Self-contained derivation of $\langle S_z(t)\rangle,\ \langle S_y(t)\rangle,\ \langle S_x(t)\rangle$

Below is **one self-contained, fully-detailed derivation** that starts with the Hamiltonian

$$\langle S_z(t)\rangle, \qquad \langle S_y(t)\rangle, \qquad \langle S_x(t)\rangle$$

0 Notation used throughout

symbol	meaning
$\gamma = \dfrac{eg}{2m_e c}$	gyro-magnetic ratio
B_z	static field along $+\hat{z}$
B_1	magnitude of circularly-polarized RF field
ω	RF angular frequency
$\omega_0 = -\gamma B_z$	Larmor (Zeeman) frequency
$\tilde{\omega} = \omega - \omega_0$	detuning
$g \equiv -\dfrac{\gamma\hbar B_1}{2}$	RF coupling *energy* (δ)
$\Omega_0 = g/\hbar$	RF coupling *frequency*
$\Omega_R = \sqrt{\Omega_0^2 + (\tilde{\omega}/2)^2}$	generalized (detuned) Rabi frequency
$\alpha = \Omega_R t,\ \theta = \tilde{\omega}t,\ b = \dfrac{\tilde{\omega}}{2\Omega_R}$	shorthand

1 Hamiltonian

$$\boxed{H = -\gamma\, \boldsymbol{S}\cdot\boldsymbol{B} = \underbrace{\frac{\hbar\omega_0}{2}\,\sigma_z}_{H_0} + \underbrace{g\left(\sigma_x \cos\omega t + \sigma_y \sin\omega t\right)}_{V(t)}}$$

Spin basis: $|1\rangle \equiv |+\rangle_z,\ |2\rangle \equiv |-\rangle_z$, so $H_0 = \dfrac{\hbar\omega_0}{2}(|2\rangle\langle 2| - |1\rangle\langle 1|)$.

With $S_\pm = (\sigma_x \pm i\sigma_y)/2$,

$$V(t) = \hbar\Omega_0\big[S_+ e^{i\omega t} + S_- e^{-i\omega t}\big] = \hbar\Omega_0\big(e^{i\omega t}|2\rangle\langle 1| + e^{-i\omega t}|1\rangle\langle 2|\big).$$

2 Interaction (Dirac) picture

Let

$$|\psi_I(t)\rangle = C_1(t)|1\rangle + C_2(t)|2\rangle.$$

Schrodinger equation:

$$i\hbar\dot{C}_1 = g\,e^{i\tilde{\omega}t}C_2, \qquad i\hbar\dot{C}_2 = g\,e^{-i\tilde{\omega}t}C_1 \tag{I, II}$$

Eliminate one amplitude:

$$\ddot{C}_2 + i\tilde{\omega}\dot{C}_2 + \Omega_0^2\,C_2 = 0,$$

whose eigen-values are $\tilde{\omega}/2 \pm \Omega_R$.

3 Probability amplitudes

Initial state: $C_1(0) = 1,\ C_2(0) = 0$:

$$\boxed{\begin{aligned} C_2(t) &= -\frac{i\Omega_0}{\Omega_R}\, e^{-i\theta/2} \sin\alpha, \\ C_1(t) &= e^{i\theta/2}\Big[\cos\alpha - i\, b \sin\alpha\Big]. \end{aligned}} \tag{1}$$

Check: $|C_1|^2 + |C_2|^2 = 1$.

4 Operator matrices

$$S_z = \frac{\hbar}{2}\sigma_z, \quad S_x = \frac{\hbar}{2}\sigma_x, \quad S_y = \frac{\hbar}{2}\sigma_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

5 Expectation values

5.1 Longitudinal component $\langle S_z(t)\rangle$

$$\boxed{\langle S_z\rangle = \frac{\hbar}{2}\Big[1 - \frac{2\Omega_0^2}{\Omega_R^2}\sin^2\alpha\Big]}$$

5.2 Transverse y-component $\langle S_y(t)\rangle$

$$\langle S_y\rangle = \frac{\hbar}{2}\, i\big(C_2^* C_1 - C_1^* C_2\big) = \hbar\,\mathrm{Im}\big[C_1^* C_2\big].$$

$$C_1^* C_2 = -\frac{i\Omega_0}{\Omega_R}\sin\alpha\, e^{-i\theta}\big(\cos\alpha + i\, b\sin\alpha\big), \qquad \mathrm{Im}(C_1^* C_2) = -\frac{\Omega_0}{\Omega_R}\sin\alpha\big[\cos\alpha\cos\theta + b\sin\alpha\sin\theta\big].$$

$$\boxed{\langle S_y(t)\rangle = -\frac{\hbar}{2}\,\frac{\Omega_0}{\Omega_R^2}\Big[\Omega_R\sin(2\alpha)\cos\theta + \tilde{\omega}\sin^2\alpha\sin\theta\Big]}$$

5.3 Transverse x-component $\langle S_x(t)\rangle$

$$\langle S_x\rangle = \frac{\hbar}{2}\big(C_1^* C_2 + C_2^* C_1\big) = \hbar\,\mathrm{Re}\big[C_1^* C_2\big],$$

$$\mathrm{Re}(C_1^* C_2) = \frac{\Omega_0}{\Omega_R}\sin\alpha\big[b\sin\alpha\cos\theta - \cos\alpha\sin\theta\big].$$

$$\boxed{\langle S_x(t)\rangle = \frac{\hbar}{2}\,\frac{\Omega_0}{\Omega_R^2}\Big[-\Omega_R\sin(2\alpha)\sin\theta + \tilde{\omega}\sin^2\alpha\cos\theta\Big]}$$

6 One-line summary

$$\boxed{\begin{aligned}\langle S_z\rangle &= \frac{\hbar}{2}\Big[1-\frac{2\Omega_0^2}{\Omega_R^2}\sin^2\alpha\Big],\\ \langle S_y\rangle &= -\frac{\hbar}{2}\,\frac{\Omega_0}{\Omega_R^2}\Big[\Omega_R\sin(2\alpha)\cos\theta+\tilde\omega\sin^2\alpha\sin\theta\Big],\\ \langle S_x\rangle &= \frac{\hbar}{2}\,\frac{\Omega_0}{\Omega_R^2}\Big[-\Omega_R\sin(2\alpha)\sin\theta+\tilde\omega\sin^2\alpha\cos\theta\Big].\end{aligned}}$$

Appendix A – Laboratory frame vs. rotating frame

Why "replace $\tilde\omega t$ by ωt" when we plot the Bloch vector?

1 Physical setting

Take a spin-$\frac{1}{2}$ or two-level atom in a static field $B_0\hat z$ plus a weak r.f. (microwave, laser) field of frequency ω in the transverse plane:

$$H=\underbrace{\frac{\hbar\omega_0}{2}\sigma_z}_{H_0}+\underbrace{\frac{\hbar\Omega_R}{2}\big(\sigma_x\cos\omega t+\sigma_y\sin\omega t\big)}_{V(t)},\qquad \omega_0=\gamma B_0,\ \Omega_0=\gamma B_1. \tag{1}$$

2 Schrodinger picture (laboratory frame)

Time-dependent state vector:

$$|\psi_S(t)\rangle=\begin{pmatrix}C_1(t)\\ C_2(t)\end{pmatrix},\qquad \langle\mathbf{S}\rangle_{\text{lab}}=\frac{\hbar}{2}\,\langle\psi_S|\boldsymbol{\sigma}|\psi_S\rangle. \tag{2}$$

Every transverse observable obtained from these amplitudes **rotates rapidly** at the Larmor frequency ω_0. Animating that motion is possible but visually crowded.

3 Interaction picture / rotating frame

Define a unitary that "removes" the static Zeeman term:

$$|\psi_I(t)\rangle=e^{+iH_0t/\hbar}\,|\psi_S(t)\rangle=e^{+i\omega_0t\sigma_z/2}\begin{pmatrix}C_1(t)\\ C_2(t)\end{pmatrix}, \tag{3}$$

$$V_I(t)=e^{+iH_0t/\hbar}\,V(t)\,e^{-iH_0t/\hbar}=\frac{\hbar\Omega_0}{2}\Big(\sigma_x\cos[\tilde\omega t]+\sigma_y\sin[\tilde\omega t]\Big), \tag{4}$$

$$\tilde\omega\equiv\omega-\omega_0\quad\text{(detuning)}.$$

All **fast** $e^{\pm i\omega_0 t}$ factors are gone; the Bloch vector in this frame processes only with the *slow* detuning $\tilde\omega=\omega-\omega_0$ and the Rabi frequency Ω_R.

4 Spin components in the rotating frame

Solve the Rabi problem (under the rotating–wave approximation) and write the expectation values:

$$\boxed{\begin{aligned} S_x^R(t) &= \tfrac{\hbar}{2}\frac{\Omega_0}{\Omega_R^2}\Big[-\Omega_R\sin(2\alpha)\sin\tilde{\omega}t + \tilde{\omega}\sin^2\alpha\ \cos\tilde{\omega}t\Big], \\ S_y^R(t) &= \tfrac{\hbar}{2}\frac{\Omega_0}{\Omega_R^2}\Big[-\Omega_R\sin(2\alpha)\cos\tilde{\omega}t - \tilde{\omega}\sin^2\alpha\ \sin\tilde{\omega}t\Big], \\ S_z^R(t) &= \tfrac{\hbar}{2}\Big[\cos^2\alpha - \sin^2\alpha\ \cos\tilde{\omega}t\Big], \end{aligned}} \tag{5}$$

with

$$\alpha = \tfrac{1}{2}\arctan\frac{\Omega_0}{\tilde{\omega}}, \qquad \Omega_R = \sqrt{\Omega_0^2 + (\tilde{\omega}/2)^2}.$$

These are the formulas that use the **tilde phase** $\tilde{\omega}t$.

5 Back-transform to the laboratory frame

A passive rotation by $\omega_0 t$ about $+\hat{z}$ restores the lab axes:

$$\boxed{\begin{aligned} \langle S_x\rangle_{\text{lab}} &= \cos\omega_0 t\ S_x^R - \sin\omega_0 t\ S_y^R, \\ \langle S_y\rangle_{\text{lab}} &= \sin\omega_0 t\ S_x^R + \cos\omega_0 t\ S_y^R, \\ \langle S_z\rangle_{\text{lab}} &= S_z^R. \end{aligned}} \tag{6}$$

All occurrences of $\tilde{\omega}t$ combine with the extra $\omega_0 t$ to produce

$$\omega t \ = \ \tilde{\omega}t + \omega_0 t,$$

exactly the phase that appears when you plot with *fixed* x, y, z axes. Collecting terms gives:

$$\boxed{\begin{aligned} \langle S_x\rangle_{\text{lab}} &= \frac{\hbar}{2}\frac{\Omega_0}{\Omega_R^2}\Big[-\Omega_R\sin(2\alpha)\sin\omega t + \tilde{\omega}\sin^2\alpha\ \cos\omega t\Big], \\ \langle S_y\rangle_{\text{lab}} &= -\frac{\hbar}{2}\frac{\Omega_0}{\Omega_R^2}\Big[\Omega_R\sin(2\alpha)\cos\omega t + \tilde{\omega}\sin^2\alpha\ \sin\omega t\Big], \\ \langle S_z\rangle_{\text{lab}} &= \frac{\hbar}{2}\Big[\cos^2\alpha - \sin^2\alpha\ \cos\tilde{\omega}t\Big]. \end{aligned}} \tag{7}$$

These are the ωt **versions** you found "make the code work".

6 What the animation program is really doing

- The Matplotlib axes you draw are *stationary* in real space $\rightarrow$ lab frame.
- Feeding in $S_{x,y}^R(t)$ (rotating frame) therefore describes the Bloch vector in a coordinate system that is itself spinning about z at ω_0. The apparent motion looks wrong.
- Multiply the transverse pair by the rotation matrix above (or, equivalently, replace $\tilde{\omega}t \rightarrow \omega t$) and the picture matches the exact quantum dynamics.

7 Summary

frame	state / operators	evolution phase	spin components contain
laboratory (Schrödinger)	$\lvert\psi_S(t)\rangle$	$e^{-i\omega_0 t\sigma_z/2}$	$\sin\omega t$, $\cos\omega t$
rotating / interaction	$\lvert\psi_I(t)\rangle = e^{+i\omega_0 t\sigma_z/2}\lvert\psi_S(t)\rangle$	fast Larmor term removed	$\sin\tilde{\omega}t$, $\cos\tilde{\omega}t$

Therefore **replace** $\tilde{\omega}t$ **by** ωt whenever you display results in a non-rotating plot. The mathematical step is nothing more than the inverse unitary transformation between the two standard pictures of quantum mechanics.