq \mathrm{tp}(C/M) \cup \mathrm{tp}(B/M)$, $\mathrm{tp}(AB/M) \supseteq \mathrm{tp}(A/M) \cup \mathrm{tp}(B/M)$. We can amalgamate them to a 3-type in an independent way. Equivalently: suppose $C \equiv_M C'$, $C \mathop{\smile\!\!\!\!\!\!\mid}_M A$, $C' \mathop{\smile\!\!\!\!\!\!\mid}_M B$. Then there is $\tilde{C} \mathop{\smile\!\!\!\!\!\!\mid}_M AB$ such that $\tilde{C}A \equiv_{MA} CA$ and $\tilde{C}B \equiv_{MB} C'B$.

Remark 4.29.

(1) Working over a model is necessary. **Example:** Let T be the theory of 2 equivalence classes each with infinitely many elements. There is a unique 1-type extending $x = x$ over $\emptyset$. Let $a, b \in \mathfrak{U}$ with different equivalence classes. $p_a := \{x \neq a\} \cup \{Exa\}$, $p_b := \{x \neq b\} \cup \{Exb\}$. Then p_a, p_b do not fork over $\emptyset$ and $p_a \upharpoonright \emptyset = p_b \upharpoonright \emptyset$. But $p_a \cup p_b$ is inconsistent.

(2) The condition $A \mathop{\smile\!\!\!\!\!\!\mid}_M B$ cannot be removed. **Example:** Let T be the theory of the random graph and $M \models T$. Let $a \in \mathfrak{U} \setminus M$ and $p_a := Exa$, $p'_a := \neg Exa$. Both are non-forking, but $a \not\mathop{\smile\!\!\!\!\!\!\mid}_M a$ and $p_a \cup p'_a$ is inconsistent.

Corollary 4.30.

Let T be simple. Let $(B_i)_{i \in I}$ be independent over M and let $b_i \mathop{\smile\!\!\!\!\!\!\mid}_M B_i$ and all b_i having the same type over M, i.e. $\mathrm{tp}(d/B_i) = \mathrm{tp}(b_i/B_i)$ for all i. Then there is some d with $d \mathop{\smile\!\!\!\!\!\!\mid}_M \{B_i : i \in I\}$.

Kim–Pillay characterisation

Theorem 4.31 *(Kim–Pillay).*

Let T be a complete theory and $\mathop{\smile\!\!\!\!\!\!\mid}^0$ a relation between finite tuples a and sets A, B, invariant under automorphisms and having the following properties:

(a) (Monotonicity and transitivity) $a \mathop{\smile\!\!\!\!\!\!\mid}^0_A BC$ iff $a \mathop{\smile\!\!\!\!\!\!\mid}^0_A B$ and $a \mathop{\smile\!\!\!\!\!\!\mid}^0_{AB} C$.

(b) (Symmetry) $a \mathop{\smile\!\!\!\!\!\!\mid}^0_A b$ iff $b \mathop{\smile\!\!\!\!\!\!\mid}^0_A a$.

(c) (Finite character) $a \mathop{\smile\hskip -0.9em ^| \ }^0_A B$ if $a \mathop{\smile\hskip -0.9em ^| \ }^0_A b$ for all finite tuples $b \in B^n$.

(d) (Local character) There is a cardinal κ, such that for all a and B there exists $B_0 \subseteq B$ of cardinality $< \kappa$ such that $a \mathop{\smile\hskip -0.9em ^| \ }^0_{B_0} B$.

(e) (Existence) For all a, A and C, there exists a' with $a \equiv_A a'$ and $a' \mathop{\smile\hskip -0.9em ^| \ }^0_A C$.

(f) (Independence over models) Let M be a model, $a' \equiv_M b'$ and $a \mathop{\smile\hskip -0.9em ^| \ }^0_M b$, $a' \mathop{\smile\hskip -0.9em ^| \ }^0_M a$, $b' \mathop{\smile\hskip -0.9em ^| \ }^0_M b$. Then there is some c such that $c \equiv_{Ma} a'$, $c \equiv_{Mb} b'$, and $c \mathop{\smile\hskip -0.9em ^| \ }^0_M ab$.

Then T is simple and $\mathop{\smile\hskip -0.9em ^| \ }^0 = \mathop{\smile\hskip -0.9em ^| \ }$.

Remark 4.32.
We have proved in simple theories that $\mathop{\smile\hskip -0.9em ^| \ }$ satisfies (a)–(f). And simple is equivalent to local character with $\kappa = |T|^+$. But it is indeed equivalent to local character with any κ (for dividing; see Tent–Ziegler, Lemma/Prop 7.2.5).

Proof of Theorem 4.31. **Strategy:**

(I) Show that if $a \mathop{\smile\hskip -0.9em ^| \ }^0_A b$, then $\operatorname{tp}(a/Ab)$ does not divide over A.

(II) Show if $\operatorname{tp}(a/Ab)$ does not divide over A, then $a \mathop{\smile\hskip -0.9em ^| \ }^0_A b$. Then by local character of $\mathop{\smile\hskip -0.9em ^| \ }^0$ and (I), dividing has local character, hence T is simple. And dividing = forking, by (I) and (II), $\mathop{\smile\hskip -0.9em ^| \ }^0 = \mathop{\smile\hskip -0.9em ^| \ }$.

Proof of (I). We may assume (by replacing κ with a larger cardinal) that κ is regular and $> |T| + |A|$. Assume $a \mathop{\smile\hskip -0.9em ^| \ }^0_A b$, let $(b_i)_{i<\omega}$ be an A-indiscernible sequence such that $b_i \equiv_A b$. We want to show $\bigcup_{i<\omega} p(x, b_i)$ is consistent where $p(x, y) := \operatorname{tp}(ab/A)$, hence $\operatorname{tp}(a/Ab)$ does not divide over A.

Claim 1: We can find a model M containing A such that $(b_i)_{i<\omega}$ is indiscernible over M and $b_K \mathop{\smile\hskip -0.9em ^| \ }^0_M \{b_j : j < K\}$.

Proof of Claim 1: By the standard lemma, we can find a sequence $(b'_i)_{i\le K}$ realising $\operatorname{EM}((b_i)_{i<\omega}/A)$. Since $\operatorname{tp}(b'_{i_0} \cdots b'_{i_n}/A)$ is in $\operatorname{EM}((b_i)_{i<\omega}/A)$ for $i_0 < \cdots < i_n$, any subsequence $(b'_{i_j})_{j<\omega}$ of $(b'_i)_{i\le K}$ has the same type as $(b_i)_{i<\omega}$.

With the above observation we construct inductively an ascending chain of models $A \subseteq M_0 \subseteq M_1 \subseteq \cdots$ and $(b'_j)_{j\le K}$ such that:

(i) $(b'_j)_{j\le K}$ realises $\operatorname{EM}((b_i)_{i<\omega}/A)$;

(ii) $\{b'_j : j \le i\}$ is contained in M_i for $i < K$;

(iii) $(b'_j)_{i_0 \le j < K}$ is M_i-indiscernible.

Let M_i be a model of cardinality $< \kappa$ containing $(M_j)_{j<i}$ and b'_i (and A if $i = 0$). This is possible by κ regular and $> |A| + |T|$. For some $i_0 < K$, then $\{b'_j : j \ge i_0\}$ is disjoint from M_i by Pigeonhole. By the standard lemma, we can find $(c_j : j \le K)$ realising

$\mathrm{EM}((b'_j)_{i_0 \le j \le K}/M_i)$, in particular $(c_j : j \le K)$ is M_i-indiscernible. Replace $(b'_j)_{j \le K}$ with $(b'_j : j \le i)^\frown(c_j : j \le K)$ and we continue.

By local character, there is some $i_0 < K$ such that $b'_K \mathop{\smile\!\!\!\!\!|}^0_{M_{i_0}} \{b'_j : i_0 < j < K\}$. We may take $M := M_{i_0}$ and the sequence $(b_i)_{i<\omega}$ to be $(b'_{i_0+i})_{i<\omega}$. Take an automorphism over A; we may further assume $b_0 = b$. $\square$ Claim 1.

Claim 2: We may assume $a \mathop{\smile\!\!\!\!\!|}^0_M b$.

By existence, we find $a' \equiv_{Ab} a$ such that $a' \mathop{\smile\!\!\!\!\!|}^0_{Ab} M$. Since $a' \mathop{\smile\!\!\!\!\!|}^0_A b$ (by $a'b \equiv_A ab$ and $a \mathop{\smile\!\!\!\!\!|}^0_A b$), by transitivity, $a' \mathop{\smile\!\!\!\!\!|}^0_A bM$. By monotonicity, $a' \mathop{\smile\!\!\!\!\!|}^0_M b$. (Take an automorphism again, maybe more; we may assume $(b_i)_{i<\omega}$ to $(b_i)_{i<\omega}$. We may assume $a \mathop{\smile\!\!\!\!\!|}^0_M b$.) $\square$ Claim 2.

Next, we use the Independence Theorem to find $a_i \models \bigcup_{j\le i} p(x, b_j)$ and $a_i b_i \equiv_M ab$. Let $a_0 = a$. Suppose $a_0, \dots, a_i$ has been constructed. Let a' such that $a' b_{i+1} \equiv_M ab$; then $a' \mathop{\smile\!\!\!\!\!|}^0_M b_{i+1}$, $a' \equiv_M a_i$, and $b_{i+1} \mathop{\smile\!\!\!\!\!|}^0_M \{b_j : j \le i\}$. By assumption, $a_i \mathop{\smile\!\!\!\!\!|}^0_M \{b_j : j \le i\}$. Then apply Independence Theorem (f): there is a_{i+1} such that $a_{i+1} \mathop{\smile\!\!\!\!\!|}^0_M \{b_j : j \le i+1\}$, $a_{i+1} b_{i+1} \equiv_M a' b_{i+1} \equiv_M ab$, hence $a_{i+1} \models p(x, b_{i+1})$; and in particular $a_{i+1} \models \mathrm{tp}(a_i/\{b_j : j \le i\}M)$, so $a_{i+1} \models \bigcup_{j \le i+1} p(x, b_j)$.

Therefore, $\bigcup_{j<\omega} p(x, b_j)$ is consistent and we are done.

Proof of (II). Now, we prove: if $\mathrm{tp}(a/Ab)$ does not divide over A, then $a \mathop{\smile\!\!\!\!\!|}^0_A b$. Using existence, we can construct a sequence $(b_i)_{i<\lambda}$ which is $\mathop{\smile\!\!\!\!\!|}^0$-independent and $\mathrm{tp}(b_i/A) = \mathrm{tp}(b/A)$. Then by Erdős–Rado (Lemma 4.12) and saturation, there is an A-indiscernible sequence $(b'_i)_{i<\kappa}$ such that $\mathrm{tp}(b'_i/A) = \mathrm{tp}(b_i/A)$ and $(b'_i)_{i<\kappa}$ is $\mathop{\smile\!\!\!\!\!|}^0$-independent. WMA $b'_0 = b$. Since $\mathrm{tp}(a/Ab)$ does not divide, by Lemma 3.23, there is $a' \equiv_{Ab} a$ such that $(b'_i)_{i<\kappa}$ is indiscernible over Aa'. By local character and monotonicity, $a' \mathop{\smile\!\!\!\!\!|}^0_A \{b'_i : i < i_0\}$ for some $i_0 < \kappa$. Since $b'_{i_0} \mathop{\smile\!\!\!\!\!|}^0_A \{b'_i : i < i_0\}$, by symmetry and transitivity, $a' \mathop{\smile\!\!\!\!\!|}^0_A b'_{i_0}$ and $a' \mathop{\smile\!\!\!\!\!|}^0_A b$ by $ab \equiv_A a'b \equiv_A a'b'_{i_0}$ (using $a' \equiv_{Ab} a$ and indiscernibility of (b'_i) over Aa'). Hence $a' \mathop{\smile\!\!\!\!\!|}^0_A b$, hence $a \mathop{\smile\!\!\!\!\!|}^0_A b$.

Thus, by local character of $\mathop{\smile\!\!\!\!\!|}^0$, T is simple. And dividing = forking, by (I) and (II), $\mathop{\smile\!\!\!\!\!|}^0 = \mathop{\smile\!\!\!\!\!|}$.
$\square$

Corollary 4.33 *(The theory of the random graph is simple).* The theory of the random graph is simple.

Proof. Define $A \mathop{\smile\!\!\!\!\!|}^0_B C$ by $A \cap C \subseteq B$ and apply Theorem 4.31. $\square$

5 Forking in Stable Theories (Chernikov)

5.1 The unique non-forking extension

Theorem 5.1.
Let T be stable, $M \models T$, $p \in S_n(M)$, and $A \supseteq M$. Then p has a unique extension $q \in S(A)$ with the following equivalent properties:

(1) q does not divide over M.

(2) q does not fork over M.

(3) q is definable over M.

(4) q is an heir of p.

(5) q is a coheir of p.

Proof. (1) $\Leftrightarrow$ (2): By T is simple (stable $\Rightarrow$ simple).

(3) $\Leftrightarrow$ (4): Proposition 3.20. If $p \in S_n(M)$ has a definition over M, then p has a unique global heir $q \supseteq p$, which is definable over M.

(1) $\Rightarrow$ (4) (Prop. 3.4.8): Suppose q does not divide over M. Let $\phi(x,y) \in \mathcal{L}(M)$ and $\phi(x,b) \in q$ with $b \in A^{|y|}$. We want to find $b' \in M^{|y|}$ such that $\phi(x,b') \in p$. Let $I = (b_i)_{i\in\omega}$ be a Morley sequence of a global coheir (i.e. finitely satisfiable) extension of $\mathrm{tp}(b/M)$ with $b_0 = b$. Let $a \models q \restriction Mb$. Since $\mathrm{tp}(a/Mb)$ does not divide over M, we may assume, by Lemma 3.23, that I is indiscernible over Ma. Thus $\mathfrak{U} \models \phi(a,b_i)$ for all $i \in \omega$.

Now $\mathrm{tp}(a/MI)$ is definable over MI by stability. Hence, the defining formula for ϕ-types, $d_\phi(y)$, we have $\mathfrak{U} \models d_\phi(b_i)$ for all i. Let $n \in \omega$ such that $\{b_0,\dots,b_n\}$ contains the parameters of $d_\phi(y)$ outside M. Then $d_\phi(y) = d_\phi(y;b_0,\dots,b_n)$. And $\mathfrak{U} \models d_\phi(b_{n+1};b_0,\dots,b_n)$. Since $\mathrm{tp}(b_{n+1}/Mb_0\cdots b_n)$ is finitely satisfiable in M, there is $b' \in M$ such that $\mathfrak{U} \models d_\phi(b';b_0,\dots,b_n)$. Thus $\phi(x,b') \in p$, as desired.

(4) $\Rightarrow$ (5): By symmetry of forking. If q is an heir of p, then q is definable over M by (3) $\Leftrightarrow$ (4); hence q does not fork over M. Let $a \models q \restriction Mb$, then $a \mathop{\smile\!\!\!\!\!\!|}_M b$. By symmetry, $b \mathop{\smile\!\!\!\!\!\!|}_M a$. So $\mathrm{tp}(b/Ma)$ is an heir of $\mathrm{tp}(b/M)$ by (1) $\Rightarrow$ (4). Thus $\mathrm{tp}(a/Mb)$ is a coheir of $\mathrm{tp}(a/M)$.

(5) $\Rightarrow$ (1): We have seen this.

Uniqueness is by Proposition 3.20: the global heir is unique (over models). $\square$

Lemma 5.2 *(Harrington's lemma)***.**
Let $\phi(x,y)$ be stable and $p(x)$, $q(y)$ be global types. Suppose the ϕ-type of p and the ϕ-type of q are defined by formulas $d_p\phi(y)$ and $d_q\phi(x)$ respectively. Then

$$d_p\phi(y) \in q(y) \iff d_q\phi(x) \in p(x).$$

Proof. Let A be the parameters in $d_p\phi$, $d_q\phi$ and ϕ. We define inductively a sequence $(a_ib_i : i \in \omega)$: given $(a_ib_i : i < n)$, let $b_n \models q \restriction Aa_0\cdots a_{n-1}$ and $a_n \models p \restriction Ab_0\cdots b_n$. Then for

$i < j$ we have $\mathfrak{U} \models \phi(a_i, b_j) \iff \phi(a_i, y) \in q \iff \mathfrak{U} \models d_q(a_i) \iff d_q(x) \in p(x)$, and for $j \leq i$ we have $\mathfrak{U} \models \phi(a_i, b_j) \iff \phi(x, b_j) \in p \iff \mathfrak{U} \models d_p(b_j) \iff d_p(y) \in q(y)$.

Since $\phi(x, y)$ does not have the order property, $d_p(y) \in q(y) \iff d_q(x) \in p(x)$. □

Definition 5.3 *(Canonical base)***.**
Let p be a global type. By a **canonical base** of p we mean a set of parameters A such that for any $\sigma \in \mathrm{Aut}(\mathfrak{U})$, $\sigma(p) = p \iff \sigma(A) = A$ setwise (point-wise).

Remark 5.4.
If both A and B are canonical bases of p, then $\sigma(b) = b$ for $b \in B$ and all $\sigma \in \mathrm{Aut}(\mathfrak{U})$ fixing A point-wise. By Lemma 1.2.9, $b \in \mathrm{dcl}(A)$. Thus $\mathrm{dcl}(A) = \mathrm{dcl}(B)$. And if p has a canonical base, then there is a unique definably closed one, denote it $\mathrm{cb}(p)$.

Proposition 5.5.
Assume T eliminates imaginaries. Let $p(x)$ be a definable global type. Then p has a canonical base, and in fact $\mathrm{cb}(p)$ is the smallest definably closed set over which p is definable.

Proof. As p is definable, each formula $\phi(x, y) \in \mathcal{L}$ has a definition $d_\phi(y) \in \mathcal{L}(\mathfrak{U})$. Let c_ϕ in $\mathfrak{U}$ be the canonical parameter for the definable set $d_\phi(\mathfrak{U})$ (by T has EI). Let $A := \{c_\phi : \phi(x, y) \in \mathcal{L}\}$. Then clearly $\sigma(p) = p \iff \sigma(A) = A$ point-wise for all $\sigma \in \mathrm{Aut}(\mathfrak{U})$.

And if p is invariant over A, then p is definable over A by Proposition 3.21(1). Conversely, if $a \in \mathrm{cb}(p)$ and p is definable over A, then $\sigma(p) = p$ for all $\sigma \in \mathrm{Aut}(\mathfrak{U}/A)$ and $\sigma(a) = a$, hence $a \in \mathrm{dcl}(A)$. □

Definition 5.6 *(Good definition)***.**
Let $p \in S(B)$ be a definable type, defined by $(d_\phi(y), \phi(x, y) \in \mathcal{L})$ with $d_\phi(y) \in \mathcal{L}(B)$. We say this definition $(d_\phi(y))_{\phi \in \mathcal{L}}$ is **good** if it is a definition of some global type (or equivalently, it is a definition of some type over some model M containing B).

Example 5.7.
Let T be the theory of an equivalence relation with two infinite equivalence classes. There is only one 1-type over $\emptyset$. Then the definition of $E(x, y)$ over $\emptyset$ is either $y = y$ or $y \neq y$, but neither of them can be extended to the definition of a global type.

Proposition 5.8 *(Non-forking and good definitions)***.**
Let T be stable. A type $p(x) \in S_x(B)$ does not fork over $A \subseteq B$ iff p has a good definition over $\mathrm{acl}^{\mathrm{eq}}(A)$.

Proof. If p does not fork over A, then it has a global extension $p' \in S_x(\mathfrak{U})$ non-forking over A. Let M be a model containing A; then p' does not fork over M, and by Theorem 5.1, p' is definable over M. By Proposition 5.5, $\mathrm{cb}(p') \subseteq M^{\mathrm{eq}}$. Hence $\mathrm{cb}(p') \subseteq \bigcap_{M \supseteq A} M^{\mathrm{eq}}$. And in general $\mathrm{acl}(A) = \bigcap_{M \supseteq A} M$ (since if $b \notin \mathrm{acl}(A)$, then the orbit of b under $\mathrm{Aut}(\mathfrak{U}/A)$

is not small/unbounded, we can find a small model containing A but not containing b). Thus $\mathrm{cb}(p') \subseteq \mathrm{acl}^{\mathrm{eq}}(A)$. And p' is definable over $\mathrm{acl}^{\mathrm{eq}}(A)$ by Proposition 5.5.

Conversely, if p has a good definition over $\mathrm{acl}^{\mathrm{eq}}(A)$, then some global extension p' of p is definable over $\mathrm{acl}^{\mathrm{eq}}(A)$, which implies by Proposition 3.21 that p' does not fork over $\mathrm{acl}^{\mathrm{eq}}(A)$, and any consistent type over $\mathrm{acl}(A)$ does not divide/fork over A, i.e. $a \perp\!\!\!\perp_{\mathrm{acl}(A)} B$ and $a \perp\!\!\!\perp_A \mathrm{acl}^{\mathrm{eq}}(A)$; by $a \perp\!\!\!\perp_A \mathrm{acl}^{\mathrm{eq}}(A)$, we get $a \perp\!\!\!\perp_A B$ by right transitivity. And if $a \models p' \restriction B = p$, then $a \perp\!\!\!\perp_A B$, hence p' does not divide over A. □

Definition 5.9 *(Stationary type)***.**
A type p is **stationary** if it has a unique global non-forking extension.

Corollary 5.10 *(Types over $\mathrm{acl}^{\mathrm{eq}}(A)$ are stationary)***.**
(Need Lemma 5.2.) Let T be stable. Then any type over $A = \mathrm{acl}^{\mathrm{eq}}(A)$ is stationary.

Proof. Let $A = \mathrm{acl}^{\mathrm{eq}}(A)$, let p' and p'' be two global non-forking extensions and $p \in S_x(A)$. Let $\phi(x,b) \in \mathcal{L}(\mathfrak{U})$ be an arbitrary formula and q be a global non-forking extension of $\mathrm{tp}(b/A)$. By Proposition 5.8, p', p'' and q are definable over $\mathrm{acl}^{\mathrm{eq}}(A) = A$. Applying Lemma 5.2, we have

$$\phi(x,b) \in p' \iff \mathfrak{U} \models d_{p'}\phi(b) \iff d_{p'}\phi(y) \in q \iff d_q\phi(x) \in p$$

and by Lemma 5.2 with $d_q\phi$ over A:

$$\iff d_q\phi(x) \in p \iff d_{p''}\phi(y) \in q \iff \mathfrak{U} \models d_{p''}\phi(b) \iff \phi(x,b) \in p''.$$

Thus $p' = p''$. □

Corollary 5.11 *(All types over models are stationary)***.**
In a stable theory, all types over models are stationary.

Proof. By Corollary 5.10, all types over M^{eq} are stationary. But $M^{\mathrm{eq}} \in \mathrm{dcl}^{\mathrm{eq}}(M)$, hence a type p over M uniquely determines (extends to) a type over $M^{\mathrm{eq}} = \mathrm{acl}^{\mathrm{eq}}(M)$. □

Remark 5.12 *(Explanation of (3) $\Leftrightarrow$ (4) in Theorem 5.1)***.**
(3) $\Rightarrow$ (4): Let $A \supseteq M$ and $q \in S_x(A)$ be a definable extension of p. Suppose $\phi(x,a) \in q$, then $d_q\phi(a)$ holds and $M \models d_q\phi(b)$ for some $b \in M$. Hence $\phi(x,b) \in p$. Thus q is an heir of p.

(4) $\Rightarrow$ (3): Let $r \supseteq p$ be an heir. Let $d_\phi(y)$ be any definition of p over M. Suppose $\mathfrak{U} \models d_\phi(b)$ but $\phi(x,b) \notin r$ for $b \in A^{|y|}$. Then $d_\phi(b) \wedge \neg\phi(x,b) \in r$; by r is an heir of p, there is $a \in M$ such that $d_\phi(a) \wedge \neg\phi(x,a) \in p$, contradiction. Thus, heir extensions are uniquely defined.

Lemma 5.13 *(Extensions to* $\mathrm{acl}(A)$ *are conjugate)***.**
Let $p', p'' \in S_x(\mathrm{acl}(A))$ be two extensions of $p \in S_x(A)$. Then p' and p'' are conjugate over A.

Proof. Let $a' \models p'$ and $a'' \models p''$. Since $a' \equiv_A a''$ (they both realise $p(x)$), there is an automorphism $\sigma \in \mathrm{Aut}(\mathfrak{U}/A)$ such that $\sigma(a') = \sigma(a'')$. Note that $\sigma(\mathrm{acl}(A)) = \mathrm{acl}(A)$ as a set. We get: for any $b \in \mathrm{acl}(A)$, $\psi(x,b) \in p' \iff \mathfrak{U} \models \psi(a',b) \iff \mathfrak{U} \models \psi(a'',\sigma(b)) \iff \psi(x,\sigma(b)) \in p''$. Thus $\sigma(p') = p''$. $\square$

Theorem 5.14 *(Conjugacy and boundedness)***.**
Assume T stable. Then forking satisfies:

(1) (Conjugacy) Let A be a small set of parameters. Then all global non-forking extensions of $p \in S_x(A)$ are conjugate over A.

(2) (Boundedness) Any $p \in S_x(A)$ has at most $2^{|T|}$ global non-forking extensions.

Proof. (1): Let q_1, q_2 be two non-forking extensions of p to $\mathfrak{U}$. Extend them to $\mathfrak{U}^{\mathrm{eq}}$. Then $q_1 \restriction \mathrm{acl}^{\mathrm{eq}}(A)$ and $q_2 \restriction \mathrm{acl}^{\mathrm{eq}}(A)$ are conjugate over A by Lemma 5.13 (since $p \in S_x(A)$ determines uniquely the type in $\mathcal{L}^{\mathrm{eq}}$ over A). But $q_1 \restriction \mathrm{acl}^{\mathrm{eq}}(A)$ is stationary. Suppose $\sigma(q_1) = q_2$ for some $\sigma \in \mathrm{Aut}(\mathfrak{U}^{\mathrm{eq}}/A)$. Then $\sigma(q_1)$ does not fork over $\sigma(\mathrm{acl}^{\mathrm{eq}}(A)) = \mathrm{acl}^{\mathrm{eq}}(A)$. Suppose $\sigma(q_1 \restriction \mathrm{acl}^{\mathrm{eq}}(A)) = q_2 \restriction \mathrm{acl}^{\mathrm{eq}}(A)$. Then $\sigma(q_1)$ is a non-forking extension of $q_2 \restriction \mathrm{acl}^{\mathrm{eq}}(A)$; hence by stationarity, $\sigma(q_1) = q_2$.

(2): Let $A_0 \subseteq A$ with $|A_0| \leq |T|$ such that p does not fork over A_0 (which exists by T is simple, in particular). Then by (1), p has at most as many non-forking extensions as $p \restriction A_0$ extensions to $\mathrm{acl}^{\mathrm{eq}}(A_0)$, of which there are $\leq 2^{|\mathrm{acl}^{\mathrm{eq}}(A_0)|} \leq 2^{|T|}$. $\square$

Corollary 5.15 *(Stationary iff good definition over A)***.**
Let T be stable and $p \in S(A)$ be given. Then p is stationary iff it has a good definition over A.

Proof. Let p be stationary. Let q be a global non-forking extension. Then q is definable over some $B \supseteq A$. By Lemma 5.13, all non-forking extensions of p are conjugate over A, and by stationarity, there is only one. Hence q is A-invariant. By Lemma 3.3.9 (if a type is definable and A-invariant, then it is A-definable), q is definable over A. Hence the definition of q over A is a good definition of p.

Conversely, suppose p has a good definition over A. Then p has a global extension p' definable over A, so p' is non-forking over A, and q is the unique global non-forking extension of p (since it is definable over A, but p' is invariant over A; there is only one global non-forking extension). $\square$

Definition 5.16 *(Canonical base of stationary type)***.**
Let $p \in S_x(A)$ be a stationary type. We define the **canonical base** of p, $\mathrm{cb}(p)$, as $\mathrm{cb}(q)$ for q the unique global non-forking extension of p.

Lemma 5.17 *(Non-forking and canonical base)***.**
A stationary type $p \in S_x(A)$ does not fork over $B \subseteq A$ iff $\mathrm{cb}(p) \subseteq \mathrm{acl}^{\mathrm{eq}}(B)$.

Proof. By Proposition 5.8, p does not fork over B iff p has a good definition over $\mathrm{acl}^{\mathrm{eq}}(B)$. It is enough to show $\mathrm{cb}(p) \subseteq \mathrm{acl}^{\mathrm{eq}}(B)$ iff p has a good (unique) definition over $\mathrm{acl}^{\mathrm{eq}}(B)$.

If p has a good definition over $\mathrm{acl}^{\mathrm{eq}}(B)$, then the global non-forking extension q is definable over $\mathrm{acl}^{\mathrm{eq}}(B)$, hence $\mathrm{cb}(p) = \mathrm{cb}(q) \subseteq \mathrm{acl}^{\mathrm{eq}}(B)$ by Proposition 5.5.

Conversely, if $\mathrm{cb}(p) = \mathrm{cb}(q) \subseteq \mathrm{acl}^{\mathrm{eq}}(B)$, then q is definable over $\mathrm{acl}^{\mathrm{eq}}(B)$, hence p has a good definition over $\mathrm{acl}^{\mathrm{eq}}(B)$. $\square$

Proposition 5.18 *(Canonical base and Morley sequences)***.**
Let $p \in S_x(A)$ be stationary and $(a_i)_{i\in\omega}$ be a Morley sequence of p over A. Then $\mathrm{cb}(p) \subseteq \mathrm{dcl}^{\mathrm{eq}}((a_i)_{i<\omega})$.

Proof. Let $a_0 \models p$ and $a_0 \mathop{\smile\!\!\!\!\!\!|}_A (a_i)_{i\in\omega}$ with $a_0 \models p$. Let $(a_i)_{i\leq\omega}$ be a Morley sequence over A in p. Then $\mathrm{tp}(a_0/A(a_i)_{i\in\omega})$ is non-forking over A and finitely satisfiable in $(a_i)_{i\in\omega}$ by indiscernibility of $(a_i)_{i\leq\omega}$.

Thus, take a global extension q of $\mathrm{tp}(a_\omega/A(a_i)_{i\in\omega})$ finitely satisfiable in $(a_i)_{i\in\omega}$. Then q does not fork over $(a_i)_{i\in\omega}$, in particular does not fork over $A(a_i)_{i\in\omega}$.

Take any set $B \supseteq A$, let $b \models q \restriction B$. Then $b \mathop{\smile\!\!\!\!\!\!|}_A (a_i)_{i\in\omega}$ and $b \mathop{\smile\!\!\!\!\!\!|}_{A(a_i)_{i\in\omega}} B$ since q extends $\mathrm{tp}(a_\omega/A(a_i)_{i\in\omega})$. Hence $b \mathop{\smile\!\!\!\!\!\!|}_A B\,(a_i)_{i\in\omega}$ by transitivity and symmetry.

Namely, q does not fork over A, and q is the unique global non-forking extension of p. Since q is A-invariant (by q finitely satisfiable in $(a_i)_{i\in\omega}$), q is definable over $(a_i)_{i\in\omega}$. Thus $\mathrm{cb}(p) \subseteq \mathrm{dcl}^{\mathrm{eq}}((a_i)_{i<\omega})$. $\square$

5.2 Special extensions of types in stable theories

Theorem 5.19.
Let T be a complete theory and $n \in \omega$. Then T is stable iff there is a special class of extensions of n-types, denoted by $p \sqsubseteq q$, with the following properties. (For types over small sets of parameters.)

(1) (Invariance) $\sqsubseteq$ is invariant under $\mathrm{Aut}(\mathfrak{U})$.

(2) (Local character) There is a small cardinal κ such that for any $q \in S_n(C)$, there is $C_0 \subseteq C$ of size $\leq \kappa$ such that $q \restriction C_0 \sqsubseteq q$.

(3) (Weak boundedness) For all A and $p \in S_n(A)$, there is a small cardinal μ such that

> p has at most μ special extensions $p \sqsubseteq q \in S_n(B)$ for all $B \supseteq A$.
>
> Moreover, if $\sqsubseteq$ satisfies in addition the following properties, then $\sqsubseteq$ coincides with non-forking:
>
> (4) (Existence) For all $p \in S_n(A)$ and $A \subseteq B$, there is $q \in S_n(B)$ such that $p \sqsubseteq q$.
>
> (5) (Transitivity) $p \sqsubseteq q \sqsubseteq r$ implies $p \sqsubseteq r$.
>
> (6) (Weak monotonicity) If $p \sqsubseteq r$ and $p \subseteq q \subseteq r$, then $p \sqsubseteq q$.

Proof. If T is stable, then non-forking extensions satisfy all of the listed properties. Weak boundedness by Theorem 5.14.

For transitivity: p over A, q over B and r over C with $A \subseteq B \subseteq C$. Then $a \models r$, $a \mathop{\smile\!\!\!\!\!\!|}_B C$, $a \mathop{\smile\!\!\!\!\!\!|}_A B \Rightarrow a \mathop{\smile\!\!\!\!\!\!|}_A C$ by right transitivity.

For weak monotonicity: p over A and q over B, r over C. $A \subseteq B \subseteq C$. $a \mathop{\smile\!\!\!\!\!\!|}_A C$, then $a \mathop{\smile\!\!\!\!\!\!|}_A B$ by monotonicity.

Now assume (1), (2) and (3) hold. Note that there are $\leq 2^{\kappa+|T|}$ types over $\emptyset$ in κ-many variables. For each such $r \in S_\kappa(\emptyset)$, let $A_r \models r$. Note that $|S_n(A_r)| \leq 2^{\kappa+|T|}$ since $|A_r| \leq \kappa$. Now by (3), each type $p \in S_n(A_r)$ has at most $\mu_{r,p}$ global $\sqsubseteq$-special extensions. Let $\mu' := \sup\{\mu_{r,p} : r \in S_\kappa(\emptyset),\ p \in S_n(A_r)\}$—still a small cardinal. By (3): for any set A of size $\leq \kappa$ and any $p \in S_n(A)$, there are at most μ' global special extensions of p.

Let $\mu' := \sup\{\mu_{r,p}\}$. For any set A of size λ with $\lambda = \lambda^\kappa$ and $\lambda > \max\{\kappa, 2^{|T|}, \mu'\}$:

$$|S_n(A)| \leq |A|^\kappa \times 2^{\max\{\kappa,|T|\}} \times \mu'$$

(number of subsets A_0 of A of size $\leq \kappa$) $\times$ (number of types p over A_0) $\times$ (number of global special extensions of p)

which implies $|S_n(A)| = |A|$ for all A of size λ. This implies stability as $|S_x(A)| \leq |S_n(A)|$ and by Theorem 2.3.9, hence T is λ-stable.

Assume now that (1)–(6) hold and let $p \in S_n(A)$, $q \in S_n(B)$ with $p \subseteq q$ and $A \subseteq B$. We want to show $p \sqsubseteq q$ iff q is a non-forking extension of p.

($\Rightarrow$): Assume $p \sqsubseteq q$. Let μ be the cardinal in (3) applied to p.

Claim: For any $r \in S_n(\mathfrak{U})$, if r forks over A, then r has more than μ conjugates over A.

Proof of Claim: If r forks over A, and forking = dividing (T stable), there is some formula $\phi(x, b) \in r$ dividing over A. Then there is some $k \in \omega$ and by saturation, for any small cardinal λ, we can find an A-indiscernible sequence $(b_i : i \leq \lambda)$ with $b_i \equiv_A b$ such that $\{\phi(x, b_i) : i \leq \lambda\}$ is k-inconsistent. Since each b_i is a conjugate of b over A, $\phi(x, b_i)$ belongs to some conjugate of r. If there are $\leq \lambda$ conjugates, then some conjugate contains an infinite collection of $\phi(x, b_i)$, contradicting k-inconsistency.

By (4), q has a global extension $r \in S_x(\mathfrak{U})$ such that $q \sqsubseteq r$. By (5) transitivity, $p \sqsubseteq r$. By invariance (1), $p \sqsubseteq r'$ for all A-conjugates r' of r. But if r forks over A, then r has more than μ-many conjugates, contradicts (3) weak boundedness. Thus r does not fork over A, in particular q does not fork over A. (i.e. q is a non-forking extension of p.)

($\Leftarrow$): Assume q non-forking over A. Let $r \in S_x(\mathfrak{U})$ be a global non-forking extension of q. Let $r' \in S_n(\mathfrak{U})$ be such that $p \sqsubseteq r'$ which exists by (4). Since r does not fork over A and r' does not fork over A (by $\Rightarrow$), by Theorem 5.14, we know r and r' are conjugate. By (1), $p \sqsubseteq r$, and so $p \sqsubseteq q$ by (6). $\square$

5.3 Forking and ranks in stable theories

Definition 5.20 *(Morley rank)***.**
We define the **Morley rank** of formulas in a complete theory T as:

- $\mathrm{MR}(\psi) \geq 0$ if ψ is consistent.
- $\mathrm{MR}(\psi) \geq \beta + 1$ if there is an infinite set of formulas $(\psi_i(x) : i < \omega)$ such that $T \models \forall x\,(\psi_i(x) \to \psi(x))$, the ψ_i are pairwise 2-inconsistent, and $\mathrm{MR}(\psi_i(x)) \geq \beta$ for all i.
- $\mathrm{MR}(\psi) \geq \lambda$ for a limit ordinal λ if $\mathrm{MR}(\psi) \geq \beta$ for all $\beta < \lambda$.

And $\mathrm{MR}(\psi) := \max\{\alpha : \mathrm{MR}(\psi) \geq \alpha\}$ if the maximum exists; otherwise define $\mathrm{MR}(\psi) = \infty$ and put $\max \emptyset = -\infty$.

Note: $\mathrm{MR}(\psi) = \infty$ iff $\mathrm{MR}(\psi) \geq \alpha$ for all α.

For a (partial) type p, define $\mathrm{MR}(p) := \inf\{\mathrm{MR}(\psi) : \psi \in p\} = \min\{\mathrm{MR}(\psi) : \psi \in p\}$.

For a theory T, define $\mathrm{MR}(T) := \mathrm{MR}(x = x)$ for $|x| = 1$.

Lemma 5.21 *(Morley rank and isolation)***.**
(Def. 3.8.1 / Lem. 3.8.5.) $\mathrm{MR}(p) = \alpha$ iff p is isolated in the subspace $B_{\geq\alpha} := \{q \in X : \mathrm{MR}(q) \geq \alpha\}$ (in $S_n(\mathfrak{U})$).

Proof. If p is isolated in $B_{\geq\alpha}$, then there is $\psi_0 \in p$ such that for any $\psi \leq \psi_0$, either $\psi \in p$ or $\mathrm{MR}(\psi) < \alpha$. Thus $\mathrm{MR}(\psi_0) \not\geq \alpha + 1$ and $\mathrm{MR}(\psi_0) = \alpha$.

If p is not isolated, then for any $\psi_0 \in p$, we want to show $\mathrm{MR}(\psi_0) \geq \alpha + 1$. Since p is not isolated, there is $\psi_0' \leq \psi_0$ such that $\psi_0' \in p$ and $\mathrm{MR}(\neg\psi_0' \wedge \psi_0) \geq \alpha$ (witnessed by another type in $B_{\geq\alpha}$). Now $\psi_0' \in p$ does not isolate p; hence there is $\psi_1 \leq \psi_0'$ such that $\psi_1 \in p$ and $\mathrm{MR}(\neg\psi_1 \wedge \psi_0') \geq \alpha$, and we can continue. Thus $\mathrm{MR}(\psi) \geq \alpha + 1$. $\square$

Definition 5.22 *(Totally transcendental)***.**
T is called **totally transcendental** (t.t.) if for any tuple of variables x, $\mathrm{MR}(x = x) < \infty$.

Remark 5.23.
T is totally transcendental iff $\mathrm{MR}(T) < \infty$ iff there is no infinite binary tree of consistent $\mathcal{L}(\mathfrak{U})$-formulas.

Definition 5.24 *(SU-rank)*.
Let T be a simple theory. We define $\mathrm{SU}(p) \geq \alpha$ for a type p over a small set by recursion on α:

- $\mathrm{SU}(p) \geq 0$ for all types p.
- $\mathrm{SU}(p) \geq \beta + 1$ if p has a forking extension q with $\mathrm{SU}(q) \geq \beta$.
- $\mathrm{SU}(p) \geq \lambda$ for a limit cardinal λ if $\mathrm{SU}(p) \geq \beta$ for all $\beta < \lambda$.

We define $\mathrm{SU}(p)$, the **SU-rank** of p, as the maximal α such that $\mathrm{SU}(p) \geq \alpha$. If no maximal exists, then we set $\mathrm{SU}(p) = \infty$.

Remark 5.25 *(Diamond lemma (exercise))*.
Suppose T is simple, $p \in S_x(A)$, $q \supseteq p$ non-forking, and $r \supseteq q$ any extension. Then there is an A-conjugate r' of r, and $s \supseteq r'$ non-forking, such that $s \supseteq q$.

$$\begin{array}{ccc} & s & \\ r' & & q \\ & p & \end{array}$$

where $s \supseteq r'$ (nf), $s \supseteq q$, $r' \supseteq p$, $q \supseteq p$ (nf).

Lemma 5.26 *(SU-rank and non-forking)*.
Let T be simple. Let $q \supseteq p$. Suppose $\mathrm{SU}(p) < \infty$. Then q is a non-forking extension of p iff $\mathrm{SU}(p) = \mathrm{SU}(q)$.

If $\mathrm{SU}(p) = \infty$, then so does any non-forking extension of p.

Proof. Let $q \supseteq p$. If q has a forking extension r, then r is also a forking extension of p. Thus $\mathrm{SU}(p) \geq \mathrm{SU}(q)$.

We only need to show that if q is a non-forking extension of p and if $\mathrm{SU}(p) \geq \alpha$, then $\mathrm{SU}(q) \geq \alpha$ for all α.

Induction on α. If α is a limit ordinal, this is just by IH. If $\alpha = \beta + 1$: suppose $\mathrm{SU}(p) \geq \alpha$. Then there is $r \supseteq p$ such that r is a forking extension of p and $\mathrm{SU}(r) \geq \beta$. By the diamond lemma, there is an A-conjugate r' of r where $p \in S_x(A)$ and $s \supseteq r'$ non-forking extension such that $s \supseteq q$. By IH and $\mathrm{SU}(r') \geq \beta$, we get $\mathrm{SU}(s) \geq \beta$. But s is a forking extension of q, since if not, then $s \supseteq q$ non-forking and $q \supseteq p$ non-forking, hence $s \supseteq p$ non-forking by transitivity; contradicts $r' \supseteq p$ forking. By definition, $\mathrm{SU}(q) \geq \beta + 1$. ☐

Remark 5.27.
In simple theory, the set of SU-ranks of types forms an initial segment of the ordinal numbers (possibly union ∞). By local character, every type does not fork over some set of size $\leq |T|$. Note if $aA \equiv bB$, then $\mathrm{SU}(a/A) = \mathrm{SU}(b/B)$. Hence there are $\leq 2^{|T|}$-many

different ordinal values of SU-rank. I.e. $\mathrm{SU}(p) \leq (2^{|T|})^+$ for all p.

Definition 5.28 *(Supersimple and superstable)***.**
A theory T is **supersimple** if every type $p \in S_n(B)$ does not fork over some finite set $A \subseteq B$. A stable supersimple theory is called **superstable**.

Lemma 5.29 *(Supersimple iff finite SU-rank)***.**
T is supersimple iff $\mathrm{SU}(p) < \infty$ for all $p \in S_n(B)$ for all n, B.

Remark 5.30.
In supersimple theory, $a \not\perp\!\!\!\perp_A b$ for finite tuple a iff $\mathrm{SU}(a/Ab) = \mathrm{SU}(a/A)$.

Proof of Lemma 5.29. We will show that a type $p \in S_n(B)$ has SU-rank ∞ iff there are $p = p_0 \subseteq p_1 \subseteq p_2 \subseteq \cdots$ such that each p_{i+1} is a forking extension of p_i (and hence each p_i has SU-rank ∞ as well).

Suppose such a sequence exists; then we can easily show by induction that for all α, $\mathrm{SU}(p_i) \geq \alpha$. Hence $\mathrm{SU}(p_i) = \infty$ for all i.

In particular, if $p \in S_n(A)$ forks over every finite subset of A, then we can find inductively a sequence of finite subsets $A_0 \subseteq A_1 \subseteq \cdots$ such that $p \restriction A_{i+1}$ forks over A_i. Hence $p \restriction A_i$ has SU-rank ∞.

Conversely, suppose p has SU-rank ∞; then there is a sequence $p = p_0 \subseteq p_1 \subseteq p_2 \subseteq \cdots$ such that each p_i has SU-rank ∞ and each p_{i+1} is a forking extension of p_i. This exists because the set $\{\alpha : p \text{ has a forking extension of SU-rank } \geq \alpha\}$ is unbounded, and SU-rank of all n-types is either $\leq (2^{|T|})^+$ or $= \infty$. Thus p has a forking extension of SU-rank ∞. Then the type $q := \bigcup_{i \in \omega} p_i$ forks over every finite subset. □

5.4 The stability spectrum

Theorem 5.31 *(Stability spectrum)***.**
Let T be a countable complete theory. Then one of the following holds:

(1) T is totally transcendental and T is κ-stable for all $\kappa \geq \aleph_0$.

(2) T is superstable but not totally transcendental, then T is κ-stable for all $\kappa \geq 2^{\aleph_0}$ (and not κ-stable for $\kappa < 2^{\aleph_0}$).

(3) T is stable but not superstable, then T is κ-stable for all κ such that $\kappa^{|T|} = \kappa$ (equivalently, $\kappa^{\aleph_0} = \kappa$) (and not κ-stable for other κ).

(4) T is not stable and not κ-stable for any κ.

Proof. Recall: for an infinite cardinal κ, we say T is **κ-stable** if $|S_n(A)| \leq \kappa$ for all $|A| \leq \kappa$ (equivalently, $|S_1(A)| \leq \kappa$ for all $|A| \leq \kappa$).

(1) If T is totally transcendental, fix n, and $\kappa \geq \aleph_0$. We want to show $|S_n(A)| \leq \kappa$ for all $|A| \leq \kappa$. Let us call a formula $\psi(x) \in \mathcal{L}_A$ *large* if $\psi(x)$ belongs to $> \kappa$-many types in $S_n(A)$, and call it *thin* if otherwise.

Suppose towards a contradiction that a large formula $\psi(x)$ exists. We want to build a binary tree such that each level consists of large formulas, hence contradicting t.t.

It is enough to show: if $\psi(x)$ is large, then there is $\psi'(x)$ such that $\psi(x) \wedge \psi'(x)$ is large and $\neg\psi'(x) \wedge \psi(x)$ is large. Let $\psi(x)$ be a large formula; let $S := \{p \in S_n(A) : \psi(x) \in p\}$. Then $|S| > \kappa$. There are at most κ-many $\mathcal{L}_A$-formulas (by $\mathcal{L}$ countable), and each thin formula is contained in $\leq \kappa$-many types. Hence there are $|S|$-many types in S which contain only large formulas. Take $p \neq p' \in S$ with p, p' only contain large formulas; then there is $\psi'(x) \in \mathcal{L}_A$ such that $\psi(x) \wedge \psi'(x) \in p$, hence large, and $\neg\psi'(x) \wedge \psi(x) \in p'$, hence also large.

(2) Firstly note that if T is not t.t., then there exists an infinite binary tree of consistent $\mathcal{L}(\mathfrak{U})$-formulas. Hence over the set of the parameters appeared in the tree (which there are countably many of them), there are $2^{\aleph_0}$-many types. Hence T is not κ-stable for $\kappa < 2^{\aleph_0}$.

Now if T is superstable, let A be $|A| \leq \kappa$. By definition, every type $p \in S_n(A)$ does not fork over a finite subset of A. Hence

$$|S_n(A)| \leq \underbrace{\kappa}_{\substack{\text{number of finite} \\ \text{subsets } E \text{ of } A}} \times \underbrace{2^{\aleph_0}}_{|S_n(E)| \text{ where } E \subseteq A \text{ finite}} \times \underbrace{2^{\aleph_0}}_{\substack{\text{number of non-forking} \\ \text{extensions of a given } p \in S_n(E)}} = \kappa + 2^{\aleph_0}.$$

Thus $|S_n(A)| = \kappa + 2^{\aleph_0}$ for $|A| = \kappa$, and κ-stable for $\kappa \geq 2^{\aleph_0}$.

(3) T is stable iff T is κ-stable for all κ such that $\kappa^{|T|} = \kappa$. By Theorem 2.3.9 (or the last part of Section 2.2 in my notes).

If T is not superstable, we want to find $|A| = \kappa$ such that $|S_n(A)| \geq \kappa^{\aleph_0}$.

Claim: Suppose $p \in S_n(B)$ has SU-rank ∞ and $|B| \leq \kappa$. Then there are $B' \supseteq B$ of size $|B'| \leq \kappa$ and κ-many distinct forking extensions $(p_\alpha)_{\alpha<\kappa}$ of p to B', each of SU-rank ∞.

Proof of Claim: Let p with $\mathrm{SU}(p) = \infty$. Then by Lemma 5.29 (the proof), p has a forking extension p' of SU-rank ∞. Let $q \supseteq p'$ be a global non-forking extension of p'. Then q forks over B (since p' does). By the proof of Theorem 5.14, q has unbounded-many B-conjugates. Let $(q_\alpha)_{\alpha<\kappa}$ be a sequence of distinct B-conjugates of q. Let $B' \supseteq B$ be such that $|B'| \leq \kappa$ and $(p_\alpha := q_\alpha \restriction B')_{\alpha<\kappa}$ are pairwise distinct. Now $\mathrm{SU}(p_\alpha) \geq \mathrm{SU}(q_\alpha) = \mathrm{SU}(q) = \mathrm{SU}(p') = \infty$, since q_α is a non-forking extension of p' and apply Lemma 5.26.

Suppose T is not superstable. Then T has a type $p \in S_n(B)$ with $\mathrm{SU}(p) = \infty$. Let $p_0 := p \restriction \emptyset$; then $\mathrm{SU}(p_0) \geq \mathrm{SU}(p) = \infty$. By Claim, there is $A_1 \supseteq \emptyset$ and $(p_\alpha)_{\alpha<\kappa}$ distinct extensions of p_0, each of SU-rank ∞.

Now suppose A_n has been constructed and $p_{\alpha_1 \cdots \alpha_n}$ with $\alpha_1, \ldots, \alpha_i < \kappa$ and $A_1 \subseteq A_2 \subseteq$

$\cdots \subseteq A_n$ of size $\leq \kappa$, and for $i \leq n$, there are $p_{\alpha_1 \cdots \alpha_i}$ such that $(p_{\alpha_1 \cdots \alpha_{i-1} \beta})_{\beta < \kappa}$ are distinct extensions of $p_{\alpha_1 \cdots \alpha_{i-1}}$ to $B_{\alpha_1 \cdots \alpha_n}$, each of SU-rank ∞.

Now let given $p_{\alpha_1 \cdots \alpha_n}$, for each $\alpha_i < \kappa$ and $i \leq n$, by the Claim, there is $B_{\alpha_1 \cdots \alpha_n} \supseteq A_n$ such that $|B_{\alpha_1 \cdots \alpha_n}| \leq \kappa$ and $(p_{\alpha_1 \cdots \alpha_n \beta})_{\beta < \kappa}$ extensions of $p_{\alpha_1 \cdots \alpha_n}$ to $B_{\alpha_1 \cdots \alpha_n}$, each of SU-rank ∞. Let $A_{n+1} := \bigcup_{\alpha_1 \cdots \alpha_n \in \kappa^n} B_{\alpha_1 \cdots \alpha_n}$. Then $|A_{n+1}| \leq \kappa$.

Let $A := \bigcup_n A_n$. Now for any path β, we have a type $p_{\alpha_1 \cdots \alpha_i \cdots}$ over A which is distinct. Hence $|S_n(A)| \geq \kappa^{\aleph_0}$ and $|A| = \kappa$, as desired. $\square$

6 Stable Groups

6.1 Stability of abelian groups

The goal of this section is to show every abelian group G in the language of groups $(G, +, 0)$ is stable.

Definition 6.1 *(R-modules)*.
Let R be a (possibly non-commutative) ring with 1. An R-**module** M is a structure in a language

$$\mathcal{L}_{\text{mod}}^{R} := \{0, +, -, (r)_{r \in R}\}$$

with r unary function symbols, such that:

(1) $(M, +, -, 0)$ is an abelian group.

(2) $\forall x, y \;\; r(x + y) = r(x) + r(y)$.

(3) $\forall x \;\; (r + s)(x) = r(x) + s(x)$.

(4) $\forall x \;\; (rs)(x) = r(s(x))$.

(5) $\forall x \;\; 1(x) = x$.

for all $r, s \in R$. We will write rx instead of $r(x)$.

Definition 6.2 *(Equations and pp-formulas)*.

(1) An **equation** is an $\mathcal{L}_{\text{mod}}^{R}$-formula $\phi(\bar{x})$ of the form $r_1 x_1 + \cdots + r_m x_m = 0$, over $\emptyset$.

(2) A **positive primitive formula** (a **pp-formula**) is of the form

$$\psi(\bar{x}) = \exists \bar{y} \, (\phi_1(\bar{x}, \bar{y}) \wedge \cdots \wedge \phi_n(\bar{x}, \bar{y}))$$

where $\phi_i(\bar{x}, \bar{y})$ are equations.

(3) A **pp-formula over** $\emptyset$ is of the form $\psi(\bar{x}, \bar{a})$ where $\psi(\bar{x}, \bar{y})$ is a pp-formula over $\emptyset$.

Remark 6.3.

(1) The class of pp-formulas is closed under $\wedge$. ($\exists \bar{y}\, \phi(\bar{x},\bar{y}) \wedge \exists \bar{z}\, \phi'(\bar{x},\bar{z})$ is equivalent to $\exists \bar{y}\, \bar{z}\, (\phi(\bar{x},\bar{y}) \wedge \phi'(\bar{x},\bar{z}))$ by making $\bar{y},\bar{z}$ disjoint.)

(2) A pp-formula $\phi(x_1,\dots,x_n)$ (without parameters) defines a subgroup of $(M^n,+,0)$, since: (a) $M \models \phi(0,\dots,0)$; (b) if $M \models \phi(\bar{x})$, then $M \models \phi(-\bar{x})$; (c) if $M \models \phi(\bar{x})$ and $M \models \phi(\bar{y})$, then there are $\bar{z}$ and $\bar{z}'$ such that $M \models \bigwedge_i \phi_i(\bar{x},\bar{z}) \wedge \bigwedge_i \phi_i(\bar{y},\bar{z}')$ where $\phi(\bar{x})$ is $\exists \bar{z} \bigwedge_i \phi_i(\bar{x},\bar{z})$. Thus $M \models \bigwedge_i \phi_i(\bar{x}+\bar{y},\bar{z}+\bar{z}')$, i.e. $M \models \phi(\bar{x}+\bar{y})$.

(3) If $\phi(x,y)$ is a pp-formula and $a \in M^{|y|}$, then $\phi(M,a)$ is either empty or a coset of $\phi(M,0)$. Since if $g \in \phi(M,a)$ and $h \notin \phi(M,0)$, then $g+\phi(M,0) \subseteq \phi(M,a)$, so $\phi(M,a)$ contains the coset $g+\phi(M,0)$; and $-g+\phi(M,a) \subseteq \phi(M,0)$, so $\phi(M,a)$ is contained in the coset $g+\phi(M,0)$.

Theorem 6.4 *(Baur–Monk)*.
For every ring R and any R-module M, every $\mathcal{L}^R_{\mathrm{mod}}$-formula is equivalent (modulo the theory of M) to a Boolean combination of pp-formulas.

Corollary 6.5 *(Abelian groups are stable)*.
Every theory of an R-module is stable (by compactness, the theory of all R-modules is stable as well). In particular, every abelian group G in the language of groups $(G,+,0)$ is stable.

Proof. We have proved stable formulas are closed under Boolean combinations. Hence it is enough to show that all pp-formulas are stable by Theorem 6.4 and Lemma 2.8. By Remark 6.3(3), if $\phi(x,y)$ is a pp-formula, then for any $b,b' \in M^{|y|}$, either $\phi(M,a)=\phi(M,b)$ or $\phi(M,a)\cap\phi(M,b)=\emptyset$. There cannot be $(a_i,b_i)_{i\in\omega}$ such that $M \models \phi(a_i,b_j) \leftrightarrow i<j$, since $(\phi(M,b_j))_{j\in\omega}$ must be distinct cosets and each a_i can only be in at most one of them.
$\square$

Definition 6.6 *(Stable group)*.
Let $(G,\cdot,1,\dots)$ be a group possibly with additional structure in the language $\mathcal{L}$ (which extends the group language). We call $(G,\cdot,1,\dots)$ a **stable group** if $\mathrm{Th}_{\mathcal{L}}(G)$ is stable.

Examples of stable groups.

(1) Abelian groups (in pure group language).

(2) Groups definable/interpretable in stable theories:

  (i) Let K be an algebraically closed field. Then $\mathrm{GL}_n(K)$, $\mathrm{SL}_n(K)$.

  (ii) More generally: affine algebraic groups $(G,\cdot,1)$ where $G \subseteq K^n$ is an affine variety, i.e. given by polynomial equations, and both $\cdot : G^2 \to G$, $(\ )^{-1} : G \to G$ are regular maps, i.e. given by polynomials $(f_1,\dots,f_n)$.

(3) Let $(K, 0, 1, \delta)$ be a differentially closed field of characteristic 0. (Fact: DCF_0 is superstable.) Then all groups definable there are superstable.

Remark 6.7 *(Weil–Hrushovski)***.**
Any definable group in ACF can be equipped with the structure of an algebraic group.

Cherlin–Zilber conjecture: Infinite totally transcendental / ω-stable simple groups (in a countable language expanding the language of groups) are simple algebraic groups over algebraically closed fields. (Proved for $\mathrm{MR}(G) \leq 3$ by Frécon, 2016.)

6.2 Chain conditions in stable groups

Definition 6.8 *(Uniform definable family, SOP)***.**

- A **uniformly definable family** of subsets $(X_i, i \in I)$ is a family of subsets such that $X_i = \phi(\mathfrak{U}^{|x|}, a_i)$ for some formula $\phi(x, y)$ and $a_i \in \mathfrak{U}^{|y|}$.
- A **uniformly definable family of subgroups** is a uniformly definable family where each subset is a subgroup.
- A formula $\phi(x, y)$ has the **strict order property** (SOP) if there is an infinite sequence $(b_i)_{i \in \omega}$ such that $\phi(\mathfrak{U}^{|x|}, b_i) \subsetneq \phi(\mathfrak{U}^{|x|}, b_j)$ for all $i < j \in \omega$.
- A theory T is **NSOP** if no formula has SOP.

Fact 6.9 *(Shelah)***.**
T is stable iff T is NSOP and NIP.

Remark 6.10.
If $\phi(x, y)$ has SOP, then take $a_i \in \phi(\mathfrak{U}^{|x|}, b_i) \setminus \phi(\mathfrak{U}^{|x|}, b_{i-1})$; then $\mathfrak{U} \models \phi(a_i, b_j) \iff i \leq j$, i.e. $\phi(x, y)$ has the order property.

Lemma 6.11 *(Chain condition for NSOP groups)***.**
Let G be an NSOP group. For every formula $\phi(x, y)$ there is some $n = n(\phi) \in \omega$ such that every chain $H_1 \subseteq H_2 \subseteq \cdots$ of subgroups of G, uniformly definable by ϕ, has length at most n. Same for descending chains $H_1 \supseteq H_2 \supseteq \cdots$.

Proof. From $\phi(x, y)$: G is NSOP, hence has no SOP. □

Lemma 6.12 *(Chain condition for NIP groups)***.**
Let G be an NIP group. For every formula $\phi(x, y)$ there is some number $m = m(\phi) \in \omega$ such that if I is finite and $(H_i : i \in I)$ is a uniformly definable family of subgroups of G of the form $H_i = \phi(\mathfrak{U}, a_i)$ for some parameters a_i, then $\bigcap_{i \in I} H_i = \bigcap_{i \in I_0} H_i$ for some $I_0 \subseteq I$ with $|I_0| \leq m$.

Proof. Otherwise, for each $m \in \omega$, there are some subgroups $(H_i : i \leq m)$ such that $H_i = \phi(\mathfrak{U}, a_i)$ and

$$\bigcap_{i \leq m} H_i \neq \bigcap_{\substack{i \leq m \\ i \neq j}} H_i \quad \text{for all } j \leq m.$$

Let $b_j \in \bigcap_{\substack{i \leq m \\ i \neq j}} H_i$ and $b_j \notin \bigcap_{i \leq m} H_i$. Note that $b_j \notin H_j$ and $b_j \in H_i$ for all $i \neq j$.

For $I \subseteq \{0, 1, \ldots, m\}$, let $b_I := \prod_{j \in I} b_j$. Then $\mathfrak{U} \models \phi(b_I, a_j) \iff j \notin I$. Since if $j \in I$, then $b_I = b_j \cdot \prod_{i \in I, i \neq j} b_i$ with $b_i \in H_j$ for $i \neq j$ and $b_j \notin H_j$. Hence $b_I \notin H_j$, i.e. $\mathfrak{U} \models \neg\phi(b_I, a_j)$. If $j \notin I$, then for all $i \in I$, $b_i \in H_j$. Hence $b_I \in H_j$. Namely, $\mathfrak{U} \models \phi(b_I, a_j)$.

This is IP for ϕ, contradiction. $\square$

Theorem 6.13 *(Baldwin–Saxl)*. Let G be a definable group. Then for any stable formula $\phi(x, y)$ there is some $k = k(\phi) \in \omega$ such that any intersection of ϕ-definable subgroups is equal to a sub-intersection of size at most k.

Remark 6.14. If $(H_i)_{i \in \omega}$ is a uniformly definable family of descending subgroups, then each finite intersection is equal to one m-intersection of H_i's, but $\bigcap_{i \in \omega} H_i$ may not equal any H_i.

Proof of Theorem 6.13. Consider the lattice of finite intersections of finitely many subgroups defined by $\phi(x, y)$. Let $(H_i)_{i \in I}$ be a family of subgroups defined by $\phi(x, y)$. Namely, $H_{I_0} := \bigcap_{i \in I_0} H_i$ for $I_0 \subseteq I$ finite.

By Lemma 6.12, each H_{I_0} is an m-intersection of H_i's in I_0. Hence $(H_{I_0})_{I_0 \subseteq_{\text{fin}} I}$ are uniformly definable by $\psi(x, \bar{y}) := \bigwedge_{j \leq m} \phi(x, y_j)$. By Lemma 6.11, the descending chain of $(H_{I_0})_{I_0 \subseteq_{\text{fin}} I}$ can be only of length k' for some fixed $k' \in \omega$. Since given I_0, I_1 with $I_0 \subseteq_{\text{fin}} I$, we have $H_{I_0 \cup I_1} \subseteq H_{I_0}$ and $H_{I_0 \cup I_1} \subseteq H_{I_1}$. The lattice $(H_{I_0})_{I_0 \subseteq_{\text{fin}} I}$ has a minimal element H_J, since the descending chain has length $\leq k'$. Thus $H_J \subseteq H_i$ for all $i \in I$ and $\bigcap_{i \in I} H_i$ is indeed an m-intersection of H_i's. $\square$

Corollary 6.15 *(Centraliser of an arbitrary subset)*. Let G be a stable group and $A \subseteq G$ be arbitrary subset, where $C_G(A) = \{g \in G : ga = ag \text{ for all } a \in A\} = \bigcap_{a \in A} C_G(a)$. Then there is some finite $A_0 \subseteq A$ such that $C_G(A) = C_G(A_0)$.

Corollary 6.16 *(Definable abelian subgroup containing A)*. Let G be a stable group and $A \leq G$ be an abelian subgroup. Then there is a definable abelian subgroup $A' \supseteq A$ of G.

Proof. Consider $Z(C_G(A))$; clearly $A \leq C_G(A)$ since A is abelian. By definition of $C_G(A)$, $A \leq Z(C_G(A))$. Hence $A' := Z(C_G(A))$ is a definable abelian subgroup of G containing A, where A' is definable by Corollary 6.15. $\square$

Definition 6.17 *(Connected group)***.**
We say a group $(G, \cdot, \dots)$ is **connected** if there is no proper definable subgroup of finite index.

Corollary 6.18 *(Additive group of a stable field)***.**
Suppose $(K, +, \cdot, 0, 1, \dots)$ is stable. Then the additive group $(K, +, 0, \dots)$ is connected and infinite.

Proof. Suppose K has a definable subgroup H of finite index. If $a \in K^*$, then $[K : aH] = [aK : aH] = [K : H] < \infty$. Namely aH is a subgroup of finite index.

By Theorem 6.13 (Baldwin–Saxl), the group $I_* = \bigcap_{a \in K^*} aH$ is a definable subgroup of K of finite index. Note that I_* is an ideal: $aI_* \subseteq a(a^{-1}H) = H$ for all $a \in K^*$. Hence $aI_* \subseteq I_*$ for all $a \in K^*$. But a field has no proper non-zero ideal, since if $b \in I_*$ and $b \neq 0$, then $b^{-1}I_*$ contains 1. Thus aI_* contains a for all $a \in K^*$. Thus $I_* = K$ and $H = K$. $\square$

6.3 Connected component of a stable group

Definition 6.19 *(G^0_ϕ and G^0)***.**
Let $G = G(\mathfrak{U})$ be a stable group.

(1) For $\phi(x, y) \in \mathcal{L}$, let

$$G^0_\phi := \bigcap \{H \leq G : H = \phi(\mathfrak{U}, a) \text{ for some } a \in G \text{ and } [G : H] < \infty\}.$$

(2) Let $G^0 := \bigcap_{\phi \in \mathcal{L}} G^0_\phi$—the **connected component** of G. It is normal (since definable subgroups of finite index in G are closed under conjugation) and type-definable over $\emptyset$.

Remark 6.20.

(1) By Theorem 6.13, G^0_ϕ is a definable subgroup of finite index in G. It is $\mathrm{Aut}(\mathfrak{U})$-invariant. Hence it is definable over $\emptyset$.

(2) Note that $[G : G^0] \leq 2^{|T|}$, since there are $|T|$-many G^0_ϕ and if one fixes a coset for each G^0_ϕ, one fixes the coset of G^0 in G.

(3) G is connected iff $G^0 = G$.

(4) Suppose G is an algebraic group; then G^0 is the connected component of G in the Zariski topology.

(5) By the chain condition on ω-stable groups, G^0 is definable.

7 Fundamental Theorem of Stable Groups

Reference: Bruno Poizat, *Stable Groups*, Section 5.1; Lectures 2, 3 by David Marker, *Lectures on Large Stable Fields*.

Let G be a stable group.

Definition 7.1 *(Generic set and type)***.**
A definable subset $X \subseteq G$ is called **generic** if for some n, there are $a_1, \ldots, a_n \in G$ and $b_1, \ldots, b_n \in G$ such that

$$G = a_1 X \cup \cdots \cup a_n X \cap b_n.$$

We say X is **left generic** if for some n there are $a_1, \ldots, a_n \in G$ such that $G = a_1 X \cup \cdots \cup a_n X$. Similarly X is **right generic** if $G = Xa_1 \cup \cdots \cup Xa_n$.

We say a type p (partial) is **generic** if all definable sets given by formulas in p are generic.

Goal: Generic types exist in stable groups, and generic is equivalent to left/right generic, which implies non-forking over $\emptyset$.

Lemma 7.2.
Let $A \subseteq G$ be definable. If A is not left generic, then $G \setminus A$ is right generic.

Proof. Suppose not; then for any $a_1, \ldots, a_n \in G$, there is $x \notin \bigcup_{i \leq n} (G \setminus A) a_i^{-1}$, i.e. $xa_i \notin G \setminus A$, i.e. $xa_i \in A$. And for any $b_1, \ldots, b_n \in G$, there is $y \notin \bigcup_{i \leq n} b_i^{-1} A$, i.e. $b_i y \notin A$ for all $i \leq n$.

We build $c_1, c_1', c_2, c_2', \ldots$ and $d_1, d_1', d_2, d_2', \ldots$ as follows: let c_1 be arbitrary. Choose suppose $c_1, \ldots, c_n, d_1, \ldots, d_{n-1}$ have been chosen. Choose d_n such that $c_i d_n \notin A$ for all $1 \leq i \leq n$ (using (2)). Then choose c_{n+1} such that $c_{n+1} d_i \in A$ for all $1 \leq i \leq n$ (using (1)).

Then $c_i d_j \in A \iff i > j$. And the formula $xy \in A$ has the order property, contradicting stability. $\square$

Corollary 7.3.
If $A, B \subseteq G$ are definable and $A \cup B$ is generic, then either A or B is generic.

Proof. Suppose $G = \bigcup_{i=1}^n a_i (A \cup B) b_i = (\bigcup_{i=1}^n a_i A b_i) \cup (\bigcup_{i=1}^n a_i B b_i)$. Then by Lemma 7.2, either $\bigcup_{i=1}^n a_i A b_i$ is left generic, or $\bigcup_{i=1}^n a_i B b_i$ is right generic.

Suppose $\bigcup_{i=1}^n a_i A b_i$ is left generic; then there are $(c_j)_{j \leq m}$ such that $G = \bigcup_{j=1}^m \bigcup_{i=1}^n c_j a_i A b_i$, and A is generic. Similarly for $\bigcup a_i B b_i$ right generic. $\square$

Corollary 7.4 *(Generic types exist)***.**
Let $A \subseteq G$ be generic. Then there is a complete generic type p extending $\{x \in A\}$. Given a set C of parameters, in particular generic types exist.

Proof. Let $\mathcal{P} := \{X \supseteq A \cup B : G \setminus B \text{ is not generic}\}$. If $A_1, \ldots, A_n \in \mathcal{P}$, then $\bigcup_{i=1}^n (G \setminus A_i)$ is not generic by Corollary 7.3. Hence $\bigcap_{i=1}^n A_i = G \setminus \bigcup_{i=1}^n (G \setminus A_i)$ is in $\mathcal{P}$. By the fact that

G is generic and Corollary 7.3, any set $B \in \mathcal{P}$ is generic. Hence it is consistent. Thus $\mathcal{P}$ is consistent. And any complete type extending $\mathcal{P}$ would be generic by Corollary 7.3 again. (In another word, the set of non-generics forms an ideal, and we will see that it contains the forking ideal over $\emptyset$.) $\square$

Lemma 7.5 *(Generic type and translates)***.**
Suppose p is a (complete) generic type and A a definable set. Then A is generic iff some translate cAd is in p.

Proof. ($\Rightarrow$): If A is generic, then there are $(a_i, b_i)_{i \leq n}$ in $G(\mathfrak{U})$ such that $G = \bigcup_{i=1}^{n} a_i A b_i$. Since p is a complete type over $\mathfrak{U}$, some $a_i A b_i \in p$.

($\Leftarrow$): If cAd is in p, then cAd is generic. And $G = \bigcup_{i \leq N} a_i(cAd)b_i = \bigcup a_i c \cdot A \cdot d b_i$ for some $(a_i, b_i)_{i \leq n}$. $\square$

Lemma 7.6 *(Set of generics is definable)***.**
For every formula $\phi(x, y)$, the set $\{b : \phi(x, b) \text{ is generic}\}$ is definable.

Proof. Let p be a generic global type (which exists by Corollary 7.4). Then $\phi(x, b)$ is generic iff there are $c, d \in G(\mathfrak{U})$ such that $\phi(cxd, b) \in p$, by Lemma 7.5. Let M be a small model. Let $\psi(x; y, u, v) := \phi(uxv, y)$. By definability of p, there is $d_p\psi(y, u, v)$ such that $\psi(x, b, c, d) \in p$ iff $d_p\psi(b, c, d)$ holds in $\mathfrak{U}$.

Hence $\exists u\, \exists v\, d_p\psi(y, u, v) \iff \phi(x, b)$ is generic. Note that the set $\{b : \phi(x, b) \text{ is generic}\}$ is invariant under all automorphisms, where $\phi(x, y)$ is defined over. Hence it is definable over the parameters of $\phi(x, y)$. $\square$

Corollary 7.7 *(Bounded covering by translates)***.**
For every formula $\phi(x, y)$, there is $N \in \omega$ such that $\phi(x, b)$ defines a generic set iff G is covered by N two-sided translates of $\phi(x, b)$.

Proof. Let $\psi_n(y)$ be: $\exists u_1 \cdots u_n\, \exists v_1 \cdots v_n \left(\bigwedge_{i=1}^{n} \phi(u_i x v_i, y)\right)$. Then $\phi(x, b)$ is generic iff $\mathfrak{U} \models \bigvee_{n \in \omega} \psi_n(b)$. By Lemma 7.6, the set $\{b : \phi(x, b) \text{ is generic}\}$ is definable, which equals $\bigcup_{n \in \omega} \psi_n(\mathfrak{U})$. By compactness, some $\psi_N(\mathfrak{U})$ contains all the elements b such that $\phi(x, b)$ is generic. $\square$

Lemma 7.8 *(Non-forking extension of generic type)***.**
If p is generic and q a non-forking extension of p, then q is also generic.

Proof. Suppose p is over the set of parameters C. Let $M \supseteq C$ be a small model. Since p is generic, there is an extension $r \supseteq p$ which is over M and generic. By the diamond lemma, there is a C-conjugate r' of r (over M', which is a conjugate of M), and s a non-forking extension of r' which also extends q.

Since r is generic, r' is also generic and M' is a model. It is enough to show s is generic since it extends q. Suppose not; then there is $\phi(x,b) \in s$ such that $\phi(x,b)$ is non-generic. Let where $\phi(x,y)$ has parameters in M'. Let $\psi(y)$ be the defining formula for "$\exists y : \phi(x,y)$ is generic". Then $\psi(y)$ has parameters in M' and $\phi(x,b) \wedge \neg\psi(b) \in s$.

Since r' is definable over M' and s is a non-forking extension, s is uniquely determined by the definition of r' over M'. In particular, there is a formula $\chi(y)$ over M' such that $\mathfrak{U} \models \chi(d) \iff \phi(x,d) \wedge \neg\psi(d) \in s$ for good d (a parameter in the parameter space of s).

Now that $\mathfrak{U} \models \exists d\, \chi(d)$, we have $M' \models \exists a\, \chi(a)$. Hence some $\phi(x,a) \wedge \neg\psi(a) \in r'$ for $a \in M'$. But $\psi(y)$ defines the generics of $\phi(x,y)$; hence $\phi(x,a)$ is non-generic, contradicting r' is generic.

Now we aim to show that if p is a generic type over a set of parameters $C \subseteq \mathfrak{U}$, then p does not fork over $\emptyset$. □

Definition 7.9 *(Shelah's ∞-rank)***.**
Let $\mathfrak{U} \models T$ be a monster model and Δ a collection of formulas $\phi(x;y)$ with fixed length of x. Let $\Phi(x)$ be a partial type over a small set of parameters. Define $R^\infty_\Delta(\Phi) \geq \alpha$ inductively as:

(1) $R^\infty_\Delta(\Phi) \geq 0$ if $\Phi(x)$ is consistent.

(2) $R^\infty_\Delta(\Phi) \geq \alpha + 1$ if there are $(\psi_i(x,b_i))_{i\in\omega}$ with $\psi_i(x,y_i) \in \Delta$ which are 2-inconsistent and $R^\infty_\Delta(\Phi \cup \{\psi_i(x,b_i)\}) \geq \alpha$.

(3) For α a limit ordinal, $R^\infty_\Delta(\Phi(x)) \geq \alpha$ if $R^\infty_\Delta(\Phi(x)) \geq \beta$ for all $\beta < \alpha$.

Remark 7.10.
Let Δ be the set of all formulas in variable x without parameters (with x fixed length). Then $R^\infty_\Delta(\Phi(x))$ is the Morley rank of $\Phi(x)$.

Theorem 7.11 *(Local rank and forking (Prop. 3.8.2, Prop. 3.8.6))***.**
Let Δ be a finite set of stable formulas.

(1) There is $N \in \omega$ such that $R^\infty_\Delta(p) \leq N$ for all Δ-types p, where Δ-types are collections of $\phi(x,b)$ where $\phi \in \Delta$.

(2) If $A \subseteq B$, $p \in S_\Delta(B)$, then p forks over A iff $R^\infty_\Delta(p) < R^\infty_\Delta(p \restriction A)$.

Corollary 7.12 *(Generic types don't fork over ∅)***.**
Let p be a generic type of G. Then p does not fork over $\emptyset$.

Proof. Let $\phi(x,b) \in p$. Suppose $\phi(x,b)$ forks over $\emptyset$. Let $\psi(x;u,v,y) := \phi(u \cdot x \cdot v, y)$. Then $\phi(x,b) = \psi(x;1,1,b)$ is a forking formula. Let $q := p \restriction \psi \restriction \emptyset$ be the ψ-type of p restricted to $\emptyset$. Extend q to a global ψ-type q' such that (let $\Delta := \{\psi\}$) $R^\infty_\Delta(q') = R^\infty_\Delta(q)$. (This exists since formulas of smaller rank form an ideal.) Let X be any definable set given by p_ψ (the ψ-type of p). We will show that $R^\infty_\Delta(X) \geq R^\infty_\Delta(q') \geq R^\infty_\Delta(q) = R^\infty_\Delta(q')$, which contradicts Theorem 7.11(2) (since p_ψ forks over $\emptyset$).

Since p is generic, in particular X is generic. There are $(a_i, b_i)_{i\leq N}$ such that $G = \bigcup_{i\leq N} a_i X b_i$. Now q' is a global type; there is some $a_i X b_i \in q'$. (This uses: if X is ψ-definable, then $a_i X b_i$ is also ψ-definable.)

Now observe that $R^\infty_\Delta(X) = R^\infty_\Delta(a_i X b_i)$ (since if $(\psi_j)_{j\leq N}$ are disjoint ψ-definable subsets of X, then $(a_i \psi_j b_i)_{j\leq N}$ are disjoint ψ-definable subsets of $a_i X b_i$). Thus $R^\infty_\Delta(X) = R^\infty_\Delta(a_i X b_i) \geq R^\infty_\Delta(q') = R^\infty_\Delta(q)$ (since $a_i X b_i \in q'$). □

Corollary 7.13 *(Bounded number of global generic types).*
There are a bounded number of global generic types.

Proof. Since generic types do not fork over $\emptyset$, generic types are non-forking global extensions. There are $\leq 2^{|T|}$-many types over $\emptyset$, and let $p \in S(\emptyset)$ be a type over $\emptyset$; then there are $\leq 2^{|T|}$-many global non-forking extensions of p. In total, there are $\leq 2^{|T|}$-many global types which do not fork over $\emptyset$. By Corollary 7.12, there are $\leq 2^{|T|}$-many global generic types. □

Corollary 7.14 *(Finitely many generic ψ-types).*
Let $\psi(x, y)$ be a formula. Then the set of global generic ψ-types is finite.

Proof. Suppose the set of global generic ψ-types is infinite. We will show that there are unboundedly many generic ψ-types; in particular there are unboundedly many generic types, contradicting Corollary 7.13.

Let I be an arbitrary small index set. By Prop. 2.3.7, all ψ-types are uniformly definable, say by $\chi(y, \bar{z})$. Let $\mathcal{P}((\bar{z}_i)_{i\in I})$ be the partial type saying:

(1) $\chi(y, \bar{z}_i)$ defines a generic ψ-type, by: $\forall y\,(\chi(y, \bar{z}_i) \to \psi(x, y)$ is generic), $\forall y\,(\neg\chi(y, \bar{z}_i) \to \neg\psi(x, y)$ is generic).

(2) $\chi(y, \bar{z}_i)$ and $\chi(y, \bar{z}_j)$ define different generic ψ-types: for $i \neq j$, $\neg\forall y\,(\chi(y, \bar{z}_i) \leftrightarrow \chi(y, \bar{z}_j))$.

Then $\mathcal{P}((\bar{z}_i)_{i\in I})$ is consistent, since for any finite set of $(\bar{z}_i)_{i\leq N}$, one can take b $(c_i)_{i\leq N}$ such that $\chi(y, c_i)$ defines a generic ψ-type p_i. They exist by assumption that there are infinitely many generic ψ-types. By saturation, there are $|I|$-many (small) generic ψ-types. □

Lemma 7.15 *(Independence and generic product).*
Let A be a small set of parameters. If g is generic over A and $a \mathop{\not\downarrow}_A b$, then $a \cdot b$ is generic over Ab and $a \cdot b \mathop{\not\downarrow}_A b$.

Proof. Since $\operatorname{tp}(a/A)$ is generic and $\operatorname{tp}(a/Ab)$ is a non-forking extension of $\operatorname{tp}(a/A)$, hence $\operatorname{tp}(a/Ab)$ is also generic (Lemma 7.8).

Let $X \in \operatorname{tp}(a \cdot b/Ab)$; then $ab \in X$ and $a \in Xb^{-1}$; namely $Xb^{-1} \in \operatorname{tp}(a/Ab)$, which is generic. Thus X is generic. (Translation of generic sets are generic: if $G = \bigcup a_i X b_i$, then $G = \bigcup a_i X g(g^{-1} b_i)$.) Namely $\operatorname{tp}(a \cdot b/Ab)$ is generic and does not fork over $\emptyset$. I.e. $a \cdot b \perp\!\!\!\perp_{\emptyset} Ab$. Hence $a \cdot b \perp\!\!\!\perp_A b$. ☐

Corollary 7.16 *(Every element is a product of two generics).*
Let A be a small set of parameters. Then any element $g \in G$ is a product of two generics.

(In general, if G is an $(|A| + |T|)^+$-saturated model, then every element in G is a product of two generics.)

Proof. Let $g \in G$. Take $h \in G$ such that h is generic and $h \perp\!\!\!\perp_A g$ (this exists by: take a generic type over A or by taking a generic type over Ag, then it does not fork over $\emptyset$, in particular does not fork over A). Then $h^{-1}g$ is generic over A by Lemma 7.15. And h is also generic (if X is generic, then X is generic). Thus $g = h \cdot (h^{-1}g)$. ☐

Corollary 7.17 *(Generic iff left-generic iff right-generic).*
Let $X \subseteq G$ be a definable set. If X is generic, then X is left-generic. Thus, generic $\Leftrightarrow$ left-generic ($\Leftrightarrow$ right generic).

Proof. Let M be a small model over which X is defined. By saturation of G, it is enough to show that $G = \bigcup_{g \in M} gX$ (then G is a finite union of left translates of X; if otherwise by saturation, one can find $\bar{g} \in G$ such that $\bar{g} \notin \bigcup_{g \in M} gX$).

Let $b \in G$. We will first show that there is $c \in G$ such that $b \perp\!\!\!\perp_M c$ and $b \in cX$. Then we use the property of non-forking over models to get $c' \in M$.

Let $a \in X$ such that $\operatorname{tp}(a/Mb)$ is generic (exists by Corollary 7.4). By generic type does not fork over $\emptyset$, we get $a \perp\!\!\!\perp_M b$. By Lemma 7.15, $a \cdot b^{-1} \perp\!\!\!\perp_M b$ and $\operatorname{tp}(ab^{-1}/Mb)$ is generic. Let $c := (ba^{-1})^{-1} = ab^{-1}$. Then $b = (ba^{-1})(a) \in c^{-1} \cdot X$, i.e. $b \in c^{-1}X$. And $c \perp\!\!\!\perp_M b$ (since $c = ab^{-1} \perp\!\!\!\perp_M b$). By symmetry of $\perp\!\!\!\perp$, $b \perp\!\!\!\perp_M c$. Hence $\operatorname{tp}(b/Mc)$ is a non-forking extension of $\operatorname{tp}(b/M)$, thus $\operatorname{tp}(b/Mc)$ is finitely satisfiable over M (by Theorem 5.1).

In particular, since $x \in c^{-1}X$ is in $\operatorname{tp}(b/Mc)$, there is $d \in M$ such that $x \in d \cdot X$. Namely $b \in d \cdot X$. Therefore $G = \bigcup_{g \in M} g \cdot X$ and we are done.

For generic $\Leftrightarrow$ left generic: note that left generic implies generic by definition. Hence they are equivalent. Same for right generic. ☐

7.1 The stabilizer of a type

Let $G = G(\mathfrak{U})$ be a stable group and $\mathfrak{U}$ a monster model. Then G acts on the space of 1-types $S(\mathfrak{U})$ by:

$$(g, p) \mapsto g \cdot p := \{g \cdot X : X \in p\} \quad \text{for } g \in G,\ p \in S(\mathfrak{U}).$$

Note that $g \cdot p$ is a complete type, since it is clearly consistent and if $X \notin g \cdot p$, then $g^{-1} \cdot X \notin p$, hence $G \setminus g^{-1} \cdot X \in p$. Note that $G \setminus g^{-1} \cdot X = g^{-1}(G \setminus X)$, thus $G \setminus X \in g \cdot p$.

Remark 7.18.
This works for $G = G(M)$ where $M \models T$ and G acts on $S(M)$.

Now we consider the **stabilizer** of a type:

$$\mathrm{Stab}(p) := \{g \in G : g \cdot p = p\};$$

it is a subgroup of G.

Aim:

(1) $\mathrm{Stab}(p)$ is type-definable and $\mathrm{Stab}(p) \subseteq G^0$, where G^0 is the connected component.

(2) $\mathrm{Stab}(p) = G^0$ iff p is generic.

Lemma 7.19 *(Generic iff translation doesn't fork)***.**
Let M be a $|T|^+$-saturated model. Then any $p \in S(M)$ is generic iff $g \cdot p$ does not fork over $\emptyset$ for all $g \in M$.

Proof. We only need to show the converse. Suppose $p \in S(M)$ and $g \cdot p$ does not fork over $\emptyset$ for all $g \in M$. Let M_0 be a submodel of size $|T|$ such that $g, h \in M$ and p is definable over M_0. Let $h \models p$; g generic over M_0, h, and $g \cdot h$ is generic over M_0 by Lemma 7.15. Then $g \cdot p$ does not fork over $\emptyset$.

We claim: $\mathrm{tp}(g \cdot h/M_0) \subseteq g \cdot p$. Then $g \cdot p$ does not fork over M_0; thus is a non-forking extension of the generic type, thus is generic. Therefore p is generic.

To see the claim, suppose $X \in \mathrm{tp}(g \cdot h/M_0)$; we want to show $g^{-1} \cdot X \in p$, i.e. $g^{-1} \cdot X \in \mathrm{tp}(h/M_0, g)$. Let $d_\psi(y)$ be the defining formula for the ψ-type of p, where $\psi(x) := y^{-1} \cdot X$. Since $\mathrm{tp}(h/M_0, g)$ is the unique non-forking extension of $\mathrm{tp}(h/M_0)$, $d_\psi(g)$ holds and $g^{-1} \cdot X \in p$ as desired. □

We will use definability of types to show $\mathrm{Stab}(p)$ is type-definable. So we look at ϕ-types. But G does not act on all ϕ-types.

Definition 7.20 *(Robust formula)***.**
Let $\psi(x, y)$ be a formula over $\emptyset$ with $|x| = 1$. We call $\psi(x, y)$ **robust** if subsets of G defined by instances of ψ are closed under left-translation: namely, for all a, if X is defined by $\psi(x, a)$, then there is b such that $g \cdot X$ is defined by $\psi(x, b)$.

For ψ robust, G acts on ψ-types: $(g, x) \mapsto g \cdot x$,

$$(g, p_\psi) \mapsto g \cdot p_\psi := \{g \cdot X : X \in p \text{ and } X \text{ defined by } \psi \text{ or } \neg\psi\}.$$

Define $\mathrm{Stab}_\psi(p) := \{g \in G : g \cdot p_\psi = p_\psi\}$.

Lemma 7.21 *($\mathrm{Stab}(p)$ as intersection over robust formulas)***.**
$\mathrm{Stab}(p) = \bigcap_{\psi \text{ robust}} \mathrm{Stab}_\psi(p)$.

Proof. Clearly $\mathrm{Stab}(p) \subseteq \bigcap_{\psi \text{ robust}} \mathrm{Stab}_\psi(p)$.

Conversely, if $g \in \bigcap_{\psi \text{ robust}} \mathrm{Stab}_\psi(p)$ and $X = \phi(x, a)$, we will change $\phi(x, y)$ to a robust formula. Let $\psi(x; u, y) := \phi(u \cdot x, y)$. Then if $Y = \psi(x; b, a)$, we have $g \cdot Y = \psi(g^{-1} \cdot x; b, a) = \psi(x; bg^{-1}, a)$. Hence $\psi(x; u, y)$ is robust and X is defined by $\psi(x; 1, a)$. Thus $X \in p$ iff $X \in p_\psi$ iff $g \cdot X \in p_\psi$ (by $g \in \mathrm{Stab}_\psi(p)$) iff $g \cdot X \in p$. Namely $g \cdot p = p$. □

Corollary 7.22 *(Stabilizer is type-definable)***.**
Let $g(\phi)(x, y) := \psi(x; u, y) := \phi(u \cdot x, y)$. Then

$$\mathrm{Stab}(p) = \bigcap_{\phi(x,y)} \mathrm{Stab}_{g(\phi)}(p).$$

For any $\phi(x, y)$ over $\emptyset$, the group $\mathrm{Stab}_{g(\phi)}(p)$ is definable, and thus $\mathrm{Stab}(p)$ is type-definable.

Proof. The first statement is immediate. Given $\phi(x, y)$, let $\psi(x; u, y) := \phi(u \cdot x, y)$ be $g(\phi)$. By the fact that p_ψ is definable, there is $d_p\psi(u, y)$ such that $\mathfrak{U} \models d_p d_p \psi(c, b, a) \iff \psi(x; b, a) \in p$.

Thus $\mathrm{Stab}_\psi(p) = \{g \in G : \forall b \forall a\ (d_p\psi(b, a) \leftrightarrow d_p\psi(bg^{-1}, a))\}$. □

Now we aim to show $\mathrm{Stab}(p) \subseteq G^0$.

Recall that $G^0_\phi = \bigcap\{H : H \text{ defined by some } \phi(x, b) \text{ and } [G : H] < \infty\}$ and $G^0 = \bigcap_\phi G^0_\phi$.

Each G^0_ϕ is not necessarily normal, and we want to write G^0 as intersection of definable normal subgroups of G. We do the same trick. Given $\phi(x, y)$, define

$$\theta(\phi)(x; u, y) := \phi(u \cdot x \cdot u^{-1}, y).$$

If H is defined by $\theta(\phi)(x; b, a) = \phi(b \cdot x \cdot b^{-1}, a)$, then $H^g := g^{-1}Hg$ is defined by $\phi(b \cdot g \cdot x \cdot g^{-1} \cdot b^{-1}, a) = \theta(\phi)(x; b \cdot g, a)$.

Hence $G^0_{\theta(\phi)}$ is a definable normal subgroup of G of finite index and $G^0 = \bigcap_\phi G^0_{\theta(\phi)}$. And

$$G/G^0 = \varprojlim\{G/G^0_{\theta(\phi)} : \phi \in \mathcal{L}\}.$$

Lemma 7.23 *(Stabilizer contained in G^0).*
For any global type p, we have $\mathrm{Stab}(p) \subseteq G^0$.

Proof. It is enough to show $\mathrm{Stab}(p) \subseteq G^0_{\theta(\phi)}$ for all ϕ. Let $g \in \mathrm{Stab}(p)$ and H be defined by $G^0_{\theta(\phi)}$. Then H is a definable normal subgroup of finite index in G. Since $G = \bigcup_{i \le N} b_i H$ and p is a complete type, there is some b_i such that $b_i H \in p$. Let M be a small model containing g and b_i. By assumption, $g \in \mathrm{Stab}(p)$, thus $g \cdot b_i \cdot H \in p$.

Let $a \models p \restriction M$; then both $b_i \cdot H$ and $g \cdot b_i \cdot H$ are in $\mathrm{tp}(a/M)$, i.e. $a \in b_i H$ and $a \in g \cdot b_i \cdot H$. Let $a = b_i\alpha = g \cdot b_i \cdot \beta$ with $\alpha, \beta \in H$. Then $g = b_i \cdot \alpha \cdot \beta^{-1} \cdot b_i^{-1} \in b_i \cdot H \cdot b_i^{-1} = H$ (since H is normal in G). Thus $g \in H$, and we are done. $\square$

Lemma 7.24 *(Generic implies* $\mathrm{Stab}(p) \supseteq G^0$*).*
If p is generic, then $\mathrm{Stab}(p) \supseteq G^0$. In particular, $\mathrm{Stab}_{g(\phi)}(p) \supseteq G^0$ (where $\mathrm{Stab}_{g(\phi)}(p)$ is as above) for generic p.

Proof. By Corollary 7.22, $\mathrm{Stab}(p) = \bigcap_\phi \mathrm{Stab}_{g(\phi)}(p)$ where each $\mathrm{Stab}_{g(\phi)}(p)$ is a definable subgroup of G. To show $\mathrm{Stab}(p) \supseteq G^0$ for generic p, it is enough to show $\mathrm{Stab}_{g(\phi)}(p)$ has finite index in G.

Let $H = \mathrm{Stab}_{g(\phi)}(p)$. G acts on the $g(\phi)$-types and H is the stabilizer of a $g(\phi)$-type of this action. By the orbit–stabilizer theorem, $[G : H] = |G \cdot p_{g(\phi)}|$, the size of the orbit $\{g \cdot p_{g(\phi)} : g \in G\}$ where $p_{g(\phi)}$ is the $g(\phi)$-type of p.

Since p is generic, $p_{g(\phi)}$ is generic and so is $g \cdot p_{g(\phi)}$ for all $g \in G$. By Corollary 7.14, the set of global generic $g(\phi)$-types is finite. Hence $|G \cdot p_{g(\phi)}|$ is a finite set and $[G : H] < \infty$ as desired. $\square$

Lemma 7.25 *(Unique generic type in G^0 over a model).*
There is a unique global generic type in G^0 (all the formulas in G^0) over M. For any small model M, there is a unique generic type containing G^0 over M.

Proof. It is enough to show that for any small model M, there is a unique generic type in G^0 over M. Suppose p and q are generic types in G^0 over M. Let $a \models p$ and $b \models q$ with $a \not\downarrow_M b$; let q' be the global non-forking extension of q. Then q' is generic. By Lemma 7.24, since $a \in G^0$ and q' generic, $a \cdot q' = q'$.

Claim: $\mathrm{tp}(a \cdot b/M) = q$. Suppose $X \in \mathrm{tp}(a \cdot b/M)$; then $a^{-1} \cdot X \in \mathrm{tp}(b/M, a)$. Since $b \not\downarrow_M a$ and q' is the unique non-forking extension of $\mathrm{tp}(b/M)$, thus $a^{-1} \cdot X \in q'$. And $X \in a \cdot q' = q'$. Hence $X \in q$.

We can consider the right action of G on $S(\mathfrak{U})$ and consider stabilizers. Then similarly we get $p' \cdot b = p'$ for p' global non-forking extension of p. Thus $\mathrm{tp}(a \cdot b/M) = p$.

In conclusion, $\mathrm{tp}(a \cdot b/M)$: $p = q$. $\square$

Lemma 7.26 (Stab$(p) = G^0$ *implies p is generic*)**.**
If $\mathrm{Stab}(p) = G^0$, then p is generic. Let p be a type in G over a model M.

Proof. Let q be a generic type in G^0 over M. Let $a \models p$ and $b \models q$ such that $a \not\downarrow_M b$. Let $\tilde{p}$ be the non-forking extension of p to the global type. We know $b \cdot a$ is generic over M (since b is generic). Thus p is generic.

Claim: $\mathrm{tp}(b \cdot a/M) = p$. If $X \in \mathrm{tp}(b \cdot a/M)$, then $b^{-1} \cdot X \in \mathrm{tp}(a/M, b)$ and $X \in b \cdot \tilde{p}$.

We claim $\mathrm{Stab}(\tilde{p}) = G^0$ (then $b \cdot \tilde{p} = \tilde{p}$, since $b \in G^0$ and $X \in b \cdot \tilde{p} = \tilde{p}$).

Note that $\bigcap_\phi \mathrm{Stab}_{g(\phi)}(\tilde{p}) = \bigcap_\phi \{g \in G : \forall b \forall a \ (d_p g(\phi)(b, a) \leftrightarrow d_p g(\phi)(bg^{-1}, a))\}$ and $d_p g(\phi)$ defines both the $g(\phi)$-type of p and $\tilde{p}$ (since $\tilde{p}$ is the unique non-forking extension of p). $\square$

Theorem 7.27 (*The Fundamental Theorem of Stable Groups*)**.**

(1) G^0 is of bounded index.

(2) There is a unique generic type in every coset of G^0.

(3) G/G^0 is a profinite group acting transitively on the generics.

(4) p is generic iff $\mathrm{Stab}(p) = G^0$.

Proof. For (2): there is a unique generic type in G^0 by Lemma 7.25, and if $p, q \in a \cdot G^0$ are generics, then $a^{-1} \cdot p$ and $a^{-1} \cdot q$ are generics in G^0; thus $a^{-1} \cdot p = a^{-1} \cdot q$ and $p = q$.

(3): Note that $G^0 = \bigcap_\phi G^0_{\theta(\phi)}$ where $G^0_{\theta(\phi)}$ are all normal subgroups of G of finite index. And $G/G^0 = \varprojlim\{G/G^0_{\theta(\phi)} : \phi \in \mathcal{L}\}$. $\square$

Corollary 7.28 (*Multiplicative group of stable field is connected*)**.**
Let $(K, +, \cdot, 0, 1, \dots)$ be an infinite stable field. Then the multiplicative group $(K^*, \cdot, \dots)$ is connected.

Proof. By Corollary 6.18, the additive group $(K, +, 0, \dots)$ is connected. For any $b \in K^*$, the map $k \mapsto b \cdot k$ is a group automorphism of $(K, +, 0, \dots)$. Hence if p is the unique generic type in $(K, +, 0, \dots)$, then $b \cdot p$ is generic, so $b \cdot p = p$ and thus $b \in \mathrm{Stab}(p)$.

Note that p is also a complete type viewed in $(K^*, \cdot, 1, \dots)$. Hence $\mathrm{Stab}(p) = K^*$ in the multiplicative group $(K^*, \cdot, 1, \dots)$. Thus $(K^*)^0 = K^*$ (since $\mathrm{Stab}(p) \subseteq (K^*)^0$ for all p, and p is the unique generic type in $(K^*, \cdot, 1, \dots)$ as well). $\square$

Remark 7.29.
In $(\mathbb{R}, +, \cdot, 0, 1)$, the multiplicative group is certainly not connected. $(\mathbb{R}^{>0}, \cdot, 1)$ is a subgroup of finite index.

8 Superstable Fields

Aim: Superstable fields are algebraically closed.

Lemma 8.1 *(Surjectivity of finite-to-one homomorphisms).*
Let $(G, \cdot, 1, \dots)$ be a connected superstable group. If $\sigma : G \to G$ is a definable finite-to-one group homomorphism, then σ is surjective. (Finite-to-one: $\sigma^{-1}(6(g))$ is finite for all $g \in G$.)

Proof. Let M be a small model and p the unique generic type over M. Let $g \models p$. Consider g and $\sigma(g)$. They are interalgebraic, hence $\mathrm{SU}(g/M) = \mathrm{SU}(\sigma(g)/M)$. (Recall: $\mathrm{SU}(a/B) := \mathrm{SU}(\mathrm{tp}(a/B))$.)

Since generics in superstable groups are exactly types of the maximal SU-rank, we get $\mathrm{tp}(\sigma(g)/M)$ is also generic.

In particular $\sigma(G) \in \mathrm{tp}(\sigma(g)/M)$ is a generic set, and $[G : \sigma(G)] < \infty$. But G is connected, so we have $\sigma(G) = G$.

(Alternatively, to prove $[G : \sigma(G)] < \infty$: suppose not. Let $a \in M$ and $a \notin \mathrm{acl}(\emptyset)$. Consider $\mathrm{tp}(a \cdot \sigma(g)/M)$. Then $\mathrm{SU}(a \cdot \sigma(g)/M) = \mathrm{SU}(\sigma(g)/M)$ as $a \in M$. The formula $\psi(x, a) := x \in a \cdot \sigma(G)$ is contained in $\mathrm{tp}(a \cdot \sigma(g)/M)$. Assuming G is $\emptyset$-defined over $\emptyset$, then $\psi(x, a)$ is a forking formula over $\emptyset$ since $[G : \sigma(G)] = \infty$. Thus $\mathrm{SU}(a \cdot \sigma(g)/M) < \mathrm{SU}(\sigma(g)/M) \leq \mathrm{SU}(g/M)$—but g is generic, contradiction.) $\square$

Corollary 8.2 *(Power maps and Frobenius in superstable fields).*
Suppose $(K, +, \cdot, 0, 1, \dots)$ is an infinite superstable field. Then:

(1) The map $x \mapsto x^n$ is surjective, for all n.

(2) If $\mathrm{char}(K) = p$, then the map $x \mapsto x^p - x$ is surjective.

Proof. The map $x \mapsto x^n$ is a finite-to-one homomorphism of $(K^*, \cdot, 1)$ and $(K^*, \cdot, 1, \dots)$ is connected by Corollary 7.28.

In characteristic $p > 0$, the map $x \mapsto x^p - x$ is a finite-to-one homomorphism of the additive group $(K, 0, +, \dots)$ (since $x^p - x + y^p - y = (x + y)^p - (x + y)$). And $(K, 0, +, \dots)$ is connected by Corollary 6.18. $\square$

Corollary 8.3 *(Superstable fields are perfect).*
An infinite superstable field is perfect, hence every algebraic extension is separable.

Lemma 8.4 *(Finite Galois extensions are definable)*.
Let L/K be a finite Galois extension of degree n. Then L is definable in $(K^n, +, \cdot, 0, 1)$ (as a definable subset of K^n).

Proof. Since L/K is Galois (hence separable), by the primitive element theorem, $L = K(\alpha)$. Let $\sum_{1 \leq i \leq n} a_i \alpha^i$ be the minimal polynomial of α over K. Then L is generated by $\{1, \alpha, \alpha^2, \dots, \alpha^{n-1}\}$ as a K-vector space. Thus we can identify L with K^n as a set.

It is enough to define the multiplication of L on K^n using the ring language on K, i.e. let $\eta : L \to K^n$, $\delta \mapsto (b_0, \dots, b_{n-1})$ where $\delta = \sum_{i<n} b_i \alpha^i$. We want to define $*$ on K^n such that $\eta(\delta \cdot \varepsilon) = \eta(\delta) * \eta(\varepsilon)$. (Note that $\eta(\delta + \varepsilon) = \eta(\delta) + \eta(\varepsilon)$ in the vector space K^n, hence addition is already defined.)

Compute: $(\sum_{i<n} b_i \alpha^i) \cdot (\sum_{j<n} c_j \alpha^j) = \sum_{0 \leq m \leq 2n-2} (\sum_{i+j=m} b_i c_j) \alpha^m$. For $m < n$ it is fine. For $m \geq n$ we want to express α^m in terms of linear combinations of $\{1, \alpha, \dots, \alpha^{n-1}\}$. We use: $0 = \sum_{1 \leq i \leq n} a_i \alpha^{i-1}$, i.e. $\alpha^n = -\sum_{1 \leq i \leq n} a_i \alpha^{i-1}$.

Inductively:

$$\begin{aligned}
\alpha^n &= -\sum_{1 \leq i \leq n} a_i \alpha^{i-1}, \\
\alpha^{n+1} &= -\sum_{1 \leq i \leq n} a_i \alpha^{n-i+1} = -\sum_{1 < i \leq n} a_i \alpha^{n-i+1} - a_1 \alpha^n \\
&= -\sum_{1 < i \leq n} a_i \alpha^{n-i+1} + a_1 \left(\sum_{1 \leq j \leq n} a_j \alpha^{j-1} \right),
\end{aligned}$$

and so on. For all $m \leq 2n-2$, the expression α^m in terms of $\{1, \alpha, \dots, \alpha^{n-1}\}$ is given. Hence the operation $*$ can be defined.

More explicitly: $(b_0, \dots, b_{n-1}) * (c_0, \dots, c_{n-1}) = (b_0 I + b_1 M_\alpha + \dots + b_{n-1} M_\alpha^{n-1}) \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix}$

where $M_\alpha = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ 0 & 1 & \cdots & 0 & -a_{n-2} \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & -a_1 \end{pmatrix}$ is the companion matrix. □

Fact 8.5 *(Kummer and Artin–Schreier extensions)*.

(1) Suppose that L/K is a cyclic Galois extension of degree n, where n is relatively prime to the characteristic of K and K contains all nth roots of unity. The minimal polynomial of α is $x^n - a$ for some $a \in K$. (Kummer extensions.)

(2) Suppose that $\mathrm{char}(K) = p > 0$ and L/K is a Galois extension of degree p. The mini-

mal polynomial of L/K is $x^p - x - a$ for some $a \in K$. (Artin–Schreier extensions.)

Lemma 8.6 *(No Galois extensions)***.**
Suppose an infinite superstable field K contains all mth-roots of unity for all $m \leq n$. Then K has no Galois extension of degree n.

Proof. Suppose L/K is a Galois extension of degree $n > 0$. Let $p|n$ with p prime. Since $|\mathrm{Gal}(L/K)| = n$ and p is a prime dividing n, by Cauchy's theorem, $\mathrm{Gal}(L/K)$ has an element of order p. Let H be the subgroup of $\mathrm{Gal}(L/K)$ generated by this element. Let F be the fixed field of H. Then by the fundamental theorem of Galois theory, L/F is a Galois extension with $\mathrm{Gal}(L/F) = H$. Then L/F is a cyclic extension of degree p.

Since L/K is separable and $K \subseteq F \subseteq L$, we have F/K is a separable extension of finite degree; by Lemma 8.4, F is definable in some K^m, hence is superstable.

Let $L' = K(\alpha)$, $L = F(\alpha)$. If $p \neq \mathrm{char}(F)$, by Fact 8.5(1), L/F is a Kummer extension (since K contains all pth-roots of unity, hence so does F) and the minimal polynomial of α is $x^p - a$ for some $a \in F$. But F is superstable, so the map $x \mapsto x^p$ is surjective, and $x^p - a$ is not irreducible, a contradiction.

If $p = \mathrm{char}(F)$, then by Fact 8.5(2), L/F is an Artin–Schreier extension, and the minimal polynomial of α is $x^p - x - a$ for some $a \in F$. But the map $x \mapsto x^p - x$ is surjective as well. We get a contradiction. □

Corollary 8.7 *(Superstable fields contain all roots of unity)***.**
Let K be an infinite superstable field. Then K contains all nth-roots of unity.

Proof. Induction on n. Suppose K contains all mth-roots of unity for $m < n$. By Lemma 8.6, K has no proper Galois extension of degree $< n$.

Let ζ be a primitive nth-root of unity, i.e. $(1, \zeta, \zeta^2, \ldots, \zeta^{n-1})$ are the nth roots of unity. Then $K(\zeta)/K$ is a Galois extension of degree $\leq n-1$ (since $x^n - 1$ splits $(x-1)(x^{n-1} + \cdots + 1)$, so the degree $\leq n-1$). And all nth-roots of unity are distinct, so $K(\zeta)/K$ is separable. It is clearly normal, since $K(\zeta)$ contains all nth-roots of unity.

Thus K contains a primitive nth-root of unity, hence contains all of them. □

Theorem 8.8 *(Cherlin–Shelah)***.**
Every infinite superstable field is algebraically closed.

Proof. Let $(K, +, \cdot, 0, 1, \ldots)$ be an infinite superstable field. By Corollary 8.7, K contains all nth-roots of unity. By Lemma 8.6, K has no proper Galois extension, hence has no proper separable extension. By Corollary 8.3, K is perfect. Thus K is algebraically closed.
□