- We built the coordinate vectors of the items in the span, put them in the columns of the matrix V and then used numpy.linalg.qr to get an orthonormal matrix Q with the same column space.

$$V = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 2 & -1 & 0 & 1 \\ 2 & -1 & 0 & 1 \\ 0 & 0 & 3 & 3 \end{bmatrix}$$

$$Q = \begin{bmatrix} \frac{1}{3} & 0.94280904 & 0 \\ \frac{2}{3} & -0.23570226 & 0 \\ \frac{2}{3} & -0.23570226 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

As such, the orthonormal base is $\left\{ \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & 0 \end{pmatrix}, \begin{pmatrix} 0.94280904 & -0.23570226 \\ -0.23570226 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

- We calculated the matrix A representing the multiplication and then used our code to find a base for $span \left\{1, x, x^2\right\}$.

$$\langle 1, 1 \rangle = 1 + 1 + 1 = 3$$

$$\langle 1, x \rangle = 0 + 2 - 1 = 1$$

$$\left\langle 1, x^2 \right\rangle = 0 + 4 + 1 = 5$$

$$\langle x, x \rangle = 0 + 4 + 1 = 5$$

$$\left\langle x, x^2 \right\rangle = 0 + 8 - 1 = 7$$

$$\left\langle x^2, x^2 \right\rangle = 0 + 16 + 1 = 17$$

$$A = \begin{bmatrix} 3 & 1 & 5 \\ 1 & 5 & 7 \\ 5 & 7 & 17 \end{bmatrix}$$

$$V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0.57735027 & -0.15430335 & -0.80178373 \\ 0 & 0.46291005 & -0.71269665 \\ 0 & 0 & 0.62360956 \end{bmatrix}$$

Which means our base is

$$\left\{0.57735027, -0.15430335 + 0.46291005x, -0.80178373 - 0.71269665x + 0.62360956x^2\right\}$$