Homework 2: Due April 12

Entanglement and Tensor Networks, Spring 2016, Prof. White

1. Write a program that diagonalizes a Heisenberg spin chain (using $J = 1$) for any (small) even length N using the built-in Julia Lanczos eigenvalue routine "eigs". The exact energy for an infinite chain, per spin, is $E = J(1/4 - \ln 2)$. (Hans Bethe figured this out in 1931!) Compare your energies with this by subtracting the energies of successive system sizes in order to cancel edge energies, i.e. $(E(N = 4) - E(N = 2))/2$, $(E(N = 6) - E(N = 4))/2$, etc.

2. Consider two different wavefunctions $|\psi_1\rangle$ and $|\psi_2\rangle$, each describing a system of two spin 1/2 spins (labeled as 1 and 2):

$$\begin{aligned} |\psi_1\rangle &= \Big(\frac{1}{\sqrt{2}}\,|{\uparrow_1}\rangle + \frac{1}{\sqrt{2}}\,|{\downarrow_1}\rangle\Big) \otimes |{\uparrow_2}\rangle \\ &= \frac{1}{\sqrt{2}}\,|{\uparrow_1}{\uparrow_2}\rangle + \frac{1}{\sqrt{2}}\,|{\downarrow_1}{\uparrow_2}\rangle \qquad (1) \\ |\psi_2\rangle &= \frac{1}{\sqrt{2}}\,|{\uparrow_1}{\uparrow_2}\rangle + \frac{1}{\sqrt{2}}\,|{\downarrow_1}{\downarrow_2}\rangle \qquad (2) \end{aligned}$$

(a) Observe that $|\psi_2\rangle$ is already in Schmidt-decomposition form. Use this observation to compute its entanglement. Based on the fact that $|\psi_1\rangle$ is a product state as shown on the first line above, what is its entanglement?

(b) Show that $\langle S_1^z\rangle = 0$ for both wavefunctions. Here S_1^z is the S^z operator acting on the first spin.

(c) Now compute $\langle \vec{S}_1\rangle$ for both wavefunctions. That is, compute

$$\langle \vec{S}_1\rangle = \langle S_1^x\rangle\,\hat{x} + \langle S_1^y\rangle\,\hat{y} + \langle S_1^z\rangle\,\hat{z}\ . \qquad (3)$$

What is the length of each of these vectors?

(d) Consider an arbitrary normalized wavefunction of a *single* spin 1/2, that is a state

$$|\phi\rangle = a\,|{\uparrow}\rangle + b\,|{\downarrow}\rangle \qquad (4)$$

such that $|a|^2 + |b|^2 = 1$ but a and b are otherwise arbitrary complex numbers. For such a single-spin pure state, prove by direct calculation that $\langle \vec{S}\rangle$ is never zero.

Note that since $|\psi_1\rangle$ is a product state, its two spins each act like they are an isolated spin, so this result applies to $\langle\psi_1|\,\vec{S}_1\,|\psi_1\rangle$. On the other hand, our result for $\langle\psi_2|\,\vec{S}_1\,|\psi_2\rangle$ shows that $|\psi_2\rangle$ cannot be written as a product state in any basis, for otherwise the first spin would have $\langle\vec{S}_1\rangle \neq 0$. Thus we can conclude that $|\psi_2\rangle$ must have non-zero entanglement.

3. Do the following using Julia, for Heisenberg spin chains (using $J = 1$) of even length $N = 2, \ldots, 12$: Find the ground state ψ of the chain; splitting the system down the middle, perform an SVD that does the Schmidt decomposition of ψ; and calculate the entanglement entropy S. From this data make a plot of $S(N)$ versus N. Include in your plot the maximum possible S for that N, $\frac{N}{2}\ln 2$. Optional: the asymptotic behavior of $S(N)$ is primarily a constant (the Area Law), but there is a correction proportional to $\ln N$ because this system does not have an energy gap to the first excited state as $N \to \infty$. Can you find evidence for this correction, and estimate its coefficient?

4. Similar to problem 3: for a fixed size (18 or 20 if you can do it, otherwise N=12 is fine), calculate S for splitting the ground state into unequal sizes. Plot S versus the location of the split x.

5. Optional problem: calculate S for a few eigenstates in the middle of the energy spectrum. Lanczos won't work for the middle, so you will need a full diagonalization and thus a smaller system. You should find that S is much bigger, scaling with a "Volume Law", for states in the middle of the spectrum.