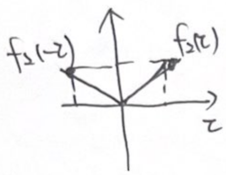
4.12).

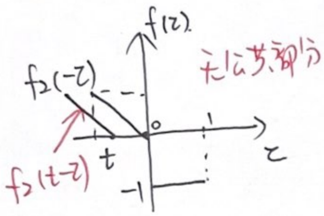
$$f_1(t)=\begin{cases}-1, & 0<t\le 1\\ 0 & \text{其他}.\end{cases}\qquad f_2(t)=\begin{cases}t, & 0\le t\le 1\\ 0 & \text{其他}.\end{cases}$$

$$f_1(t)*f_2(t)=\int_{-\infty}^{+\infty}f_1(\tau)f_2(t-\tau)d\tau.$$

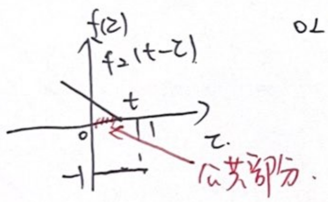
① 先作反褶，对 $f_2(t)$.



② 考虑平移.



$t<0$ $\quad \int_{-\infty}^{+\infty}f_1(\tau)f_2(t-\tau)d\tau=0.$

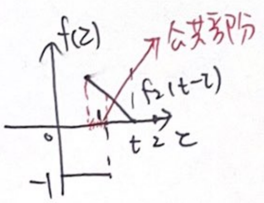


$0<t<1$

$$\int_{-\infty}^{+\infty}f_1(\tau)f_2(t-\tau)d\tau$$

$$=\int_0^t -1\cdot(t-\tau)d\tau.$$

$$=-\frac{1}{2}t^2$$

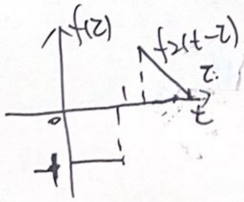


$1<t<2$

$$\int_{-\infty}^{+\infty}f_1(\tau)f_2(t-\tau)d\tau$$

$$=\int_{t-1}^{1}-1\,(t-\tau)\,d\tau.$$

$$=\frac{1}{2}t^2-t.$$



$t>2$ $\quad \int_{-\infty}^{+\infty}f_1(\tau)f_2(t-\tau)d\tau=0$

$$\therefore\ f_1*f_2=\begin{cases}0, & t\le 0\\ -\frac{1}{2}t^2, & 0<t\le 1\\ \frac{1}{2}t^2-t, & 1<t\le 2\\ 0, & t>2\end{cases}$$

(1). $f(t) = \sin\omega_0 t \cdot u(t)$. $\mathcal{F}(e^{j\omega_0 t} f(t)) = F(\omega-\omega_0)$.

解：法一．可应用线性性质和像平移性质．

由 $\sin\omega_0 t = \frac{1}{2j}(e^{j\omega_0 t} - e^{-j\omega_0 t})$.

则 $\mathcal{F}(f(t)) = \frac{1}{2j}\left(\mathcal{F}[e^{j\omega_0 t}u(t)] - \mathcal{F}[e^{-j\omega_0 t}u(t)]\right)$

$= \frac{1}{2j}\left(U(\omega-\omega_0) - U(\omega+\omega_0)\right)$

这里 $U(\omega)$ 是 $u(t)$ 的 Fourier 变换，已知 $\mathcal{F}[u(t)] = \frac{1}{j\omega} + \pi\delta(\omega)$.

$= \frac{1}{2j}\left(\frac{1}{j(\omega-\omega_0)} + \pi\delta(\omega-\omega_0) - \frac{1}{j(\omega+\omega_0)} - \pi\delta(\omega+\omega_0)\right)$.

$= \frac{\omega_0}{\omega-\omega_0^2} + \frac{\pi}{2j}\left(\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right)$.

法二．也可应用卷积Thm．与δ函数的卷积性质．$\delta(t-t_0) * h(t) = h(t-t_0)$.

令 $f_1(t) = \sin\omega_0 t$．对应的Fourier变换为 $F_1(\omega) = j\pi\left(\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right)$

$f_2(t) = u(t)$ $F_2(\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$.

由卷积Thm. $\mathcal{F}(f_1(t)\cdot f_2(t)) = \frac{1}{2\pi}F_1(\omega) * F_2(\omega)$.

$= \frac{1}{2\pi}\cdot j\pi\left(\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right) * \left(\frac{1}{j\omega} + \pi\delta(\omega)\right)$.

$\cancel{= \frac{1}{2}\left(\frac{1}{\omega+\omega_0} - \frac{1}{\omega-\omega_0}\right)}$

$\left(\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right) * \frac{1}{\omega}$.

$= \frac{1}{2}\cdot\left(\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right) * \delta(\omega)$.

$+ \frac{j\pi}{2}\left(\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right)$.

$= \frac{1}{2}\left(-\frac{1}{\omega+\omega_0} + \frac{1}{\omega-\omega_0}\right) + \frac{j\pi}{2}\left(\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right)$.

$= \frac{\omega_0}{\omega-\omega_0} + \frac{j\pi}{2}\left(\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right)$.

(2) $f(t) = e^{j\omega_0 t} \cdot t \cdot u(t)$.

解：应用象函数的微分性质.

$$F^{(n)}(\omega) = (-j)^n \mathscr{F}(t^n f(t)).$$

以及象平移性质.

$f(t)$ 可看作 $t \cdot g(t)$, $g(t) = e^{j\omega_0 t} \cdot u(t)$.

$$(-j)^1 \cdot \mathscr{F}(t^1 \cdot g(t)) = G'(\omega).$$

$$\mathscr{F}(t g(t)) = j G'(\omega).$$

因此，只需要求 $g(t)$ 的 Fourier 变换 $G(\omega)$.

$$G(\omega) = \mathscr{F}(e^{j\omega_0 t} u(t)) = \frac{1}{j(\omega-\omega_0)} + \pi\delta(\omega-\omega_0).$$

$$\therefore \mathscr{F}(t g(t)) = j \cdot \left(\frac{1}{j(\omega-\omega_0)} + \pi(\delta(\omega-\omega_0))\right)'$$

$$= -\frac{1}{(\omega-\omega_0)^2} + \pi j \delta'(\omega-\omega_0).$$