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A Spinorial Approach to Riemannian and Conformal Geometry

European Mathematical Society

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Introduction

Spin Geometry is the hidden facet of Riemannian Geometry. It arises from the representation theory of the special orthogonal group SO_n, more precisely, from the spinor representation, a certain representation of the Lie algebra $\mathfrak{so}_n$ which is *not* a representation of SO_n. Spinors can always be constructed locally on a given Riemannian manifold, but globally there are topological obstructions for their existence.

Spin Geometry lies therefore at the cross-road of several subfields of modern Mathematics. Algebra, Geometry, Topology, and Analysis are subtly interwoven in the theory of spinors, both in their definition and in their applications. Spinors have also greatly influenced Theoretical Physics, which is nowadays one of the main driving forces fostering their formidable development. The Noncommutative Geometry of Alain Connes has at its core the Dirac operator on spinors. The same Dirac operator is at the heart of the Atiyah–Singer index formula for elliptic operators on compact manifolds, linking in a spectacular way the topology of a manifold to the space of solutions to elliptic equations. Significantly, the classical Riemann–Roch formula and its generalization by Hirzebruch; the Gauß–Bonnet formula and its extension by Chern; and finally Hirzebruch's topological signature theorem, provide most of the examples in the index formula, but it was the Dirac operator acting on spinors which turned out to be the keystone of the index formula, both in its formulation and in its subsequent developments. The Dirac operator appears to be the primordial example of an elliptic operator, while spinors, although younger than differential forms or tensors, illustrate once again that *aux âmes bien nées, la valeur n'attend point le nombre des années*.

Our book aims to provide a comprehensive introduction to the theory of spinors on oriented Riemannian manifolds, including some (but by no means all) recent developments illustrating their effectiveness. Our primordial interest comes from Riemannian Geometry, and we adopt the point of view that spinors are some sort of "forgotten" tensors, which we study in depth.

Several textbooks related to this subject have been published in the last two or three decades. Without trying to list them all, we mention Lawson and Michelson's foundational *Spin geometry* [LM89], the monograph by Berline, Getzler and Vergne centered on the heat kernel proof of the Atiyah–Singer index theorem [BGV92], and more recently the books by Friedrich [Fri00] and Ginoux [Gin09] devoted to the spectral aspects of the Dirac operator. One basic issue developed in our book which is not present in the aforementioned works is the interplay between spinors and special geometric structures on Riemannian manifolds. This aspect becomes particularly evident in small dimensions $n \leq 8$, where the spin group acts transitively on the unit sphere of the real (half-) spin representation, i.e., all spinors are *pure*. In this way, a non-vanishing (half-) spinor is equivalent to a SU_2-structure for $n = 5$, a SU_3-structure for $n = 6$, a G_2-structure for $n = 7$, and a Spin_7-structure for $n = 8$. Many recent contributions in low-dimensional geometry (e.g., concerning hypo, half-flat, or co-calibrated G_2 structures) are actually avatars of the very same phenomena which have more natural interpretation in spinorial terms.

One further novelty of the present book is the simultaneous treatment of the spin, Spin^c, conformal spin, and conformal Spin^c geometries. We explain in detail the relationship between almost Hermitian and Spin^c structures, which is an essential aspect of the Seiberg–Witten theory, and in the conformal setting, we develop the theory of weighted spinors, as introduced by N. Hitchin and P. Gauduchon, and derive several fundamental identities, e.g., the conformal Schrödinger–Lichnerowicz formula.

The Clifford algebra

We introduce spinors via the standard construction of the Clifford algebra Cl_n of a Euclidean vector space $\mathbb{R}^n$ with its standard positive-definite scalar product. The definition of the Clifford algebra in every dimension is simple to grasp: it is the unital algebra generated by formal products of vectors, with relations implying that vectors anti-commute up to their scalar product:

$$u \cdot v + v \cdot u = -2\langle u, v\rangle.$$

In low dimensions $n = 1$ and $n = 2$, the Clifford algebra is just $\mathbb{C}$, respectively $\mathbb{H}$, the quaternion algebra. Already for $n = 1$, we encounter the extravagant idea of "imaginary numbers", which complete the real numbers and which were accepted only as late as the eighteenth century. Quaternions took another century to be devised, while their generalization to higher-dimensional Clifford algebras is rather evident. The same construction, but with the bilinear form replaced by 0, gives rise to the exterior algebra, while the universal enveloping algebra of a Lie algebra is closely related. The Clifford algebra acts transitively on the exterior algebra, and thus in particular it is nonzero. This algebraic construction has unexpected applications in Topology, as it is directly related to the famous vector field problem on spheres, namely finding the maximal number of everywhere linearly independent vector fields on a sphere.

The spin group

Inside the group of invertible elements of Cl_n we distinguish the subgroup formed by products of an even number of unit vectors, called the *spin group* Spin_n. It is a compact Lie group, simply connected for $n \geq 3$, endowed with a canonical orthogonal action on $\mathbb{R}^n$ defining a non-trivial $2:1$ cover $\mathrm{Spin}_n \to \mathrm{SO}_n$. In other words, Spin_n is the universal cover of SO_n for $n \geq 3$. Every representation of SO_n is of course also a representation space for Spin_n, but there exists a *fundamental representation* of Spin_n which does not come from SO_n, described below. It is this "shadow orthogonal representation" which gives rise to spinor fields.

The complex Clifford algebra $\mathbb{C}\mathrm{l}_n$ is canonically isomorphic to the matrix algebra $\mathbb{C}(2^{n/2})$ for n even, respectively to the direct sum of two copies of $\mathbb{C}(2^{(n-1)/2})$ for n odd, and thus has a standard irreducible representation on $\mathbb{C}^{2^{[n/2]}}$ for n even and two non-equivalent representations on $\mathbb{C}^{2^{[n/2]}}$ for n odd. In the first case, the restriction to Spin_n of this representation splits as a direct sum of two inequivalent representations of the same dimension (the so-called *half-spin representations*), whereas for n odd the restrictions to Spin_n of the two representations of $\mathbb{C}\mathrm{l}_n$ are equivalent and any of them is referred to as the *spin representation*.

The classification of real Clifford algebras is slightly more involved, and is based on the algebra isomorphisms $\mathrm{Cl}_{n+8} = \mathbb{R}(16) \otimes \mathrm{Cl}_n$. Note that the periodicity of real and complex Clifford algebras is intimately related to the Bott periodicity of real, respectively complex K-theory, which provides a very convenient algebraic setting for the index theorem.

Spinors

The geometric idea of spinors on an n-dimensional Riemannian manifold (M^n, g) is to consider the spin group as the structure group in a principal fibration extending, in a natural sense, the orthonormal frame bundle of M. The existence of such a fibration, called a *spin structure*, is not always guaranteed, since it is equivalent to a topological condition, the vanishing of the second Stiefel–Whitney class $w_2 \in H^2(M, \mathbb{Z}/2\mathbb{Z})$. Typical examples of manifolds which do not admit spin structures are the complex projective spaces of even dimensions. When it is non-empty, the set of spin structures is an affine space modeled on $H^1(M, \mathbb{Z}/2\mathbb{Z})$. For instance, on a compact Riemann surface of genus g, there exist 2^{2g} inequivalent spin structures. In this particular case, a spin structure amounts to a holomorphic square root of the holomorphic tangent bundle $\mathrm{T}^{1,0}M$.

Once we fix a spin structure on M, *spinors* are simply sections of the vector bundle associated to the principal Spin_n-bundle via the fundamental spin representation. In general, we may think of spinors as *square roots of exterior forms*. More to the point, on Kähler manifolds in every dimension, spin structures correspond to holomorphic square roots of the canonical line bundle.

The Dirac operator

In the same way as the Laplacian $\Delta = d^*d$ is naturally associated to every Riemannian metric, on spin manifolds there exists a prominent first-order elliptic differential operator, the Dirac operator. The first instance of this operator appeared indeed in Dirac's work as a *differential* square root of the Laplacian on Minkowski space-time $\mathbb{R}^4$, by allowing the coefficients to be matrices, more precisely the Pauli matrices, which satisfy the Clifford anti-commutation relations. The generalization of Dirac's operator to arbitrary spin manifolds was given by Atiyah–Singer and Lichnerowicz. The spin bundle inherits the Levi-Civita connection from the frame bundle, and hence the spinor bundle is endowed with a natural covariant derivative ∇. Moreover, vector fields act on spinors by Clifford multiplication, denoted by γ. The Dirac operator is then defined as the Levi-Civita connection composed with the Clifford multiplication:

$$\mathcal{D} = \gamma \circ \nabla \colon \Gamma(\Sigma M) \longrightarrow \Gamma(\Sigma M), \quad \mathcal{D}\Psi = \sum_{i=1}^{n} e_i \cdot \nabla_{e_i} \Psi,$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame. The Dirac operator satisfies the fundamental Schrödinger–Lichnerowicz formula:

$$\mathcal{D}^2 = \nabla^*\nabla + \frac{1}{4}\mathrm{Scal}, \tag{1}$$

where Scal is the scalar curvature function. A spectacular and elementary application of this formula is the Lichnerowicz theorem which says that if the manifold is closed and the scalar curvature is positive, then there is no harmonic spinor, i.e., $\operatorname{Ker} \mathcal{D} = 0$. On the other hand, Atiyah and Singer had computed the index of the Dirac operator on compact even-dimensional spin manifolds. They noted that the spinor bundle on such a manifold is naturally $\mathbb{Z}_2$-graded, and the Dirac operator is odd with respect to this grading. The operator $\mathcal{D}^+$ is defined as the restriction of $\mathcal{D}$ to positive spinors, $\mathcal{D}^+ \colon \Gamma(\Sigma^+ M) \to \Gamma(\Sigma^- M)$, and its index is the Fredholm index, namely the difference of the dimensions of the spaces of solutions for $\mathcal{D}^+$ and its adjoint. The Atiyah–Singer formula gives this index in terms of a characteristic class, the $\widehat{A}$ genus. It follows that M does not admit metrics of positive scalar curvature if $\widehat{A}(M) \neq 0$. It also follows that $\widehat{A}(M)$ is an integer class when M is a spin manifold. The Atiyah–Singer formula for twisted Dirac operators was also a crucial ingredient in the recent classification of inner symmetric spaces and positive quaternion-Kähler manifolds with weakly complex tangent bundle [GMS11].

Another unexpected application of the Dirac operator is Witten's proof of the positive mass theorem for spin manifolds. In this book we present a variant of Witten's approach by Ammann and Humbert, which works on locally conformally flat compact spin manifolds. The idea is to express the mass as the logarithmic term in the diagonal expansion of the Green kernel for the conformal Laplacian, then use a scalar-flat conformal metric to construct a harmonic spinor on the complement of a fixed point, and finally to use the Schrödinger–Lichnerowicz formula (1) and integration by parts to deduce that the mass is non-negative.

Elliptic theory and representation theory

A main focus in this text is on the eigenvalues of the Dirac operator on compact spin manifolds. It appeared desirable to include a self-contained treatment of several (by now, classical) facts about the spectrum of an elliptic differential operator. Once these are established, as a bonus we apply the corresponding results for the Laplacian on functions to prove the Peter–Weyl theorem. Then we develop the representation theory of semisimple compact Lie groups, in order to compute explicitly the spectra of the Dirac operator on certain compact symmetric spaces.

The lowest eigenvalues of the Dirac operator and special spinors

As already mentioned, the Atiyah–Singer index theorem illustrates how the spectrum of the Dirac operator encodes subtle information on the topology and the geometry of the underlying manifold. Another seminal work on the subject is due to N. Hitchin and concerns harmonic spinors; see [Hit74]. He first discovered that, in contrast with the Laplacian on exterior forms, the dimension of the space of harmonic spinors is a conformal invariant which can (dramatically) change with the metric.

A significant part of the book is devoted to a detailed study of the relationship between the spectrum of the Dirac operator and the geometry of closed spin manifolds with positive scalar curvature.

In that context, the Schrödinger–Lichnerowicz formula not only states that on a closed spin manifold with positive scalar curvature there is no harmonic spinor, but also that there is a gap in the spectrum of the square of the Dirac operator. The first important achievement is Friedrich's inequality, which says that the first eigenvalue of the Dirac operator is bounded from below by that of the sphere, the model space of such a family of manifolds. The original proof of Th. Friedrich involved the notion of "modified connection." The inequality may also be proved by using elementary arguments in representation theory, but the simplest proof relies on the Schrödinger–Lichnerowicz formula together with the spinorial Cauchy–Schwarz inequality:

$$|\nabla\Psi|^2 \geq \frac{1}{n}|\mathcal{D}\Psi|^2, \quad \text{for any spinor field } \Psi.$$

Another remarkable consequence of this point of view is that how far this inequality is from being an equality is precisely measured by $|\mathcal{P}\Psi|^2$, where $\mathcal{P}$ is the Penrose operator (also called the Twistor operator).

Closed manifolds for which the first eigenvalue of the Dirac operator (in absolute value) satisfies the limiting case of Friedrich's inequality are called *limiting manifolds*. They are characterized by the existence of a spinor field in the kernel of the Penrose operator (called *twistor-spinor*) which is also an eigenspinor of $\mathcal{D}$. These special spinors are called *real Killing spinors* (this terminology is due to the fact that the associated vector field is Killing).

It has been observed by Hijazi and Lichnerowicz that manifolds having real Killing spinors cannot carry non-trivial parallel forms, hence there is no real Killing spinor on a manifold with "special" holonomy. With this in mind, Friedrich's inequality could be improved in different directions.

First, by relaxing the assumption on the positivity of the scalar curvature. For instance, based on the conformal covariance of the Dirac operator, a property shared with the Yamabe operator (the conformal scalar Laplacian), it is surprising to note that if one considers the Schrödinger–Lichnerowicz formula over a closed spin manifold for a conformal class of metrics, then for a specific choice of the conformal factor, basically a first eigenfunction of the Yamabe operator, it follows that the square of the first eigenvalue of the Dirac operator is, up to a constant, at least the first eigenvalue of the Yamabe operator (this is known as *the Hijazi inequality*). Again the limiting case is characterized by the existence of a real Killing spinor. Another approach is to consider a deformation (which generalizes that introduced by Friedrich) of the spinorial covariant derivative by Clifford multiplication by the symmetric endomorphism of the tangent bundle associated with the energy–momentum tensor corresponding to the eigenspinor. This approach is of special interest in the setup of extrinsic spin geometry.

Secondly, a natural question is to improve Friedrich's inequality for manifolds with "special" holonomy. By the Berger–Simons classification, one knows that among all compact spin manifolds with positive scalar curvature, we have Kähler manifolds and quaternion-Kähler manifolds. They carry a parallel 2-form (the Kähler form) and a parallel 4-form (the Kraines form), respectively. Model spaces of such manifolds are respectively the complex projective space (note that complex projective spaces of even complex dimension are not spin, but are standard examples of Spin^c manifolds) and the quaternionic projective space. It is natural to expect that for these manifolds the first eigenvalue of the Dirac operator is at least equal to that of the corresponding model space. This is actually the case (with the restriction that for Kähler manifolds of even complex dimension, the lower bound turns out to be given by the first eigenvalue of the product of the complex projective space with the 2-dimensional real torus). These lower bounds are due to K.-D. Kirchberg in the Kähler case and to W. Kramer, U. Semmelmann, and G. Weingart in the quaternion-Kähler setup. There are different proofs of these inequalities, but it is now clear that representation theory of the holonomy group plays a central role in the approach. Roughly speaking, the proofs presented here are based on the use of Penrose-type operators given by the decomposition of the spinor bundle under the action, via Clifford multiplication, of the geometric parallel forms characterizing the holonomy.

Thirdly, a natural task is to classify closed spin manifolds M of positive scalar curvature admitting real Killing spinors and characterize limiting manifolds of Kähler or quaternion-Kähler type. For the first family of manifolds, the classification (obtained by C. Bär) is based on the cone construction, i.e., the manifold $\bar{M}$ defined as a warped product of M with the interval $(0, +\infty)$. This warped product is defined in such a way that for $M = \mathbb{S}^n$ then $\bar{M}$ is isometric to $\mathbb{R}^{n+1}\backslash\{0\}$. The cone construction was used by S. Gallot in order to characterize the sphere as being the only limiting manifold for the Laplacian on exterior forms. M. Wang characterized complete simply connected spin manifolds $\bar{M}$ carrying parallel spinors by their possible holonomy groups. Bär has shown that Killing spinors on M are in one-to-one correspondence with parallel spinors on $\bar{M}$, hence he deduced a list of possible holonomies for $\bar{M}$, and consequently a list of possible geometries for M.

The classification of limiting manifolds of Kähler type is due to A. Moroianu. The key idea is to interpret any limiting spinor as a Killing spinor on the unit canonical bundle of the manifold. It turns out that in odd complex dimensions, limiting manifolds of dimension $4l - 1$ are exactly twistor spaces associated to quaternion-Kähler manifolds of positive scalar curvature, and those of dimension $4l + 1$ are the complex projective spaces. In even complex dimensions $m = 2l \geq 4$, the universal cover $\tilde{M}$ of a limiting manifold M^{2m} is either isometric to the Riemannian product $\mathbb{C}\mathrm{P}^{m-1} \times \mathbb{R}^2$, for l odd, or to the Riemannian product $N^{2m-2} \times \mathbb{R}^2$, for l even, where N is a limiting manifold of odd complex dimension. A spectacular application of these classification results is an alternative to LeBrun's proof of the fact that every contact positive Kähler–Einstein manifold is the twistor space of a positive quaternion-Kähler manifold.

Finally, in the case of compact spin quaternion-Kähler manifolds M^{4m}, it was proved by W. Kramer, U. Semmelmann, and G. Weingart, that the only limiting manifold is the projective space $\mathbb{H}\mathrm{P}^m$. We present here a more elementary proof.

Dirac spectra of model spaces

As pointed out previously, it is important to compute the Dirac spectrum for some concrete examples, and the archetypal examples in geometry are given by symmetric spaces. For these manifolds, computing the spectrum is a purely algebraic problem which can be theoretically solved by classical harmonic analysis methods, involving representation theory of compact Lie groups. Those methods were already well known in case of the Laplacian.

In the last part of the book we give a short self-contained review of the representation theory of compact groups. Our aim is to provide a sort of practical guide in order to understand the techniques involved. From this point of view, representations of the standard examples (unitary groups, orthogonal and spin groups, symplectic groups) are described in detail. In the same spirit, we give an elementary introduction to symmetric spaces. This point of view is also followed to explain the general procedure for an explicit computation of the spectrum of the Dirac operator of compact symmetric spaces, and then the spectrum of the standard examples (spheres, complex and quaternionic projective spaces) is computed in order to illustrate the method.

Part I

Basic spinorial material

Chapter 1

Algebraic aspects

This chapter is devoted to the basic algebraic ingredients of spin geometry: Clifford algebras and spin groups. Clifford algebras are naturally associated to vector spaces endowed with symmetric bilinear forms. The Clifford algebra of the n-dimensional Euclidean space contains the spin group Spin_n, which is the 2-fold cover of the special orthogonal group SO_n for $n \geq 3$. We study the complex representations of the spin groups induced by algebra representations of the Clifford algebras and highlight some of their important properties.

1.1 Clifford algebras

1.1.1 Definitions

Definition 1.1. Let V be an n-dimensional vector space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and q a bilinear symmetric form on V. The *Clifford algebra* $\mathrm{Cl}(V, q)$ associated with (V, q) is the associative algebra with unit, defined by

$$\mathrm{Cl}(V, q) := V^{\otimes} \big/ \mathcal{I}(V, q)$$

where $V^{\otimes} = \oplus_{i \geq 0} V^{\otimes i}$ is the tensor algebra of V and $\mathcal{I}(V, q)$ the two-sided ideal generated by all elements of the form $x \otimes x + q(x, x)1$, for $x \in V$. The product in the Clifford algebra will be denoted by "$\cdot$".

Remark 1.2. There is a natural map $\iota\colon V \to \mathrm{Cl}(V, q)$ obtained by considering the natural embedding $V \hookrightarrow V^{\otimes}$, followed by the projection $V^{\otimes} \to \mathrm{Cl}(V, q)$. By Proposition 1.5 below, this map is an embedding. Viewing V as a subset of $\mathrm{Cl}(V, q)$ in that way, the algebra $\mathrm{Cl}(V, q)$ is generated by V (and the unit 1), subject to the relations

$$v \cdot v = -q(v, v)1.$$

The Clifford algebra can be characterized by the following universal property.

Proposition 1.3. *Let $\mathcal{A}$ be an associative algebra with unit and $f: V \to \mathcal{A}$ a linear map such that for all $v \in V$*

$$f(v)^2 = -q(v,v)1_{\mathcal{A}}. \tag{1.1}$$

Then there exists a unique $\mathbb{K}$-algebra homomorphism

$$\tilde{f}: \mathrm{Cl}(V,q) \longrightarrow \mathcal{A}$$

satisfying

$$\tilde{f} \circ \iota = f.$$

Furthermore, if $\mathcal{C}$ is an associative $\mathbb{K}$-algebra with unit carrying a linear map $\iota': V \to \mathcal{C}$ satisfying $\iota'(v)^2 = -q(v,v)1_{\mathcal{C}}$, with the property above, then $\mathcal{C}$ is isomorphic to $\mathrm{Cl}(V,q)$.

Proof. If $i: V \hookrightarrow V^{\otimes}$ is the natural inclusion, then by the universal property of the tensor algebra, there exists a unique algebra homomorphism F such that $f = F \circ i$. We have to show that $\mathcal{I}(V,q) \subset \operatorname{Ker} F$. Indeed, for any $v \in V$, we have

$$\begin{aligned} F(v \otimes v + q(v,v)1) &= F(v) \cdot F(v) + q(v,v)1_{\mathcal{A}} \\ &= f(v) \cdot f(v) + q(v,v)1_{\mathcal{A}} \\ &= 0, \end{aligned}$$

hence F factors through the quotient and we have the commutative diagram

$$\begin{tikzcd} V^{\otimes} \arrow[r, "\mathrm{Proj}"] \arrow[dr, "F"] & \mathrm{Cl}(V,q) \arrow[d, "\tilde{f}"] \\ V \arrow[u, "i"] \arrow[r, "f"'] & \mathcal{A} \end{tikzcd}$$

where Proj is the natural projection. We thus have

$$f = \tilde{f} \circ \mathrm{Proj} \circ i = \tilde{f} \circ \iota.$$

Since Proj is onto, the uniqueness of F implies that of $\tilde{f}$. Applying this to $f = \iota$ shows that the identity is the unique algebra homomorphism a of $\mathrm{Cl}(V,q)$ which satisfies $a \circ \iota = \iota$.

Let $\mathcal{C}$ be any associative $\mathbb{K}$-algebra with unit, with a linear map $\iota': V \to \mathcal{C}$, verifying the same universal property as $\mathrm{Cl}(V,q)$. Applying the universal properties of $\mathcal{C}$ and $\mathrm{Cl}(V,q)$ to the maps ι and ι' yields algebra homomorphisms $\tilde{\iota}: \mathcal{C} \to \mathrm{Cl}(V,q)$ and $\tilde{\iota}': \mathrm{Cl}(V,q) \to \mathcal{C}$ satisfying $\tilde{\iota} \circ \iota' = \iota$ and $\tilde{\iota}' \circ \iota = \iota'$. Moreover, the composition $a := \tilde{\iota} \circ \tilde{\iota}'$ is an algebra morphism of $\mathrm{Cl}(V,q)$ satisfying $a \circ \iota = \iota$, so by the above remark a is the identity. Similarly, $\tilde{\iota}' \circ \tilde{\iota}$ is the identity of $\mathcal{C}$. □

From now on we will only consider Clifford algebras of real Euclidean spaces (i.e., with a positive definite bilinear form q) or $\mathbb{C}$-vector spaces endowed with a non-degenerate quadratic form.

Remarks 1.4. (1) The Clifford algebra $\mathrm{Cl}(V,q)$ can be abstractly defined as the algebra generated by $n+1$ elements $\{v_0, v_1, \ldots, v_n\}$, subject to the relations

$$v_0 \cdot v_i = v_i \cdot v_0 = v_i, \quad v_0^2 = v_0, \quad v_i^2 = -v_0 \quad (i \geq 1),$$

$$v_i \cdot v_j = -v_j \cdot v_i \quad (1 \leq i \neq j \geq 1).$$

Of course, v_0 corresponds to 1 and $\{v_1, \ldots, v_n\}$ to a q-orthonormal basis of V.

(2) If $\{e_1, \ldots, e_n\}$ is a q-orthonormal basis of V, then the system

$$\{1, e_{i_1} \cdots e_{i_k};\ 1 \leq i_1 < \cdots < i_k \leq n, 1 \leq k \leq n\} \tag{1.2}$$

spans $\mathrm{Cl}(V,q)$ as vector space, thus $\dim \mathrm{Cl}(V,q) \leq 2^n$. It will be shown in the next proposition that this system is actually a basis.

(3) Note that $\mathrm{Cl}(V,0)$ is nothing else than $\Lambda^* V$, the exterior algebra of V.

It turns out that the exterior algebra and the Clifford algebra can be naturally identified as vector spaces (see Chevalley [Che62] and Kähler [Käh62]).

Proposition 1.5. *There is a canonical isomorphism of vector spaces between the exterior algebra and the Clifford algebra of* (V,q)*, which, in terms of an orthonormal basis* $\{e_1, \ldots, e_n\}$ *of* V*, is given by*

$$\Lambda^* V \longrightarrow \mathrm{Cl}(V,q), \quad e_{i_1} \wedge \cdots \wedge e_{i_k} \longmapsto e_{i_1} \cdots e_{i_k}.$$

Proof. For any vector $v \in V$, let $v\wedge$ and $v \lrcorner$ be the exterior multiplication, respectively the interior multiplication by v relative to q, and consider the map

$$f : V \longrightarrow \mathrm{End}(\Lambda^* V), \quad v \longmapsto f(v)(\omega) = v \wedge \omega - v \lrcorner \omega.$$

Then one can easily check that the homomorphism f satisfies condition (1.1), with $\mathcal{A} = \mathrm{End}(\Lambda^* V)$, hence by the universal property, f uniquely extends to a homomorphism

$$\tilde{f} : \mathrm{Cl}(V,q) \longrightarrow \mathrm{End}(\Lambda^* V).$$

Let us define the linear map

$$F : \mathrm{Cl}(V,q) \longrightarrow \Lambda^* V, \quad \varphi \longmapsto (\tilde{f}(\varphi))(1_{\Lambda^* V}).$$

If $\{e_1, \ldots, e_n\}$ is a q-orthonormal basis of V, then for every $i_1 < \cdots < i_k$, we obtain

$$F(e_{i_1} \cdots e_{i_k}) = e_{i_1} \wedge \cdots \wedge e_{i_k},$$

so F is onto. On the other hand, Remark 1.4 (2) shows that $\dim \mathrm{Cl}(V,q) \leq 2^n = \dim \Lambda^* V$, thus F is an isomorphism. $\square$

Remark 1.6. The Clifford algebra and the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ are both defined as quotients of the tensor algebra. A classical result in Lie algebra theory is the Poincaré–Birkhoff–Witt theorem stating that $\mathcal{U}(\mathfrak{g})$ has a basis formed by ordered monomials in the elements of a well-ordered basis of $\mathfrak{g}$. In this light, Proposition 1.5 or even the embedding of V in $\mathrm{Cl}(V)$ are perhaps less obvious than they might seem at first sight. We note that there is a gap in the proof of this result in [LM89], Proposition 1.2, p. 10.

Clifford algebras are endowed with the following fundamental automorphisms.

(1) Using the universal property, the injective morphism $V \hookrightarrow \mathrm{Cl}(V,q)$ induced by the map $-\mathrm{Id}\colon v \mapsto -v$, gives rise to the automorphism

$$\alpha\colon \mathrm{Cl}(V,q) \longrightarrow \mathrm{Cl}(V,q), \quad e_{i_1}\cdots e_{i_k} \longmapsto (-1)^k e_{i_1}\cdots e_{i_k}.$$

As $\alpha^2 = \mathrm{Id}$, we get the decomposition

$$\mathrm{Cl}(V,q) = \mathrm{Cl}^0(V,q) \oplus \mathrm{Cl}^1(V,q),$$

where

$$\mathrm{Cl}^i(V,q) := \{\varphi \in \mathrm{Cl}(V,q);\ \alpha(\varphi) = (-1)^i\varphi\}. \tag{1.3}$$

Clearly, for $i, j \in \mathbb{Z}/2\mathbb{Z}$, we have

$$\mathrm{Cl}^i(V,q)\cdot \mathrm{Cl}^j(V,q) \subset \mathrm{Cl}^{i+j}(V,q).$$

Thus the Clifford algebra $\mathrm{Cl}(V,q)$ is a $\mathbb{Z}/2\mathbb{Z}$-graded algebra, i.e., a superalgebra. The subspace $\mathrm{Cl}^0(V,q)$ (resp. $\mathrm{Cl}^1(V,q)$) is called the *even* (resp. *odd*) part of $\mathrm{Cl}(V,q)$.

(2) Consider the $\mathbb{K}$-algebra anti-automorphism defined by

$${}^t\colon V^{\otimes} \longrightarrow V^{\otimes}, \quad x_{i_1}\otimes\cdots\otimes x_{i_k} \longmapsto x_{i_k}\otimes\cdots\otimes x_{i_1}.$$

Since ${}^t(\mathcal{I}(V,q)) \subset \mathcal{I}(V,q)$, there is an induced automorphism

$${}^t\colon \mathrm{Cl}(V,q) \longrightarrow \mathrm{Cl}(V,q), \quad x_{i_1}\cdots x_{i_k} \longmapsto x_{i_k}\cdots x_{i_1}.$$

An immediate application of the universal property shows that if there exists a $\mathbb{K}$-isomorphism f between two $\mathbb{K}$-vector spaces V and V', endowed respectively with two bilinear symmetric forms q and q', such that $f^*q' = q$, then f uniquely extends to a $\mathbb{K}$-algebra isomorphism between $\mathrm{Cl}(V,q)$ and $\mathrm{Cl}(V',q')$.

Lemma 1.7. *Let $f\colon (V,q) \to (V',q')$ be an isometry. Then f uniquely extends to a $\mathbb{K}$-algebra isomorphism*

$$\tilde{f}\colon \mathrm{Cl}(V,q) \longrightarrow \mathrm{Cl}(V',q').$$

By the above remark, we may restrict to the following canonical examples.

Definition 1.8. The Clifford algebra $\mathrm{Cl}_n := \mathrm{Cl}(\mathbb{R}^n, q^{\mathbb{R}})$ (resp. $\mathbb{C}\mathrm{l}_n := \mathrm{Cl}(\mathbb{C}^n, q^{\mathbb{C}})$) associated with the canonical Euclidean scalar product $q^{\mathbb{R}}$ (resp. $q^{\mathbb{C}}$) defined by $q^{\mathbb{R}}(x,y) = \sum_i x_i y_i$ (resp. $q^{\mathbb{C}}(z,w) = \sum_i z_i w_i$) is called the *n-dimensional real* (resp. *complex*) *Clifford algebra.*

Examples 1.9. If $\{e_1, \dots, e_n\}$ denotes the canonical orthonormal basis of $\mathbb{R}^n$, then the relations

$$e_i \cdot e_j + e_j \cdot e_i = -2\delta_{ij}. \tag{1.4}$$

hold in Cl_n. Hence, we have the following fact.

(1) A basis of Cl_1 is given by $\{1, e_1\}$. Since $e_1^2 = -1$, one has $\mathrm{Cl}_1 \cong \mathbb{C}$.

(2) A basis of Cl_2 is given by $\{1, e_1, e_2, e_1 \cdot e_2\}$. Since the three vectors e_1, e_2, and $e_1 \cdot e_2$ verify the same multiplication rules as the standard basis of imaginary quaternions, one has $\mathrm{Cl}_2 \cong \mathbb{H}$.

(3) The volume element $\omega := e_1 \cdot e_2 \cdot e_3 \in \mathrm{Cl}_3$ is central and $\omega \cdot \omega = 1$. Define $\pi^{\pm} := \frac{1}{2}(1 \pm \omega)$. Since

$$\pi^+ + \pi^- = 1,$$

$$(\pi^{\pm})^2 = \pi^{\pm},$$

$$\pi^- \cdot \pi^+ = \pi^+ \cdot \pi^- = 0,$$

one has the algebra decomposition $\mathrm{Cl}_3 = \mathrm{Cl}_3^+ \oplus \mathrm{Cl}_3^-$ where $\mathrm{Cl}_3^{\pm} = \pi^{\pm} \cdot \mathrm{Cl}_3 = \mathrm{Cl}_3 \cdot \pi^{\pm}$. A basis of $\mathrm{Cl}_3^{\pm}$ is given by $\{\pi^{\pm}, \pi^{\pm} \cdot e_1, \pi^{\pm} \cdot e_2, \pi^{\pm} \cdot e_1 \cdot e_2\}$, whence $\mathrm{Cl}_3^{\pm} \cong \mathbb{H}$ and $\mathrm{Cl}_3 \cong \mathbb{H} \oplus \mathbb{H}$.

The following important results can be derived from the universal property.

Proposition 1.10. *The complex Clifford algebra is isomorphic to the complexification of the real Clifford algebra, i.e.,*

$$\mathbb{C}\mathrm{l}_n \cong \mathrm{Cl}_n \otimes_{\mathbb{R}} \mathbb{C}.$$

Proof. By the universal property, the inclusion $\mathbb{R}^n \hookrightarrow \mathbb{C}^n \subset \mathbb{C}\mathrm{l}_n$ can be extended to a morphism $\imath \colon \mathrm{Cl}_n \to \mathbb{C}\mathrm{l}_n$ which induces the morphism

$$\imath^{\mathbb{C}} \colon \mathrm{Cl}_n \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow \mathbb{C}\mathrm{l}_n, \quad a \otimes z \longmapsto \imath(a)z.$$

On the other hand, the $\mathbb{C}$-linear map

$$\jmath^{\mathbb{C}} \colon \mathbb{C}^n \longrightarrow \mathrm{Cl}_n \otimes_{\mathbb{R}} \mathbb{C}, \quad v \longmapsto \mathrm{Re}(v) \otimes 1 + \mathrm{Im}(v) \otimes i$$

satisfies condition (1.1), i.e.,

$$\jmath^{\mathbb{C}}(v)^2 = -q^{\mathbb{C}}(v,v)1,$$

hence it can be extended to a morphism $\mathbb{C}\mathrm{l}_n \to \mathrm{Cl}_n \otimes_{\mathbb{R}} \mathbb{C}$, which is the inverse of $\imath^{\mathbb{C}}$. □

Proposition 1.11. *The n-dimensional real (resp. complex) Clifford algebra is isomorphic to the even part of the $(n+1)$-dimensional real (resp. complex) Clifford algebra, i.e.,*

$$\mathrm{Cl}_n \cong \mathrm{Cl}^0_{n+1} \quad \textit{and} \quad \mathbb{C}\mathrm{l}_n \cong \mathbb{C}\mathrm{l}^0_{n+1}\,. \tag{1.5}$$

Proof. Denote by $\{e_1,\dots,e_n\}$ and $\{e_1,\dots,e_{n+1}\}$ the canonical bases of $\mathbb{K}^n$ and $\mathbb{K}^{n+1}$, $\mathbb{K}=\mathbb{R}$ (resp. $\mathbb{C}$). This suggests to identify $\mathbb{K}^n$ with the image of its canonical injection in $\mathbb{K}^{n+1}$ as the subspace generated by the first n vectors. Define the linear map

$$f\colon \mathbb{K}^n \longrightarrow \mathrm{Cl}^0_{n+1} \ (\text{resp. } \mathbb{C}\mathrm{l}^0_{n+1}), \quad e_i \longmapsto e_i \cdot e_{n+1}.$$

By the definition of f, we have $f(e_i)^2 = -1$, thus f extends to

$$\tilde{f}\colon \mathrm{Cl}_n \longrightarrow \mathrm{Cl}^0_{n+1} \quad (\text{resp. } \mathbb{C}\mathrm{l}_n \longrightarrow \mathbb{C}\mathrm{l}^0_{n+1}).$$

Clearly, $\tilde{f}$ is an injective linear map between vector spaces of the same dimension, thus the map $\tilde{f}$ is an algebra isomorphism. □

Proposition 1.12. *For any couple of positive integers (m,n), the $(m+n)$-dimensional real (resp. complex) Clifford algebra is isomorphic to the graded tensor product of the m-dimensional real (resp. complex) Clifford algebra with the n-dimensional real (resp. complex) Clifford algebra:*

$$\mathrm{Cl}_{m+n} \simeq \mathrm{Cl}_m \,\hat{\otimes}\, \mathrm{Cl}_n \quad \textit{and} \quad \mathbb{C}\mathrm{l}_{m+n} \simeq \mathbb{C}\mathrm{l}_m \,\hat{\otimes}\, \mathbb{C}\mathrm{l}_n,$$

the "graded tensor product" being the usual tensor product endowed with the multiplication

$$(a\otimes b)\cdot(a'\otimes b') = (-1)^{\deg(b)\deg(a')} a\cdot a' \otimes b\cdot b',$$

where $\deg\colon \mathrm{Cl}(V,q) \to \{0,1\}$ *is defined by the grading* (1.3).

Proof. Let $(e_i)_{i=1,\dots,m+n}$ be the standard basis of $\mathbb{K}^{m+n}$, $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$. Identify $\mathbb{K}^m$ (resp. $\mathbb{K}^n$) with the subspace of $\mathbb{K}^{m+n}$ spanned by the e_i, $i=1,\dots,m$ (resp. the e_i, $i=m+1,\dots,n$). Define the linear map

$$f\colon \mathbb{K}^{m+n} \longrightarrow \mathrm{Cl}(\mathbb{K}^m, q^{\mathbb{K}}) \hat{\otimes}\, \mathrm{Cl}(\mathbb{K}^n, q^{\mathbb{K}})$$

by

$$f(e_i) = \begin{cases} e_i \otimes 1 & \text{if } i = 1, \ldots, m, \\ 1 \otimes e_i & \text{if } i = m+1, \ldots, n. \end{cases}$$

Since $f(e_i) \cdot f(e_j) + f(e_j) \cdot f(e_i) = -2\delta_{ij}$, the map f extends to a $\mathbb{K}$-algebra homomorphism

$$\mathrm{Cl}(\mathbb{K}^{m+n}, q^{\mathbb{K}}) \longrightarrow \mathrm{Cl}(\mathbb{K}^m, q^{\mathbb{K}}) \widehat{\otimes} \mathrm{Cl}(\mathbb{K}^n, q^{\mathbb{K}}).$$

Considering generators of $\mathrm{Cl}(\mathbb{K}^{m+n}, q^{\mathbb{K}})$, it is clear that this linear map is onto between vector spaces of the same dimension, hence an isomorphism. □

By Proposition 1.5, there exists a canonical vector space isomorphism between Cl_n and $\Lambda^* \mathbb{R}^n$ (resp. $\mathbb{C}\mathrm{l}_n$ and $\Lambda^* \mathbb{C}^n$). The following proposition gives the relation between the product in the Clifford algebra in terms of the exterior and interior products in the exterior algebra.

Proposition 1.13. *For all $v \in \mathbb{K}^n$, and all*

$$\varphi \in \Lambda^p \mathbb{K}^n \subset \Lambda^* \mathbb{K}^n \cong \begin{cases} \mathrm{Cl}_n & \text{if } \mathbb{K} = \mathbb{R}, \\ \mathbb{C}\mathrm{l}_n & \text{if } \mathbb{K} = \mathbb{C}, \end{cases}$$

we have

$$v \cdot \varphi \cong v \wedge \varphi - v \lrcorner \varphi \quad \text{and} \quad \varphi \cdot v \cong (-1)^p (v \wedge \varphi + v \lrcorner \varphi),$$

where $\wedge$ denotes the exterior product and $\lrcorner$ the interior product.

Proof. Let $v = e_j$ and $\varphi = e_{i_1} \cdots e_{i_p}$ with $i_1 < \cdots < i_p$. If there exists i_k such that $j = i_k$, then $v \wedge \varphi = 0$ and

$$\begin{aligned} v \lrcorner \varphi &= (-1)^{k-1} e_{i_1} \wedge \cdots \wedge e_{i_{k-1}} \wedge e_{i_{k+1}} \wedge \cdots \wedge e_{i_p} \\ &\cong (-1)^{k-1} e_{i_1} \cdots e_{i_{k-1}} \cdot e_{i_{k+1}} \cdots e_{i_p} \\ &= -v \cdot \varphi \\ &= (-1)^p \varphi \cdot v. \end{aligned}$$

If $j \notin \{i_1, \ldots, i_p\}$, then $v \lrcorner \varphi = 0$ and

$$\begin{aligned} v \wedge \varphi &= e_j \wedge e_{i_1} \wedge \cdots \wedge e_{i_p} \\ &\cong e_j \cdot e_{i_1} \cdots e_{i_p} \\ &= v \cdot \varphi \\ &= (-1)^p \varphi \cdot v. \end{aligned}$$

We then conclude by bilinearity. □

We will also need the following interesting feature of the Clifford product viewed on $\Lambda^*\mathbb{K}^n$.

Proposition 1.14. *For any*

$$\varphi \in \Lambda^p\mathbb{K}^n \subset \Lambda^*\mathbb{K}^n \cong \begin{cases} \mathrm{Cl}_n & \text{if } \mathbb{K} = \mathbb{R}, \\ \mathbb{C}\mathrm{l}_n & \text{if } \mathbb{K} = \mathbb{C}, \end{cases}$$

and for any orthonormal basis $\{e_1, \ldots, e_n\}$ *we have*

$$\sum_{j=1}^{n} e_j \cdot \varphi \cdot e_j = (-1)^{p+1}(n-2p)\varphi.$$

Proof. It suffices to consider $\varphi = e_{i_1} \ldots e_{i_p}$. Then, by equation (1.4) and for a fixed $j \in \{1, \ldots, n\}$, it is straightforward to see that

$$e_j \cdot \varphi \cdot e_j = \begin{cases} (-1)^p\varphi & \text{if } j \in \{i_1, \ldots, i_p\}, \\ (-1)^{p+1}\varphi & \text{if } j \notin \{i_1, \ldots, i_p\}, \end{cases}$$

hence

$$\begin{aligned} \sum_{j=1}^{n} e_j \cdot \varphi \cdot e_j &= [(-1)^p p + (-1)^{p+1}(n-p)]\varphi \\ &= (-1)^{p+1}(n-2p)\varphi. \end{aligned}$$

□

1.1.2 Classification of Clifford algebras

Since in the following we mainly deal with complex Clifford algebras, we now give some basic facts which lead to their classification. We also mention the tools which lead to the classification of real Clifford algebras. Henceforth, if $\mathbb{K}$ is a field, $\mathbb{K}(n)$ denotes the algebra of $n \times n$ matrices with coefficients in $\mathbb{K}$.

Theorem 1.15 (Classification of complex Clifford algebras). *Let* $n \in \mathbb{N}$.

(1) *There exists a canonical algebra isomorphism* $\mathbb{C}\mathrm{l}_{n+2} \cong \mathbb{C}\mathrm{l}_n \otimes \mathbb{C}\mathrm{l}_2$.

(2) *We have the following algebra isomorphisms:*

$$\mathbb{C}\mathrm{l}_{2n+1} \cong \mathbb{C}(2^n) \oplus \mathbb{C}(2^n),$$

and

$$\mathbb{C}\mathrm{l}_{2n+2} \cong \mathbb{C}(2^{n+1}).$$

Proof. (1) Denote by $\{e_1, \ldots, e_{n+2}\}$ the canonical basis of $\mathbb{C}^{n+2}$ and consider the identifications $\mathbb{C}^n = \operatorname{span}\{e_1, \ldots, e_n\}$, $\mathbb{C}^2 = \operatorname{span}\{e_{n+1}, e_{n+2}\}$. One can easily check that the linear map

$$f\colon \mathbb{C}^{n+2} \longrightarrow \mathbb{C}\mathrm{l}_n \otimes \mathbb{C}\mathrm{l}_2, \quad e_k \longmapsto \begin{cases} e_k \otimes i\, e_{n+1} \cdot e_{n+2}, & \text{for } k \leq n, \\ 1 \otimes e_k, & \text{for } k > n, \end{cases}$$

satisfies (1.1). By Proposition 1.3, f can be extended to a homomorphism

$$\mathbb{C}\mathrm{l}_{n+2} \longmapsto \mathbb{C}\mathrm{l}_n \otimes \mathbb{C}\mathrm{l}_2 \,.$$

With the help of the standard bases (1.2) of the Clifford algebras, one checks that this homomorphism is onto, and hence that it is an isomorphism, since $\dim_{\mathbb{C}} \mathbb{C}\mathrm{l}_{n+2} = \dim_{\mathbb{C}} \mathbb{C}\mathrm{l}_n \otimes \mathbb{C}\mathrm{l}_2$.

(2) By (1), using the isomorphism $\mathbb{C}(k) \otimes \mathbb{C}(l) \cong \mathbb{C}(kl)$, induced by the identification $\mathbb{C}^k \otimes \mathbb{C}^l \cong \mathbb{C}^{kl}$, it is sufficient to prove the assertion for $n = 0$. We already noted that $\mathrm{Cl}_1 \cong \mathbb{C}$, hence $\mathbb{C}\mathrm{l}_1 \cong \mathrm{Cl}_1 \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$, the last isomorphism being given on decomposable elements by $z_1 \otimes z_2 \mapsto (z_1 z_2, \bar{z}_1 z_2)$.

We also noted that $\mathrm{Cl}_2 \cong \mathbb{H}$. It is easy to verify that the identification $\mathbb{H} \cong \mathbb{C}^2$ given by the $\mathbb{R}$-isomorphism

$$a\mathbf{1} + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k} = (a\mathbf{1} + b\boldsymbol{i}) + \boldsymbol{j}(c - d\boldsymbol{i}) \longmapsto (\lambda = a + ib, \mu = c - id),$$

induces the injective $\mathbb{R}$-homomorphism of algebras

$$\mathbb{H} \longrightarrow \mathbb{C}(2), \quad q = \lambda + \boldsymbol{j}\mu \longmapsto \begin{pmatrix} \lambda & -\bar{\mu} \\ \mu & \bar{\lambda} \end{pmatrix}. \tag{1.6}$$

Extending this injective homomorphism to an injective $\mathbb{C}$-homomorphism

$$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow \mathbb{C}(2),$$

we conclude by a dimensional argument that $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}(2)$. □

The classification of real Clifford algebras is obtained by considering the Clifford algebra

$$\mathrm{Cl}'_n := \mathrm{Cl}(\mathbb{R}^n, -q^{\mathbb{R}^n}).$$

Using the universal property, we can prove the following proposition.

Proposition 1.16. *For all $n \in \mathbb{N}$,*

$$\mathrm{Cl}_{n+2} \cong \mathrm{Cl}_n \otimes \mathrm{Cl}'_2 \quad \textit{and} \quad \mathrm{Cl}'_{n+2} \cong \mathrm{Cl}'_n \otimes \mathrm{Cl}_2 \,.$$

Using this result and the explicit description of Cl'_2, one deduces the first row of the following table.

Table 1

n	1	2	3	4
Cl_n	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{H}(2)$
$\mathbb{C}\mathrm{l}_n$	$\mathbb{C} \oplus \mathbb{C}$	$\mathbb{C}(2)$	$\mathbb{C}(2) \oplus \mathbb{C}(2)$	$\mathbb{C}(4)$
n	5	6	7	8
Cl_n	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{R}(16)$
$\mathbb{C}\mathrm{l}_n$	$\mathbb{C}(4) \oplus \mathbb{C}(4)$	$\mathbb{C}(8)$	$\mathbb{C}(8) \oplus \mathbb{C}(8)$	$\mathbb{C}(16)$

Finally, using again Proposition 1.16 together with the results of the above table, one deduces the following proposition.

Proposition 1.17. *For all $n \in \mathbb{N}$,*

$$\mathrm{Cl}_{n+8} \cong \mathrm{Cl}_n \otimes_{\mathbb{R}} \mathrm{Cl}_8 .$$

This gives the complete classification of real Clifford algebras.

1.2 Spin groups and their representations

In this section we define certain subgroups of the multiplicative group of units in the real or the complex Clifford algebra. We are mainly interested in the real and complex spin groups and the conformal spin groups. We then establish explicit realizations of the corresponding Lie algebras. Finally, we study the irreducible representations of these Lie groups.

1.2.1 Spin groups

We begin by the following remark. Let v be any non-zero vector of an Euclidean space (V, q). The symmetry σ_v with respect to the hyperplane orthogonal to v is defined by

$$\sigma_v(x) = x - 2\frac{q(x,v)}{q(v,v)}v.$$

Viewing V as a subset of $\mathrm{Cl}(V,q)$, we can write

$$\sigma_v(x) = x - (x \cdot v + v \cdot x) \cdot v^{-2} \cdot v = -v \cdot x \cdot v^{-1} = -\mathrm{Ad}(v)(x). \tag{1.7}$$

Since any element of the group $\mathrm{SO}(V,q)$ is the product of an even number of symmetries of that type, $\mathrm{SO}(V,q)$ is the image under the map Ad of a certain subgroup of invertible elements in $\mathrm{Cl}(V,q)$. We are thus led to having a closer look at the group of units in $\mathrm{Cl}(V,q)$.

We consider the map

$$N\colon \mathrm{Cl}(V,q) \longrightarrow \mathrm{Cl}(V,q), \quad a \longmapsto \alpha({}^t a)\cdot a.$$

Note that $N(v) = |v|^2$, for all $v \in V$. Denote by $\mathrm{Cl}^*(V,q)$ the group of invertible elements of $\mathrm{Cl}(V,q)$. It can be easily checked that $N(\alpha({}^t(a^{-1})))\cdot N(a) = 1$. This implies $N(\mathrm{Cl}^*(V,q)) \subset \mathrm{Cl}^*(V,q)$. Since $v\cdot v = -|v|^2$, it is clear that any nonzero vector $v \in V \subset \mathrm{Cl}(V,q)$ lies in $\mathrm{Cl}^*(V,q)$, and $v^{-1} = -v/|v|^2$. For each $a \in \mathrm{Cl}^*(V,q)$, we consider the inner automorphism

$$\mathrm{Ad}_a\colon \mathrm{Cl}(V,q) \longrightarrow \mathrm{Cl}(V,q), \quad b \longmapsto a\cdot b\cdot a^{-1},$$

and the map

$$\widetilde{\mathrm{Ad}}_a\colon \mathrm{Cl}(V,q) \longrightarrow \mathrm{Cl}(V,q), \quad b \longmapsto \alpha(a)\cdot b\cdot a^{-1}.$$

As we pointed out in (1.7), for any nonzero vector v, the vector space $V \subset \mathrm{Cl}(V,q)$ is stable under $\widetilde{\mathrm{Ad}}_v$:

$$\widetilde{\mathrm{Ad}}_v|_V = \sigma_v. \tag{1.8}$$

For any $\mathbb{K}$-vector space V endowed with a symmetric bilinear form q, let $\mathrm{P}(V,q)$ be the group defined by

$$\mathrm{P}(V,q) = \{a \in \mathrm{Cl}^*(V,q);\ \widetilde{\mathrm{Ad}}_a(V) \subset V\}.$$

As before, we will restrict to the case $(V,q) = (\mathbb{K}^n, q^{\mathbb{K}})$. (For a non-degenerate symmetric bilinear form q, this choice is not restrictive in case $\mathbb{K} = \mathbb{C}$, whereas in case $\mathbb{K} = \mathbb{R}$, it means that we will not deal with non-degenerate bilinear forms with nonzero signature).

Lemma 1.18. (i) *The kernel of the map*

$$\widetilde{\mathrm{Ad}}\colon \mathrm{P}(V,q) \longrightarrow \mathrm{GL}_n(\mathbb{K})$$

is $\mathbb{K}^*$.

(ii) $N(\mathrm{P}(V,q)) \subset \mathbb{K}^*$.

(iii) *The map*

$$N|_{\mathrm{P}(V,q)} : \mathrm{P}(V,q) \longmapsto \mathbb{K}^*$$

is a group homomorphism.

Proof. (i) Consider $a \in \mathrm{Cl}(V, q)$ such that

$$\alpha(a) \cdot v = v \cdot a, \quad v \in \mathbb{K}^n \tag{1.9}$$

and, as above, denote by $\{e_1, \ldots, e_n\}$ the canonical basis of $\mathbb{K}^n$. By Proposition 1.5, for each fixed i, $1 \leq i \leq n$, any element $a \in \mathrm{P}(V, q)$ can be uniquely written as $a = b + e_i \cdot c$, where b and c are linear combinations of elements in the basis (1.2) that do not involve e_i. Condition (1.9), written for $v = e_i$, is equivalent to

$$\alpha(b) \cdot e_i = e_i \cdot b \quad \text{and} \quad \alpha(c) \cdot e_i = -e_i \cdot c.$$

The last relation implies $c = 0$. Since the argument holds for each e_i, it follows that a is a scalar, and since $a \in \mathrm{Cl}^*(V, q)$, it should be a nonzero scalar.

(ii) For all $v \in \mathbb{K}^n$ and $a \in \mathrm{P}(V, q)$ we have $\widetilde{\mathrm{Ad}}_a(v) \in \mathbb{K}^n$, hence

$$\widetilde{\mathrm{Ad}}_a(v) = {}^t(\widetilde{\mathrm{Ad}}_a(v)).$$

This implies $\widetilde{\mathrm{Ad}}_{N(a)}(v) = v$, hence from (i) we conclude that $N(a) \in \mathbb{K}^*$.

(iii) By (ii),

$$N(a \cdot b) = \alpha({}^t b \cdot {}^t a) \cdot a \cdot b = \alpha({}^t b) \cdot N(a) b = N(a) N(b). \qquad \square$$

Definition 1.19. (i) The *spin group* Spin_n is the subgroup of $\mathrm{P}(\mathbb{R}^n, q^{\mathbb{R}})$ generated by elements of the form $v_1 \cdots v_{2k}$, with $k \geq 1$ and $\|v_i\| = 1$, for $1 \leq i \leq 2k$.

(ii) The *conformal spin group* CSpin_n is the group $\mathrm{Spin}_n \times \mathbb{R}^*_+$.

Proposition 1.20. *For $n \geq 2$, the homomorphism*

$$\xi := \widetilde{\mathrm{Ad}}|_{\mathrm{Spin}_n}$$

is a nontrivial double covering of the special orthogonal group SO_n. In particular for $n \geq 3$, the group Spin_n is the universal cover of SO_n.

Proof. By (1.8) we know that the image of a nonzero vector by the map

$$\widetilde{\mathrm{Ad}}\colon \mathbb{R}^n \setminus \{0\} \subset \mathrm{Cl}_n^* \longrightarrow \mathrm{GL}_n$$

is the symmetry with respect to the hyperplane orthogonal to this vector. The image of $\widetilde{\mathrm{Ad}}|_{\mathrm{Spin}_n}$ is then the group of even products of such symmetries, which by the Cartan–Dieudonné theorem, is exactly the group SO_n. Using (iii) of Lemma 1.18, any element a of Spin_n satisfies $N(a) = 1$. By (i) of Lemma 1.18, we conclude that the kernel of $\widetilde{\mathrm{Ad}}\colon \mathrm{Spin}_n \to \mathrm{SO}_n$ is $\{\pm 1\}$.

To show that the covering is nontrivial it is sufficient to check that 1 and -1 belong to the same connected component of Spin_n. To see this, choose unit orthogonal vectors $v, w \in \mathbb{R}^n$ ($n \geq 2$) and note that the curve

$$\begin{aligned} c\colon [0, \pi/2] &\longrightarrow \mathrm{Spin}_n, \\ t &\longmapsto c(t) = (v \sin t + w \cos t) \cdot (v \sin t - w \cos t), \end{aligned}$$

satisfies $c(0) = 1$, $c(\pi/2) = -1$. Finally, the last assertion follows from the fact that, for $n \geq 3$, we have $\pi_1(\mathrm{SO}_n) = \mathbb{Z}/2\mathbb{Z}$. □

Remark 1.21. Note that since $\mathrm{P}(\mathbb{R}^n, q^{\mathbb{R}})$ is a closed subgroup of the Lie group Cl_n^*, it is actually a Lie group. Hence $\mathrm{Spin}_n = \widetilde{\mathrm{Ad}}^{-1}(\mathrm{SO}_n)$, being a closed subgroup of $\mathrm{P}(\mathbb{R}^n, q^{\mathbb{R}})$, is also a Lie group. Furthermore, the condition that Spin_n is generated by elements of the form $v_1 \cdots v_{2k}$, with $\|v_i\| = 1$, for $1 \leq i \leq 2k$, implies that it is a compact Lie group.

Examples 1.22. With obvious notations, we consider

(1) the unitary group

$$\mathrm{U}_n := \{A \in \mathbb{C}(n);\ {}^t A \bar{A} = \mathrm{Id}_n\};$$

(2) the special unitary group

$$\mathrm{SU}_n := \{A \in \mathrm{U}_n;\ \det A = 1\};$$

(3) the symplectic group[1]

$$\begin{aligned} \mathrm{Sp}_n := \{&A \in \mathrm{Hom}_{\mathbb{H}}(\mathbb{H}^n, \mathbb{H}^n); \\ &\langle A \cdot x, A \cdot y \rangle = \langle x, y \rangle,\ (x, y) \in \mathbb{H}^n \times \mathbb{H}^n\}, \end{aligned}$$

where

$$\langle x, y \rangle := \sum_{i=1}^{n} x_i \bar{y}_i,$$

and conjugation is defined, for any element $v = v_0\mathbf{1} + v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k} \in \mathbb{H} \cong \mathbb{R}^4$, by

$$\bar{v} := v_0\mathbf{1} - v_1\boldsymbol{i} - v_2\boldsymbol{j} - v_3\boldsymbol{k};$$

(4) the unit sphere

$$\mathbb{S}^n := \{x \in \mathbb{R}^{n+1};\ \langle x, x \rangle = 1\}.$$

[1] In order to represent elements of Sp_n by matrices with the usual convention for products, $\mathbb{H}^n$ is viewed as a *right* vector space over $\mathbb{H}$.

Then it is not difficult to describe the spin groups in low dimensions; see [LM89].

(1) $\mathrm{Spin}_1 \subset \mathrm{Cl}_1^0 \cong \mathrm{Cl}_0 \cong \mathbb{R}$, thus $\mathrm{Spin}_1 \cong \{\pm 1\} = \mathbb{Z}/2\mathbb{Z}$ and

$$\xi\colon \mathrm{Spin}_1 \cong \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathrm{SO}_1 \cong \{1\}$$

is the map $t \mapsto t^2$.

(2) One has $\mathrm{Spin}_2 \subset \mathrm{Cl}_2^0 \cong \mathrm{Cl}_1 \cong \mathbb{C}$. It is easy to verify that $\mathbb{S}^1 \cong \mathrm{U}_1 \cong \mathrm{Spin}_2$ via the map

$$\begin{aligned} \mathrm{e}^{i\theta} &\overset{\cong}{\longmapsto} \Big(\cos\frac{\theta}{2}\, e_1 + \sin\frac{\theta}{2}\, e_2\Big)\cdot\Big(-\cos\frac{\theta}{2}\, e_1 + \sin\frac{\theta}{2}\, e_2\Big) \\ &= \cos\theta + \sin\theta\, e_1 \cdot e_2 . \end{aligned}$$

The map

$$\xi\colon \mathrm{U}_1 \cong \mathrm{Spin}_2 \longrightarrow \mathrm{SO}_2$$

is then given by

$$\mathrm{e}^{i\theta} \longmapsto \begin{pmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{pmatrix},$$

hence, under the identification $\mathrm{SO}_2 \cong \mathrm{U}_1$, ξ is the "square" map

$$\mathrm{U}_1 \longrightarrow \mathrm{U}_1, \quad z \longmapsto z^2 .$$

The algebra Cl_3^0 is isomorphic to $\mathbb{H}$ by the isomorphism

$$\Phi\colon \begin{cases} 1 \longmapsto \mathbf{1}, \\ e_1 \cdot e_2 \longmapsto \boldsymbol{i}, \\ e_2 \cdot e_3 \longmapsto \boldsymbol{j}, \\ e_3 \cdot e_1 \longmapsto \boldsymbol{k}. \end{cases}$$

By this isomorphism, $\mathrm{Spin}_3 \subset \mathrm{Cl}_3^0$ is identified with a subgroup of $\mathbb{H}^*$. If $a \in \mathrm{Cl}_3^0$ and $q = \Phi(a)$, then $N(a) = 1 \implies q\bar{q} = 1$, so $\Phi(\mathrm{Spin}_3) \subset \mathrm{Sp}_1$, and since these two groups have the same dimension, we have $\Phi(\mathrm{Spin}_3) = \mathrm{Sp}_1$. Under those identifications, the covering map ξ is then given by

$$\mathrm{Spin}_3 \cong \mathrm{Sp}_1 \overset{\xi}{\longrightarrow} \mathrm{SO}_3, \quad q \longmapsto (x \in \mathrm{Im}(\mathbb{H}) \longmapsto qxq^{-1} = qx\bar{q}).$$

Furthermore, it is easy to verify that the injective homomorphism of algebras $\mathbb{H} \to \mathbb{C}(2)$ given in (1.6), induces an isomorphism between the groups Sp_1 and SU_2. So we have the isomorphisms

$$\mathrm{Spin}_3 \cong \mathrm{Sp}_1 (\cong \mathbb{S}^3) \cong \mathrm{SU}_2 .$$

(3) To have a description of Spin_4, we consider the map

$$\xi\colon \mathrm{Sp}_1 \times \mathrm{Sp}_1 \longrightarrow \mathrm{GL}(\mathbb{H}), \quad (q,q') \longmapsto (x \mapsto qxq'^{-1} = qx\overline{q'}).$$

It is easy to check that ξ is a group homomorphism with values in O_4 and even SO_4, since $\mathrm{Sp}_1 \times \mathrm{Sp}_1$ is connected. It is also straightforward that its kernel is the group $\{\pm(\mathbf{1},\mathbf{1})\} \cong \mathbb{Z}/2\mathbb{Z}$. Finally $\xi(\mathrm{Sp}_1 \times \mathrm{Sp}_1) = \mathrm{SO}_4$, since the two groups have the same dimension. So ξ is a two-fold covering of SO_4, hence

$$\mathrm{Spin}_4 \cong \mathrm{Sp}_1 \times \mathrm{Sp}_1 \cong \mathrm{SU}_2 \times \mathrm{SU}_2\,.$$

Note that, identifying $\mathbb{H}$ with a subspace of $\mathbb{C}(2)$ by the injective homomorphism given in (1.6), the covering

$$\xi\colon \mathrm{SU}_2 \times \mathrm{SU}_2 \longmapsto \mathrm{SO}_4$$

is given by the map

$$(A,B) \longmapsto \left(X = \begin{pmatrix} x & -\bar{y} \\ y & \bar{x} \end{pmatrix} \in \mathbb{H} \longmapsto AX^t\overline{B} \right).$$

We now introduce the conformal spin group $\mathrm{CSpin}_n := \mathrm{Spin}_n \times \mathbb{R}^*_+$. Recall that the conformal group CO_n^+ is identified with $\mathrm{SO}_n \times \mathbb{R}^*_+$ via the canonical isomorphism

$$\begin{aligned} \phi\colon \mathrm{SO}_n \times \mathbb{R}^*_+ &\longrightarrow \mathrm{CO}_n^+, \\ (A,t) &\longmapsto tA. \end{aligned}$$

Hence, by Proposition 1.20 it is clear that, for $n \geq 3$, the morphism

$$\begin{aligned} \zeta\colon \mathrm{CSpin}_n &\longrightarrow \mathrm{CO}_n^+, \\ (a,t) &\longmapsto t\xi(a), \end{aligned}$$

is the universal cover of the conformal group. So we have the fundamental exact sequences

$$\{1\} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathrm{Spin}_n \xrightarrow{\xi} \mathrm{SO}_n \longrightarrow \{1\}$$

and

$$\{1\} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathrm{CSpin}_n \xrightarrow{\zeta} \mathrm{CO}_n^+ \longrightarrow \{1\}.$$

Consider next the group homomorphism

$$\begin{aligned} \jmath\colon \mathrm{Spin}_n \times \mathrm{U}_1 &\longrightarrow \mathbb{C}\mathrm{l}_n^*, \\ (a,z) &\longmapsto \imath(a)z, \end{aligned}$$

where $\imath\colon \mathrm{Cl}_n \to \mathbb{C}\mathrm{l}_n \cong \mathrm{Cl}_n \otimes_{\mathbb{R}} \mathbb{C}$ is the standard inclusion of Cl_n in $\mathbb{C}\mathrm{l}_n$. For simplicity, we shall identify Cl_n with its image by $\imath$. Since the kernel of $\jmath$ is $\{(1,1),(-1,-1)\} \cong \mathbb{Z}/2\mathbb{Z}$, its image is isomorphic to $(\mathrm{Spin}_n \times \mathrm{U}_1)/(\mathbb{Z}/2\mathbb{Z})$, which will be *denoted* by $\mathrm{Spin}_n \times_{\mathbb{Z}/2\mathbb{Z}} \mathrm{U}_1$.

Definition 1.23. (i) The *complex spin group* is the group

$$\mathrm{Spin}_n^{\mathrm{c}} := \mathrm{Spin}_n \times_{\mathbb{Z}/2\mathbb{Z}} \mathrm{U}_1,$$

which, via $\jmath$, is identified with a subgroup of $\mathbb{C}\mathrm{l}_n^*$.

(ii) The *conformal complex spin group* is the group

$$\mathrm{CSpin}_n^{\mathrm{c}} := (\mathrm{Spin}_n \times_{\mathbb{Z}/2\mathbb{Z}} \mathrm{U}_1) \times \mathbb{R}_+^*,$$

which via $\jmath \times \mathrm{Id}$, is identified with a subgroup of $\mathbb{C}\mathrm{l}_n^* \times \mathbb{R}_+^*$.

Example 1.24 (The group $\mathrm{Spin}_4^{\mathrm{c}}$). As we have seen, the spin group Spin_4 is isomorphic to $\mathrm{SU}_2 \times \mathrm{SU}_2$, hence it can be realized as

$$\mathrm{Spin}_4 = \left\{ \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right);\ a, b \in \mathrm{SU}_2 \right\}.$$

The complex spin group can thus be realized as

$$\mathrm{Spin}_4^{\mathrm{c}} = \left\{ \left(\begin{smallmatrix} za & 0 \\ 0 & zb \end{smallmatrix}\right);\ a, b \in \mathrm{SU}_2, z \in \mathrm{U}_1 \right\}.$$

An element of the complex (resp. conformal complex) spin group can be considered as an equivalence class $[a, z]$, $a \in \mathrm{Spin}_n$ (resp. $a \in \mathrm{CSpin}_n$), $z \in \mathrm{U}_1$, modulo $[a, z] = [-a, -z]$. The homomorphisms ξ and ζ induce double coverings

$$\xi^{\mathrm{c}}\colon \mathrm{Spin}_n^{\mathrm{c}} \longrightarrow \mathrm{SO}_n \times \mathrm{U}_1, \quad [a, z] \longmapsto (\xi(a), z^2)$$

and

$$\zeta^{\mathrm{c}}\colon \mathrm{CSpin}_n^{\mathrm{c}} \longrightarrow \mathrm{CO}_n^+ \times \mathrm{U}_1, \quad [b, z] \longmapsto (\zeta(b), z^2).$$

We have the following commutative diagram of exact sequences relating ξ to ξ^{c}:

$$\begin{tikzcd}
 & \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[dd, "\wr"] & \mathrm{Spin}_n \times \mathrm{U}_1 \arrow[r, "\xi \times \mathrm{Id}"] \arrow[dd, "\jmath"] & \mathrm{SO}_n \times \mathrm{U}_1 \arrow[dd, "\mathrm{Id} \times \mathrm{sq}"'] \arrow[dr] & \\
\{1\} \arrow[ur] \arrow[dr] & & & & \{1\} \\
 & \mathbb{Z}/2\mathbb{Z} \arrow[r] & \mathrm{Spin}_n^{\mathrm{c}} \arrow[r, "\xi^{\mathrm{c}}"'] & \mathrm{SO}_n \times \mathrm{U}_1 \arrow[ur] &
\end{tikzcd}$$

where sq is the square map

$$\mathrm{sq}\colon \mathrm{U}_1 \longrightarrow \mathrm{U}_1, \quad z \longmapsto z^2.$$

There is also a similar commutative diagram corresponding to the conformal setup relating ζ to ζ^{c}. We shall use these exact sequences to get the explicit infinitesimal action of the morphisms ξ, ξ^{c}, ζ and ζ^{c}. Since the spin group sits in the Clifford algebra Cl_n, its Lie algebra, $\mathfrak{spin}_n$, is a Lie subalgebra of Cl_n, seen as a Lie algebra by setting

$$[a,b] := a\cdot b - b\cdot a.$$

Theorem 1.25. *The Lie algebras of the Lie groups* Spin_n, $\mathrm{Spin}_n^{\mathrm{c}}$, CSpin_n *and* $\mathrm{CSpin}_n^{\mathrm{c}}$ *and the differentials of the corresponding representations* ξ, ξ^{c}, ζ *and* ζ^{c} *satisfy the following relations:*

$$\begin{aligned}
\mathfrak{spin}_n &\cong \mathfrak{so}_n, & \xi_*(e_i\cdot e_j) &= 2e_i\wedge e_j,\\
\mathfrak{spin}_n^{\mathrm{c}} &\cong \mathfrak{so}_n \oplus i\,\mathbb{R}, & \xi_*^{\mathrm{c}} &= \xi_* \oplus 2\mathrm{Id}_{i\mathbb{R}},\\
\mathfrak{cspin}_n &\cong \mathfrak{so}_n \oplus \mathbb{R}, & \zeta_* &= \xi_* \oplus \mathrm{Id}_{\mathbb{R}},\\
\mathfrak{cspin}_n^{\mathrm{c}} &\cong \mathfrak{so}_n \oplus \mathbb{R} \oplus i\,\mathbb{R}, & \zeta_*^{\mathrm{c}} &= \xi_* \oplus \mathrm{Id}_{\mathbb{R}} \oplus 2\mathrm{Id}_{i\mathbb{R}},
\end{aligned}$$

where $e_i \wedge e_j \in \mathfrak{so}_n \subset \mathfrak{gl}_n$ *is defined by*

$$(e_i\wedge e_j)(e_k) = \delta_{ik}e_j - \delta_{jk}e_i.$$

Proof. Recall that $\Lambda^2\mathbb{R}^n$ is identified with a vector subspace of Cl_n by

$$e_i\wedge e_j \longmapsto e_i\cdot e_j, \quad 1\le i<j\le n.$$

This subspace lies in $\mathfrak{spin}_n$, since $e_i\cdot e_j$ is the tangent vector at $t=0$ to the curve in Spin_n given by

$$\begin{aligned}
c(t) &= \Big(e_i \sin\frac{t}{2} - e_j\cos\frac{t}{2}\Big)\cdot\Big(e_i\sin\frac{t}{2} + e_j\cos\frac{t}{2}\Big)\\
&= \cos t + e_i\cdot e_j \sin t.
\end{aligned}$$

Since $\dim_{\mathbb{R}} \mathfrak{spin}_n = \dim_{\mathbb{R}} \mathfrak{so}_n = \dim_{\mathbb{R}} \Lambda^2\mathbb{R}^n$, we get $\Lambda^2\mathbb{R}^n \cong \mathfrak{spin}_n$. The other assertions follow directly.

Finally, for all i, j, k, $1 \le i < j \le n$ and $1 \le k \le n$ we have $\xi(c(t))(e_k) = c(t) \cdot e_k \cdot c(t)^{-1}$, hence

$$\begin{aligned}
\xi_*(e_i \cdot e_j)(e_k) &= \frac{d}{dt}\Big|_{t=0} \xi(c(t))(e_k) \\
&= \frac{d}{dt}\Big|_{t=0} (\cos t + e_i \cdot e_j \sin t) \cdot e_k \cdot (\cos t - e_i \cdot e_j \sin t) \\
&= e_i \cdot e_j \cdot e_k - e_k \cdot e_i \cdot e_j = 2\delta_{ik} e_j - 2\delta_{jk} e_i \\
&= 2(e_i \wedge e_j)(e_k).
\end{aligned}$$

The other relations are immediate consequences of the above exact sequences. □

1.2.2 Representations of spin groups

In order to introduce the spinor representation of the spin group, which plays a central role in the sequel, we give a short description of the irreducible representations of Spin_n (details may be found in Chapter 12).

We only consider *complex* finite-dimensional representations of Spin_n, i.e., (continuous) Lie groups homomorphisms

$$\rho \colon \mathrm{Spin}_n \longrightarrow \mathrm{GL}(V),$$

where V is a finite-dimensional $\mathbb{C}$-vector space. Recall that any such representation is actually *real analytic* (cf. for instance [God82]). Note also that any representation of the group SO_n is obtained as a quotient of a representation ρ of the covering group Spin_n verifying the condition $\rho(\pm 1) = \mathrm{Id}$.

The group Spin_n being compact, any of its representations is totally reducible, cf. Theorem 12.1, hence we may restrict to irreducible representations.

Furthermore, the group Spin_n being simply connected and compact, the general theory (see the comments following Theorem 12.61) asserts that all irreducible representations of Spin_n may be described by considering a finite family of "basic" irreducible representations called "fundamental."[2]

Now the fundamental irreducible representations of the group Spin_n are given by considering the standard irreducible representations of the group SO_n, together with the *spinor representations*, induced by irreducible representations of the corresponding Clifford algebras.

[2] In the sense that each irreducible representation is a summand in some tensor product of fundamental representations.

1.2.2.1 The standard irreducible representations of the Group SO_n, $n \geq 3$. Let us consider the standard complex representation ρ of the group SO_n in the space $\mathbb{C}^n$ induced by the injective homomorphism $\mathrm{SO}_n \hookrightarrow \mathrm{SO}_{n,\mathbb{C}}$.

Theorem 1.26. *For any positive integer k such that $k \leq m-1$, if $n = 2m$, and $k \leq m$, if $n = 2m+1$, the representation $\Lambda^k \rho$ in the space $\Lambda^k \mathbb{C}^n$ is an irreducible representation of the group* SO_n.

Proof. Let $\{e_1, \dots, e_n\}$ be the canonical basis of $\mathbb{R}^n$, identified with the canonical basis of $\mathbb{C}^n$. The canonical basis of $\Lambda^k \mathbb{C}^n$ is then given by the vectors

$$e_I = e_{i_1} \wedge \cdots \wedge e_{i_k},$$

where $I = \{i_1 < \cdots < i_k\}$ runs through the set of k-element subsets of $\{1, \dots, n\}$.

Let V be a SO_n-invariant subspace of $\Lambda^k \mathbb{C}^n$ and v a non-zero vector in V. One has $v = \sum_I c_I e_I$, where $c_I \in \mathbb{C}$. We will prove by induction on the number of non-zero coefficients c_I that $V = \Lambda^k \mathbb{C}^n$. If there is only one such non-zero coefficient, say c_I, then $e_I \in V$. But, up to a sign, all the e_J are obtained from e_I by a convenient permutation of the vectors $e_1, \dots, e_n$. Substituting $-e_1$ for e_1 in the case of an odd permutation, every e_J is then the image of e_I under the action of an element of SO_n, so all the e_J are in V, and hence $V = \Lambda^k \mathbb{C}^n$.

Suppose now that v has at least two non-zero coefficients c_I and c_J. Under the hypothesis made on k, and since $I \neq J$, it is always possible to find a couple (i, j) such that $i \in I$, $i \notin J$, $j \notin I \cup J$. Consider the transformation $g \in \mathrm{SO}_n$ such that

$$g \cdot e_i = -e_i, \quad g \cdot e_j = -e_j, \quad g \cdot e_k = e_k, \qquad k \neq i, j.$$

Then

$$g \cdot e_I = -e_I, \quad g \cdot e_J = e_J, \quad g \cdot e_K = \pm e_K, \quad K \neq I, J.$$

Hence $v + g \cdot v \in V$ has fewer non-zero coefficients than v, but still at least one and the proof follows by induction. $\square$

Remark 1.27. Let k be a positive integer such that $k \leq m-1$, if $n = 2m$, and $k \leq m$, if $n = 2m+1$. Then the $\mathbb{C}$-linear extension of the Hodge $*$-operator defines an isomorphism between $\Lambda^{n-k}\rho$ and $\Lambda^k \rho$.

If $n = 2m$ and $k = m$, then the representation $\Lambda^m \rho$ is reducible: since the restriction of the $*$-operator to $\Lambda^m \rho$ squares to $(-1)^m$ times the identity, there is a decomposition of $\Lambda^m \mathbb{C}^{2m}$ into eigenspaces corresponding to the eigenvalues ± 1 for m even and $\pm i$ for m odd. By the SO_n-invariance of the $*$-operator, these eigenspaces are invariant under the action of SO_n.

1.2.2.2 The spinor representations. In this section we will show that by considering the standard complex representations of the Clifford algebras, the spin group Spin_n inherits two irreducible complex representations if n is even and one if n is odd. As those representations do not descend to the group SO_n, they are called *spinor representations*. These representations are of fundamental importance since, together with the standard irreducible representations of the group SO_n, they constitute the "fundamental" (in the sense mentioned before) irreducible representations of the spin group.

We first consider complex representations of the Clifford algebras. Since every complex representation of Cl_n induces a complex representation of $\mathbb{C}\mathrm{l}_n$ and conversely, we shall only consider complex Clifford algebras.

By a theorem of Burnside, up to equivalence, the only complex irreducible representation of the matrix algebra $\mathbb{C}(N)$ is the canonical representation on $\mathbb{C}^N$, hence by the classification of the complex Clifford algebras (Theorem 1.15) we have at once the following theorem.

Theorem 1.28. *The complex Clifford algebra $\mathbb{C}\mathrm{l}_n$ has a unique irreducible representation*

$$\chi_n \colon \mathbb{C}\mathrm{l}_n \longrightarrow \mathrm{End}(\Sigma_n),$$

for n even, and two inequivalent irreducible representations

$$\chi_n^{\pm} \colon \mathbb{C}\mathrm{l}_n \longrightarrow \mathrm{End}(\Sigma_n),$$

for n odd, where Σ_n is a complex vector space of complex dimension $N := 2^{[\frac{n}{2}]}$. (*Here* [ ] *stands for the integer part*).

For a better understanding of the structure of the Clifford algebras we introduce the *complex volume element*

$$\omega^{\mathbb{C}} := i^{[\frac{n+1}{2}]} e_1 \cdots e_n \in \mathbb{C}\mathrm{l}_n \,. \tag{1.10}$$

It is straightforward to check the relations

$$(\omega^{\mathbb{C}})^2 = 1,$$

and

$$x \cdot \omega^{\mathbb{C}} = (-1)^{n-1} \omega^{\mathbb{C}} \cdot x, \quad x \in \mathbb{R}^n \subset \mathbb{C}\mathrm{l}_n \,. \tag{1.11}$$

We now prove the following two propositions.

Proposition 1.29. *For n odd,*

$$\mathbb{C}\mathrm{l}_n = \mathbb{C}\mathrm{l}_n^+ \oplus \mathbb{C}\mathrm{l}_n^-,$$

where $\mathbb{C}\mathrm{l}_n^{\pm} = \pi^{\pm} \cdot \mathbb{C}\mathrm{l}_n = \mathbb{C}\mathrm{l}_n \cdot \pi^{\pm}$ and $\pi^{\pm} = \frac{1}{2}(1 \pm \omega^{\mathbb{C}})$. Moreover, $\alpha(\mathbb{C}\mathrm{l}_n^{\pm}) = \mathbb{C}\mathrm{l}_n^{\mp}$.

Proof. Since $(\omega^{\mathbb{C}})^2 = 1$, one has

$$\pi^+ + \pi^- = 1,$$
$$(\pi^\pm)^2 = \pi^\pm,$$
$$\pi^- \cdot \pi^+ = \pi^+ \cdot \pi^- = 0.$$

Since n is odd, $\omega^{\mathbb{C}}$ and $\pi^\pm$ are central in $\mathbb{C}\mathrm{l}_n$. It is then clear that $\mathbb{C}\mathrm{l}_n^\pm = \pi^\pm \cdot \mathbb{C}\mathrm{l}_n$ are two ideals of $\mathbb{C}\mathrm{l}_n$ and $\mathbb{C}\mathrm{l}_n = \mathbb{C}\mathrm{l}_n^+ \oplus \mathbb{C}\mathrm{l}_n^-$.

The volume element being odd, i.e., $\omega^{\mathbb{C}} \in \mathbb{C}\mathrm{l}_n^1$, we have $\alpha(\pi^\pm) = \pi^\mp$, hence $\alpha(\mathbb{C}\mathrm{l}_n^\pm) = \mathbb{C}\mathrm{l}_n^\mp$ and the two subalgebras are isomorphic. □

Proposition 1.30. *For n odd, let χ_n be a complex irreducible representation of* $\mathbb{C}\mathrm{l}_n$*. Then either* $\chi_n(\omega^{\mathbb{C}}) = \mathrm{Id}$*, or* $\chi_n(\omega^{\mathbb{C}}) = -\mathrm{Id}$*. The two possibilities occur and they are inequivalent.*

Proof. Since $(\omega^{\mathbb{C}})^2 = 1$ and $\chi(\omega^{\mathbb{C}})^2 = \mathrm{Id}$, the vector space Σ_n can be written as the sum of the eigenspaces $\Sigma_n^\pm$ associated with the ± 1-eigenvalues, i.e., $\Sigma_n = \Sigma_n^+ \oplus \Sigma_n^-$. The volume element $\omega^{\mathbb{C}}$ being central, the eigenspaces $\Sigma_n^\pm$ are $\mathbb{C}\mathrm{l}_n$-invariant. The representation being irreducible, we conclude that $\Sigma_n = \Sigma_n^+$ or $\Sigma_n = \Sigma_n^-$. It is then clear that these two representations are inequivalent. By considering the action of $\mathbb{C}\mathrm{l}_n$ on $\mathbb{C}\mathrm{l}_n^\pm$ by left multiplication, we see that the two possibilities occur. □

Definition 1.31. The map

$$\begin{aligned} \mathbb{C}\mathrm{l}_n \otimes \Sigma_n &\longrightarrow \Sigma_n, \\ \sigma \otimes \psi &\longmapsto \sigma \cdot \psi := \begin{cases} \chi_n(\sigma)(\psi) & \text{if } n \text{ is even}, \\ \chi_n^+(\sigma)(\psi) & \text{if } n \text{ is odd}, \end{cases} \end{aligned}$$

is called *Clifford multiplication* of σ with ψ.

Proposition 1.32. (i) *For n even, the restriction of χ_n to* Spin_n (*resp.* $\mathbb{C}\mathrm{l}_n^0$) *splits into $\Sigma_n = \Sigma_n^+ \oplus \Sigma_n^-$, where Σ_n^+ and Σ_n^- are complex inequivalent irreducible representations of* Spin_n (*resp.* $\mathbb{C}\mathrm{l}_n^0$).

(ii) *For n odd, the restrictions of $\chi_n^\pm$ to* Spin_n (*resp.* $\mathbb{C}\mathrm{l}_n^0$) *are irreducible and equivalent.*

(iii) *For n even, for all $x \in \mathbb{R} \setminus \{0\}$, the linear maps $\chi_n(x)\colon \Sigma_n^\pm \to \Sigma_n^\mp$ are isomorphisms. Moreover, under the isomorphism $\mathbb{C}\mathrm{l}_n^0 \cong \mathbb{C}\mathrm{l}_{n-1}$, the vector spaces $\Sigma_n^\pm$ correspond to the two inequivalent irreducible representations of* $\mathbb{C}\mathrm{l}_{n-1}$.

Proof. Note that the complex subalgebra generated by $\mathrm{Spin}_n \subset \mathbb{C}\mathrm{l}_n$ is the even part $\mathbb{C}\mathrm{l}_n^0$ of $\mathbb{C}\mathrm{l}_n$. Hence, two representations of $\mathbb{C}\mathrm{l}_n^0$ are irreducible or equivalent if and only if it is the case for their restrictions to Spin_n. It is therefore sufficient to prove the assertions for $\mathbb{C}\mathrm{l}_n^0$ instead of Spin_n.

(i) Let $n = 2m$. Since $\omega^{\mathbb{C}}$ commutes with $\mathbb{C}\mathrm{l}_n^0$, the restriction of χ_n to $\mathbb{C}\mathrm{l}_n^0 \cong \mathbb{C}\mathrm{l}_{n-1}$ splits into $\Sigma_n = \Sigma_n^+ \oplus \Sigma_n^-$, where

$$\Sigma_n^{\pm} = \chi_n\Big(\frac{1 \pm \omega^{\mathbb{C}}}{2}\Big)(\Sigma_n).$$

By Theorem 1.28, it is sufficient to prove that Σ_n^+ and Σ_n^- are nontrivial vector spaces. Since the linear map χ_n is an isomorphism, $\chi_n(\omega^{\mathbb{C}})$ cannot be equal to $\pm\mathrm{Id}_{\Sigma_n} = \chi_n(\pm 1)$.

(ii) Again, we make use of the isomorphism $\mathbb{C}\mathrm{l}_n^0 \cong \mathbb{C}\mathrm{l}_{n-1}$. For n odd, we know that $\alpha(\mathbb{C}\mathrm{l}_n^{\pm}) = \mathbb{C}\mathrm{l}_n^{\mp}$. Note that the even part $\mathbb{C}\mathrm{l}_n^0$ sits diagonally in the decomposition $\mathbb{C}\mathrm{l}_n = \mathbb{C}\mathrm{l}_n^+ \oplus \mathbb{C}\mathrm{l}_n^-$. In fact, since $\omega^{\mathbb{C}} \in \mathbb{C}\mathrm{l}_n^1$, we have $\omega^{\mathbb{C}}\colon \mathbb{C}\mathrm{l}_n^0 \to \mathbb{C}\mathrm{l}_n^1$, hence $\mathbb{C}\mathrm{l}_n^0 \cap \mathbb{C}\mathrm{l}_n^{\pm} = \{0\}$. More precisely, we have

$$\mathbb{C}\mathrm{l}_n^0 = \{u^{\pm} + \alpha(u^{\pm});\ u^{\pm} \in \mathbb{C}\mathrm{l}_n^{\pm}\}.$$

Since the two inequivalent irreducible representations of $\mathbb{C}\mathrm{l}_n$ are distinguished by the isomorphism α, by restriction to $\mathbb{C}\mathrm{l}_n^0$, they become equivalent.

(iii) By (1.11), $x \cdot \omega^{\mathbb{C}} = -\omega^{\mathbb{C}} \cdot x$. Then,

$$\begin{aligned}\chi_n(x)\Sigma_n^{\pm} &= \chi_n(x)\chi_n\Big(\frac{1 \pm \omega^{\mathbb{C}}}{2}\Big)(\Sigma_n)\\ &= \chi_n\Big(\frac{1 \mp \omega^{\mathbb{C}}}{2}\Big)\chi_n(x)(\Sigma_n)\\ &\subset \Sigma_n^{\mp}.\end{aligned}$$

The linear map $\chi_n(x)$ is in fact an isomorphism since $\chi_n(x)^2 = -\|x\|^2\mathrm{Id}$.

For the last statement, it is sufficient to note that the isomorphism (1.5) maps the volume element $\omega_{n-1}^{\mathbb{C}} := i^{[\frac{n}{2}]}e_1 \cdots e_{n-1}$ of $\mathbb{C}\mathrm{l}_{n-1}$ to the volume element $\omega_n^{\mathbb{C}} := i^{[\frac{n+1}{2}]}e_1 \cdots e_{n-1} \cdot e_n$ of $\mathbb{C}\mathrm{l}_n$. □

Definition 1.33. The representation

$$\rho_n := \begin{cases} \chi_n|_{\mathrm{Spin}_n} & \text{for } n \text{ even},\\ \chi_n^+|_{\mathrm{Spin}_n} & \text{for } n \text{ odd}, \end{cases}$$

is called the *canonical complex spin representation* and is denoted by (ρ_n, Σ_n).

If n is odd, ρ_n is irreducible, whereas if n is even, it splits into two irreducible components $\rho_n^{\pm}$. Note that since $\rho_n(-1) = -\mathrm{Id}$, the canonical complex spin representation does not descend to the group SO_n, and neither do its irreducible components $\rho_n^{\pm}$ if n is even.

Definition 1.34. For n even, the Clifford multiplication by the complex volume element

$$\omega^{\mathbb{C}} \cdot \colon \Sigma_n = \Sigma_n^+ \oplus \Sigma_n^- \longrightarrow \Sigma_n = \Sigma_n^+ \oplus \Sigma_n^-,$$
$$\psi = \psi^+ + \psi^- \longmapsto \overline{\psi} := \omega^{\mathbb{C}} \cdot \psi = \psi^+ - \psi^-,$$

is called the *conjugation map*.

The existence of a real or quaternionic structure on the canonical complex spin representation will be discussed below. Note that, looking to the table of real Clifford algebras we see that Cl_n has, up to isomorphism, one irreducible representation for $n \equiv 0, 1, 2 \mod 4$, and two irreducible representations for $n \equiv 3 \mod 4$. In this last case, the two representations become equivalent when restricted to Spin_n. For all n, we thus get a representation of Spin_n called the *real spin representation*.

Note that since the algebra generated by the groups Spin_n and $\mathrm{Spin}_n^{\mathrm{c}}$ in $\mathbb{C}\mathrm{l}_n$ is exactly $\mathbb{C}\mathrm{l}_n^0$, it is clear that the preceding facts are also true for the group $\mathrm{Spin}_n^{\mathrm{c}}$. In fact, observe that the complex representations of Spin_n are the same as that of the group $\mathrm{Spin}_n^{\mathrm{c}}$. Hence the restriction of χ_n (resp. χ_n^+) to $\mathrm{Spin}_n^{\mathrm{c}}$, also denoted by ρ_n^{c}, satisfies Proposition 1.32 too.

For each $k \in \mathbb{R}$, we consider the representation ρ_n^k (resp. $\rho_n^{\mathrm{c},k}$) of CSpin_n (resp. $\mathrm{CSpin}_n^{\mathrm{c}}$) on Σ_n defined, for all $a \in \mathrm{Spin}_n$ (resp. $a \in \mathrm{Spin}_n^{\mathrm{c}}$) and $t \in \mathbb{R}_+^*$, by

$$\rho_n^k((a,t)) = t^k \rho_n(a) \quad (\text{resp. } \rho_n^{\mathrm{c},k}((a,t)) = t^k \rho_n^{\mathrm{c}}(a)). \tag{1.12}$$

As before, one checks that these representations are irreducible for n odd, and are the direct sum of two inequivalent irreducible representations for n even.

Now since it is inherited from a representation of the complex Clifford algebra, the canonical complex spin representation (ρ_n, Σ_n) enjoys the following important property.

Proposition 1.35 (the natural Hermitian product on the spinor space). *There exists a* Hermitian product $(\cdot\,,\cdot)$ *on* Σ_n *such that*

$$(\sigma_1, \sigma_2) = (x \cdot \sigma_1, x \cdot \sigma_2), \quad \textit{for all } x \in \mathbb{R}^n, \|x\| = 1, \sigma_1, \sigma_2 \in \Sigma_n. \tag{1.13}$$

A Hermitian product on Σ_n *with the above property is unique up to a positive constant.*

Proof. Let Γ_n be the multiplicative subgroup of Cl_n^* generated by a g-orthonormal basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$. Using the relations $(-1)^2 = 1$, $e_i^2 = -1$, and $e_i \cdot e_j = -e_j \cdot e_i$, $1 \leq i, j \leq n$, $i \neq j$, we see that Γ_n is finite and $|\Gamma_n| = 2^{n+1}$. Now we choose an arbitrary Hermitian product $\langle \cdot, \cdot \rangle$ on Σ_n and define for $\sigma_1, \sigma_2 \in \Sigma_n$

$$(\sigma_1, \sigma_2) = \frac{1}{|\Gamma_n|} \sum_{\nu \in \Gamma_n} \langle \nu \cdot \sigma_1, \nu \cdot \sigma_2 \rangle.$$

First, for $e_i \in \Gamma_n$, one has that

$$\begin{aligned}
(e_i \cdot \sigma_1, e_i \cdot \sigma_2) &= \frac{1}{|\Gamma_n|} \sum_{\nu \in \Gamma_n} \langle \nu \cdot e_i \cdot \sigma_1, \nu \cdot e_i \cdot \sigma_2 \rangle \\
&= \frac{1}{|\Gamma_n|} \sum_{\nu \in \Gamma_n} \langle \nu \cdot \sigma_1, \nu \cdot \sigma_2 \rangle \\
&= (\sigma_1, \sigma_2).
\end{aligned}$$

Then, for $x \in \mathbb{R}^n$ with $\|x\| = 1$, i.e., $x = \sum_{i=1}^n x_i e_i$, with $\sum_{i=1}^n x_i^2 = 1$, we get

$$\begin{aligned}
(x \cdot \sigma_1, x \cdot \sigma_2) &= \sum_i x_i^2 (e_i \cdot \sigma_1, e_i \cdot \sigma_2) + \sum_{i \neq j} x_i x_j (e_i \cdot \sigma_1, e_j \cdot \sigma_2) \\
&= \sum_i x_i^2 (\sigma_1, \sigma_2) + \sum_{i<j} x_i x_j ((e_i \cdot \sigma_1, e_j \cdot \sigma_2) + (e_j \cdot \sigma_1, e_i \cdot \sigma_2)) \\
&= (\sigma_1, \sigma_2)
\end{aligned}$$

since, for $i < j$,

$$\begin{aligned}
(e_i \cdot \sigma_1, e_j \cdot \sigma_2) &= (e_i \cdot e_i \cdot \sigma_1, e_i \cdot e_j \cdot \sigma_2) \\
&= -(\sigma_1, e_i \cdot e_j \cdot \sigma_2) \\
&= -(e_j \cdot \sigma_1, e_j \cdot e_i \cdot e_j \cdot \sigma_2) \\
&= (e_j \cdot \sigma_1, e_i \cdot e_j \cdot e_j \cdot \sigma_2) \\
&= -(e_j \cdot \sigma_1, e_i \cdot \sigma_2).
\end{aligned}$$

Assume now that $(\cdot, \cdot)'$ is another Hermitian product on Σ_n satisfying (1.13). Then $(\cdot, \cdot)' = (A(\cdot), \cdot)$ for some Cl_n-invariant automorphism A. The representation of the Clifford algebra on Σ_n being irreducible, A has to be a constant multiple of the identity by the Schur lemma. $\square$

Corollary 1.36. *For all $x \in \mathbb{R}^n$ and for all $\sigma_1, \sigma_2 \in \Sigma_n$, we have*

$$(x \cdot \sigma_1, \sigma_2) = -(\sigma_1, x \cdot \sigma_2).$$

Proof. Let x be in $\mathbb{R}^n \setminus \{0\}$. Then

$$(x \cdot \sigma_1, \sigma_2) = \left(x \cdot \frac{x}{\|x\|} \cdot \sigma_1, \frac{x}{\|x\|} \cdot \sigma_2\right)$$

and

$$(x \cdot \sigma_1, \sigma_2) = \frac{1}{\|x\|^2}(x \cdot x \cdot \sigma_1, x \cdot \sigma_2) = -(\sigma_1, x \cdot \sigma_2). \qquad \square$$

The following corollary is an immediate consequence of Proposition 1.35.

Corollary 1.37. *The representations (ρ_n, Σ_n) and $(\rho_n^{\mathrm{c}}, \Sigma_n)$ of* Spin_n *and* $\mathrm{Spin}_n^{\mathrm{c}}$*, respectively, are unitary for the above Hermitian product on Σ_n.*

Finally, it is often useful to have an explicit description of the spinor representation. We only give it for n even, since the description for n odd may then be simply derived.

For this, let $\{e_1, \dots, e_m, e_{m+1}, \dots, e_{2m}\}$ be the canonical orthonormal basis of $\mathbb{R}^{2m}$ and $\{z_j, \bar{z}_j\}_{j=1,\dots,m}$, the corresponding Witt basis of $\mathbb{R}^{2m} \otimes_{\mathbb{R}} \mathbb{C}$, i.e.,

$$z_j := \frac{1}{2}(e_j \otimes 1 - e_{j+m} \otimes i) \quad \text{and} \quad \bar{z}_j := \frac{1}{2}(e_j \otimes 1 + e_{j+m} \otimes i).$$

These vectors satisfy the relations

$$q^{\mathbb{C}}(z_j, z_k) = q^{\mathbb{C}}(\bar{z}_j, \bar{z}_k) = 0,$$

and

$$q^{\mathbb{C}}(\bar{z}_j, z_k) = q^{\mathbb{C}}(z_j, \bar{z}_k) = \frac{1}{2}\delta_{jk},$$

which yield

$$z_j \cdot z_k + z_k \cdot z_j = 0, \tag{1.14a}$$

$$\bar{z}_j \cdot \bar{z}_k + \bar{z}_k \cdot \bar{z}_j = 0, \tag{1.14b}$$

$$z_j \cdot \bar{z}_k + \bar{z}_k \cdot z_j = -\delta_{jk}, \tag{1.14c}$$

since $x \cdot y + y \cdot x = -2q^{\mathbb{C}}(x, y)$ for $x, y \in \mathbb{C}^{2m} \subset \mathbb{C}\mathrm{l}_{2m}$. It is convenient to introduce the notations $z_{L_r} := z_{l_1} \cdots z_{l_r}$, for $1 \le l_1 < \cdots < l_r \le m$. Then, by relations (1.14), it is straightforward to show that, as a vector space, we have

$$\mathbb{C}\mathrm{l}_{2m} = \mathrm{span}\{z_{J_p} \cdot \bar{z}_{K_q};\ 1 \le j_1 < \cdots < j_p \le m, 1 \le k_1 < \cdots < k_q \le m,$$
$$z_{J_0} = \bar{z}_{K_0} = 1,\ 0 \le p, q \le m\}.$$

Note that, if we *define* the element $\bar{\omega}$ by

$$\bar{\omega} := \bar{z}_1 \cdots \bar{z}_m,$$

then again by (1.14) one has $\bar{z}_k \cdot \bar{\omega} = 0$ for $1 \leq k \leq m$. Define the complex vector space Σ_{2m} by

$$\Sigma_{2m} = \operatorname{span}\{z_{L_r} \cdot \bar{\omega};\ 1 \leq l_1 < \cdots < l_r \leq m, 0 \leq r \leq m\}.$$

Clearly the complex dimension of Σ_{2m} is at most 2^m. Now consider the map

$$\begin{aligned} \rho\colon \qquad & \mathbb{C}\mathrm{l}_{2m} \longrightarrow \operatorname{End}(\Sigma_{2m}), \\ v = z_{J_p} \cdot \bar{z}_{K_q} & \longmapsto \rho(v) = (z_{L_r} \cdot \bar{\omega} \mapsto z_{J_p} \cdot \bar{z}_{K_q} \cdot z_{L_r} \cdot \bar{\omega}). \end{aligned}$$

This is obviously a representation of $\mathbb{C}\mathrm{l}_{2m}$, which, by Theorem 1.28, is isomorphic to the unique irreducible representation of $\mathbb{C}\mathrm{l}_{2m}$. Thus, the spinor complex representation of Spin_{2m} can be described by considering the restriction $\rho|_{\mathrm{Spin}_{2m}}$.

1.2.3 Real and quaternionic structures

In this section we investigate the existence of real or quaternionic structures of the canonical complex spin representation ρ_n (see Definition 1.33).

Definition 1.38. Let $\rho\colon G \to \Sigma$ be a complex representation of a Lie group G. A *real*, resp. *quaternionic* structure of ρ is a $\mathbb{C}$-anti-linear G-equivariant automorphism j of Σ satisfying $\mathrm{j}^2 = 1$, resp. $\mathrm{j}^2 = -1$.

In order to avoid possible confusions we emphasize here that the complex spin representation ρ_n (defined as the restriction to Spin_n of an irreducible representation of $\mathbb{C}\mathrm{l}_n$) *is not* in general the complexification of the real spin representation (defined as the restriction to Spin_n of an irreducible representation of Cl_n).

In fact, for dimensional reasons, using the table of real Clifford algebras (Table 1 in Section 1.1.2), it is clear that the restriction to Cl_n of an irreducible representation of $\mathbb{C}\mathrm{l}_n$ is again irreducible for $n \equiv 1, 2, 3, 4, 5 \mod 8$, and therefore in these dimensions the real and complex spin representations coincide. In the remaining dimensions, $n \equiv 6, 7, 8 \mod 8$, the complex spin representation is the complexification of the real spin representation.

Since ρ_n is irreducible for n odd and reducible for n even, we need to distinguish two cases.

Case I: *n* odd. Since Spin_n is a subset of $\mathrm{Cl}_n^0 \cong \mathrm{Cl}_{n-1}$, Table 1 in Section 1.1.2 shows that the complex spin representation ρ_n on Σ_n has a real structure for $n \equiv 1$ and $n \equiv 7 \mod 8$, and a quaternionic structure for $n \equiv 3$ and $n \equiv 5 \mod 8$. We denote this structure by j.

We would like to understand the behavior of j with respect to the Clifford action on Σ_n. Clearly j is $\mathbb{C}$-anti-linear and Cl_n^0-equivariant. It remains to see whether j commutes or anti-commutes with the Clifford product by odd elements of Cl_n. Recall that the complex volume element $\omega^{\mathbb{C}}$, defined in an orthonormal basis by

$$\omega^{\mathbb{C}} = i^{[\frac{n+1}{2}]} e_1 \cdots e_n, \tag{1.15}$$

acts on Σ_n as the identity, so obviously commutes with j. On the other hand, $\omega^{\mathbb{C}}$ (which has odd degree), is real for $n \equiv 3$ and $n \equiv 7 \mod 8$, and purely imaginary for $n \equiv 1$ and $n \equiv 5 \mod 8$. Thus j commutes (resp. anti-commutes) with the Clifford product by real vectors for $n \equiv 3$ and $n \equiv 7 \mod 8$ (resp. $n \equiv 1$ and $n \equiv 5 \mod 8$). This, by the way, corroborates the fact (visible on the table of Clifford algebras) that the representation of the whole complex Clifford algebra is quaternionic for $n \equiv 3 \mod 8$, and real for $n \equiv 7 \mod 8$.

Case II: *n* even. The complex spin representation splits into the direct sum of spin half-representations $\Sigma_n = \Sigma_n^+ \oplus \Sigma_n^-$. Looking at Table 1 in Section 1.1.2 again, we deduce the following. For $n \equiv 2$ and $n \equiv 6 \mod 8$, $\mathrm{Cl}_n^0 \cong \mathrm{Cl}_{n-1}$ is the algebra of complex matrices $\mathbb{C}(2^{[\frac{n-1}{2}]})$, acting irreducibly on $E = \mathbb{C}^{2^{[\frac{n-1}{2}]}}$. Taking the tensor product with $\mathbb{C}$ yields a representation of Cl_{n-1} on $E \otimes_{\mathbb{R}} \mathbb{C} \cong E \oplus \bar{E}$, showing that Σ_n^+ can be identified with E and Σ_n^- with $\bar{E}$. They carry no real or quaternionic structure separately, but their direct sum carries both a real and a quaternionic structure, canonically defined by

$$\mathrm{j}_{\mathbb{R}} = \begin{pmatrix} 0 & f^{-1} \\ f & 0 \end{pmatrix}$$

and

$$\mathrm{j}_{\mathbb{H}} = \begin{pmatrix} 0 & f^{-1} \\ -f & 0 \end{pmatrix},$$

where $f : E \to \bar{E}$ denotes the ($\mathbb{C}$-anti-linear) identification of E and $\bar{E}$. Since Cl_n is an algebra of quaternionic matrices for $n \equiv 2 \mod 8$, and of real matrices for $n \equiv 6 \mod 8$, we obtain that $\mathrm{j}_{\mathbb{H}}$ is Cl_n-equivariant for $n \equiv 2 \mod 8$ and $\mathrm{j}_{\mathbb{R}}$ is Cl_n-equivariant for $n \equiv 6 \mod 8$.

Finally, for $n \equiv 4 \mod 8$ (resp. $n \equiv 8 \mod 8$), Cl_{n-1} is the direct sum of two copies of some algebra of quaternionic (resp. real) matrices, so each half-spin representation $\Sigma_n^{\pm}$ carries a quaternionic (resp. real) structure $\mathrm{j}_{\pm}$. Moreover, the direct sum $\mathrm{j}_+ \oplus \mathrm{j}_-$ acting on Σ_n is Cl_n-equivariant since Cl_n has the same quaternionic (resp. real) nature as Cl_{n-1}.

Summarizing, we have proved the following theorem.

Theorem 1.39. *The complex spin representation Σ_n has a real structure $\mathfrak{j}_{\mathbb{R}}$ for $n \equiv 1, 2, 6, 7, 8 \mod 8$, and a quaternionic structure $\mathfrak{j}_{\mathbb{H}}$ for $n \equiv 2, 3, 4, 5, 6 \mod 8$. The Clifford product with real vectors commutes with $\mathfrak{j}_{\mathbb{R}}$ for $n \equiv 6, 7, 8 \mod 8$, and anti-commutes with it for $n \equiv 1, 2 \mod 8$. The Clifford product with real vectors commutes with $\mathfrak{j}_{\mathbb{H}}$ for $n \equiv 2, 3, 4 \mod 8$, and anti-commutes with it for $n \equiv 5, 6 \mod 8$.*

Chapter 2

Geometrical aspects

This chapter is devoted to the general properties of the connections induced on the complex spinor bundle in several natural geometric settings: Riemannian, conformal, spin and Spin^c. We study the corresponding Dirac operators and derive the well-known Bochner-type formulas relating the square of the Dirac operator and the rough Laplacian. We also study spinors in particular geometric instances: hypersurfaces, warped products, and Riemannian submersions. This turns out to have important consequences in the study of manifolds with special spinor fields.

2.1 Spinorial structures

In this section we introduce the basic concepts needed to deal geometrically with spinors. We review them in decreasing order of generality, starting with the puzzling concept of a *spin structure*. We refer to Chapter 3 for a complete discussion on the existence of such structures and for their interaction with the global structure of a manifold (see also [Bou05] for a short introduction).

2.1.1 Spin structures and spinorial metrics

Let M^n be an oriented n-dimensional manifold and let P be a principal bundle over M with group G. Recall that every representation $\rho\colon G \to \mathrm{Aut}(V)$ defines an *associated vector bundle* E, denoted by $E = P \times_\rho V$, defined as the quotient of $P \times V$ by the right G-action

$$g \cdot (u, X) := (ug, \rho(g^{-1})(X)).$$

The equivalence class of (u, X) is denoted by $[u, X]$ and the space of smooth sections of E is denoted by $\Gamma(E)$.

The principal bundle of positive linear frames of M will be denoted by

$$P_{\mathrm{GL}_n^+} M \longrightarrow M.$$

Let

$$\Xi \colon \widetilde{\mathrm{GL}}_n^+ \longrightarrow \mathrm{GL}_n^+$$

denote the universal covering of GL_n^+.

Definition 2.1. A *spin structure* on an n-dimensional manifold M is given by a principal $\widetilde{\mathrm{GL}}_n^+$-bundle $P_{\widetilde{\mathrm{GL}}_n^+}M$ together with a projection

$$\Theta \colon P_{\widetilde{\mathrm{GL}}_n^+}M \longrightarrow P_{\mathrm{GL}_n^+}M,$$

making the diagram

$$\begin{tikzcd}
\widetilde{\mathrm{GL}}_n^+ \arrow[r, "a\mapsto \tilde{u}a"] \arrow[dd, "\Xi"'] & P_{\widetilde{\mathrm{GL}}_n^+}M \arrow[dd, "\Theta"] \arrow[dr] & \\
 & & M \\
\mathrm{GL}_n^+ \arrow[r, "A\mapsto \Theta(\tilde{u})A"'] & P_{\mathrm{GL}_n^+}M \arrow[ur] &
\end{tikzcd}$$

commute for each $\tilde{u} \in P_{\widetilde{\mathrm{GL}}_n^+}M$.

Equivalently, Θ is fibre-preserving and satisfies $\Theta(\tilde{u}a) = \Theta(\tilde{u})\Xi(a)$, for all $\tilde{u} \in P_{\widetilde{\mathrm{GL}}_n^+}M$ and $a \in \widetilde{\mathrm{GL}}_n^+$.

Consider now a Riemannian metric g on M and let $P_{\mathrm{SO}_n}M$ be the principal SO_n-bundle of positive g-orthonormal frames over M. We denote by ι the representation of SO_n on $\mathbb{C}l_n$ obtained by extending every linear map $A \colon \mathbb{R}^n \to \mathbb{R}^n$ of SO_n to an algebra morphism of $\mathbb{C}l_n$ (see Corollary 1.7). We denote by $\mathbb{C}l(M)$ the vector bundle associated to $P_{\mathrm{SO}_n}M$ for the representation

$$\mathbb{C}l(M) = P_{\mathrm{SO}_n}M \times_\iota \mathbb{C}l_n \,.$$

Recall that $\mathbb{C}l(M)$ (which is called the *Clifford bundle* of (M,g)) is the set of equivalence classes $[u,a]$ of pairs ($u \in P_{\mathrm{SO}_n}M$, $a \in \mathbb{C}l_n$) with respect to the equivalence relation $[u,a] = [uA, \iota(A^{-1})(a)]$, for all $A \in \mathrm{SO}_n$. Each fibre of $\mathbb{C}l(M)$ has a natural structure of a complex algebra, the product being defined by $[u,a]\cdot[u,b] := [u, a\cdot b]$ (actually the fibre of $\mathbb{C}l(M)$ at some point $x \in M$ is just the Clifford algebra $\mathbb{C}l(\mathrm{T}_xM, g_x)$). Consequently, the sections of $\mathbb{C}l(M)$ form a complex algebra over $\mathcal{C}^\infty(M)$.

The algebraic results of the first chapter yield directly the following proposition.

Proposition 2.2. *The tangent bundle of M is a sub-bundle of $\mathbb{C}\mathrm{l}(M)$. The Clifford bundle of M is a complex vector bundle of rank 2^n, canonically isomorphic to $\Lambda^* M \otimes \mathbb{C}$, the vector bundle of complex-valued exterior forms on M.*

Note that this construction can be carried over to E-objects, i.e., to the case where one considers a general linear representation μ of SO_n on a vector space E and the associated bundles

$$E_g M = \mathrm{SO}_g\, M \times_\mu E.$$

Definition 2.3. A *spinorial metric* on a Riemannian manifold (M, g) of dimension n is given by a principal Spin_n-bundle $P_{\mathrm{Spin}_n} M$ together with a projection

$$\theta\colon P_{\mathrm{Spin}_n} M \longrightarrow P_{\mathrm{SO}_n} M,$$

making the diagram

$$\begin{tikzcd}
\mathrm{Spin}_n \arrow[r, "a \mapsto \tilde{u}a"] \arrow[d, "\xi"'] & P_{\mathrm{Spin}_n} M \arrow[d, "\theta"] \arrow[dr] & \\
 & & M \\
\mathrm{SO}_n \arrow[r, "A \mapsto \theta(\tilde{u})A"'] & P_{\mathrm{SO}_n} M \arrow[ur] &
\end{tikzcd}$$

commute for each $\tilde{u} \in P_{\mathrm{Spin}_n} M$.

The spinorial metric is said to be *subordinated* to a spin structure P if $P_{\mathrm{Spin}_n} M$ is a reduction to Spin_n of the $\widetilde{\mathrm{GL}_n^+}$-principal bundle $P_{\widetilde{\mathrm{GL}_n^+}} M$.

Proposition 2.4. *On a manifold endowed with a spin structure, each Riemannian metric gives rise to a spinorial metric.*

The proof is clear. The Riemannian metric g on M determines the subbundle $P_{\mathrm{SO}_n} M$ inside $P_{\mathrm{GL}_n^+} M$. Since the group Spin_n is the inverse image of SO_n in $\widetilde{\mathrm{GL}_n^+}$ under Ξ, the inverse image of $P_{\mathrm{SO}_n} M$ in $P_{\widetilde{\mathrm{GL}_n^+}} M$ is a Spin_n-principal bundle, which determines the associated spinorial metric.

By Proposition 2.4, it is natural to give up the not so widely used term *spinorial metric*, which will be used later in the book, for the more classical term, in the context of Riemannian geometry, of a *spin manifold.*

To $P_{\mathrm{Spin}_n} M$ we associate a complex vector bundle $\Sigma M = P_{\mathrm{Spin}_n} M \times_{\rho_n} \Sigma_n$, called the *spinor bundle*. The equivariance of θ shows that there exists a representation of $\mathcal{C}^\infty(M)$-algebras

$$\mathbb{C}\mathrm{l}(M) \longrightarrow \mathrm{End}(\Sigma M),$$

given by

$$[u,a]([\tilde{u},\psi]) = [\tilde{u},\rho_n(a)(\psi)]$$

for each $\tilde{u} \in P_{\mathrm{Spin}_n}M$, $u = \theta(\tilde{u}) \in P_{\mathrm{SO}_n}M$, $a \in \mathbb{C}\mathrm{l}_n$ and $\psi \in \Sigma_n$. This action of $\mathbb{C}\mathrm{l}(M)$ on ΣM is called the *Clifford product* and is denoted by

$$(\sigma,\Psi) \longmapsto \sigma \cdot \Psi.$$

Local sections of $P_{\mathrm{Spin}_n}M$ are called *local spinorial gauges* and sections of ΣM *spinor fields* or simply *spinors*.

Proposition 2.5. *There exists a canonical Hermitian product $\langle \cdot,\cdot \rangle$ on ΣM with respect to which the Clifford product with every vector of unit norm is unitary.*

Proof. Let $(\cdot,\cdot)$ denote the Hermitian product on Σ_n introduced in Proposition 1.13. If $\Psi = [\tilde{u},\psi]$ and $\Phi = [\tilde{u},\varphi]$, define $\langle \Psi,\Phi \rangle := (\psi,\varphi)$ and use Corollary 1.37 to check that this does not depend on the choice of the spinorial frame $\tilde{u}$. □

Similarly, Theorem 1.39 of Chapter 1, yields directly the following theorem.

Theorem 2.6. *If M is a spin manifold of real dimension n, the complex spinor bundle ΣM carries a real structure $\mathfrak{j}_{\mathbb{R}}$, for $n \equiv 1,2,6,7,8 \mod 8$, and a quaternionic structure $\mathfrak{j}_{\mathbb{Q}}$, for $n \equiv 2,3,4,5,6 \mod 8$. The Clifford product with real tangent vectors commutes with $\mathfrak{j}_{\mathbb{R}}$, for $n \equiv 6,7,8 \mod 8$, and anti-commutes with $\mathfrak{j}_{\mathbb{R}}$, for $n \equiv 1,2 \mod 8$. The Clifford product with real tangent vectors commutes with $\mathfrak{j}_{\mathbb{Q}}$, for $n \equiv 2,3,4 \mod 8$, and anti-commutes with $\mathfrak{j}_{\mathbb{Q}}$, for $n \equiv 5,6 \mod 8$.*

2.1.2 Spinorial connections and curvatures

We now develop more advanced tools to deal with spinors.

The Levi-Civita connection on $P_{\mathrm{SO}_n}M$ induces on $P_{\mathrm{Spin}_n}M$ a canonical connection as follows (for full details, see [KN63], Section 2.6). When viewed as a $\mathfrak{so}_n$-valued equivariant 1-form on $P_{\mathrm{SO}_n}M$, the connection form ω of the Levi-Civita connection can be lifted, through the map

$$\theta\colon P_{\mathrm{Spin}_n}M \longrightarrow P_{\mathrm{SO}_n}M$$

defining the spinorial metric, to a $\mathfrak{so}_n$-valued 1-form on $P_{\mathrm{Spin}_n}M$. This can be made into a Spin_n-connection form by using the inverse isomorphism between the Lie algebras of Spin_n and SO_n.

One can then, in a very canonical way, define a covariant derivative on the spinor bundle. We denote by ∇ the covariant derivative on $\mathrm{T}M$, as on any Riemannian bundle such as $\mathbb{C}\mathrm{l}(M)$, as well as that on ΣM, induced by the Levi-Civita connection. These covariant derivatives satisfy the following Leibniz rule relative to the Clifford product:

$$\nabla_X(\sigma \cdot \tau) = (\nabla_X \sigma) \cdot \tau + \sigma \cdot \nabla_X \tau, \quad \sigma, \tau \in \Gamma(\mathbb{C}\mathrm{l}(M)),$$

and

$$\nabla_X(\sigma \cdot \Psi) = (\nabla_X \sigma) \cdot \Psi + \sigma \cdot \nabla_X \Psi, \quad \sigma \in \Gamma(\mathbb{C}\mathrm{l}(M)),\ \Psi \in \Gamma(\Sigma M).$$

It is easy to see that the canonical Hermitian product on ΣM defined in Lemma 2.5 is parallel with respect to ∇.

Let R be the Riemannian curvature tensor, which can be seen as a $\Lambda^2 M$-valued 2-form on M. We then have the following theorem.

Theorem 2.7. *Let $\Psi = [\tilde{u}, \psi]$ be a (local) spinor field on M written in the gauge $\tilde{u}$ and let $\theta(\tilde{u}) = (X_1, \dots, X_n)$ be the local orthonormal frame on M corresponding to $\tilde{u}$. Then the covariant derivative of Ψ can be computed by the formula*

$$\nabla_X \Psi = [\tilde{u}, X(\psi)] + \frac{1}{2} \sum_{i<j} g(\nabla_X X_i, X_j) X_i \cdot X_j \cdot \Psi. \tag{2.1}$$

Moreover, the spin curvature operator

$$\mathcal{R}_{X,Y} := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$$

of ∇ on ΣM satisfies

$$\mathcal{R}_{X,Y} \Psi = \frac{1}{2} R(X, Y) \cdot \Psi, \quad \Psi \in \Gamma(\Sigma M).$$

Proof. Let $\omega \in \Omega^1(P_{\mathrm{SO}_n} M, \mathfrak{so}_n)$ be the connection form of the Levi-Civita connection of M and let $\tilde{\omega} = \xi_*^{-1} \circ \theta^* \omega$ be the connection form induced on $P_{\mathrm{Spin}_n} M$. By definition, if X is a vector field on M, $u\colon U \to P_{\mathrm{SO}_n} M$ is a local section and $v\colon U \to \mathbb{R}^n$ is a local function, then

$$\nabla_X [u, v] = [u, X(v) + \omega(u_* X)(v)]. \tag{2.2}$$

Similarly, if $\tilde{u}\colon U \to P_{\mathrm{Spin}_n} M$ is a local section and $\psi\colon U \to \Sigma_n$ is a local function, then

$$\nabla_X [\tilde{u}, \psi] = [\tilde{u}, X(\psi) + \tilde{\omega}(\tilde{u}_* X) \cdot \psi].$$

We fix a spinor Ψ locally written $\Psi = [\tilde{u}, \psi]$ and denote by $u = (X_1, \ldots, X_n)$ the projection of $\tilde{u}$ by θ ($X_i = [u, e_i]$). We let $v = e_i$ in (2.2) and take the scalar product with X_j to obtain

$$\omega(u_* X) = \sum_{i<j} e_{ij} g(\nabla_X X_i, X_j),$$

where $e_{ij} \in \mathfrak{so}_n$ is defined by $e_{ij}(e_k) = \delta_{ik} e_j - \delta_{jk} e_i$. We then get

$$\begin{aligned}
\tilde{\omega}(\tilde{u}_* X) &= \xi_*^{-1}(\omega(\theta_* \tilde{u}_* X)) \\
&= \xi_*^{-1}(\omega(u_* X)) \\
&= \xi_*^{-1} \sum_{i<j} e_{ij} g(\nabla_X X_i, X_j) = \frac{1}{2} \sum_{i<j} g(\nabla_X X_i, X_j) e_i \wedge e_j \\
&= \frac{1}{2} \sum_{i<j} g(\nabla_X X_i, X_j) e_i \cdot e_j,
\end{aligned}$$

thus proving (2.1). Let now $\Omega \in \Omega^2(P_{\mathrm{SO}_n} M, \mathfrak{so}_n)$ be the curvature form of ω and let $\widetilde{\Omega} = \xi_*^{-1} \circ \theta^* \Omega$ be the curvature form on $P_{\mathrm{Spin}_n} M$. With the notations above we have

$$R_{X,Y} X_i = R_{X,Y}[u, e_i] = [u, \Omega(u_* X, u_* Y)(e_i)],$$

so that

$$\Omega(u_* X, u_* Y) = \sum_{i<j} e_{ij} g(R_{X,Y} X_i, X_j),$$

and finally

$$\begin{aligned}
\mathcal{R}_{X,Y} \Psi &= [\tilde{u}, \widetilde{\Omega}(\tilde{u}_* X, \tilde{u}_* Y) \cdot \psi] \\
&= [\tilde{u}, \xi_*^{-1}(\Omega(u_* X, u_* Y) \cdot \psi] \\
&= \Big[\tilde{u}, \frac{1}{2} \sum_{i<j} g(R_{X,Y} X_i, X_j) e_i \cdot e_j \cdot \psi\Big] \\
&= \frac{1}{2} R(X, Y) \cdot \Psi. \qquad \square
\end{aligned}$$

Corollary 2.8. *If X_i is an orthonormal frame at $x \in M$ and $X \in \mathrm{T}_x M$, then*

$$\sum_{i=1}^{n} X_i \cdot \mathcal{R}_{X,X_i} \Psi = -\frac{1}{2} \mathrm{Ric}(X) \cdot \Psi. \tag{2.3}$$

Proof. According to Theorem 2.7 together with the first Bianchi identity and the relation in $\mathbb{C}\mathrm{l}(M)$

$$X_j \cdot X_k \cdot X_i = X_i \cdot X_j \cdot X_k - 2X_j\delta_{ik} + 2X_k\delta_{ij} \tag{2.4}$$

we may write

$$\begin{aligned}
\sum_{i=1}^{n} X_i \cdot \mathcal{R}_{X,X_i}\Psi &= \frac{1}{2}\sum_{i=1}^{n} X_i \cdot R(X,X_i)\cdot\Psi \\
&= \frac{1}{4}\sum_{i,j,k} g(R(X,X_i)X_j,X_k)X_i\cdot X_j\cdot X_k\cdot\Psi \\
&= \frac{1}{12}\sum_{i,j,k}(g(R(X,X_i)X_j,X_k)X_i\cdot X_j\cdot X_k\cdot\Psi \\
&\qquad + g(R(X,X_j)X_k,X_i)X_j\cdot X_k\cdot X_i\cdot\Psi \\
&\qquad + g(R(X,X_k)X_i,X_j)X_k\cdot X_i\cdot X_j\cdot\Psi) \\
&= \frac{1}{12}\sum_{i,j,k}((-2X_j\delta_{ik}+2X_k\delta_{ij})g(R(X,X_j)X_k,X_i) \\
&\qquad + (-2X_j\delta_{ik}+2X_i\delta_{kj})g(R(X,X_k)X_i,X_j))\cdot\Psi \\
&= \frac{1}{12}(-6\,\mathrm{Ric}(X))\cdot\Psi.
\end{aligned}$$

□

2.2 Spin$^{\mathrm{c}}$ and conformal structures

2.2.1 Spin$^{\mathrm{c}}$ structures

Definition 2.9. Let (M,g) be an oriented Riemannian manifold. A Spin$^{\mathrm{c}}$ *structure* on (M,g) is given by a principal $\mathrm{Spin}_n^{\mathrm{c}}$-bundle $P_{\mathrm{Spin}_n^{\mathrm{c}}}M$, a principal U_1-bundle $P_{\mathrm{U}_1}M$ called the *auxiliary line bundle*, and a principal bundle morphism

$$\theta = (\theta_1,\theta_2)\colon P_{\mathrm{Spin}_n^{\mathrm{c}}}M \longmapsto P_{\mathrm{SO}_n}M \times P_{\mathrm{U}_1}M,$$

making the diagram

$$\begin{tikzcd}
\mathrm{Spin}_n^{\mathrm{c}} \arrow[r, "a\mapsto\tilde{u}a"] \arrow[d, "\xi^{\mathrm{c}}"'] & P_{\mathrm{Spin}_n^{\mathrm{c}}}M \arrow[d, "\theta"] \arrow[dr] & \\
 & & M \\
\mathrm{SO}_n\times\mathrm{U}_1 \arrow[r, "(A,z)\mapsto\theta(\tilde{u})(A,z)"'] & P_{\mathrm{SO}_n}M\times P_{\mathrm{U}_1}M \arrow[ur] &
\end{tikzcd}$$

commute for every $\tilde{u}\in P_{\mathrm{Spin}_n^{\mathrm{c}}}M$.

Remark 2.10. In this definition, $P_{\mathrm{SO}_n}M \times P_{\mathrm{U}_1}M$ denotes the Whitney product, that is, the inverse image of the Cartesian product of these principal bundles under the diagonal inclusion of M into $M \times M$.

Given a $\mathrm{Spin}^{\mathrm{c}}$ structure on M, we introduce as before the bundle of complex spinors $\Sigma M = P_{\mathrm{Spin}^{\mathrm{c}}_n}M \times_{\xi^{\mathrm{c}}} \Sigma_n$, on which $\mathbb{C}l(M)$ acts, on the left, by Clifford multiplication. Suppose now that we are given a connection A on $P_{\mathrm{U}_1}M$. This induces, together with the Levi-Civita connection of $P_{\mathrm{SO}_n}M$, a connection on $P_{\mathrm{Spin}^{\mathrm{c}}_n}M$, and hence a covariant derivative, denoted by ∇^A, on ΣM, which satisfies the Leibniz rule

$$\nabla^A_X(\sigma \cdot \Psi) = (\nabla_X \sigma) \cdot \Psi + \sigma \cdot \nabla^A_X \Psi, \quad \sigma \in \Gamma(\mathbb{C}l(M)), \Psi \in \Gamma(\Sigma M),$$

and preserves the canonical Hermitian product on ΣM.

In general, by a $\mathrm{Spin}^{\mathrm{c}}$ *manifold* we will understand a 5-tuple $(M, g, \mathcal{S}, L, A)$, where (M, g) is an oriented Riemannian manifold, $\mathcal{S}$ is a $\mathrm{Spin}^{\mathrm{c}}$ structure, L is the complex line bundle associated with the auxiliary bundle of $\mathcal{S}$ for the canonical representation ι of U_1 on $\mathbb{C}$, and A is a connection on L.

Consider a local spinor field $\Psi = [\tilde{u}, \psi]$, where $\tilde{u}$ is a local section of $P_{\mathrm{Spin}^{\mathrm{c}}_n}M$, and let $\theta(\tilde{u}) = (u, \gamma)$, where γ is a local section of $P_{\mathrm{U}_1}M$ and $u = (X_1, \dots, X_n)$ is a local section of $P_{\mathrm{SO}_n}M$. If $i\omega \in \Omega^1(P_{\mathrm{U}_1}M, i\mathbb{R})$ denotes the connection form of A and $i\,d\omega \in \Omega^2(P_{\mathrm{U}_1}M, i\mathbb{R})$ is its curvature form (which is projectable to a 2-form $G \in \Omega^2(M, i\mathbb{R})$), we have the following analogue to Theorem 2.7

Theorem 2.11. *The covariant derivative of the spinor field* Ψ *is given by*

$$\nabla^A_X \Psi = [\tilde{u}, X(\psi)] + \frac{1}{2}\sum_{i<j} g(\nabla_X X_i, X_j) X_i \cdot X_j \cdot \Psi + \frac{1}{2} i\omega(\gamma_* X)\Psi.$$

The curvature operator

$$\mathcal{R}^A_{X,Y} := [\nabla^A_X, \nabla^A_Y] - \nabla^A_{[X,Y]}$$

of ∇^A *has the following expression:*

$$\mathcal{R}^A_{X,Y}\Psi = \frac{1}{2} R(X,Y) \cdot \Psi + \frac{1}{2} G(X,Y) \cdot \Psi.$$

Note that L is isomorphic to the complex line bundle associated to $P_{\mathrm{Spin}^{\mathrm{c}}_n}M$ for the representation θ_2 of $\mathrm{Spin}^{\mathrm{c}}_n$ on $\mathbb{C}$,

$$L = P_{\mathrm{U}_1}M \times_\iota \mathbb{C} = P_{\mathrm{Spin}^{\mathrm{c}}_n}M \times_{\theta_2} \mathbb{C}.$$

The line bundle L is called the *determinant line bundle* of the $\mathrm{Spin}^{\mathrm{c}}$ structure, and we denote by $\nabla^{L,A}$ the covariant derivative induced by A on L.

Lemma 2.12. *A* Spin^c *structure on M with trivial determinant line bundle can be canonically identified with a spin structure. If moreover, the connection A is the canonical flat connection, then ∇^A corresponds to the spinorial connection ∇ through the identification of spinor bundles.*

Proof. If L is trivial, the same holds for $P_{\mathrm{U}_1}M$, so we may find a global section γ of the latter. Then the pull-back (denoted $P_{\mathrm{Spin}_n}M$) of $P_{\mathrm{SO}_n}M \times \gamma$ by θ is a spin structure on M, thus proving the first statement. Moreover, if A is the canonical flat connection, γ may be chosen parallel, so the connection on $P_{\mathrm{Spin}^c_n}M$ restricts to the Levi-Civita connection on $P_{\mathrm{Spin}_n}M$. □

Consequently, all the results concerning Spin^c structures obtained below will hold, in particular, for usual spin structures.

The next theorem justifies the above definitions, in the sense that it shows that spin and Spin^c structures come up naturally when one deals geometrically with real and complex spinors.

Theorem 2.13. *Let (M^n, g) be an oriented Riemannian manifold and let E be a $\mathbb{C}\mathrm{l}(M)$-module (resp. a $\mathrm{Cl}(M)$-module), that is, a vector bundle over M such that there exists a bundle morphism $\mathbb{C}\mathrm{l}(M) \times E \to E$ (resp. $\mathrm{Cl}(M) \times E \to E$) compatible with the Clifford product of $\mathbb{C}\mathrm{l}(M)$ (resp. $\mathrm{Cl}(M)$). Furthermore, suppose that for each $x \in M$, E_x is irreducible as a $\mathbb{C}\mathrm{l}(M)_x$-module (resp. as a $\mathrm{Cl}(M)_x$-module). Then E is isomorphic to the spinor bundle of a Spin^c structure (resp. spin structure) on M.*

Proof. We will not treat here the real case, which is similar to the complex one, but just algebraically more involved.

Let $\langle\cdot,\cdot\rangle$ be a Hermitian product on E with the property that the action of unit vector fields on E is unitary. The existence of such a Hermitian product follows from Proposition 1.35, using a partition of unity. We consider first the case where the dimension n of M is even. A frame $u \in (P_{\mathrm{SO}_n}M)_x$ can be seen as an isomorphism

$$u\colon \mathbb{C}\mathrm{l}_n \longrightarrow \mathbb{C}\mathrm{l}(M)_x, \quad \psi \longmapsto [u, \psi],$$

satisfying $u(\mathrm{Ad}_b(a)) = u\xi_1^c(b)(a)$, for all $b \in \mathrm{Spin}_n$, $a \in \mathbb{C}\mathrm{l}_n$, where $\xi_1^c(b)$ denotes the projection of $\xi^c(b) \in \mathrm{SO}_n \times \mathrm{U}_1$ on SO_n. For each $x \in M$, we define

$$P_x = \{v\colon \Sigma_n \to E_x \text{ unitary; there exists } u \in (P_{\mathrm{SO}_n}M)_x, v(a\cdot\psi) = u(a)\cdot v(\psi), \text{ for all } a \in \mathbb{C}\mathrm{l}_n\},$$

as well as the projection $\tau\colon P_x \to (P_{\mathrm{SO}_n}M)_x$ which maps v to u. Of course, u is unique because the representation of SO_n on $\mathbb{C}\mathrm{l}_n$ is faithful.

For fixed u, the elements of $\tau^{-1}(u)$ are exactly the $\mathbb{C}l_n$-module isomorphisms $\Sigma_n \to E_x$ preserving the Hermitian products of these two spaces. Note that the $\mathbb{C}l_n$-module structure on E_x is obtained from the identification by u of $\mathbb{C}l(M)_x$ and $\mathbb{C}l_n$. The group Spin_n^c acts freely on the right on P_x by $v \mapsto vb$, where

$$vb(\psi) := v(b \cdot \psi)$$

for every $b \in \mathrm{Spin}_n^c$. Indeed, on the one hand, vb is unitary, by Corollary 1.37. On the other hand, if $B = \xi_1^c(b) \in \mathrm{SO}_n$, then for all $a \in \mathbb{C}l_n,\ \psi \in \Sigma_n$

$$\begin{aligned} v \cdot b(a\psi) &= v(b \cdot a \cdot \psi) \\ &= u(b \cdot a) \cdot v(\psi) = u(b \cdot a) \cdot u(b^{-1}) \cdot v(b \cdot \psi) \\ &= u(\mathrm{Ad}_b(a)) \cdot vb(\psi) \\ &= uB(a) \cdot vb(\psi), \end{aligned}$$

thus showing that $vb \in P_x$ and

$$\tau(vb) = \tau(v)\xi_1^c(b). \tag{2.5}$$

Theorem 1.28 implies that (for every fixed u) the $\mathbb{C}l_n$-representations Σ_n and E_x are equivalent, so there exists an isomorphism of $\mathbb{C}l_n$-modules $\Sigma_n \to E_x$. Corollary 1.37 shows that such an isomorphism is necessarily unitary, hence P_x is not empty.

Finally, we check that the action of Spin_n^c on P_x is transitive. According to (2.5), every orbit of this action intersects each fiber $\tau^{-1}(u)$, so it is enough to show that every two elements of the same fiber belong to the same orbit. Indeed, if $v_1, v_2 \in \tau^{-1}(u)$, then $v_2^{-1} \circ v_1$ is a unitary endomorphism f of Σ_n which commutes with every element of $\mathbb{C}l_n$. This shows that $f \in \mathrm{U}_1$, since the center of the endomorphism algebra of any complex vector space is $\mathbb{C}$. This shows that there exists $z \in \mathrm{U}_1 \subset \mathrm{Spin}_n^c$ such that $v_1 = v_2 z$.

Consider now local trivializations of $P_{\mathrm{SO}_n}M$ and E. The above discussion shows that the union P of all sets P_x is a principal Spin_n^c-bundle over M. Moreover, (2.5) implies that P is actually a Spin^c structure. Indeed, one defines

$$P_{\mathrm{U}_1}M = P \times_{\xi_2^c} \mathrm{U}_1$$

and

$$\theta \colon P \longmapsto P_{\mathrm{SO}_n}M \times P_{\mathrm{U}_1}M, \quad \theta(u) = (\tau(u), [u, 1]),$$

which satisfy

$$\theta(va) = (\tau(va), [va, 1]) = (\tau(v)\xi_1^c(a), [v, \xi_2^c(a)]) = \theta(v)\xi^c(a)\ .$$

By construction, the spinor bundle associated with this Spin^c structure is isomorphic to E as a $\mathbb{C}l(M)$-module.

For n odd, we note that $\mathrm{End}(E)$ is isomorphic to $\mathbb{C}l^+(M)$ or to $\mathbb{C}l^-(M)$, according to whether the complex volume element acts by 1 or -1 on E. Replacing $\mathbb{C}l(M)$ by $\mathbb{C}l^\pm(M)$, the above argument then holds verbatim. □

The results in this section were independently obtained by Th. Friedrich and A. Trautman in the framework of Lipschitz structures [FT00].

2.2.2 Weyl structures

Let M^n be an oriented manifold and let $P_{\mathrm{GL}_n^+}M$ be the principal GL_n^+-bundle of oriented frames on M. We define a family of oriented real line bundles $\mathcal{L}^k$ on M by

$$\mathcal{L}^k = P_{\mathrm{GL}_n^+}M \times_{\det^{k/n}} \mathbb{R} .$$

Obviously, we have $\mathcal{L}^k \otimes \mathcal{L}^l \simeq \mathcal{L}^{k+l}$. The bundle $\mathcal{L}^k$ is called the *density bundle* of weight k. For each $k \in \mathbb{R}$, the complement of the zero section in $\mathcal{L}^k$ has two connected components, defined by

$$\mathcal{L}^k_\pm = P_{\mathrm{GL}_n^+}M \times_{\det^{k/n}} \mathbb{R}^*_\pm .$$

A *conformal structure* on M is a section c of $\mathrm{Sym}^2(\mathrm{T}^*M)\otimes\mathcal{L}^2$ such that $c(X,X) \in \mathcal{L}^2_+$ for every non-zero vector X on M. The existence of such a structure yields a reduction $P_{\mathrm{CO}_n^+}M$ of the positive frame bundle of M to the conformal group CO_n^+. If M is a conformal manifold, we also can write

$$\mathcal{L}^k = P_{\mathrm{CO}_n^+}M \times_{\det^{k/n}} \mathbb{R}.$$

Definition 2.14. A *Weyl structure* on an oriented conformal manifold (M^n, c) is a torsion-free connection on $P_{\mathrm{CO}_n^+}M$, or equivalently, a torsion-free connection on $P_{\mathrm{GL}_n^+}M$ whose induced covariant derivative on sections of $\mathrm{Sym}^2(\mathrm{T}^*M)\otimes\mathcal{L}^2$ preserves c.

The foundational theorem of conformal geometry, due to H. Weyl, says that there is a bijective correspondence between Weyl structures and linear connections on $\mathcal{L} := \mathcal{L}^1$. Indeed, every Weyl structure induces a connection on the principal $\mathbb{R}^*_+$-bundle

$$\mathcal{L}_+ := P_{\mathrm{GL}_n^+}M \times_{\det^{k/n}} \mathbb{R}^*_+ ,$$

hence a linear connection on $\mathcal{L}$. Conversely, if D and $D^{\mathcal{L}}$ are the covariant derivatives induced by a Weyl structure on $\mathrm{T}M$ and on $\mathcal{L}$, respectively, then the following analogue of the Koszul formula holds:

$$\begin{aligned}2c(D_X Y, Z) = D^{\mathcal{L}}_X(c(Y,Z)) + D^{\mathcal{L}}_Y(c(X,Z)) - D^{\mathcal{L}}_Z(c(X,Y))\\ + c(Z,[X,Y]) - c(Y,[X,Z]) - c(X,[Y,Z]).\end{aligned}$$

This shows the existence and the uniqueness of D, given $D^{\mathcal{L}}$.

One usually calls the above covariant derivative D on $\mathrm{T}M$ a Weyl structure, too. A Weyl structure D is called *closed* (resp. *exact*) if $D^{\mathcal{L}}$ is flat (resp. if $\mathcal{L}$ carries a global $D^{\mathcal{L}}$-parallel section). As Riemannian metrics of the conformal class c correspond to sections of $\mathcal{L}_+$, it follows immediately that D is closed (resp. exact) if and only of D is locally (resp. globally) the Levi-Civita connection of a metric $g \in c$.

For every metric g in the conformal class c there exists a unique non-vanishing section l of $\mathcal{L}_+$ (trivializing $\mathcal{L}$) such that

$$c = g \otimes l^2. \tag{2.6}$$

Let $D^{\mathcal{L}}$ be a torsion-free connection on $\mathcal{L}$ and let $\theta \in \Omega^1(M,\mathbb{R})$ be its connection form, written in the gauge l:

$$D^{\mathcal{L}}_X l = \theta(X) l, \quad X \in \mathrm{T}M. \tag{2.7}$$

An easy calculation using (2.6), together with the Koszul formula above, gives

$$D_X Y = \nabla_X Y + \theta(X)Y + \theta(Y)X - g(X,Y)T, \quad X, Y \in \mathrm{T}M, \tag{2.8}$$

where ∇ is the Levi-Civita covariant derivative of g and T is the vector field dual to θ with respect to g. The 1-form θ is called the *Lee form* of (D,g).

Consider a conformal change of the metric $\bar{g} = e^{-2f} g$, where f is a function on M. Then $\bar{l} = e^f l$, and (2.7) shows that the Lee form $\bar{\theta}$ associated with $(D,\bar{g})$ is related to θ by

$$\bar{\theta} = \theta + df.$$

Consequently, a Weyl structure is closed (resp. exact) if and only if its Lee form is closed (resp. exact) for one metric in the conformal class – and thus for all of them.

We now study the curvature tensor of a Weyl structure D, defined as usual for vector fields X, Y, and Z by

$$R^D_{X,Y} Z := [D_X, D_Y]Z - D_{[X,Y]}Z.$$

Let F denote the curvature tensor of the connection $D^{\mathcal{L}}$, viewed as a 2-form on M and called the *Faraday form* of D. In contrast to the Riemannian case, the image of a pair of vectors by R^D is an endomorphism of $\mathrm{T}M$ which is not always skew-symmetric. Of course, the symmetric part of $R^D_{X,Y}$ is just $F(X,Y)\mathrm{Id}$, where Id denotes the identity of $\mathrm{T}M$. We denote by $R^D_{\mathrm{a}}(X,Y)$ the skew-symmetric part of $R^D_{X,Y}$, that is

$$R^D = R^D_{\mathrm{a}} + F \otimes \mathrm{Id}. \tag{2.9}$$

In Chapter 9 we will need the following lemma.

Lemma 2.15. *We have*

$$\underset{X,Y,Z}{\mathfrak{S}} c(R^D_{\mathrm{a}}(T,X)Y,Z) = \underset{X,Y,Z}{\mathfrak{S}} (F(Y,X)c(Z,T)). \tag{2.10}$$

Proof. Using formula (2.9) and the first Bianchi identity

$$\underset{T,X,Y}{\mathfrak{S}} R^D_{T,X}Y = 0\,,$$

we compute

$$\begin{aligned}\underset{X,Y,Z}{\mathfrak{S}} c(R^D_{T,X}Y,Z) &= \underset{X,Y,Z}{\mathfrak{S}} (c(R^D_{Y,X}T,Z) + c(R^D_{T,Y}X,Z)) \\ &= \underset{X,Y,Z}{\mathfrak{S}} (-c(R^D_{Y,X}Z,T) + 2F(Y,X)c(Z,T) \\ &\qquad - c(R^D_{T,Y}Z,X) + 2F(T,Y)c(X,Z)).\end{aligned}$$

Thus

$$\underset{X,Y,Z}{\mathfrak{S}} c(R^D_{T,X}Y,Z) = \underset{X,Y,Z}{\mathfrak{S}} (F(Y,X)c(Z,T) + F(T,Y)c(X,Z)) \tag{2.11}$$

which, together with (2.9), yields (2.10). □

We define the *Ricci tensor* of D by

$$\mathrm{Ric}^D(X,Y) = \mathrm{tr}\{V \mapsto R^D_{\mathrm{a}}(V,X)Y\}$$

and the *scalar curvature* Scal^D as the conformal trace of Ric^D. In particular, Scal^D is not a function, but a density of weight -2 (as c maps $\mathrm{T}^*M \otimes \mathrm{T}^*M$ to $\mathcal{L}^{-2}$).

Lemma 2.16. *The skew-symmetric part* $\mathrm{Ric}^D_{\mathrm{a}}$ *of the Ricci tensor* Ric^D *is equal to* $\frac{2-n}{2}F$.

Proof. Let $\widetilde{\mathrm{Ric}}^D(X,Y) = \mathrm{tr}\{V \mapsto R^D_{V,X}Y\}$. The Bianchi identity applied to R^D easily yields

$$\widetilde{\mathrm{Ric}}^D(X,Y) - \widetilde{\mathrm{Ric}}^D(Y,X) = -nF(X,Y). \tag{2.12}$$

On the other hand, (2.9) shows that

$$\widetilde{\mathrm{Ric}}^D(X,Y) - \mathrm{Ric}^D(X,Y) = \mathrm{tr}\{V \mapsto F(V,X)Y\} = -F(X,Y),$$

which, together with (2.12), proves the desired result. □

Definition 2.17. A Weyl structure is called *Weyl–Einstein* if the trace-free part of the symmetric part of the Ricci tensor Ric^D vanishes, or, equivalently, if

$$\mathrm{Ric}^D = \frac{1}{n}\mathrm{Scal}^D c + \frac{2-n}{2}F.$$

Weyl–Einstein structures will play an important role in Section 9.2.

2.2.3 Spin and $\mathrm{Spin}^{\mathrm{c}}$ conformal manifolds

Definition 2.18. A *spin structure* on an oriented conformal manifold (M, c) is given by a principal CSpin_n-bundle $P_{\mathrm{CSpin}_n}M$ together with a projection

$$\theta\colon P_{\mathrm{CSpin}_n}M \longrightarrow P_{\mathrm{CO}_n^+}M$$

making the diagram

$$\begin{tikzcd}
\mathrm{CSpin}_n \arrow[r, "a\mapsto\tilde{u}a"] \arrow[d, "\xi"'] & P_{\mathrm{CSpin}_n}M \arrow[d, "\theta"] \arrow[dr] & \\
 & & M \\
\mathrm{CO}_n^+ \arrow[r, "A\mapsto\theta(\tilde{u})A"'] & P_{\mathrm{CO}_n^+}M \arrow[ur] &
\end{tikzcd}$$

commute for every $\tilde{u} \in P_{\mathrm{CSpin}_n}M$.

Definition 2.19. A $\mathrm{Spin}^{\mathrm{c}}$ *structure* on (M, c) is given by a principal $\mathrm{CSpin}_n^{\mathrm{c}}$-bundle $P_{\mathrm{CSpin}_n^{\mathrm{c}}}M$, a principal U_1-bundle $P_{\mathrm{U}_1}M$, and a projection

$$\theta = (\theta_1, \theta_2)\colon P_{\mathrm{CSpin}_n^{\mathrm{c}}}M \longrightarrow P_{\mathrm{CO}_n^+}M \times P_{\mathrm{U}_1}M$$

making the diagram

$$\begin{tikzcd}
\mathrm{CSpin}_n^{\mathrm{c}} \arrow[r, "a\mapsto\tilde{u}a"] \arrow[d, "\xi^{\mathrm{c}}"'] & P_{\mathrm{CSpin}_n^{\mathrm{c}}}M \arrow[d, "\theta"] \arrow[dr] & \\
 & & M \\
\mathrm{CO}_n^+ \times \mathrm{U}_1 \arrow[r, "(A,z)\mapsto\theta(\tilde{u})(A,z)"'] & P_{\mathrm{CO}_n^+}M \times P_{\mathrm{U}_1}M \arrow[ur] &
\end{tikzcd}$$

commute for every $\tilde{u} \in P_{\mathrm{CSpin}_n^{\mathrm{c}}}M$.

Like before, every spin structure can be canonically identified with a Spin^c structure with trivial determinant bundle.

Given a Spin^c structure on (M, c), and $k \in \mathbb{R}$, we define the spinor bundles of weight k by

$$\Sigma^{(k)} M = P_{\mathrm{CSpin}_n^c} M \times_{\rho_n^{c,k}} \Sigma_n,$$

where $\rho_n^{c,k}$ were defined in (1.12). Note that $\Sigma^{(k)} M$ is canonically isomorphic to $\Sigma^{(0)} M \otimes_{\mathbb{R}} \mathcal{L}^k$ for every k. Since we deal with complex bundles, it is more convenient to complexify the density bundle $\mathcal{L}$ by defining $\mathcal{L}_{\mathbb{C}} = \mathcal{L} \otimes_{\mathbb{R}} \mathbb{C}$, so that $\Sigma^{(k)} M = \Sigma^{(0)} M \otimes_{\mathbb{C}} \mathcal{L}_{\mathbb{C}}^k$. From now on all tensor products are taken over the field of complex numbers.

If g is any choice of metric in the conformal class c, the usual spinor bundle $\Sigma^g M$ may be identified with any of the weighted spinor bundles $\Sigma^{(k)} M$ by restricting the frame bundle of the latter to $P_{\mathrm{Spin}_n^c} M$. This induces a family of isomorphisms

$$\Phi^{(k)} \colon \Sigma^g M \longrightarrow \Sigma^{\bar{g}} M\ , \quad [s, u] \longmapsto [s\phi, \phi^{-k} u],$$

where $\bar{g} = \phi^{-2} g$, $s \in P^g_{\mathrm{Spin}_n^c} M$ and u is the expression of a spinor field in the spinor frame s. It is easily checked that

$$|\Phi^{(k)} \psi|^2_{\bar{g}} = \phi^{-2k} |\psi|^2_g \tag{2.13}$$

for any spinor ψ.

Corollary 2.20. *There exists a canonical Hermitian product on $\Sigma^{(0)} M$, and more generally a Hermitian pairing*

$$\Sigma^{(k)} M \otimes \Sigma^{(l)} M \longrightarrow \mathcal{L}_{\mathbb{C}}^{k+l},$$

satisfying

$$\langle X \cdot \Psi, \Phi \rangle = -\langle \Psi, X \cdot \Phi \rangle \in \mathcal{L}_{\mathbb{C}}^{k+l-1}$$

for $X \in \mathrm{T}M$, $\Psi \in \Sigma^{(k)} M$, and $\Phi \in \Sigma^{(l)} M$.

Similarly to the Riemannian case, if M has a spin structure, then any Weyl structure D induces a connection on $P_{\mathrm{CSpin}_n} M$, and therefore a covariant derivative, denoted by $D^{(k)}$, on $\Sigma^{(k)} M$. If M has a Spin^c structure, then any Weyl structure on M and any linear connection A on the determinant bundle of the Spin^c structure, induce together a covariant derivative, denoted by $D^{A,(k)}$, on $\Sigma^{(k)} M$. The Hermitian pairing defined above is parallel with respect to this connection.

In contrast to the Riemannian case, on a conformal manifold there is no Clifford bundle containing the tangent bundle as a sub-bundle. Some care to define the Clifford product is thus required. One possibility would be to view c as a metric on $\mathrm{T}M \otimes \mathcal{L}^{-1}$ and define the Clifford bundle associated with this Euclidean vector bundle, as well as the Clifford product acting on weighted spinors. Nevertheless, we will not adopt this point of view since it does not seem well adapted to spinorial calculus.

A better idea is the following: rather than having a Clifford product on each $\Sigma^{(k)}M$ by abstract objects which cannot be identified with forms or vectors on M, we define a Clifford product by usual forms (or vectors), which no longer preserves the weight of spinors.

More precisely, if $\Psi = [\tilde{u}, \psi]$ is a spinor of weight k and $\omega = [u, \tau]$ a p-form (where the conformal frame u is the projection of the spin frame $\tilde{u}$), then we define the Clifford product $\omega \cdot \Psi$ as the spinor of weight $k - p$ given by $\omega \cdot \Psi = [\tilde{u}, \tau \cdot \psi]$. It is easy to see that this definition does not depend on the representatives of Ψ and ω.

One similarly defines the Clifford product of a spinor by a vector, which raises the conformal weight by one. The Clifford product defined in this way respects the Leibniz rule

$$D_X^{(k-p)}(\omega \cdot \Psi) = (D_X\omega) \cdot \Psi + \omega \cdot D_X^{(k)}\Psi$$

for $X \in \mathrm{T}M$, $\omega \in \Omega^p M$, and $\Psi \in \Gamma(\Sigma^{(k)}M)$, and

$$D_X^{(k+1)}(Y \cdot \Psi) = (D_X Y) \cdot \Psi + Y \cdot D_X^{(k)}\Psi,$$

for $X \in \mathrm{T}M$, $Y \in \Gamma(\mathrm{T}M)$, and $\Psi \in \Gamma(\Sigma^{(k)}M)$.

We now fix a metric g in the conformal class. As already mentioned, for each $k \in \mathbb{R}$ there exists a canonical isomorphism

$$I^k \colon \Sigma^{(k)}M \longrightarrow \Sigma M$$

identifying the spinor bundle of weight k of (M^n, c) with the spinor bundle of (M^n, g). This isomorphism is compatible with the Clifford product, that is, $I^{k-p}(\omega \cdot \Psi) = \omega \cdot I^k(\Psi)$, for all $\omega \in \Lambda^p M$, $\Psi \in \Sigma^{(k)}M$. To simplify notations, we will from now on omit I^k when there is no risk of confusion. Let D be a Weyl structure on M and let θ be the Lee form of (D, g).

Theorem 2.21. *The covariant derivatives induced on ΣM by the Levi-Civita connection of g and on $\Sigma^{(k)}M$ by the Weyl structure D are related by*

$$D_X^{(k)}\Phi = \nabla_X\Phi - \frac{1}{2}X \cdot \theta \cdot \Phi + \Big(k - \frac{1}{2}\Big)\theta(X)\Phi, \quad \Phi \in \Gamma(\Sigma^{(k)}M). \tag{2.14}$$

Proof. Let $\omega \in \Omega^1(P_{\mathrm{SO}_n}M, \mathfrak{so}_n)$ be the connection form of the Levi-Civita connection of (M, g) and let $\tilde{\omega} = \xi_*^{-1} \circ \theta^*\omega$ be the induced connection form on $P_{\mathrm{Spin}_n}(M, g)$. We denote by $\omega^D \in \Omega^1(P_{\mathrm{CO}_n^+}M, \mathfrak{co}_n)$ the connection form of D and by $\tilde{\omega}^D = \xi_*^{-1} \circ \theta^*\omega^D \in \Omega^1(P_{\mathrm{CSpin}_n}M, \mathfrak{co}_n)$, the induced connection form on $P_{\mathrm{CSpin}_n}M$. We will identify for the rest of the proof $P_{\mathrm{SO}_n}M$ and $P_{\mathrm{Spin}_n}(M, g)$ with sub-bundles of $P_{\mathrm{CO}_n^+}M$ and $P_{\mathrm{CSpin}_n}(M, c)$ respectively. For each point of M, there exist a neighborhood U, a spin frame $\tilde{u}\colon U \to P_{\mathrm{Spin}_n}(M, g)$, and a function $\phi\colon U \to \Sigma_n$, such that $\Phi = [u, \phi]$ over U. Let $\{e_1, \dots, e_n\}$ be the standard basis of $\mathbb{R}^n$, $u = \theta \circ \tilde{u}\colon U \to P_{\mathrm{SO}_n}M$, and denote $X_i = [u, e_i]$, so that $u = \{X_1, \dots, X_n\}$. If θ_i are the components of θ with respect to this basis, (2.8) yields

$$
\begin{aligned}
[u, (\omega^D - \omega)(u_* X_i)e_j] = D_{X_i}X_j - \nabla_{X_i}X_j &= \theta_i X_j + \theta_j X_i - T\delta_{ij} \\
&= \Big[u, \theta_i e_j + \theta_j e_i - \sum_{l=1}^n \theta_l e_l \delta_{ij}\Big] \\
&= \Big[u, \Big(\theta_i \mathrm{I}_n + \sum_{l=1}^n \theta_l e_{li}\Big) e_j\Big],
\end{aligned}
$$

so

$$
(\omega^D - \omega)(u_* X_i) = \theta_i \mathrm{I}_n + \sum_{l=1}^n \theta_l e_{li} \in \mathbb{R}\mathrm{I}_n \oplus \mathfrak{so}_n \simeq \mathfrak{co}_n.
$$

According to Theorem 1.25 we obtain

$$
\begin{aligned}
(\tilde{\omega}^D - \tilde{\omega})(\tilde{u}_* X_i) &= \zeta_*^{-1}((\omega^D - \omega)(u_* X_i)) \\
&= \theta_i + \frac{1}{2}\sum_{l=1}^n \theta_l (e_l \cdot e_i + \delta_{li}),
\end{aligned}
$$

and consequently

$$
\begin{aligned}
D_{X_i}\Phi - \nabla_{X_i}\Phi &= [u, (\rho_n^k)_*(\tilde{\omega}^D - \tilde{\omega})(\tilde{u}_* X_i)\phi] \\
&= [u, k\theta_i \phi] + \Big[u, \frac{1}{2}\sum_{l=1}^n \theta_l (e_l \cdot e_i + \delta_{li})\phi\Big] \\
&= k\theta(X_i)\Phi + \frac{1}{2}(\theta \cdot X_i \cdot \Phi + \theta(X_i)\Phi),
\end{aligned}
$$

which is equivalent to (2.14). $\square$

2.3 Natural operators on spinors

There are different approaches to define first-order differential operators acting on a given $\mathfrak{so}(n)$ irreducible vector bundle. We first mention the classical approach based on representation theory.

We start by introducing the general notion of *Stein–Weiss operators*, sometimes known as *gradients* cf. [SW68], [Bra96], and [Bra97]. If (M, g) is an orientable n-dimensional Riemannian manifold, every finite-dimensional SO_n representation on some vector space V induces a vector bundle $\mathbf{V}$ over M. If $\mathbf{V}$ and $\mathbf{W}$ are two such vector bundles, denote by $\mathrm{Diff}(\mathbf{V}, \mathbf{W})$ the space of smooth-coefficient linear differential operators from sections of $\mathbf{V}$ to sections of $\mathbf{W}$. Take an SO_n-irreducible bundle $\mathbf{U}$, and consider the covariant derivative

$$\nabla \in \mathrm{Diff}(\mathbf{U}, \mathrm{T}^*M \otimes \mathbf{U}).$$

The bundle on the right-hand side splits up into SO_n-irreducible bundles:

$$\mathrm{T}^*M \otimes \mathbf{U} := \mathbf{V} = \mathbf{V}_0 \oplus \cdots \oplus \mathbf{V}_N,$$

where of course the integer N depends on $\mathbf{U}$. It is well known [Feg87] that this is a *multiplicity free* decomposition:

$$\mathbf{V}_i \cong_{\mathrm{SO}_n} \mathbf{V}_j \implies i = j.$$

Thus we can project onto the $\mathbf{V}_j$ summand and define the gradients

$$\nabla^j := \mathrm{Proj}^j_{\mathbf{V}} \circ \nabla.$$

The Dirac and twistor operators, and the exterior and interior derivatives d and δ are all (constant multiples of) Stein–Weiss operators. The point of view in [SW68] is to consider gradients as generalizations of the two-dimensional Cauchy–Riemann operator.

The second approach which we shall develop in this section is due to P. Gauduchon [Gau95a] and uses the decomposition of a vector bundle into eigenbundles associated with the principal symbol morphism of the corresponding first-order differential operator. Of course, these two points of view are equivalent, but the second one has the advantage of being explicit and elementary.

2.3.1 General algebraic setting

Let V and W be two finite-dimensional $\mathbb{K}$-vector spaces endowed with an Euclidean or Hermitian scalar products depending on whether $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Lemma 2.22. *Let H be a nontrivial homomorphism*

$$\begin{aligned} H\colon V &\longrightarrow W, \\ \varphi &\longmapsto H(\varphi) = \phi, \end{aligned}$$

and H^* its metric adjoint, mapping W to V. We denote by $0 < \mu_1 < \cdots < \mu_N$ the positive eigenvalues of the endomorphism $H \circ H^*$ and by V_j (resp. W_j) the associated eigenspaces of $H^* \circ H$ (resp. $H \circ H^*$).

Then, for all $j \in \{1, \ldots, N\}$, the map

$$\frac{1}{\sqrt{\mu_j}} H\colon V_j \longrightarrow W_j$$

is an isometry.

Proof. First note that the endomorphisms $H^* \circ H$ and $H \circ H^*$ have the same positive eigenvalues. It is clear that the restriction of H to the eigenspace V_j is an isomorphism from V_j to W_j whose inverse is $\frac{1}{\mu_j} H^*$. Hence, the composition

$$V_j \xrightarrow{\frac{1}{\sqrt{\mu_j}} H} W_j \xrightarrow{\frac{1}{\sqrt{\mu_j}} H^*} V_j$$

is the identity of V_j. $\square$

Let

$$V_0 := \operatorname{Ker} H \quad \text{and} \quad W_0 := \operatorname{Ker} H^*.$$

It is interesting to note that we have the diagram

$$\begin{array}{ccccccccc} V = V_0 \oplus & V_1 & \oplus \cdots \oplus & V_j & \oplus \cdots \oplus & V_N & \\ & H \Big\downarrow \Big\uparrow \mu_1^{-1} H^* & & H \Big\downarrow \Big\uparrow \mu_j^{-1} H^* & & H \Big\downarrow \Big\uparrow \mu_N^{-1} H^* & \\ & W_1 & \oplus \cdots \oplus & W_j & \oplus \cdots \oplus & W_N & \oplus W_0 \end{array}$$

Now, for $k \in \{0, 1, \ldots, N\}$, define the two projections Proj_V^k and Proj_W^k, by

$$\begin{aligned} \operatorname{Proj}_V^k\colon V = V_0 \oplus V_1 \oplus \cdots \oplus V_N &\longrightarrow V_k, \\ \varphi = \varphi_0 \oplus \varphi_1 \oplus \cdots \oplus \varphi_N &\longmapsto \varphi_k, \end{aligned}$$

and

$$\begin{aligned} \operatorname{Proj}_W^k\colon W = W_0 \oplus W_1 \oplus \cdots \oplus W_N &\longrightarrow W_k, \\ \phi = \phi_0 \oplus \phi_1 \oplus \cdots \oplus \phi_N &\longmapsto \phi_k. \end{aligned}$$

Lemma 2.23. (1) *For all* $\varphi = \varphi_0 + \varphi_1 + \cdots + \varphi_N \in V$, *one has*

$$|\varphi|^2 \geq \sum_{j=1}^{N} \frac{1}{\mu_j} |\operatorname{Proj}_W^j \circ H(\varphi)|^2,$$

and equality holds if and only if $\varphi_0 = 0$, *i.e.,* $\varphi \in (\operatorname{Ker} H)^{\perp}$.

(2) *Let* $\mu = \mu_N$ *be the maximum eigenvalue of the endomorphism* $H \circ H^*$. *One has*

$$|\varphi|^2 \geq \frac{1}{\mu} |H\varphi|^2, \quad \varphi \in V, \tag{2.15}$$

and

$$|\varphi|^2 = \frac{1}{\mu}|H\varphi|^2 \iff \begin{cases} \varphi_0 = 0 & \textit{and} \\ \operatorname{Proj}_W^j \circ H(\varphi) = 0, & \textit{for } j \in \{1, \ldots, N-1\}. \end{cases}$$

In other words, equality holds in (2.15) *if and only if*

$$H^*(H\varphi) = \mu\varphi.$$

Proof. By Lemma 2.22, for all $\varphi \in V$, one has

$$\begin{aligned} |\varphi|^2 &= |\varphi_0|^2 + |\varphi_1|^2 + \cdots + |\varphi_N|^2 \\ &= |\varphi_0|^2 + \sum_{j=1}^{N} \frac{1}{\mu_j} |\operatorname{Proj}_W^j \circ H(\varphi)|^2. \end{aligned}$$

This gives the first part of the lemma. For the last statement, it is sufficient to note that

$$\begin{aligned} |\varphi|^2 &= |\varphi_0|^2 + \sum_{j=1}^{N} \frac{1}{\mu_j} |\operatorname{Proj}_W^j \circ H(\varphi)|^2 \\ &\geq |\varphi_0|^2 + \frac{1}{\mu} \sum_{j=1}^{N} |\operatorname{Proj}_W^j \circ H(\varphi)|^2 \\ &\geq \frac{1}{\mu} |H(\varphi)|^2. \end{aligned}$$

Moreover, equality in (2.15) holds if and only if $\varphi \in V_N$, i.e., $H^*(H\varphi) = \mu\varphi$. □

2.3.2 First-order differential operators

In this section we apply the preceding algebraic set-up to $\mathbb{K}$-vector bundles $\mathbf{U}$, $\mathbf{V}$, and $\mathbf{W}$ on an n-dimensional Riemannian manifold (M^n, g) induced by finite-dimensional SO_n representations on U, V, and W, endowed with real or Hermitian scalar products. For any $\mathbb{K}$-vector bundle $\mathbf{U}$ of finite rank, let $\mathbf{V} := \mathrm{T}^*M \otimes_{\mathbb{R}} \mathbf{U}$, where T^*M is the cotangent (real) vector bundle. In case $\mathbb{K} = \mathbb{C}$, the vector bundle $\mathbf{V}$ is considered to be a complex vector bundle by the action of complex numbers on $\mathbf{U}$.

Let

$$\mathbf{H}\colon \Gamma(\mathbf{V}) := \Gamma(\mathrm{T}^*M \otimes_{\mathbb{R}} \mathbf{U}) \longrightarrow \Gamma(\mathbf{W})$$

be a bundle morphism induced by some SO_n-equivariant map

$$H\colon V := \mathbb{R}^n \otimes U \longrightarrow W,$$

and denote by $\mathbf{H}^*$, its metric adjoint. The positive eigenvalues of the endomorphism $H \circ H^*$, denoted by $\mu_1 < \cdots < \mu_N$, are also the eigenvalues of the bundle endomorphism $\mathbf{H} \circ \mathbf{H}^*$ of $\Gamma(\mathbf{W})$ and coincide with the positive eigenvalues of the bundle endomorphism $\mathbf{H}^* \circ \mathbf{H}$ of $\Gamma(\mathbf{V})$. Moreover, the associated eigenspaces $\mathbf{W}_j$ and $\mathbf{V}_j$ are sub-bundles of $\mathbf{W}$ and $\mathbf{V}$, $j \in \{1, \dots, N\}$.

For any linear connection ∇ acting on sections of $\mathbf{U}$, consider the first-order differential operator defined by

$$\mathbb{D} := \mathbf{H} \circ \nabla.$$

Note that the homomorphism $\mathbf{H}$ is equal (up to some factor i, depending on the convention used) to the principal symbol of the differential operator $\mathbb{D}$. For each $j \in \{1, \dots, N\}$, we define the operator $\mathbb{D}^j$ by

$$\mathbb{D}^j := \mathrm{Proj}_{\mathbf{W}_j} \circ \mathbb{D}$$

and for each $k \in \{0, 1, \dots, N\}$, the differential operator ∇^k by

$$\nabla^k := \mathrm{Proj}_{\mathbf{V}_k} \circ \nabla.$$

The relation between these operators is expressed by the diagram

$$\begin{tikzcd}
 & \Gamma(\mathbf{V}) \arrow[r, "\mathbf{H}"] \arrow[dd, "\mathrm{Proj}_{\mathbf{V}_j}"'] & \Gamma(\mathbf{W}) \arrow[dd, "\mathrm{Proj}_{\mathbf{W}_j}"] \\
\Gamma(\mathbf{U}) \arrow[ur, "\nabla"] \arrow[dr, "\nabla^j"'] & & \\
 & \Gamma(\mathbf{V}_j) & \Gamma(\mathbf{W}_j). \arrow[l, "\frac{1}{\mu_j}\mathbf{H}^*"]
\end{tikzcd}$$

In fact, for all $j \in \{1, \dots, N\}$, we have the relations

$$\nabla^j = \frac{1}{\mu_j} \mathbf{H}^* \circ \mathbb{D}^j, \quad \mathbb{D} = \sum_{j=1}^N \mathbb{D}^j, \quad \nabla = \nabla^0 + \widetilde{\mathbb{D}},$$

where the operator $\widetilde{\mathbb{D}}$, called the *suspension* of $\mathbb{D}$, is given by

$$\widetilde{\mathbb{D}} := \sum_{j=1}^N \nabla^j.$$

By definition, we thus have

$$\begin{aligned} \widetilde{\mathbb{D}} \colon \Gamma(\mathbf{U}) &\longrightarrow (\operatorname{Ker} \mathbf{H})^\perp \subset \Gamma(\mathrm{T}^* M \otimes \mathbf{U}), \\ \psi &\longmapsto \varphi = \sum_{j=1}^N \nabla^j \psi, \end{aligned}$$

which could be identified with the operator $\mathbb{D}$ via the isomorphism

$$\widetilde{\mathbf{H}} = \mathbf{H}|_{(\operatorname{Ker} \mathbf{H})^\perp} \colon (\operatorname{Ker} \mathbf{H})^\perp \longmapsto \operatorname{Im} \mathbf{H},$$

so that

$$\mathbb{D} = \widetilde{\mathbf{H}} \circ \widetilde{\mathbb{D}}.$$

The operator

$$\nabla^0 := \nabla - \sum_{j=1}^N \frac{1}{\mu_j} \mathbf{H}^* \circ \mathbb{D}^j \tag{2.16}$$

is called the *Penrose operator*. It can be characterized by

$$\mathbf{H} \circ \nabla^0 = 0.$$

For each point $x \in M$, we can apply Lemma 2.23 to get

Lemma 2.24. (1) *For all $\psi \in \Gamma(\mathbf{U})$ and at any $x \in M$,*

$$|\nabla \psi|^2 = |\nabla^0 \psi|^2 + \sum_{j=1}^N \frac{1}{\mu_j} |\mathbb{D}^j \psi|^2.$$

(2) *For all* $\psi \in \Gamma(\mathbf{U})$ *and at any* $x \in M$,

$$|\nabla\psi|^2 \geq \sum_{j=1}^{N} \frac{1}{\mu_j} |\mathbb{D}^j \psi|^2,$$

with equality if and only if

$$\nabla^0 \psi = 0.$$

(3) *Let* $\mu = \mu_N$ *be the largest eigenvalue of the endomorphism* $\mathbf{H} \circ \mathbf{H}^*$*; then for all* $\psi \in \Gamma(\mathbf{U})$ *and at any* $x \in M$*, we have*

$$|\nabla\psi|^2 \geq \frac{1}{\mu} |\mathbb{D}\psi|^2 \tag{2.17}$$

and

$$|\nabla\psi|^2 = \frac{1}{\mu} |\mathbb{D}\psi|^2 \iff \begin{cases} \nabla^0 \psi = 0 & \textit{and} \\ \mathbb{D}^j \psi = 0, & \textit{for } j \in \{1, \ldots, N-1\}. \end{cases}$$

In other words, equality holds in (2.17) *if and only if*

$$\mathbf{H}^*(\mathbb{D}\psi) = \mu \nabla\psi.$$

2.3.3 Basic differential operators on spinor fields

We now apply the results of Section 2.3.2 to the spinor bundle ΣM of an n-dimensional Riemannian spin manifold (M^n, g). This will lead to two natural differential operators, the Dirac and the Penrose operators. We then examine their basic properties and establish their conformal covariance. More precisely, we take $\mathbf{U} = \mathbf{W} = \Sigma M$, and ∇ the Levi-Civita connection acting on sections of ΣM. The morphism $\mathbf{H} = \gamma|_{\mathrm{T}^*M \otimes \Sigma M}$, denoted by the same symbol γ, where

$$\begin{aligned} \gamma \colon \mathbb{C}\mathrm{l}(M) \otimes \Sigma M &\longrightarrow \Sigma M, \\ \sigma \otimes \Psi &\longmapsto \sigma \cdot \Psi, \end{aligned}$$

is the pointwise Clifford multiplication. In this setup, the operator $\mathbb{D}$ will be denoted by $\mathcal{D}$. We have

Definition 2.25. The *Dirac operator* is the first-order differential operator acting on sections of the spinor bundles, given by

$$\mathcal{D} := \gamma \circ \nabla.$$

Locally, on an open set $U \subset M$, we get

$$\begin{aligned} \mathcal{D}\colon \Gamma(\Sigma M) &\xrightarrow{\nabla} \Gamma(\mathrm{T}^* M \otimes \Sigma M) \xrightarrow{\gamma} \Gamma(\Sigma M), \\ \Psi &\longmapsto \sum_{i=1}^{n} e_i^* \otimes \nabla_{e_i} \Psi \longmapsto \sum_{i=1}^{n} e_i \cdot \nabla_{e_i} \Psi, \end{aligned}$$

where $\{e_1, \ldots, e_n\} \in \Gamma_U(P_{\mathrm{SO}_n} M)$ is a local orthonormal frame of the tangent bundle and $\{e_1^*, \ldots, e_n^*\}$ the dual frame.

Lemma 2.26. *The metric adjoint*

$$\gamma^*\colon \Sigma M \longrightarrow \mathrm{T}^* M \otimes \Sigma M$$

of the Clifford multiplication γ *is given, on any element* $\Psi \in \Sigma M$*, by*

$$\gamma^*(\Psi) = -\sum_{i=1}^{n} e_i \otimes e_i \cdot \Psi,$$

and

$$\gamma \circ \gamma^* = n \mathrm{Id}_{\Sigma M}.$$

Proof. Since Clifford multiplication is pointwise and bilinear, it is sufficient to make use of Corollary 1.36, to get for all $\Psi, \Phi \in \Sigma M$ and $X \in \mathrm{T}M$, the relations

$$\begin{aligned} \langle \gamma(X \otimes \Phi), \Psi \rangle &= \langle X \cdot \Phi, \Psi \rangle \\ &= -\langle \Phi, X \cdot \Psi \rangle \\ &= -\sum_{i=1}^{n} X^i \langle \Phi, e_i \cdot \Psi \rangle \\ &= \Big\langle X \otimes \Phi, -\sum_{i=1}^{n} e_i \otimes e_i \cdot \Psi \Big\rangle. \end{aligned}$$

□

From Lemma 2.26 it follows that the endomorphism $\gamma \circ \gamma^*$ has n as the unique positive eigenvalue and hence, the suspension $\widetilde{\mathcal{D}}$ of the Dirac operator is given, for any tangent vector field X and any spinor field Ψ, by

$$\widetilde{\mathcal{D}}_X \Psi = -\frac{1}{n} X \cdot \mathcal{D}\Psi,$$

and the Penrose (or twistor) operator $\mathcal{P}$, i.e., the complement of the Dirac operator, by

$$\mathcal{P}_X \Psi = \nabla_X \Psi + \frac{1}{n} X \cdot \mathcal{D}\Psi.$$

From Lemma 2.24 it follows that for any spinor field Ψ, one has

$$|\nabla\Psi|^2 = |\mathcal{P}\Psi|^2 + \frac{1}{n}|\mathcal{D}\Psi|^2. \tag{2.18}$$

2.3.4 The Dirac operator: basic properties and examples

We saw that associated with a spin structure of a Riemannian manifold (M^n, g), there are three essential data sets.

i) The spinor bundle

$$\Sigma M = P_{\mathrm{Spin}_n} M \times_{\rho_n} \Sigma_n$$

with the Clifford multiplication

$$\begin{aligned} \gamma\colon \mathrm{T}M \otimes \Sigma M &\longrightarrow \Sigma M, \\ X \otimes \Psi &\longmapsto X \cdot \Psi := \chi_n(X)\Psi, \end{aligned}$$

where χ_n is the Clifford multiplication and ρ_n is the spinor representation defined in (1.33). This multiplication extends to

$$\begin{aligned} \gamma\colon \Lambda^p(\mathrm{T}M) \otimes \Sigma M, &\longrightarrow \Sigma M, \\ \alpha \otimes \Psi &\longmapsto \sum_{1 \le i_1 < \cdots < i_p \le n} \alpha_{i_1 \ldots i_p} e_{i_1} \cdots e_{i_p} \cdot \Psi, \end{aligned}$$

where locally

$$\alpha = \sum_{1 \le i_1 < \cdots < i_p \le n} \alpha_{i_1 \ldots i_p} e^*_{i_1} \wedge \cdots \wedge e^*_{i_p},$$

and $e^*_i = g(e_i, \cdot)$ is the dual basis of e_i.

ii) The natural Hermitian product $\langle \cdot, \cdot \rangle$ on sections of ΣM.

iii) The Levi-Civita connection ∇ on ΣM.

As we saw, these data satisfy the compatibility conditions

$$\begin{aligned} \langle X \cdot \Psi, \Phi \rangle + \langle \Psi, X \cdot \Phi \rangle &= 0, \\ X \langle \Psi, \Phi \rangle - \langle \nabla_X \Psi, \Phi \rangle - \langle \Psi, \nabla_X \Phi \rangle &= 0, \end{aligned}$$

and

$$\nabla_X (Y \cdot \Psi) - \nabla_X Y \cdot \Psi - Y \cdot \nabla_X \Psi = 0,$$

for all $X, Y \in \Gamma(\mathrm{T}M)$, $\Psi, \Phi \in \Gamma(\Sigma M)$.

Lemma 2.27. *The commutator of the Dirac operator with the action, by pointwise multiplication on the spinor bundle, of a function*

$$f\colon M \longrightarrow \mathbb{C},$$

is given by

$$[\mathcal{D}, f]\Psi := df \cdot \Psi, \quad \Psi \in \Gamma(\Sigma M).$$

Proof. A local calculation shows that

$$\begin{aligned}
[\mathcal{D}, f]\Psi &= (\mathcal{D} f - f\mathcal{D})\Psi \\
&= \sum_{i=1}^{n} e_i \cdot \nabla_{e_i}(f\Psi) - f\mathcal{D}\Psi \\
&= \sum_{i=1}^{n} df(e_i)e_i \cdot \Psi + f\mathcal{D}\Psi - f\mathcal{D}\Psi \\
&= df \cdot \Psi.
\end{aligned}$$

□

Lemma 2.28. *The Dirac operator is a first-order differential operator, which is*

(i) *elliptic and*

(ii) *formally self-adjoint on compactly-supported spinors with respect to the L^2 inner product $\int_M \langle\cdot,\cdot\rangle v_g$, where v_g denotes the volume element.*

Proof. (i) Let $x \in M$, $\xi \in \mathrm{T}_x^*M \setminus \{0\}$ and $f \in \mathcal{C}^\infty(M, \mathbb{R})$ such that $(df)_x = \xi$. Then the *principal symbol*

$$\sigma_\xi(\mathcal{D})\colon \Sigma_x M \longrightarrow \Sigma_x M$$

is given by

$$\begin{aligned}
\sigma_\xi(\mathcal{D})\big(\Psi(x)\big) &:= \mathcal{D}[(f - f(x))\Psi](x) \\
&= (f\mathcal{D}\Psi + df \cdot \Psi - f(x)\mathcal{D}\Psi)(x) \\
&= (df)_x \cdot \Psi(x) \\
&= \xi \cdot \Psi(x),
\end{aligned}$$

i.e., $\sigma_\xi(\mathcal{D})$ is Clifford multiplication by ξ. To see that $\mathcal{D}$ is elliptic, we have to check that, for all $\xi \in \mathrm{T}^*M \setminus \{0\}$, $\sigma_\xi(\mathcal{D})\colon \Sigma_x M \to \Sigma_x M$ is an isomorphism. Indeed,

$$\xi \cdot \Psi = 0 \implies \xi \cdot \xi \cdot \Psi = 0 \implies -\|\xi\|^2\Psi = 0 \implies \Psi = 0.$$

(ii) Let Φ, Ψ be smooth spinors on M, one of them (say Φ) with compact support. To show that $\mathcal{D}$ is formally self-adjoint, we choose *normal coordinates* at $x \in M$, i.e.,

$$(\nabla_{e_i} e_j)(x) = 0, \quad 1 \leq i, j \leq n,$$

and compute first

$$\begin{aligned}
\langle \mathcal{D}\Psi, \Phi \rangle &= \Big\langle \sum_{i=1}^n e_i \cdot \nabla_{e_i} \Psi, \Phi \Big\rangle \\
&= -\sum_{i=1}^n \langle \nabla_{e_i} \Psi, e_i \cdot \Phi \rangle \\
&= -\sum_{i=1}^n [e_i \langle \Psi, e_i \cdot \Phi \rangle - \langle \Psi, \nabla_{e_i}(e_i \cdot \Phi) \rangle] \\
&= -\sum_{i=1}^n e_i \langle \Psi, e_i \cdot \Phi \rangle + \langle \Psi, \mathcal{D}\Phi \rangle.
\end{aligned}$$

Here, we note that the sum on the right-hand side is the divergence of a complex vector field. To see this, consider the two vector fields $X_1, X_2 \in \Gamma(\mathrm{T}M)$ defined for all $Y \in \mathrm{T}M$ by

$$g(X_1, Y) + i g(X_2, Y) = \langle \Psi, Y \cdot \Phi \rangle.$$

Clearly, X_1, X_2 have compact support. Therefore,

$$\begin{aligned}
\operatorname{div} X_1 + i \operatorname{div} X_2 &= \sum_{k=1}^n g(\nabla_{e_k} X_1, e_k) + i \sum_{l=1}^n g(\nabla_{e_l} X_2, e_l) \\
&= \sum_{k=1}^n (e_k g(X_1, e_k) - g(X_1, \nabla_{e_k} e_k)) \\
&\qquad + i \sum_{l=1}^n (e_l g(X_2, e_l) - g(X_2, \nabla_{e_l} e_l)) \\
&= \sum_{k=1}^n e_k (g(X_1, e_k) + i g(X_2, e_k)) \\
&= \sum_{k=1}^n e_k \langle \psi, e_k \cdot \varphi \rangle.
\end{aligned}$$

Finally we have

$$\langle \mathcal{D}\Psi, \Phi\rangle = -\operatorname{div} X_1 - i \operatorname{div} X_2 + \langle \Psi, \mathcal{D}\Phi\rangle.$$

This equation no longer depends on the choice of the coordinates, so we can integrate over M and use Stokes theorem for the compactly supported vector fields X_1, X_2 to obtain

$$\int_M \langle \mathcal{D}\Psi, \Phi\rangle v_g = \int_M \langle \Psi, \mathcal{D}\Phi\rangle v_g,$$

since M has no boundary. □

Lemma 2.29. *Let $n = 2m$. Then*

(i) $\mathcal{D}\colon \Gamma(\Sigma^{\pm} M) \to \Gamma(\Sigma^{\mp} M)$*, i.e., the Dirac operator exchanges positive and negative spinors and*

(ii) *the eigenvalues of $\mathcal{D}$ are symmetric with respect to the origin.*

Proof. (i) Recall that

$$\Sigma_n^{\pm} := \frac{1}{2}(1 \pm \omega^{\mathbb{C}}) \cdot \Sigma_n,$$

so Σ_n^+ is the subspace on which multiplication by $\omega^{\mathbb{C}}$ is the identity, and Σ_n^- the one on which the multiplication by $\omega^{\mathbb{C}}$ is minus the identity. We therefore get, for $\Psi^+ \in \Gamma(\Sigma^+ M)$,

$$\begin{aligned}
\omega^{\mathbb{C}} \cdot \mathcal{D}\Psi^+ &= \omega^{\mathbb{C}} \cdot \sum_{i=1}^{n} e_i \cdot \nabla_{e_i} \Psi^+ \\
&= -\sum_i e_i \cdot \omega^{\mathbb{C}} \cdot \nabla_{e_i} \Psi^+ \\
&= -\sum_{i=1}^{n} e_i \cdot \nabla_{e_i} (\omega^{\mathbb{C}} \cdot \Psi^+) \\
&= -\mathcal{D}\Psi^+.
\end{aligned}$$

(ii) Let Ψ be an eigenspinor for $\mathcal{D}$, i.e., $\mathcal{D}\Psi = \lambda\Psi$ for $\lambda \in \mathbb{R}$, with $\Psi = \Psi^+ + \Psi^-$. Then $\mathcal{D}\Psi^+ + \mathcal{D}\Psi^- = \lambda\Psi^- + \lambda\Psi^+$, yields with i): $\mathcal{D}\Psi^{\pm} = \lambda\Psi^{\mp}$. So the spinor $\bar{\Psi} := \Psi^+ - \Psi^-$ is an eigenspinor of $\mathcal{D}$ associated with the eigenvalue $-\lambda$, since

$$\mathcal{D}\bar{\Psi} = \mathcal{D}(\Psi^+ - \Psi^-) = \lambda\Psi^- - \lambda\Psi^+ = -\lambda(\Psi^+ - \Psi^-) = -\lambda\bar{\Psi}.$$ □

Examples 2.30. (i) For $M = \mathbb{R}^n$, $\Sigma\mathbb{R}^n = \mathbb{R}^n \times \mathbb{C}^N$, $N = 2^{[\frac{n}{2}]}$, so every spinor $\Psi \in \Gamma(\Sigma\mathbb{R}^n)$ is in fact a map $\Psi\colon \mathbb{R}^n \to \mathbb{C}^N$. The Dirac operator is given by

$$\mathcal{D} = \sum_{i=1}^{n} e_i \cdot \partial_i,$$

and acts on differentiable maps from $\mathbb{R}^n$ to $\mathbb{C}^N$, where $\partial_i = \nabla_{e_i}$. Then

$$\begin{aligned}
\mathcal{D}^2 &= \Big(\sum_{i=1}^{n} e_i \cdot \partial_i\Big)\Big(\sum_{j=1}^{n} e_j \cdot \partial_j\Big) \\
&= \sum_{i,j} e_i \cdot e_j \cdot \partial_i \partial_j \\
&= -\sum_i \partial_i^2 + \sum_{i<j} e_i \cdot e_j \cdot \partial_i \partial_j + \sum_{i>j} e_i \cdot e_j \cdot \partial_i \partial_j \\
&= -\sum_i \partial_i^2 + \sum_{i<j} e_i \cdot e_j \cdot \partial_i \partial_j + \sum_{i<j} e_j \cdot e_i \cdot \partial_j \partial_i \\
&= -\sum_i \partial_i^2 + \sum_{i<j} e_i \cdot e_j \cdot (\partial_i \partial_j - \partial_j \partial_i) \\
&= -\sum_{i=1}^{n} \partial_i^2 \\
&= \begin{pmatrix} \Delta & & \\ & \ddots & \\ & & \Delta \end{pmatrix}.
\end{aligned}$$

(ii) In the case $M = \mathbb{R}^2$, we have $\mathbb{C}\mathrm{l}_2 = \mathbb{C}(2)$, the complex volume element is $\omega^{\mathbb{C}} = i e_1 \cdot e_2$, and one can identify the spinor bundle $\Sigma_2 = \Sigma_2^+ \oplus \Sigma_2^- = \mathbb{C} \oplus \mathbb{C}$, with $\Sigma_2^+ = \mathrm{span}_{\mathbb{C}}(e_1 + i e_2)$ and $\Sigma_2^- = \mathrm{span}_{\mathbb{C}}(1 - i e_1 \cdot e_2)$. Then, each spinor field $\Psi \in \Gamma(\Sigma M)$ is given by two complex functions $f, g\colon \mathbb{R}^2 \to \mathbb{C}$, such that

$$\Psi = f(e_1 + i e_2) + g(1 - i e_1 \cdot e_2).$$

The Dirac operator acting on Ψ, is then

$$\begin{aligned}
\mathcal{D}\Psi &= (e_1 \cdot \partial_1 + e_2 \cdot \partial_2)[(e_1 + i e_2) f + (1 - i e_1 \cdot e_2) g] \\
&= -(\partial_1 + i \partial_2) f (1 - i e_1 \cdot e_2) + (\partial_1 - i \partial_2) g (e_1 + i e_2) \\
&= 2(-\partial_{\bar z} f (1 - e_1 \cdot e_2) + \partial_z g (e_1 + i e_2)),
\end{aligned}$$

where $\partial_{\bar z} = \frac{1}{2}(\partial_1 + i\partial_2)$ and $\partial_z = \frac{1}{2}(\partial_1 - i\partial_2)$. That is,

$$\mathcal{D} = \begin{pmatrix} 0 & 2\partial_z \\ -2\partial_{\bar z} & 0 \end{pmatrix}$$

in the basis $\{e_1 + ie_2, 1 - ie_1 \cdot e_2\}$ of Σ_2. Hence the Dirac operator $\mathcal{D}$ can be considered as a generalization of the Cauchy–Riemann operator.

(iii) The *Clifford bundle* $\mathrm{Cl}(M)$. For a Riemannian manifold (M^n, g), recall that the vector bundle $\mathrm{Cl}(M)$ defined by

$$(\mathrm{Cl}(M))_x = \mathrm{Cl}(\mathrm{T}_x M, g_x)$$

can be seen as a vector bundle associated to $P_{\mathrm{SO}_n}M$. Indeed, by the universal property (1.7), one can extend the standard representation

$$\mathrm{SO}_n \longrightarrow \mathrm{Aut}(\mathbb{R}^n) \quad \text{to} \quad \iota\colon \mathrm{SO}_n \longrightarrow \mathrm{Aut}(\mathrm{Cl}_n),$$

so that

$$\mathrm{Cl}(M) = P_{\mathrm{SO}_n}M \times_\iota \mathrm{Cl}_n\,.$$

From (1.13) we have

$$v \cdot \Psi \simeq v \wedge \Psi - v \lrcorner \Psi$$

under the isomorphism $\mathrm{Cl}(\mathrm{T}_x M, g_x) \xrightarrow{\simeq} \Lambda^*(\mathrm{T}_x M)$. The differential and its adjoint can be locally written as

$$d = \sum_{i=1}^n e_i \wedge \nabla_{e_i} \quad \text{and} \quad \delta = -\sum_{i=1}^n e_i \lrcorner \nabla_{e_i}\,.$$

If we define the Dirac operator as before, we have

$$\mathcal{D} := \sum_{i=1}^n e_i \cdot \nabla_{e_i} \simeq d + \delta.$$

This is the “square root” of the Laplacian, since on $\Lambda^*(\mathrm{T}M)$

$$\mathcal{D}^2 \simeq (d+\delta)^2 = d\delta + \delta d = \Delta.$$

2.3.5 Conformal covariance of the Dirac and Penrose operators

Let (M^n, g) be a Riemannian spin manifold. For a conformal change of the metric

$$\bar{g} := e^{2u} g, \quad u\colon M \longrightarrow \mathbb{R},$$

the given spin structure on (M^n, g) induces a spin structure on $(M^n, \overline{g})$. An isomorphism of the two SO_n-principal fibre bundles is given by

$$\begin{aligned} G_u\colon \qquad \mathrm{SO}_g\, M &\longrightarrow \mathrm{SO}_{\bar{g}}\, M, \\ \{X_1, \dots, X_n\} &\longmapsto \{e^{-u} X_1, \dots, e^{-u} X_n\}. \end{aligned}$$

Note that in this section we denote by $\mathrm{SO}_g\, M$ (resp. $\mathrm{Spin}_g M$), the SO_n-principal (resp. Spin_n-principal) bundle associated with (M^n, g) instead of $P_{\mathrm{SO}_n} M$ (resp. $P_{\mathrm{Spin}_n} M$).

The spin structure induced on $(M^n, \bar{g})$ is defined up to isomorphism by the commutative diagram

$$\begin{tikzcd} \mathrm{Spin}_g \arrow[r, "\tilde{G}_u"] \arrow[d, "\eta_g"'] & \mathrm{Spin}_{\bar{g}} M \arrow[d, "\eta_{\bar{g}}"] \\ \mathrm{SO}_g\, M \arrow[r, "G_u"'] & \mathrm{SO}_{\bar{g}}\, M. \end{tikzcd}$$

All the arrows in the diagram are supposed to be invariant under the group action. Let $\rho_n\colon \mathrm{Spin}_n \to \mathrm{Aut}(\Sigma_n)$ be the spinor representation. An isomorphism of the associated spinor bundles is explicitly given by

$$\begin{aligned} \tilde{\mathbf{G}}_u\colon \Sigma M = \mathrm{Spin}_g M \times_{\rho_n} \Sigma_n &\longrightarrow \Sigma \bar{M} = \mathrm{Spin}_{\bar{g}} M \times_{\rho_n} \Sigma_n, \\ \Psi = [s, \psi] &\longmapsto \bar{\Psi} = [\tilde{G}_u(s), \psi]. \end{aligned}$$

This map is an isometry with respect to the Hermitian product on the spinor bundles. Together with the corresponding isometry of the tangent bundle, given by $X \mapsto \overline{X} := e^{-u} X$, we have the following relation between the associated Clifford multiplications "$\cdot$" and "$\bar{\cdot}$"

$$\overline{X}\, \bar{\cdot}\, \overline{\Psi} = \overline{X \cdot \Psi}$$

for every $X \in \Gamma(\mathrm{T}M)$ and $\Psi \in \Gamma(\Sigma M)$.

We are ready now to prove explicitly the conformal covariance of the Dirac operator, established by N. Hitchin [Hit74] (see also [Bau81] and [Hij86b]).

Proposition 2.31. *Let (M^n, g) be an n-dimensional Riemannian spin manifold and $\bar{g} = e^{2u} g$ a conformal change of the metric, then one has*

$$\bar{\mathcal{D}}(e^{-\frac{n-1}{2}u}\bar{\Psi}) = e^{-\frac{n+1}{2}u}\overline{\mathcal{D}\Psi} \tag{2.19}$$

for every $\Psi \in \Gamma(\Sigma M)$.

Remark 2.32. It follows therefore, that the dimension of the space of harmonic spinors

$$\mathcal{H} := \{\Psi \in \Gamma(\Sigma M);\ \mathcal{D}\Psi = 0\}$$

is conformally invariant. However, in contrast to the Laplace–Beltrami operator on p-forms, the dimension of $\mathcal{H}$ for $\mathcal{D}$ is not topologically invariant (see [Hit74] and [Bär96]).

In order to prove Proposition 2.31, we start by relating the Levi-Civita connections associated with two conformally related metrics:

Proposition 2.33. *Let (M^n, g) be a Riemannian spin manifold and consider a conformal change of the metric given by $\bar{g} = e^{2u} g$. Then*

$$\bar{\nabla}_X \bar{\Psi} = \overline{\nabla_X \Psi} - \frac{1}{2}\overline{X \cdot du \cdot \Psi} - \frac{1}{2}X(u)\bar{\Psi} \tag{2.20}$$

for all $\Psi \in \Gamma(\Sigma M)$, $X \in \Gamma(\mathrm{T}M)$.

Proof. This is a direct consequence of Theorem 2.21 applied to the Weyl structure $D = \bar{\nabla}$ acting on spinors of weight 0 whose Lee form with respect to g is $\theta = du$. Nevertheless, we will also provide a direct proof for the convenience of the reader.

By the Koszul formula and the fact that ∇ and $\bar{\nabla}$ are torsion free, their connection forms with respect to local g-orthonormal bases $u := \{X_1, \dots, X_n\}$ and $\bar{u} := \{\bar{X}_1, \dots, \bar{X}_n\}$, are

$$\bar{\omega}_{ij}(X_k) = \omega_{ij}(X_k) + X_i(u)\delta_{jk} - X_j(u)\delta_{ik}, \quad i, j, k = 1, \dots, n.$$

Let $\tilde{u}$ be a local section of $\mathrm{Spin}_g M$ such that $\eta_g(\tilde{u}) = u$. Then $\widetilde{G}_u(\tilde{u})$ is a local section of $\mathrm{Spin}_{\bar{g}} M$ such that $\eta_{\bar{g}}(\widetilde{G}_u(\tilde{u})) = \bar{u}$. Consider the local spinor $\Psi := [\tilde{u}, \psi] \in \Gamma(\Sigma M)$ defined by some constant vector $\psi \in \Sigma_n$ and the corresponding spinor $\bar{\Psi} := [\widetilde{G}_u(\tilde{u}), \psi] \in \Gamma(\Sigma\bar{M})$. Theorem 2.11 shows that

$$\begin{aligned}\bar{\nabla}_{X_k}\bar{\Psi} &= \frac{1}{4}\sum_{i,j}\bar{\omega}_{ij}(X_k)\overline{X_i \cdot X_j \cdot \Psi}\\ &= \overline{\nabla_{X_k}\Psi} - \frac{1}{2}\overline{X_k \cdot \mathrm{grad}(u) \cdot \Psi} - \frac{1}{2}X_k(u)\bar{\Psi}.\end{aligned}$$

□

Lemma 2.34. *The Dirac operator associated with the metric $\bar{g}$ satisfies*

$$\bar{\mathcal{D}}(f\bar{\Psi}) = f\bar{\mathcal{D}}\bar{\Psi} + e^{-u}\overline{df \cdot \Psi}. \tag{2.21}$$

Proof.

$$\begin{aligned}\bar{\mathcal{D}}(f\bar{\Psi}) &= \sum_i \bar{X}_i \,\bar{\cdot}\, \bar{\nabla}_{\bar{X}_i}(f\bar{\Psi}) \\ &= \sum_i \bar{X}_i \,\bar{\cdot}\, [\bar{X}_i(f)\bar{\Psi} + f\bar{\nabla}_{\bar{X}_i}\bar{\Psi}] \\ &= f\bar{\mathcal{D}}\bar{\Psi} + e^{-u}\overline{df \cdot \Psi}.\end{aligned}$$

Recall that $\bar{X}_i = e^{-u}X_i$. □

Lemma 2.35. *The Dirac operators $\mathcal{D}$ and $\bar{\mathcal{D}}$ are related by*

$$\bar{\mathcal{D}}\bar{\Psi} = e^{-u}\Big[\overline{\mathcal{D}\Psi} + \frac{n-1}{2}\overline{du \cdot \Psi}\Big]. \tag{2.22}$$

Proof. We have

$$\begin{aligned}\bar{\mathcal{D}}\bar{\Psi} &= \sum_i \bar{X}_i \,\bar{\cdot}\, \bar{\nabla}_{\bar{X}_i}\bar{\Psi} \\ &= e^{-u}\sum_i \bar{X}_i \,\bar{\cdot}\, \bar{\nabla}_{X_i}\bar{\Psi} \\ &= e^{-u}\sum_i \bar{X}_i \,\bar{\cdot}\Big[\overline{\nabla_{X_i}\Psi} - \frac{1}{2}\overline{X_i \cdot \mathrm{grad}(u) \cdot \Psi} - \frac{1}{2}X_i(u)\bar{\Psi}\Big] \\ &= e^{-u}\Big[\overline{\mathcal{D}\Psi} + \frac{n-1}{2}\overline{du \cdot \Psi}\Big]\end{aligned}$$

□

Proof of Proposition 2.31. Let

$$f := e^{-\frac{n-1}{2}u}.$$

Then

$$\begin{aligned}\bar{\mathcal{D}}(f\bar{\Psi}) &= f\bar{\mathcal{D}}\bar{\Psi} + e^{-u}\overline{df \cdot \Psi} \\ &= fe^{-u}\overline{\mathcal{D}\Psi} + fe^{-u}\frac{n-1}{2}\overline{du \cdot \Psi} + e^{-u}\overline{df \cdot \Psi} \\ &= fe^{-u}\overline{\mathcal{D}\Psi},\end{aligned}$$

since

$$df = -\frac{n-1}{2}f\,du.$$

□

We now show that the Penrose operator is also conformally covariant [Lic88]. More precisely, we have the following proposition.

Proposition 2.36. *For any spinor field Ψ, the Penrose operators $\mathcal{P}$ and $\overline{\mathcal{P}}$ associated with the conformally related metrics g and $\bar{g} = e^{2u} g$ satisfy*

$$\overline{\mathcal{P}}(e^{\frac{u}{2}}\overline{\Psi}) = e^{\frac{u}{2}}\overline{\mathcal{P}\Psi}. \tag{2.23}$$

Proof. By the definition of the Penrose operator, for any tangent vector field X,

$$\overline{\mathcal{P}}_X(e^{\frac{u}{2}}\overline{\Psi}) = \overline{\nabla}_X(e^{\frac{u}{2}}\overline{\Psi}) + \frac{1}{n} X \,\bar{\cdot}\, \overline{\mathcal{D}}(e^{\frac{u}{2}}\overline{\Psi}). \tag{2.24}$$

By (2.20),

$$\begin{aligned} \overline{\nabla}_X(e^{\frac{u}{2}}\overline{\Psi}) &= e^{\frac{u}{2}}\Big[\overline{\nabla}_X\overline{\Psi} + \frac{1}{2}X(u)\overline{\Psi}\Big] \\ &= e^{\frac{u}{2}}\Big[\overline{\nabla_X\Psi} - \frac{1}{2}\overline{X\cdot du\cdot\Psi}\Big]. \end{aligned} \tag{2.25}$$

On the other hand, using (2.21) and (2.22) it follows that

$$\begin{aligned} \overline{\mathcal{D}}(e^{\frac{u}{2}}\overline{\Psi}) &= e^{\frac{u}{2}}\overline{\mathcal{D}}\overline{\Psi} + \frac{1}{2}e^{-\frac{u}{2}}\overline{du\cdot\Psi} \\ &= e^{-\frac{u}{2}}\Big[\overline{\mathcal{D}\Psi} + \frac{n-1}{2}\overline{du\cdot\Psi}\Big] + \frac{1}{2}e^{-\frac{u}{2}}\overline{du\cdot\Psi} \\ &= e^{-\frac{u}{2}}\Big[\overline{\mathcal{D}\Psi} + \frac{n}{2}\overline{du\cdot\Psi}\Big]. \end{aligned} \tag{2.26}$$

We get (2.23) by inserting (2.25) and (2.26) into (2.24) and using the fact that for any tangent vector field X and spinor field Φ, the following relation holds

$$X \,\bar{\cdot}\, \overline{\Phi} = e^{u}\overline{X\cdot\Phi}.$$ □

2.3.6 Conformally covariant powers of the Dirac operator

Let us mention without details a recent generalization [GMP12] of the conformal covariance of the Dirac operator. Namely, on a Riemannian spin manifold (M, g) of dimension n there exist conformally covariant differential operators $\mathcal{L}_{2k+1}$ of odd order $2k+1$ for every k if n is odd, respectively for $k < n/2$ if n is even, with principal symbol equal to that of $\mathcal{D}^{2k+1}$. Moreover, for $k = 1$ the expression of the operator is explicit.

Proposition 2.37. *Let (M, g) be a Riemannian spin manifold of dimension $n \geq 3$, and denote by* Scal, Ric, *and* $\mathcal{D}$ *the scalar curvature, the Ricci curvature, and the Dirac operator of (M, g). Let γ denote the Clifford multiplication. Then the operator $\mathcal{L}_3$ defined by*

$$\mathcal{L}_3 := \mathcal{D}^3 - \frac{\gamma(d(\mathrm{Scal}))}{2(n-1)} - \frac{2\gamma \circ \mathrm{Ric} \circ \nabla}{n-2} + \frac{\mathrm{Scal}}{(n-1)(n-2)}\mathcal{D}$$

is a natural conformally covariant differential operator:

$$e^{\frac{n+3}{2}u}\overline{\mathcal{L}_3}\,\overline{\Phi} = \overline{\mathcal{L}_3(e^{\frac{n-3}{2}u}\Phi)}$$

if $\overline{\mathcal{L}_3}$ is defined in terms of the conformal metric $\bar{g} = e^{2u}g$. The Ricci tensor is identified via g with an endomorphism of the cotangent bundle and acts on $\mathrm{T}^*M \otimes \Sigma M$ *in the above formula by* $\mathrm{Ric}(\alpha \otimes \Psi) := \mathrm{Ric}(\alpha) \otimes \Psi$.

These conformally covariant operators arise as residues of a certain meromorphic family of pseudodifferential operators on M, the scattering operators for the Dirac operator on some Poincaré-Einstein manifold X with conformal infinity M. Since we do not define these metrics in this book, we refer to Section 4 in [GMP12] for the proofs.

2.4 Spinors in classical geometrical contexts

In this section we obtain relations between the covariant derivatives acting on spinors on two Spin^c manifolds M and N, where N is

(1) a hypersurface of M or

(2) a warped Riemannian product $N = M \times_f B$, where B is some other Spin^c manifold and f is a function on B or

(3) the total space of a Riemannian submersion $N \to M$ with totally geodesic one-dimensional fibers.

2.4.1 Restrictions of spinors to hypersurfaces

Let (M^n, g) be a Spin^c manifold with auxiliary line bundle L and let N be an oriented hypersurface of M endowed with the Riemannian metric g^N induced by g. We denote by V the unitary vector field compatible with the orientations of M and N which spans the normal bundle of N in M (compatibility means that $(X_1, \ldots, X_{n-1}, V)$ is a positively oriented basis of $\mathrm{T}M$ for every positively oriented basis $(X_1, \ldots, X_{n-1})$ of $\mathrm{T}N$). Let ι be the inclusion of N in M.

Lemma 2.38. *There exists a* Spin^c *structure on* N *canonically induced by that of* M*, whose auxiliary line bundle is the restriction of* L *to* N*. For* n *even (resp. odd), there exists an identification of the pull-back bundle* $\iota^*(\Sigma^+ M)$ *(resp.* $\iota^*(\Sigma M)$*) with* ΣN *such that the Clifford product on* ΣN *satisfies*

$$X \cdot \iota^* \Psi = \iota^*(X \cdot V \cdot \Psi), \quad X \in \mathrm{T}N. \tag{2.27}$$

The notation ι^* *stands for the restriction of objects – spinors, bundles, etc. – from* M *to* N*.*

Proof. The isomorphism $\mathrm{Cl}_{n-1} \to \mathrm{Cl}_n^0$ given by Proposition 1.11 induces by complexification an isomorphism $\mathbb{C}\mathrm{l}_{n-1} \to \mathbb{C}\mathrm{l}_n^0$. This identifies Spin_{n-1}^c with the inverse image of $\mathrm{SO}_{n-1} \times \mathrm{U}_1$ by $\xi^c \colon \mathrm{Spin}_n^c \to \mathrm{SO}_n \times \mathrm{U}_1$. We obtain in this way a complex representation ρ of $\mathbb{C}\mathrm{l}_{n-1}$ on Σ_n which satisfies $\rho(e_i) = \rho_n(e_i \cdot e_n)$, for any $1 \le i \le n-1$.

For n odd, Theorem 1.28 shows that this representation is isomorphic to $(\rho_{n-1}, \Sigma_{n-1})$.

For n even, ρ preserves the decomposition $\Sigma_n = \Sigma_n^+ \oplus \Sigma_n^-$ (see the proof of Proposition 1.32) and its restriction to Σ_n^+ is isomorphic to $(\rho_{n-1}^+, \Sigma_{n-1})$ since they have the same dimension and $\rho(\omega_{n-1}^{\mathbb{C}}) = \rho_n(\omega_n^{\mathbb{C}}) = \mathrm{Id}_{\Sigma_n^+}$ (easy computation).

The map $(X_1, \dots, X_{n-1}) \mapsto (X_1, \dots, X_{n-1}, V)$ identifies $P_{\mathrm{SO}_{n-1}} N$ with a subbundle P of $\iota^*(P_{\mathrm{SO}_n} M)$. Thus, the inverse image $P_{\mathrm{Spin}_{n-1}^c} N$ of $P \times \iota^*(P_{\mathrm{U}_1} M)$ by

$$\iota^*\theta \colon \iota^*(P_{\mathrm{Spin}_n^c} M) \longrightarrow \iota^*(P_{\mathrm{SO}_n} M \times P_{\mathrm{U}_1} M)$$

defines a Spin^c structure on N with auxiliary line bundle $\iota^* L$.

The algebraic discussion above shows that the spinor bundle associated with this Spin^c structure is isomorphic to $\iota^*(\Sigma M)$ for n odd and to $\iota^*(\Sigma^+ M)$ for n even, and that the Clifford product satisfies (2.27). □

Given a connection A on $P_{\mathrm{U}_1} M$ (inducing a covariant derivative ∇^A on ΣM), we obtain a connection $\iota^* A$ on $\iota^*(P_{\mathrm{U}_1} M)$ inducing, together with the Levi-Civita connection of N, a covariant derivative, denoted $\nabla^{\iota^* A}$, on the spinors of N. Let $\Psi = [\tilde{u}, \psi]$ be a spinor of M, where we have chosen the local section $\tilde{u}$ of $P_{\mathrm{Spin}_n^c} M$ in such a manner that its projection by θ_1 on $P_{\mathrm{SO}_n} M$ be of the form

$$u = (X_1, \dots, X_{n-1}, V).$$

Then $\iota^* \Psi = [\tilde{u}, \iota^* \psi]$, and an easy calculation using (2.27) and Theorem 2.11 implies the following result.

Theorem 2.39. *The covariant derivative of a spinor Ψ on M and that of its restriction to N are related by*

$$\iota^*(\nabla^A_X\Psi) = \nabla^{\iota^*A}_X(\iota^*\Psi) + \frac{1}{2}\mathrm{II}(X)\cdot(\iota^*\Psi), \quad X \in \mathrm{T}N \subset \mathrm{T}M, \tag{2.28}$$

where II *is the second fundamental form of N.*

2.4.2 Spinors on warped products

We treat here a special case of warped product, namely the Riemannian cone, which will be of particular interest for us in Chapter 8. Let (M^n, g) be a Riemannian manifold and let

$$(N, g_N) := (M \times \mathbb{R}^*_+, r^2 g + dr^2)$$

be the Riemannian cone over M. We denote by $\pi\colon N \to M$ the canonical projection and identify in the sequel the vector fields on M and the *projectable* vector fields on N. In particular, every vector $X \in \mathrm{T}_x M$ induces a vector field (also denoted by X) along the geodesic line $\{x\} \times \mathbb{R}^*_+$. The following result is quite similar to Lemma 2.38:

Lemma 2.40. *A* $\mathrm{Spin^c}$ *structure on the manifold M endowed with an auxiliary line bundle L induces, in a canonical manner, a* $\mathrm{Spin^c}$ *structure on N whose auxiliary line bundle is the pull-back π^*L of L. For n even, the spinor bundle $\pi^*(\Sigma M)$ can be canonically identified with ΣN so that the Clifford product satisfies*

$$\frac{1}{r}X\cdot\partial_r\cdot(\pi^*\Psi) = \pi^*(X\cdot\Psi), \quad X \in \mathrm{T}M. \tag{2.29}$$

For n odd, the equation

$$\frac{1}{r}X\cdot\partial_r\cdot(\pi^*\Psi) = \pm\pi^*(X\cdot\Psi), \quad X \in \mathrm{T}M. \tag{2.30}$$

defines an identification of $\pi^*(\Sigma M)$ and $\Sigma^\pm N$ as $\mathbb{C}\mathrm{l}^0(N)$-modules.

Proof. The oriented orthonormal frame bundle of N can be canonically identified with the inverse image by $\pi\colon N \to M$ of the extension $P_{\mathrm{SO}_{n+1}}M$ of $P_{\mathrm{SO}_n}M$ to SO_{n+1}. This identification is given by

$$([u_x, A], r) \in \pi^*(P_{\mathrm{SO}_{n+1}}M) \longmapsto \Big(\frac{1}{r}\pi^*u_{(x,r)}, \partial_r\Big)A \in P_{\mathrm{SO}_{n+1}}N. \tag{2.31}$$

By extension of the structure group of $P_{\mathrm{Spin}^c_n}M$ to Spin^c_{n+1}, we obtain an equivariant morphism of principal bundles

$$\theta\colon P_{\mathrm{Spin}^c_{n+1}}M \longrightarrow P_{\mathrm{SO}_{n+1}}M \times P_{\mathrm{U}_1}M,$$

whose inverse image by π yields a Spin^c structure on N

$$\pi^*\theta \colon P_{\mathrm{Spin}^c_{n+1}} N \longrightarrow P_{\mathrm{SO}_{n+1}} N \times \pi^*(P_{\mathrm{U}_1} M),$$

with auxiliary line bundle $\pi^* L$. Equation (2.29) follows directly from (2.31) and from the isomorphism of Σ_n and Σ_{n+1} (resp. Σ^+_{n+1} for n odd) as $\mathbb{C}\mathrm{l}_n$-representations described by Lemma 2.38. $\square$

The covariant derivatives ∇ and $\overline{\nabla}$ defined by the Levi-Civita connections of g and g^N satisfy the warped product formulas (see [O'N83], p. 206)

$$\overline{\nabla}_{\partial_r} \partial_r = 0, \tag{2.32}$$

$$\overline{\nabla}_{\partial_r} X = \overline{\nabla}_X \partial_r = \frac{1}{r} X, \tag{2.33}$$

$$\overline{\nabla}_X Y = \nabla_X Y - r g(X, Y) \partial_r. \tag{2.34}$$

Let A be a connection on $P_{\mathrm{U}_1} M$ and ∇^A the covariant derivative induced on the spinors on M. We denote by $\pi^* A$ the induced connection on $P_{\mathrm{U}_1} N \simeq \pi^* P_{\mathrm{U}_1} M$ and by $\overline{\nabla}^A$ the covariant derivative induced by $\overline{\nabla}$ and $\pi^* A$ on the spinors on N. Using (2.29)–(2.34) and Theorem 2.11, we obtain directly the formula which relates ∇^A and $\overline{\nabla}^A$:

Theorem 2.41. *Let Ψ be a spinor field on M and $\pi^*\Psi$ (resp. $(\pi^*\Psi)^\pm$) the induced spinor field on ΣN for n even (resp. $\Sigma^\pm N$ for n odd). Then*

$$\overline{\nabla}^A_X(\pi^*\Psi) = \pi^*\Big(\nabla^A_{\pi_* X}\Psi - \frac{1}{2}(\pi_* X)\cdot\Psi\Big), \quad X \in \mathrm{T}N \ (n \textit{ even}),$$

$$\overline{\nabla}^A_X(\pi^*\Psi)^\pm = \pi^*\Big(\nabla^A_{\pi_* X}\Psi \mp \frac{1}{2}(\pi_* X)\cdot\Psi\Big), \quad X \in \mathrm{T}N \ (n \textit{ odd}).$$

and

$$\overline{\nabla}^A_{\partial_r}(\pi^*\Psi)^\pm = 0.$$

2.4.3 Spinors on Riemannian submersions

The third geometrical situation that we treat here is that of a Riemannian submersion $\pi \colon S \to M$, with totally geodesic fibers diffeomorphic to the circle $\mathbb{S}^1$, where (M^n, g) is a Spin^c manifold and $(S^{n+1}, \bar{g})$ is an oriented Riemannian manifold. By definition, π is called a Riemannian submersion if for all $s \in S$, the restriction of $\pi_* \colon \mathrm{T}_s S \to \mathrm{T}_{\pi(s)} M$ maps the orthogonal complement (with respect to $\bar{g}$) of $\mathrm{Ker}(\pi_*)$ isometrically onto $(\mathrm{T}_{\pi(s)} M, g)$.

For each $s \in S$, let $V_s \in \mathrm{T}_s S$ be the unit vertical vector (i.e., contained in $\mathrm{Ker}(\pi_*)$) compatible with the orientations of M and S (see the beginning of this section for the definition of compatibility). We obtain in this way a vector field on S, called the *standard vertical vector field*.

For each vector $X_x \in \mathrm{T}_x M$ and for each point $y \in \pi^{-1}(x)$ we define the horizontal lift $X^*_y \in \mathrm{Ker}(\pi_*)^{\perp}_y \subset \mathrm{T}_y M$ of X at y to be the inverse image of X by the above isomorphism. It is clear that every vector field X on M induces a horizontal vector field X^* on S that projects to X. The vector fields on S obtained in this way are called *standard horizontal vector fields*.

Lemma 2.42. *The fibers of π are circles of constant length, denoted by ℓ.*

Proof. Let ∇ and $\bar{\nabla}$ be the covariant derivatives on (M, g) and $(S, \bar{g})$; let X be a vector field on M and let X^* be its horizontal lift to S. Since X^* projects to X and V onto 0, it follows that $[X^*, V]$ projects to 0, in other words it is vertical. On the other hand, $\bar{g}([X^*, V], V) = -\bar{g}(\bar{\nabla}_V X^*, V) = \bar{g}(X^*, \bar{\nabla}_V V) = 0$, so finally

$$[X^*, V] = 0. \tag{2.35}$$

Let ϕ_t be the local group of diffeomorphisms of S induced by X^*. Equation (2.35) shows that ϕ_t preserves the fibers of π and satisfies $(\phi_t)_* V = V$. Consequently, the restriction of ϕ_t to the fibers of π is an isometry for each (small) t, so the fibers are isometric, since X was arbitrary. $\square$

By Lemma 2.42, the flow of $\frac{\ell}{2\pi} V$ induces a free U_1-action on S which turns π into a principal U_1-bundle. The horizontal distribution of S is invariant with respect to this action because of (2.35), so it defines a connection. We denote by F the 2-form on M with the property that $2\pi i F$ is the projection on M of the curvature form G of this connection (we identify $\mathfrak{u}_1$ and $i\mathbb{R}$, so that the curvature form can be seen as an imaginary-valued two-form on S).

Let T be the field of endomorphisms on M defined by $g(TX, Y) = F(X, Y)$ and let $i\omega \in \Omega^1(S, i\mathbb{R})$ be the connection form. As $\frac{\ell}{2\pi} V$ is the standard vertical vector field induced by $i \in \mathfrak{u}_1$ on S, we deduce that $\omega(\frac{\ell}{2\pi} V) = 1$, and consequently, $\omega(Z) = \frac{2\pi}{\ell} \bar{g}(Z, V)$, $Z \in \mathrm{T}S$.

We now derive the relationship between $\bar{\nabla}$ and ∇. It is enough to compute $\bar{g}(\bar{\nabla}_A B, C)$ for any (horizontal or vertical) standard vector fields A, B, C. First, the Koszul formula shows that

$$\bar{g}(\bar{\nabla}_{X^*} Y^*, Z^*) = g(\nabla_X Y, Z), \quad X, Y, Z \in \Gamma(\mathrm{T}M). \tag{2.36}$$

Next, we have

$$\bar{g}(\bar{\nabla}_{X^*} V, V) = \bar{g}(\bar{\nabla}_V X^*, V) = -\bar{g}(\bar{\nabla}_V V, X^*) = 0, \quad X \in \Gamma(\mathrm{T}M).$$

Because of (2.35), it only remains to compute $\bar{g}(\bar{\bar{\nabla}}_{X^*}Y^*, V)$, or, equivalently, to compute $\bar{\bar{\nabla}}_{X^*}V$.

With the notations above, for any vector fields X, Y on M

$$\begin{aligned}
2\pi i\,\bar{g}((TX)^*, Y^*) &= 2\pi i\,g(TX, Y) \\
&= 2\pi i F(X, Y) \\
&= G(X^*, Y^*) \\
&= i\,d\omega(X^*, Y^*) \\
&= -i\,\omega([X^*, Y^*]) \\
&= -\frac{2\pi i}{\ell}\bar{g}([X^*, Y^*], V) \\
&= -\frac{4\pi i}{\ell}\bar{g}(\bar{\bar{\nabla}}_{X^*}Y^*, V) \\
&= \frac{4\pi i}{\ell}\,\bar{g}(Y^*, \bar{\bar{\nabla}}_{X^*}V);
\end{aligned}$$

hence

$$\bar{\bar{\nabla}}_{X^*}V = \frac{\ell}{2}(TX)^*, \quad X \in \mathrm{T}M. \tag{2.37}$$

Conversely, let $\pi\colon S \to M$ be a principal U_1-bundle with connection form $i\omega \in \Lambda^1(S, i\mathbb{R})$ and curvature form $G \in \Lambda^2(S, i\mathbb{R})$ projecting over $2\pi i F$ for some 2-form F on M. Define a 1-parameter family of metrics on S by

$$g_S^t(X, Y) = g(\pi_*(X), \pi_*(Y)) + t^2\omega(X)\omega(Y)$$

and let ∇ and $\bar{\bar{\nabla}}^t$ denote the covariant derivatives of the Levi-Civita connections of g and g_S^t.

Denote by V^t the unit vertical vector field on S defined by $\omega(V^t) = 1/t$ and, for $X \in \mathrm{T}M$, let X^* denote its horizontal lift to $\mathrm{T}S$.

From the very construction, the projection $(S, g_S^t) \to (M, g)$ is a Riemannian submersion. Each fiber has constant length $2\pi t$, and is totally geodesic. Indeed, V^t has constant norm so $g_S^t(\bar{\bar{\nabla}}^t_{V^t}V^t, V^t) = 0$. Moreover, from the invariance of the horizontal distribution of the connection, one gets $[X^*, V^t] = 0$, for every X,

$$g_S^t(\bar{\bar{\nabla}}^t_{V^t}V^t, X^*) = -g_S^t(V^t, \bar{\bar{\nabla}}^t_{V^t}X^*) = -g_S^t(V^t, \bar{\bar{\nabla}}^t_{X^*}V^t) = 0.$$

We thus proved the following proposition.

Proposition 2.43. *Let (M, g) be an oriented Riemannian manifold. There exists a one-to-one correspondence between oriented Riemannian submersions $S \to M$ with totally geodesic one-dimensional fibers, on the one hand, and U_1-connections on principal U_1-bundles $S \to M$, on the other hand. The parameter t defining the metric on the total space of the bundle is defined by the formula $2\pi t = \ell$, where ℓ is the length of the fibers of the submersion. The the covariant derivative $\bar{\bar{\nabla}}$ on S and the curvature form G of the U_1-connection are related by*

$$g^S(\bar{\bar{\nabla}}_{X^*} V, Y^*) = \frac{t}{2i} G(X^*, Y^*), \tag{2.38}$$

where V denotes the unit vertical vector field compatible with the orientations of M and S.

Note that (2.38) is equivalent to

$$d(i\omega) = G,$$

which is just the structure equation of the connection.

We are now ready to consider the spinorial aspect of this geometrical situation. For the sake of simplicity, we will restrict ourselves to the case where the dimension of M is even, which is the only case to be considered in the sequel.

Lemma 2.44. *If M and S are as above, then the* $\mathrm{Spin}^{\mathrm{c}}$ *structure on M induces canonically a* $\mathrm{Spin}^{\mathrm{c}}$ *structure on S whose auxiliary line bundle is the pull back $\pi^* L$ of L. The associated spinor bundle ΣS can be identified with $\pi^*(\Sigma M)$, and with respect to this identification, the Clifford multiplication is given by*

$$X^* \cdot (\pi^* \Psi) = \pi^*(X \cdot \Psi), \quad X \in \mathrm{T}M \tag{2.39}$$

and

$$V \cdot (\pi^* \Psi) = i \pi^*(\bar{\Psi}), \tag{2.40}$$

where

$$\bar{\Psi} := \Psi^+ - \Psi^-.$$

Proof. We first point out that the oriented orthonormal frame bundle on S can be canonically identified with the pull-back by π of the extension $P_{\mathrm{SO}_{n+1}} M$ of $P_{\mathrm{SO}_n} M$ to SO_{n+1}. This identification is given by

$$\begin{aligned} \pi^*(P_{\mathrm{SO}_{n+1}} M) &\longrightarrow P_{\mathrm{SO}_{n+1}} S, \\ ([(X_1, \dots, X_n)_{\pi(s)}, a], s) &\longmapsto (X_1^*, \dots, X_n^*, V)_s a. \end{aligned}$$

Then, an argument similar to that used in the proof of Lemma 2.40 shows that the pull-back by π of $P_{\mathrm{U}_1}M$ and of the extension to $\mathrm{Spin}^{\mathrm{c}}_{n+1}$ of the structure group of $P_{\mathrm{Spin}^{\mathrm{c}}_n}M$ defines a $\mathrm{Spin}^{\mathrm{c}}$ structure on S.

For the second part of the lemma, we observe that the representation ρ of $\mathbb{C}\mathrm{l}_{n+1}$ on Σ_n given by

$$\rho(e_k)(\psi) := \begin{cases} e_k \cdot \psi & \text{for } 1 \le k \le n \\ i\bar{\psi} & \text{for } k = n+1 \end{cases}$$

is equivalent to (ρ_+, Σ_{n+1}) and thus defines an isomorphism between ΣS and $\pi^*(\Sigma M)$ satisfying (2.39) and (2.40). $\square$

Let A be a connection on $P_{\mathrm{U}_1}M$ with connection form $\alpha \in \Omega^1(P_{\mathrm{U}_1}M, i\mathbb{R})$ and let ∇^A be the covariant derivative that it defines on the spinors of M. We denote by π^*A the induced connection on $P_{\mathrm{U}_1}S \simeq \pi^*P_{\mathrm{U}_1}M$ (with connection form $\pi^*\alpha$) and by $\bar{\bar{\nabla}}^A$ the covariant derivative defined by $\bar{\bar{\nabla}}$ and π^*A on the spinors of S. Using (2.37)–(2.40) and Theorem 2.11 we deduce the formulas relating ∇^A and $\bar{\bar{\nabla}}^A$.

Theorem 2.45. *Let Ψ be a spinor field on M and $\pi^*\Psi$ the projectable spinor field that it induces on ΣS. Then*

$$\bar{\bar{\nabla}}^A_{X^*}(\pi^*\Psi) = \pi^*\Big(\nabla^A_X\Psi - \frac{i\ell}{4}TX \cdot \bar{\Psi}\Big), \qquad X \in \Gamma(\mathrm{T}M)$$

and

$$\bar{\bar{\nabla}}^A_V(\pi^*\Psi) = \frac{\ell}{4}\pi^*(F \cdot \Psi).$$

Proof. Let $u = (X_1, \dots, X_n)$ and γ be local sections of $P_{\mathrm{SO}_n}M$ and $P_{\mathrm{U}_1}M$ respectively and let $\tilde{u}$ be a local section of $P_{\mathrm{Spin}^{\mathrm{c}}_n}M$ such that $\theta(\tilde{u}) = (u, \gamma)$. The spinor Ψ can be written locally as $\Psi = [\tilde{u}, \psi]$, where ψ is a local function of M with values in Σ_n.

Let $v = (X_1^*, \dots, X_n^*, V)$ be the local frame of $P_{\mathrm{SO}_{n+1}}S$ induced by u and let $\delta = \pi^*\gamma$ be the local section of $P_{\mathrm{U}_1}S = \pi^*P_{\mathrm{U}_1}M$ induced by γ (which satisfies $\pi \circ \delta = \gamma$). We define a local section

$$\tilde{v} := \pi^*(\tilde{u})$$

of the $\mathrm{Spin}^{\mathrm{c}}$ structure of S which projects over $v \times \delta$. By definition, we have

$$\pi^*(\Psi) = [\tilde{v}, \psi \circ \pi]$$

and Theorem 2.11 applied to Ψ and $\pi^*(\Psi)$, together with (2.36)–(2.40) yield

$$\begin{aligned}
\bar{\nabla}^A_{X^*}(\pi^*\Psi) &= [\tilde{v}, X^*(\psi \circ \pi)] + \frac{1}{2}\sum_{i<j} \bar{g}(\bar{\nabla}_{X^*}X_i^*, X_j^*)X_i^* \cdot X_j^* \cdot (\pi^*\Psi) \\
&\quad + \frac{1}{2}\sum_i \bar{g}(\bar{\nabla}_{X^*}X_i^*, V)X_i^* \cdot V \cdot (\pi^*\Psi) + \frac{1}{2}\pi^*\alpha(\delta_* X^*)(\pi^*\Psi) \\
&= [\tilde{v}, X(\psi) \circ \pi] + \frac{1}{2}\sum_{i<j} g(\nabla_X X_i, X_j)X_i^* \cdot X_j^* \cdot (\pi^*\Psi) \\
&\quad - \frac{\ell}{4}\sum_i g(TX, X_i)X_i^* \cdot V \cdot (\pi^*\Psi) + \frac{1}{2}\alpha(\gamma_* X)(\pi^*\Psi) \\
&= \pi^*\Big([\tilde{u}, X(\psi)] + \frac{1}{2}\sum_{i<j} g(\nabla_X X_i, X_j)X_i \cdot X_j \cdot \Psi \\
&\qquad - \frac{i\ell}{4}\sum_i g(TX, X_i)X_i \cdot \bar{\Psi} + \frac{1}{2}\alpha(\gamma_* X)\Psi\Big) \\
&= \pi^*\Big(\nabla^A_X \Psi - \frac{i\ell}{4} TX \cdot \bar{\Psi}\Big).
\end{aligned}$$

The proof of the second relation is similar. □

2.5 The Schrödinger–Lichnerowicz formula

Definition 2.46 (Extension of $\langle\cdot,\cdot\rangle$ and ∇). i) Extend the natural Hermitian scalar product $\langle\cdot,\cdot\rangle$ on $\Gamma(\Sigma M)$ to sections of $\mathrm{T}^*M \otimes \Sigma M$ by

$$\begin{aligned}
\langle\cdot,\cdot\rangle\colon \Gamma(\mathrm{T}^*M \otimes \Sigma M) \times \Gamma(\mathrm{T}^*M \otimes \Sigma M) &\longrightarrow \mathcal{C}^\infty(M, \mathbb{C}), \\
\big(\alpha \otimes \Psi, \beta \otimes \Phi\big) &\longmapsto g(\alpha, \beta)\langle\Psi, \Phi\rangle,
\end{aligned}$$

where the metric g extends to covectors by means of the isomorphism $\mathrm{T}^*M \simeq \mathrm{T}M$ induced by g. Thus, for $\omega, \eta \in \Gamma(\mathrm{T}^*M \otimes \Sigma M)$, we get

$$\langle\omega, \eta\rangle = \sum_{j=1}^n \langle\omega(X_j), \eta(X_j)\rangle$$

for any orthonormal basis $\{X_1, \ldots, X_n\}$ of $\mathrm{T}_x M$.

ii) Assume that M is compact and define ∇^* to be the formal adjoint of ∇, i.e.,

$$\nabla^*\colon \Gamma(\mathrm{T}^*M \otimes \Sigma M) \longrightarrow \Gamma(\Sigma M)$$

with $\langle\nabla^*\Theta, \Phi\rangle_{L^2} = \langle\Theta, \nabla\Phi\rangle_{L^2}$ for all $\Theta \in \Gamma(\mathrm{T}^*M \otimes \Sigma M)$ and $\Phi \in \Gamma(\Sigma M)$.

Lemma 2.47. *In local normal coordinates* $(X_1, \dots, X_n)$ *at* $x \in M$*, we have*

$$\nabla^*\nabla\Psi = -\sum_{j=1}^{n} \nabla_{X_j}\nabla_{X_j}\Psi$$

for all $\Psi \in \Gamma(\Sigma M)$.

Proof. By the definition of ∇^*, for any spinor fields Ψ, Φ, we have

$$\langle\nabla^*\nabla\Psi, \Phi\rangle_{L_2} = \langle\nabla\Psi, \nabla\Phi\rangle_{L^2} = \sum_{i=1}^{n}\langle\nabla_{X_i}\Psi, \nabla_{X_i}\Phi\rangle_{L^2}.$$

Consider the two vector fields $V_1, V_2 \in \Gamma(\mathrm{T}M)$ defined for all $X \in \mathrm{T}M$ by

$$g(V_1, X) + i g(V_2, X) = \langle\Psi, X \cdot \Phi\rangle.$$

Then

$$\begin{aligned}
\operatorname{div} V_1 + i \operatorname{div} V_2 &= \sum_{k=1}^{n} g(\nabla_{X_k} V_1, X_k) + i \sum_{l=1}^{n} g(\nabla_{X_l} V_2, X_l) \\
&= \sum_{k=1}^{n}(X_k g(V_1, X_k) - g(V_1, \nabla_{X_k} X_k)) \\
&\qquad + i \sum_{l=1}^{n}(X_l g(V_2, X_l) - g(V_2, \nabla_{X_l} X_l)) \\
&= \sum_{k=1}^{n} X_k (g(V_1, X_k) + i g(V_2, X_k)) \\
&= \sum_{k=1}^{n} X_k \langle\Psi, X_k \cdot \Phi\rangle.
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{k=1}^{n}\langle\nabla_{X_k}\Psi, \nabla_{X_k}\Phi\rangle &= \sum_{k=1}^{n} X_k\langle\nabla_{X_k}\Psi, \Phi\rangle - \langle\nabla_{X_k}\nabla_{X_k}\Psi, \Phi\rangle \\
&= \operatorname{div} V_1 + i \operatorname{div} V_2 - \sum_{k=1}^{n}\langle\nabla_{X_k}\nabla_{X_k}\Psi, \Phi\rangle,
\end{aligned}$$

The sum in the last term is the divergence of a complex vector field. By integration, it gives the required condition for ∇^* to be the formal adjoint of ∇. $\square$

Proposition 2.48. *Let*

$$\widetilde{\mathcal{R}} := \frac{1}{2}\sum_{i,j=1}^{n} X_i \cdot X_j \cdot \mathcal{R}_{X_i,X_j},$$

where $\mathcal{R}$ is the spinorial curvature. The square of the Dirac operator is given by

$$\mathcal{D}^2 = \nabla^*\nabla + \widetilde{\mathcal{R}}.$$

Proof. Take normal coordinates at $x \in M$. Then

$$\begin{aligned}
\mathcal{D}^2 &= \Big(\sum_{i=1}^{n} X_i \cdot \nabla_{X_i}\Big)\Big(\sum_{j=1}^{n} X_j \cdot \nabla_{X_j}\Big)\\
&= \sum_{i,j=1}^{n} X_i \cdot X_j \cdot \nabla_{X_i}\nabla_{X_j}\\
&= -\sum_{i=1}^{n} \nabla_{X_i}\nabla_{X_i} + \sum_{i\neq j=1}^{n} X_i \cdot X_j \cdot \nabla_{X_i}\nabla_{X_j}\\
&= -\sum_{i=1}^{n} \nabla_{X_i}\nabla_{X_i} + \sum_{i<j}^{n} X_i \cdot X_j \cdot \big(\nabla_{X_i}\nabla_{X_j} - \nabla_{X_j}\nabla_{X_i}\big)\\
&= \nabla^*\nabla + \frac{1}{2}\sum_{i,j=1}^{n} X_i \cdot X_j \cdot \mathcal{R}_{X_i,X_j}\\
&= \nabla^*\nabla + \widetilde{\mathcal{R}}.
\end{aligned}$$

□

Theorem 2.49 (Schrödinger–Lichnerowicz formula). *Let* Scal *be the scalar curvature of M. Then*

$$\mathcal{D}^2 = \nabla^*\nabla + \frac{1}{4}\,\mathrm{Scal}.$$

Proof. By Proposition 2.48, it is sufficient to show that

$$\widetilde{\mathcal{R}} = \frac{1}{4}\,\mathrm{Scal}.$$

By Corollary 2.8,

$$\begin{aligned}
\widetilde{\mathcal{R}} &= \frac{1}{2}\sum_{i,j=1}^{n} X_i \cdot X_j \cdot \mathcal{R}_{X_i,X_j} \\
&= \frac{1}{2}\sum_{i=1}^{n} X_i \cdot \Big(\sum_j X_j \cdot \mathcal{R}_{X_i,X_j}\Big) \\
&= \frac{1}{2}\sum_{i=1}^{n} X_i \cdot \Big(-\frac{1}{2}\mathrm{Ric}(X_i)\cdot\Big) \\
&= \frac{1}{4}\mathrm{Scal}.
\end{aligned}$$

□

Remark 2.50. It is useful to point out that the Ricci identity (2.8),

$$-\frac{1}{2}\,\mathrm{Ric}(X)\cdot = \sum_{i=1}^{n} X_i \cdot \mathcal{R}_{X,X_i}, \quad X \in \Gamma(\mathrm{T}M), \tag{2.41}$$

can be recast as

$$-\frac{1}{2}\mathrm{Ric}\,(X)\cdot = [\nabla_X, \mathcal{D}].$$

To see this, it is sufficient to consider normal coordinates at some point so that

$$\begin{aligned}
-\frac{1}{2}\,\mathrm{Ric}(X)\cdot &= \sum_{i=1}^{n} X_i \cdot \mathcal{R}_{X,X_i} \\
&= \sum_{i=1}^{n} X_i \cdot (\nabla_X \nabla_{X_i} - \nabla_{X_i}\nabla_X) \\
&= \sum_{i=1}^{n} \big(\nabla_X \big(X_i \cdot \nabla_{X_i}\big) - X_i \cdot \nabla_{X_i}\nabla_X\big) \\
&= \nabla_X \mathcal{D} - \mathcal{D}\nabla_X.
\end{aligned}$$

Chapter 3

Topological aspects

In this chapter we discuss the interaction between spin structures and the topology of the underlying manifold. There are two sides to this: one concerns existence, and the other classification. The most natural tools to deal with these issues are based on describing principal bundles using local trivializations and cocycle conditions. We will show that spin and Spin^c structures can be described in terms of Čech cohomology groups with respect to the good cover associated to any triangulation. In particular, the corresponding cohomology groups are independent of the triangulation. We therefore do not need to worry about independence on the cover, since it follows for free.

3.1 Topological aspects of spin structures

Let M be a smooth manifold of dimension n. A k-dimensional *simplex* is an embedding in M of the standard k-simplex

$$\Delta_k = \{(x_1, \ldots, x_{k+1}) \in \mathbb{R}^{k+1};\ x_1 + \cdots + x_{k+1} = 1, x_j \geq 0 \text{ for all } j\}.$$

The hyperfaces of Δ_k (i.e., the intersections of Δ_k with some subspace $\{x_j = 0\}$) are canonically identified with Δ_{k-1}. Inductively, a simplex defines other simplices (namely, its faces) of every lower dimension. The image of a simplex $\sigma\colon \Delta_k \hookrightarrow M$ is also the image of $\sigma\circ\pi$, where the permutation $\pi \in S_{k+1}$ acts on Δ_{k+1} by permuting the coordinates. We identify all these simplices with their common image in M. An orientation on a simplex is an equivalence class of such geometric realizations modulo the alternating group A_{k+1}. A *triangulation* $\mathcal{T}$ of M is a decomposition of M into a union of n-simplices, each two of them intersecting along a sub-simplex or the empty set. Moreover, we require each vertex (i.e., 0-simplex) to belong to a finite number of n-simplices in the triangulation. More precisely, $\mathcal{T}$ is the set of n-simplices and of their faces of every dimension. A standard result of Whitehead asserts that every smooth manifold admits triangulations.

Consider a triangulation $\mathcal{T}$ of M, let $\mathcal{T}_l$ be the set of l-simplices, and Star(p) the *open star* of $p \in \mathcal{T}_0$, defined as the union of the interiors of all simplices in the triangulation containing p. After replacing the triangulation by its barycentric subdivision if necessary, the sets

$$S_j := \mathrm{Star}(p_j)$$

form a *good open cover* $\mathcal{S}$ of M, in the sense that all intersections are contractible. For every l-simplex $\sigma = (p_{j_0}, \dots, p_{j_l}) \in \mathcal{T}_j$, let

$$\mathrm{Star}(\sigma) := \bigcap_{s=0}^{l} S_{j_s}.$$

Then $S_i \cap S_j$ is empty unless $(p_i, p_j) \in \mathcal{T}_1$ is a 1-simplex in $\mathcal{T}$, in which case $S_i \cap S_j$ retracts onto the open segment (p_i, p_j). Similarly, $S_i \cap S_j \cap S_k$ is empty unless $(p_i, p_j, p_k) \in \mathcal{T}_2$ is a 2-simplex in $\mathcal{T}$, in which case $S_i \cap S_j \cap S_k$ retracts onto the open triangle (p_i, p_j, p_k). More generally, for every l-simplex σ, the open set Star(σ) retracts onto the interior of σ.

We fix in the rest of this chapter the triangulation $\mathcal{T}$ of M and the associated good cover $\mathcal{S}$. The results below will express certain topological properties of M in terms of cohomology groups associated to the cover; clearly then, those groups do not depend on the cover as well. It is of course well known that Čech cohomology can be computed using any good open cover, but in order to be as self-contained as possible we do not need to use this fact.

3.1.1 Čech cohomology and principal bundles

Let $\mathcal{F}$ be a sheaf of commutative groups over M (in this book, we in fact use only sheaves of sections of locally trivial fiber bundles). We emphasize that the sheaves must be commutative in order for the construction below to make sense. A Čech cochain of degree j is a collection of maps ϕ associating to each *ordered* j-simplex $(\sigma, \mathcal{O})$ in $\mathcal{T}$ a section in $\mathcal{F}$ over the contractible open set Star(σ). If we reverse the orientation, we require that $\phi_{(\sigma,\mathcal{O}^{-1})} = -\phi_{(\sigma,\mathcal{O})}$. Denote $\check{C}^j(M, \mathcal{F})$ the set of cochains of degree j. The boundary map

$$\partial_j \colon \check{C}^j(M, \mathcal{F}) \longrightarrow \check{C}^{j+1}(M, \mathcal{F})$$

is defined by

$$(\partial_j \phi)_{i_0 \dots i_{j+1}} := \sum_{l=0}^{j} (-1)^l \phi_{i_0 \dots \hat{\imath}_l \dots i_j}.$$

We have omitted the obvious restriction map from the above formula.

When $\mathcal{F}$ is noncommutative, we can still extend this definition as follows, but only for $j = 0, 1$:

$$(\partial_0\phi)_{ij} := \phi_i^{-1}\phi_j, \quad (\partial_1\phi)_{ijl} := \phi_{ij}\phi_{jl}\phi_{li}.$$

Henceforth we make the assumption that either $\mathcal{F}$ is commutative, or $j \leq 1$, and we note that in these cases $\partial_j \circ \partial_{j-1} = 0$ (or 1 in the multiplicative notation). The Čech cohomology group $\check{H}^j(M, \mathcal{F})$ is defined for $\mathcal{F}$ commutative or for $j = 1$ as the group

$$\check{H}^j(M, \mathcal{F}) := \operatorname{Ker} \partial_j / \operatorname{Im} \partial_{j-1}.$$

The 0$^{\text{th}}$ cohomology group $\check{H}^0$ is just the group of global sections in $\mathcal{F}$. When $\mathcal{F}$ is noncommutative, we can still define $\check{H}^1(M, \mathcal{F})$ as the *set* of equivalence classes of 1-cocycles modulo the relation

$$\phi' \cong \phi \iff \text{there exists } a \in \check{C}^0(M, \mathcal{F}) \text{ such that } \phi'_{ij} = a_i^{-1}\phi_{ij}a_j.$$

Let G be a Lie group. A G-principal bundle over M is a locally trivial fiber bundle $P_G \to M$ with a right action of G which is free and transitive on each fiber. Let $\mathcal{C}_G^\infty$ denote the sheaf of smooth G-valued functions on M.

Proposition 3.1. *The set of isomorphism classes of principal G-bundles over M is in natural bijection with $\check{H}^1(M, \mathcal{C}_G^\infty)$.*

Proof. Let P be a G-principal bundle over M. Since the open sets S_j are contractible, there exist sections $s_j : S_j \to P$. Let $\phi_{ij} : S_i \cap S_j \to G$ be the unique smooth function such that $s_j = s_i\phi_{ij}$. Then clearly the functions ϕ_{ij} form a 1-cocycle. By changing the local sections to $s'_j = s_j a_j$, $a_j : S_j \to G$ we get an equivalent cocycle $\phi'_{ij} = a_i^{-1}\phi_{ij}a_j$. Conversely, any cocycle ϕ_{ij} can be used to construct a G-principal bundle P_G, defined as the trivial bundle $S_j \times G$ over S_j modulo the identifications $(x, g) \sim (y, g\phi_{ij})$ for all $x \in S_i$, $y \in S_j$ with $x = y$. This bundle has by construction trivializations over S_j, and the associated cocycle is ϕ. □

The trivial bundle corresponds to a distinguished element in the set $\check{H}^1(M, \mathcal{C}_G^\infty)$, denoted 0. When G is commutative, the set of equivalence classes of G-bundles is therefore a group, called the Picard group when $G = \mathrm{U}_1$. Recall that a sheaf of commutative groups over M is called *soft* if every germ over a closed set can be realized as the germ of a global section. Soft sheaves are acyclic (their cohomology vanishes in degree $j \geq 1$). Typical examples are sheaves of smooth real functions, in particular $\check{H}^j(M, \mathcal{C}_{\mathbb{R}}^\infty) = 0$ for $j \geq 1$.

Proposition 3.2. *For every $j \geq 1$ there exists a natural isomorphism*

$$\check{H}^j(M, \mathcal{C}_{\mathrm{U}_1}^\infty) \to \check{H}^{j+1}(M, \mathbb{Z}).$$

Proof. Consider the short exact sequence $0 \to \mathcal{C}^\infty_{\mathbb{Z}} \to \mathcal{C}^\infty_{\mathbb{R}} \to \mathcal{C}^\infty_{\mathrm{U}_1} \to 1$, where the first map is the inclusion and the last the exponential $x \mapsto \exp(2\pi i x)$. The sheaf $\mathcal{C}^\infty_{\mathbb{Z}}$ consists of locally constant functions on M with values in $\mathbb{Z}$, so by a slight abuse of notation we denote it by $\mathbb{Z}$. Since $\check{H}^j(M, \mathcal{C}^\infty_{\mathbb{R}}) = 0$ for $j \geq 1$, the corresponding long exact sequence of cohomology groups splits into the short exact sequence

$$0 \longrightarrow \mathcal{C}^\infty(M, \mathbb{R})/\mathbb{Z} \longrightarrow \mathcal{C}^\infty(M, \mathrm{U}_1) \longrightarrow \check{H}^1(M, \mathbb{Z}) \longrightarrow 0,$$

and isomorphisms

$$\delta_j \colon \check{H}^j(M, \mathcal{C}^\infty_{\mathrm{U}_1}) \longrightarrow \check{H}^{j+1}(M, \mathbb{Z}), \quad j \geq 1. \qquad \square$$

For any Hermitian (complex) line bundle L, let $[L]$ denote its representative in $\check{H}^1(M, \mathcal{C}^\infty_{\mathrm{U}_1})$ given by Proposition 3.1. The class $\delta_1[L] \in \check{H}^2(M, \mathbb{Z})$ is called the *first Chern class* of L, and is denoted by $c_1(L)$.

Remark 3.3. If $\xi \colon G \to G'$ is a homomorphism of Lie groups, we get the induced map

$$\xi_* \colon \check{H}^1(M, \mathcal{C}^\infty_G) \longrightarrow \check{H}^1(M, \mathcal{C}^\infty_{G'}).$$

For any G-principal bundle $P_G M$, there exist a G'-principal bundle $P_{G'} M$, unique up to isomorphism, and a bundle map $P_G M \to P_{G'} M$ compatible with ξ. The corresponding cohomology class $[P_{G'} M] \in \check{H}^1(M, \mathcal{C}^\infty_{G'})$ is just $\xi_*[P_G M]$.

3.1.2 Lifting principal bundles via central extensions

A homomorphism of Lie groups $\xi \colon \widetilde{G} \to G$ is an *extension of* G if ξ is onto, and a *central extension* if $Z := \operatorname{Ker} \pi$ is a central subgroup of $\widetilde{G}$. Let $p \colon P_G M \to M$ be a G-principal bundle over M. A $\widetilde{G}$-lift of P_G is a $\widetilde{G}$-principal bundle $\tilde{p} \colon P_{\widetilde{G}} M \to M$ together with a bundle map $\pi \colon P_{\widetilde{G}} M \to P_G M$ compatible with ξ, in the sense that

$$\pi(x\tilde{g}) = \pi(x)\xi(\tilde{g})$$

for all $x \in P_{\widetilde{G}} M$, $\tilde{g} \in \widetilde{G}$.

Assume we have a G-principal bundle $p \colon P_G M \to M$ and a central extension ξ. Let $s_j \colon S_j \to P_G M$ be local sections and $\phi = (\phi_{ij})$ the associated $\mathcal{C}^\infty_G$-valued 1-cocycle of transition maps. Lift ϕ_{ij} to some maps $\tilde{\phi}_{ij} \colon S_i \cap S_j \to \widetilde{G}$ so that $\xi \circ \tilde{\phi}_{ij} = \phi_{ij}$ (this is possible since $S_i \cap S_j$ is contractible). Then the products $\psi_{ijk} := \tilde{\phi}_{ij} \tilde{\phi}_{jk} \tilde{\phi}_{ki}$ take values in $Z = \operatorname{Ker} \xi$ since (ϕ_{ij}) is ∂_1-closed. Moreover, as Z commutes with $\widetilde{G}$, the 2-cochain ψ obtained in this way is closed. Different choices of trivializations $s'_i := s_i g_i$ and of lifts $\tilde{\phi}'_{ij} = z_{ij} \tilde{g}_i^{-1} \tilde{\phi}_{ij} \tilde{g}_j$ lead to the cochain $\partial_1 z + \psi$. Hence, a G-principal bundle P_G and a central extension ξ define a cohomology class $\delta(P_G) \in \check{H}^2(M, \mathcal{C}^\infty_Z)$ independent of choices.

Proposition 3.4. *Let $p\colon P_G M \to M$ be a G-principal bundle over M and let $\xi\colon \widetilde{G} \to G$ be a central extension with kernel $Z \subset \widetilde{G}$.*

(1) *There exists a lift*

$$\pi\colon P_{\tilde{G}} M \longrightarrow P_G M \longrightarrow M$$

compatible with ξ if and only if the class $\delta(P_G)$ vanishes in $\check{H}^2(M, \mathcal{C}_Z^\infty)$.

(2) *If $\delta(P_G) = 0$, the group $\check{H}^1(M, \mathcal{C}_Z^\infty)$ acts freely and transitively on the set of isomorphism classes of lifts of p via ξ.*

Proof. (1) If a lift $P_{\tilde{G}} M$ exists, let $\widetilde{s_j}$ be local sections over S_j projecting onto s_j. Then $\widetilde{s_j} = \widetilde{s_i}\tilde{\phi}_{ij}$ for some $\widetilde{G}$-valued cocycle $\tilde{\phi}_{ij}$ covering ϕ_{ij}, so the class $\delta(P_G)$ clearly vanishes. Conversely, let $\delta(P_G) = 0$ in cohomology. In terms of arbitrary lifts $\tilde{\phi}_{ij}\colon S_j \to \widetilde{G}$ of ϕ_{ij} this means that $\delta_1\tilde{\phi} = \delta_1 z$ for some $\mathcal{C}_Z^\infty$-valued 1-cocycle z_{ij}. Since Z is central in G, the cochain $\tilde{\phi}'_{ij}$ defined by $\tilde{\phi}'_{ij} := z_{ij}^{-1}\tilde{\phi}_{ij}$ forms a cocycle. Let $P_{\tilde{G}} M$ be the bundle defined by this cocycle. By definition, it is trivialized over S_j, so we can define local bundle maps to $P_G M$ by sending the canonical section $\tilde{s}_j$ to s_j. These maps are compatible on overlaps and automatically compatible with ξ, hence $P_{\tilde{G}} M$ is a lift of $P_G M$.

(2) Assume there exists a lift $P_{\tilde{G}} M$ of $P_G M$. Fix local sections $\tilde{s}_j\colon S_j \to P_{\tilde{G}} M$ and let $s_j\colon S_j \to P_G M$ be their projections. For every other lift $P^1_{\tilde{G}} M$, take $\tilde{s}^1_j\colon S_j \to P^1_{\tilde{G}} M$ to be lifts of s_j, and $\tilde{\phi}^1_{ij}$ to be the associated cocycle. Then we have $\xi((\tilde{\phi}_{ij})^{-1}\tilde{\phi}^1_{ij}) = 1$, hence $z_{ij} := (\tilde{\phi}_{ij})^{-1}\tilde{\phi}^1_{ij}$ defines a $\mathcal{C}_Z^\infty$-valued 1-chain. Since both $\tilde{\phi}_{ij}$, $\tilde{\phi}^1_{ij}$ are cocycles and Z commutes with $\widetilde{G}$, it follows that z_{ij} is a cocycle. Different choices of $\tilde{\phi}^1_{ij}$ lead to a cohomologous cocycle, hence we get a map Φ from the set of $\widetilde{G}$-lifts of $P_G M$ to $\check{H}^1(M, \mathcal{C}_Z^\infty)$.

In the reverse direction, $\check{H}^1(M, \mathcal{C}_Z^\infty)$, which is the group of Z-principal bundles on M, acts on the set of lifts of $P_G M$ by the Z cross-product:

$$P_Z M \times P^1_{\tilde{G}} M \longmapsto P^1_{\tilde{G}} M \times_Z P_Z M$$

where the right-hand side is the quotient of the fibered product by the equivalence relation $(pz, q) \sim (p, qz)$ for all $p \in P^1_{\tilde{G}} M$, $q \in P_Z M$, $z \in Z$. A direct application of the definitions shows that the map $\check{H}^1(M, \mathcal{C}_Z^\infty) \ni z \mapsto P_{\tilde{G}} M \times_Z z$ (using the fixed lift $P_{\tilde{G}} M$) is a right and left inverse of the map Φ constructed above, so the action is free and transitive. $\square$

3.1.3 Stiefel–Whitney classes

To avoid introducing unnecessary notions, we restrict the discussion to degrees 1 and 2. For $i = 1, 2$, the i^{th} *Stiefel–Whitney class* w_i of the tangent bundle of M is a $\mathbb{Z}/2\mathbb{Z}$-cohomology class on M measuring (in the sense explained below) an obstruction to finding $n - i + 1$ linearly independent sections in $\mathrm{T}M$ over the i-skeleton of $\mathcal{T}$, i.e., the union of all closed simplices of dimension at most i. It is defined as follows. Choose $n - i + 1$ linearly independent sections in $\mathrm{T}M$ over the $(i - 1)$-skeleton (this is possible for $i = 2$ since the space of $(n - 1)$-tuples of linearly independent vectors in $\mathbb{R}^n$ is connected). For $i = 1$ or $n \geq 3$, associate to each i-simplex $\sigma \in \mathcal{T}_i$ either 0 or 1, depending on whether these sections can (resp. cannot) be extended over σ in a linearly independent way. In the special case $i = 2, n = 2$ we have a nonzero vector field along the 1-skeleton, and since $\pi_1(\mathbb{R}^2 \setminus \{0\}) = \mathbb{Z}$ we obtain an integer k_σ for each oriented triangle σ in $\mathcal{T}$. We associate to the open set $\mathrm{Star}(\sigma)$ the mod 2 residue $k_\sigma + 2\mathbb{Z}$, thus obtaining an i-dimensional $\mathbb{Z}/2\mathbb{Z}$-valued Čech cochain. Since $\pi_0(\mathrm{GL}_n(\mathbb{R})) = \mathbb{Z}/2\mathbb{Z}$, the cochain w_1 is closed, and its cohomology class is independent of the choices. For $i = 2$ and $n \geq 3$ we have $\pi_1(\mathrm{GL}_n^+(\mathbb{R})) = \mathbb{Z}/2\mathbb{Z}$, hence w_2 is closed, while for $n = 2$ any 2-cocycle is trivially closed. Hence, we get well-defined classes $w_1 \in \check{H}^1(M, \mathbb{Z}/2\mathbb{Z})$ and $w_2 \in \check{H}^2(M, \mathbb{Z}/2\mathbb{Z})$.

Lemma 3.5. *Assume M is orientable and let $P_{\mathrm{GL}_n^+}M$ be the oriented frame bundle. The cohomology class $\delta(P_{\mathrm{GL}_n^+}M) \in \check{H}^2(M, \mathbb{Z}/2\mathbb{Z})$ corresponding to the central extension*

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \widetilde{\mathrm{GL}_n^+} \xrightarrow{\xi} \mathrm{GL}_n^+ \longrightarrow 1$$

is the second Stiefel–Whitney class w_2. Here $\xi\colon \widetilde{\mathrm{GL}_n^+} \to \mathrm{GL}_n^+$ is the universal cover of GL_n^+ for $n \geq 3$, the two-folded cover $\mathrm{sq}\colon \mathrm{GL}_2^+ \to \mathrm{GL}_2^+$ for $n = 2$, respectively $|\cdot|\colon \mathbb{R}^* \to \mathbb{R}_+^*$ for $n = 1$.

Proof. Choose a section s in the oriented frame bundle in a closed, sufficiently small neighborhood of the 0-skeleton, and extend it smoothly over the 1-skeleton Sk^1 using the fact that GL_n^+ is connected. For each j, extend $s_{|\mathrm{Sk}^1 \cap S_j}$ to local sections $s_j\colon \overline{S_j} \to P_{\mathrm{GL}_n^+}M$ and let $\phi_{ij}\colon \overline{S_i} \cap \overline{S_j} \to \mathrm{GL}_n^+$ be the transition maps. By construction, $\phi_{ij} = 1$ over the 1-simplex $[p_i\, p_j]$, and $\phi_{ij}(p_k) = 1$ if $[p_i\, p_j\, p_k] \in \mathcal{T}_2$. Let $\tilde{\phi}_{ij}$ be the unique lift of ϕ_{ij} to $\widetilde{\mathrm{GL}_n^+}$ which equals 1 on $[p_i\, p_j]$. For each triangle $\sigma = [p_i\, p_j\, p_k] \in \mathcal{T}_2$, $\tilde{\phi}_{ij}(p_k)$ maps to 1 via ξ, thus it belongs to $\{\pm 1\} = \operatorname{Ker} \xi$. Moreover, by continuity, $\tilde{\phi}_{ij}\tilde{\phi}_{jk}\tilde{\phi}_{ki}$ is constant and equals $\tilde{\phi}_{ij}(p_k)$ (we see this by evaluating at p_k). We claim that this is precisely $w_2(\mathrm{Star}(\sigma))$ from the above definition of w_2. When $n \geq 3$, this is so because s is null-homotopic along the boundary of σ if and only if ϕ_{ij} is null-homotopic on $[p_j\, p_k]$ relative to the boundary.

For $n = 2$, the parity of the homotopy class of s along $\partial\sigma$ equals $\tilde{\phi}_{ij}(p_k)$. Hence, with the above choices, $\delta(P_{\mathrm{GL}_n^+}M)$ and w_2 are represented by the same cocycle. $\square$

Proposition 3.6. *A spin structure exists on M if and only if the Stiefel–Whitney classes w_1 and w_2 of $\mathrm{T}M$ vanish. In this case, the set of spin structures (up to isomorphism) is a free and transitive $\check{H}^1(M, \mathbb{Z}/2\mathbb{Z})$-module.*

Proof. The vanishing of w_1 means that the frame bundle of M is trivial over the 1-skeleton, which is clearly equivalent to the orientability of M and hence to the existence of the oriented frame bundle $P_{\mathrm{GL}_n^+}M$. The necessary and sufficient condition for the existence of a spin structure follows from Proposition 3.4 and Lemma 3.5. The classification of spin structures is contained in Proposition 3.4. $\square$

In particular, 2-connected manifolds (i.e., $\pi_1(M) = \pi_2(M) = 1$) always admit spin structures.

3.2 Topological classification of Spin$^{\mathrm{c}}$ structures

Any Spin$^{\mathrm{c}}$ structure (Definition 2.19) induces a morphism of principal bundles

$$\begin{tikzcd}
\mathrm{Spin}_n^{\mathrm{c}} \arrow[r] \arrow[d, "\xi^{\mathrm{c}}"'] & P_{\mathrm{Spin}_n^{\mathrm{c}}}M \arrow[d, "\theta"] \arrow[dr] & \\
 & & M. \\
\mathrm{SO}_n \arrow[r] & P_{\mathrm{SO}_n} \arrow[ur] &
\end{tikzcd}$$

Conversely, such a diagram defines a Spin$^{\mathrm{c}}$ structure, the auxiliary bundle being obtained from $P_{\mathrm{Spin}_n^{\mathrm{c}}}M$ using the group morphism $\mathrm{sq}\colon \mathrm{Spin}_n^{\mathrm{c}} \to \mathrm{U}_1$.

Proposition 3.7. *A* Spin$^{\mathrm{c}}$ *structure on M exists if and only if $w_1(M) = 0$ and the second Stiefel–Whitney class $w_2(M)$ is the reduction modulo 2 of an integer class. If non-empty, the set of* Spin$^{\mathrm{c}}$ *structures is acted upon freely and transitively by $H^2(M, \mathbb{Z})$.*

Proof. The bundle $P_{\mathrm{SO}_n}M$ exists if and only if $w_1 = 0$. Consider the central extension

$$1 \longrightarrow \mathrm{U}_1 \longrightarrow \mathrm{Spin}_n^{\mathrm{c}} \longrightarrow \mathrm{SO}_n \longrightarrow 1.$$

By Proposition 3.4, there exists a $\mathrm{Spin}^{\mathrm{c}}_n$-lift of $P_{\mathrm{SO}_n}M$ if and only if the cohomology class $\delta(P_{\mathrm{SO}_n}M)$ vanishes in $\check{H}^2(M, \mathcal{C}^\infty_{\mathrm{U}_1})$. Since the subgroup $\mathrm{Spin}_n \subset \mathrm{Spin}^{\mathrm{c}}_n$ surjects over SO_n, the lifts of the transition maps $\tilde{\phi}_{ij}$ in the definition of $\delta(P_{\mathrm{SO}_n}M)$ can be chosen with values in Spin_n, hence the resulting $\{\pm 1\}$-valued 2-cocycle $\partial_1\tilde{\phi}$ is precisely a representative of w_2. Consequently, $\delta(P_{\mathrm{SO}_n}M) = \iota_* w_2 \in \check{H}^2(M, \mathcal{C}^\infty_{\mathrm{U}_1})$, where $\iota\colon \{\pm 1\} \to \mathrm{U}_1$ is the inclusion. The existence part follows from Lemma 3.8 below. The classification of $\mathrm{Spin}^{\mathrm{c}}_n$ structures (when they exist) follows directly from Proposition 3.4. □

Lemma 3.8. *A class $w \in \check{H}^2(M, \{\pm 1\})$ maps to $0 \in \check{H}^2(M, \mathcal{C}^\infty_{\mathrm{U}_1})$ via ι_* if and only if it is the reduction modulo 2 of an integer class in $\check{H}^2(M, \mathbb{Z})$.*

Proof. The group homomorphism

$$\exp\colon \mathbb{R} \longrightarrow \mathrm{U}_1, \quad k \longmapsto \exp(\pi i k)$$

leads to an inclusion of short exact sequences of commutative groups

$$\begin{tikzcd}
0 \arrow[r] & 2\mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "\exp"] \arrow[d] & \{\pm 1\} \arrow[r] \arrow[d, "\iota"] & 1 \\
0 \arrow[r] & 2\mathbb{Z} \arrow[r] & \mathbb{R} \arrow[r, "\exp"'] & \mathrm{U}_1 \arrow[r] & 1.
\end{tikzcd}$$

Consider the sheaves of smooth functions on M with values in the above groups. Since the sheaf $\mathcal{C}^\infty_{\mathbb{R}}$ is soft, the commutative diagram induced in cohomology has the form

$$\begin{tikzcd}
H^2(M, 2\mathbb{Z}) \arrow[r] & H^2(M, \mathbb{Z}) \arrow[r, "\exp"] \arrow[d] & H^2(M, \mathbb{Z}/2\mathbb{Z}) \arrow[r, "\delta"] \arrow[d, "\iota_*"'] & H^3(M, 2\mathbb{Z}) \arrow[d, "1"] & \\
& 0 \arrow[r] & H^2(M, \mathcal{C}^\infty_{\mathrm{U}_1}) \arrow[r, "\delta"'] & H^3(M, 2\mathbb{Z}) \arrow[r] & 0.
\end{tikzcd}$$

Thus, the class $[w_2] \in H^2(M, \{\pm 1\})$ is mapped to $0 \in H^2(M, \mathcal{C}^\infty_{\mathrm{U}_1})$ if and only if it lives in the image of $\exp_*\colon H^2(M, \mathbb{Z}) \to H^2(M, \{\pm 1\})$. □

Proposition 3.9. *Let M be orientable and $P_{\mathrm{U}_1}M \to M$ a U_1-principal bundle. A $\mathrm{Spin}^{\mathrm{c}}$ structure over the bundle $P_{\mathrm{SO}_n}M \times_M P_{\mathrm{U}_1}M$ exists if and only if the reduction modulo 2 of $c_1(P_{\mathrm{U}_1}M)$ equals the Stiefel–Whitney class w_2. If non-empty, the set of such $\mathrm{Spin}^{\mathrm{c}}_n$ structures is a free and transitive $\check{H}^1(M, \mathbb{Z}/2\mathbb{Z})$-module.*

Proof. The classification of lifts (when they exist) follows again from Proposition 3.4. Assume $P_{\mathrm{Spin}_n^c}M$ exists and consider the Spin_n^c-valued 1-cocycle $\tilde{\phi}_{ij}$ associated to local trivializations of $P_{\mathrm{Spin}_n^c}M$ over the open cover $\mathcal{S}$. Over the contractible sets $S_i \cap S_j$ lift $\tilde{\phi}_{ij}$ to a $\mathrm{Spin}_n \times \mathrm{U}_1$-cochain $(\tilde{\psi}_{ij}, z_{ij})$. Then $\partial\tilde{\psi}$ and ∂z are both $\{\pm 1\}$-valued 2-cocycles, and their sum is 0 since $\tilde{\phi}_{ij}$ is a 1-cocycle. It follows that (z_{ij}^2) is a 1-cocycle, hence it defines a U_1-principal bundle $P_{\mathrm{U}_1}M$. Note that this bundle, called the determinant bundle of $P_{\mathrm{Spin}_n^c}M$, is induced from $P_{\mathrm{Spin}_n^c}M$ using the group homomorphism $\mathrm{sq}\colon \mathrm{Spin}_n^c \to \mathrm{U}_1$. Its first Chern class $c_1(P_{\mathrm{U}_1})$ is defined as $\partial(f_{ij})$, where $(f_{ij}) \in \check{C}^1(M, \mathcal{C}_{\mathbb{R}}^{\infty})$ is a logarithm of (z_{ij}^2), i.e., $\exp(2\pi i f_{ij}) = z_{ij}^2$. The mod 2 reduction of any integer k is just $\exp(i\pi k)$. Hence the mod 2 reduction of the integer class $c_1(P_{\mathrm{U}_1})$ is precisely $\partial \exp(\pi i f_{ij}) = \partial(z_{ij})$, which represents w_2.

Conversely, assume that w_2 is the mod 2 reduction of the first Chern class of a U_1-bundle L. By Proposition 3.7, there exists a principal Spin_n^c bundle $P_{\mathrm{Spin}_n^c}M$ lifting $P_{\mathrm{SO}_n}M$. Let P_{U_1} be the determinant bundle of $P_{\mathrm{Spin}_n^c}M$. We have seen in the first part of the proof that $c_1(P_{\mathrm{U}_1}) \equiv w_2 \mod 2$, hence by hypothesis we have $c_1(L) - c_1(P_{\mathrm{U}_1}) \equiv 0 \mod 2$. By exactness of the top row in the cohomology diagram from Lemma 3.8, $c_1(L) - c_1(P_{\mathrm{U}_1}) = 2c'$ for some integer 2-class c'. Let E be a U_1-bundle with $c_1(E) = c'$ and define a new Spin_n^c-bundle by

$$P'_{\mathrm{Spin}_n^c}M := P_{\mathrm{Spin}_n^c}M \times_{\mathrm{U}_1} E.$$

The determinant bundle of $P'_{\mathrm{Spin}_n^c}M$ has the first Chern class $c_1(P_{\mathrm{U}_1}) + 2c' = c_1(L)$, thus it is isomorphic to L as required. □

3.3 Spin structures in low dimensions

3.3.1 Dimension 1

Every 1-dimensional manifold M is orientable and has $w_2 = 0$ by dimensional reasons, hence it is spin. By parametrizing using arc-length, every connected Riemannian manifold in this dimension is isometric either to the circle $\mathbb{S}^1$ with a constant multiple of its standard metric, or to a (possibly infinite) interval inside the real line. The group Spin_1 is $\{\pm 1\}$ and the spinor representation is the canonical (i.e., non-trivial) representation of this group on $\mathbb{C}$. For the real line there exists just one spin structure, while on $\mathbb{S}^1$ there are two such structures because $H^1(\mathbb{S}^1, \mathbb{Z}/2\mathbb{Z}) = \{\pm 1\}$. The orthonormal frame bundle of $\mathbb{S}^1$ is identified with $\mathbb{S}^1$. The two spin structures are covering spaces of $\mathbb{S}^1$; one consists of two copies of $\mathbb{S}^1$ (the so-called trivial spin structure), while the second one is the two-sheeted cover

$$\mathbb{S}^1 \longrightarrow \mathbb{S}^1, \quad z \longmapsto z^2.$$

In dimension 1 the spinor bundle is thus of rank 1 and trivial (every complex vector bundle on $\mathbb{S}^1$ is trivial since it admits a non-zero real section). Let X be the positively oriented unit vector field on M. Since the volume element $\omega^{\mathbb{C}} = iX$ defined in (1.10) acts as 1 on the spinor representation, it follows that Clifford multiplication by X on spinors equals $-i$. The vector field X is clearly parallel with respect to the Levi-Civita connection on M, hence every local lift $\widetilde{X}$ to $P_{\mathrm{Spin}_1} M$ is horizontal. The Dirac operator $\mathcal{D}$ is therefore a scalar differential operator. Except for the case of the non-trivial spin structure on the circle, the lift of X exists globally on M and it trivializes the spinor bundle. In this trivialization, $\mathcal{D} = -iX$. In particular, the Dirac operator of the trivial spin structure on $\mathbb{S}^1$ is $-i\partial_t$ if we parametrize the circle by $z = e^{it}$. The spectrum of this operator consists of the integers $k \in \mathbb{Z}$, with eigenspinors e^{kit}. Theorem 4.35 states in this case that $L^2(\mathbb{S}^1)$ is spanned by this orthogonal system of functions, thus we recover the classical decomposition of periodic functions in Fourier series.

For the non-trivial spin structure, the Levi-Civita connection on spinors is flat, but has non-trivial holonomy -1, because the lift $\widetilde{X}$ satisfies $\widetilde{X}(t+2\pi) = e^{\pi i}\widetilde{X}(t)$. Hence a trivialization of the spinor bundle is provided by the global spinor which equals $s = [\widetilde{X}, e^{it/2}]$ above $e^{it} \in \mathbb{S}^1$. In this trivialization, $\mathcal{D} = -iX + \frac{1}{2}$, and so the spectrum of $\mathcal{D}$ is $\frac{1}{2} + \mathbb{Z}$; in particular $\mathcal{D}$ is invertible, unlike for the trivial spin structure.

The non-trivial spin structure on $\mathbb{S}^1$ is the restriction of the unique spin structure on $\mathbb{R}^2$, as explained below.

3.3.2 Dimension 2

Every oriented surface M with a Riemannian metric g inherits an almost complex structure J, defined as follows. For $0 \neq X \in \mathrm{T}_x M$, JX is the unique vector of the same length as X, orthogonal to X and such that $\{X, JX\}$ is a positively oriented frame. This almost complex structure is integrable, and thus defines a Kähler structure, because of the existence of local isothermal coordinates (x, y) such that $g = e^{f(x,y)}(dx^2 + dy^2)$. Indeed, the atlas consisting of oriented isothermal charts is a holomorphic atlas, since the transition maps are oriented and conformal. If M is compact, then its tangent bundle is stably trivial, thus the second Stiefel–Whitney class w_2 vanishes. If M is non-compact, then its holomorphic tangent bundle is trivial, so $w_2 = 0$ as well. Thus in both cases M is spin.

Let us examine in more detail some particular cases. On $\mathbb{R}^2$ there exists a unique spin structure $P_{\mathrm{Spin}_2}\mathbb{R}^2 \to P_{\mathrm{SO}_2}\mathbb{R}^2$. By fixing the canonical frame on $\mathrm{T}\mathbb{R}^2$, the bundles $P_{\mathrm{Spin}_2}\mathbb{R}^2$ and $P_{\mathrm{SO}_2}\mathbb{R}^2$ are both identified with the trivial bundle $\mathbb{R}^2 \times \mathbb{S}^1$, and the covering map becomes the square $z \mapsto z^2$. Let X be the unit tangent vector to the circle, so that (X, JX) is an orthonormal frame in $\mathrm{T}\mathbb{R}^2$ along $\mathbb{S}^1$. Its lift to the spinor bundle is not a closed loop, hence the restriction to the boundary of the spin structure is the connected (or non-trivial) spin structure on the circle.

More generally, consider an oriented compact surface M of genus g with a connected boundary. The spin structure induced along the boundary ∂M is again the non-trivial one. Indeed, let N be the closed surface obtained by gluing a disk along ∂M. By Proposition 3.6, every spin structure on M is the restriction of a spin structure from N, since $H^1(M, \mathbb{Z}/2\mathbb{Z})$ and $H^1(N, \mathbb{Z}/2\mathbb{Z})$ are both isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{2g}$.

Now let M be a compact oriented surface of negative Euler characteristic, let $p \in M$ and endow $M \setminus \{p\}$ with a complete hyperbolic metric of finite volume. It was proved in [Bär00] that the spectrum of $\mathcal{D}$ on $M \setminus \{p\}$ is discrete, as in the case of compact manifolds. The key remark is that the continuous spectrum is related to the zero-modes of the Dirac operator along the boundary, and that the induced Dirac operator along the boundary is invertible. Moreover, in [Mor08] it is proved that the eigenvalues of $\mathcal{D}$ grow according to the usual Weyl law (valid for closed surfaces): the number of eigenvalues smaller in absolute value than λ is asymptotically $c|\chi(M)|\lambda^2$, where c is a universal constant. The same fact is actually valid for more general complete metrics in every dimension, under the hypothesis that the induced Dirac operator along the model boundary is invertible.

Closed oriented surfaces of genus g are Kähler. By the results of Section 6.2, the spinor bundle decomposes as $\Sigma^+ M = L$, $\Sigma^- M = L \otimes \Lambda^{0,1} M$, where L is any of the 2^{2g} square roots of the canonical bundle $K = \Lambda^{1,0} M$. Moreover, the Dirac operator is self-adjoint, and its positive component is $\mathcal{D}^+ = \overline{\partial}$. The Riemann–Roch formula in this case computes the Euler characteristic of L to be 0, since L must be of degree $g - 1 = \deg(\mathrm{T}M)/2$:

$$\chi(L) = \deg(L) + 1 - g = 0.$$

The Euler characteristic of L coincides with the index of $\mathcal{D}^+$ (see Theorem 4.32), which by the Atiyah–Singer index theorem 3.10 is a Pontryagin number, so it vanishes in dimension 2.

3.3.3 Dimensions 3 and 4

Every orientable 3-manifold M is parallelizable, in particular it is a spin manifold. To see that $\mathrm{T}M$ is trivial one must in fact check that the Stiefel–Whitney classes w_2 and w_3 vanish (of course, the vanishing of the former implies the existence of spin structures). The proof involves some non-elementary ingredients of algebraic-topological nature (Steenrod squares), so we will not attempt to reproduce it here.

Along the same lines, the second Stiefel–Whitney class of an oriented 4-manifold is the reduction mod 2 of an integer class, thus every oriented 4-manifold admits Spin^c structures.

Let us stress that the above statements are true regardless of whether M is compact or not.

3.4 Examples of obstructed manifolds

The first dimension in which there exist non-spin orientable manifolds is 4. The standard example is $\mathbb{C}\mathrm{P}^2$; we have

$$H^2(\mathbb{C}\mathrm{P}^2, \mathbb{Z}) \simeq \mathbb{Z}, \quad H^2(\mathbb{C}\mathrm{P}^2, \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z},$$

and $w_2(\mathbb{C}\mathrm{P}^2)$ corresponds to 1 under this isomorphism. In higher complex dimensions, $\mathbb{C}\mathrm{P}^n$ admits exactly one spin structure if n is odd, and none if n is even.

An obstruction to the existence of spin structures on compact manifolds is provided by the $\widehat{A}$-genus, a certain characteristic class with rational coefficients which we review below. This obstruction is a direct consequence of the index theorem for the Dirac operator, due to Atiyah and Singer.

Let (M^n, g) be a closed Riemannian manifold, and $R \in \Gamma(\Lambda^2 M \otimes \mathrm{End}(\mathrm{T}M))$ its curvature tensor. We define

$$\mathrm{tr}(R^k) \in \Gamma(\Lambda^{2k} M)$$

as the $\mathrm{End}(\mathrm{T}M)$-trace of the composition $R \circ \cdots \circ R$ of k endomorphisms with values in the (commutative) even exterior algebra (by skew-symmetry, for k odd the above trace is zero). The result of the trace is a closed differential form of degree $2k$, whose cohomology class is independent of the choice of the metric g on M. This allows one to define $\mathrm{tr}(F(R))$ for any polynomial (or even formal series) F in one variable. Let

$$F(X) := \frac{X/2}{\sinh(X/2)}$$

be such a series, and set

$$\widehat{A}(R) := \exp\left(\frac{1}{2}\,\mathrm{tr}\left(\log\left(F\left(\frac{1}{2\pi i}R\right)\right)\right)\right).$$

Formally one can interpret this differential form as the square root of the determinant of $F(R/(2\pi i))$. One could also describe the cohomology class of $\widehat{A}(R)$ in terms of a polynomial in the Pontryagin classes of M with rational coefficients.

The *index* of the Dirac operator $\mathcal{D}^+$ on M is by definition

$$\mathrm{Ind}(\mathcal{D}^+) = \dim \mathrm{Ker}(\mathcal{D}^+) - \dim \mathrm{Coker}(\mathcal{D}^+) \in \mathbb{Z}.$$

Both quantities are finite by Theorem 4.32 below. Although we recall certain analytical aspects of the theory of elliptic operators in Chapter 4, a full treatment of the index theorem would be too far from the spirit of the present book. We limit ourselves to stating the index theorem, referring to the books of Lawson and Michelson [LM89] or Berline, Getzler, and Vergne [BGV92] for the proof.

Theorem 3.10 (Atiyah–Singer index theorem). *The index of the Dirac operator $\mathcal{D}^+$ on M equals the integral of $\widehat{A}(R)$ over M.*

Thus if the top-dimensional component of the rational cohomology class $\widehat{A}(R)$ is not integer, we obtain an obstruction to the existence of spin structures on M. More geometrical consequences of Theorem 3.10 appear in Chapter 5.

Chapter 4

Analytical aspects

We use in this book, in an essential way, some properties of the Dirac and Laplace operators which are typical for more general elliptic differential operators. In this chapter we provide a self-contained treatment of pseudo-differential operators on compact manifolds without boundary, which allows us to prove the following facts.

- The space of harmonic spinors is finite-dimensional.
- The Dirac operator has a discrete set of real eigenvalues of finite multiplicity, accumulating to $\pm\infty$, such that the corresponding eigenspinors are smooth and form a complete set in $L^2(M, \Sigma)$.
- For every $k \in \mathbb{N}$, every smooth spinor can be approximated in $\mathcal{C}^k$ norm by finite sums of eigenspinors.
- The eigenvalues of the Laplacian form a discrete subset in $[0, \infty)$. The corresponding eigenspaces are of finite dimension and consist of smooth functions.
- Every smooth function can be approximated in $\mathcal{C}^k$ norm by finite sums of eigenfunctions of the Laplacian.

These facts are nowadays well known, but we felt that assuming them without proof would have obscured the analytic facet of spinors and of the Dirac operator. These facts are also used in Chapter 12 in our self-contained proof of the Peter–Weyl theorem. The theory of pseudo-differential operators was developed by Calderón and Zygmund, and in its modern form by Kohn, Nirenberg and Hörmander, see for instance [Hor87]. Our presentation follows closely Melrose's yet unpublished lecture notes [Mel05].

4.1 Fourier transform

A smooth function $f\colon \mathbb{R}^n \to \mathbb{C}$ is said to belong to the *Schwartz class* if for any multi-indices $j = (j_1, \ldots, j_n), k = (k_1, \ldots, k_n) \in \mathbb{N}^n$, there exists a constant $c_{j,k} \in \mathbb{R}$ such that

$$|x^j \partial_x{}^k f(x)| \le c_{j,k}.$$

Schwartz functions are clearly absolutely integrable, thus for every $\xi \in \mathbb{R}^n$ the integral

$$\mathcal{F}(f)(\xi) := \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx$$

is absolutely convergent (here $x \cdot \xi$ denotes the Euclidean scalar product in $\mathbb{R}^n$). By Lebesgue's dominated convergence theorem, $\mathcal{F}(f)$, called the *Fourier transform* of f, is continuous and bounded in absolute value by $\|f\|_{L^1}$. Using integration by parts, one can easily see that $\mathcal{F}(f)$ is itself Schwartz. Sometimes $\mathcal{F}(f)$ will also be denoted by $\hat{f}$.

Lemma 4.1. *The Fourier transform of* $e^{-|x|^2/2}$ *is* $(2\pi)^{n/2}e^{-|\xi|^2/2}$.

Proof. On every horizontal strip $a \leq \Im(w) \leq b$, the complex function e^{-w^2} is bounded and decays exponentially as $|\Re(w)| \to \infty$. Cauchy's integral formula on the contour $\{\Im(w) = 0\} \cup \{\Im(w) = \xi\}$ and the identity

$$\int_{\mathbb{R}} e^{-x^2/2}dx = (2\pi)^{\frac{1}{2}}$$

imply the lemma. $\square$

Set

$$\mathfrak{g}(x) := e^{-|x|^2/2} \quad \text{and} \quad \mathfrak{g}_\varepsilon(x) := \mathfrak{g}(\varepsilon x)$$

for every $\varepsilon > 0$. As above, one computes $\mathcal{F}(\mathfrak{g}_\varepsilon) = \varepsilon^{-n}(2\pi)^{n/2}\mathfrak{g}_{1/\varepsilon}$. In particular,

$$\int_{\mathbb{R}^n} \mathfrak{g}(x)dx = (2\pi)^{n/2}.$$

Lemma 4.2. *The Fourier transform preserves the* L^2 *inner product on Schwartz functions up to the multiplicative factor* $(2\pi)^n$.

Proof. Let $f, g \in \mathcal{S}(\mathbb{R}^n)$. Since $|\mathfrak{g}(\xi)| \leq 1$ and $\lim_{\varepsilon\to 0}\mathfrak{g}_\varepsilon = 1$ pointwise, by Lebesgue's dominated convergence we have

$$\begin{aligned}
\langle \hat{f}, \hat{g} \rangle &= \int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi \\
&= \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} \mathfrak{g}_\varepsilon(\xi)\hat{f}(\xi)\bar{\hat{g}}(\xi)d\xi \\
&= \lim_{\varepsilon\to 0} \int_{\mathbb{R}^{3n}} e^{-\varepsilon^2|\xi|^2/2 - i(x-x')\cdot\xi} f(x)\overline{g(x')}dxdx'd\xi \\
&= (2\pi)^{n/2} \lim_{\varepsilon\to 0} \varepsilon^{-n} \int_{\mathbb{R}^{2n}} \mathfrak{g}_{1/\varepsilon}(x - x')f(x)\overline{g(x')}dxdx'
\end{aligned}$$

(by changing the contour of integration as in Lemma 4.1)

$$= (2\pi)^{n/2} \lim_{\varepsilon\to 0} \int_{\mathbb{R}^{2n}} \mathfrak{g}(Y) f(x) \overline{g(x-\varepsilon Y)} dx dY$$
$$= (2\pi)^n \langle f, g\rangle$$

(again by dominated convergence and by evaluating the integral of $\mathfrak{g}$). □

Let $L^2(\mathbb{R}^n)$ denote the completion of $\mathcal{S}(\mathbb{R}^n)$ with respect to the L^2 inner product. It follows from the above lemma that $\mathcal{F}$ extends as a bounded operator on $L^2(\mathbb{R}^n)$. Moreover, the adjoint of $\mathcal{F}$ is the operator $\check{\mathcal{F}}$ defined by

$$\check{\mathcal{F}}(f)(\xi) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(x) dx,$$

and Lemma 4.2 implies that $\check{\mathcal{F}}\mathcal{F} = (2\pi)^n I$. In precisely the same way, we get $\mathcal{F}\check{\mathcal{F}} = (2\pi)^n I$ and hence $(2\pi)^{-n/2}\mathcal{F}$ is an isometry of $L^2(\mathbb{R}^n)$.

4.2 Pseudo-differential calculus

4.2.1 Symbols

Fix $n \in \mathbb{N}$. The Euclidean space $\mathbb{R}^n$ is the interior of a manifold with boundary $\overline{\mathbb{R}^n}$ defined by radial compactification. More precisely, define

$$\overline{\mathbb{R}^n} := \left(\mathbb{R}^n \bigsqcup (0,\infty) \times \mathbb{S}^{n-1}\right)/\sim,$$

where $\sim$ identifies $0 \neq x = r\sigma \in \mathbb{R}^n$, $\sigma \in \mathbb{S}^{n-1}$, with $(1/r, \sigma)$. We shall use the boundary-defining function $\rho := (1 + r^2)^{-1/2}$, where $r = |x|$ is the Euclidean distance to the origin.

Let V be a real vector space (in practice, $V = 0$, $\mathbb{R}^n$, or $\mathbb{R}^{2n}$). For any compact manifold X, with or without boundary, let $\mathcal{C}^\infty_V(X)$ denote the space of smooth functions $a(v, x)$ on $V \times X$ such that for every $k \in \mathbb{N}^{\dim(V)}$ and every differential operator P on X,

$$|\partial_v^k P a(v, x)| \leq c_{k,P}, \quad (v, x) \in V \times X. \tag{4.1}$$

If X has non-empty boundary, then restriction to ∂X evidently defines a surjective map

$$\mathcal{C}^\infty_V(X) \longrightarrow \mathcal{C}^\infty_V(\partial X).$$

If ν is any vector field on X, it also defines a map

$$\nu\colon \mathcal{C}^\infty_V(X) \longmapsto \mathcal{C}^\infty_V(X).$$

By iterating ν followed by restriction to ∂X, we get a map into formal power series

$$\mathcal{C}_V^\infty(X) \longrightarrow \mathcal{C}_V^\infty(\partial X)[\![t]\!]. \tag{4.2}$$

When ν is transverse to ∂X, we obtain in this way the Taylor series at the boundary with respect to ν.

One particular case is obtained when X is a point; we denote by $\mathcal{C}_V^\infty$ the space of functions on V with bounded partial derivatives of every order. At the other extreme, when $V = 0$ we recover $\mathcal{C}^\infty(X)$, the space of smooth functions on the compact manifold X.

Definition 4.3. The space of *symbols of order* 0 on $\mathbb{R}^n$ with coefficients in V, denoted $S^0(V;\mathbb{R}^n)$, is the set $\mathcal{C}_V^\infty(\overline{\mathbb{R}^n})$. For $z \in \mathbb{C}$, the space of *symbols of order* z on $\mathbb{R}^n$ with coefficients in V, denoted $S^z(V;\mathbb{R}^n)$, is

$$S^z(V;\mathbb{R}^n) = \rho^{-z} S^0(V;\mathbb{R}^n) = (1+|x|^2)^{z/2} S^0(V;\mathbb{R}^n).$$

Clearly the second definition of S^0, obtained by setting $z = 0$, agrees with the first. The space of symbols of order 0 forms an algebra, while in general

$$S^z \cdot S^w \subset S^{z+w}.$$

Let us also define $S^{-\infty}(V;\mathbb{R}^n)$ as the subset of S^0 consisting of symbols which vanish at the boundary of $V \times \overline{\mathbb{R}^n}$ in Taylor series, or equivalently the kernel of the map (4.2) when ν is the radial vector field ∂_ρ.

Remark 4.4. For $V = 0$, the space $S^{-\infty}(0;\mathbb{R}^n)$ is the same as the Schwartz space $\mathcal{S}(\mathbb{R}^n)$.

Lemma 4.5. *For every $z \in \mathbb{C}$ the intersection*

$$\bigcap_{j=0}^{\infty} S^{z-j}(V;\mathbb{R}^n)$$

equals $S^{-\infty}(V;\mathbb{R}^n)$.

Proof. For $z = 0$ this is just the definition, since the first j Taylor coefficients of a symbol of order $-j$ are 0. In general, note that for every $z \in \mathbb{C}$, multiplication by ρ^{-z} preserves $S^{-\infty}(V;\mathbb{R}^n)$. □

4.2.2 Asymptotic summation

Lemma 4.6. *Let $(a_j)_{j\geq 0}$ be a sequence of complex numbers. There exists a smooth function $f\colon \mathbb{R} \to \mathbb{C}$ whose Taylor series at $x = 0$ is given by*

$$f(x) \sim \sum_{j=0}^{\infty} a_j x^j.$$

Of course, when the series is absolutely convergent, one could define f as its sum.

Proof. Let $\chi\colon \mathbb{R} \to \mathbb{R}$ be a non-negative, smooth function with compact support in $(-1, 1)$, and identically 1 on $[-\frac{1}{2}, \frac{1}{2}]$. For $j \geq 1$, take $b_j := \max\{j, 4|a_j|^{2/j}\}$ and set

$$f(x) := \sum_{j=0}^{\infty} a_j x^j \chi(b_j x).$$

The sum is locally finite near every $x \neq 0$ because $\chi(b_j x) = 0$ whenever $|x| > 1/j$. By direct computation, one checks that for all $p \geq 0$ there exists a constant c_p such that for every $j > 2p$ we have the uniform bound in $x \in \mathbb{R}$

$$|a_j|\,|\partial_x{}^p(x^j \chi(b_j x))| \leq c_p \frac{j^p}{2^j}.$$

This means that the series is absolutely convergent in the $\mathcal{C}^p$ topology for every $p \geq 0$, hence the sum of the series is smooth. □

Lemma 4.7. *Let $z \in \mathbb{C}$ and consider a sequence $s_j \in S^{z-j}(V; \mathbb{R}^n)$, $j \geq 0$. Then there exists $s \in S^z(V; \mathbb{R}^n)$ such that for every $k \geq 0$*

$$s - \sum_{j=0}^{k} s_j \in S^{z-k-1}(V; \mathbb{R}^n).$$

The symbol s is called the *asymptotic sum* of the (generally divergent) series $\sum_{j=0}^{\infty} s_j$.

Proof. We will prove more generally that for every compact manifold with boundary X and for every sequence $a_j \in \mathcal{C}_V^{\infty}(\partial X)$, there exists $f \in \mathcal{C}_V^{\infty}(X)$ with Taylor series $f \sim \sum_{j=0}^{\infty} a_j \rho^j$, where ρ is a boundary-defining function for ∂X; in other words, the map (4.2) is onto. The proof is based on a diagonal subsequence procedure adapted to the proof of Lemma 4.6. We claim that there exists a countable set of differential operators $\mathcal{A}$ on X so that in the definition of $\mathcal{C}_V^{\infty}(X)$ it is enough to check inequality (4.1) for $P \in \mathcal{A}$. Such a sequence can be obtained as follows. Take a finite number of vector fields $U_1, \dots, U_m$ which generate $\mathrm{T}_x X$ at each $x \in X$, including at boundary points. Then every differential operator P on X can be written as a polynomial in $U_1, \dots, U_m$ with coefficients in $\mathcal{C}^{\infty}(X)$. By compactness of X, the desired countable set can thus be chosen to be the monomials in the variables $U_1, \dots, U_m$.

The set of multi-indices k appearing in (4.1) is also countable; therefore, we must choose the constants b_j as in Lemma 4.6 to satisfy a countable set of inequalities. In other words, we have a double-index sequence

$$a_{j,m} := \sup\{|\partial_v^{k_m} P_m a_j(v, x)|;\ v \in V, x \in X\}$$

and we would like the series

$$f_m(\rho) := \sum_{j=0}^{\infty} a_{j,m}\rho^j \chi(b_j \rho)$$

to be absolutely convergent in $\mathcal{C}^p$ norm for all m, p. This is achieved by setting $b_j := \max\{b_{j,1}, \dots, b_{j,j}\}$, where $b_{j,m} := \max\{j, 4|a_{j,m}|^{2/j}\}$ is defined in terms of $a_{j,m}$ as in Lemma 4.6. □

4.3 Pseudo-differential operators

Let $V = \mathbb{R}^{2n}$ and $a = a(x, x', \xi) \in S^z(\mathbb{R}^{2n}; \mathbb{R}^n)$. We wish to define a linear operator

$$A = \mathrm{Op}_a \colon \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$$

defined as

$$Af(x) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{i(x-x')\cdot\xi} a(x, x', \xi) f(x') dx' d\xi. \tag{4.3}$$

The motivation for such a definition is the Fourier transform expression of a differential operator,

$$P(x, i^{-1}\partial_x) f = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{i(x-x')\cdot\xi} P(x, \xi) f(x') dx' d\xi,$$

where P depends polynomially on ξ. The integral (4.3) is absolutely convergent for $\Re(z) < -n$. For other z, one must perform first the x' integration, whose result is bounded from above by every power of $\rho = (1 + |\xi|^2)^{-1/2}$.

Lemma 4.8. *Let $a \in S^z(\mathbb{R}^{2n}; \mathbb{R}^n)$ and $f \in \mathcal{S}(\mathbb{R}^n)$. Then*

$$u_{a,f}(x, \xi) := \int_{\mathbb{R}^n} e^{i(x-x')\cdot\xi} a(x, x', \xi) f(x') dx'$$

is uniformly of order $O(1 + |\xi|^2)^{-j}$ for every $j > 0$.

Proof. Since the integral in x' is uniformly convergent, we have a priori

$$u_{a,f}(x, \xi) = O(\rho^{-\Re(z)}).$$

Write $a = (1 + |\xi|^2)b$ with $b = \rho^{-2}a \in S^{z-2}$. Integrating by parts, for every $c \in S^w$, we get

$$u_{\xi_k c, f} = -i u_{\frac{\partial c}{\partial x'_k}, f} - i u_{c, \frac{\partial f}{\partial x'_k}} = O(\rho^{-\Re(w)}).$$

Thus $u_{\rho c, f} = O(\rho^{-\Re(w)})$ and by the lemma follows by induction. □

We remark that directly from the definition,

$$\partial_{x_j} u_{a,f} = u_{i\xi_j a,f}, \quad x_j u_{a,f} = i u_{\partial_{\xi_j} a,f} + i u_{a,\xi_j' f},$$

so by iteration we get $u_{a,f} \in S^{-\infty}(\mathbb{R}^n;\mathbb{R}^n)$. This implies that $Af \in \mathcal{S}(\mathbb{R}^n)$, so A defines a linear operator on $\mathcal{S}(\mathbb{R}^n)$.

Let $\widetilde{\Psi}^z(\mathbb{R}^n)$ be the space of linear operators on $\mathcal{S}(\mathbb{R}^n)$ defined by the oscillating integral (4.3) in terms of symbols in $S^z(\mathbb{R}^{2n};\mathbb{R}^n)$. Define also the subspaces

$$\Psi^z_{\mathrm{l}}(\mathbb{R}^n) = \{\mathrm{Op}_a;\ a = a(x,\xi) \in S^z(\mathbb{R}^n;\mathbb{R}^n)\},$$

and

$$\Psi^z_{\mathrm{r}}(\mathbb{R}^n) = \{\mathrm{Op}_a;\ a = a(x',\xi) \in S^z(\mathbb{R}^n;\mathbb{R}^n)\}.$$

If $\Re(z) < -n$, the integral (4.3) is absolutely convergent; thus, the operator A is defined by a continuous integral kernel:

$$Af(x) = \int_{\mathbb{R}^n} \kappa_A(x,x') f(x') dx',$$

$$\kappa_A(x,x') = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x-x')\cdot\xi} a(x,x',\xi) d\xi.$$

If $\Re(z) < -n-k$, then $\kappa_A(x,x')$ is of class $\mathcal{C}^k$. In fact, $\kappa_A(x,x')$ is smooth outside the diagonal $\{x = x'\}$:

Lemma 4.9. *Let χ be a cut-off function as in the proof of Lemma 4.6. For every symbol $a \in S^z(\mathbb{R}^{2n};\mathbb{R}^n)$, the kernel*

$$\kappa(x,x') := (1 - \chi(|x-x'|))\kappa_A(x,x')$$

is smooth on $\mathbb{R}^{2n}$.

Proof. We will prove that $\kappa(x,x')$ belongs to $\mathcal{C}^\infty_{\mathbb{R}^{2n}}$, in other words, all its partial derivatives are bounded. Since $\chi(|x-x'|) \in \mathcal{C}^\infty_{\mathbb{R}^{2n}}$, the product $b(x,x',\xi) := (1-\chi(|x-x'|))a(x,x',\xi)$ is also a symbol in $S^z(\mathbb{R}^{2n};\mathbb{R}^n)$. Using $\kappa(x,x) = 0$ write

$$\begin{aligned} b(x,x',\xi) &= \int_0^1 \partial_t b(x' + t(x-x'), x', \xi) dt \\ &= (x-x') \cdot \int_0^1 (\partial_x b)(x' + t(x-x'), x', \xi) dt, \end{aligned} \tag{4.4}$$

therefore

$$b = \sum_{j=1}^n (x-x')_j b_j, \quad b_j \in S^z(\mathbb{R}^{2n};\mathbb{R}^n).$$

Integrating by parts with respect to ξ_j, we see that $(x - x')_j b_j$ defines the same operator as $-\partial_{\xi_j} b_j \in S^{z-1}(\mathbb{R}^{2n}; \mathbb{R}^n)$. By iterating this argument, we conclude that $\kappa(x, x')$ is the integral kernel of an operator defined by symbols of arbitrarily low order, hence it is smooth. $\square$

Let $\mathcal{R}$ denote the space of integral kernels $\kappa(x, x')$ such that

$$(x, x') \longmapsto \kappa(x, x - x')$$

belongs to $S^{-\infty}(\mathbb{R}^n; \mathbb{R}^n)$.

Proposition 4.10. *The space of integral kernels of operators in each of the three spaces* $\widetilde{\Psi}^{-\infty}(\mathbb{R}^n)$, $\Psi_{\mathrm{r}}^{-\infty}(\mathbb{R}^n)$, *and* $\Psi_{\mathrm{l}}^{-\infty}(\mathbb{R}^n)$ *is* $\mathcal{R}$.

Proof. It is rather clear that for every $a(x, x', \xi) \in S^{-\infty}(\mathbb{R}^{2n}; \mathbb{R}^n)$ the associated integral kernel is Schwartz as a function of $x - x'$ and has bounded derivatives in the variables x_j. It is thus enough to prove that the right and left quantization maps $\mathrm{Op}_{\mathrm{r}}, \mathrm{Op}_{\mathrm{l}}$ from $S^{-\infty}(\mathbb{R}^n; \mathbb{R}^n)$ to $\mathcal{R}$ are onto. That this is indeed the case is a consequence of the fact that the Fourier transformation is an isomorphism from $\mathcal{S}$ to itself. $\square$

An operator with smooth integral kernel is called *smoothing*; its action clearly extends from $\mathcal{S}(\mathbb{R}^n)$ to spaces of functions with less regularity, for instance, to $L^1(\mathbb{R}^n)$.

Proposition 4.11. *For every* $z \in \mathbb{C} \sqcup \{-\infty\}$, *the spaces of operators* $\widetilde{\Psi}^z(\mathbb{R}^n)$, $\Psi_{\mathrm{r}}^z(\mathbb{R}^n)$, *and* $\Psi_{\mathrm{l}}^z(\mathbb{R}^n)$ *are equal.*

Proof. A priori, it is clear that $\Psi_{\mathrm{r}}^z \subset \widetilde{\Psi}^z$. We will prove that

$$\widetilde{\Psi}^z(\mathbb{R}^n) \subset \Psi_{\mathrm{r}}^z(\mathbb{R}^n) + \widetilde{\Psi}^{z-1}(\mathbb{R}^n).$$

Let χ be a cut-off function and let $a \in S^z(\mathbb{R}^{2n}; \mathbb{R}^n)$. Write as in (4.4)

$$a(x, x', \xi) - a(x', x', \xi) = \sum_{j=1}^{n} (x - x')_j a_j, \quad a_j \in S^z(\mathbb{R}^{2n}; \mathbb{R}^n).$$

Integrating by parts with respect to ξ_j, the terms in the right-hand side define operators of order $z - 1$. Hence $\mathrm{Op}(a) = \mathrm{Op}(a(x', x', \xi)) + \mathrm{Op}(b)$, $\mathrm{Op}(b) \in \widetilde{\Psi}^{z-1}(\mathbb{R}^n)$. By iterating this argument, $\widetilde{\Psi}^z(\mathbb{R}^n) \subset \Psi_{\mathrm{r}}^z(\mathbb{R}^n) + \widetilde{\Psi}^{z-k}(\mathbb{R}^n)$. By asymptotic summation, there exists a symbol $b \in S^z(\mathbb{R}^n; \mathbb{R}^n)$ such that $\mathrm{Op}(a) - \mathrm{Op}(b(x', \xi)) =: C \in \widetilde{\Psi}^{-\infty}(\mathbb{R}^n)$. Proposition 4.10 shows that $\widetilde{\Psi}^{-\infty}(\mathbb{R}^n) = \Psi_{\mathrm{r}}^{-\infty}(\mathbb{R}^n)$, thus $\widetilde{\Psi}^z(\mathbb{R}^n) = \Psi_{\mathrm{r}}^z(\mathbb{R}^n)$. The equality with $\Psi_{\mathrm{r}}^z(\mathbb{R}^n)$ is analogous. $\square$

Definition 4.12. A *pseudo-differential operator* of order $z \in \mathbb{C}$ on $\mathbb{R}^n$ is an operator defined by an oscillatory integral (4.3) for some symbol in $S^z(\mathbb{R}^{2n}; \mathbb{R}^n)$.

For every pseudo-differential operator A, there exist unique symbols $a_{\mathrm{l}}(x, \xi)$, resp. $a_{\mathrm{r}}(x', \xi)$ whose quantization Op defines A. It is moreover clear from Proposition 4.11 that for every $\xi \in \mathbb{R}^n \setminus \{0\}$,

$$\lim_{r\to\infty} r^{-z} a_{\mathrm{l}}(x, r\xi) = \lim_{r\to\infty} r^{-z} a_{\mathrm{r}}(x, r\xi).$$

This common limit is called the *principal symbol* of A, denoted $\sigma_z(A)$. The principal symbol is radially constant, hence it defines an element in $\mathcal{C}^\infty_{\mathbb{R}^n}(\mathbb{S}^{n-1})$. From Proposition 4.11, the principal symbol of $\mathrm{Op}(a(x, x', \xi))$ equals $\sigma_z(A)(x, \xi) = \lim\limits_{r\to\infty} r^{-z} a_{\mathrm{l}}(x, x, r\xi)$.

4.4 Composition of pseudo-differential operators

Let A, A' be two pseudo-differential operators on $\mathbb{R}^n$ of orders $z, z' \in \mathbb{C}$. We claim that the composition $A \circ A' : \mathcal{S} \to \mathcal{S}$ is again pseudo-differential, and

$$\sigma_{z+z'}(AA') = \sigma_z(A)\sigma_{z'}(A').$$

Indeed, represent A as the left quantization of the symbol $a(x, \xi)$, and A' as the right quantization of $a'(x', \xi)$. Their composition is

$$AA'f(x) = (2\pi)^{-2n} \int e^{i(x-y)\cdot\xi} a(x, \xi) e^{i(y-x')\cdot\xi'} a'(x', \xi') f(x') dx' d\xi' dy d\xi.$$

Formally, the integral in y is the Fourier transform of the constant function 1, which is the delta distribution in the variable $\xi - \xi'$. This can be computed without using distributions. To be able to reverse the order of integration, introduce a Gaussian weight as in Section 4.1. Namely, set

$$\mathfrak{g}(y) := e^{-|y|^2/2}, \quad \text{and} \quad \mathfrak{g}_\varepsilon(y) := \mathfrak{g}(\varepsilon y)$$

for every $\varepsilon > 0$. Then

$$\begin{aligned} &AA'f(x) \\ &\quad = (2\pi)^{-2n} \lim_{\varepsilon\to 0} \int \mathfrak{g}_\varepsilon(y) e^{iy\cdot(\xi'-\xi)} e^{ix\cdot\xi - ix'\cdot\xi'} a(x, \xi) a'(x', \xi') f(x') dx' d\xi' dy d\xi. \end{aligned}$$

Now the integral is absolutely convergent in y, and is computed by exchanging the order of integration, as in Section 4.1:

$$\begin{aligned} AA'f(x) &= (2\pi)^{-3n/2}\lim_{\varepsilon\to 0}\varepsilon^{-n}\int \mathfrak{g}_{1/\varepsilon}(\xi-\xi')e^{ix\cdot\xi-ix'\cdot\xi'} \\ &\qquad\qquad a(x,\xi)a'(x',\xi')f(x')dx'd\xi'd\xi \\ &= (2\pi)^{-3n/2}\lim_{\varepsilon\to 0}\int e^{-|X|^2/2}e^{ix\cdot\xi-ix'\cdot(\xi+\varepsilon X)} \\ &\qquad\qquad a(x,\xi)a'(x',\xi+\varepsilon X)f(x')dx'dXd\xi \\ &= (2\pi)^{-n}\int e^{ix\xi-ix'\xi}a(x,\xi)a'(x',\xi)f(x')dx'd\xi. \end{aligned}$$

Thus AA' is a pseudo-differential operator defined by the symbol $a(x,\xi)a'(x',\xi)$.

In particular, the space of smoothing operators forms a bilateral ideal inside the pseudo-differential calculus, in the sense that $\Psi^z\circ\Psi^{-\infty}=\Psi^{-\infty}\circ\Psi^z=\Psi^{-\infty}$.

4.5 Action of diffeomorphisms on pseudo-differential operators

Let $\Omega\subset\mathbb{R}^n$ be an open set, and $\Phi\colon\Omega\to\Omega'\subset\mathbb{R}^n$ a diffeomorphism. For every symbol $a(x,x',\xi)$ of order z with support in $K\times K\times\mathbb{R}^n$ with K compact in Ω', let A be the associated pseudo-differential operator on $\mathcal{S}$.

Proposition 4.13. *The pull-back of A by the diffeomorphism Φ is again pseudo-differential of order z, and its principal symbol is*

$$\sigma_z(\Phi^*A)(x,\xi)=\sigma_z(A)(\Phi(x),{}^tD\Phi_x^{-1}\xi).$$

Proof. By definition, $\Phi^*Af=\Phi^{-1}\circ A(f\circ\Phi)$. In coordinates,

$$\begin{aligned} \Phi^*Af(x) &= (2\pi)^{-n}\int_{\mathbb{R}^{2n}} e^{i(\Phi(x)-x')\cdot\xi}a(\Phi(x),x',\xi)f(\Phi^{-1}(x'))dx'd\xi \\ &= (2\pi)^{-n}\int_{\mathbb{R}^{2n}} e^{i(\Phi(x)-\Phi(y))\cdot\xi}a(\Phi(x),\Phi(y),\xi)f(y)J_\Phi(y)dyd\xi, \end{aligned}$$

where $J_\Phi(y)=\det D_y\Phi$. Write $\Phi(x)-\Phi(y)=U_{x,y}(x-y)$ for some matrix $U_{x,y}$ obtained as in (4.4) via Taylor expansion near $x=y$. This implies that $U_{x,x}=D_x\Phi$ is invertible, hence $U_{x,y}$ is also invertible for x sufficiently close to y, say for $|x-y|<\varepsilon$ whenever $x\in K$.

Recall that outside the diagonal the integral kernel of a pseudo-differential operator is smooth; such a kernel clearly pulls back via Φ to a smooth kernel, hence to a smoothing operator in $\Psi^{-\infty}$. Thus it is enough to analyze what happens near the diagonal. By multiplying a with a suitable cut-off function $\psi(x-y)$, we can assume that $U_{x,y}$ is invertible wherever $a(x,y,\xi) \neq 0$. Hence we write

$$\begin{aligned}
\Phi^* A f(x) &= (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{i U_{x,y}(x-y)\cdot\xi} a(\Phi(x), \Phi(y), \xi) f(y) J_\Phi(y) dy d\xi \\
&= (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot {}^t U_{x,y}\xi} a(\Phi(x), \Phi(y), \xi) f(y) J_\Phi(y) dy d\xi \\
&= (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\Xi} a(\Phi(x), \Phi(y), {}^t U_{x,y}^{-1}\Xi) \\
&\qquad\qquad f(y) J_\Phi(y) \det U_{x,y}^{-1} dy d\Xi.
\end{aligned}$$

Thus $\Phi^* A$ is a pseudo-differential operator with symbol

$$a(\Phi(x), \Phi(y), {}^t U_{x,y}^{-1}\Xi) f(y) J_\Phi(y) \det U_{x,y}^{-1}.$$

The principal symbol is computed by extracting the leading asymptotic term in $a(\Phi(x), \Phi(x), {}^t D_x\Phi^{-1}\Xi)$. □

4.6 Pseudo-differential operators on vector bundles

Let A be a $m \times k$ matrix of pseudo-differential operators with symbol $a(x, x', \xi)$, $a = (a_{ij})$, acting from $\mathcal{S}(\mathbb{R}^n) \otimes \mathbb{R}^m$ to $\mathcal{S}(\mathbb{R}^n) \otimes \mathbb{R}^k$. Let also

$$G' \in \mathcal{C}^\infty_{\mathbb{R}^n} \otimes \mathrm{GL}_m \quad \text{and} \quad G \in \mathcal{C}^\infty_{\mathbb{R}^m} \otimes \mathrm{GL}_k,$$

i.e., both G' and G are square matrices of functions on $\mathbb{R}^n$ with bounded partial derivatives. Then $G \circ A \circ G'$ is again a matrix of pseudo-differential operators, with the symbol

$$G(x) a(x, x', \xi) G'(x'). \tag{4.5}$$

Let $E, F \to M$ be complex vector bundles over a compact manifold M. A *smoothing operator* on $(M; E, F)$ is an operator

$$R: \mathcal{C}^\infty(M, E) \longrightarrow \mathcal{C}^\infty(M, F)$$

defined by a smooth integral kernel $r(x, x')|dx'| \in \mathcal{C}^\infty(M \times M; E^* \boxtimes F \otimes \Omega_M)$, where Ω_M is the bundle of 1-densities on M:

$$Rf(x) = \int_M r(x, x') f(x') |dx'|.$$

The notation $E^* \boxtimes F$ means the tensor product of the bundles over $M \times M$ obtained by pulling back F from the first copy of M, respectively E^* from the second copy.

Definition 4.14. Let M be a compact manifold without boundary, and $E, F \to M$ vector bundles of respective rank m, k. A pseudo-differential operator

$$A\colon \mathcal{C}^\infty(M,E) \longrightarrow \mathcal{C}^\infty(M,F)$$

is called *pseudo-differential of order* $z \in \mathbb{C}$ if there exist

- a smoothing operator $R\colon \mathcal{C}^\infty(M,E) \to \mathcal{C}^\infty(M,F)$,
- a finite cover of $M = U_1 \cup \cdots \cup U_p$ by domains of charts $\phi_j\colon U_j \to V_j \subset \mathbb{R}^n$,
- an associated partition of unity χ_j, and a family of smooth functions with compact support $\chi'_j\colon U_j \to \mathbb{R}$ (equal 1 on $\operatorname{supp}(\chi_j)$), such that $\chi_j \chi'_j = \chi_j$,
- trivializations γ_j, γ'_j of E, F over U_j, and
- for each j, a $m \times k$ matrix A_j of pseudo-differential operators of order z on V_j, such that
$$A = R + \sum_{j=1}^{p} \chi' \phi_j^* A_j (\phi_j^{-1})^* \chi_j.$$

From the invariance of the class of pseudo-differential operators under diffeomorphisms (Proposition 4.13) and linear transformations, it follows immediately that a pseudo-differential operator on M is of the form in Definition 4.14 in *any* system of charts, with respect to any local trivializations of E and F. Moreover, it is clear from the local composition results that pseudo-differential operators form a *calculus,* in the following sense:

$$\Psi^{z_1}(M;E',E) \circ \Psi^{z_2}(M;E'',E') \subset \Psi^{z_1+z_2}(M;E'',E)$$

where $z_1, z_2 \in \mathbb{C} \sqcup \{-\infty\}$ (by definition, $\Psi^{-\infty}$ denotes the space of smoothing operators).

The principal symbol of a pseudo-differential operator $A \in \Psi^z(M;E,F)$ is defined once we choose a chart $\phi\colon U \to \mathbb{R}^n$ and local trivializations of E and F over U. Using (4.5), $\sigma_z(A)$ is in fact a smooth section over $U \times \mathbb{R}^n$ in the bundle $E^* \otimes F$, i.e., it does not depend on the trivializations. By Proposition 4.13, if we identify $U \times \mathbb{R}^n$ with T^*U using the chart ϕ, the resulting principal symbol

$$\sigma_z(A) \in \mathcal{C}^\infty(\mathrm{T}^*U \setminus \{0\}, E^* \otimes F)$$

is independent of ϕ altogether. Here a bundle over U is pulled back to T^*U using the bundle projection. Moreover $\sigma_z(A)$ is homogeneous of degree z along the fibers of T^*U.

Thus to each pseudo-differential operator $A \in \Psi^z(M; E, F)$, we associate a section $\sigma_z(A)$ in $\pi^*(E^* \otimes F)$ over $\mathrm{T}^*M \setminus \{0\}$, homogeneous of degree z. This map is compatible with composition of operators and of symbols:

$$\sigma_{z_1+z_2}(AB) = \sigma_{z_1}(A)\sigma_{z_2}(B)$$

for every $A \in \Psi^{z_1}(M; E', E)$, $B \in \Psi^{z_2}(M; E'', E')$.

For every manifold M, let

$$\mathrm{S}^*M \longrightarrow M$$

denote the bundle of $(n-1)$-dimensional spheres associated to T^*X, with fiber $(\mathrm{T}^*_pX \setminus \{0\})/\mathbb{R}_+$ over $p \in X$. By fixing a Riemannian metric on M, this sphere bundle is identified with the unit sphere bundle in T^*M. Homogeneous functions of degree z on $\mathrm{T}^*M \setminus \{0\}$ are in bijection with smooth functions on S^*M.

Lemma 4.15. *For every z, the principal symbol map*

$$\sigma_z\colon \Psi^z(M; E, F) \longrightarrow \mathcal{C}^\infty(\mathrm{S}^*M, \pi^*(E^* \otimes F))$$

is onto.

Proof. By using a partition of unity, it is enough to prove the surjectivity of the principal symbol map from scalar pseudo-differential operators on $\mathbb{R}^n$ onto compactly supported smooth functions on $\mathrm{S}^*\mathbb{R}^n \sim \partial(\mathbb{R}^n \times \overline{\mathbb{R}}^n)$. This follows for operators of order 0 from the definition of symbols as smooth functions on the compactification of $\mathrm{T}^*\mathbb{R}^n$ and from the fact that for every manifold X with boundary ∂X, the restriction map $\mathcal{C}^\infty_c(X) \to \mathcal{C}^\infty_c(\partial X)$ between the spaces of compactly-supported functions is onto. For symbols of order $z \in \mathbb{C}$, the statement is reduced to the case $z = 0$ by multiplying with $(1 + |\xi^2|)^{-z/2}$. □

Every smooth function f on M defines a multiplication operator on $\mathcal{C}^\infty(M, E)$. This operator is pseudo-differential of order 0, since in local coordinates and trivializations it is given by the 0-order symbol $f(x)$, independent of ξ. More generally, every differential operator is pseudo-differential (of the same degree), with polynomial symbol.

If A is any operator of order z, the commutator $[A, f]$ is pseudo-differential of order z by the composition theorem. Its principal symbol is the commutator $[\sigma_z(A), f] = 0$ (because multiplication by f commutes with fiberwise morphisms form E to F). Thus, $[A, f] \in \Psi^{z-1}$.

Proposition 4.16. *Let $A \in \mathrm{Diff}^1(M; E, F)$ be a first-order* differential *operator of order* 1 *on M. Let $f \in \mathcal{C}^\infty(M)$ define multiplication operators on $\mathcal{C}^\infty(M, E)$ and $\mathcal{C}^\infty(M, F)$. Then the principal symbol of A evaluated on $(x, d_x f) \in \mathrm{T}^*_xM$ equals the commutator $\frac{1}{i}[A, f]$ evaluated at x (the commutator is a 0-order differential operator, hence a bundle morphism).*

Using this proposition we can compute the principal symbols of the spin Dirac operator $\mathcal{D}$ and of the de Rham differential. Namely,

$$\sigma_1(\mathcal{D})(x,\xi) = \frac{1}{i}\xi\cdot\colon \Sigma_x M \longrightarrow \Sigma_x M$$

and

$$\sigma_1(d)(x,\xi) = \frac{1}{i}\xi\wedge\colon \Lambda_x^* M \longrightarrow \Lambda_x^* M,$$

where $\xi\cdot$ denotes Clifford multiplication by ξ.

4.7 Elliptic operators

Definition 4.17. An operator $A \in \Psi^z(M; E, F)$ is called *elliptic* if for every $p \in M$ and $\xi \in \mathrm{S}_p^* M$, the map

$$\sigma_z(A)(\xi)\colon E_p \longrightarrow F_p$$

is invertible.

If A is elliptic, then of course the two bundles must have the same rank. Examples of elliptic differential operators are the spin Dirac operator $\mathcal{D}$, the de Rham Dirac operator $d+\delta$ where δ is the adjoint of d with respect to some Riemannian metric, and the Laplace operator on differential forms, $\Delta = d\delta + \delta d$ (see Lemma 2.28).

Proposition 4.18. *Let $A \in \Psi^z(M; E, F)$ be an elliptic pseudo-differential operator. Then there exists $G \in \Psi^{-z}(M; F, E)$ such that*

$$AG - I_F = R' \in \Psi^{-\infty}(M; F)$$

and

$$GA - I_E = R \in \Psi^{-\infty}(M; E)$$

Such an operator G is called a *parametrix* for A.

Proof. Find first $G_0 \in \Psi^{-z}(M; F, E)$ with $\sigma_{-z}(G_0)(x,\xi) = (\sigma_z(A)(x,\xi))^{-1}$. This is possible by ellipticity and by surjectivity of the principal symbol map. Then $\sigma_0(AG_0)(x,\xi) = I_{F_x}$, so $\sigma_0(AG_0 - I_F) = 0$. Therefore, there exists $R_1 \in \Psi^{-1}(M, F)$ with $AG_0 - I_F = R_1$. Choose $G_1 \in \Psi^{-z-1}(M; F, E)$ with symbol $\sigma_{-z-1}(G_1) = -\sigma_z(A)^{-1}\sigma_{-1}(R_1)$ (again, this is possible since $\sigma_z(A)$ is invertible and of the symbol map is surjective). Then $R_2 := A(G_0 + G_1) - I_F \in \Psi^{-2}(M; F)$. Continuing this procedure we get a sequence $G_j \in \Psi^{-z-j}(M; F, E)$ with $A(G_0 + \cdots + G_j) - 1_F \in \Psi^{-j-1}(M; F)$. Let $G \in \Psi^{-z}(M; F, E)$ denote any asymptotic sum of the series $\sum_{j=0}^{\infty} G_j$. Then $R' := AG - 1_F \in \Psi^{-\infty}(M; F)$;

in other words, G is a right inverse for A modulo smoothing operators. Similarly, we construct a left inverse $G' \in \Psi^{-z}(M; F, E)$ with $R'' := G'A - I_E \in \Psi^{-\infty}(M; E)$. Then $G'R' - G'AG = G'$ and $GR'' - G'AG = G$, and since the smoothing operators form an ideal, $G - G'$ is smoothing and hence G is itself a left inverse modulo smoothing operators. □

It is clear from the proof that the parametrix G is unique up to $\Psi^{-\infty}(M; F, E)$.

4.8 Adjoints

Lemma 4.19. *Let M be a compact manifold, $A \in \Psi^z(M; E, F)$, and ν_M a volume density on M. There exists a pseudo-differential operator $A^* \in \Psi^z(M; F^*, E^*)$ such that for every $e \in \mathcal{C}^\infty(M, E)$ and $f \in \mathcal{C}^\infty(M, F^*)$ we have*

$$\int_M \langle Ae, f\rangle \nu_M = \int_M \langle e, A^* f\rangle \nu_M.$$

Clearly the operator A^* is unique if it exists. For differential operators, A^* is the formal adjoint obtained by integration by parts.

Proof. The adjoint of a smoothing operator with integral kernel $\kappa(x, y)$ exists and is also smoothing with kernel $(x, y) \mapsto \kappa(y, x)^*$, where $*$ denotes the adjoint endomorphism. Using partitions of unity as in Definition 4.14, it is enough to show the existence of adjoints for a matrix of pseudo-differential operators with symbol $a(x, y, \xi)$, with compact support in x, y. The adjoint of $\mathrm{Op}(a(x, y, \xi))$ is just $\mathrm{Op}(a(y, x, \xi)^*)$. □

Hence, in particular, the principal symbol map is a $*$-morphism, in the sense that is commutes with the operation of taking adjoints.

4.9 Sobolev spaces

To understand the topology of the Fréchet space of smooth functions on a compact manifold M we need to introduce some Hilbert spaces. The main one is L^2.

Definition 4.20. Let $E \to M$ be a vector bundle with a Hermitian metric, and ν_M a density on M. The L^2 inner product on $\mathcal{C}^\infty(M, E)$ is

$$\langle e, e'\rangle_{L^2} = \int_M \langle e(x), e'(x)\rangle \nu_M(x).$$

The space $L^2(M, E)$ of square-integrable sections in E is defined as the closure of $\mathcal{C}^\infty(M, E)$ with respect to the L^2 inner product.

Since M is compact, we obtain an equivalent L^2 inner product, hence the same L^2 space, if we start with a different inner product on E and a different density.

Theorem 4.21. *Every operator $A \in \Psi^z(M; E, F)$ with $\Re(z) \leq 0$ is bounded with respect to the L^2 norms on $\mathcal{C}^\infty(M, E)$ and $\mathcal{C}^\infty(M, F)$.*

As a consequence, A extends as a Hilbert space operator

$$A: L^2(M, E) \longrightarrow L^2(M, F).$$

Proof. The statement is clear for smoothing operators; using Fubini's formula we can check immediately that if K is defined by a continuous integral kernel $\kappa(x, y)$, then

$$\|Ke\|_{L^2(M,F)} \leq \sup_{x,y \in M} \|\kappa(x, y)\| \operatorname{Vol}(M) \|e\|_{L^2(M,E)}.$$

Thus it remains to check the statement locally in $\mathbb{R}^n$ for operators with symbol of compact support in x, y and acting on the trivial bundles $E = \mathbb{R}^k$, $F = \mathbb{R}^m$. Since boundedness of A is equivalent to boundedness for each coefficient A_{ij}, we can assume that A is a scalar operator. The condition of compact support implies that there exists $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ with $\psi \circ A = A = A \circ \psi$, where ψ denotes the operator of multiplication by ψ on $\mathcal{S}(\mathbb{R}^n)$.

Decompose A into an operator with symbol $a(x, \xi)$ (i.e., a is independent of x') and an operator S with smooth kernel, i.e., $A = \operatorname{Op}(a(x, \xi)) + S$. We deduce $A = \psi \operatorname{Op}(a)\psi + \psi S \psi$. The operator $\psi S \psi$ is defined by a compactly supported smooth integral kernel, hence it is clearly L^2-bounded. The operator of multiplication by ψ is also clearly bounded. To finish the proof, it is enough to show that a scalar operator of order z, $\Re(z) \leq 0$, with left symbol $a(x, \xi)$ with compact support in x, is bounded in L^2 sense.

Assume that $A \in \Psi^z(\mathbb{R}^n)$ with $\Re(z) < -n$. Then the symbol $a(x, \xi)$ is absolutely integrable in ξ, so for every $f, g \in \mathcal{S}(\mathbb{R}^n)$,

$$|\langle Af, Ag \rangle_{L^2}| \leq (2\pi)^{-2n} \sup_{x \in \mathbb{R}^n} \left(\int_{\mathbb{R}^n} |a(x, \xi)| d\xi \right)^2 \langle \hat{f}, \hat{g} \rangle.$$

Since the Fourier transform is an L^2 isometry up to a dilation (Lemma 4.2), the above inequality implies that A is bounded as an operator in L^2 with domain $\mathcal{S}$, thus it extends to a unique bounded operator.

Assume now that $z = 0$. Let $c > \sup_{(x,\xi) \in \mathrm{T}^*\mathbb{R}^n} |a(x, \xi)|^2$ (such a $c < \infty$ exists since a is of order 0). By the definition of symbols, $\lambda - a$ is a strictly positive smooth function on $\mathbb{R}^n \times \overline{\mathbb{R}}^n$. The square root $(c - |a(x, \xi)|^2)^{1/2}$ is also smooth with compact support on $\mathbb{R}^n \times \overline{\mathbb{R}}^n$, hence it defines a strictly positive symbol b_0 of order 0. Let $B_0 \in \Psi^0$ be an operator with symbol b_0. By replacing it with $(B_0 + B_0^*)/2$ if necessary, we can assume $B_0^* = B_0$. Since $\sigma_0(B_0^2) = \sigma_0(c - A^*A)$, there exists $R_1 \in \Psi^{-1}$ with

$$B_0^2 = c - A^*A + R_1.$$

Let $r_1 = \sigma_{-1}(R_1)$ be the principal symbol of R_1. Since B_0, c, and A^*A are self-adjoint, it follows that r_1 is real. Choose $C_1 \in \Psi^{-1}$ to be an operator with principal symbol $b_1 = -b_0^{-1}r_1/2$. Again, C_1 can be assumed to be self-adjoint. Set

$$B_1 := B_0 + C_1$$

Then $B_1^2 - B_0^2$ belongs to Ψ^{-1} and its principal symbol is $-r_1$. It follows that $B_1^2 = c - A^*A + R_2$ for some self-adjoint $R_2 \in \Psi^{-2}$. Continuing in this way, we find $B_n \in \Psi^0$ with

$$B_n^2 - (c - A^*A) = R_{n+1} \in \Psi^{-n-1}.$$

Let $\phi \in \mathcal{S}(\mathbb{R}^n)$. Then to bound the L^2 norm of $A\phi$ we write

$$\|A\phi\|_{L^2}^2 = \langle A^*A\phi, \phi\rangle_{L^2} = c\|\phi\|_{L^2}^2 + \langle R_{n+1}\phi, \phi\rangle_{L^2} - \|B_n\phi\|_{L^2}^2.$$

The term containing B_n is negative, while the term containing R_{n+1} is bounded in terms of $|\phi|^2$ by the first part of the proof.

Finally, for arbitrary z with $\Re(z) \leq 0$, let A' be the operator with symbol $a(x,\xi)(1+\xi^2)^{-z/2}$, and $D = \mathrm{Op}((1+\xi^2)^{z/2})$. By the composition theorem, $A'D = A$. The operator D is clearly L^2 bounded because after conjugation with the Fourier transform it becomes the multiplication operator by the bounded function $(1+\xi^2)^{z/2}$. The operator A' was shown above to be bounded. □

Let $A \in \Psi^z(M; E, F)$ be a pseudo-differential operator. Define an inner product on $\mathcal{C}^\infty(M, E)$ by

$$\langle \phi, \psi\rangle_A := \langle \phi, \psi\rangle_{L^2} + \langle A\phi, A\psi\rangle_{L^2}.$$

If $\Re(z) \leq 0$, by Theorem 4.21 the resulting norm is equivalent to the L^2 norm.

Lemma 4.22. *Let $A \in \Psi^z(M; E, F)$, $A' \in \Psi^{z'}(M; E, F')$ be two pseudo-differential operators with injective principal symbols, such that $\Re(z) = \Re(z') > 0$. Then the norms $\|\cdot\|_A^2$ and $\|\cdot\|_{A'}^2$ are equivalent.*

Proof. In finite dimensions, a linear map L between Euclidean vector spaces is injective if and only if L^*L is an isomorphism. Accordingly, if we choose metrics on M, E, F, F' in order to define adjoints, the operators

$$U := A^*A \quad \text{and} \quad U' := (A')^*A'$$

are elliptic. Let G be a parametrix for U, so $UG = I - R$ with $R \in \Psi^{-\infty}(M, E)$. Then $U' = UGU' + RU' = U(GU') + R'$ where $R' \in \Psi^{-\infty}(M, E)$ and

$GU' \in \Psi^{2z'-2z}(M, E)$. By the hypothesis and Theorem 4.21, GU' is bounded on L^2; since R' is also bounded, we see that $\|\cdot\|_A^2$ dominates $\|\cdot\|_{A'}^2$. The reverse inequality is analogous. $\square$

Based on this lemma, for every $k \geq 0$ we define the Sobolev space $H^k(M, E)$ as the completion of $\mathcal{C}^\infty(M, E)$ with respect to the inner product $\langle\cdot,\cdot\rangle_A$, for some pseudo-differential operator $A \in \Psi^z(M; E, F)$, $\Re(z) = k$, with injective principal symbol. The Hilbert space H^k itself is independent of the choice of A, while the inner product clearly depends on A.

With this definition, it is useful to show that the Sobolev spaces are subspaces of L^2.

Lemma 4.23. *For every $k \geq 0$, $H^k(M, E) \subset L^2(M, E)$.*

Proof. Fix $A \in \Psi^z(M; E, F)$ and let f_j be a Cauchy sequence for the norm $\|\cdot\|_A^2$. Then clearly the sequence f_j is also Cauchy with respect to the L^2 norm. Let $f \in L^2(M, E)$ denote its limit, and denote by $F = [(f_j)_{j\geq 1})]$ its limit in H^k. We claim that the map $F \mapsto f$ is injective. To prove injectivity, assume that $f = 0$ and fix $\varepsilon > 0$. Then there exists j_1 such that for $i, j \geq j_1$ we have $\|A(f_i - f_j)\|^2 < \varepsilon$. For $j \geq j_1$ fixed we write

$$\varepsilon > \|A(f_i - f_j)\|_{L^2}^2 = \|Af_i\|_{L^2}^2 - 2\Re\langle f_i, A^*Af_j\rangle + \|Af_j\|_{L^2}^2,$$

so, since $f_i \to f = 0$ as $i \to \infty$, we have $\limsup \|Af_i\|_{L^2}^2 < \varepsilon$. Since ε was arbitrary, $Af_i \to 0$, so the limit F of $(f_j)_{j\geq 1}$ in H^k is 0. $\square$

A pseudo-differential operator $A \in \Psi^z(M; E)$, $\Re(z) \geq 0$ is called *non-negative* if for every $\phi \in \mathcal{C}^\infty(M, E)$ one has

$$\langle A\phi, \phi\rangle \geq 0.$$

In this case, we clearly have $A^* = A$, which implies that $z \in \mathbb{R}$. A non-negative operator is called *positive* if there exists some $c > 0$ such that

$$\langle A\phi, \phi\rangle \geq c\|\phi\|^2, \quad \phi \in \mathcal{C}^\infty(M, E).$$

For every elliptic positive operator of order k, the norm $\|A\phi\|_{L^2}^2$ dominates ϕ^2, hence it is equivalent to the Sobolev norm $\|\cdot\|_A$.

Lemma 4.24. *Let $k \geq 0$ be an integer and $A \in \Psi^k(M; E, F)$ an elliptic, non-negative pseudo-differential operator. Then the operator*

$$I + A\colon H^k(M, E) \longrightarrow L^2(M, E)$$

is an isomorphism of Hilbert spaces.

Proof. We already know that $I + A$ is continuous as an operator from H^k to L^2. In fact, by defining the inner product on H^k using A, the non-negativity of A implies that $I + A$ increases norms, in the sense that

$$\|(I + A)\phi\|_{L^2}^2 \geq \|\phi\|_A^2. \tag{4.6}$$

Immediately this implies that the range of $I + A$ is closed. Let G be a parametrix for $I + A$, so $G(I + A) = I + R$, $(I + A)G = I + R'$ with smoothing error terms R, R'. Then every $\phi \in H^k$ which is mapped to 0 by $1 + A$ must satisfy $\phi + R\phi = 0$, thus ϕ is smooth. By the non-negativity of A, this implies that $\phi = 0$. Now, let $\psi \in L^2(M, E)$ be orthogonal to the range of $I + A$. Then ψ must be orthogonal to the range of $I + R'$, in particular $\psi \perp (I + R')\phi$ for all $\phi \in \mathcal{C}^\infty(M, E)$. This can be interpreted as $\phi \perp (I + (R')^*)\psi$, and since smooth sections are dense in L^2, $(I + (R')^*)\psi = 0$. Hence $\psi = -(R')^*\psi$ is smooth. But a smooth section orthogonal to the range of $I + A$ must be in the kernel of $I + A$, hence it must be 0. Thus $I + A$ is continuous and bijective. The inverse is clearly continuous thanks to (4.6), or alternately thanks to the bounded inverse theorem. □

The above lemma can be extended to every elliptic positive operator of real non-negative order k instead of $I + A$; the modifications in the proof are minor.

Elliptic non-negative pseudo-differential operators are easily constructed; for instance, for any $B \in \Psi^{k/2}(M, E, F)$ with injective symbol, the operator

$$A := B^* B$$

is both elliptic and non-negative.

Lemma 4.25. *Let $A \in \Psi^z(M; E, F)$ be an elliptic pseudo-differential operator with $k := \Re(z) \geq 0$. Assume that*

$$A\colon H^k(M, E) \longrightarrow L^2(M, F)$$

is a linear isomorphism. Then the inverse operator

$$A^{-1}\colon L^2(M, F) \longrightarrow H^k(M, E)$$

belongs to $\Psi^{-z}(M; F, E)$.

Proof. Choose a parametrix G with $GA = I + R$, $AG = I + R'$ for some smoothing errors R, R'. Multiplying by A^{-1} we get $G = A^{-1} + RA^{-1} = A^{-1} + A^{-1}R'$ as operators on L^2. This yields

$$A^{-1} = G - RA^{-1} = G - A^{-1}R' = RA^{-1}R' - RG + G.$$

The last two terms are in $\Psi^{-\infty}$, resp. Ψ^{-z}. We claim that the operator $RA^{-1}R'$ is given by a smooth integral kernel $\kappa(x, x')$. For every $x, x' \in M$ define $\kappa(x, x') = \langle r(x, \cdot), A^{-1}r'(\cdot, x')\rangle$. Since A^{-1} is bounded in L^2, κ is continuous. Applying the same reasoning for the smoothing operators PR, resp. $R'P'$ where P, P' are arbitrary differential operators on M, we see that $P\kappa P'$ is again continuous, so κ is smooth. □

Thus, the Sobolev spaces H^k are isomorphic to L^2, the isomorphism and its inverse being both given by pseudo-differential operators. From this we deduce that for $k > k' > 0$ we have $H^k(M, E) \subset H^{k'}(M, E) \subset L^2(M, E)$. For $k > 0$, define $H^{-k}(M, E)$ as the dual of the Hilbert space $H^k(M, E)$, i.e., the space of continuous linear functionals on $H^k(M, E)$. It clearly does not depend on the choice of norm on $H^k(M, E)$. It follows that for all $k, k' \in \mathbb{R}$, $k > k'$ we have canonical inclusions

$$H^k(M, E) \hookrightarrow H^{k'}(M, E).$$

Hence, every pseudo-differential operator $B \in \Psi^z(M; E, F)$ acts boundedly from $H^k(M, E)$ to $H^{k-\Re(z)}(M, F)$ for every $k \in \mathbb{R}$. Moreover, $\Psi^{-\infty}(M; E, F)$ maps $H^k(M, E)$ into $\mathcal{C}^\infty(M, F)$.

Proposition 4.26 (Elliptic Regularity). *Let $A \in \Psi^z(M; E, F)$ be elliptic, and $k \in \mathbb{R}$. If $\phi \in H^h(M, E)$ for some $h \in \mathbb{R}$ such that $A\phi \in H^k(M, F)$, then $\phi \in H^{k+\Re(z)}(M, E)$. If $A\phi \in \mathcal{C}^\infty(M, F)$, then $\phi \in \mathcal{C}^\infty(M, E)$.*

Proof. Let $G \in \Psi^{-z}(M; E, F)$ be a parametrix,

$$GA = I + R.$$

Then $\phi = GA\phi - R\phi$. Since R is smoothing, $R\phi$ is smooth and so it belongs to every Sobolev space. If $A\phi \in H^k$, then from the mapping properties of pseudo-differential operators it follows that $GA\phi \in H^{k+\Re(z)}$. Assume now $A\phi$ to be smooth. Then $GA\phi$ is smooth by the definition of pseudo-differential operators as acting on smooth functions; since $R\phi$ is also smooth, we have $\phi \in \mathcal{C}^\infty(M, F)$. □

4.10 Compact operators

Fix a Hilbert space H and let $B(H)$ be the algebra of bounded linear endomorphisms of H. Let $\mathcal{F}$ denote in this section the bilateral ideal of operators of finite rank. Let $\mathcal{C}(H)$ denote the norm-closure of $\mathcal{F}$ in $B(H)$. It is again a bilateral ideal.

Definition 4.27. A *compact* operator on H is an operator which can be approximated in norm by finite-rank operators.

Clearly, the space of compact operators is just the ideal $\mathcal{C}(H)$. Alternately, an operator in $B(H)$ is compact if and only if it maps bounded sets into relatively compact sets in H.

Lemma 4.28. *Every smoothing operator $R \in \Psi^{-\infty}(M, F)$ is compact as an operator on $L^2(M, F)$.*

Proof. By the Stone–Weierstraß theorem, every continuous function $r(x, x')$ on $M \times M$ can be $\mathcal{C}^0$-approximated by finite sums of terms of the form $f(x)g(x')$ (such a term defines a rank 1 operator). The operator norm of R is bounded by (in fact, equivalent to) the sup norm of its integral kernel r. Thus R can be approximated in norm by finite-rank operators. □

Definition 4.29. A bounded linear operator $A\colon H \to H'$ between Hilbert spaces is called *Fredholm* if

(1) $\dim \operatorname{Ker}(A) < \infty$ and

(2) $\dim \operatorname{Coker}(A) < \infty$, where the cokernel of A, $\operatorname{Coker}(A)$, is the linear space $H'/A(H)$.

The second condition implies that the range of A is closed in H'. Indeed, after factoring A through $H/\operatorname{Ker}(A)$, we can assume that A is injective. Let V be a finite-dimensional complement in H' to $A(H)$, i.e., $H' = V \oplus A(H)$. Every finite-dimensional subspace in H' is a Hilbert space. Then the linear map

$$\Phi\colon V \oplus H \longrightarrow H', \quad (v, x) \longmapsto v + A(x),$$

is bounded and bijective between Hilbert spaces, hence by the open mapping theorem it maps the closed subspace H bijectively onto another closed subspace, which is precisely $A(H)$.

Proposition 4.30. *An operator $A\colon H \to H'$ is Fredholm if and only if it admits an inverse $G\colon H' \to H$ modulo compact operators, i.e.,*

$$GA = I_H + R, \quad \textit{and} \quad AG = I_{H'} + R'$$

with $R \in \mathcal{C}(H)$, $R' \in \mathcal{C}(H')$.

Definition 4.31. The *index* of a Fredholm operator $A\colon H \to H'$ is the difference

$$\operatorname{Ind}(A) := \dim \operatorname{Ker}(A) - \dim \operatorname{Coker}(A) \in \mathbb{Z}.$$

Theorem 4.32. *Let $A \in \Psi^k(M; E, F)$ be an elliptic (pseudo-)differential operator on a compact manifold M,*

$$A\colon \mathcal{C}^\infty(M, E) \longrightarrow \mathcal{C}^\infty(M, F).$$

Then A has finite-dimensional kernel and cokernel.

Proof. Let $G \in \Psi^{-k}(M; F, E)$ be a parametrix for A. By elliptic regularity, the kernel of

$$A\colon H^k(M, E) \longrightarrow L^2(M, F)$$

is precisely the kernel of $A\colon \mathcal{C}^\infty(M, E) \to \mathcal{C}^\infty(M, F)$. Let now $\psi \in L^2(M, F)$ be orthogonal to $A(H^k(M, E))$. Then $\psi \perp AG\phi$ for every $\phi \in \mathcal{C}^\infty(M, F)$, where $AG = I + R'$, $R' \in \Psi^{-\infty}(M, F)$. This means that $(I + R')\psi$ is orthogonal to ϕ, and since ϕ was arbitrary we have $\psi = -R'\psi \in \mathcal{C}^\infty(M, F)$. This shows that the orthogonal complement $A(H^k(M, E))^\perp$ is $\mathrm{Ker}(A^*)$, which consists of smooth sections thanks to elliptic regularity. Thus $L^2(M, F)$ decomposes into the orthogonal direct sum

$$L^2(M, F) = A(H^k(M, E)) \oplus \mathrm{Ker}(A^*), \quad \mathrm{Coker}(A) \simeq \mathrm{Ker}(A^*) \subset \mathcal{C}^\infty(M, F).$$

Hence every $\psi \in L^2(M, F)$ which is orthogonal to the finite-dimensional space $\mathrm{Coker}(A)$ is of the form $A(\phi)$ with $\phi \in H^k(M, E)$. By elliptic regularity, if ψ is smooth, so is ϕ. Let $f_1, \dots, f_m$ be an orthonormal basis consisting of smooth sections in the finite-dimensional space $\mathrm{Ker}(A^*)$. The proof is done by noting that every $f \in \mathcal{C}^\infty(M, F)$ can be L^2-orthogonally decomposed as

$$f = a_1 f_1 + \cdots + a_m f_m + f',$$

with $a_j = \langle f, f_j \rangle_{L^2}$ and $f' \perp \mathrm{Ker}(A^*)$, $f' \in \mathcal{C}^\infty(M, F)$. □

The proof also shows that the kernel and cokernel of

$$A\colon H^s(M, E) \longrightarrow H^{k-s}(M, F)$$

are independent of $s \in \mathbb{R}$ and consist of smooth sections.

Corollary 4.33. *The eigenspaces of any elliptic operator $A \in \Psi^k(M; E, F)$ with $k > 0$ are finite dimensional.*

Proof. If $k > 0$, then for every $\lambda \in \mathbb{C}$ the operator $A - \lambda I$ is elliptic. Apply Theorem 4.32 to $A - \lambda I$. □

Lemma 4.28 is in fact valid for operators of negative order.

Lemma 4.34. *Let $A \in \Psi^z(M; E, F)$, $\Re(z) < 0$. Then*

$$A\colon L^2(M, E) \longrightarrow L^2(M, F)$$

is compact.

Another way of stating this is that the canonical inclusion

$$H^k(M, F) \hookrightarrow L^2(M, F)$$

is compact; this is known as the Rellich lemma.

Proof. Since operators in Ψ^{iy} act boundedly on L^2 for $y \in \mathbb{R}$, $\Psi^{-\infty}$ consists of compact operators, and the compact operators form an ideal in $B(L^2)$, it is enough to prove the claim for *one* elliptic operator in $\Psi^k(M, E)$, $k < 0$. Since multiplication by cut-off functions is bounded on L^2, we can localize the proof to a relatively compact domain $\Omega \subset \mathbb{R}^n$, for scalar operators. It is enough to show that the operator $\mathrm{Op}((1 + |\xi|^2)^{k/2})$ is compact from $L^2(\Omega)$ to $L^2(\mathbb{R}^n)$. By the Cauchy–Schwarz inequality, $\hat{f}$ is smooth and there exist c_0, c_1 depending on Ω such that for every $f \in L^2(\Omega)$,

$$\|\hat{f}\|_{\mathcal{C}^0} \leq c_0 \|f\|_{L_2} \quad \text{and} \quad \|\partial_{\xi_j} \hat{f}\|_{\mathcal{C}^0} \leq c_1 \|f\|_{L_2}$$

(for instance, $c_0 = \mathrm{vol}(\Omega)^{1/2}$). Let $(f_j)_{j \geq 1}$ be a sequence in $L^2(\Omega)$, $\|f_j\|_{L^2} \leq 1$. Since $\mathrm{Op}((1 + |\xi|^2)^{k/2}) f$ is the Fourier transform of $(1 + |\xi|^2)^{k/2} \hat{f}$, it is enough to show that the sequence $(1 + |\xi|^2)^{k/2} \hat{f}_j$ admits a convergent subsequence.

Take a dense countable subset $p_1, p_2, \ldots$ in $\mathbb{R}^n$ (like the points with rational coordinates). Since $\hat{f}_j(p_k)$ is bounded by c_0, by the local compactness of $\mathbb{C}$ we can extract a subsequence such that $\hat{f}_{j_s}(p_1)$ is convergent. Extract then a sub-subsequence such that the evaluations at p_2 are also convergent, and so on. The diagonal procedure applied to this nested sequence of sub-sequences produces a subsequence $(\hat{f}_i)_{i \in \mathcal{I}}$ with $\hat{f}(p_k)$ convergent for all k. Since the derivatives of $\hat{f}_j$ are uniformly bounded by c_1 and the p_k's are dense, it follows that $\hat{f}_i(\xi)$ is convergent to some $u(\xi)$ for all $\xi \in \mathbb{R}^n$. By the Lebesgue dominated convergence theorem, this function is square-integrable on every ball $B_R(0)$ in $\mathbb{R}^n$ (i.e., it is locally L^2). Thus, after relabeling the sequence, it follows that $\hat{f}_j$ is a Cauchy sequence in $L^2(B_R(0))$ for every R. On the other hand, the norm of $\hat{f}_j(1 + |\xi|^2)^{k/2}$ in $L^2(\mathbb{R}^n \setminus B_R(0))$ is uniformly bounded above by $c_0(1 + R^2)^{k/2}$, hence small for large R since k is negative. This means that $\hat{f}_j(1 + |\xi|^2)^{k/2}$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$. □

4.11 Eigenvalues of self-adjoint elliptic operators

We recall that a pseudo-differential operator $A \in \Psi^k(M, E)$ is called here *self-adjoint* if M is endowed with a volume density, E with a Hermitian metric, and

$$\langle Af, f' \rangle_{L^2} = \langle f, Af' \rangle_{L^2}, \quad f, f' \in \mathcal{C}^\infty(M, E).$$

By continuity, the same identity holds for $f \in H^s(M, E)$, $f' \in H^{k-s}(M, E)$. This is not the usual definition of self-adjoint unbounded operators, but for our purposes it is adequate.

Theorem 4.35. *Let $A \in \Psi^k(M, E)$ be elliptic and self-adjoint, with $k > 0$. Then A has finite-dimensional eigenspaces, with discrete eigenvalues accumulating to $\pm\infty$. Moreover, the space $L^2(M, E)$ decomposes as an orthogonal Hilbert direct sum of the eigenspaces of A.*

Proof. Let $f_1, \dots, f_m$ be an orthonormal basis in $\mathrm{Ker}(A)$. It consists of smooth sections in E, so the integral kernel

$$p(x, x') := \sum_{j=1}^{m} f_j(x) \otimes f_j(x')$$

is smooth and defines the orthogonal projector onto $\mathrm{Ker}(A)$ in $L^2(M, E)$. Hence the operator $A + P_{\mathrm{Ker}\,A}$ is pseudo-differential, elliptic, self-adjoint and invertible. It follows from Lemma 4.25 that $B := (A + P_{\mathrm{Ker}\,A})^{-1}$ belongs to $\Psi^{-k}(M, E)$. The operator B is bounded on $L^2(M, E)$ by Theorem 4.21, and in fact compact by Lemma 4.34. It is also clearly self-adjoint.

Hence B shares the properties of every self-adjoint compact operator on a Hilbert space: the eigenspaces of B are finite dimensional, mutually orthogonal, and they span a dense subspace of L^2. The eigenvalues are real, and they accumulate to 0. Since B is invertible, 0 is not an eigenvalue.

Every eigenspace of B of eigenvalue λ is also an eigenspace of $A + P_{\mathrm{Ker}\,A}$, of eigenvalue λ^{-1}. Moreover, both A and $A + P_{\mathrm{Ker}\,A}$ preserve the orthogonal splitting $L^2 = \mathrm{Ker}(A) \oplus \mathrm{Ker}(A)^\perp$, hence they have essentially the same eigenspaces with the same eigenvalues, with the exception of $\mathrm{Ker}(A)$, which is an eigenspace of eigenvalue 1 for $A + P_{\mathrm{Ker}\,A}$, resp. 0 for A. The conclusion follows. □

Since the Dirac operator $\mathcal{D}$ is elliptic and self-adjoint, we obtain directly

Corollary 4.36. *Let (M, g) be a closed Riemannian spin manifold and ΣM the spinor bundle. There exists an orthonormal basis in the Hilbert space $L^2(M, \Sigma M)$ consisting of smooth eigenspinors. Each eigenspace is finite dimensional, and the eigenvalues accumulate to $\pm\infty$.*

The discrete set of eigenvalues of $\mathcal{D}$ considered with their multiplicities is called the *spectrum* of $\mathcal{D}$.

Similarly, the Laplace operator $\Delta = d^*d$ on $\mathcal{C}^\infty(M)$ is self-adjoint and elliptic. Thus we deduce

Corollary 4.37. *Let (M, g) be a closed connected Riemannian manifold. There exists an orthonormal basis in the Hilbert space $L^2(M)$ consisting of smooth eigenfunctions of Δ. Each eigenspace is finite dimensional, and the eigenvalues accumulate to ∞. The kernel of Δ is the space of constant functions on M.*

Only the last statement deserves some explanation. If $\Delta f = 0$, then by integrating against f we get

$$0 = \langle \Delta f, f \rangle_{L^2} = \langle d^* df, f \rangle_{L^2} = \langle df, df \rangle_{L^2} = \|df\|^2_{L^2(M,\Lambda^1 M)}$$

and so $df = 0$, which is equivalent to $f =$ constant.

Corollary 4.38 (Sobolev embedding). *Let f be a smooth function (resp. a smooth spinor) on a compact Riemannian (resp. spin) manifold (M, g). Let*

$$f = \sum f_j$$

be the orthogonal L^2 decomposition of f into eigenfunctions of Δ, resp. eigenspinors. Then the above series converges in $\mathcal{C}^k$ norm for every $k \geq 0$.

Proof. Let $\{e_i\}$ be an orthonormal basis of L^2 consisting of eigensections of $D = \Delta$ or $\mathcal{D}$ with eigenvalue λ_i, and write $f = \sum a_i e_i$. Then $\langle Df, e_i \rangle = \langle f, De_i \rangle = \lambda_i a_i$, so the decomposition of Df in the basis $\{e_i\}$ is $Df = \sum a_i \lambda_i e_i$. It follows that the L^2 decomposition of f into eigenmodes of D converges in every H^s norm. We now prove that the $\mathcal{C}^k$ norm is dominated by any H^s norm with $s > k + n/2$ where $n = \dim M$. This can be checked locally, thus for compactly supported functions on $\mathbb{R}^n$. In that setting, it is a consequence of the fact that $(1 + |\xi|^2)^{\frac{k-s}{2}}$ is square-integrable under our hypothesis $k - s < -n/2$. □

Corollary 4.39. *Let (M, g) be a closed connected Riemannian manifold. Then every square-integrable function $f \in L^2(M, \nu_g)$ of mean 0 with respect to the Riemannian volume density ν_g (i.e., $\int_M f \nu_g = 0$) belongs to the image of the Laplacian. If f belongs to the Sobolev space H^k, $k \geq 0$, then $f = \Delta u$ with $u \in H^{k+2}$ unique up to an additive constant. If f is smooth, then the solution u is also smooth.*

Proof. The condition $\int_M f \nu_g = 0$ is equivalent to f being L^2-orthogonal to the constant function 1. But we have seen in Corollary 4.37 that $\mathrm{Ker}(\Delta)$ is spanned by the constant functions. Thus f decomposes as a series $f = \sum_{j=1}^{\infty} f_j \phi_j$ in terms of an orthonormal basis of eigenfunctions ϕ_j for Δ, of eigenvalues $\lambda_j > 0$. Set $u := \sum_{j=1}^{\infty} \lambda_j^{-1} f_j \phi_j$. Since the sequence λ_j is increasing, the series defining u is absolutely convergent, hence it defines an L^2 function. The regularity properties follow from Proposition 4.26. □

Part II

Lowest eigenvalues of the Dirac operator on closed spin manifolds

Chapter 5

Lower eigenvalue bounds on Riemannian closed spin manifolds

In this chapter we start by recalling the following fundamental result of A. Lichnerowicz: on a closed (compact without boundary) Riemannian spin manifold there exists a topological obstruction to the existence of metrics with positive scalar curvature. This result is obtained as a corollary of a (non-optimal) eigenvalue estimate for the Dirac operator. We then derive Friedrich's (optimal) inequality, which can be seen as the spinorial Cauchy–Schwarz inequality. In fact, we shall see that the defect in such an equality is measured by the Penrose operator. We also give necessary conditions for the existence of "special spinor fields" satisfying the limiting case of this inequality. Finally, we show that the conformal covariance of the Dirac operator and that of the Yamabe operator lead to a relation between their first eigenvalues.

5.1 The Lichnerowicz theorem

An important consequence of the Schrödinger–Lichnerowicz formula is the following gap phenomenon in the spectrum of the Dirac operator [Lic63].

Theorem 5.1. *On a compact Riemannian spin manifold with positive scalar curvature* Scal*, the Dirac operator has no eigenvalues in the closed interval*

$$\Big[-\frac{1}{2}\sqrt{\mathrm{Scal}_0}, +\frac{1}{2}\sqrt{\mathrm{Scal}_0}\Big],$$

where Scal_0 *denotes the infimum of the scalar curvature.*

Proof. If ∇^* denotes the formal adjoint of the Levi-Civita connection with respect to the natural Hermitian scalar product $\langle \cdot, \cdot \rangle$ on the spinor bundle, then applying the Schrödinger–Lichnerowicz formula

$$\mathcal{D}^2 = \nabla^*\nabla + \frac{1}{4}\mathrm{Scal}, \tag{5.1}$$

to a spinor field Ψ, taking its scalar product with Ψ, and integrating over the compact manifold M with respect to the volume element v_g, yields

$$\int_M |\nabla\Psi|^2 v_g = \int_M |\mathcal{D}\Psi|^2 v_g - \frac{1}{4}\int_M \mathrm{Scal}|\Psi|^2 v_g. \tag{5.2}$$

Let λ be any eigenvalue of $\mathcal{D}$ and Ψ a corresponding eigenspinor. Then since $|\nabla\Psi|^2 \geq 0$, (5.2) yields

$$\lambda^2 \geq \frac{1}{4}\mathrm{Scal}_0.$$

If equality holds, then the scalar curvature is constant and $\nabla\Psi \equiv 0$. Hence, $\mathcal{D}\Psi = 0$ and

$$0 = \lambda^2 = \frac{1}{4}\mathrm{Scal}_0 = \frac{1}{4}\mathrm{Scal},$$

which contradicts the curvature assumption. Therefore,

$$\lambda^2 > \frac{1}{4}\mathrm{Scal}_0 \tag{5.3}$$

which concludes the proof. □

A fundamental consequence of this result is the famous Lichnerowicz theorem. Before stating this result, let us recall from Chapter 4 that the *analytical index* of an elliptic operator D is defined as

$$\mathrm{Ind}\,\mathcal{D} = \dim \mathrm{Ker}\,\mathcal{D} - \dim \mathrm{Coker}\,\mathcal{D} \tag{5.4}$$

(by Theorem 4.32, both quantities in the right-hand side are finite). Since the Dirac operator $\mathcal{D}$ is elliptic and formally self-adjoint, its analytical index is zero. As we have seen, on an even-dimensional manifold, under the action of the volume element, the spinor bundle splits into the direct sum of positive and negative spinors. The corresponding splitting of the Dirac operator is given by $\mathcal{D} = \mathcal{D}^+ + \mathcal{D}^-$, where $\mathcal{D}^+$ (resp. $\mathcal{D}^-$) interchanges positive (resp. negative) and negative (resp. positive) spinors. Since $(\mathcal{D}^+)^* = \mathcal{D}^-$, the analytical index of the elliptic operator $\mathcal{D}^+$ is given by

$$\mathrm{Ind}\,\mathcal{D}^+ = \dim \mathrm{Ker}\,\mathcal{D}^+ - \dim \mathrm{Ker}\,\mathcal{D}^-.$$

By the Atiyah–Singer index theorem 3.10, the analytical index of $\mathcal{D}^+$ is equal to the $\widehat{A}$-genus of M (see Section 3.4 for the definition), hence

Theorem 5.2. *On a compact Riemannian spin manifold with non-vanishing $\widehat{A}$-genus, there is no metric with positive scalar curvature.*

Proof. By Theorem 5.1, if the scalar curvature is positive, then

$$\mathrm{Ker}(\mathcal{D}^+) = \mathrm{Ker}(\mathcal{D}^-) = \{0\},$$

hence $\mathrm{Ind}\,\mathcal{D}^+ = 0$. □

The importance of this result resides in the surprising fact that a weak geometrical assumption ($\mathrm{Scal} > 0$) imposes severe restrictions on the topology of the manifold. Important extensions of positive scalar curvature obstructions were obtained by Hitchin, Gromov and Lawson, and Stolz [Hit74], [GL80b], [GL80a], [GL83], [Sto90], and [Sto92]. More recently, in a series of papers by Ammann, Dahl, and Humbert [ADH09a], [AH08], and [ADH09b], the authors obtained some refined topological obstructions. See also [Gin09] for a short survey of these results.

We are ready now to give a uniform approach, based on the use of the Penrose operators introduced in Section 2.3, yielding to basic optimal refinements of inequality (5.3). For metrics with positive scalar curvature, three families of compact spin manifolds are of interest: Riemannian, Kählerian, and quaternion-Kähler manifolds. It turns out that on the spectral level, via the twistor construction, the three structures are related. The round sphere, the complex projective space (of odd complex dimension), and the quaternionic projective space are the model spaces of such families.

We will show that in the three cases, if λ_1 denotes the lowest eigenvalue of the Dirac operator, then the ratio $\lambda_1^2/\mathrm{Scal}_0$ for the given manifold is at least equal to the corresponding ratio for the model space.

It is worth noting that the size of the gap in the spectrum of the Dirac operator increases with the rigidity of the geometry of the compact Riemannian spin manifold. As we have seen in Section 2.3, it is sufficient to determine the Penrose operators of manifolds with special holonomy group.

5.2 The Friedrich inequality

The next step is to improve the Lichnerowicz inequality (5.3) so that equality could be achieved. We have the following result of Th. Friedrich [Fri80].

Theorem 5.3. *On a compact Riemannian spin manifold any eigenvalue λ of the Dirac operator satisfies*

$$\lambda^2 \geq \frac{n}{4(n-1)}\mathrm{Scal}_0. \tag{5.5}$$

Moreover, if equality holds, then the manifold is Einstein.

Remark 5.4. Of course inequality (5.5) is trivial if $\mathrm{Scal}_0 \leq 0$, so it is interesting only for the case $\mathrm{Scal}_0 > 0$. In this case, denote by

$$\lambda_1^2 := \lambda_1^2(M^n, g)$$

the lowest eigenvalue of $\mathcal{D}^2$ on a compact Riemannian spin manifold (M^n, g) with positive scalar curvature and let ν_1 be the ratio

$$\frac{\lambda_1^2}{\mathrm{Scal}_0}.$$

Since

$$\lambda_1^2(\mathbb{S}^n, g_0) = \frac{n^2}{4}$$

(see [Sul79]), equation (5.5) can be recast as

$$\nu_1(M^n, g) \geq \nu_1(\mathbb{S}^n, g_0), \tag{5.6}$$

where $(\mathbb{S}^n, g_0)$ denotes the standard n-dimensional sphere with its canonical structure.

Proof. The first proof we give here, due to Sylvestre Gallot, is based on the Cauchy–Schwarz inequality, like for the corresponding problem for the Laplacian on functions and differential forms; see [Lic58] and [GM75]. For an arbitrary spinor field $\Psi \in \Gamma(\Sigma M)$ we have

$$\begin{aligned}
|\mathcal{D}\Psi|^2 &= \left| \sum_{i=1}^{n} e_i \cdot \nabla_{e_i}\Psi \right|^2 \\
&\leq \Big(\sum_{i=1}^{n} |e_i \cdot \nabla_{e_i}\Psi| \Big)^2 \\
&= \Big(\sum_{i=1}^{n} 1 |\nabla_{e_i}\Psi| \Big)^2 \\
&\leq n \sum_{i=1}^{n} |\nabla_{e_i}\Psi|^2 \\
&= n|\nabla\Psi|^2.
\end{aligned} \tag{5.7}$$

This inequality combined with the Schrödinger–Lichnerowicz formula implies that

$$\frac{1}{n}\int_M |\mathcal{D}\Psi|^2 \nu_g \leq \int_M |\nabla\Psi|^2 \nu_g = \int_M |\mathcal{D}\Psi|^2 \nu_g - \frac{1}{4}\int_M \mathrm{Scal}|\Psi|^2 \nu_g,$$

whence

$$\Big(1 - \frac{1}{n}\Big)\int_M |\mathcal{D}\Psi|^2 \nu_g \geq \frac{1}{4}\int_M \mathrm{Scal}|\Psi|^2 \nu_g.$$

For an eigenspinor $\Psi \in \Gamma(\Sigma M)$, it follows that

$$\lambda^2 \int_M |\Psi|^2 \nu_g \geq \frac{1}{4}\frac{n}{n-1}\int_M \mathrm{Scal}|\Psi|^2 \nu_g \geq \frac{1}{4}\frac{n}{n-1}\mathrm{Scal}_0 \int_M |\Psi|^2 \nu_g.$$

For the last statement, see Proposition 5.11 below. □

Remark 5.5. If $\Psi \in \Gamma(\Sigma M)$ is an eigenspinor for which equality holds, that is

$$\lambda_0^2 = \frac{n}{4(n-1)}\mathrm{Scal}_0,$$

then, for all $X \in \Gamma(\mathrm{T}M)$, the spinor field Ψ satisfies the *twistor equation*

$$\nabla_X \Psi + \frac{1}{n} X \cdot \mathcal{D}\Psi = 0,$$

which specializes to the *Killing equation*

$$\nabla_X \Psi + \frac{\lambda_0}{n} X \cdot \Psi = 0,$$

since $\mathcal{D}\Psi = \lambda_0 \Psi$.

To see this, note that if equality is achieved in (5.7), then by Cauchy–Schwarz, the two vectors are proportional, i.e.,

$$e_i \cdot \nabla_{e_i} \Psi = e_j \cdot \nabla_{e_j} \Psi, \quad i, j \in \{1, \dots, n\},$$

which implies, for all $i \in \{1, \dots, n\}$,

$$\mathcal{D}\Psi = n e_i \cdot \nabla_{e_i} \Psi,$$

i.e.,

$$\nabla_{e_i} \Psi + \frac{1}{n} e_i \cdot \mathcal{D}\Psi = 0.$$

For further considerations, it is useful to prove inequality (5.5) by considering an optimal decomposition of the gradient of a spinor. From Section 2.3, we know that the Penrose (or twistor) operator $\mathcal{P}$, the complement of the Dirac operator in the Levi-Civita covariant derivative, is locally given by

$$\begin{aligned}
\mathcal{P}\colon \Gamma(\Sigma M) &\xrightarrow{\nabla} \Gamma(\mathrm{T}^*M \otimes \Sigma M) \xrightarrow{\pi} \operatorname{Ker}\gamma \subset \Gamma(\mathrm{T}^*M \otimes \Sigma M), \\
\Psi &\longmapsto \sum_{i=1}^{n} e_i^* \otimes \nabla_{e_i} \Psi \longmapsto \sum_{i=1}^{n} e_i \otimes \Big(\nabla_{e_i} \Psi + \frac{1}{n} e_i \cdot \mathcal{D}\Psi\Big),
\end{aligned}$$

hence,

$$\mathcal{P}_X \Psi := \nabla_X \Psi + \frac{1}{n} X \cdot \mathcal{D}\Psi, \quad X \in \Gamma(\mathrm{T}M), \Psi \in \Gamma(\Sigma M).$$

Note that by definition, the Penrose operator satisfies

$$\sum_{i=1}^{n} e_i \cdot \mathcal{P}_{e_i}\Psi = 0, \quad \Psi \in \Gamma(\Sigma M),$$

and recall that, see (2.18),

$$|\nabla\Psi|^2 = |\mathcal{P}\Psi|^2 + \frac{1}{n}|\mathcal{D}\Psi|^2. \tag{5.8}$$

It is worth noting that (5.8) can be easily established as follows. Since Clifford multiplication by vectors is skew-symmetric, one has

$$\begin{aligned}
|\mathcal{P}\Psi|^2 &= \sum_{i=1}^{n} \langle \mathcal{P}_{e_i}\Psi, \mathcal{P}_{e_i}\Psi\rangle \\
&= \sum_{i=1}^{n} \left\langle \mathcal{P}_{e_i}\Psi, \nabla_{e_i}\Psi + \frac{1}{n}e_i \cdot \mathcal{D}\Psi\right\rangle \\
&= \sum_{i=1}^{n} \langle \mathcal{P}_{e_i}\Psi, \nabla_{e_i}\psi\rangle \\
&= \sum_{i=1}^{n} \left\langle \nabla_{e_i}\Psi + \frac{1}{n}e_i \cdot \mathcal{D}\Psi, \nabla_{e_i}\Psi\right\rangle \\
&= |\nabla\Psi|^2 - \frac{1}{n}|\mathcal{D}\Psi|^2.
\end{aligned}$$

Combining (5.8) with (5.2), we obtain

$$\int_M |\mathcal{P}\Psi|^2 v_g = \frac{n-1}{n}\int_M |\mathcal{D}\Psi|^2 v_g - \frac{1}{4}\int_M \mathrm{Scal}|\Psi|^2 v_g, \tag{5.9}$$

which implies (5.5). If equality holds in (5.5), i.e., if there exists a spinor field Ψ such that $\mathcal{D}\Psi = \lambda_0\Psi$, with $\lambda_0^2 = \frac{n}{4(n-1)}\mathrm{Scal}_0$, then

$$\nabla_X\Psi = -\frac{\lambda_0}{n}X \cdot \Psi, \quad X \in \Gamma(\mathrm{T}M).$$

Remark 5.6. In fact the Dirac and the twistor operators are given by the composition of the covariant derivative with the projections on the Spin_n-irreducible components of $\Gamma(\mathrm{T}^*M \otimes \Sigma M)$. This is a special case of a general approach developed in [SW68]. In [Feg87] it is shown that any such differential operator is conformally covariant (see Section 2.3.5, where this fact is proved explicitly for the Dirac and twistor operators). We refer to [Bra97] for further results in this direction, based on a systematic use of representation theory.

5.3 Special spinor fields

We now examine some necessary conditions for the existence of certain special spinor fields which arise naturally in the setup of eigenvalue estimates for the Dirac operator.

Definition 5.7. A non-trivial spinor $\Psi \in \Gamma(\Sigma M)$ is called

(i) a *twistor-spinor* if $\mathcal{P}\Psi = 0$, i.e., for all $X \in \Gamma(\mathrm{T}M)$,

$$\nabla_X \Psi + \frac{1}{n} X \cdot \mathcal{D}\Psi = 0;$$

(ii) a *real Killing spinor* if for some non-trivial real-valued function f, for all $X \in \Gamma(\mathrm{T}M)$,

$$\nabla_X \Psi + \frac{f}{n} X \cdot \Psi = 0. \tag{5.10}$$

Remark 5.8. In Definition 5.7 of real Killing spinors the function f is supposed to be real. In fact, A. Lichnerowicz showed that if the function f is complex-valued, then it should be either real or purely imaginary. In the second case, such spinors are called *imaginary Killing spinors*. Manifolds with imaginary Killing spinors have been classified by H. Baum in the case where the function is constant and by H. B. Rademacher for a general function. We refer to the book [BFGK91] for such classifications. In the real case, the function f is always constant (cf. Proposition 5.11 below).

The notion of a twistor-spinor, as an element in the kernel of the twistor operator, was introduced by A. Lichnerowicz, who showed that if such a spinor has no zero, then it can be conformally related to an imaginary Killing spinor. Since then, twistor-spinors and their zeros have been extensively studied by many people including Th. Friedrich, K. Habermann, and W. Kühnel and H.B. Rademacher; see [BFGK91].

For a spinor field $\Psi \in \Gamma(\Sigma M)$ one can define its *associated vector field* by

$$X_\Psi := i \sum_{j=1}^{n} \langle \Psi, e_j \cdot \Psi \rangle e_j .$$

The following result explains the terminology of Killing spinors.

Lemma 5.9. *For a real Killing spinor Ψ, the associated vector field X_Ψ is a* Killing vector field, *i.e.,* $\pounds_{X_\Psi} g = 0$, *where $\pounds_{X_\Psi}$ denotes the Lie derivative in the direction of X_Ψ.*

Proof. For arbitrary vector fields X, Y, and Z one has

$$\begin{aligned}(\pounds_X g)(Y,Z) := {} & Xg(Y,Z) - g(\pounds_X Y, Z) - g(Y, \pounds_X Z)\\ = {} & g(\nabla_X Y, Z) + g(Y, \nabla_X Z) - g([X,Y],Z) - g(Y,[X,Z])\\ = {} & g(\nabla_Y X, Z) + g(Y, \nabla_Z X)\end{aligned}$$

since ∇ is metric and torsion free. On the other hand, by the definition of the associated vector field, in normal coordinates at the point $x \in M$ it holds that

$$\begin{aligned}\nabla_Y X_\Psi &= i\sum_{j=1}^n [\langle \nabla_Y \Psi, e_j \cdot \Psi\rangle e_j + \langle \Psi, e_j \cdot \nabla_Y \Psi\rangle] e_j\\ &= -i\frac{f}{n}\sum_{j=1}^n \langle \Psi, (e_j \cdot Y - Y \cdot e_j)\cdot \Psi\rangle e_j\end{aligned}$$

since Ψ is a real Killing spinor. Now, for any vector fields Y and Z, we get

$$\begin{aligned}&(\pounds_{X_\Psi} g)(Y,Z)\\ &= -i\frac{f}{n}(\langle \Psi, (Z\cdot Y - Y\cdot Z)\cdot\Psi\rangle + \langle \Psi, (Y\cdot Z - Z\cdot Y)\cdot \Psi\rangle)\\ &= 0.\end{aligned}$$

□

Another important property of Killing spinors is the following:

Lemma 5.10. *The square length $|\Psi|^2$ of a real Killing spinor Ψ is constant. Hence, a non-trivial Killing spinor has no zeros.*

Proof. For this, we prove that for any vector field X, the Lie derivative of $|\Psi|^2$ in the direction of X is zero. By (5.10), it follows

$$\begin{aligned}X|\Psi|^2 &= \langle \nabla_X \Psi, \Psi\rangle + \langle \Psi, \nabla_X \Psi\rangle\\ &= -\frac{f}{n}(\langle X\cdot\Psi, \Psi\rangle + \langle \Psi, X\cdot \Psi\rangle)\\ &= 0.\end{aligned}$$

□

We now examine some necessary conditions for the existence of real Killing spinors.

Proposition 5.11. *Let $\Psi \in \Gamma(\Sigma M)$ be a real Killing spinor on a Riemannian spin manifold (M^n, g). Then the manifold is Einstein with positive scalar curvature* Scal_0 *and the function f is constant, given by*

$$f^2 = \lambda_0^2 = \frac{n}{4(n-1)}\mathrm{Scal}_0.$$

If (M, g) is complete, then M compact.

Proof. Since Ψ is a Killing spinor, for all $X, Y \in \Gamma(\mathrm{T}M)$ we have

$$\nabla_Y \Psi = -\frac{f}{n} Y \cdot \Psi,$$

$$\begin{aligned}\nabla_X \nabla_Y \Psi &= -\frac{1}{n} X(f) Y \cdot \Psi - \frac{f}{n} Y \cdot \nabla_X \Psi - \frac{f}{n}(\nabla_X Y) \cdot \Psi \\ &= -\frac{1}{n} X(f) Y \cdot \Psi + \frac{f^2}{n^2} Y \cdot X \cdot \Psi - \frac{f}{n}(\nabla_X Y) \cdot \Psi,\end{aligned}$$

$$\nabla_Y \nabla_X \Psi = -\frac{1}{n} Y(f) X \cdot \Psi + \frac{f^2}{n^2} X \cdot Y \cdot \Psi - \frac{f}{n}(\nabla_Y X) \cdot \Psi,$$

and

$$\mathcal{R}_{X,Y} \Psi = -\frac{1}{n}(X(f)Y - Y(f)X) \cdot \Psi + \frac{f^2}{n^2}(Y \cdot X - X \cdot Y) \cdot \Psi.$$

On the other hand, using the Ricci identity (2.41) we get

$$\begin{aligned}-\frac{1}{2}\mathrm{Ric}(X) \cdot \Psi = &-\frac{1}{n}\sum_i (X(f) e_i \cdot e_i \cdot \Psi - e_i(f) e_i \cdot X \cdot \Psi) \\ &+ \frac{f^2}{n^2}\sum_i (e_i \cdot e_i \cdot X - e_i \cdot X \cdot e_i) \cdot \Psi.\end{aligned}$$

Using the fact that $e_i \cdot e_i = -1$ and $\sum_i e_i \cdot X \cdot e_i = (n-2)X$, we further have

$$\begin{aligned}-\frac{1}{2}\mathrm{Ric}(X) \cdot \Psi &= -\frac{1}{n}(-nX(f) \cdot \Psi - df \cdot X \cdot \Psi) \\ &\quad + \frac{f^2}{n^2}(-nX - (n-2)X) \cdot \Psi \\ &= X(f)\Psi + \frac{1}{n} df \cdot X \cdot \Psi - \frac{2(n-1)}{n^2} f^2 X \cdot \Psi.\end{aligned} \tag{5.11}$$

Note that for a k-form α and any spinor Ψ,

$$\langle \alpha \cdot \Psi, \Psi \rangle = (-1)^{\frac{k(k+1)}{2}} \overline{\langle \alpha \cdot \Psi, \Psi \rangle}.$$

Hence, for any vector X and any spinor Ψ,

$$\langle X \cdot \Psi, \Psi \rangle = -\overline{\langle X \cdot \Psi, \Psi \rangle},$$

so the function $\langle X \cdot \Psi, \Psi \rangle$ is purely imaginary. For $X := \operatorname{grad} f$, then it follows that

$$\begin{aligned}\Big\langle -\frac{1}{2}\mathrm{Ric}(X) \cdot \Psi + \frac{2(n-1)}{n^2} f^2 X \cdot \Psi, \Psi \Big\rangle &= X(f)|\Psi|^2 + \frac{1}{n}\langle df \cdot X \cdot \Psi, \Psi \rangle \\ &= df(X)|\Psi|^2 - \frac{1}{n}|df|^2\,|\Psi|^2 \\ &= \Big(1 - \frac{1}{n}\Big)|df|^2\,|\Psi|^2.\end{aligned}$$

The left-hand side being purely imaginary, we get $|df|^2 = 0$, since by Lemma 5.10, $|\Psi|^2 \neq 0$. This yields

$$X(f)\Psi = \frac{1}{n} df \cdot X \cdot \Psi = 0, \quad X \in \Gamma(\mathrm{T}M),$$

which when combined with (5.11), implies that for all $X \in \Gamma(\mathrm{T}M)$

$$\mathrm{Ric}(X) = \frac{4(n-1)}{n^2} f^2 X.$$

Thus $\frac{4(n-1)}{n^2} f^2 = \frac{\mathrm{Scal}_0}{n}$, by the definition of the scalar curvature. The last statement follows directly from Myers theorem. □

It should be mentioned that real Killing spinors first appeared in mathematical physics in the context of supergravity and then in superstring theories. In our context, they appear as eigenspinors associated with the minimal eigenvalues of the Dirac operator. The isometry groups of compact manifolds with Killing spinors have special properties; see [Mor00]. The classification of compact manifolds carrying real Killing spinors has been obtained by C. Bär and will be discussed in detail in Chapter 8 (cf. Theorem 8.37).

5.4 The Hijazi inequality

In the remaining part of this section we give a qualitative improvement of Theorem 5.3 based on the conformal covariance of the Dirac operator; cf. [Hit74] and [Hij86b].

Theorem 5.12. *On a compact Riemannian spin manifold* (M^n, g) *any eigenvalue* λ *of the Dirac operator satisfies*

$$\lambda^2 \geq \frac{n}{4(n-1)} \sup_u \inf_M \left(\overline{\mathrm{Scal}}\, e^{2u}\right), \tag{5.12}$$

where $\overline{\mathrm{Scal}}$ *is the scalar curvature of the metric* $\bar{g} = e^{2u} g$.

Proof. We use the notations and the facts concerning conformal covariance of the Dirac operator established in Section 2.3.5. For a metric $\bar{g} = e^{2u} g$, in the conformal class of g, the associated Dirac operators are related by the identity

$$\bar{\mathcal{D}}(e^{-\frac{(n-1)}{2}u}\bar{\Psi}) = e^{-\frac{(n+1)}{2}u}\overline{\mathcal{D}\Psi}.$$

Note that, by (2.19),

$$\mathcal{D}\Psi = \lambda\Psi \implies \bar{\mathcal{D}}\bar{\Phi} = \lambda e^{-u}\bar{\Phi}, \quad \text{with } \Phi := e^{-\frac{(n-1)}{2}u}\Psi. \tag{5.13}$$

With respect to the metric $\bar{g} = e^{2u} g$, apply (5.9) to the spinor field $\bar{\Phi}$ satisfying (5.13) to get

$$\begin{aligned}
&\frac{n}{n-1}\int_M |\bar{\mathcal{P}}\bar{\Phi}|^2 v_{\bar{g}} \\
&\quad = \int_M |\bar{\mathcal{D}}\bar{\Phi}|^2 v_{\bar{g}} - \frac{n}{4(n-1)}\int_M \overline{\mathrm{Scal}}\,|\bar{\Phi}|^2 v_{\bar{g}} \\
&\quad = \int_M \left(\lambda^2 - \frac{n}{4(n-1)}\overline{\mathrm{Scal}}\, e^{2u}\right) e^{-2u}\,|\bar{\Phi}|^2 v_{\bar{g}} \\
&\quad \geq 0.
\end{aligned}$$

□

If equality holds in (5.12), then

$$\bar{\mathcal{P}}\bar{\Phi} \equiv 0, \quad \text{with } \bar{\mathcal{D}}\bar{\Phi} = \lambda e^{-u}\bar{\Phi},$$

that is

$$\bar{\nabla}_{\bar{X}}\bar{\Phi} + \frac{\lambda e^{-u}}{n}\bar{X}\,\bar{\cdot}\,\bar{\Phi} = 0, \quad X \in \Gamma(\mathrm{T}M),$$

which, for $\lambda \neq 0$, by Proposition 5.11 implies that the function u is constant, hence the spinor field $\Psi = e^{\frac{n-1}{2}u}\Phi$ is a real Killing spinor.

Definition 5.13 (Yamabe operator). On a Riemannian manifold (M^n, g) of dimension $n \geq 3$, the *Yamabe operator* (also called *conformal Laplacian*) L_g is the second-order operator defined (see [Bes87], for example) as

$$\mathrm{L}_g = 4\frac{n-1}{n-2}\Delta + \mathrm{Scal}.$$

This operator, like the Dirac operator, is conformally covariant in the following sense: for g and a conformally related metric $\bar{g} = e^{2u}g$, the associated Yamabe operators L_g and $\mathrm{L}_{\bar{g}}$ are related by

$$\mathrm{L}_{\bar{g}} = e^{-\frac{n+2}{2}u}\mathrm{L}_g e^{\frac{n-2}{2}u}. \tag{5.14}$$

The conformal Laplacian is related to the Yamabe problem: on a compact Riemannian manifold (M^n, g), is there a metric conformal to g whose scalar curvature is constant? Indeed, $\mathrm{Scal} = \mathrm{L}_g(1)$ (here 1 is the constant function), hence for instance g is conformally scalar-flat if we can solve the equation $\mathrm{L}_g v = 0$, for some positive smooth function v.

We now show that the Yamabe operator is intimately related to the Dirac operator.

Corollary 5.14. *On a compact Riemannian spin manifold (M^n, g) with positive scalar curvature, any eigenvalue λ of the Dirac operator satisfies the following properties:*

(a) *for $n \geq 3$ (see* [Hij86b]*),*

$$\lambda^2 \geq \frac{n}{4(n-1)}\mu_1(M^n, g), \tag{5.15}$$

where $\mu_1(M^n, g)$ stands for the lowest eigenvalue of the scalar conformal Laplacian L_g;

(b) *for $n = 2$ (see* [Bär92b]*),*

$$\lambda^2 \geq \frac{2\pi\chi(M^2)}{\mathrm{Area}(M^2, g)}, \tag{5.16}$$

where $\chi(M^2)$ is the Euler characteristic.

In case of equality in (a) *or* (b)*, the eigenspinors associated with λ are real Killing spinors.*

Proof. (a) It is clear that $\mu_1(M^n, g) \geq \inf_M \mathrm{Scal}$, and equality is achieved if and only if the scalar curvature is constant. Let $u\colon M \to \mathbb{R}$ be an arbitrary function. The scalar curvature transforms according to the formula (cf. [Bes87])

$$\overline{\mathrm{Scal}}\, e^{2u} = \mathrm{Scal} + 2(n-1)\Delta u - (n-1)(n-2)|du|^2. \tag{5.17}$$

For $n \geq 3$, define the positive function h by

$$h^{\frac{4}{n-2}} = e^{2u}. \tag{5.18}$$

Then (5.17) simplifies to

$$\overline{\mathrm{Scal}}\, h^{\frac{4}{n-2}} = h^{-1} \mathrm{L}_g h. \tag{5.19}$$

By the maximum principle, one knows that an eigenfunction h_1 associated with the lowest eigenvalue μ_1 of L_g can be chosen to be positive. Moreover, it is known that μ_1 is positive if and only if there exists a metric of positive scalar curvature in the conformal class of g. Thus $h_1 > 0$ and

$$h_1^{-1} \mathrm{L}_g h_1 = \mu_1 > 0.$$

Now, take u_1 associated with h_1 by (5.18). From (5.19) we get

$$\overline{\mathrm{Scal}}\, e^{2u_1} = \overline{\mathrm{Scal}}\, h_1^{\frac{4}{n-2}} = h_1^{-1} \mathrm{L}_g h_1 = \mu_1.$$

It can be shown that if $\overline{\mathrm{Scal}}\, e^{2u_1}$ is constant, then it equals $\sup_u \inf_M (\overline{\mathrm{Scal}}\, e^{2u})$; see [Hij91]. Estimate (5.15) thus follows from (5.12).

(b) For $n = 2$, the transformation formula (5.17) of the scalar curvature under conformal change of the metric becomes

$$\overline{\mathrm{Scal}}\, e^{2u} = \mathrm{Scal} + 2\Delta u.$$

Now for any function u we can write

$$\begin{aligned}
\inf_M (\overline{\mathrm{Scal}}\, e^{2u}) &\leq \frac{1}{\mathrm{Area}(M^2, g)} \int_M \overline{\mathrm{Scal}}\, e^{2u} v_g \\
&= \frac{1}{\mathrm{Area}(M^2, g)} \int_M (\mathrm{Scal} + 2\Delta u) v_g \\
&\leq \frac{1}{\mathrm{Area}(M^2, g)} \int_M \mathrm{Scal}\, v_g \\
&= \frac{4\pi \chi(M^2)}{\mathrm{Area}(M^2, g)}.
\end{aligned}$$

by Stokes and the Gauß–Bonnet formula. Let u_1 be a smooth function satisfying

$$2\Delta u_1 = \frac{1}{\mathrm{Area}(M^2, g)} \int_M \mathrm{Scal}\, v_g - \mathrm{Scal}$$

(the function on the right-hand side has zero integral over M so belongs to the image of the Laplacian by Corollary 4.39). Then we finally get

$$\overline{\mathrm{Scal}}\, e^{2u_1} = \mathrm{Scal} + 2\Delta u_1 = \frac{1}{\mathrm{Area}(M^2, g)} \int_M \mathrm{Scal}\, v_g = \frac{4\pi \chi(M^2)}{\mathrm{Area}(M^2, g)},$$

and we conclude as before by (5.12). $\square$

Remark 5.15. For $\mu_1 > 0$, with the help of the conformal covariance of the Dirac operator and of the Sobolev embedding theorem for pseudo-differential operators, J. Lott [Lot86] established the existence of a qualitative conformal lower bound for the eigenvalues of the Dirac operator.

A second corollary of Theorem 5.12 could be obtained by comparing $\mu_1(M^n, g)$ to the Yamabe number, denoted by $\mu(M^n, [g])$. This is the scalar conformal invariant defined by

$$\begin{aligned}\mu(M^n,[g]) &\underset{n\geq 2}{:=} \inf_{\bar g\in[g]} \frac{\displaystyle\int_M \overline{\mathrm{Scal}}\, v_{\bar g}}{(\mathrm{Vol}(M,\bar g))^{(n-2)/n}} \\ &\underset{n\geq 3}{:=} \inf_{h>0} \frac{\displaystyle\int_M h\mathrm{L}_g(h) v_g}{\|h\|^2_{2n/(n-2)}}.\end{aligned}$$

By a direct application of the Hölder inequality, we get the following theorem.

Theorem 5.16. *On a compact Riemannian spin manifold (M^n, g) with positive scalar curvature and dimension $n \geq 2$, any eigenvalue λ of the Dirac operator satisfies*

$$\lambda^2\,(\mathrm{Vol}(M^n,g))^{\frac{2}{n}} \geq \frac{n}{4(n-1)}\mu(M^n,[g]).$$

In the case $n = 2$, by the Gauß–Bonnet formula, one gets $\mu = 4\pi\chi(M)$, where $\chi(M)$ is the Euler–Poincaré characteristic number. We point out that Bär's proof of (5.16), see [Bär92b], uses modified connections which correspond to the modified connections $\overline{\nabla}^f$ (see Proposition 5.11) associated with a conformal class of metrics, on which the original proof of (5.16) is based; see [Hij86b] and [Hij91].

We also mention that inequality (5.15) is also true for the second-order differential operators introduced in [HL88], and there is a similar inequality for the twistor operator [Lic89]. Moreover, we point out that extensions of (5.15) to other first order differential operators and vanishing results were obtained in [BH97] and [BH00].

5.5 The action of harmonic forms on Killing spinors

We will now show that the topology and the geometry of manifolds with real Killing spinors are quite special. The first step was made in [Hij86b], and then extended by A. Lichnerowicz [Lic87]. Moreover, we point out that extensions of (5.15) to other first order differential operators and vanishing results were obtained in [BH97] and [BH00].

Theorem 5.17. (a) *If Ψ is a real Killing spinor on a compact n-dimensional Riemannian spin manifold M and $\alpha \in \Omega^k(M)$ an arbitrary harmonic form of degree $k \neq 0, n$, then*

$$\alpha \cdot \Psi = 0. \tag{5.20}$$

(b) *If M admits a non-trivial Killing spinor, then there exists no non-trivial parallel form $\alpha \in \Omega^k(M)$ for $k \notin \{0, n\}$.*

Proof. (a) The Killing spinor Ψ is an eigenspinor associated with the smallest eigenvalue λ_0 of the Dirac operator $\mathcal{D}$. Using now the formulas

$$d = \sum_{i=1}^{n} e_i \wedge \nabla_{e_i}, \quad \text{and} \quad \delta = -\sum_{i=1}^{n} e_i \lrcorner \nabla_{e_i},$$

and the compatibility of the covariant derivative with Clifford multiplication together with the identity

$$\sum_{i=1}^{n} e_i \cdot \alpha \cdot e_i = (-1)^{k-1}(n-2k)\alpha,$$

for any $\alpha \in \Omega^k(M)$, we get

$$\begin{aligned}
\mathcal{D}(\alpha \cdot \Psi) &= \sum_{i=1}^{n} e_i \cdot \nabla_{e_i}(\alpha \cdot \Psi) \\
&= \sum_{i=1}^{n} e_i \cdot ((\nabla_{e_i}\alpha) \cdot \Psi + \alpha \cdot \nabla_{e_i}\Psi) \\
&= ((d+\delta)\alpha) \cdot \Psi + \sum_{i=1}^{n} e_i \cdot \alpha \cdot \Big(-\frac{\lambda_0}{n} e_i \cdot \Psi\Big) \\
&= \underbrace{((d+\delta)\alpha)}_{=0} \cdot \Psi + \underbrace{(-1)^k \lambda_0 \Big(1 - \frac{2k}{n}\Big)}_{:=\lambda} \underbrace{\alpha \cdot \Psi}_{:=\varphi},
\end{aligned}$$

since on compact manifolds $\Delta\alpha = 0 \iff d\alpha = \delta\alpha = 0$. Now $|\lambda| < |\lambda_0|$ implies by Friedrich's inequality (5.5)

$$\alpha \cdot \Psi = 0.$$

(b) Let α be a parallel k-form. Then by (5.20), $\alpha \cdot \Psi = 0$. Now for any vector field X,

$$0 = \nabla_X(\alpha \cdot \Psi) = (\underbrace{\nabla_X \alpha}_{=0}) \cdot \Psi + \alpha \cdot \nabla_X \Psi.$$

Since Ψ is a Killing spinor, for all $X \in \Gamma(\mathrm{T}M)$, one has that

$$\alpha \cdot X \cdot \Psi = 0.$$

On the other hand, by Proposition 1.13, that

$$\alpha \cdot X = (-1)^k (X \cdot \alpha + 2X \lrcorner \alpha),$$

and so for any vector field X

$$(X \lrcorner \alpha) \cdot \Psi = 0,$$

which by induction on k, shows that $\Psi = 0$. □

This theorem has immediate topological consequences. For example, if there is a non-trivial real Killing spinor on a compact manifold, then the first de Rham cohomology group is zero. Since the Kähler form $\Omega(X, Y) := g(JX, Y)$ of a Kähler manifold is parallel, compact Kähler manifolds of real dimension $n \geq 4$ carry no real Killing spinors.

5.6 Other estimates of the Dirac spectrum

Theorem 5.17 together with Remark 5.4 show that Friedrich's inequality (5.5) cannot be sharp on manifolds with geometric structures which support parallel forms. Sharp improvements of (5.5) on Kähler and quaternionic Kähler manifolds have been obtained by Kirchberg [Kir86] and [Kir90], Hijazi [Hij94], by Kramer, Semmelmann, and Weingart [HM95b], [KSW99], [KSW98a], and [KSW98b], and by Hijazi and Milhorat [HM97]. They will be treated in detail in Chapters 6 and 7.

Another recent improvement of Friedrich inequality in this direction was obtained by Alexandrov, Grantcharov, and Ivanov [AGI98]. They proved that the existence of a parallel 1-form on a compact Riemannian spin manifold (M^n, g), $n \geq 3$, implies that every eigenvalue λ of the Dirac operator satisfies the inequality

$$\lambda^2 \geq \frac{n-1}{4(n-2)} \mathrm{Scal}_0. \tag{5.21}$$

The universal covering space of the manifolds appearing in the limiting case is also described.

5.6.1 Moroianu–Ornea's estimate

In this section we discuss the generalization of the above result obtained by A. Moroianu and L. Ornea [MO04], who showed that (5.21) can be derived from the existence of a *harmonic* 1-form *of constant length*. Note that until now, this is the unique improvement of Friedrich inequality which does not assume a holonomy reduction.

Theorem 5.18 ([MO04]). *Inequality* (5.21) *holds on any compact spin manifold*

$$(M^n, g), \quad n \geq 3,$$

admitting a non-trivial harmonic 1*-form* θ *of unit length. The limiting case is obtained if and only if* θ *is parallel and the eigenspinor* Ψ *corresponding to the smallest eigenvalue of the Dirac operator satisfies*

$$\nabla_X \Psi + \frac{\lambda}{n-1}(X \cdot \Psi - \langle X, \theta \rangle \theta \cdot \Psi) = 0.$$

The condition that the norm of the 1-form being constant is essential, in the sense that the topological constraint alone – the existence of a non-trivial harmonic 1-form – does not allow any improvement of Friedrich's inequality.

Indeed, Bär and Dahl [BD04] have constructed on any compact spin manifold M^n and for every positive real number ε, a metric g_ε on M with the property that $\mathrm{Scal}_{g_\varepsilon} \geq n(n-1)$ and such that the first eigenvalue of the Dirac operator satisfies $\lambda_1^2(\mathcal{D}_\varepsilon) \leq \frac{n^2}{4} + \varepsilon$. This construction clearly shows that no improvement of Friedrich's inequality can be obtained under purely topological restrictions.

Proof of Theorem 5.18. Let θ be a 1-form of unit length and let Ψ be an arbitrary spinor field on M. Consider the "twistor-like" operator

$$T \colon \Gamma(\mathrm{T}M \otimes \Sigma M) \longrightarrow \Gamma(\Sigma M),$$

defined as follows:

$$T_X \Psi = \nabla_X \Psi + \frac{1}{n-1} X \cdot \mathcal{D}\Psi - \frac{1}{n-1}\langle X, \theta \rangle \theta \cdot \mathcal{D}\Psi - \langle X, \theta \rangle \nabla_\theta \Psi.$$

A simple calculation yields

$$|T\Psi|^2 = |\nabla\Psi|^2 - \frac{1}{n-1}|\mathcal{D}\Psi|^2 - |\nabla_\theta \Psi|^2 + \frac{2}{n-1}\langle \mathcal{D}\Psi, \theta \cdot \nabla_\theta \Psi \rangle. \tag{5.22}$$

From now on we will suppose that θ is harmonic, M is compact with positive scalar curvature Scal, v_g is the volume element, and Ψ is an eigenspinor of the Dirac operator $\mathcal{D}$ of M corresponding to the least eigenvalue (in absolute value), say λ. We let $\{e_i\}$, $i = 1, \ldots, n$ denote a local orthonormal frame on M.

The harmonicity of θ implies the useful relation

$$\mathcal{D}(\theta \cdot \Psi) = -\theta \cdot \mathcal{D}\Psi - 2\nabla_\theta \Psi. \tag{5.23}$$

Indeed,

$$\begin{aligned}
\mathcal{D}(\theta \cdot \Psi) &= \sum_{i=1}^{n} e_i \cdot \nabla_{e_i}(\theta \cdot \Psi) \\
&= \sum_{i=1}^{n} e_i \cdot (\nabla_{e_i}\theta) \cdot \Psi + e_i \cdot \theta \cdot \nabla_{e_i}\Psi \\
&= (d\theta + \delta\theta) \cdot \Psi + e_i \cdot \theta \cdot \nabla_{e_i}\Psi \\
&= -\theta \cdot \sum_{i=1}^{n} e_i \cdot \nabla_{e_i}\Psi - 2\langle e_i, \theta\rangle \nabla_{e_i}\Psi \\
&= -\theta \cdot \mathcal{D}\Psi - 2\nabla_\theta \Psi.
\end{aligned}$$

Taking the square norm in (5.23) yields

$$|\mathcal{D}(\theta \cdot \Psi)|^2 = |\theta \cdot \mathcal{D}\Psi|^2 + 4|\nabla_\theta \Psi|^2 - 4\langle \mathcal{D}\Psi, \theta \cdot \nabla_\theta \Psi\rangle. \tag{5.24}$$

By integration of (5.22) over M, using (5.24) to express the last term in the right-hand side of (5.22), and the Schrödinger–Lichnerowicz formula, we get

$$\begin{aligned}
\int_M |T\Psi|^2 v_g = \int_M \Big\{\frac{n-2}{n-1}|\mathcal{D}\Psi|^2 - \frac{1}{4}\,\mathrm{Scal}\,|\Psi|^2 - \frac{n-3}{n-1}|\nabla_\theta \Psi|^2 \\
- \frac{1}{2(n-1)}(|\mathcal{D}(\theta \cdot \Psi)|^2 - |\theta \cdot \mathcal{D}\Psi|^2)\Big\} v_g.
\end{aligned} \tag{5.25}$$

The term in the last bracket of the integrand is clearly positive since, due to the choice of λ to be minimal, we have from the classical Rayleigh inequality that

$$\lambda^2 \leq \frac{\displaystyle\int_M |\mathcal{D}\Phi|^2 v_g}{\displaystyle\int_M |\Phi|^2 v_g}$$

for every Φ. In particular, for $\Phi = \theta \cdot \Psi$ this reads

$$\begin{aligned}\int_M |\mathcal{D}(\theta \cdot \Psi)|^2 \nu_g &\geq \lambda^2 \int_M |\theta \cdot \Psi|^2 \nu_g \\ &= \lambda^2 \int_M |\Psi|^2 \nu_g \\ &= \int_M |\mathcal{D}\Psi|^2 \nu_g \\ &= \int_M |\theta \cdot \mathcal{D}\Psi|^2 \nu_g .\end{aligned}$$

Thus (5.25) gives

$$\begin{aligned}&\int_M \left(\frac{n-2}{n-1} \lambda^2 - \frac{1}{4} \operatorname{Scal} \right) |\Psi|^2 \nu_g \\ &\quad = \int_M \left[|T\Psi|^2 + \frac{n-3}{n-1} |\nabla_\theta \Psi|^2 \right] \nu_g + \frac{1}{2(n-1)} \int_M [|\mathcal{D}(\theta \cdot \Psi)|^2 - |\theta \cdot \mathcal{D}\Psi|^2] \nu_g \\ &\quad \geq 0,\end{aligned}$$

which immediately implies the first statement of Theorem 5.18.

The limiting case. Suppose now that equality is reached in (5.21) for the eigenvalue λ with corresponding eigenspinor Ψ. Then $T\Psi = 0$. Contracting with e_i:

$$\sum_{i=1}^n e_i \cdot \nabla_{e_i} \Psi + \frac{\lambda}{n-1} \sum_{i=1}^n e_i \cdot e_i \cdot \Psi - \frac{\lambda}{n-1} \sum_{i=1}^n e_i \cdot \langle e_i, \theta \rangle \theta \cdot \Psi - \sum_{i=1}^n e_i \cdot \langle e_i, \theta \rangle \nabla_\theta \Psi = 0,$$

gives $\theta \cdot \nabla_\theta \Psi = 0$, so $\nabla_\theta \Psi = 0$ (for $n > 3$ this follows directly from the vanishing of the integral $\int_M \frac{n-3}{n-1} |\nabla_\theta \Psi|^2 \nu_g$). Thus Ψ satisfies the Killing type equation

$$\nabla_X \Psi = aX \cdot \Psi - a\langle X, \theta \rangle \theta \cdot \Psi, \qquad a = -\frac{\lambda}{n-1}.$$

In order to show that θ is parallel, we first compute the spin curvature operator $\mathcal{R}_{Y,X} = [\nabla_Y, \nabla_X] - \nabla_{[Y,X]}$ acting on Ψ. We have successively

$$\begin{aligned}\frac{1}{a} \nabla_Y \nabla_X \Psi = {}& \nabla_Y X \cdot \Psi + aX \cdot (Y - \langle Y, \theta \rangle \theta) \cdot \Psi - \langle \nabla_Y X, \theta \rangle \theta \cdot \Psi \\ & - \langle X, \theta \rangle \nabla_Y \theta \cdot \Psi - \langle X, \nabla_Y \theta \rangle \theta \cdot \Psi - a\langle X, \theta \rangle \theta \cdot (Y - \langle Y, \theta \rangle \theta) \cdot \Psi\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{a}\mathcal{R}_{Y,X}\Psi &= a(X\cdot Y - Y\cdot X)\cdot\Psi \quad - a(\langle Y,\theta\rangle X\cdot\theta - \langle X,\theta\rangle Y\cdot\theta)\cdot\Psi \\
&\quad + (\langle Y,\nabla_X\theta\rangle - \langle X,\nabla_Y\theta\rangle)\theta\cdot\Psi + (\langle Y,\theta\rangle\nabla_X\theta - \langle X,\theta\rangle\nabla_Y\theta)\cdot\Psi \\
&\quad + a(\langle Y,\theta\rangle\theta\cdot X - \langle X,\theta\rangle\theta\cdot Y)\cdot\Psi \\
&= a(X\cdot Y - Y\cdot X)\cdot\Psi + 2a(\langle Y,\theta\rangle\theta\cdot X - \langle X,\theta\rangle\theta\cdot Y)\cdot\Psi \\
&\quad + (\langle Y,\nabla_X\theta\rangle - \langle X,\nabla_Y\rangle)\theta\cdot\Psi + (\langle Y,\theta\rangle\nabla_X\theta - \langle X,\theta\rangle\nabla_Y\theta)\cdot\Psi.
\end{aligned}$$

Using again the harmonicity of θ, we easily check that

$$\begin{aligned}
\frac{1}{2a}\mathrm{Ric}(X)\cdot\Psi &= \frac{1}{a}\sum_{i=1}^{n} e_i\cdot\mathcal{R}_{e_i,X}\Psi \\
&= 2(n-2)a(X - \langle X,\theta\rangle\theta)\cdot\Psi + 2\theta\cdot\nabla_X\theta\cdot\Psi - 2\langle X,\nabla_\theta\theta\rangle\Psi.
\end{aligned} \tag{5.26}$$

But $\nabla_\theta\theta = 0$, since θ is closed and has unit length. Indeed, for any vector field Y,

$$\langle Y,\nabla_\theta\theta\rangle = d\theta(\theta,Y) + \langle\theta,\nabla_Y\theta\rangle = 0.$$

Hence, taking $X = \theta$ in (5.26), we obtain $\mathrm{Ric}(\theta) = 0$. Then, as θ is harmonic, the Bochner formula ensures that θ is parallel. This completes the proof of Theorem 5.18. □

Since the 1-form θ has to be parallel in the limiting case, we can apply Theorem 3.1 in [AGI98] to determine the universal cover of M. In fact, as proved in [AGI98], this is isometric to a Riemannian product $\mathbb{R}\times N$, where N is a spin manifold carrying a real Killing spinor, hence can be described by Bär's classification [Bär93] (see Theorem 8.37 below). Finally, M turns out to be a mapping torus of an isometry of N/Γ (a finite quotient of N).

5.7 Further developments

Recall that for a closed (compact without boundary) Riemannian spin manifold (M^n, g), the Schrödinger–Lichnerowicz formula yields by integration over M

$$\int_M |\mathcal{D}\Psi|^2\nu_g = \frac{1}{4}\int_M \mathrm{Scal}|\Psi|^2\nu_g + \int_M |\nabla\Psi|^2\nu_g. \tag{5.27}$$

As we have seen above, the first applications of this fundamental integral formula are the Lichnerowicz vanishing theorem and the optimal eigenvalues lower bounds for the Dirac operator. In this section we give an overview of further applications. For the proofs, we refer to the original papers or to N. Ginoux [Gin09] for a complete survey of such applications with a brief presentation of the ideas of the proofs.

5.7.1 The energy–momentum tensor

Recall that the lower bounds in question are based on the observations that

$$|\nabla\Psi|^2 \geq \frac{1}{n}|\mathcal{D}\Psi|^2, \quad \Psi \in \Gamma(\Sigma M), \tag{5.28}$$

together with an argument based on the conformal covariance of the Dirac operator. As we have pointed out, inequality (5.28) is the spinorial Cauchy–Schwarz inequality.

Now we use the fact that, for any $u, v \in \mathbb{R}^n$, with u a unit vector, one has that $|v| \geq |\langle v, u\rangle|$. Hence, for any spinor field Ψ and for any unit vector field X, on $\mathcal{Z}_\Psi$, the complement set of zeros of Ψ, one has

$$\begin{aligned}\frac{|\nabla_X\Psi|}{|\Psi|} &= \frac{|X\cdot\nabla_X\Psi|}{|\Psi|}\\ &\geq \left|\left\langle\frac{X\cdot\nabla_X\Psi}{|\Psi|}, \frac{\Psi}{|\Psi|}\right\rangle\right|\\ &\geq \Re\left\langle\frac{X\cdot\nabla_X\Psi}{|\Psi|}, \frac{\Psi}{|\Psi|}\right\rangle,\end{aligned}$$

whence

$$|\nabla\Psi|^2 \geq |Q_\Psi|^2|\Psi|^2, \tag{5.29}$$

where the energy–momentum tensor Q_Ψ is the symmetric 2-tensor associated with the quadratic form defined on $\mathcal{Z}_\Psi$ by

$$Q_\Psi(X) = \Re\left\langle X\cdot\nabla_X\Psi, \frac{\Psi}{|\Psi|^2}\right\rangle.$$

Assume now that Ψ is an eigenspinor of $\mathcal{D}$ corresponding to the eigenvalue λ. Via the Schrödinger–Lichnerowicz formula, inequality (5.29) translates to (see [Hij95])

$$\lambda^2 \geq \inf_{\mathcal{Z}_\Psi}\left(\frac{1}{4}\mathrm{Scal} + |Q_\Psi|^2\right). \tag{5.30}$$

Note that the set of zeros of Ψ is contained in an $(n-2)$-dimensional submanifold (see [Bär99]) and, by the Cauchy–Schwarz inequality,

$$|Q_\Psi|^2 \geq \frac{(\mathrm{Tr}\, Q_\Psi)^2}{n} = \frac{\lambda^2}{n},$$

inequality (5.30) implies (5.5). N. Ginoux and G. Habib showed (see [GH10] Ex. 6.4) that Heisenberg manifolds satisfy equality in (5.30), but not in (5.5).

We also point out that it has been observed by G. Habib [Hab07] that inequality (5.30) is still true for any 2-tensor (see also [GH08], [HR09], and [GH10] for further results on foliated spin manifolds).

It is also straightforward to study the conformal behavior of the energy–momentum tensor in order to get the inequality (see [Hij95])

$$\lambda^2 \geq \frac{1}{4}\mu_1 + \inf_{\mathcal{Z}_\Psi}(|Q_\Psi|^2), \tag{5.31}$$

which by Cauchy–Schwarz implies inequality (5.15). Finally, it is interesting to mention that the energy-momentum tensor associated with a spinor field appears naturally in the study of the variation of the eigenvalues of the Dirac operator under metric variation (see Bourguignon and Gauduchon [BG92]).

5.7.2 Witten's proof of the positive mass theorem and applications

A classical survey on the Yamabe Problem and on Witten's spinorial proof [Wit81] of the positive mass theorem is the paper by Lee and Parker [LP87] (see also [PT82]). The theorem reads:

Let (M^n, g) be an asymptotically flat spin manifold of dimension $n \geq 3$ (whose mass $m(g)$ is well defined), with non-negative scalar curvature, then $m(g) \geq 0$. Moreover, $m(g) = 0$ if and only if the manifold (M^n, g) is isometric to the Euclidean space.

We shall recall the definition of the mass in the next section. Roughly speaking, Witten's idea is to observe that the mass of an asymptotically flat manifold can be seen as the limit at infinity of a boundary term in (5.27), when applied to compact manifolds with boundary and for a suitable choice of an asymptotically constant spinor field.

The conclusion is based on the fact that one can solve the problem of existence of a harmonic spinor field Ψ (i.e., $\mathcal{D}\Psi = 0$) which is asymptotically constant. The equality case is then easily deduced from the fact that such spinor fields are parallel, hence the manifold has a maximum number of parallel spinor fields, i.e., $(M^n, g) = (\mathbb{R}^n, \langle\cdot,\cdot\rangle)$.

Since we have not developed in this book the necessary analytic tools on asymptotically Euclidean spaces, we will present at the end of this section the proof of Ammann and Humbert, which has the advantage of using only elliptic theory on compact manifolds.

We now present some other applications of Witten's argument which rely on a careful study of the boundary term and adapted boundary value problems. To define the Dirac–Witten operator and the hypersurface Dirac operator, we consider restrictions of spinor fields to oriented hypersurfaces and the Gauß spinorial formula (see Lemma 2.38 and Theorem 2.39, in the special case of spin manifolds).

Let M be an $(n+1)$-dimensional Riemannian spin manifold with non-empty boundary ∂M. Denote by $\langle\cdot,\cdot\rangle$ its scalar product and by ∇ its corresponding Levi-Civita connection on the tangent bundle $\mathrm{T}M$. We fix a spin structure (and so a corresponding orientation) on the manifold M and we denote by ΣM the associated spinor bundle, which is a complex vector bundle of rank $2^{[\frac{n+1}{2}]}$, and by

$$\gamma\colon \mathbb{C}\mathrm{l}(M) \longrightarrow \mathrm{End}_{\mathbb{C}}(\Sigma M)$$

the Clifford multiplication, defined on the Clifford bundle $\mathbb{C}\mathrm{l}(M)$. As we have seen in Chapter 2, there exist a natural Hermitian metric and a compatible spinorial Levi-Civita connection on the spinor bundle ΣM, also denoted by $\langle\cdot,\cdot\rangle$ and ∇. The (classical) Dirac operator $\mathcal{D}$ on M is given locally by

$$\mathcal{D} = \sum_{i=1}^{n+1} \gamma(e_i)\nabla_{e_i},$$

where $\{e_1,\ldots,e_{n+1}\}$ is a local orthonormal frame of $\mathrm{T}M$.

The boundary hypersurface ∂M is also an oriented Riemannian manifold, with the orientation and the metric induced from the ambient space. Since its normal bundle is trivial, the Riemannian manifold ∂M is also a spin manifold and so we have a corresponding spinor bundle $\Sigma\partial M$, a Clifford multiplication $\gamma^{\partial M}$, a spinorial Levi-Civita connection $\nabla^{\partial M}$, and an intrinsic Dirac operator $\mathcal{D}^{\partial M}$. It is not difficult to check (see for example Lemma 2.38 or [Bur93], [Tra93], [BFGK91], [Bär98], [HMZ01a], [HMZ01b], and [Mon05]) that the restriction of the spinor bundle of M to its boundary is related to the intrinsic Hermitian spinor bundle $\Sigma\partial M$ by

$$\mathbb{S} := \Sigma M|_{\partial M} \cong \begin{cases} \Sigma\partial M & \text{if } n \text{ is even,} \\ \Sigma\partial M \oplus \Sigma\partial M & \text{if } n \text{ is odd.} \end{cases}$$

For any spinor field $\Psi \in \Gamma(\mathbb{S})$ on the boundary hypersurface ∂M and for any vector field $X \in \Gamma(\mathrm{T}\partial M)$, define on the restricted bundle $\mathbb{S}$ the Clifford multiplication $\gamma^{\mathbb{S}}$ and the connection $\nabla^{\mathbb{S}}$ by (see Theorem 2.39)

$$\gamma^{\mathbb{S}}(X)\Psi := \gamma(X)\gamma(N)\Psi$$

and

$$\nabla^{\mathbb{S}}_X\Psi := \nabla_X\Psi - \frac{1}{2}\gamma^{\mathbb{S}}(AX)\Psi = \nabla_X\Psi - \frac{1}{2}\gamma(AX)\gamma(N)\Psi,$$

where N denotes the outward unit normal vector field on ∂M and $A := -\mathrm{II}$ is the Weingarten tensor. Then, $\gamma^{\$}$ and $\nabla^{\$}$ satisfy the same compatibility relations as ∇ and γ, together with the additional relation

$$\nabla^{\$}_X(\gamma(N)\Psi) = \gamma(N)\nabla^{\$}_X\Psi.$$

Taking into account the relation between the Hermitian bundles $\$$ and $\Sigma\partial M$, one can see that

$$\nabla^{\$} \cong \begin{cases} \nabla^{\partial M} & \text{if } n \text{ is even,} \\ \nabla^{\partial M} \oplus \nabla^{\partial M} & \text{if } n \text{ is odd} \end{cases}$$

and

$$\gamma^{\$} \cong \begin{cases} \gamma^{\partial M} & \text{if } n \text{ is even,} \\ \gamma^{\partial M} \oplus -\gamma^{\partial M} & \text{if } n \text{ is odd.} \end{cases}$$

On the space of smooth sections $\Psi \in \Gamma(\$)$, we have a Dirac operator $\mathcal{D}^{\$}$ associated with the connection $\nabla^{\$}$ and the Clifford multiplication $\gamma^{\$}$, locally given by

$$\mathcal{D}^{\$}\Psi = \sum_{j=1}^{n}\gamma^{\$}(e_j)\nabla^{\$}_{e_j}\Psi = \frac{n}{2}H\Psi - \gamma(N)\sum_{j=1}^{n}\gamma(e_j)\nabla_{e_j}\Psi,$$

where $\{e_1,\ldots,e_n\}$ is a local orthonormal frame tangent to the boundary ∂M and $H = \mathrm{tr}(A)/n$ is its mean curvature function. In this formula, the last term

$$\sum_{j=1}^{n}\gamma(e_j)\nabla_{e_j}$$

is called *Dirac–Witten operator* and $\mathcal{D}^{\$}$ the hypersurface Dirac operator.

In the particular case where the field $\Psi \in \Gamma(\$)$ is the restriction of a spinor field $\Psi \in \Gamma(\Sigma M)$ on M, one gets

$$\mathcal{D}^{\$}\Psi - \frac{n}{2}H\Psi = -\gamma(N)\mathcal{D}\Psi - \nabla_N\Psi.$$

With the above identifications, the hypersurface Dirac operator is related to the intrinsic Dirac operator $\mathcal{D}^{\partial M}$ of the boundary by

$$\mathcal{D}^{\$} \cong \begin{cases} \mathcal{D}^{\partial M} & \text{if } n \text{ is even,} \\ \mathcal{D}^{\partial M} \oplus -\mathcal{D}^{\partial M} & \text{if } n \text{ is odd.} \end{cases}$$

Note that we always have the anticommutativity property

$$\mathcal{D}^{\$}\gamma(N) = -\gamma(N)\mathcal{D}^{\$}$$

and so, when ∂M is compact, the spectrum of $\mathcal{D}^{\$}$ is symmetric with respect to zero and coincides with the spectrum of $\mathcal{D}^{\partial M}$ if n is even, and with $\operatorname{Spec}(\mathcal{D}^{\partial M}) \cup -\operatorname{Spec}(\mathcal{D}^{\partial M})$ if n is odd.

With this, we are now able to give the formula corresponding to (5.27) in the case of a compact spin manifold M with boundary. In fact, the Green formula applied to (5.27) is precisely

$$\begin{aligned}\int_M \Big(|\nabla\Psi|^2 + \frac{1}{4}\mathrm{Scal}|\Psi|^2 - |\mathcal{D}\Psi|^2\Big)\nu_g &= -\int_{\partial M} \langle \gamma(N)\mathcal{D}\Psi + \nabla_N\Psi, \Psi\rangle \nu_g \\ &= \int_{\partial M} \langle \mathcal{D}^{\$}\Psi - \frac{n}{2}H\Psi, \Psi\rangle \nu_g .\end{aligned}$$

In terms of the Penrose operator (see (2.18)), one has that

$$\begin{aligned}&\int_M \Big(|\mathcal{P}\Psi|^2 + \frac{1}{4}\mathrm{Scal}|\Psi|^2 - \frac{n-1}{n}|\mathcal{D}\Psi|^2\Big)\nu_g \\ &\quad = \int_{\partial M} \Big\langle \mathcal{D}^{\$}\Psi - \frac{n}{2}H\Psi, \Psi\Big\rangle \nu_g .\end{aligned} \tag{5.32}$$

5.7.2.1 The Dirac–Witten operator. As a first application, consider the case of a space-like spin hypersurface (M^n, g) on a Lorentzian manifold $\widetilde{M}$, whose metric g satisfies the Einstein field equations and whose energy–momentum tensor satisfies the dominant energy condition. Then the square of the eigenvalues of the Dirac–Witten operator satisfy lower bounds of type (5.5), (5.30), and (5.31) (see [HZ03]). We also refer to [Mae08] for the case of bounded domains with smooth boundary and to [CHZ12] for the case of higher codimensions.

5.7.2.2 Compact spin manifolds with boundary. In what follows, we will see that the fundamental formula (5.32) can be exploited in two directions: solve appropriate boundary value problems for the Dirac operator in order to control the sign of the left-hand side or the right-hand side of (5.32).

The spectrum of the classical Dirac operator on compact Riemannian spin manifolds with non-empty boundary can be studied under four different boundary conditions (see [HMR02]):

(1) the global Atiyah–Patodi–Singer (APS) condition associated with the spectral resolution of the intrinsic Dirac operator on the boundary hypersurface;

(2) the local condition associated with a chirality (CHI) operator on the manifold (for example, if its dimension is even or if it is a space-like hypersurface in a Lorentzian manifold);

(3) the Riemannian version of the so-called (local) MIT bag condition; and

(4) a global boundary condition obtained by a suitable modification of the APS condition (mAPS).

These four conditions satisfy ellipticity criteria and three of them, APS, CHI, and mAPS, make $\mathcal{D}$ a self-adjoint operator and so the corresponding spectra are real sequences tending to $+\infty$ and $-\infty$. Under the MIT condition, the spectrum of the Dirac operator is an unbounded discrete set of complex numbers with positive imaginary part. Finally one has, under the four boundary conditions, the same lower bound (5.5) in terms of the infimum of the scalar curvature as in the closed case, provided that the mean curvature of the boundary hypersurface is non-negative (note that under the APS and the CHI conditions a different proof of this fact is given in [HMZ01a] and [HMZ01b].)

It is interesting to note that the four conditions behave differently with respect to the equality case. In fact, such an equality is never achieved for the APS and MIT conditions, it occurs for the CHI boundary condition if and only if the manifold is a half-sphere, and it is achieved for the mAPS condition if and only if the manifold has a non-trivial real Killing spinor field and the boundary is a minimal hypersurface. For example, all domains enclosed in a sphere by embedded minimal hypersurfaces have the same first eigenvalue for the Dirac operator under the mAPS condition.

Here, we also mention that under the MIT bag boundary condition, a lower bound for the eigenvalues of the Dirac operator on a compact domain of a Riemannian spin manifold is established in [Rau05], where the limiting case is characterized by the existence of an imaginary Killing spinor. Moreover, in [Rau06] the Hijazi-type inequality on compact Riemannian spin manifolds is proved under two boundary conditions: the condition associated with a chirality operator, and the Riemannian version of the MIT bag condition. The limiting case is characterized as being a half-sphere for the first condition, whereas the equality cannot be achieved for the second.

Finally, we point out that regularity for a class of boundary value problems for first-order elliptic systems, with boundary conditions determined by spectral decompositions, under weaker differentiability conditions on the coefficients is proved in [BC05]. Fredholm properties for Dirac-type equations with these boundary conditions are also established. Moreover, boundary value problems for linear first-order elliptic differential operators of order one are examined in [BB12], where the underlying manifold may be noncompact, but the boundary is assumed to be compact. A symmetry property, satisfied by Dirac type operators, of the principal symbol of the operator along the boundary is required.

5.7.2.3 The hypersurface Dirac operator. With the help of equation (5.32), one can obtain lower bounds for the eigenvalues of the hypersurface Dirac operator in terms of the Yamabe number, the energy–momentum tensor, and the mean curvature; see [Zha98], [Zha99], and [HZ01a]. In the limiting case, the hypersurface is an Einstein manifold with constant mean curvature.

Using the identifications given in [Bär98] for the restriction of the spinor bundle to submanifolds, generalizations of [HZ01a] were obtained (see [HZ01b]) for the submanifold Dirac operator. In particular, one has optimal lower bounds for the submanifold Dirac operator in terms of the mean curvature and other geometric invariants, such as the Yamabe number or the energy–momentum tensor. In the limiting case, the submanifold is Einstein if the normal bundle is flat (see also [GM02] for further extensions).

Now, let $\Sigma = \partial M$ be the boundary of a compact $(n+1)$-dimensional Riemannian spin manifold M with non-negative scalar curvature. If the mean curvature H of Σ is also non-negative, then the lowest non-negative eigenvalue λ_1 of the intrinsic hypersurface Dirac operator $\mathcal{D}^{\$}$ satisfies (see [HMZ01a])

$$\lambda_1 \geq \frac{n}{2} H_0, \tag{5.33}$$

where $H_0 = \inf_\Sigma H$. This estimate improves previous ones. In particular, it is valid even if the scalar curvature of the boundary is negative. In fact, if the Einstein tensor of the manifold M is non-negative, then by the Gauß formula and the Cauchy–Schwarz inequality one has

$$\frac{n}{2} H_0 \geq \frac{n}{4(n-1)} R_0^\Sigma,$$

where R_0^Σ is the infimum of the scalar curvature of Σ. Note that the inequality can be strict (for example, take Σ to be a revolution torus in $\mathbb{R}^3$). It then became clear that one can get subtle information on a spin manifold via extrinsic invariants.

A spinorial simple proof for the classical Alexandrov theorem can be deduced from the equality case in (5.33), where corresponding spinor fields are the restrictions to the boundary of parallel spinor fields; see [HMZ01a]. Another application of the equality case is to prove that minimal compact hypersurfaces bounding a compact domain admitting a parallel spinor are totally geodesic.

Inequality (5.33) can be improved by using, as in the intrinsic setup, the conformal covariance of the hypersurface Dirac operator. We need three key ingredients. First, consider the Yamabe problem for manifolds with boundary, cf. [Esc92]: if B denotes the linear scalar operator which relates the mean curvatures of two metrics in the same conformal class on the boundary Σ, then the eigenvalue problem

$$\begin{cases} \mathrm{L}u = 0 & \text{on } \Omega, \\ Bu - \nu u = 0 & \text{on } \Sigma = \partial\Omega \end{cases}$$

has smooth solutions. Moreover, if the ambient scalar curvature is non-negative, the first eigenvalue $\nu_1(B)$ of B is a finite number (whose sign is conformally invariant). In addition, it is shown that $\nu_1(B)$ is positive if and only if there exists on Ω a metric in the conformal class with zero scalar curvature and positive mean curvature.

The second key ingredient is to find appropriate conformal local elliptic boundary conditions. Finally, a suitable choice of the metric in a conformal class and a corresponding spinor field in the Schrödinger–Lichnerowicz type formula (5.32) yield the following result.

Theorem 5.19 ([HMZ02]). *Let Σ be a compact hypersurface of dimension $n \geq 2$ of a Riemannian spin manifold M bounding a compact domain Ω. The lowest non-negative eigenvalue λ_1 of the intrinsic Dirac operator on Σ satisfies*

$$\lambda_1 \geq \frac{n}{2}\nu_1(B),$$

where $\nu_1(B)$ is the first eigenvalue of the boundary linear operator B on Σ, acting on functions f defined on Ω with $\mathrm{L}f = 0$, where L is the conformal Laplacian of M. Moreover, if equality holds, then Ω is conformally equivalent to a Riemannian spin manifold with nontrivial parallel spinors and the eigenspace corresponding to λ_1 is isomorphic to the space of restrictions to Σ of these parallel spinors.

Using the Hölder inequality, one can establish a conformal extrinsic lower bound for the non-negative eigenvalues of the intrinsic Dirac operator on the boundary Σ.

While in the original scalar Reilly inequality, the Ricci curvature of the ambient manifold is assumed to be non-negative, in the spinorial set-up it is sufficient to assume the non-negativity of the ambient scalar curvature. If the ambient scalar curvature is bounded from below by a negative constant (normalized to be the scalar curvature of a unit hyperbolic space) we have the following result.

Theorem 5.20 ([HMR03]). *Let Σ be the boundary of an $(n+1)$-dimensional compact domain in a Riemannian spin manifold whose scalar curvature is bounded from below by $-n(n+1)$. Assume that the inner mean curvature $H \geq 1$. Let λ_1 be the first eigenvalue of the intrinsic Dirac operator $\mathcal{D}^{\$}$. Then*

$$|\lambda_1| \geq \frac{n}{2} \inf_\Sigma \sqrt{H^2 - 1}.$$

The limiting case is then characterized by the existence on the domain of imaginary Killing spinors. As a consequence, a spinorial proof of the (hyperbolic) Alexandrov theorem could be obtained; see [Mon99].

5.7.2.4 Ammann–Humbert version of Witten's positive mass theorem. A nice adaptation of Witten's spinorial proof was provided by Ammann and Humbert in [AH05]. Consider a compact, locally conformally flat Riemannian manifold (M, g) of positive Yamabe invariant (i.e., there exist metrics in the conformal class of g having positive scalar curvature).

Recall the definition of the Yamabe operator:

$$\mathrm{L}_g = 4\frac{n-1}{n-2}\Delta + \mathrm{Scal}.$$

If g has positive scalar curvature, L_g is strictly positive, hence invertible. Thanks to the conformal covariance of L_g (see (5.14)), for every other metric conformal to g the corresponding conformal Laplacian is invertible. By the results of Chapter 4, there exists a bounded self-adjoint operator G on $L^2(M)$, pseudo-differential of order -2, which is a right and left inverse of L_g. As such, there exists a smooth function κ_G, the Schwartz kernel of G, defined on the complement of the diagonal in $M \times M$, which is annihilated by L_g. This function has an asymptotic expansion near the diagonal, and the *mass* at a point $p \in M$ is defined as the constant term in this expansion. We give below a more elementary description of this invariant.

Fix $p \in M$ and choose a metric g in the given conformal class which is flat in a neighborhood U of p. In polar coordinates with respect to this flat metric near p, assuming $n = \dim(M) \geq 3$, the function r^{2-n} is harmonic on U. Choose a cut-off function χ with support on U and which equals 1 near p. Then

$$f := r^{2-n}\chi(r) \in \mathcal{C}^\infty(M \setminus \{p\})$$

satisfies $\mathrm{L}_g f = 0$ near p, because $\mathrm{Scal} = 0$ on U. It follows that $\mathrm{L}_g f$ is a smooth function on M. Since L_g is invertible in $\mathcal{C}^\infty(M)$ (by Theorem 4.32 and Lemma 4.24), we can solve the equation $\mathrm{L}_g u = \mathrm{L}_g f$ for a unique $u \in \mathcal{C}^\infty(M)$. Let $\Gamma = \Gamma_p \in \mathcal{C}^\infty(M \setminus \{p\})$ be the difference

$$\Gamma := f - u.$$

This function is annihilated by L_g (outside p, where it is not defined). Near p it behaves like

$$\Gamma = r^{2-n} + m(p) + O(r) = r^{2-n}(1 + m(p)r^{n-2} + O(r^{n-1})).$$

Up to a positive multiplicative constant, the mass at p is by definition the constant term $m(p)$. Since only the sign of $m(p)$ is of interest to us, we shall ignore normalization constants.

Theorem 5.21 ([AH05]). *Let (M, g) be a compact connected locally conformally flat spin manifold of positive Yamabe invariant, of dimension $n \geq 3$, such that g is flat near $p \in M$. Then the mass $m(p)$ is non-negative, and is equal to zero if and only if M is conformal to the standard sphere.*

Proof. As in the construction of the solution f for L_g, we can build a harmonic spinor on $M \setminus \{p\}$ as follows. Choose a Euclidean coordinate system near p, and let ψ_0 be a constant (i.e., parallel) spinor on U in these coordinates, of norm 1. Then by direct computation,

$$\psi(x) := r^{-n} x \cdot \psi_0$$

is a harmonic spinor on $U \setminus \{p\}$, i.e., it is annihilated by the Dirac operator $\mathcal{D}$. It follows that $\mathcal{D}(\chi\psi)$ is a smooth spinor on M (its support does not contain the singular point p for ψ). For any metric g_0 of positive scalar curvature in the conformal class of g, the Dirac operator is positive by the Schrödinger–Lichnerowicz formula (5.1), hence it is invertible on the space of smooth spinors (Theorem 4.32 and Lemma 4.24). Using the conformal covariance of $\mathcal{D}$ with respect to conformal changes of metric, we deduce that one can solve uniquely the equation $\mathcal{D}\tilde{\psi} = \mathcal{D}(\chi\psi)$ for a smooth spinor $\tilde{\psi}$ on M. It follows that

$$\phi := \chi\psi - \tilde{\psi}$$

is a harmonic spinor on $M \setminus \{p\}$, of the form

$$\phi = r^{-n} x \cdot \psi_0 + O(1)$$

near p, where the error term is in fact a smooth spinor.

The function Γ lives in the null-space of L_g. This suggests a conformal factor (singular at p) such that the resulting metric is scalar-flat:

$$\bar{g} := \Gamma^{\frac{4}{n-2}} g, \quad \mathrm{Scal}_{\bar{g}} = \mathrm{L}_{\bar{g}}(1) = \Gamma^{-\frac{2(n+2)}{n-2}} \mathrm{L}_g(\Gamma) = 0.$$

By conformal covariance, the spinor

$$\Phi := \Gamma^{-\frac{2(n-1)}{n-2}} \phi$$

is harmonic (in the null space of the Dirac operator with respect to $\bar{g}$). We now gather the asymptotic behavior of the various functions, metrics, and spinors near p in polar coordinates with respect to g:

$$\begin{aligned} g &= dr^2 + r^2 d\theta^2; \\ \nu_g &= r^{n-1} dr \wedge \nu_\theta; \\ \phi &= r^{-n}(x \cdot \psi_0 + O(r^n)); \end{aligned}$$

$$\Gamma = r^{2-n}(1 + m(p)r^{n-2} + O(r^{n-1}));$$

$$\bar{g} = \big(1 + \tfrac{4}{n-2}m(p)r^{n-2} + O(r^{n-1})\big)(r^{-4}dr^2 + r^{-2}d\theta^2);$$

$$\nu_{\bar{g}} = r^{-n-1}(1 + O(r^{n-2}))\nu_g;$$

$$\|\Phi\|^2 = (1 + O(r^{n-1}))\big(1 - \tfrac{2(n-1)}{n-2}m(p)r^{n-2} + O(r^{n-1})\big);$$

$$\overline{\partial_r} = r^2(1 + O(r^{n-2}))\partial_r.$$

The vector field in the last line is the unit vector field (with respect to $\bar{g}$) orthogonal to the circles $\{r = \text{constant}\}$. The idea of the proof consists in exploiting the fact that Φ is harmonic on $M \setminus \{p\}$ together with the Schrödinger–Lichnerowicz formula, the fact that $\bar{g}$ is scalar-flat, and integration by parts on the compact manifolds with boundary $M_\epsilon := \{r \geq \epsilon\}$:

$$\begin{aligned} 0 &= \int_{M_\epsilon} \langle \bar{D}^2\Phi, \Phi\rangle \nu_{\bar{g}} \\ &= \int_{M_\epsilon} \langle \bar{\nabla}^*\bar{\nabla}\Phi, \Phi\rangle \nu_{\bar{g}} \\ &= \int_{M_\epsilon} \langle \bar{\nabla}\Phi, \bar{\nabla}\Phi\rangle \nu_{\bar{g}} + \int_{\partial M_\epsilon} \langle \bar{\nabla}_{\overline{\partial_r}}\Phi, \Phi\rangle \nu_{\bar{g}}, \end{aligned}$$

which, by Stokes' formula, yields

$$0 \geq -\int_{M_\epsilon} \langle \bar{\nabla}\Phi, \bar{\nabla}\Phi\rangle \nu_{\bar{g}} = \int_{\partial M_\epsilon} \langle \bar{\nabla}_{\overline{\partial_r}}\Phi, \Phi\rangle \nu_{\bar{g}} = \frac{1}{2}\int_{\partial M_\epsilon} \overline{\partial_r}\|\Phi\|^2 \nu_{\bar{g}}.$$

Using the asymptotic expansions for $\overline{\partial_r}$, $\|\Phi\|^2$, and $\nu_{\bar{g}}$ near p, we can write the integrand $\overline{\partial_r}\|\Phi\|^2\nu_{\bar{g}}$ in the form

$$\overline{\partial_r}\|\Phi\|^2\nu_{\bar{g}} = \Big(-\frac{2(n-1)}{n-2}m(p) + O(r^{n-2})\Big)\nu_\theta.$$

After integration over the sphere of radius ϵ, the leading term in the expansion as $\epsilon \to 0$ is non-positive if and only if $m(p) \geq 0$. Thus the mass $m(p)$ is non-negative, as claimed.

The equality $m(p) = 0$ implies that Φ is parallel on $M \setminus \{p\}$. The existence of a nonzero parallel spinor entails the vanishing of Ricci curvature, hence $\bar{g}$ is actually flat since by hypothesis g is locally conformally flat. Moreover $(M \setminus \{p\}, \bar{g})$ is complete (from the form of $\bar{g}$ near $r = 0$) and has an Euclidean end. It follows that $M \setminus \{p\}$ is an isometric quotient of $\mathbb{R}^n$, and since it contains an embedded Euclidean end, it must be isometric to $\mathbb{R}^n$. The inverse of the stereographic projection maps then (M, g) conformally onto $\mathbb{S}^n$. $\square$

5.7.3 Further applications

We finally mention several directions of active research during the last years. For details we refer to the original articles.

- Conformal spin invariants were introduced by Ammann, Dahl, Humbert, and Morel in [ADH09a, ADH09b], [AH08], and [AHM06], who also obtained several vanishing theorems. The non-compact case was studied by Große in [Gro06], [Gro11], and [Gro12].
- The theory of spin hypersurfaces was studied by Friedrich [Fri98], Morel [Mor05], Roth [Rot10] and [Rot11], and Lawn and Roth [LR10] and [LR11].
- Probably the most spectacular applications of Spin^c geometry were obtained in dimension 4 by means of the Seiberg–Witten equations, cf. Seiberg and Witten [Wit94], LeBrun [LeB97] [GL98], and [Fri00], and Ginoux [Gin04].
- Dirac operators on Lagrangian submanifolds have been studied by Hijazi, Montiel, and Urbano [HMU06]. Spin^c geometry of Kähler manifolds and the Hodge Laplacian on minimal Lagrangian submanifolds have been investigated by Nakad [Nak10], [Nak11a], and [Nak11b] and Nakad and by Roth [NR12].

We synthesize these and several other recent results in Table 2.

Table 2

Riemannian manifolds	**Dirac lower bounds in terms of the**	**References**
spin closed	first eigenvalue of the Yamabe operator and the energy–momentum tensor (EMT) scalar curvature, the EMT, and other terms scalar curvature and the Weyl tensor scalar curvature and the Ricci tensor scalar curvature and Codazzi tensors	 [Hij95] [FK01] [FK02b] [FK02a] [FK08]
spin compact foliated	scalar curvature and the generalized EMT	[Hab07], [HR09]
spin compact with boundary	first eigenvalue of the Yamabe operator and the EMT scalar and mean curvatures first eigenvalue of the Yamabe operator (Dirac–Witten operator) intrinsic and extrinsic curvatures and the EMT	 [HMZ01b] [Rau05] [Rau06] [Mae08]
embedded hypersurfaces of compact domains in Riemannian spin manifolds	mean curvature first eigenvalue of the Steklov–Escobar operator	[HMZ01a] [HMZ02]
compact space-like hypersurfaces of Lorentzian spin manifolds	(Dirac–Witten operator) scalar curvature and the EMT	 [HZ03]
compact space-like submanifolds of pseudo-Riemannian spin manifolds	(Dirac–Witten operator) intrinsic/extrinsic curvatures and the EMT	 [CHZ12], [GM02]
compact hypersurfaces	intrinsic and extrinsic curvatures	[HZ01a]
compact submanifolds		[GM02], [HZ01b]
complete non-compact spin of finite volume open spin (not necessary complete)	first eigenvalue of the Yamabe operator conformal or topological invariants	[Gro06], [Gro11] [Bär09]
Spin^c compact without boundary	first eigenvalue of the perturbed Yamabe operator and the EMT	[HM99] [Nak10], [Nak11a]
Spin^c non-compact of finite volume	first eigenvalue of the perturbed Yamabe operator and the EMT	 [Nak11b]

Chapter 6

Lower eigenvalue bounds on Kähler manifolds

This chapter is devoted to the proof of Kirchberg's inequalities [Kir86] and [Kir90] which say that on a compact Kähler spin manifold, the square of the lowest eigenvalues of the Dirac operator is not less than the infimum of the scalar curvature times an explicit constant depending on the dimension.

In fact, this constant depends on the parity of the complex dimension. We shall see that, in many respects, the odd-dimensional case is quite different from the even-dimensional one. There are also different proofs of these inequalities. First, the odd-dimensional case was proved by Kirchberg [Kir86] using the decomposition of the spinor bundle under the action by Clifford multiplication of the Kähler form together with the notion of modified connections. Then, in [Kir90] both inequalities were proved by introducing the Kählerian Penrose operators. Later, another proof of inequality (6.1), which does not rely on the decomposition (6.4) below, was given in [Hij94].

Here we present the approach of P. Gauduchon [Gau95a], based on the systematic algebraic setting introduced in Section 2.3.

6.1 Kählerian spinor bundle decomposition

Recall that a Kähler structure on a Riemannian manifold (M^n, g) of dimension $n = 2m$ with Levi-Civita connection ∇ is a ∇-parallel almost complex structure J compatible with the metric g in the sense that $g(JX, JY) = g(X, Y)$ for all tangent vectors X, Y. The associated Kähler form Ω is the parallel 2-form defined, for any tangent vector fields X and Y, by

$$\Omega(X, Y) := g(JX, Y) = -g(X, JY).$$

Observe that the volume form v_g defining the orientation of (M^{2m}, g) is related to the Kähler form Ω by

$$v_g = \frac{1}{m!} \underbrace{\Omega \wedge \cdots \wedge \Omega}_{m \text{ times}}.$$

We have seen that a compact Kähler spin manifold (M^{2m}, g, J, γ) with positive scalar curvature cannot carry (real) Killing spinors. In other words, on such manifolds any eigenvalue λ satisfies

$$\lambda^2(M^{2m}, g, J, \gamma) > \frac{n}{4(n-1)}\mathrm{Scal}_0,$$

where Scal_0 denotes as usual the infimum of the scalar curvature of M. This shows that if a compact spin manifold has a Kähler structure, then Friedrich's inequality can be improved. Indeed, we have the following result.

Theorem 6.1. *On a compact Kähler spin manifold (M^n, g, J, γ), $n = 2m$, with positive scalar curvature* Scal*, any eigenvalue λ of the Dirac operator satisfies*

$$\lambda^2 \geq \frac{m+1}{4m}\mathrm{Scal}_0, \quad \textit{if } m \textit{ is odd,} \tag{6.1}$$

and

$$\lambda^2 \geq \frac{m}{4(m-1)}\mathrm{Scal}_0, \quad \textit{if } m \textit{ is even.} \tag{6.2}$$

The first proof of (6.1) could be found in [Kir86]. In the same paper, Kirchberg showed that if equality in (6.1) is achieved, then the manifold is of odd complex dimension. The idea of the proof is to decompose the spinor bundle as the sum of eigenspaces associated to the eigenvalues of the Kähler form and to define, on an arbitrary eigenbundle, an appropriate modified connection. We point out that a proof of this inequality is given in [Hij94] using a natural modified connection which corresponds to the one introduced by Friedrich.

The proof we shall give here is based on the use of the *Kählerian twistor operators* which we shall introduce. This follows the general principle introduced in Section 2.3, cf. [Lic90] and [Hij94]).

At any point x in M, choose normal coordinates at x so that $(\nabla e_i)(x) = 0$, for all $i \in \{1, \dots, n\}$. All the computations will be carried out in such charts.

First, we need to establish some lemmas, most of which are due to Kirchberg; cf. [Kir86], [Kir88], and [Kir90].

Lemma 6.2. *As an endomorphism of the spinor bundle, the Kähler form is given locally by*

$$\Omega = \frac{1}{2}\sum_{k=1}^{n} e_k \cdot Je_k = -\frac{1}{2}\sum_{k=1}^{n} Je_k \cdot e_k.$$

Proof. By the definition of the 2-form Ω,

$$\begin{aligned}\Omega &= \frac{1}{2}\sum_{k,l=1}^{n} \Omega_{kl} e_k \wedge e_l \\ &= \frac{1}{2}\sum_{k,l=1}^{n} g(Je_k, e_l) e_k \cdot e_l \\ &= \frac{1}{2}\sum_{k=1}^{n} e_k \cdot Je_k. \end{aligned}$$ □

Lemma 6.3. *For any vector field X,*

$$[\Omega, X] = 2JX, \quad \textit{and} \quad [\Omega, JX] = -2X. \tag{6.3}$$

Proof. Since $J^2 = -\mathrm{Id}$, the second identity follows from the first. By linearity, it is sufficient to take $X = e_k$, for $1 \le k \le n$, fixed. We have

$$\begin{aligned}\Omega \cdot e_k &= \frac{1}{2}\sum_{l=1}^{n} e_l \cdot Je_l \cdot e_k \\ &= -\frac{1}{2}\sum_{l=1}^{n} e_l \cdot \big(e_k \cdot Je_l + 2g(Je_l, e_k)\big) \\ &= Je_k + \frac{1}{2}\sum_{l=1}^{n} \big(e_k \cdot e_l + 2\delta_{kl}\big) Je_l \\ &= 2Je_k + e_k \cdot \Omega. \end{aligned}$$ □

Lemma 6.4. *Under the action of the Kähler form, the spinor bundle splits into an orthogonal sum*

$$\Sigma M = \bigoplus_{r=0}^{m} \Sigma_r M, \tag{6.4}$$

where each sub-bundle $\Sigma_r M$ is an eigenbundle of Ω with eigenvalue

$$i\mu_r := i(2r - m)$$

and has rank $\binom{m}{r}$.

An orthonormal local coframe $\{e_1, \dots, e_m, e_{m+1}, \dots, e_{2m}\}$ is said to be *adapted* if for all $1 \le j \le m$, one has $J(e_j) = e_{m+j}$ and $J(e_{m+j}) = -e_j$. The action of J on 1-forms is defined by the identification between 1-forms and vectors via the metric. Let

$$p_+(e_j) := \bar{z}_j := \frac{e_j - ie_{m+j}}{2}$$

and

$$p_-(e_j) := z_j := \frac{e_j + i e_{m+j}}{2}.$$

It is a straightforward verification that $\{z_1, \ldots, z_m, \bar{z}_1, \ldots, \bar{z}_m\}$ is a Witt basis with respect to the complex extension of the Riemannian metric g to $\Lambda^1 M$. In fact, one has

$$g(z_j, z_k) = g(\bar{z}_j, \bar{z}_k) = 0,$$

and

$$g(z_j, \bar{z}_k) = g(\bar{z}_j, z_k) = \frac{1}{2}\delta_{jk}.$$

The complex Clifford algebra $\mathbb{C}\mathrm{l}_x\, M$ is generated by the above Witt basis, subject to the relations

$$z_j \cdot z_k + z_k \cdot z_j = \bar{z}_j \cdot \bar{z}_k + \bar{z}_k \cdot \bar{z}_j = 0 \tag{6.5a}$$

and

$$z_j \cdot \bar{z}_k + \bar{z}_j \cdot z_k = -\delta_{jk}. \tag{6.5b}$$

6.2 The canonical line bundle

Lemma 6.5. *The vector subspace*

$$L_x := \{\Psi \in \Sigma_x M;\ z_j \cdot \Psi = 0, \textit{ for all } j \in \{1, \ldots, m\}\}$$

of $\Sigma_x M$ *has complex dimension* 1.

Proof. The vector space L_x is non-empty since there exists at least one spinor ψ such that

$$\Psi := z_1 \cdot z_2 \cdots z_m \cdot \psi \neq 0$$

(the Clifford action is faithful) and clearly $\Psi \in L_x$. Suppose $\Psi_1, \Psi_2 \in L_x$. There exists an element $a \in \mathbb{C}\mathrm{l}_x\, M$ such that $\Psi_2 = a \cdot \Psi_1$. By (6.5), a can be written as $a = \bar{z}_1 \cdot b + c$, where b and c do not contain $\bar{z}_1$. Then z_1 anti-commutes with b and c, so

$$0 = z_1 \cdot \Psi_2 = z_1 \cdot a \cdot \Psi_1 = -b \cdot \Psi_1 - \bar{z}_1 \cdot z_1 \cdot b \cdot \Psi_1 + z_1 \cdot c \cdot \Psi_1 = -b \cdot \Psi_1,$$

as $z_1 \cdot \Psi_1 = 0$. Thus $\Psi_2 = c \cdot \Psi_1$, and c does not contain $\bar{z}_1$. Repeating this argument, we may assume that c does not contain $\bar{z}_j$ for any j. Thus

$$c = c_0 + c_j z_j + c_{jk} z_j \cdot z_k + \ldots$$

and clearly $\Psi_2 = c \cdot \Psi_1 = c_0 \Psi_1$. So L_x is one dimensional. $\square$

Let us fix a point $x \in M$ and a non-zero element $\Psi \in L_x$. Since every element of $\mathbb{C}\mathrm{l}_x M$ can be written as a linear combination of

$$\bar{z}_{j_1} \cdots \bar{z}_{j_p} \cdot z_{k_1} \cdots z_{k_q},$$

the set

$$\{\bar{z}_{j_1} \cdots \bar{z}_{j_r} \cdot \Psi;\ 1 \leq j_1 < \cdots < j_r \leq m, 0 \leq r \leq m\},$$

spans $\Sigma_x M$. It is actually a basis, for dimensional reasons.

The Clifford action of Ω can be expressed in terms of the Witt basis as

$$\begin{aligned}
\Omega &= \sum_{k=1}^{m} e_k \cdot Je_k \\
&= i \sum_{k=1}^{m} (z_k + \bar{z}_k) \cdot (\bar{z}_k - z_k) \\
&= -i \sum_{k=1}^{m} (2\bar{z}_k \cdot z_k + 1) \\
&= -im - 2i \sum_{k=1}^{m} \bar{z}_k \cdot z_k.
\end{aligned}$$

Proof of Lemma 6.4. Denote by L the complex line bundle whose fiber at each $x \in M$ is L_x. For each $r \in \{0, \ldots, m\}$, let $\Sigma_r M$ be the complex vector subspace of ΣM spanned by

$$\{\bar{z}_{j_1} \cdots \bar{z}_{j_r} \cdot \Psi;\ 1 \leq j_1 < \cdots < j_r \leq m,\ \Psi \in L\}.$$

Consider a spinor $\Psi_r := \bar{z}_{j_1} \cdots \bar{z}_{j_r} \cdot \Psi \in \Sigma_r M$. By (6.5),

$$\begin{aligned}
\Omega \cdot \Psi_r &= -i\Big(m + 2\sum_{j=1}^{m} \bar{z}_j \cdot z_j\Big) \cdot \bar{z}_{j_1} \cdots \bar{z}_{j_r} \cdot \Psi_r \\
&= -im\Psi_r + 2ir\Psi_r \\
&= i\mu_r \Psi_r.
\end{aligned}$$

□

The above considerations show the spinor bundle ΣM is canonically isomorphic to $L \otimes (\Lambda^{0,0} \oplus \cdots \oplus \Lambda^{0,m})$.

In order to gather some information on L, we have to examine more carefully the restriction of the spin representation to the unitary group. More precisely, let $\widetilde{\mathrm{U}}_m$ be the inverse image of U_m in Spin_{2m} and let $E \subset \Sigma_{2m}$ be the complex line of spinors annihilated by $(1,0)$-forms (in other words, E is the algebraic analog of L). Then for every $v \in \mathbb{R}^{2m}$ and $\psi \in E$,

$$(v + iJv) \cdot \psi = 0,$$

whence

$$Jv \cdot \psi = iv \cdot \psi. \tag{6.6}$$

Denote by ρ_{2m} the spin representation and by ξ the projection $\mathrm{Spin}_{2m} \to \mathrm{SO}_{2m}$. We claim that for every $a \in \widetilde{\mathrm{U}}_m$, the homomorphism $\rho_{2m}(a)$ acts on E by a complex scalar whose square is the determinant of $\xi(a)$, viewed as a unitary matrix. This clearly implies that the square of the line bundle L, which is associated with the representation E of $\widetilde{\mathrm{U}}_m$, is isomorphic to the line bundle associated with the determinant representation of U_m on $\mathbb{C}$, which is nothing else than the canonical bundle $K := \Lambda^{m,0}M$ of M.

Now, it is an elementary fact that U_m is generated by rotations of complex lines of $(\mathbb{R}^{2m}, J)$, so $\widetilde{\mathrm{U}}_m$ is generated (as multiplicative group) by products $v \cdot w$ in the (real) Clifford algebra Cl_{2m} of unit vectors belonging to the same complex line of $(\mathbb{R}^{2m}, J)$. In order to prove the claim, it is thus sufficient to take $a = v \cdot (\alpha v + \beta J v)$, with $\alpha^2 + \beta^2 = 1$. Then $\xi(a)(v) = a \cdot v \cdot a^{-1}$ acts as the identity on the orthogonal complement of v and Jv, and is equal to $(\alpha I + \beta J)^2$ on the subspace spanned by v and Jv. Thus $\det_{\mathbb{C}}(\xi(a)) = (\alpha + i\beta)^2$. On the other hand, using (6.6) we get for every $\psi \in E$

$$\rho(a)\psi = v \cdot (\alpha v + \beta J v) \cdot \psi = v \cdot (\alpha v + \beta i v) \cdot \psi = -(\alpha + i\beta)\psi.$$

Thus $\rho_{2m}(a)^2(\psi) = \det(\xi(a))\psi$, so the claim is proved.

Thus, we obtained the so-called *Hitchin representation* of the spinor bundle of any almost Hermitian manifold as

$$\Sigma M \cong K^{\frac{1}{2}} \otimes \Lambda^{0,*}, \tag{6.7}$$

where $K^{\frac{1}{2}}$ denotes a square root of the canonical bundle.

6.3 Kählerian twistor operators

Lemma 6.6. *For any tangent vector field X and $r \in \{0, \dots, m\}$, one has*

$$p_+(X) \cdot \Sigma_r M \subset \Sigma_{r+1} M \quad \text{and} \quad p_-(X) \cdot \Sigma_r M \subset \Sigma_{r-1} M. \tag{6.8}$$

with the convention that $\Sigma_{-1} M = \Sigma_{m+1} M = M \times \{0\}$.

Proof. The proof is a consequence of Lemma 6.3. It is also possible to use the explicit decomposition given in Lemma 6.4 to get the result. To see this, consider a nontrivial spinor $\Psi_r := \bar{z}_{j_1} \cdots \bar{z}_{j_r} \cdot \Psi \in \Sigma_r M$. By linearity, it is sufficient to show (6.8) for $X = e_k$, for any $k \in \{1, \ldots, m\}$. Lemma 6.4 and (6.5) imply that $\bar{z}_k \cdot \Psi_r = 0$ if $k \in \{j_1, \ldots, j_r\}$ and $\bar{z}_k \cdot \Psi_r \in \Sigma_{r+1} M$ if $k \notin \{j_1, \ldots, j_r\}$. The second inclusion can be seen similarly. $\square$

We now show that Lemma 6.6 implies that the image of the restriction of the Dirac operator to sections of the subbundle $\Sigma_r M$ lies in the space of sections of $\Sigma_{r-1} M \oplus \Sigma_{r+1} M$.

Associated with J there is an elliptic formally self-adjoint operator $\widetilde{\mathcal{D}}$ defined locally by

$$\widetilde{\mathcal{D}} = \sum_{i=1}^{n} J e_i \cdot \nabla_{e_i} = -\sum_{i=1}^{n} e_i \cdot \nabla_{J e_i};$$

see [Hit74], [Mic80], and [Kir86]. Since the almost complex structure J is parallel, Lemma 6.3 yields the following lemma.

Lemma 6.7. *The operators $\mathcal{D}$ and $\widetilde{\mathcal{D}}$ satisfy the relations*

$$[\Omega, \mathcal{D}] = 2\widetilde{\mathcal{D}}, \tag{6.9}$$

$$[\Omega, \widetilde{\mathcal{D}}] = -2\mathcal{D}, \tag{6.10}$$

$$[\Omega, \mathcal{D}^2] = 0, \tag{6.11}$$

$$\widetilde{\mathcal{D}}\mathcal{D} + \mathcal{D}\widetilde{\mathcal{D}} = 0, \tag{6.12}$$

$$\widetilde{\mathcal{D}}^2 = \mathcal{D}^2. \tag{6.13}$$

Proof. Identities (6.9) and (6.10) follow from (6.3). Relation (6.11) can be established by using the Schrödinger–Lichnerowicz formula and the fact that Ω is parallel. Equation (6.12) is then a consequence of the first three identities.

By (6.9),

$$2\mathcal{D}^2 = \mathcal{D}[\widetilde{\mathcal{D}}, \Omega] = \mathcal{D}\widetilde{\mathcal{D}}\Omega - \mathcal{D}\Omega\widetilde{\mathcal{D}} \tag{6.14}$$

and

$$2\widetilde{\mathcal{D}}^2 = \widetilde{\mathcal{D}}[\Omega, \mathcal{D}] = \widetilde{\mathcal{D}}\Omega\mathcal{D} - \widetilde{\mathcal{D}}\mathcal{D}\Omega. \tag{6.15}$$

Equations (6.14), (6.15), and (6.12) yield

$$2\widetilde{\mathcal{D}}^2 - 2\mathcal{D}^2 = \widetilde{\mathcal{D}}\Omega\mathcal{D} + \mathcal{D}\Omega\widetilde{\mathcal{D}}. \tag{6.16}$$

On the other hand, by (6.9) and (6.12),

$$\widetilde{\mathcal{D}}\Omega\mathcal{D} + \mathcal{D}\Omega\widetilde{\mathcal{D}} = (\Omega\widetilde{\mathcal{D}} + 2\mathcal{D})\mathcal{D} + (\Omega\mathcal{D} - 2\widetilde{\mathcal{D}})\widetilde{\mathcal{D}} \tag{6.17}$$
$$= 2\mathcal{D}^2 - 2\widetilde{\mathcal{D}}^2.$$

Equations (6.16) and (6.17) imply (6.13). □

Lemma 6.6 and Lemma 6.7 immediately imply the following result.

Lemma 6.8. *Define the two operators $\mathcal{D}_-$ and $\mathcal{D}_+$ by*

$$p_-(\mathcal{D}) := \mathcal{D}_- := \frac{1}{2}(\mathcal{D} + i\widetilde{\mathcal{D}}) \tag{6.18a}$$

and

$$p_+(\mathcal{D}) := \mathcal{D}_+ := \frac{1}{2}(\mathcal{D} - i\widetilde{\mathcal{D}}). \tag{6.18b}$$

We have the differential complexes

$$\Gamma(\Sigma_m M) \xrightarrow{\mathcal{D}_-} \cdots \xrightarrow{\mathcal{D}_-} \Gamma(\Sigma_r M) \xrightarrow{\mathcal{D}_-} \Gamma(\Sigma_{r-1} M) \xrightarrow{\mathcal{D}_-} \cdots \xrightarrow{\mathcal{D}_-} \Gamma(\Sigma_0 M),$$

and

$$\Gamma(\Sigma_0 M) \xrightarrow{\mathcal{D}_+} \cdots \xrightarrow{\mathcal{D}_+} \Gamma(\Sigma_r M) \xrightarrow{\mathcal{D}_+} \Gamma(\Sigma_{r+1} M) \xrightarrow{\mathcal{D}_+} \cdots \xrightarrow{\mathcal{D}_+} \Gamma(\Sigma_m M).$$

Now we are ready to define Kählerian twistor operators. On a Kähler spin manifold (M^{2m}, g, J), for each $r \in \{0, \dots, m\}$, the Kählerian twistor operator $\mathcal{P}^{(r)}$ is the composition of the restriction of the covariant derivative to the subbundle $\Gamma(\Sigma_r M)$ with the projection on $\operatorname{Ker}\gamma_r$:

$$\mathcal{P}^{(r)} \colon \Gamma(\Sigma_r M) \xrightarrow{\nabla} \Gamma(\mathrm{T}^* M \otimes \Sigma_r M) \xrightarrow{\pi_r} \operatorname{Ker}\,\gamma_r,$$

where γ_r is the restricted Clifford multiplication defined by

$$\begin{aligned}\gamma_r := \gamma_r^- \oplus \gamma_r^+ \colon \Gamma(\mathrm{T}^* M \otimes \Sigma_r M) &\longrightarrow \Gamma(\Sigma_{r-1} M) \oplus \Gamma(\Sigma_{r+1} M),\\ (X, \Psi_r) &\longmapsto p_-(X)\cdot\Psi_r \oplus p_+(X)\cdot\Psi_r,\end{aligned}$$

and π_r is the projection on $\operatorname{Ker}\gamma_r \subset \Gamma(\mathrm{T}^* M \otimes \Sigma_r M)$.

Lemma 6.9. *The Clifford multiplications $\gamma_r^{\pm}$ satisfy the identities*

$$\gamma_r^+ \circ (\gamma_r^-)^* = 0,$$
$$\gamma_r^- \circ (\gamma_r^+)^* = 0,$$
$$\gamma_r^+ \circ (\gamma_r^+)^* = 2(r+1)\mathrm{Id}_{\Sigma_{r+1} M},$$
$$\gamma_r^- \circ (\gamma_r^-)^* = 2(m-r+1)\mathrm{Id}_{\Sigma_{r-1} M}.$$

Proof. For any spinor field $\Psi_{r+1} \in \Gamma(\Sigma_{r+1}M)$, one gets

$$\begin{aligned}
\gamma_r^+ \circ (\gamma_r^+)^* \Psi_{r+1} &= \gamma_r^+ \Big(-\sum_{j=1}^{n} e_j \otimes p_-(e_j) \cdot \Psi_{r+1} \Big) \\
&= -\sum_{j=1}^{n} e_j \cdot p_+(e_j) \cdot p_-(e_j) \cdot \Psi_{r+1} \\
&= -2 \sum_{j=1}^{m} z_j \cdot \overline{z_j} \cdot \Psi_{r+1} \\
&= (-i\Omega + m) \cdot \Psi_{r+1} \\
&= 2(r+1)\Psi_{r+1}.
\end{aligned}$$

The other relations are obtained similarly. $\square$

Remark 6.10. Identities (6.19) immediately yield

$$\gamma_r \circ (\gamma_r)^* = 2(r+1) \operatorname{Proj}_r^+ + 2(m-r+1) \operatorname{Proj}_r^-, \tag{6.19}$$

where Proj_r^+ and Proj_r^- are respectively the orthogonal projections on $\Sigma_{r+1}M$ and $\Sigma_{r-1}M$, and by convention, $\operatorname{Proj}_0^- = \operatorname{Proj}_m^+ = 0$. Therefore, the linear map $\gamma_r \circ (\gamma_r)^*$ has two different eigenvalues $2(r+1)$ and $2(m-r+1)$ for $0 < r < \frac{m}{2}$ and $\frac{m}{2} < r < m$, while for $2r = m$, $r = 0$ or $r = m$, it has only one eigenvalue.

Using the general approach of Section 2.3 and relations (2.16) and (2.24), we immediately get the following:

Proposition 6.11. *For each $r \in \{0, \dots, m\}$ and for any spinor field $\Psi_r \in \Gamma(\Sigma_r M)$, the local expression of the Kählerian twistor operator $\mathcal{P}^{(r)}$ is given by*

$$\begin{aligned}
\mathcal{P}^{(r)}\Psi_r = \sum_{j=1}^{2m} e_j \otimes \Big[\nabla_{e_j}\Psi_r &+ \frac{1}{2(r+1)} p_-(e_j) \cdot \mathcal{D}_+\Psi_r \\
&+ \frac{1}{2(m-r+1)} p_+(e_j) \cdot \mathcal{D}_-\Psi_r \Big]
\end{aligned} \tag{6.20}$$

Moreover,

$$|\nabla\Psi_r|^2 \geq \frac{1}{2(r+1)} |\mathcal{D}_+\Psi_r|^2 + \frac{1}{2(m-r+1)} |\mathcal{D}_-\Psi_r|^2.$$

Note that by the definition of the Kählerian twistor operator $\mathcal{P}^{(r)}$, one has

$$\mathcal{P}^{(r)}_X \colon \Gamma(\Sigma_r M) \longrightarrow \Gamma(\Sigma_r M), \quad X \in \Gamma(\mathrm{T}M), \tag{6.21}$$

and, for any spinor field $\Psi_r \in \Gamma(\Sigma_r M)$,

$$\sum_{j=1}^{n} e_j \cdot \mathcal{P}^{(r)}_{e_j} \Psi_r = 0. \tag{6.22}$$

Moreover, we have the following lemma.

Lemma 6.12. *For each $r \in \{0, \dots, m\}$ and any spinor field $\Psi_r \in \Gamma(\Sigma_r M)$, the Kählerian twistor operator $\mathcal{P}^{(r)}$ satisfies*

$$\sum_{j=1}^{n} J(e_j) \cdot \mathcal{P}^{(r)}_{e_j} \Psi_r = 0. \tag{6.23}$$

Proof. Identities (6.3), (6.21), and (6.22) imply that

$$\begin{aligned}
0 &= \Omega \cdot \Big(\sum_{j=1}^{n} e_j \cdot \mathcal{P}^{(r)}_{e_j} \Psi_r \Big), \\
&= \sum_{j=1}^{n} (e_j \cdot \Omega + 2 J e_j) \cdot \mathcal{P}^{(r)}_{e_j} \Psi_r \\
&= i \mu_r \sum_{j=1}^{n} e_j \cdot \mathcal{P}^{(r)}_{e_j} \Psi_r + 2 \sum_{j=1}^{n} J(e_j) \cdot \mathcal{P}^{(r)}_{e_j} \Psi_r \\
&= 2 \sum_{j=1}^{n} J(e_j) \cdot \mathcal{P}^{(r)}_{e_j} \Psi_r .
\end{aligned}$$

□

6.4 Proof of Kirchberg's inequalities

Proof of Theorem 6.1. Consider the space

$$\Sigma^{\lambda} M := \{ \Psi \in \Gamma(\Sigma M);\ \mathcal{D}^2 \Psi = \lambda^2 \Psi \}.$$

Since $[\Omega, \mathcal{D}^2] = 0$, it follows that there exists an orthogonal decomposition of $\Sigma^{\lambda} M$ as

$$\Sigma^{\lambda} M = \bigoplus_{r=0}^{m} \Sigma^{\lambda}_r M, \tag{6.24}$$

where $\Sigma_r^\lambda M \subset \Sigma_r M$. In decomposition (6.24), take r to be the smallest integer for which $\Sigma_r^\lambda M \neq \{0\}$. Then $0 \le r < m$. Since $[\mathcal{D}^2, \mathcal{D}_\pm] = 0$, then for $0 \neq \Psi_r \in \Sigma_r^\lambda M$, one has $\mathcal{D}_- \Psi_r = 0$, i.e., $\mathcal{D}\Psi_r = \mathcal{D}_+\Psi_r$. Identities (6.22) and (6.23) imply that

$$\begin{aligned}\int_M |\mathcal{P}^{(r)}\Psi_r|^2 \nu_g &= \int_M \sum_{j=1}^n |\mathcal{P}^{(r)}_{e_j}\Psi_r|^2 \nu_g \\ &= \int_M \sum_{j=1}^n (\mathcal{P}^{(r)}_{e_j}\Psi_r, \nabla_{e_j}\Psi_r)\nu_g \\ &= \int_M \sum_{j=1}^n \Big(\nabla_{e_j}\Psi_r + \frac{1}{2(r+1)} p_-(e_j)\cdot \mathcal{D}_+\Psi_r, \nabla_{e_j}\Psi_r\Big)\nu_g.\end{aligned}$$

Since Clifford multiplication by X or JX is skew-symmetric, it follows that

$$\int_M |\mathcal{P}^{(r)}\Psi_r|^2 \nu_g = \int_M |\nabla\Psi_r|^2\nu_g - \frac{1}{2(r+1)}\int_M |\mathcal{D}\Psi_r|^2 \nu_g. \tag{6.25}$$

Equation (6.25) combined with the Schrödinger–Lichnerowicz formula yields

$$\lambda^2 \ge \frac{2(r+1)}{2r+1}\frac{\mathrm{Scal}_0}{4}, \qquad 0 \le r < m. \tag{6.26}$$

On the other hand, if one takes R to be the largest integer for which $\Sigma_R^\lambda M \neq 0$, then as above one gets

$$\lambda^2 \ge \frac{2(m-R)+2}{2(m-R)+1}\frac{\mathrm{Scal}_0}{4}, \qquad 0 < R \le m. \tag{6.27}$$

Note that if we denote by $f(r)$ the coefficient of $\frac{\mathrm{Scal}_0}{4}$ in the right-hand side of (6.26), then the corresponding coefficient in inequality (6.27) is $f(m-R)$. By definition, one has $R = r + j$, for some integer j, $1 \le j < m$. Since the function f is strictly decreasing, it follows from (6.26) and (6.27) that the number

$$\inf_{r,j}[\sup(f(r), f(m-j-r))] = \begin{cases} f\Big(\dfrac{m-1}{2}\Big) = \dfrac{m+1}{m}, & \text{if } m \text{ is odd,} \\ f\Big(\dfrac{m-2}{2}\Big) = \dfrac{m}{m-1}, & \text{if } m \text{ is even,}\end{cases}$$

gives the optimal coefficient. This proves Theorem 6.1. □

6.5 The limiting case

We will now study the limiting cases of (6.1) and (6.2). We assume $\mathrm{Scal}_0 > 0$, since otherwise the limiting case corresponds (by the Schrödinger–Lichnerowicz formula) to the existence of a parallel spinor and is settled by Theorem 8.1 below.

We start with the case where $m = 2k + 1$ is odd. Let $\Psi \in \Gamma(\Sigma^\lambda M)$ be an eigenspinor and assume that $\lambda^2 = \frac{m+1}{4m}\mathrm{Scal}_0$. From the above proof we see that if this equality holds, Scal is constant, the projection Ψ_r of Ψ on $\Sigma_r M$ vanishes for $r \le k-1$ and $r \ge k+2$, and

$$\mathcal{P}^{(k)}\Psi_k = 0 \quad \text{and} \quad \mathcal{P}^{(k+1)}\Psi_{k+1} = 0. \tag{6.28}$$

By (6.11),

$$\mathcal{D}^2\Psi_k = \lambda^2\Psi_k \quad \text{and} \quad \mathcal{D}^2\Psi_{k+1} = \lambda^2\Psi_{k+1}.$$

Now, by (6.12), $\mathcal{D}_-\Psi_k \in \Gamma(\Sigma_{k-1}M)$ and $\mathcal{D}^2(\mathcal{D}_-\Psi_k) = \lambda^2\mathcal{D}_-\Psi_k$. The argument above shows that $\mathcal{D}_-\Psi_k = 0$. Similarly, $\mathcal{D}_+\Psi_{k+1} = 0$. From (6.18) we then get $\mathcal{D}_+\Psi_k = \mathcal{D}\Psi_k$ and $\mathcal{D}_-\Psi_{k+1} = \mathcal{D}\Psi_{k+1}$. From (6.20) and (6.28) we obtain

$$\nabla_X\Psi_k + \frac{1}{m+1}p_-(X)\cdot\mathcal{D}\Psi_k = 0 \tag{6.29}$$

and

$$\nabla_X\Psi_{k+1} + \frac{1}{m+1}p_+(X)\cdot\mathcal{D}\Psi_{k+1} = 0, \tag{6.30}$$

for all tangent vectors X. On the other hand, since $\Psi = \Psi_k + \Psi_{k+1} \in \Gamma(\Sigma^\lambda M)$, we deduce

$$\mathcal{D}\Psi_k = \lambda\Psi_{k+1} \quad \text{and} \quad \mathcal{D}\Psi_{k+1} = \lambda\Psi_k.$$

Since $\bar\Psi = (-1)^k\Psi_k + (-1)^{k+1}\Psi_{k+1}$, (6.29) and (6.30) yield

$$\nabla_X\Psi + \frac{\lambda}{m+1}X\cdot\Psi + (-1)^k\frac{\lambda}{m+1}JX\cdot\bar\Psi = 0 \tag{6.31}$$

Definition 6.13. Let (M^{4k+2}, g, J) be a Kähler spin manifold of odd complex dimension $m = 2k+1$. A *Kählerian Killing spinor* is a section Ψ of the spinor bundle of M satisfying (6.31) for some real constant λ.

We have proved the following result.

Theorem 6.14 ([Kir86]). *If (M^{4k+2}, g, J) is a compact Kähler spin manifold satisfying the limiting case in Kirchberg's inequality (6.1), then M carries a non-trivial Kählerian Killing spinor.*

Consider now the case where $m = 2k$ is even. Let $\Psi \in \Gamma(\Sigma^\lambda M)$ be an eigenspinor and assume that $\lambda^2 = \frac{m}{4(m-1)}\mathrm{Scal}_0$. From the proof of Theorem 6.1 we see that if this equality holds, then Scal is constant, the projection Ψ_r of Ψ on $\Sigma_r M$ vanishes for $r \leq k-2$ and $r \geq k+2$, and

$$\mathcal{P}^{(k-1)}\Psi_{k-1} = 0, \quad \mathcal{P}^{(k)}\Psi_k = 0, \quad \mathcal{P}^{(k+1)}\Psi_{k+1} = 0. \tag{6.32}$$

Decomposing the equality $\mathcal{D}\Psi = \lambda\Psi$ yields

$$\mathcal{D}_+\Psi_k = \lambda\Psi_{k+1}, \quad \mathcal{D}_-\Psi_k = \lambda\Psi_{k-1}, \quad \mathcal{D}_+\Psi_{k-1} + \mathcal{D}_-\Psi_{k+1} = \lambda\Psi_k, \tag{6.33}$$

and

$$\mathcal{D}_+\Psi_{k+1} = 0, \quad \mathcal{D}_-\Psi_{k-1} = 0. \tag{6.34}$$

Since $\lambda \neq 0$, this shows that Ψ_{k-1} and Ψ_{k+1} cannot vanish simultaneously. By changing the orientation of M if necessary, we may assume that $\Psi_{k+1} \neq 0$.

From (6.20) and (6.32) we obtain

$$\nabla_X\Psi_{k+1} + \frac{1}{m}p_+(X)\cdot\mathcal{D}\Psi_{k+1} = 0, \qquad X \in \Gamma(\mathrm{T}M).$$

Taking (6.33) and (6.34) into account, we arrive at the following result.

Theorem 6.15 ([Gau95a, Hij94], [Kir90], and [Lic90]). *A compact Kähler spin manifold of real dimension $n = 2m = 4k$ is a limiting manifold if and only if (up to an orientation change) there exists a nonzero spinor $\Psi_{k+1} \in \Gamma(\Sigma_{k+1}M)$ satisfying*

$$\nabla_X\Psi_{k+1} = -\frac{1}{n}(X - iJX)\cdot\mathcal{D}\Psi_{k+1} \tag{6.35}$$

and

$$\mathcal{D}^2\Psi_{k+1} = \lambda^2\Psi_{k+1}.$$

Chapter 7

Lower eigenvalue bounds on quaternion-Kähler manifolds

In this chapter we establish lower bounds for the square of the eigenvalues of the Dirac operator on compact quaternion-Kähler spin manifolds with positive scalar curvature. By definition, a Riemannian manifold is said to be quaternion-Kähler if its restricted holonomy group is contained in $\mathrm{Sp}_1 \cdot \mathrm{Sp}_m$. This holonomy condition may also be characterized by the existence of a parallel 4-form Ω of special algebraic type (cf. (7.5) below), hence Friedrich's inequality cannot be an equality in this case (cf. Chapter 5).

The model space of this family of manifolds is the quaternionic projective space $\mathbb{H}\mathrm{P}^m$ (cf. Section 13.3). From the computation of the Dirac spectrum of $\mathbb{H}\mathrm{P}^m$ (cf. Section 15.4), it was conjectured that any eigenvalue λ of the Dirac operator on such a manifold should satisfy

$$\lambda^2 \geq \frac{m+3}{m+2}\frac{\mathrm{Scal}}{4}. \tag{7.1}$$

(Note that since any quaternion-Kähler manifold is Einstein, the scalar curvature Scal is a constant). Several results in this direction were obtained in [HM95b] and [HM97], where the first eigenvalues estimates were given. Those estimates relied on the decomposition of the spinor bundle ΣM into parallel subbundles under the action of the group $\mathrm{Sp}_1 \cdot \mathrm{Sp}_m$,

$$\Sigma M = \bigoplus_{r=0}^{m} \Sigma_r M.$$

In [HM95b], these lower bounds were obtained using a twistor operator which is the *quaternionic* analogue of the Kählerian twistor operator introduced by K.-D. Kirchberg in [Kir90].

Better estimates were given in [HM97], using the same approach as in the Kähler case (cf. Chapter 6), based on the splitting of the bundle $\mathrm{T}M \otimes \Sigma_r M$ into the orthogonal sum $\mathrm{Ker}\,\gamma_r \oplus (\mathrm{Ker}\,\gamma_r)^{\perp}$, where γ_r is the restriction of the Clifford multiplication γ to $\mathrm{T}M \otimes \Sigma_r M$. The quaternion-Kähler twistor operator was introduced as the composition of the Levi-Civita covariant derivative ∇ with the projection on $\operatorname{Ker} \gamma_r$. Applying this operator to a convenient eigenspinor Ψ of $\mathcal{D}^2$, eigenvalue estimates were obtained by means of a Weitzenböck formula.

These results were not totally satisfactory, inequality (7.1) could only be deduced under the assumption that some eigenspinor corresponding to the first eigenvalue λ of the Dirac operator has a non-trivial component in $\Sigma_r M$, $r = 0$ or $r \geq [m/2] + 1$ (this is always true in quaternionic dimension $m = 2$ and $m = 3$, [HM97]). In particular, this condition is satisfied by $\mathbb{H}\mathrm{P}^m$: there exists an eigenspinor associated with the first eigenvalue of the Dirac operator which has a component in $\Sigma_0 M$ (this can be easily deduced using the results of Section 15.4).

The lower bound (7.1) was proved by W. Kramer, U. Semmelmann, and G. Weingart, using a new and conceptually simple method; see [KSW99], [KSW98a], and [KSW98b]. It is based on the decomposition of the two isomorphic bundles $(\mathrm{T}M \otimes \mathrm{T}M) \otimes \Sigma M \simeq \mathrm{T}M \otimes (\mathrm{T}M \otimes \Sigma M)$ into parallel subbundles (corresponding to the irreducible components of the corresponding representation) under the action of the group $\mathrm{Sp}_1 \cdot \mathrm{Sp}_m$. The result is proved with the help of a Weitzenböck formula, obtained by considering the isotypical parts of the two splittings of $\nabla^2 \Psi$, for a convenient choice of the eigenspinor Ψ of $\mathcal{D}^2$, cf. [KSW99] and [KSW98a]. However, this method leads to cumbersome computations (see [Bra98] and [Bra05]).

Later, using a new point of view, U. Semmelmann and G. Weingart gave a more general and systematic approach to state eigenvalue estimates for various natural operators and obtained inequality (7.1) as a special case; see [SW01] and [SW02].

Similar results were obtained recently by Y. Homma [Hom06] with a different point of view, based on the generalization in that context of Branson's "optimal Bochner–Weitzenböck formulas for gradients."

In order to study the limiting case in (7.1) (which cannot be derived from the two general systematic approaches quoted before), we will present here a simpler alternative to the proof given in [KSW99].

We will then describe the general method of Semmelmann and Weingart to produce eigenvalues estimates for various natural operators.

We start by giving some preliminaries on quaternion-Kähler manifolds.

7.1 The geometry of quaternion-Kähler manifolds

An introduction to quaternion-Kähler manifolds could be found in [Bes87], Chapter 14, or in the survey [Sal99].

Definition 7.1. A Riemannian manifold (M^{4m}, g) is said to be *quaternion-Kähler* if its bundle of oriented orthonormal frames $P_{\mathrm{SO}_{4m}} M$ admits a reduction P to the subgroup $\mathrm{Sp}_1 \cdot \mathrm{Sp}_m \subset \mathrm{SO}_{4m}$, compatible with the Levi-Civita connection ∇ (that is, such that ∇ reduces to a connection on P).

The group $\mathrm{Sp}_1 \cdot \mathrm{Sp}_m$ can be defined[1] as the image of the homomorphism

$$\begin{aligned} \alpha\colon \mathrm{Sp}_1 \times \mathrm{Sp}_m &\longrightarrow \mathrm{SO}_{4m}, \\ (q, A) &\longmapsto (x \mapsto Ax\bar{q}), \end{aligned} \tag{7.2}$$

where $\mathbb{R}^{4m} = \mathbb{H}^m$, is viewed as a right vector space over $\mathbb{H}$, the group Sp_m acting on it by $(m \times m)$ quaternion matrices. Indeed, α defines a two-fold covering of the group $\mathrm{Sp}_1 \cdot \mathrm{Sp}_m$ with $\operatorname{Ker} \alpha = \{(1, \mathrm{Id}_m), (-1, -\mathrm{Id}_m)\}$. Hence, any (irreducible) representation of this group comes from an (irreducible) representation of $\mathrm{Sp}_1 \times \mathrm{Sp}_m$ which factors through the quotient. Because $\mathrm{Sp}_1 \cdot \mathrm{Sp}_1$ is just the group SO_4, we will only consider quaternion-Kähler manifolds of dimension $4m \geq 8$.

Let Q be the 3-dimensional $\mathbb{R}$-vector space spanned by the $\mathbb{R}$-endomorphisms I, J, K of $\mathbb{H}^m$ defined respectively by

$$I\colon x \longmapsto -x\boldsymbol{i}, \quad J\colon x \longmapsto -x\boldsymbol{j}, \quad K\colon x \longmapsto -x\boldsymbol{k}, \tag{7.3}$$

so that they satisfy the same commutation rules as the standard basis of imaginary quaternions $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$. The action of the group $\mathrm{Sp}_1 \cdot \mathrm{Sp}_m$ on $\mathrm{End}(\mathbb{H}^m)$ leaves the vector space Q invariant, hence induces a representation of the group on Q, which can also be described as follows. Identify $\mathbb{R}^3 = \mathrm{span}\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ with Q, by $\boldsymbol{i} \mapsto I$, $\boldsymbol{j} \mapsto J$, $\boldsymbol{k} \mapsto K$, and endow the vector space Q with the natural Euclidean structure which makes this isomorphism an isometry. The unitary representation of $\mathrm{Sp}_1 \times \mathrm{Sp}_m$ on Q, such that Sp_m acts trivially and Sp_1 acts through the two fold covering

$$\begin{aligned} \mathrm{Sp}_1 &\longrightarrow \mathrm{SO}_3 \simeq \mathrm{SO}(\mathrm{span}\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}), \\ q &\longmapsto (x \mapsto qx\bar{q}), \end{aligned} \tag{7.4}$$

obviously factors through the quotient, and defines the aforementioned representation of the group $\mathrm{Sp}_1 \cdot \mathrm{Sp}_m$.

By definition, the quaternionic structure on M is given by the rank 3-bundle

$$\mathcal{Q}M = P \times_{\mathrm{Sp}_1 \cdot \mathrm{Sp}_m} Q.$$

Note that $\mathcal{Q}M$ is globally parallel under the Levi-Civita connection ∇, since Q is invariant under the group $\mathrm{Sp}_1 \cdot \mathrm{Sp}_m$.

Any local section σ of the bundle P defines a local frame $(J_\alpha)_{\alpha=1,2,3}$ of the bundle $\mathcal{Q}M$ by

$$\begin{aligned} J_1 x &= [\sigma(x), I], \\ J_2 x &= [\sigma(x), J], \\ J_3 x &= [\sigma(x), K]. \end{aligned}$$

[1] The notation $\mathrm{Sp}_m \cdot \mathrm{Sp}_1$ is also often used to emphasize that Sp_1 acts by right multiplication.

This local frame is such that

(a) the metric g is Hermitian for the three almost complex structures J_α, $\alpha = 1, 2, 3$;

(b) we have

$$J_\alpha \circ J_\beta = -\delta_{\alpha\beta}\mathrm{Id} + \varepsilon^{123}_{\alpha\beta\gamma} J_\gamma,$$

where

$$\varepsilon^{123}_{\alpha\beta\gamma} = \begin{cases} +1 & \text{if } (\alpha, \beta, \gamma) \text{ is an even permutation of } (1, 2, 3), \\ -1 & \text{if } (\alpha, \beta, \gamma) \text{ is an odd permutation of } (1, 2, 3), \\ 0 & \text{otherwise;} \end{cases}$$

(c) for any $\alpha = 1, 2, 3$, the covariant derivative ∇J_α is a linear combination of J_1, J_2 and J_3.

Such a local frame is said to be *standard*. Note that for the natural Euclidean structure on $\mathcal{Q}M$ carried from the fibre Q, any standard local frame is orthonormal.

The holonomy condition can also be characterized by the existence of a parallel 4-form Ω, called fundamental (see [Bon67] and [Kra66]), defined locally by

$$\Omega = \sum_{\alpha=1}^{3} \Omega_\alpha \wedge \Omega_\alpha, \tag{7.5}$$

where the Ω_α are the local 2-forms associated via the metric with the local almost complex structures J_α of a standard local frame of $\mathcal{Q}M$:

$$\Omega_\alpha(X, Y) = g(J_\alpha X, Y), \quad X, Y \in \mathrm{T}M.$$

Any quaternion-Kähler manifold is Einstein (cf. [Ber66], [Ale68], or [Ish74]; two proofs of this result may be found in [Bes87] Theorem 14.39), hence its scalar curvature is constant.

Note that if the manifold is supposed to be connected and complete (this will be the case in all what follows) then, by Myers' theorem, the positivity of the scalar curvature implies that the manifold is compact. Any such quaternion-Kähler manifold is said to be *positive*.

Any positive quaternion-Kähler manifold has a unique spin structure if m is even, whereas the only positive quaternion-Kähler spin manifold with m odd is the quaternionic projective space [Sal82].

The condition for a quaternion-Kähler manifold to be positive is rather restrictive: C. LeBrun and S. Salamon proved that, for a given integer $m \geq 2$, up to a homothetic change of the metric, there are only finitely many positive quaternion-Kähler manifolds of dimension $4m$; see [LS94]. The only known examples are

Wolf's symmetric spaces (cf. [Wol65] or Theorem 14.51 in [Bes87]). It is conjectured that there are no other examples [LS94].

Other (important) properties of quaternion-Kähler manifolds are briefly mentioned in Section 10.1.

7.2 Quaternion-Kähler spinor bundle decomposition

From now on (M^{4m}, g) is a positive quaternion-Kähler spin manifold (with $4m \geq 8$). A key ingredient in the proof of the Dirac eigenvalue estimate is the decomposition of the spinor bundle ΣM into parallel subbundles corresponding to the irreducible parts of the decomposition of the spin representation.

We first consider the following parameterization of the irreducible representations of the group $\mathrm{Sp}_1 \times \mathrm{Sp}_m$. Since this group can be identified with a subgroup of Sp_{m+1} (cf. Section 13.3), and since the standard maximal torus T of Sp_{m+1} is also a maximal torus of $\mathrm{Sp}_1 \times \mathrm{Sp}_m$, irreducible representations can be parametrized by their dominant weight relatively to this maximal torus T. Such a dominant weight has the form

$$(r, \ell_1, \dots, \ell_m),$$

where r and the ℓ_i are integers verifying the conditions $r \geq 0$, and $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_m \geq 0$, (cf. Section 15.4).

Indeed, any such $(m+1)$-tuple may be viewed as the dominant weight of a tensor product of the form $\lambda \otimes \rho$, where λ (resp. ρ) is an irreducible representation of Sp_1 (resp. Sp_m), relative to the product of the standard tori of Sp_1 and Sp_m, respectively.

In the following we often identify an irreducible representation of $\mathrm{Sp}_1 \times \mathrm{Sp}_m$ with the dominant weight $(r, \ell_1, \dots, \ell_m)$ with such a tensor product $\lambda \otimes \rho$.

More precisely, let $H := \mathbb{C}^2 \simeq \mathbb{H}$ and $E := \mathbb{C}^{2m} \simeq \mathbb{H}^m$ be the standard complex representations of the groups Sp_1 and Sp_m, respectively. Then λ is the natural representation in the space $\mathrm{Sym}^r H$ of symmetric tensors of order r over H (see Theorem 12.41), and ρ an irreducible representation of Sp_m with dominant weight $(\ell_1, \dots, \ell_m)$ (cf. Section 12.4.3).

The irreducible representations of Sp_m which are often considered are the so-called fundamental ones (cf. Section 12.4.3), whose dominant weights are given by

$$(1, \dots, 1, \underbrace{0, \dots, 0}_{\ell}),$$

where $0 \leq \ell \leq m-1$. They appear as the standard representations on the spaces $\Lambda_\circ^{m-\ell} E$, $0 \leq \ell \leq m-1$, where $\Lambda_\circ^{m-\ell} E$ denotes the orthogonal complement of the space $\omega \wedge (\Lambda^{m-\ell-2} E)$ in the space $\Lambda^{m-\ell} E$. Here, ω is the standard symplectic 2-form on E.

Example 7.2. One can easily verify that the representation of $\mathrm{Sp}_1 \times \mathrm{Sp}_m$ in the space $Q_{\mathbb{C}} := Q \otimes_{\mathbb{R}} \mathbb{C}$ is irreducible with dominant weight $(2, 0, \ldots, 0)$. Hence $Q_{\mathbb{C}} \simeq \mathrm{Sym}^2 H$ and the bundle $\mathcal{Q}_{\mathbb{C}} M := \mathcal{Q}M \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic to the bundle $P \times_{\mathrm{Sp}_1 \cdot \mathrm{Sp}_m} \mathrm{Sym}^2 H$.

Since $\mathrm{Sp}_1 \times \mathrm{Sp}_m$ is simply connected, the homomorphism

$$\alpha \colon \mathrm{Sp}_1 \times \mathrm{Sp}_m \longrightarrow \mathrm{Sp}_1 \cdot \mathrm{Sp}_m$$

lifts to a unique homomorphism

$$\tilde{\alpha} \colon \mathrm{Sp}_1 \times \mathrm{Sp}_m \longrightarrow \mathrm{Spin}_{4m}$$

(see Examples 14.13). The group $\mathrm{Sp}_1 \times \mathrm{Sp}_m$ acts on the spinor space Σ_{4m} by the representation $\widetilde{\rho_{4m}} = \rho_{4m} \circ \tilde{\alpha}$. The decomposition of this representation may be obtained by determining its maximal vectors (see Section 12.3 and [BS83], [Wan89], and [HM95a]),

$$\Sigma_{4m} = \bigoplus_{r=0}^{m} \Sigma_{4m,r}, \tag{7.6}$$

where $\Sigma_{4m,r}$ is the space of the irreducible representation with dominant weight

$$(r, 1, \ldots, 1, \underbrace{0, \ldots, 0}_{r}).$$

Thus

$$\Sigma_{4m,r} \simeq \mathrm{Sym}^r H \otimes \Lambda_\circ^{m-r} E.$$

This algebraic decomposition induces the following decomposition of the spinor bundle ΣM:

$$\Sigma M = \bigoplus_{r=0}^{m} \Sigma_r M, \tag{7.7}$$

where $\Sigma_r M$ is a globally parallel subbundle of rank $(r+1)\left(\binom{2m}{m-r} - \binom{2m}{m-r-2}\right)$.

Indeed, if m is even, all the $\mathrm{Sp}_1 \times \mathrm{Sp}_m$-representations in the spaces $\Sigma_{4m,r}$ descend to $\mathrm{Sp}_1 \cdot \mathrm{Sp}_m$. In this case ΣM can be identified with the well-defined bundle $P \times_{\mathrm{Sp}_1 \cdot \mathrm{Sp}_m} \Sigma_{4m}$, and one gets the decomposition by considering the bundles $\Sigma_r M := P \times_{\mathrm{Sp}_1 \cdot \mathrm{Sp}_m} \Sigma_{4m,r}$. If m is odd, then the spin condition implies the existence of a lifting of P to a $\mathrm{Sp}_1 \times \mathrm{Sp}_m$-principal bundle $\widetilde{P}$; see [Sal82]. In this case, the spinor bundle is isomorphic to $\widetilde{P} \times_{\mathrm{Sp}_1 \times \mathrm{Sp}_m} \Sigma_{4m}$ and one gets the decomposition by considering the bundles $\Sigma_r M := \widetilde{P} \times_{\mathrm{Sp}_1 \times \mathrm{Sp}_m} \Sigma_{4m,r}$.

Viewed as an endomorphism of the spinor bundle, the fundamental form Ω has the following local expression (cf. [HM95b]):

$$\Omega = \sum_{\alpha=1}^{3} \Omega_\alpha \cdot \Omega_\alpha + 6m\mathrm{Id}. \tag{7.8}$$

Note that, as endomorphisms of the spinor bundle, the local 2-forms Ω_α are defined by

$$\Omega_\alpha = \frac{1}{2}\sum_{i=1}^{4m} e_i \cdot J_\alpha e_i. \tag{7.9}$$

Moreover, it is easy to see that, for all $\alpha, \beta = 1, 2, 3$, they satisfy the relations

$$\begin{aligned} [\Omega_\alpha, \Omega_\beta] &= 4\varepsilon^{123}_{\alpha\beta\gamma}\Omega_\gamma, \\ 4\Omega_\alpha &= \sum_{\beta,\gamma} \varepsilon^{123}_{\alpha\beta\gamma}\Omega_\beta \cdot \Omega_\gamma, \end{aligned} \tag{7.10}$$

and

$$[\Omega, \Omega_\alpha] = 0. \tag{7.11}$$

Since Ω is $(\mathrm{Sp}_1 \cdot \mathrm{Sp}_m)$-invariant, the restriction of its action on each subbundle $\Sigma_r M$ is a scalar multiple of the identity, and one has [HM95a]

$$\Omega|_{\Sigma_r M} = (6m - 4r(r+2))\,\mathrm{Id}. \tag{7.12}$$

A second step in the proof is the decomposition of the "twistor" bundle $\mathrm{T}M^{\mathbb{C}} \otimes \Sigma M$ into parallel subbundles. By the above result, this amounts to the decomposition of the bundles $\mathrm{T}M^{\mathbb{C}} \otimes \Sigma_r M$, which are induced by the decompositions of the corresponding $\mathrm{Sp}_1 \times \mathrm{Sp}_m$-representations in the spaces $\mathbb{C}^{4m} \otimes \Sigma_{4m,r}$.

The tensor product $\mathbb{C}^{4m} \otimes \Sigma_{4m,r}$ splits as (cf. [KSW99] or [Mil99])

$$\begin{aligned} \mathbb{C}^{4m} \otimes \Sigma_{4m,r} = W_{r-1,\bar{r}} \oplus W_{r+1,\bar{r}} \oplus W_{r-1,r+1} \oplus W_{r+1,r-1} \\ \oplus W_{r-1,r-1} \oplus W_{r+1,r+1}, \end{aligned} \tag{7.13}$$

denoting by $W_{r,s}$ the space of the irreducible representation of $\mathrm{Sp}_1 \times \mathrm{Sp}_m$ with dominant weight

$$(r, 1, \dots, 1, \underbrace{0, \dots, 0}_{s}),$$

and by $W_{r,\bar{s}}$, the space of the irreducible representation of $\mathrm{Sp}_1 \times \mathrm{Sp}_m$ with dominant weight

$$(r, 2, 1, \dots, 1, \underbrace{0, \dots, 0}_{s}),$$

setting $W_{r,s} = \{0\}$ if $r < 0$ or $s > m$, and $W_{r,\bar{s}} = \{0\}$ if $r < 0$ or $s > m-1$. Note that the last two terms in (7.13) are respectively isomorphic to $\Sigma_{4m,r-1}$ and $\Sigma_{4m,r+1}$.

Exactly as for the spinor bundle, this decomposition induces the decomposition of the twistor bundle into globally parallel subbundles:

$$\begin{aligned} \mathrm{T}M^{\mathbb{C}} \otimes \Sigma_r M = {} & W_{r-1,\bar{r}}M \oplus W_{r+1,\bar{r}}M \oplus W_{r-1,r+1}M \oplus W_{r+1,r-1}M \\ & \oplus W_{r-1,r-1}M \oplus W_{r+1,r+1}M; \end{aligned} \tag{7.14}$$

the two last bundles in the sum are respectively isomorphic to $\Sigma_{r-1}M$ and $\Sigma_{r+1}M$.

7.3 The main estimate

Let $\Psi \in \Gamma(\Sigma_r M)$. According to the decomposition (7.14), $\nabla\Psi$ splits into

$$\begin{aligned} \nabla\Psi = {} & (\nabla\Psi)_{r-1,\bar{r}} + (\nabla\Psi)_{r+1,\bar{r}} + (\nabla\Psi)_{r-1,r+1} + (\nabla\Psi)_{r+1,r-1} \\ & + (\nabla\Psi)_{r-1,r-1} + (\nabla\Psi)_{r+1,r+1}. \end{aligned} \tag{7.15}$$

Since the direct sum in (7.14) is orthogonal, we get

$$\begin{aligned} |\nabla\Psi|^2 = {} & |(\nabla\Psi)_{r-1,\bar{r}}|^2 + |(\nabla\Psi)_{r+1,\bar{r}}|^2 + |(\nabla\Psi)_{r-1,r+1}|^2 + |(\nabla\Psi)_{r+1,r-1}|^2 \\ & + |(\nabla\Psi)_{r-1,r-1}|^2 + |(\nabla\Psi)_{r+1,r+1}|^2. \end{aligned} \tag{7.16}$$

As mentioned before, the last two terms in (7.15) are sections of bundles respectively isomorphic to $\Sigma_{r-1}M$ and $\Sigma_{r+1}M$. Using elementary arguments in representation theory, we first note that the last two terms in (7.16) may be expressed in terms of the square norms $|\mathcal{D}_-\Psi|^2$ and $|\mathcal{D}_+\Psi|^2$, where $\mathcal{D}_-\Psi$ and $\mathcal{D}_+\Psi$ are the only two components of $\mathcal{D}\Psi$ given by (7.7).

Lemma 7.3. *Let γ_r be the restriction of the Clifford multiplication γ to*

$$\mathbb{C}^{4m} \otimes \Sigma_{4m,r}.$$

One has

$$\operatorname{Ker}\gamma_r = W_{r-1,\bar{r}} \oplus W_{r+1,\bar{r}} \oplus W_{r-1,r+1} \oplus W_{r+1,r-1}.$$

The restriction of γ_r to $W_{r-1,r-1}$ (resp. $W_{r+1,r+1}$) is an isomorphism onto $\Sigma_{4m,r-1}$ (resp. $\Sigma_{4m,r+1}$).

Furthermore,

$$|\gamma(w)|^2 = \frac{2(r+1)(m-r+1)}{r}|w|^2, \quad w \in W_{r-1,r-1}, \tag{7.17}$$

and

$$|\gamma(w)|^2 = \frac{2(r+1)(m+r+3)}{r+2}|w|^2, \quad w \in W_{r+1,r+1}. \tag{7.18}$$

(*The Hermitian product $\langle\cdot,\cdot\rangle$ on $\mathbb{C}^{4m}\otimes\Sigma_{4m}$ is the natural one, constructed from the Hermitian products of $\mathbb{C}^{4m}$ and Σ_{4m}, respectively*).

Proof. The first part is a simple consequence of the Schur lemma. For the proof of (7.17), for simplicity, denote by γ_- the restriction of γ_r to $W_{r-1,r-1}$. This is an $(\mathrm{Sp}_1\times\mathrm{Sp}_m)$-equivariant isomorphism onto $\Sigma_{4m,r-1}$.

Now let $(\cdot,\cdot)$ be the Hermitian scalar product on $W_{r-1,r-1}$ defined by

$$(w,w') = \langle\gamma_-(w),\gamma_-(w')\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the usual Hermitian scalar product on Σ_{4m}. This is an $(\mathrm{Sp}_1\times\mathrm{Sp}_m)$-invariant scalar product on $W_{r-1,r-1}$ (since the spin representation is unitary for the scalar product $\langle\cdot,\cdot\rangle$).

By the Schur lemma, the two $(\mathrm{Sp}_1\times\mathrm{Sp}_m)$-invariant scalar products $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ on $W_{r-1,r-1}$ have to be proportional: $(\cdot,\cdot) = c\langle\cdot,\cdot\rangle$. The value of the constant c is obtained from the computation of (w_-,w_-) and $\langle w_-,w_-\rangle$ respectively, where w_- is (for instance) the maximal vector of $W_{r-1,r-1}$. The proof of (7.18) is similar to (7.17). □

Considering now the restriction of the Dirac operator $\mathcal{D}$ to the space $\Gamma(\Sigma_r M)$ of sections of the globally parallel subbundle $\Sigma_r M$, we may conclude from the above lemma that

$$\mathcal{D}\colon \Gamma(\Sigma_r M) \xrightarrow{\nabla} \Gamma(\mathrm{T}M\otimes\Sigma_r M) \xrightarrow{\gamma} \Gamma(\Sigma_{r-1}M\oplus\Sigma_{r+1}M).$$

Hence, for any $\Psi\in\Gamma(\Sigma_r M)$, the spinor field $\mathcal{D}\Psi$ splits as

$$\mathcal{D}\Psi = \mathcal{D}_-\Psi + \mathcal{D}_+\Psi,$$

where

$$\mathcal{D}_-\Psi := (\mathcal{D}\Psi)_{r-1,r-1} \quad \text{and} \quad \mathcal{D}_+\Psi := (\mathcal{D}\Psi)_{r+1,r+1}.$$

Moreover, by (7.17) and (7.18), we also have

Lemma 7.4. *For any* $\Psi \in \Gamma(\Sigma_r M)$*, the following relations hold:*

$$|(\nabla\Psi)_{r-1,r-1}|^2 = \frac{r}{2(r+1)(m-r+1)}|\mathcal{D}_-\Psi|^2$$

and

$$|(\nabla\Psi)_{r+1,r+1}|^2 = \frac{r+2}{2(r+1)(m+r+3)}|\mathcal{D}_+\Psi|^2.$$

We now consider the endomorphism

$$\mathcal{Q}\colon \mathbb{C}^{4m} \otimes \Sigma_{4m} \longrightarrow \mathbb{C}^{4m} \otimes \Sigma_{4m},$$

defined on decomposable tensors $x \otimes \psi$, $x \in \mathbb{C}^{4m}$, $\psi \in \Sigma_{4m}$, by

$$\mathcal{Q}(x \otimes \psi) = Ix \otimes (\Omega_I \cdot \psi) + Jx \otimes (\Omega_J \cdot \psi) + Kx \otimes (\Omega_K \cdot \psi),$$

where (I, J, K) is the canonical basis of Q (cf. (7.3)), and where $\Omega_I, \Omega_J, \Omega_K$ are the 2-forms associated with the three endomorphisms I, J, K. More precisely,

$$\Omega_I(x,y) = \langle Ix, y\rangle, \quad \Omega_J(x,y) = \langle Jx, y\rangle, \quad \Omega_K(x,y) = \langle Kx, y\rangle.$$

Note that, by the SO_3-invariance of the expression $I \otimes \Omega_I + J \otimes \Omega_J + K \otimes \Omega_K$, we can consider any standard basis of Q in the definition of $\mathcal{Q}$.

This indeed implies the $(\mathrm{Sp}_1 \cdot \mathrm{Sp}_m)$-equivariance of the endomorphism $\mathcal{Q}$, which naturally extends to a bundle endomorphism, also denoted by $\mathcal{Q}$:

$$\mathcal{Q}\colon \Gamma(\mathrm{T}M \otimes \Sigma M) \longrightarrow \Gamma(\mathrm{T}M \otimes \Sigma M),$$

defined locally, for all $X \in \Gamma(\mathrm{T}M)$ and $\Psi \in \Gamma(\Sigma M)$, by

$$\mathcal{Q}(X \otimes \Psi) = \sum_{\alpha=1}^{3} J_\alpha X \otimes (\Omega_\alpha \cdot \Psi),$$

where $(J_\alpha)_{\alpha=1,2,3}$ is a standard local frame of the bundle $\mathcal{Q}M$.

Lemma 7.5. *The restriction of* $\mathcal{Q}$ *to sections of* $\mathrm{T}M \otimes \Sigma_0 M$ *vanishes identically.*

Proof. Let J_α be a standard local frame of $\mathcal{Q}M$. For any $\Psi \in \Gamma(\mathrm{T}M \otimes \Sigma_0 M)$, one has

$$\sum_\alpha |\Omega_\alpha \cdot \Psi|^2 = -\Big\langle \sum_\alpha \Omega_\alpha \cdot \Omega_\alpha \cdot \Psi, \Psi\Big\rangle = 0,$$

since the restriction of $\sum_\alpha \Omega_\alpha \cdot \Omega_\alpha$ to $\Sigma_0 M$ vanishes identically. □

Now, the endomorphism $\mathcal{Q}$ being $(\mathrm{Sp}_1 \cdot \mathrm{Sp}_m)$-equivariant, its restriction to any irreducible component in decomposition (7.13) is a scalar multiple of the identity. This implies that the bundle endomorphism $\mathcal{Q}$ acts as a scalar multiple of the identity on sections of the subbundles occurring in decomposition (7.14). Computing (for instance) the image of a maximal vector in each component in decomposition (7.13), one gets the following lemma.

Lemma 7.6. *The restriction of the bundle endomorphism $\mathcal{Q}$ to any subbundle in decomposition (7.14) of $\mathrm{T}M \otimes \Sigma M$ is given by*

$$\begin{aligned} \mathcal{Q}|_{W_{r-1,\bar{r}}M} &= 2(r+2)\mathrm{Id}, & \mathcal{Q}|_{W_{r+1,\bar{r}}M} &= -2r\mathrm{Id}, \\ \mathcal{Q}|_{W_{r-1,r+1}M} &= 2(r+2)\mathrm{Id}, & \mathcal{Q}|_{W_{r+1,r-1}M} &= -2r\mathrm{Id}, \\ \mathcal{Q}|W_{r-1,r-1}M &= 2(r+2)\mathrm{Id}, & \mathcal{Q}|_{W_{r+1,r+1}M} &= -2r\mathrm{Id}. \end{aligned}$$

Now let $\Psi \in \Gamma(\Sigma_r M)$. From (7.15) we obtain

$$\begin{aligned} \mathcal{Q}(\nabla\Psi) = 2(r+2)(\nabla\Psi)_{r-1,\bar{r}} - 2r(\nabla\Psi)_{r+1,\bar{r}} + 2(r+2)(\nabla\Psi)_{r-1,r+1} \\ - 2r(\nabla\Psi)_{r+1,r-1} + 2(r+2)(\nabla\Psi)_{r-1,r-1} - 2r(\nabla\Psi)_{r+1,r+1}. \end{aligned} \tag{7.19}$$

Hence,

$$\begin{aligned} |\mathcal{Q}(\nabla\Psi)|^2 = 4(r+2)^2|(\nabla\Psi)_{r-1,\bar{r}}|^2 + 4r^2|(\nabla\Psi)_{r+1,\bar{r}}|^2 \\ + 4(r+2)^2|(\nabla\Psi)_{r-1,r+1}|^2 + 4r^2|(\nabla\Psi)_{r+1,r-1}|^2 \\ + 4(r+2)^2|(\nabla\Psi)_{r-1,r-1}|^2 + 4r^2|(\nabla\Psi)_{r+1,r+1}|^2. \end{aligned}$$

This further yields

$$\begin{aligned} &|(\nabla\Psi)_{r-1,\bar{r}}|^2 + |(\nabla\Psi)_{r-1,r+1}|^2 + |(\nabla\Psi)_{r-1,r-1}|^2 \\ &\quad = \frac{1}{4(r+2)^2}|\mathcal{Q}(\nabla\Psi)|^2 \\ &\qquad - \frac{r^2}{(r+2)^2}(|(\nabla\Psi)_{r+1,\bar{r}}|^2 + |(\nabla\Psi)_{r+1,r-1}|^2 + |(\nabla\Psi)_{r+1,r+1}|^2). \end{aligned}$$

Hence, we may write equation (7.16) as

$$\begin{aligned} &|\nabla\Psi|^2 \\ &\quad = \frac{1}{4(r+2)^2}|\mathcal{Q}(\nabla\Psi)|^2 \\ &\qquad + \frac{4(r+1)}{(r+2)^2}(|(\nabla\Psi)_{r+1,\bar{r}}|^2 + |(\nabla\Psi)_{r+1,r-1}|^2 + |(\nabla\Psi)_{r+1,r+1}|^2). \end{aligned} \tag{7.20}$$

Note that, by Lemma 7.5, (7.20) reduces to (7.16) for $\Psi \in \Gamma(\Sigma_0)$.

On the other hand, the operator $\mathcal{Q}$ satisfies the following property.

Lemma 7.7. *For any $\Psi \in \Gamma(\Sigma_r M)$, one has*

$$\int_M |\mathcal{Q}(\nabla\Psi)|^2 v_g = 4r(r+2)\int_M \Big(|\nabla\Psi|^2 + \frac{\mathrm{Scal}}{2(m+2)}|\Psi|^2\Big) v_g.$$

Proof. Let p be an arbitrary point in M. On a neighborhood U of p, let $\{e_1, \ldots, e_{4m}\}$ be a local orthonormal frame of $\mathrm{T}M$, such that

$$\nabla_{e_i} e_j(p) = 0,$$

and let (J_α) be a standard local frame of $\mathcal{Q}M$. For any $\Psi \in \Gamma(\Sigma_r)$, using (7.8), (7.12), and (7.10), at the point p, one gets

$$\begin{aligned}
|\mathcal{Q}(\nabla\Psi)|^2 &= \sum_{\alpha,\beta,i,j} \langle J_\alpha e_i \otimes \Omega_\alpha \cdot \nabla_{e_i}\Psi, J_\beta e_j \otimes \Omega_\beta \cdot \nabla_{e_j}\Psi\rangle \\
&= \sum_{\alpha,\beta,i,j} \langle J_\alpha e_i, J_\beta e_j\rangle \langle \Omega_\alpha \cdot \nabla_{e_i}\Psi, \Omega_\beta \cdot \nabla_{e_j}\Psi\rangle \\
&= \sum_{\alpha,\beta,i,j} \langle e_i, J_\alpha J_\beta e_j\rangle \langle \nabla_{e_i}\Psi, \Omega_\alpha \cdot \Omega_\beta \cdot \nabla_{e_j}\Psi\rangle \\
&= -\sum_{\alpha,i,j} \delta_{ij} \langle \nabla_{e_i}\Psi, \Omega_\alpha \cdot \Omega_\alpha \cdot \nabla_{e_j}\Psi\rangle \\
&\qquad + \sum_{\alpha,\beta,\gamma,i,j} \langle e_i, J_\gamma e_j\rangle \langle \nabla_{e_i}\Psi, \varepsilon_{\alpha\beta\gamma}\Omega_\alpha \cdot \Omega_\beta \cdot \nabla_{e_j}\Psi\rangle \\
&= 4r(r+2)|\nabla\Psi|^2 + 4\sum_i \Big\langle \nabla_{e_i}\Psi, \sum_{\gamma,j} \langle e_i, J_\gamma e_j\rangle \Omega_\gamma \cdot \nabla_{e_j}\Psi\Big\rangle.
\end{aligned} \tag{7.21}$$

Now, since the expression $\sum_\alpha J_\alpha(\cdot) \cdot \Omega_\alpha$ is parallel (it comes from an algebraic $(\mathrm{Sp}_1 \cdot \mathrm{Sp}_m)$-invariant object), we may write the second term in the right-hand side of (7.21) as

$$4\sum_i e_i\Big(\Big\langle \Psi, \sum_{\gamma,j} \langle e_i, J_\gamma e_j\rangle \Omega_\gamma \cdot \nabla_{e_j}\Psi\Big\rangle\Big) - 4\Big\langle \Psi, \sum_{\gamma,i,j} \langle e_i, J_\gamma e_j\rangle \Omega_\gamma \cdot \nabla_{e_i}\nabla_{e_j}\Psi\Big\rangle. \tag{7.22}$$

The first term in (7.22) being a divergence term, it gives zero by integration over M, so we only need to consider the second term. We have

$$
\begin{aligned}
& -4\sum_{\gamma,i,j}\langle e_i, J_\gamma e_j\rangle \Omega_\gamma \cdot \nabla_{e_i}\nabla_{e_j}\Psi \\
&= -2\sum_{\gamma,i,j}\langle e_i, J_\gamma e_j\rangle \Omega_\gamma \cdot \nabla_{e_i}\nabla_{e_j}\Psi - 2\sum_{\gamma,j,i}\langle e_j, J_\gamma e_i\rangle \Omega_\gamma \cdot \nabla_{e_j}\nabla_{e_i}\Psi \\
&= -2\sum_{\gamma,i,j}\langle e_i, J_\gamma e_j\rangle \Omega_\gamma \cdot (\nabla_{e_i}\nabla_{e_j} - \nabla_{e_j}\nabla_{e_i})\Psi \\
&= -2\sum_{\gamma,i,j}\langle e_i, J_\gamma e_j\rangle \Omega_\gamma \cdot \mathcal{R}_{e_i,e_j}\Psi \\
&= -2\sum_{\gamma,j}\Omega_\gamma \cdot \mathcal{R}_{J_\gamma e_j,e_j}\Psi \\
&= \frac{1}{2}\sum_{\gamma,j,k,l} R(e_j, J_\gamma e_j, e_k, e_l)\Omega_\gamma \cdot e_k \cdot e_l \cdot \Psi.
\end{aligned}
$$

Since the last expression is tensorial, we may now consider an *adapted* local frame on U, that is an orthonormal frame of the form

$$(e_1, J_1 e_1, J_2 e_1, J_3 e_1, \dots, e_m, J_1 e_m, J_2 e_m, J_3 e_m).$$

Since any e_j has the form $\pm J_\gamma e_i$, the Bianchi identity gives

$$\sum_{j,k} R(e_j, J_\gamma e_j)e_k = 2\sum_{i,k} R(e_i, e_k)J_\gamma e_i.$$

Therefore,

$$-4\sum_{\gamma,i,j}\langle e_i, J_\gamma e_j\rangle \Omega_\gamma \cdot \nabla_{e_i}\nabla_{e_j}\Psi = \sum_{\gamma,i,k,l} R(e_i, e_k, J_\gamma e_i, e_l)\Omega_\gamma \cdot e_k \cdot e_l \cdot \Psi.$$

Now, for any quaternion-Kähler manifold, the following basic equality holds; cf. for instance [Ish74], [Bes87], or, [HM95b]:[2]

$$[R(X,Y), J_\alpha] = \frac{\mathrm{Scal}}{4m(m+2)}\sum_{\beta,\gamma}\varepsilon^{123}_{\alpha\beta\gamma}\Omega_\beta(X,Y)J_\gamma. \tag{7.23}$$

[2]With a change of sign depending on the definition of the curvature.

From this equality, we deduce that

$$
\begin{aligned}
&\sum_{\gamma,i,k,l} R(e_i, e_k, J_\gamma e_i, e_l)\Omega_\gamma \cdot e_k \cdot e_l \cdot \Psi \\
&\quad = \sum_{\gamma,i,k,l} \langle J_\gamma(R(e_i, e_k)e_i), e_l\rangle \Omega_\gamma \cdot e_k \cdot e_l \cdot \Psi \\
&\qquad + \frac{\mathrm{Scal}}{4m(m+2)} \sum_{\substack{\gamma,\alpha,\beta\\ i,k,l}} \varepsilon^{123}_{\gamma\alpha\beta}\Omega_\alpha(e_i, e_k)\langle J_\beta e_i, e_l\rangle \Omega_\gamma \cdot e_k \cdot e_l \cdot \Psi \\
&\quad = -\sum_{\gamma,i,k,l} \langle R(e_i, e_k)e_i, J_\gamma e_l\rangle \Omega_\gamma \cdot e_k \cdot e_l \cdot \Psi \\
&\qquad + \frac{\mathrm{Scal}}{4m(m+2)} \sum_{\substack{\gamma,\alpha,\beta\\ i,k,l}} \varepsilon^{123}_{\gamma\alpha\beta}\langle J_\alpha e_i, e_k\rangle\langle J_\beta e_i, e_l\rangle \Omega_\gamma \cdot e_k \cdot e_l \cdot \Psi \\
&\quad = \sum_{\gamma,i,k,l} R(e_i, e_k, J_\gamma e_l, e_i)\Omega_\gamma \cdot e_k \cdot e_l \cdot \Psi \\
&\qquad + \frac{\mathrm{Scal}}{4m(m+2)} \sum_{\gamma,\alpha,\beta,i} \varepsilon^{123}_{\gamma\alpha\beta}\Omega_\gamma \cdot J_\alpha e_i \cdot J_\beta e_i \cdot \Psi \\
&\quad = \sum_{\gamma,k,l} \mathrm{Ric}(e_k, J_\gamma e_l)\Omega_\gamma \cdot e_k \cdot e_l \cdot \Psi \\
&\qquad + \frac{\mathrm{Scal}}{4m(m+2)} \sum_{\gamma,\alpha,\beta,i} \varepsilon^{123}_{\gamma\alpha\beta}\Omega_\gamma \cdot J_\alpha e_i \cdot J_\beta e_i \cdot \Psi.
\end{aligned}
$$

Using that the manifold is Einstein, and applying once again the fact that any e_i has the form $\pm J_\alpha e_j$, we obtain

$$
\begin{aligned}
&\sum_{\gamma,i,k,l} R(e_i, e_k, J_\gamma e_i, e_l)\Omega_\gamma \cdot e_k \cdot e_l \cdot \Psi \\
&\quad = \sum_{\gamma,k,l} \frac{\mathrm{Scal}}{4m}\langle e_k, J_\gamma e_l\rangle \Omega_\gamma \cdot e_k \cdot e_l \cdot \Psi \\
&\qquad - \frac{\mathrm{Scal}}{4m(m+2)} \sum_{\gamma,\alpha,\beta,i} \varepsilon^{123}_{\gamma\alpha\beta}\Omega_\gamma \cdot e_i \cdot J_\beta J_\alpha e_i \cdot \Psi \\
&\quad = \sum_{\gamma,l} \frac{\mathrm{Scal}}{4m}\Omega_\gamma \cdot J_\gamma e_l \cdot e_l \cdot \Psi + \frac{\mathrm{Scal}}{4m(m+2)} \sum_{\gamma,\alpha,\beta,\lambda,i} \varepsilon^{123}_{\alpha\beta\gamma}\varepsilon^{123}_{\alpha\beta\lambda}\Omega_\gamma \cdot e_i \cdot J_\lambda e_i \cdot \Psi.
\end{aligned}
$$

Hence by (7.9), it follows that

$$\begin{aligned}
&\sum_{\gamma,i,k,l} R(e_i, e_k, J_\gamma e_i, e_l)\Omega_\gamma \cdot e_k \cdot e_l \cdot \Psi \\
&\quad = -\frac{\mathrm{Scal}}{2m}\sum_\gamma \Omega_\gamma \cdot \Omega_\gamma \cdot \Psi + \frac{\mathrm{Scal}}{m(m+2)}\sum_\gamma \Omega_\gamma \cdot \Omega_\gamma \cdot \Psi \\
&\quad = -\frac{\mathrm{Scal}}{2(m+2)}\sum_\gamma \Omega_\gamma \cdot \Omega_\gamma \cdot \Psi \\
&\quad = 4r(r+2)\frac{\mathrm{Scal}}{2(m+2)}\Psi,
\end{aligned}$$

which yields the result. □

Plugging this last result into the integrated version of equation (7.20), we obtain

$$\begin{aligned}
&\int_M |\nabla\Psi|^2 v_g \\
&\quad = r\frac{\mathrm{Scal}}{4(m+2)}\int_M |\Psi|^2 v_g \\
&\qquad + \frac{2(r+1)}{r+2}\int_M (|(\nabla\Psi)_{r+1,\bar{r}}|^2 + |(\nabla\Psi)_{r+1,r-1}|^2 + |(\nabla\Psi)_{r+1,r+1}|^2) v_g.
\end{aligned} \tag{7.24}$$

This yields the following inequality:

$$\int_M |\nabla\Psi|^2 v_g - r\frac{\mathrm{Scal}}{4(m+2)}\int_M |\Psi|^2 v_g - \frac{2(r+1)}{r+2}\int_M |(\nabla\Psi)_{r+1,r+1}|^2 v_g \geq 0.$$

But by Lemma 7.4, this inequality can be written as

$$\int_M |\nabla\Psi|^2 v_g - r\frac{\mathrm{Scal}}{4(m+2)}\int_M |\Psi|^2 v_g - \frac{1}{m+r+3}\int_M |\mathcal{D}_+\Psi|^2 v_g \geq 0.$$

The final "trick" consists in applying the equality above to a convenient spinor field Ψ in $\Gamma(\Sigma_r M)$.

Let λ be the first eigenvalue of the Dirac operator. Consider the space

$$E^\lambda M := \{\Psi \in \Gamma(\Sigma M) : \mathcal{D}^2\Psi = \lambda^2\Psi\}.$$

Since the action of the 4-form Ω on $\Gamma(\Sigma M)$ commutes with $\mathcal{D}^2$ (this results from the Schrödinger–Lichnerowicz formula), this space splits into the orthogonal decomposition

$$E^\lambda M = \bigoplus_{r=0}^{m} E_r^\lambda M, \quad E_r^\lambda M := E^\lambda M \cap \Gamma(\Sigma_r M).$$

Lemma 7.8. *Let $r_{\min}$ be the smallest integer r for which $E^\lambda_r M \neq \{0\}$. For any non-trivial $\Psi \in E^\lambda_{r_{\min}} M$, the following holds.*

(i) $\mathcal{D}_-\Psi = 0$, *thus* $\mathcal{D}\Psi = \mathcal{D}_+\Psi$.

(ii) $\Phi := \mathcal{D}\Psi$ *is a non-trivial spinor field in* $E^\lambda_{r_{\min}+1}M$, *such that* $\mathcal{D}_+\Phi = 0$. *Thus* $0 \leq r_{\min} \leq m-1$.

Proof. (i) If $\mathcal{D}_-\Psi \neq 0$, then it should be a non-trivial spinor in $E^\lambda_{r_{\min}-1}M$, contradicting the definition of $r_{\min}$.

(ii) By the Lichnerowicz theorem, since the scalar curvature is assumed to be positive, there is no harmonic spinor on M, thus Φ is non-trivial. Furthermore, since $\mathcal{D}\Psi = \mathcal{D}_+\Psi$, $\Phi \in E^\lambda_{r_{\min}+1}M$, and since $\mathcal{D}\Phi = \mathcal{D}^2\Psi = \lambda^2\Psi \in E^\lambda_{r_{\min}}M$, one has $\mathcal{D}_+\Phi = 0$. □

With the notations of Lemma 7.8, let Ψ be a non-trivial spinor field in $E^\lambda_{r_{\min}}M$. Since $\mathcal{D}_-\Psi = 0$ and $\mathcal{D}_+\Psi = \mathcal{D}\Psi$, the above inequality gives

$$\int_M \Big(\frac{m + r_{\min} + 2}{m + r_{\min} + 3}\lambda^2 - \frac{m + r_{\min} + 2}{m+2}\frac{\mathrm{Scal}}{4}\Big)|\Psi|^2 v_g \geq 0.$$

Hence we obtain the inequality

$$\lambda^2 \geq \frac{m + r_{\min} + 3}{m+2}\frac{\mathrm{Scal}}{4}. \tag{7.25}$$

Since $0 \leq r_{\min} \leq m-1$, we conclude that

$$\lambda^2 \geq \frac{m+3}{m+2}\frac{\mathrm{Scal}}{4}.$$

7.4 The limiting case

Assume that the first eigenvalue λ of the Dirac operator satisfies

$$\lambda^2 = \frac{m+3}{m+2}\frac{\mathrm{Scal}}{4}.$$

Then (7.25) implies that necessarily $r_{\min} = 0$. By Lemma 7.8, there exists a spinor field Ψ_0 such that

$$\Psi_0 \in \Gamma(\Sigma_0 M) \quad \text{and} \quad \mathcal{D}^2\Psi_0 = \frac{m+3}{m+2}\frac{\mathrm{Scal}}{4}\Psi_0.$$

Moreover, the spinor field

$$\Psi_1 := \mathcal{D}\Psi_0$$

satisfies

$$\Psi_1 \in \Gamma(\Sigma_1 M) \quad \text{and} \quad \mathcal{D}^2\Psi_1 = \frac{m+3}{m+2}\frac{\mathrm{Scal}}{4}\Psi_1.$$

Now, the covariant derivatives of Ψ_0 and Ψ_1 can be obtained from the proof of the lower bound, and this is a first step in the study of the limiting case.

Applying equation (7.24) to the spinor field Ψ_0 gives

$$\int_M |\nabla\Psi_0|^2 \nu_g = \int_M (|(\nabla\Psi_0)_{1,\bar{0}}|^2 + |(\nabla\Psi_0)_{1,1}|^2)\nu_g.$$

By Lemma 7.4, we get

$$|(\nabla\Psi_0)_{1,1}|^2 = \frac{1}{m+3}|\mathcal{D}\Psi_0|^2,$$

so

$$\begin{aligned}\int_M |(\nabla\Psi_0)_{1,1}|^2\nu_g &= \frac{1}{m+3}\int_M |\mathcal{D}\Psi_0|^2\nu_g \\ &= \frac{\lambda^2}{m+3}\int_M |\Psi_0|^2\nu_g \\ &= \frac{1}{m+2}\frac{\mathrm{Scal}}{4}\int_M |\Psi_0|^2\nu_g.\end{aligned}$$

On the other hand, by the Schrödinger–Lichnerowicz formula,

$$\begin{aligned}\int_M |\nabla\Psi_0|^2\nu_g &= \Big(\lambda^2 - \frac{\mathrm{Scal}}{4}\Big)\int_M |\Psi_0|^2\nu_g \\ &= \frac{1}{m+2}\frac{\mathrm{Scal}}{4}\int_M |\Psi_0|^2\nu_g.\end{aligned}$$

Therefore,

$$\int_M |(\nabla\Psi_0)_{1,\bar{0}}|^2\nu_g = 0,$$

and so

$$(\nabla\Psi_0)_{1,\bar{0}} = 0.$$

Hence, the splitting of $\nabla\Psi_0$ under decomposition (7.14) has only one component:

$$\nabla\Psi_0 = (\nabla\Psi_0)_{1,1}.$$

Since the restriction of the Clifford multiplication γ to $W_{1,1}M$ gives an isomorphism onto $\Sigma_1 M$, this suggests that one may express $\nabla\Psi_0$ as the inverse image of $\gamma(\nabla\Psi_0) = \mathcal{D}\Psi_0 = \Psi_1$.

Lemma 7.9. *For any vector field X, denoting by $(X \cdot \Psi_1)_0$ the component of $X \cdot \Psi_1$ belonging to $\Gamma(\Sigma_0 M)$, one gets*

$$\begin{aligned}\nabla_X \Psi_0 &= -\frac{1}{m+3}(X \cdot \Psi_1)_0 \\ &= -\frac{1}{4(m+3)}\Big(X \cdot \Psi_1 + \frac{1}{2}\sum_\alpha J_\alpha X \cdot \Omega_\alpha \cdot \Psi_1\Big).\end{aligned}$$

Proof. Let $\{e_1, \dots, e_{4m}\}$, be a local orthonormal frame of $\mathrm{T}M$. One has

$$\begin{aligned}&\sum_i \Big|\nabla_{e_i}\Psi_0 + \frac{1}{m+3}(e_i \cdot \Psi_1)_0\Big|^2 \\ &\quad = \sum_i |\nabla_{e_i}\Psi_0|^2 + \frac{2}{m+3}\mathrm{Re}\sum_i \langle \nabla_{e_i}\Psi_0, e_i \cdot \Psi_1\rangle \\ &\qquad + \frac{1}{(m+3)^2}\sum_i \langle e_i \cdot \Psi_1, (e_i \cdot \Psi_1)_0\rangle \\ &\quad = |\nabla \Psi_0|^2 - \frac{2}{m+3}\mathrm{Re}\langle \mathcal{D}\Psi_0, \Psi_1\rangle - \frac{1}{(m+3)^2}\Big\langle \Psi_1, \sum_i e_i \cdot (e_i \cdot \Psi_1)_0\Big\rangle \\ &\quad = |\nabla \Psi_0|^2 - \frac{2}{m+3}|\Psi_1|^2 - \frac{1}{(m+3)^2}\Big\langle \Psi_1, \sum_i e_i \cdot (e_i \cdot \Psi_1)_0\Big\rangle.\end{aligned}$$

Now, denoting $(e_i \cdot \Psi_1)_2$ the component of $e_i \cdot \Psi_1$ belonging to $\Gamma(\Sigma_2 M)$, one gets by (7.12)

$$\Omega \cdot e_i \cdot \Psi_1 = 6m(e_i \cdot \Psi_1)_0 + (6m - 32)(e_i \cdot \Psi_1)_2.$$

Hence

$$(e_i \cdot \Psi_1)_0 = \frac{1}{32}((32 - 6m)e_i \cdot \Psi_1 + \Omega \cdot e_i \cdot \Psi_1).$$

Using the fact that

$$\sum_i e_i \cdot \Omega \cdot e_i = (8 - 4m)\Omega,$$

this implies

$$\sum_i e_i \cdot (e_i \cdot \Psi_1)_0 = -(m+3)\Psi_1.$$

Hence, since

$$|\nabla\Psi_0|^2 = |(\nabla\Psi_0)_{1,1}|^2 = \frac{1}{m+3}|\Psi_1|^2,$$

one obtains

$$\sum_i \left|\nabla_{e_i}\Psi_0 + \frac{1}{m+3}(e_i\cdot\Psi_1)_0\right|^2 = 0,$$

hence the first result. To conclude the second one, it is easy to verify that

$$\begin{aligned}(X\cdot\Psi_1)_0 &= \frac{1}{32}((32-6m)X\cdot\Psi_1 + \Omega\cdot X\cdot\Psi_1)\\ &= \frac{1}{32}(20X\cdot\Psi_1 + [\Omega, X]\cdot\Psi_1)\\ &= \frac{1}{4}\Big(X\cdot\Psi_1 + \frac{1}{2}\sum_\alpha J_\alpha X\cdot\Omega_\alpha\cdot\Psi_1\Big).\end{aligned}$$

□

We now attempt to apply the above arguments to the spinor field Ψ_1. Using equation (7.24) for the spinor field Ψ_1 yields

$$\begin{aligned}&\int_M |\nabla\Psi_1|^2 \nu_g\\ &\quad = \frac{\mathrm{Scal}}{4(m+2)}\int_M |\Psi_1|^2\nu_g\\ &\qquad + \frac{4}{3}\int_M (|(\nabla\Psi_1)_{0,\bar{1}}|^2 + |(\nabla\Psi_1)_{2,0}|^2 + |(\nabla\Psi_1)_{2,2}|^2)\nu_g.\end{aligned}$$

Since the left-hand side of the above equation is equal to

$$\frac{\mathrm{Scal}}{4(m+2)}\int_M |\Psi_1|^2\nu_g,$$

this implies that

$$(\nabla\Psi_1)_{0,\bar{1}} = (\nabla\Psi_1)_{2,0} = (\nabla\Psi_1)_{2,2} = 0,$$

which together with (7.19) gives

$$\mathcal{Q}(\nabla\Psi_1) = 6\nabla\Psi_1,$$

hence implying the following result.

Lemma 7.10. *On the domain of definition of a standard local frame* (J_α) *of* $\mathcal{Q}M$*, let* $\mathcal{D}_\alpha$*,* $\alpha = 1, 2, 3$*, be the local operators defined by*

$$\mathcal{D}_\alpha = \sum_{i=1}^{4m} J_\alpha e_i \cdot \nabla_{e_i}, \tag{7.26}$$

where (e_i) *is a local orthonormal frame of the bundle* $\mathrm{T}M$*. Then*

$$\sum_\alpha \Omega_\alpha \cdot \mathcal{D}_\alpha \Psi_1 = 0 \tag{7.27}$$

and

$$8\mathcal{D}_\alpha \Psi_1 = \sum_{\beta,\gamma} \varepsilon^{123}_{\alpha\beta\gamma} \Omega_\beta \cdot \mathcal{D}_\gamma \Psi_1. \tag{7.28}$$

Proof. Since $\gamma(\mathcal{Q}(\nabla\Psi_1)) = 6\gamma(\nabla\Psi_1) = 6\mathcal{D}\Psi_1$, we get

$$\sum_{\alpha,i} J_\alpha e_i \cdot \Omega_\alpha \cdot \nabla_{e_i} \Psi_1 = 6\mathcal{D}\Psi_1.$$

For any spinor field Ψ, any tangent vector field X, and for $\alpha = 1, 2, 3$, it is easy to see that

$$\Omega_\alpha \cdot X \cdot \Psi = X \cdot \Omega_\alpha \cdot \Psi + 2J_\alpha X \cdot \Psi. \tag{7.29}$$

This yields

$$\sum \Omega_\alpha \cdot \mathcal{D}_\alpha \Psi_1 + 6\mathcal{D}\Psi_1 = 6\mathcal{D}\Psi_1,$$

hence (7.27).

To prove (7.28), we first express locally the equality $\mathcal{Q}(\nabla\Psi_1) = 6\nabla\Psi_1$:

$$\sum_\alpha \sum_i J_\alpha e_i \otimes \Omega_\alpha \cdot \nabla_{e_i} \Psi_1 = 6 \sum_i e_i \otimes \nabla_{e_i} \Psi_1.$$

Since for any $\alpha = 1, 2, 3$, the sum $\sum_i J_\alpha e_i \otimes \Omega_\alpha \cdot \nabla_{e_i} \Psi_1$ is a tensorial expression in the e_i, we may assume that the local frame (e_i) is adapted to the corresponding almost structure J_α: each e_j is up to a sign equal to some $J_\alpha e_i$. We get

$$\sum_i J_\alpha e_i \otimes \Omega_\alpha \cdot \nabla_{e_i} \Psi_1 = -\sum_i e_i \otimes \Omega_\alpha \cdot \nabla_{J_\alpha e_i} \Psi_1.$$

Hence

$$\sum_i e_i \otimes \Big(-\sum_\alpha \Omega_\alpha \cdot \nabla_{J_\alpha e_i} \Psi_1 - 6\nabla_{e_i} \Psi_1 \Big) = 0,$$

so

$$\nabla_{e_i} \Psi_1 = -\frac{1}{6} \sum_\alpha \Omega_\alpha \cdot \nabla_{J_\alpha e_i} \Psi_1.$$

We then obtain

$$\begin{aligned}
6\mathcal{D}_\alpha \Psi_1 &= 6 \sum_{\alpha,i} J_\alpha e_i \cdot \nabla_{e_i} \Psi_1 \\
&= -\sum_{\alpha,\beta,i} J_\alpha e_i \cdot \Omega_\beta \cdot \nabla_{J_\beta e_i} \Psi_1 \\
&= \sum_{\alpha,\beta,i} J_\alpha J_\beta e_i \cdot \Omega_\beta \cdot \nabla_{e_i} \Psi_1 \\
&= -\sum_{\alpha,i} e_i \cdot \Omega_\alpha \cdot \nabla_{e_i} \Psi_1 + \sum_{\alpha,\beta,i} \varepsilon^{123}_{\alpha\beta\gamma} J_\gamma e_i \cdot \Omega_\beta \cdot \nabla_{e_i} \Psi_1 \\
&= -\sum_\alpha \Omega_\alpha \cdot \mathcal{D}\Psi_1 + 2\sum_\alpha \mathcal{D}_\alpha \Psi_1 + \sum_{\alpha,\beta,\gamma} \varepsilon^{123}_{\alpha\beta\gamma} \Omega_\beta \cdot \mathcal{D}_\gamma \Psi_1 \\
&\quad - 2 \sum_{\alpha,\beta,\gamma} \varepsilon^{123}_{\alpha\beta\gamma} J_\beta J_\gamma e_i \cdot \nabla_{e_i} \Psi_1.
\end{aligned}$$

Since $\mathcal{D}\Psi_1 \in \Gamma(\Sigma_0 M)$, we have $\sum_\alpha \Omega_\alpha \cdot \mathcal{D}\Psi_1 = 0$, by Lemma 7.5. Furthermore,

$$-2 \sum_{\alpha,\beta,\gamma} \varepsilon^{123}_{\alpha\beta\gamma} J_\beta J_\gamma e_i \cdot \nabla_{e_i} \Psi_1 = -2 \sum_{\alpha,\beta,\gamma} (\varepsilon^{123}_{\alpha\beta\gamma})^2 \mathcal{D}_\alpha \Psi_1 = -4\mathcal{D}_\alpha \Psi_1,$$

hence the result. □

Thus, the proof of the estimate provides the decomposition

$$\nabla\Psi_1 = (\nabla\Psi_1)_{2,\bar{1}} + (\nabla\Psi_1)_{0,2} + (\nabla\Psi_1)_{0,0},$$

where only the last term in the right-hand side may be expressed in terms of Ψ_0 by Lemma 7.4.

To overcome this difficulty, we consider the twisted bundle $\mathcal{Q}M \otimes \Sigma M$, and introduce the operator

$$\iota \colon \Gamma(\Sigma M) \longrightarrow \Gamma(\mathcal{Q}M \otimes \Sigma M),$$

locally defined on the domain of a standard local frame $(J_\alpha)_{\alpha=1,2,3}$ of $\mathcal{Q}M$ by

$$\Psi \longmapsto \sum_\alpha J_\alpha \otimes \Omega_\alpha \cdot \Psi.$$

Since this operator is induced by an $(\mathrm{Sp}_1 \cdot \mathrm{Sp}_m)$-equivariant algebraic operator between the corresponding fibres, it preserves the decomposition of

$$\Sigma M = \bigoplus_{r=0}^{m} \Sigma_r M;$$

i.e., we have

$$\imath \colon \Gamma(\Sigma_r M) \longrightarrow \Gamma(\mathcal{Q}M \otimes \Sigma_r M).$$

(this can also be obtained by using the commutation relations (7.11)).

In particular, by Lemma 7.5,

$$\imath|_{\Gamma(\Sigma_0 M)} \equiv 0.$$

Proposition 7.11. *Let $\mathcal{D}$ be the twisted Dirac operator acting on sections of the twisted bundle $\mathcal{Q}M \otimes \Sigma M$. For any spinor field $\Psi \in \Gamma(\Sigma M)$,*

$$\mathcal{D}(\imath\Psi) = \imath(\mathcal{D}\Psi) - 2\sum_\alpha J_\alpha \otimes \mathcal{D}_\alpha \Psi,$$

where the $\mathcal{D}_\alpha$ are the (*local*) *operators introduced in* (7.26).

Proof. Being induced by an $\mathrm{Sp}_1 \cdot \mathrm{Sp}_m$-equivariant operator, the operator $\imath$ is such that

$$\nabla(\imath\Psi) = \imath(\nabla\Psi), \qquad \Psi \in \Gamma(\Sigma M). \tag{7.30}$$

Let $\{e_1, \dots, e_{4m}\}$ be a local orthonormal frame of $\mathrm{T}M$, and $(J_\alpha)_{\alpha=1,2,3}$, be a standard local frame of $\mathcal{Q}M$, both defined on the same domain. Using (7.29), for any spinor field $\Psi \in \Gamma(\Sigma M)$, one has

$$\begin{aligned}
\mathcal{D}(\imath\Psi) &= \sum_i e_i \cdot \nabla_{e_i}(\imath\Psi) \\
&= \sum_i e_i \cdot \imath(\nabla_{e_i}\Psi) \\
&= \sum_i e_i \cdot \Big(\sum_\alpha J_\alpha \otimes \Omega_\alpha \cdot \nabla_{e_i}\Psi\Big) \\
&= \sum_\alpha J_\alpha \otimes \Big(\sum_i e_i \cdot \Omega_\alpha \cdot \nabla_{e_i}\Psi\Big) \\
&= \imath(\mathcal{D}\Psi) - 2\sum_\alpha J_\alpha \otimes \mathcal{D}_\alpha\Psi.
\end{aligned}$$

□

This and the fact that $\mathcal{D}\Psi_1 \in \Gamma(\Sigma_0 M)$ yield

$$\mathcal{D}(\iota\Psi_1) = -2\sum_\alpha J_\alpha \otimes \mathcal{D}_\alpha\Psi_1. \tag{7.31}$$

Now let γ be the Clifford multiplication acting on sections of $\mathcal{Q}M \otimes \Sigma M$.

By the properties of the Clifford multiplication acting on sections of $\Sigma_r M$, the restriction of γ to $\mathrm{T}M \otimes (\mathcal{Q}M \otimes \Sigma_1 M)$ takes values in $\mathcal{Q}M \otimes (\Sigma_0 M \oplus \Sigma_2 M)$. Using the Clebsch–Gordan formulas, this last bundle has a fiber isomorphic to

$$\begin{aligned}&\mathrm{Sym}^2 H \otimes (\Lambda^m_\circ \oplus (\mathrm{Sym}^2 H \otimes \Lambda^{m-2}_\circ))\\ &\quad\simeq (\mathrm{Sym}^2 H \otimes \Lambda^m_\circ) \oplus (\mathrm{Sym}^4 H \otimes \Lambda^{m-2}_\circ) \oplus (\mathrm{Sym}^2 H \otimes \Lambda^{m-2}_\circ) \oplus \Lambda^{m-2}_\circ.\end{aligned}$$

Being induced by an $\mathrm{Sp}_1 \cdot \mathrm{Sp}_m$-equivariant operator, the operator

$$\gamma \circ (\mathrm{Id} \otimes \iota) \colon \mathrm{T}M \otimes \Sigma_1 M \longrightarrow \mathcal{Q}M \otimes (\Sigma_0 M \oplus \Sigma_2 M),$$

preserves the decomposition of $\nabla\Psi_1$ as $\nabla\Psi_1 = (\nabla\Psi_1)_{2,\bar{1}} + (\nabla\Psi_1)_{0,2} + (\nabla\Psi_1)_{0,0}$, hence $\nabla\Psi_1$ and its image by $\gamma \circ (\mathrm{Id} \otimes \iota)$ must be sections of isomorphic bundles. Considering the decomposition above, this implies that

$$\gamma \circ (\mathrm{Id} \otimes \iota) \cdot (\nabla\Psi_1)_{0,\bar{1}} = 0,$$

$$\gamma \circ (\mathrm{Id} \otimes \iota) \cdot (\nabla\Psi_1)_{0,0} = 0,$$

and

$$\mathcal{D}(\iota\Psi_1) = \gamma \circ (\mathrm{Id} \otimes \iota) \cdot \nabla\Psi_1 = \gamma \circ (\mathrm{Id} \otimes \iota) \cdot (\nabla\Psi_1)_{0,2}.$$

Therefore, $\mathcal{D}(\iota\Psi_1)$ is a section of a bundle with fiber isomorphic to $\Lambda^{m-2}_\circ$.

Now one can prove with the same arguments as in Lemma 7.4 the following result.

Lemma 7.12. *For any $X \otimes \Psi \in \Gamma(\mathrm{T}M \otimes \Sigma_1 M)$, one has*

$$\left|\gamma \circ (\mathrm{Id} \otimes \iota) \cdot (X \otimes \Psi)\right|^2 = 16(m+4)|X \otimes \Psi|^2.$$

This implies the relation

$$|\mathcal{D}(\iota\Psi_1)|^2 = 4\sum_\alpha |\mathcal{D}_\alpha\Psi_1|^2 = 16(m+4)|(\nabla\Psi_1)_{0,2}|^2. \tag{7.32}$$

Lemma 7.13. *We have*

$$\begin{aligned}\int_M |\mathcal{D}(\iota\Psi_1)|^2 \nu_g &= 4\int_M \sum_\alpha |\mathcal{D}_\alpha\Psi_1|^2 \nu_g\\ &= \Big(\lambda^2 - \frac{\mathrm{Scal}}{m(m+2)}\Big)\int_M |\iota\Psi_1|^2 \nu_g\\ &= \frac{12(m+4)(m-1)}{m(m+2)}\frac{\mathrm{Scal}}{4}\int_M |\Psi_1|^2 \nu_g.\end{aligned}$$

Proof. The first equality follows from (7.31). To prove the second one, one observes that

$$\int_M |\mathcal{D}(\imath\Psi_1)|^2 \nu_g = \int_M \langle \mathcal{D}^2(\imath\Psi_1), \imath\Psi_1\rangle \nu_g.$$

But by the Schrödinger–Lichnerowicz formula for twisted bundles,

$$\mathcal{D}^2(\imath\Psi_1) = \nabla\nabla^*(\imath\Psi_1) + \frac{\mathrm{Scal}}{4}\imath\Psi_1 + \mathcal{R}^{\mathcal{Q}M}\cdot\imath\Psi_1,$$

where $\mathcal{R}^{\mathcal{Q}M}$ denotes the action of the curvature $R^{\mathcal{Q}M}$ of the bundle $\mathcal{Q}M$ on sections of the twisted bundle $\mathcal{Q}M \otimes \Sigma M$ (cf. Theorem 8.17 in [LM89]).

By (7.30),

$$\nabla\nabla^*(\imath\Psi_1) = \imath(\nabla\nabla^*\Psi_1),$$

hence

$$\nabla\nabla^*(\imath\Psi_1) + \frac{\mathrm{Scal}}{4}\imath\Psi_1 = \lambda^2\imath\Psi_1.$$

On the other hand,

$$\mathcal{R}^{\mathcal{Q}M}\cdot\imath\Psi_1 = \frac{1}{2}\sum_\alpha\sum_{i,j}(R^{\mathcal{Q}M}(e_i,e_j)J_\alpha)\otimes e_i\cdot e_j\cdot\Omega_\alpha\cdot\Psi_1.$$

But since $\mathcal{Q}M$ is a subbundle of the bundle $\mathrm{End}(\mathrm{T}M)$, its curvature $R^{\mathcal{Q}M}$ is given by

$$R^{\mathcal{Q}M}(X,Y)J = [R(X,Y),J], \quad X,Y\in\Gamma(\mathrm{T}M),\ J\in\Gamma(\mathcal{Q}M).$$

Hence, using (7.23), one gets

$$\begin{aligned}
&\frac{1}{2}\sum_\alpha\sum_{i,j}(R^{\mathcal{Q}M}(e_i,e_j)J_\alpha)\otimes e_i\cdot e_j\cdot\Omega_\alpha\cdot\Psi_1\\
&\quad= \frac{\mathrm{Scal}}{4m(m+2)}\sum_{\alpha,\beta,\gamma}\varepsilon^{123}_{\alpha\beta\gamma}J_\gamma\otimes\Omega_\beta\cdot\Omega_\alpha\Psi_1\\
&\quad= -\frac{\mathrm{Scal}}{m(m+2)}\sum_\gamma J_\gamma\otimes\Omega_\gamma\Psi_1 \quad \text{(using (7.10))}\\
&\quad= -\frac{\mathrm{Scal}}{m(m+2)}\imath\Psi_1.
\end{aligned}$$

Therefore,

$$\mathcal{D}^2(\imath\Psi_1) = \Big(\lambda^2 - \frac{\mathrm{Scal}}{m(m+2)}\Big)\imath\Psi_1. \tag{7.33}$$

Moreover,

$$|\imath\Psi_1|^2 = \sum_\alpha \langle\Omega_\alpha \cdot \Psi_1, \Omega_\alpha \cdot \Psi_1\rangle = -\Big\langle\sum_\alpha \Omega_\alpha \cdot \Omega_\alpha \Psi_1, \Psi_1\Big\rangle = 12|\Psi_1|^2,$$

hence the result. □

From (7.32) we then have

$$\int_M |(\nabla\Psi_1)_{0,2}|^2 \nu_g = \frac{3(m-1)}{4m(m+2)}\frac{\mathrm{Scal}}{4}\int_M |\Psi_1|^2 \nu_g.$$

On the other hand, using Lemma 7.4 and the fact that $\mathcal{D}_-\Psi_1 = \mathcal{D}\Psi_1$, we also have

$$\int_M |(\nabla\Psi_1)_{0,0}|^2 \nu_g = \frac{m+3}{4m(m+2)}\frac{\mathrm{Scal}}{4}\int_M |\Psi_1|^2 \nu_g.$$

This implies

$$\begin{aligned}
&\int_M \big(|(\nabla\Psi_1)_{0,2}|^2 + |(\nabla\Psi_1)_{0,0}|^2\big)\,\nu_g \\
&\quad = \frac{1}{m+2}\frac{\mathrm{Scal}}{4}\int_M |\Psi_1|^2 \nu_g \\
&\quad = \int_M |\nabla\Psi_1|^2 \nu_g \\
&\quad = \int_M (|(\nabla\Psi_1)_{2,\bar{1}}|^2 + |(\nabla\Psi_1)_{0,2}|^2 + |(\nabla\Psi_1)_{0,0}|^2)\nu_g.
\end{aligned}$$

Consequently,

$$\int_M |(\nabla\Psi_1)_{2,\bar{1}}|^2 \nu_g = 0,$$

and so

$$(\nabla\Psi_1)_{2,\bar{1}} = 0.$$

The splitting of $\nabla\Psi_1$ under the decomposition (7.14) is therefore given by

$$\nabla\Psi_1 = (\nabla\Psi_1)_{0,2} + (\nabla\Psi_1)_{0,0}.$$

This suggests that the covariant derivative of Ψ_1 may be expressed in terms of the spinor fields $\mathcal{D}\Psi_1 = \lambda^2\Psi_0$ and $\mathcal{D}(\imath\Psi_1) = \sum_\alpha J_\alpha \otimes \mathcal{D}_\alpha\Psi_1$.

Lemma 7.14. *For any vector field X,*

$$\nabla_X \Psi_1 = -\frac{\lambda^2}{4m} X \cdot \Psi_0 - \frac{1}{4(m+4)} \sum_\alpha J_\alpha X \cdot \mathcal{D}_\alpha \Psi_1 .$$

Proof. Let $\{e_1, \dots, e_{4m}\}$ be a local orthonormal frame of $\mathrm{T}M$, and $(J_\alpha)_{\alpha=1,2,3}$, be a standard local frame of $\mathcal{Q}M$, both defined on the same domain. The proof relies on the computation of

$$\sum_i \left| \nabla_{e_i} \Psi_1 + \frac{\lambda^2}{4m} e_i \cdot \Psi_0 + \frac{1}{4(m+4)} \sum_\alpha J_\alpha e_i \cdot \mathcal{D}_\alpha \Psi_1 \right|^2 .$$

One has

$$2\mathrm{Re} \sum_i \langle \nabla_{e_i} \Psi_1, e_i \cdot \Psi_0 \rangle = -2\mathrm{Re} \langle \mathcal{D}\Psi_1, \Psi_0 \rangle = -2\lambda^2 |\Psi_0|^2,$$

and

$$2\mathrm{Re} \sum_i \Big\langle \nabla_{e_i} \Psi_1, \sum_\alpha J_\alpha e_i \cdot \mathcal{D}_\alpha \Psi_1 \Big\rangle = -2 \sum_\alpha |\mathcal{D}_\alpha \Psi_1|^2 .$$

By (7.27) in Lemma 7.10, one gets

$$2\mathrm{Re} \sum_i \Big\langle e_i \cdot \Psi_0, \sum_\alpha J_\alpha e_i \cdot \mathcal{D}_\alpha \Psi_1 \Big\rangle = -4 \Big\langle \Psi_0, \sum_\alpha \Omega_\alpha \cdot \mathcal{D}_\alpha \Psi_1 \Big\rangle = 0.$$

Furthermore, using (7.28) in Lemma 7.10, it follows that

$$\begin{aligned}
&\sum_i \Big\langle \sum_\alpha J_\alpha e_i \cdot \mathcal{D}_\alpha \Psi_1, \sum_\beta J_\beta e_i \cdot \mathcal{D}_\beta \Psi_1 \Big\rangle \\
&\quad = -\sum_\alpha \Big\langle \mathcal{D}_\alpha \Psi_1, \sum_{\beta,i} J_\alpha e_i \cdot J_\beta e_i \cdot \mathcal{D}_\beta \Psi_1 \Big\rangle \\
&\quad = \sum_\alpha \Big\langle \mathcal{D}_\alpha \Psi_1, \sum_{\beta,i} e_i \cdot J_\beta J_\alpha e_i \cdot \mathcal{D}_\beta \Psi_1 \Big\rangle \\
&\quad = 4m \sum_\alpha |\mathcal{D}_\alpha \Psi_1|^2 + 2 \sum_\alpha \Big\langle \mathcal{D}_\alpha \Psi_1, \sum_{\beta,\gamma} \varepsilon^{123}_{\alpha\beta\gamma} \Omega_\beta \cdot \mathcal{D}_\gamma \Psi_1 \Big\rangle \\
&\quad = 4(m+4) \sum_\alpha |\mathcal{D}_\alpha \Psi_1|^2 .
\end{aligned}$$

Finally,

$$\begin{aligned}
&\sum_i \left| \nabla_{e_i}\Psi_1 + \frac{\lambda^2}{4m} e_i \cdot \Psi_0 + \frac{1}{4(m+4)} \sum_\alpha J_\alpha e_i \cdot \mathcal{D}_\alpha \Psi_1 \right|^2 \\
&\quad = |(\nabla\Psi_1)_{0,2}|^2 + |(\nabla\Psi_1)_{0,0}|^2 - \frac{\lambda^4}{4m}|\Psi_0|^2 - \frac{1}{4(m+4)} \sum_\alpha |\mathcal{D}_\alpha \Psi_1|^2 \\
&\quad = 0,
\end{aligned}$$

by (7.32) and Lemma 7.4. □

Remark 7.15. Since $\Psi_0 \in \Gamma(\Sigma_0 M)$ and $\iota|_{\Gamma(\Sigma_0 M)} \equiv 0$, one has $\mathcal{D}(\iota\Psi_0) = 0$. On the domain of any standard frame $(J_\alpha)_{\alpha=1,2,3}$, by Proposition 7.11, this implies that

$$\Omega_\alpha \cdot \Psi_1 = 2\mathcal{D}_\alpha \Psi_0.$$

Hence, for any vector field X, the covariant derivative of the spinor fields Ψ_0 and Ψ_1 may be written as

$$\begin{aligned}
\nabla_X \Psi_0 &= -\frac{1}{4(m+3)} X \cdot \mathcal{D}\Psi_0 - \frac{1}{4(m+3)} \sum_\alpha J_\alpha X \cdot \mathcal{D}_\alpha \Psi_0, \\
\nabla_X \Psi_1 &= -\frac{1}{4m} X \cdot \mathcal{D}\Psi_1 - \frac{1}{4(m+4)} \sum_\alpha J_\alpha X \cdot \mathcal{D}_\alpha \Psi_1.
\end{aligned}$$

Note that those equations may be seen as quaternion-Kähler analogues of the Kählerian twistor-spinors equations.

So the limiting case of the lower bound provides two spinor fields $\Psi_0 \in \Gamma(\Sigma_0 M)$ and $\Psi_1 \in \Gamma(\Sigma_1 M)$ such that

$$\Psi := \lambda\Psi_0 + \Psi_1$$

is an eigenspinor of the Dirac operator for the lowest eigenvalue

$$\lambda = \sqrt{\frac{m+3}{m+2}\frac{\mathrm{Scal}}{4}},$$

together with the two sections of the twisted bundle $\mathcal{Q}M \otimes \Sigma M$, $\iota(\Psi_1)$ and $\mathcal{D}(\iota\Psi_1)$, such that

$$\widetilde{\Psi} := \mu\iota(\Psi_1) + \mathcal{D}(\iota\Psi_1)$$

is an eigenspinor of the twisted Dirac operator for the eigenvalue

$$\mu = \sqrt{\frac{(m+4)(m-1)}{m(m+2)}\frac{\mathrm{Scal}}{4}}.$$

We will see in Chapter 11 that Ψ_0, Ψ_1, $\imath(\Psi_1)$, and $\mathcal{D}(\imath\Psi_1)$ may be identified with spinors on the canonical 3-Sasakian SO_3-principal bundle associated with $\mathcal{Q}M$, and that a convenient combination of these spinor fields will give rise to a non-trivial Killing spinor on that bundle. This will imply that the only limiting positive quaternion-Kähler manifolds are the quaternionic projective spaces.

7.5 A systematic approach

The general approach presented below is a powerful tool that allows one to provide not only simple proofs of eigenvalue estimates for the Dirac and the Laplace operators, but also results on harmonic forms giving vanishing theorems for Betti numbers, both for positive and negative scalar curvature. It is based on the following formulation of the Weitzenböck decompositions of Laplace operators (cf. Section I, Chapter 1 in [Bes87]).

7.5.1 General Weitzenböck formulas

Let (M^n, g) be an orientable n-dimensional Riemannian manifold. It is well known that the representation

$$\mathrm{Ad}\colon \mathrm{SO}_n \longrightarrow \mathrm{GL}(\mathfrak{so}_n)$$

can be identified with the 2-wedge product of the standard representation

$$\Lambda^2\rho_{\mathrm{std}}\colon \mathrm{SO}_n \longrightarrow \mathrm{GL}(\Lambda^2(\mathbb{R}^n)),$$

(the vector space $\Lambda^2(\mathbb{R}^n)$ being endowed with its natural scalar product $\langle , \rangle$), by the SO_n-invariant isomorphism

$$\begin{aligned} \Lambda^2(\mathbb{R}^n) &\longrightarrow \mathfrak{so}_n, \\ v\wedge w &\longmapsto (x \mapsto \langle x, v\rangle w - \langle x, w\rangle v). \end{aligned}$$

The above identification allows to associate to any representation (ρ, V) of SO_n the SO_n-equivariant map

$$c_\rho\colon \Lambda^2\mathbb{R}^n \simeq \mathfrak{so}_n \xrightarrow{\rho_*} \mathrm{End}(V).$$

This induces a morphism of vector bundles, also denoted by c_ρ,

$$c_\rho\colon \Lambda^2 M \longrightarrow \mathrm{End}(E_\rho),$$

where E_ρ is the associated vector bundle corresponding to the representation ρ.

Now let c_ρ^2 be the SO_n-equivariant map defined by the following composition

$$\Lambda^2\mathbb{R}^n \otimes \Lambda^2\mathbb{R}^n \xrightarrow{c_\rho \otimes c_\rho} \mathrm{End}(V) \otimes \mathrm{End}(V) \overset{\circ}{\longmapsto} \mathrm{End}(V),$$

the second map being the composition of endomorphisms. This induces a morphism of vector bundles, also denoted by c_ρ^2,

$$c_\rho^2 \colon \Lambda^2 M \otimes \Lambda^2 M \longrightarrow \mathrm{End}(E_\rho).$$

Considering the curvature tensor R as a section of the bundle $\Lambda^2 M \otimes \Lambda^2 M$, one thus gets a section $c_\rho^2(R)$ of the bundle $\mathrm{End}(E_\rho)$.

If (e_i) is a local orthonormal frame, then $c_\rho^2(R)$ is locally defined by the formula

$$c_\rho^2(R) = \frac{1}{4} \sum_{i,j,k,l} R(e_i, e_j, e_k, e_l) c_\rho(e_i \wedge e_j) \circ c_\rho(e_k \wedge e_l). \tag{7.34}$$

Note that, using the isomorphism $\mathfrak{so}_n \simeq \mathfrak{spin}_n$, the same construction holds, at least locally (globally if the manifold is spin), for any representation of Spin_n. If for instance, one considers the spin representation ρ_Σ, then the Schrödinger–Lichnerowicz formula can be written as

$$\mathcal{D}^2 = \nabla^*\nabla + 2c_{\rho_\Sigma}^2(R). \tag{7.35}$$

Consider now the twisted Dirac operator acting on sections of the vector bundle $\Sigma M \otimes E_\rho$, where E_ρ is now the associated vector bundle (may be locally defined as well as the bundle ΣM) corresponding to a representation ρ of Spin_n. The Schrödinger–Lichnerowicz formula is then given by

$$\mathcal{D}^2 = \nabla^*\nabla + \frac{\mathrm{Scal}}{4} \mathrm{Id}_{\Sigma M} \otimes \mathrm{Id}_{E_\rho} + \mathcal{R}^{E_\rho},$$

where the operator $\mathcal{R}^{E_\rho}$ acts, via the curvature R^{E_ρ} of the bundle E_ρ, on decomposable tensors of the form $\varphi \otimes s$ by

$$\mathcal{R}^{E_\rho}(\varphi \otimes s) := \frac{1}{2} \sum_{i,j} e_i \cdot e_j \cdot \varphi \otimes R^{E_\rho}(e_i, e_j)s,$$

(cf. Theorem 8.17 in [LM89]). It is easy to see that

$$\mathcal{R}^{E_\rho} = c_{\rho_\Sigma \otimes \rho}^2(R) - c_{\rho_\Sigma}^2(R) \otimes \mathrm{Id}_{E_\rho} - \mathrm{Id}_{\Sigma M} \otimes c_\rho^2(R),$$

hence using

$$c_{\rho_\Sigma}^2(R) = \frac{\mathrm{Scal}}{8} \mathrm{Id}_{\Sigma M},$$

one gets

$$\mathcal{D}^2 = \nabla^*\nabla + c^2_{\rho_\Sigma\otimes\rho}(R) + \frac{\mathrm{Scal}}{8}\mathrm{Id}_{\Sigma M}\otimes \mathrm{Id}_{E_\rho} - \mathrm{Id}_{\Sigma M}\otimes c^2_\rho(R). \tag{7.36}$$

In particular, choose for ρ the spin representation ρ_Σ. Recall the equivalence of the following SO_n-representations:

$$\rho_\Sigma\otimes\rho_\Sigma \simeq \rho_\Lambda := \begin{cases} \bigoplus_{p=0}^{n} \Lambda^p\rho^{\mathbb{C}}_{\mathrm{std}}, & \text{for } n \text{ even},\\ \bigoplus_{p=0}^{[\frac{n}{2}]} \Lambda^{2p}\rho^{\mathbb{C}}_{\mathrm{std}} \simeq \bigoplus_{p=0}^{[\frac{n}{2}]} \Lambda^{2p+1}\rho^{\mathbb{C}}_{\mathrm{std}}, & \text{for } n \text{ odd}, \end{cases}$$

cf. for instance Corollary 5.19 in [LM89].

Since $\mathcal{D}^2$ is the Hodge Laplacian Δ acting on

$$\bigoplus_{p=0}^{n} \Lambda^p\mathrm{T}^*_{\mathbb{C}}M \quad \text{for } n \text{ even},$$

and

$$\bigoplus_{p=0}^{[\frac{n}{2}]} \Lambda^{2p}\mathrm{T}^*_{\mathbb{C}}M \simeq \bigoplus_{p=0}^{[\frac{n}{2}]} \Lambda^{2p+1}\mathrm{T}^*_{\mathbb{C}}M \quad \text{for } n \text{ odd},$$

cf. [ABS64], [Hit74], and [LM89], one deduces immediately from (7.36) and

$$c^2_{\rho_\Sigma}(R) = \frac{\mathrm{Scal}}{8}\mathrm{Id}_{\Sigma M},$$

that

$$\Delta = \nabla^*\nabla + c^2_{\rho_\Lambda}(R). \tag{7.37}$$

Of course, this formula can also be stated directly; it is a basic example in [SW01] and [SW02].

We now assume that the holonomy group of (M^n, g) is contained in a closed subgroup H of SO_n.

This implies the existence of a reduction P of the bundle $P_{\mathrm{SO}_n}M$ to the group H, compatible with the Levi-Civita connection (that is, such that the Levi-Civita connection reduces to a connection on P).

Under the isomorphism $\mathrm{SO}_n \simeq \Lambda^2\mathbb{R}^n$ above, the adjoint representation of H on its Lie algebra $\mathfrak{h} \subset \mathfrak{so}_n$ can be identified with a representation on a subspace $\Lambda^2_{\mathfrak{h}}\mathbb{R}^n$ of $\Lambda^2\mathbb{R}^n$. Let $\Lambda^2_{\mathfrak{h}}M$ be the vector bundle associated with P by this representation. It is well known that the holonomy condition implies that the curvature tensor R, viewed as an operator $\Lambda^2 M \to \Lambda^2 M$, takes values in $\Lambda^2_{\mathfrak{h}} M$, and since R is a symmetric map, it can in fact be viewed as an operator $\Lambda^2_{\mathfrak{h}} M \to \Lambda^2_{\mathfrak{h}} M$. Consequently, one may define, as above, for any representation (ρ, V) of H, a section $c^2_\rho(R)$ of the bundle $\mathrm{End}(E_\rho)$, where E_ρ is the vector bundle associated with P by the representation ρ. Here again, the same construction holds, at least locally, if ρ is a representation of some finite covering of H.

Now, the decomposition of the representation ρ_Λ (resp. the spin representation ρ_Σ) into irreducible components under the action of H (or possibly some finite covering of H) induces a global decomposition of the vector bundle $\Lambda^* \mathrm{T}^*_{\mathbb{C}} M$ (resp. ΣM) into parallel subbundles.

By the property of the curvature relative to the holonomy condition, it is clear from the Weitzenböck decompositions (7.37) and (7.35) that the Hodge Laplacian Δ (resp. the square of the Dirac operator $\mathcal{D}^2$) preserves this decomposition.

The main ingredient of the approach consists in a comparison between the restriction of Δ (resp. $\mathcal{D}^2$) to these parallel subbundles and the following second-order differential operators.

Definition 7.16. Let (ρ, V) be a representation of H and let E_ρ be the associated vector bundle. The (elliptic) second-order differential operator Δ_ρ defined on the space $\Gamma(E_\rho)$ of sections of E_ρ by

$$\Delta_\rho := \nabla^* \nabla + c^2_\rho(R), \tag{7.38}$$

is called the *canonical Laplacian* induced by the representation.

Remark 7.17. Indeed there is a geometric reason to consider this operator instead of the rough Laplacian: for a symmetric space G/H, differential forms or spinor fields may be identified with particular functions on the group G, and under this identification, the operator Δ_ρ corresponds to a Casimir operator on G (see Chapter 15).

Now, let (ρ, V) be an *irreducible* complex representation of H and let E_ρ be the associated vector bundle.

The representation ρ is "contained" in the representation ρ_Λ (with the terminology of representation theory) if and only if the space $\mathrm{Hom}_H(V, \Lambda^* \mathbb{C}^n)$ of H-equivariant $\mathbb{C}$-linear homomorphisms $V \to \Lambda^* \mathbb{C}^n$ is non-trivial. Each non-trivial vector $\Pi \in \mathrm{Hom}_H(V, \Lambda^* \mathbb{C}^n)$ induces a parallel embedding $E_\rho \hookrightarrow \Lambda^* \mathrm{T}^*_{\mathbb{C}} M$, also denoted by Π.

Considering the canonical Laplacian Δ_ρ, one gets easily from the Weitzenböck decomposition (7.37) that

$$\Delta \circ \Pi = \Pi \circ \Delta_\rho. \tag{7.39}$$

Analogously, let ρ be an *irreducible* complex representation of H (or eventually some finite covering of H) contained in the spin representation ρ_Σ. This induces a parallel embedding $\Pi\colon E_\rho \hookrightarrow \Sigma M$. It is immediate from the Weitzenböck decomposition (7.35) that

$$\mathcal{D}^2 \circ \Pi = \Pi \circ \Delta_\rho + \frac{\mathrm{Scal}}{8}\Pi. \tag{7.40}$$

Note that equation (7.39) (resp. (7.40)) implies not only that the restriction of Δ (resp. $\mathcal{D}^2$) to any of the parallel subbundles considered above may be defined by considering some parallel embedding of the form $\Pi\colon E_\rho \hookrightarrow \Lambda^*\mathrm{T}^*_{\mathbb{C}}M$ (resp. of the form $\Pi\colon E_\rho \hookrightarrow \Sigma M$), but also that this restriction does not depend on this specific embedding.

Consider now the twisted Dirac operator $\mathcal{D}$ acting on sections of the vector bundle $\Sigma M \otimes E_\rho$, where E_ρ is the associated vector bundle corresponding to a complex representation (not necessarily irreducible) ρ of H (or possibly a finite covering of H).

Because of the last term $\mathrm{Id}_{\Sigma M} \otimes c_\rho^2(R)$ in the Weitzenböck decomposition (7.36), the operator $\mathcal{D}^2$ does not in general preserve the decomposition of $\Sigma M \otimes E_\rho$ into the sum of parallel subbundles corresponding to the decomposition of the representation $\rho_\Sigma \otimes \rho$ into irreducible components under the action of H (or possibly some finite covering of H). However, if this term acts by scalar multiplication, then $\mathcal{D}^2$ preserves this decomposition and satisfies the same properties relative to parallel embeddings as the Hodge Laplacian or the square of the untwisted Dirac operator. This situation occurs in the quaternion-Kähler setting for "natural" twists of the bundle ΣM, providing the key formula of this approach.

7.5.2 Application to quaternion-Kähler manifolds

Let P be a reduction of the bundle $P_{\mathrm{SO}_n}M$ to the group $\mathrm{Sp}_1 \cdot \mathrm{Sp}_m$, compatible with the Levi-Civita connection. A vector bundle associated with P via a (complex) representation $\rho\colon \mathrm{Sp}_1 \cdot \mathrm{Sp}_m \to \mathrm{GL}_{\mathbb{C}}(V)$ is denoted by P_V.

Let $\pi\colon \mathrm{Sp}_1 \to \mathrm{GL}_{\mathbb{C}}(H)$ and $\rho\colon \mathrm{Sp}_m \to \mathrm{GL}_{\mathbb{C}}(E)$ be the standard complex representations. Any irreducible representation of the group $\mathrm{Sp}_1 \times \mathrm{Sp}_m$ is contained in a tensor product of the form $(\otimes^p H) \otimes (\otimes^q E)$ (cf. Proposition 12.9 and Theorem 12.15), and then factors through the group $\mathrm{Sp}_1 \cdot \mathrm{Sp}_m = \mathrm{Sp}_1 \times_{\mathbb{Z}_2} \mathrm{Sp}_m$ if and only if $p + q$ is even. Moreover, the representation inherits a real structure in this case, induced by the quaternionic structures of π and ρ.

With the notations of Section 7.2, we consider the (irreducible) representations

$$\pi_k\colon \mathrm{Sp}_1 \times \mathrm{Sp}_m \longrightarrow \mathrm{Sp}_1 \longrightarrow \mathrm{GL}_{\mathbb{C}}(\mathrm{Sym}^k H)$$

and

$$\rho_\ell\colon \mathrm{Sp}_1 \times \mathrm{Sp}_m \longrightarrow \mathrm{Sp}_m \longrightarrow \mathrm{GL}_{\mathbb{C}}(\Lambda^\ell_\circ E),$$

and denote their tensor product by

$$\rho_{k,\ell} := \pi_k \otimes \rho_\ell : \mathrm{Sp}_1 \times \mathrm{Sp}_m \longrightarrow \mathrm{Sym}^k H \otimes \Lambda^\ell_\circ E.$$

Note that the associated vector bundle $P_{\mathrm{Sym}^k H\otimes \Lambda^\ell_\circ E}$ is well defined if $k+\ell$ is even. However, even this is not the case, its tensor product with another not well-defined bundle, may produce a well-defined bundle, hence we do not pay attention to this condition in what follows. Note moreover that, as noted in Section 7.2, the spin condition ensures that one can lift P to a $\mathrm{Sp}_1 \times \mathrm{Sp}_m$-principal bundle $\widetilde{P}$, hence in any case, we get well-defined bundles under that condition.

Lemma 7.18 ([SW01] and [SW02]). *For each representation of the form $\rho_{k,\ell}$, the endomorphism $c^2_{\rho_{k,\ell}}(R)$ acts by scalar multiplication on the vector bundle $P_{\mathrm{Sym}^k H\otimes \Lambda^\ell_\circ E}$. More precisely*

$$c^2_{\rho_{k,\ell}}(R) = \frac{\mathrm{Scal}}{8m(m+2)}(k(k+2) + \ell(2m-\ell+2))\mathrm{Id}_{P_{\mathrm{Sym}^k H\otimes \Lambda^\ell_\circ E}}.$$

Proof. We only sketch the proof (see [SW01] and [SW02] for details). It is based on the decomposition of the curvature tensor R into irreducible components under the action of the group $\mathrm{Sp}_1 \cdot \mathrm{Sp}_m$; see [Ale68] and [Sal82].

First, the two complex irreducible representations $\mathrm{Ad}_{\mathbb{C}} : \mathrm{Sp}_m \to \mathrm{GL}_{\mathbb{C}}(\mathfrak{sp}_{m,\mathbb{C}})$ and $\mathrm{Sp}_m \to \mathrm{GL}_{\mathbb{C}}(\mathrm{Sym}^2 E)$ are equivalent. To see this, recall that the complexified Lie algebra $\mathfrak{sp}_{m,\mathbb{C}}$ is the Lie algebra of the group $\mathrm{Sp}_{2m,\mathbb{C}}$, (cf. (12.28)), so it can be identified with the space $\mathrm{Sym}^2 E$ by the map

$$\begin{aligned} \mathrm{Sym}^2 E &\hookrightarrow E\otimes E \cong_{\omega_E} E\otimes E^* = \mathrm{End}(E), \\ x\vee y &\longmapsto x\otimes y + y\otimes x \cong \omega_E(x,\cdot)y + \omega_E(y,\cdot)x, \end{aligned} \tag{7.41}$$

the space E being identified with its dual E^* by means of the standard symplectic form ω_E. In the same way, the complexified Lie algebra $\mathfrak{sp}_{1,\mathbb{C}}$ is identified with $\mathrm{Sym}^2 H$.

Hence the curvature tensor R may be seen as a section of the bundle

$$\mathrm{Sym}^2(P_{\mathrm{Sym}^2 H} \oplus P_{\mathrm{Sym}^2 E}),$$

defined by the representation of $\mathrm{Sp}_1 \times \mathrm{Sp}_m$ in the space

$$\mathrm{Sym}^2(\mathrm{Sym}^2 H \oplus \mathrm{Sym}^2 E).$$

But this space splits into

$$\begin{aligned} &\mathrm{Sym}^2(\mathrm{Sym}^2 H \oplus \mathrm{Sym}^2 E) \\ &\quad = \mathrm{Sym}^2(\mathrm{Sym}^2 H) \oplus \mathrm{Sym}^2 H \vee \mathrm{Sym}^2 E \oplus \mathrm{Sym}^2(\mathrm{Sym}^2 E), \end{aligned}$$

where $\vee$ is the symmetrized product. Note that this is not a decomposition into irreducible parts. It is easy to see that the irreducible decomposition of $\mathrm{Sym}^2(\mathrm{Sym}^2 H)$ has a unique one-dimensional factor. Since the $\mathbb{C}$-linear extension $B^{\mathbb{C}}_{\mathfrak{sp}_1}$ of the Killing form of the Lie algebra $\mathfrak{sp}_1$ is Sp_1-invariant, this factor is actually $\mathbb{C}B^{\mathbb{C}}_{\mathfrak{sp}_1}$. Now the symplectic form ω_H induces an Sp_1-invariant symmetric form B_H on $\mathrm{Sym}^2 H$, using Gram's determinant. Thus $B^{\mathbb{C}}_{\mathfrak{sp}_1}$ is a scalar multiple of B_H. Choosing a particular vector in $\mathrm{Sym}^2 H$, it follows that $B^{\mathbb{C}}_{\mathfrak{sp}_1} = 8B_H$.

In the same way, it can be shown that the Sp_m-invariant symmetric form B_E on $\mathrm{Sym}^2 E$, induced by the symplectic form ω_E using Gram's determinant, is a scalar multiple of the $\mathbb{C}$-linear extension $B^{\mathbb{C}}_{\mathfrak{sp}_m}$ of the Killing form of $\mathfrak{sp}_m$ and that $B^{\mathbb{C}}_{\mathfrak{sp}_m} = 4(m+1)B_E$.

Now the invariant symmetric bilinear form B_H, (resp. B_E) induces a section R_H (resp. B_E) of the bundle $P_{\mathrm{Sym}^2(\mathrm{Sym}^2 H)}$, (resp. $P_{\mathrm{Sym}^2(\mathrm{Sym}^2 E)}$), obtained by parallel transport along the fibers.

It then turns out that R admits the following decomposition (see [Ale68], [Sal82], [KSW99], [SW01], and [SW02]):

$$R = -\frac{\mathrm{Scal}}{8m(m+2)}(R_H + R_E) + R_{\mathrm{hyper}},$$

where R_{hyper} is a section of the bundle $P_{\mathrm{Sym}^2(\mathrm{Sym}^2 E)}$. (The notation "$R_{\mathrm{hyper}}$" comes from the fact that this part of the curvature tensor behaves like the curvature tensor of a Riemannian manifold with holonomy contained in Sp_m, that is a hyper-Kähler manifold).

Since, by (7.34), the expression $c^2_{\rho_{k,\ell}}(R)$ depends linearly on R, one gets

$$c^2_{\rho_{k,\ell}}(R) = -\frac{\mathrm{Scal}}{8m(m+2)}(c^2_{\rho_{k,\ell}}(R_H) + c^2_{\rho_{k,\ell}}(R_E)) + c^2_{\rho_{k,\ell}}(R_{\mathrm{hyper}}).$$

Now because $B^{\mathbb{C}}_{\mathfrak{sp}_1} = 8B_H$, it is easy to see that $c^2_{\rho_{k,\ell}}(B_H)$ is nothing else but $-8\,\mathrm{Cas}_{\mathfrak{sp}_1}(\pi_k)$, where $\mathrm{Cas}_{\mathfrak{sp}_1}(\pi_k)$ is the Casimir operator of the representation π_k of Sp_1. It is known that $\mathrm{Cas}_{\mathfrak{sp}_1}(\pi_k)$ acts by scalar multiplication on the irreducible Sp_1-space $\mathrm{Sym}^k H$. Furthermore, this Casimir eigenvalue can be computed in terms of the dominant weight of the representation by means of the Freudenthal formula (cf. the comments following Theorem 15.10 for details). It is easy to verify that

$$\mathrm{Cas}_{\mathfrak{sp}_1}(\pi_k)|_{\mathrm{Sym}^k H} = \frac{1}{8}k(k+2)\mathrm{Id}_{\mathrm{Sym}^k H}.$$

Hence,

$$c^2_{\rho_{k,\ell}}(R_H) = -k(k+2)\mathrm{Id}_{P_{\mathrm{Sym}^k H \otimes \Lambda^\ell_\circ E}}.$$

Using analogous arguments, one can show that

$$c^2_{\rho_{k,\ell}}(B_E) = -4(m+1)\,\mathrm{Cas}_{\mathfrak{sp}_m}(\rho_\ell)$$

and

$$\mathrm{Cas}_{\mathfrak{sp}_m}(\rho_\ell)|_{\Lambda^\ell_\circ E} = \frac{1}{4(m+1)}\ell(2m-\ell+2)\mathrm{Id}_{\Lambda^\ell_\circ E},$$

giving

$$c^2_{\rho_{k,\ell}}(R_E) = -\ell(2m-\ell+2)\mathrm{Id}_{P_{\mathrm{Sym}^k H\otimes\Lambda^\ell_\circ E}}.$$

Finally, we claim that the last component $c^2_{\rho_{k,\ell}}(R_{\mathrm{hyper}})$ acts trivially on sections of $P_{\mathrm{Sym}^k H\otimes\Lambda^\ell_\circ E}$. To see this, it is sufficient to show that $c^2_{\rho_\ell}(w)$ acts trivially on $\Lambda^\ell_\circ E$, for any $w \in \mathrm{Sym}^2(\mathrm{Sym}^2 E)$. Since any $w \in \mathrm{Sym}^2(\mathrm{Sym}^2 E)$ can be expressed as a sum of terms of the form e^4, $e \in E$, it is indeed sufficient to check that $c^2_{\rho_\ell}(e^4)$ acts trivially on $\Lambda^\ell_\circ E$, for any $e \in E$. Now, using the isomorphism $\mathrm{Sym}^2 E \simeq \mathfrak{sp}_{m,\mathbb{C}}$, given in (7.41), the action of $c^2_{\rho_\ell}(e^4)$ on $\Lambda^\ell_\circ E$ is, up to a scalar,

$$c^2_{\rho_\ell}(e^4) = (e \wedge e^\sharp \lrcorner) \circ (e \wedge e^\sharp \lrcorner),$$

where $\sharp$ is the isomorphism $E \to E^*$ induced by the symplectic form ω_E. So

$$c^2_{\rho_\ell}(e^4) = -(e \wedge e \wedge) \circ (e^\sharp \lrcorner\, e^\sharp \lrcorner) = 0,$$

which completes the proof. □

Thus, the following result can be deduced from (7.36).

Proposition 7.19 ([SW01] and [SW02]). *Let $\mathcal{D}_{k,\ell}$ be the twisted Dirac operator acting on sections of the bundle $\Sigma M \otimes P_{\mathrm{Sym}^k H\otimes\Lambda^\ell_\circ E}$.*

The square $\mathcal{D}^2_{k,\ell}$ of this operator preserves the decomposition of the bundle $\Sigma M \otimes P_{\mathrm{Sym}^k H\otimes\Lambda^\ell_\circ E}$ into parallel subbundles corresponding to the decomposition of the representation $\rho_\Sigma \otimes \rho_{k,\ell}$, into irreducible components under the action of $\mathrm{Sp}_1 \times \mathrm{Sp}_m$.

Moreover, if P_{E_ϱ} is the vector bundle defined by an irreducible representation ϱ contained in $\rho_\Sigma \otimes \rho_{k,\ell}$, the restriction $\mathcal{D}^2_{k,\ell}|_\varrho$ of $\mathcal{D}^2_{k,\ell}$ to sections of the subbundle given by the parallel embedding $P_{E_\varrho} \hookrightarrow \Sigma M \otimes P_{\mathrm{Sym}^k H\otimes\Lambda^\ell_\circ E}$ does not depend on this specific embedding and satisfies (up to the obvious identification of the space of sections of P_{E_ϱ} with the space of sections of the embedding)

$$\Delta_\varrho = \mathcal{D}^2_{k,\ell}|_\varrho + \frac{\mathrm{Scal}}{8m(m+2)}(k+\ell-m)(k-\ell+m+2)\mathrm{Id}, \tag{7.42}$$

where Δ_ϱ is the canonical Laplacian induced by the representation ϱ; see (7.38).

7.5.3 Proof of the estimate

With the notations of Section 7.2, for $r = 0, \dots, m$, let

$$\Sigma_r M \simeq P_{\mathrm{Sym}^r H} \otimes P_{\Lambda_\circ^{m-r} E}$$

be one of the parallel subbundles of ΣM; see (7.7). Suppose for the moment that there exists a parallel embedding $\Sigma_r M \hookrightarrow \Sigma M \otimes P_{\mathrm{Sym}^k H \otimes \Lambda_\circ^\ell E}$. Then, combining formulas (7.40) and (7.42), one gets

$$\mathcal{D}^2|_{\Sigma_r} = \mathcal{D}^2_{\mathcal{R}^{k,\ell}}|_{\Sigma_r} + \frac{\mathrm{Scal}}{8m(m+2)}(k^2 + 2k - \ell^2 + 2\ell(m+1)). \tag{7.43}$$

Thus if $\Psi \in \Gamma(\Sigma_r)$ is an eigenspinor of $\mathcal{D}^2$ with eigenvalue λ^2, then

$$\lambda^2 \geq \frac{\mathrm{Scal}}{8m(m+2)}(k^2 + 2k - \ell^2 + 2\ell(m+1)).$$

To obtain the best estimate in this way, one has to determine the twists $P_{\mathrm{Sym}^k H \otimes \Lambda_\circ^\ell E}$ such that there is a parallel embedding $\Sigma_r M \hookrightarrow \Sigma M \otimes P_{\mathrm{Sym}^k H \otimes \Lambda_\circ^\ell E}$, (the *admissible twists*, in the terminology of [SW01] and [SW02]), and among those particular twists, the twist(s) for which the last term in the right-hand side of equation (7.43) is maximal (the *maximal twist(s)* for $\Sigma_r M$, in the terminology of [SW01] and [SW02]).

This is a purely algebraic problem. As a first step, one has to determine the (algebraic) twists $\mathrm{Sym}^k H \otimes \Lambda_\circ^\ell E$ such that the irreducible decomposition of the space $\Sigma_{4m} \otimes (\mathrm{Sym}^k H \otimes \Lambda_\circ^\ell E)$ under the group $\mathrm{Sp}_1 \times \mathrm{Sp}_m$ contains the irreducible module $\mathrm{Sym}^r H \otimes \Lambda_\circ^{m-r} E$. Then one has to determine among these twists, those for which the expression $k^2 + 2k - \ell^2 + 2\ell(m+1)$ is maximal.

In [SW01] and [SW02], U. Semmelmann and G. Weingart give a characterization of the admissible twists of the irreducible representations $\mathrm{Sym}^k H \otimes \Lambda_\top^{a,b} E$, where $\Lambda_\top^{a,b} E$ is the irreducible representation of highest weight in the tensor product[3] $\Lambda_\circ^a E \otimes \Lambda_\circ^b E$, and classify the maximal twists of those irreducible representations. We will not give (even a sketch of) the proof here, since it is rather long and requires some non-trivial knowledge on the representation theory of the group Sp_m.

It turns out that the maximal twist for $\Sigma_r M = P_{\mathrm{Sym}^r H} \otimes P_{\Lambda_\circ^{m-r} E}$ is

$$P_{\mathrm{Sym}^{m+r} H \otimes \Lambda_\circ^{m-r} E}.$$

In this case equation (7.43) reads

$$\mathcal{D}^2|_{\Sigma_r} = \mathcal{D}^2_{m+r,m-r}|_{\Sigma_r} + \frac{m+r+2}{m+2}\frac{\mathrm{Scal}}{4}.$$

[3]This irreducible representation is often called the *Cartan summand* of the tensor product.

Now let λ be the first eigenvalue of the Dirac operator. With the notations of Lemma 7.8, let Ψ be a non-trivial spinor field in $E^{\lambda}_{r_{\min}}M$. Since

$$\Phi := \mathcal{D}\Psi \in \Gamma(\Sigma_{r_{\min}+1})$$

is a non-trivial eigenspinor of $\mathcal{D}^2$ for the eigenvalue λ^2, one deduces from the above equation that

$$\lambda^2 \geq \frac{m + r_{\min} + 3}{m+2}\frac{\mathrm{Scal}}{4}.$$

Since $r_{\min} \in \{0, \ldots, m-1\}$, it follows that

$$\lambda^2 \geq \frac{m+3}{m+2}\frac{\mathrm{Scal}}{4}.$$

Part III

Special spinor fields and geometries

Chapter 8

Special spinors on Riemannian manifolds

8.1 Parallel spinors on spin and $\mathrm{Spin}^{\mathrm{c}}$ manifolds

In this section we study spin and $\mathrm{Spin}^{\mathrm{c}}$ manifolds carrying parallel spinors. The local structure of these manifolds is now completely understood (see [Wan89] and [Mor97b]), but the global situation is, to our knowledge, far from being solved. In the locally irreducible case, however, the complete list of Riemannian manifolds admitting a spin structure with parallel spinors is available in terms of their holonomy groups (see [Wan95] and [MS00]). The similar problem in the conformal setting will be studied in Section 9.2 below.

8.1.1 Parallel spinors on spin manifolds

Let (M^n, g) be a spin manifold with (complex) spinor bundle ΣM, and let ∇ denote the Levi-Civita covariant derivative on $\mathrm{T}M$, as well as that on ΣM. Parallel spinors are sections Ψ of ΣM satisfying the differential equation $\nabla\Psi \equiv 0$. They obviously correspond to fixed points (in Σ_n) of the restriction of the spin representation ρ_n to the spin holonomy group $\widetilde{\mathrm{Hol}}(M) \subset \mathrm{Spin}_n$.

We consider first the local problem. A spin manifold M is said to admit a local parallel spinor if its universal cover has a (globally defined) parallel spinor. Using the Berger–Simons Holonomy theorem, Hitchin [Hit74] and later on Wang [Wan89] obtained the following result.

Theorem 8.1. *Let (M^n, g) be a locally irreducible spin manifold of dimension $n \geq 2$. Then M carries a local parallel spinor if and only if its local holonomy group* $\mathrm{Hol}_0(M, g)$ *is one of the following:* G_2 $(n = 7)$*,* Spin_7 $(n = 8)$*,* SU_m $(n = 2m)$*,* Sp_k $(n = 4k)$.

Proof. If M carries a local parallel spinor, it cannot be locally symmetric. Indeed, M is Ricci-flat by Corollary 2.8, and Ricci-flat locally symmetric manifolds are flat. This would contradict the irreducibility hypothesis. One can thus use the Berger–Simons theorem, which states that the local holonomy group of M belongs to the following list: G_2 $(n = 7)$, Spin_7 $(n = 8)$, SU_m $(n = 2m)$, Sp_k $(n = 4k)$, U_m $(n = 2m)$, $\mathrm{Sp}_1 \cdot \mathrm{Sp}_k$ $(n = 4k)$, SO_n.

On the other hand, M carries a local parallel spinor if and only if there exists a fixed point in Σ_n of the restricted spin holonomy group $\widetilde{\mathrm{Hol}}_0(M)$, which is equivalent to the existence of a vector $\psi \in \Sigma_n$ on which the Lie algebra $\widetilde{\mathfrak{hol}}(M) = \mathfrak{hol}(M)$ of $\widetilde{\mathrm{Hol}}_0(M)$ acts trivially. It is an easy exercise of representation theory to check that this holds exactly for the first four groups from the Berger–Simons list (cf. [Wan89]). □

We now study the problem of the existence of *global* parallel spinors on a locally irreducible spin manifold M. If M is simply connected, the answer is of course given by the previous theorem. As we will need this result later on, we restate it, for the reader's convenience.

Theorem 8.2. *Let (M^n, g) be a simply connected irreducible spin manifold of dimension $n \geq 2$. Then M carries a parallel spinor if and only if its holonomy group* $\mathrm{Hol}(M, g)$ *is one of the following:* G_2 *($n = 7$),* Spin_7 *($n = 8$),* SU_m *($n = 2m$),* Sp_k *($n = 4k$).*

In the non-simply connected case, the first step is to obtain some algebraic restrictions on the holonomy group of M. In [Wan95], Wang obtains the list of all possible holonomy groups of irreducible Ricci-flat manifolds (which can be considerably reduced if M is compact; see [McI91]). Then, the following simple observation of Wang (which we state from a slightly different point of view, more convenient for our purposes), gives a criterion for a subgroup of SO_n to be the holonomy group of an n-dimensional manifold with parallel spinors:

Lemma 8.3. *Let (M^n, g) be a spin manifold admitting a parallel spinor. Then there exists an embedding*

$$\tilde{\iota}\colon \mathrm{Hol}(M) \longrightarrow \mathrm{Spin}_n$$

such that

$$\xi \circ \tilde{\iota} = \mathrm{Id}_{\mathrm{Hol}(M)},$$

where

$$\xi\colon \mathrm{Spin}_n \longrightarrow \mathrm{SO}_n\,.$$

Moreover, the restriction of the spin representation to $\tilde{\iota}(\mathrm{Hol}(M))$ *has a fixed point on Σ_n.*

Using this criterion, a case by case analysis yields

Theorem 8.4 ([Wan95]). *Let (M^n, g) be an irreducible Riemannian spin manifold which is not simply connected. If M admits a non-trivial parallel spinor, then the full holonomy group* $\mathrm{Hol}(M)$ *belongs to those in Table 3.*

Table 3

$\mathrm{Hol}_0(M)$	$\dim(M)$	$\mathrm{Hol}(M)$	N	conditions
SU_m	$2m$	SU_m	2	
		$\mathrm{SU}_m \rtimes \mathbb{Z}/2\mathbb{Z}$	1	$m \equiv 0(4)$
Sp_m	$4m$	Sp_m	$m+1$	
		$\mathrm{Sp}_m \times \mathbb{Z}/d\mathbb{Z}$	$(m+1)/d$	$d > 1$, d odd, d divides $m+1$
		$\mathrm{Sp}_m \cdot \Gamma$	see [Wan95]	$m \equiv 0(2)$
Spin_7	8	Spin_7	1	
G_2	7	G_2	1	

Here Γ is either $\mathbb{Z}/2d\mathbb{Z}$ ($d > 1$), or an infinite subgroup of $\mathrm{U}_1 \rtimes \mathbb{Z}/2\mathbb{Z}$, or a binary dihedral, tetrahedral, octahedral, or icosahedral group, and N denotes the dimension of the space of parallel spinors. If, moreover, M is compact, then only the possibilities described in Table 4 may occur.

Table 4

$\mathrm{Hol}_0(M)$	$\dim(M)$	$\mathrm{Hol}(M)$	N	conditions
SU_m	$2m$	SU_m	2	m odd
		$\mathrm{SU}_m \rtimes \mathbb{Z}/2\mathbb{Z}$	1	$m \equiv 0(4)$
Sp_m	$4m$	$\mathrm{Sp}_m \times \mathbb{Z}/d\mathbb{Z}$	$(m+1)/d$	$d > 1$, d odd, d divides $m+1$
G_2	7	G_2	1	

Let us point out that the algebraic restrictions on the holonomy group given by Wang theorem are actually sufficient for the existence of a spin structure carrying parallel spinors. The main tool is the following converse of Lemma 8.3.

Lemma 8.5. *Let (M^n, g) be a Riemannian manifold and suppose that there exists an embedding $\tilde{\iota}\colon \mathrm{Hol}(M) \to \mathrm{Spin}_n$ which makes the diagram*

$$\begin{tikzcd} & \mathrm{Spin}_n \arrow[d, "\xi"] \\ \mathrm{Hol}(M) \arrow[ur, "\tilde{\iota}"] \arrow[r, "\iota"'] & \mathrm{SO}_n \end{tikzcd}$$

commutative. Then M carries a spin structure whose holonomy group is exactly $\tilde{\iota}(\mathrm{Hol}(M))$, hence isomorphic to $\mathrm{Hol}(M)$.

Proof. Let ι be the inclusion of $\mathrm{Hol}(M)$ into SO_n and $\tilde{\iota}\colon \mathrm{Hol}(M) \to \mathrm{Spin}_n$ be such that $\xi \circ \tilde{\iota} = \iota$. We fix a frame $u \in P_{\mathrm{SO}_n}M$ and let $P \subset P_{\mathrm{SO}_n}M$ denote the holonomy bundle of M through u, which is a $\mathrm{Hol}(M)$ principal bundle (see [KN63], Chapter 2). Then there is a canonical bundle isomorphism $P \times_\iota \mathrm{SO}_n \simeq P_{\mathrm{SO}_n}M$ and it is clear that $P \times_{\tilde{\iota}} \mathrm{Spin}_n$ together with the canonical projection onto $P \times_\iota \mathrm{SO}_n$, defines a spin structure on M. The spin connection comes of course from the restriction to P of the Levi-Civita connection of M and hence the spin holonomy group is just $\tilde{\iota}(\mathrm{Hol}(M))$, as claimed. □

Now recall that Table 3 was obtained in the following way: among all possible holonomy groups of non-simply connected irreducible Ricci-flat Riemannian manifolds, one selects those whose holonomy group lifts isomorphically to Spin_n and such that the spin representation has fixed points when restricted to this lift. Using Lemma 8.5 we then deduce the converse of Theorem 8.4.

Theorem 8.6. *An oriented non-simply connected irreducible Riemannian manifold has a spin structure carrying parallel spinors if and only if its Riemannian holonomy group appears in Table* 3 (*or, equivalently, if it satisfies the conditions in Lemma* 8.3).

There is still an important point to be clarified here. Let $G = \mathrm{Hol}(M)$ be the holonomy group of a manifold such that G belongs to Table 3. Suppose that there are two lifts $\tilde{\iota}_i\colon G \to \mathrm{Spin}_n$, $i = 1, 2$, of the inclusion $G \to \mathrm{SO}_n$. By Lemma 8.5, each of these lifts gives rise to a spin structure on M carrying parallel spinors, and one may legitimately ask whether these spin structures are equivalent or not. The answer to this question is given by the following (more general) result.

Theorem 8.7. *Let $G \subset \mathrm{SO}_n$ and let P be a G-structure on M which is connected as a topological space. Then the extensions to* Spin_n *of P using two different lifts of G to* Spin_n *are not equivalent as spin structures.*

Proof. Recall that two spin structures Q and Q' are said to be *equivalent* if there exists a bundle isomorphism $F\colon Q \to Q'$ such that the diagram

$$\begin{tikzcd} Q \arrow[rr, "F"] \arrow[dr] & & Q' \arrow[dl] \\ & P_{\mathrm{SO}_n}M & \end{tikzcd}$$

commutes. Let $\tilde{\iota}_i\colon G \to \mathrm{Spin}_n$ $(i = 1, 2)$ be two different lifts of G and suppose that $P \times_{\tilde{\iota}_i} \mathrm{Spin}_n$ are equivalent spin structures on M. Assume that there exists a bundle map F which makes the diagram

$$\begin{tikzcd}
P \times_{\tilde\iota_1} \mathrm{Spin}_n \arrow[rr, "F"] \arrow[dr] & & P \times_{\tilde\iota_2} \mathrm{Spin}_n \arrow[dl] \\
& P \times_\iota \mathrm{SO}_n &
\end{tikzcd}$$

commutative. This easily implies the existence of a smooth mapping

$$f \colon P \times \mathrm{Spin}_n \longrightarrow \mathbb{Z}/2\mathbb{Z}$$

such that

$$F(u \times_{\tilde\iota_1} a) = u \times_{\tilde\iota_2} f(u,a)a, \quad u \in P, a \in \mathrm{Spin}_n. \tag{8.1}$$

As P and Spin_n are connected, we deduce that f is constant, say $f \equiv \varepsilon$. Then (8.1) immediately implies $\tilde\iota_1 = \varepsilon\tilde\iota_2$, hence $\varepsilon = 1$, since $\tilde\iota_i$ are group homomorphisms (and both map the identity element of G to the identity element of Spin_n), so $\tilde\iota_1 = \tilde\iota_2$, which contradicts the hypothesis. □

Using these results, one can construct examples of compact Riemannian manifolds with several spin structures carrying parallel spinors (see [MS00] for details).

8.1.2 Parallel spinors on Spin^c manifolds

In this section we classify all simply connected complete Spin^c manifolds $(M, g, \mathcal{S}, L, A)$ admitting parallel spinors. As before, a parallel spinor corresponds to a fixed point of the holonomy group of the Spin^c connection, but in contrast to the spin case, there is no analogue of the Berger–Simons theorem in this situation. The main idea here is to use a special complex distribution on M defined by the parallel spinor; see (8.6).

The curvature form dA of A can be viewed as an imaginary-valued 2-form on M, and the corresponding real 2-form will be denoted by $\Omega := -i\,dA$.

Remark 8.8. If we fix $(M, g, \mathcal{S}, L)$ and the curvature form $i\Omega$, there is a canonical free action of $\mathcal{C}^\infty(M)$ on the set of pairs (A, Ψ), where A is a connection on L with curvature $i\Omega$ and Ψ is a non-zero parallel spinor with respect to this connection. This action is given by

$$f.(A, \Psi) = (A - 2i\,df, e^{if}\Psi).$$

When seeking for Spin^c structures with parallel spinors, we will thus most of the time implicitly divide by this free action.

Lemma 8.9. *Suppose there exists a parallel spinor Ψ on M^n, i.e.,*

$$\nabla^A_X \Psi = 0, \quad X \in \Gamma(\mathrm{T}M). \tag{8.2}$$

Then

$$\mathrm{Ric}(X) \cdot \Psi = i X \lrcorner \Omega \cdot \Psi, \quad X \in \Gamma(\mathrm{T}M). \tag{8.3}$$

Proof. Let $\{e_1, \dots, e_n\}$ be a local orthonormal tangent frame. From (8.2) we easily obtain

$$\mathcal{R}^A_{X,Y} \Psi = 0, \quad X, Y \in \Gamma(\mathrm{T}M). \tag{8.4}$$

A local computation shows that the curvature operator on the spinor bundle is given by the formula

$$\mathcal{R}^A = \mathcal{R} + \frac{i}{2}\Omega, \tag{8.5}$$

where

$$\mathcal{R}_{X,Y} = \frac{1}{2} \sum_{j<k} R(X, Y, e_j, e_k) e_j \cdot e_k \cdot$$

Using (8.4) and (8.5) together with the Ricci identity (2.3), we get

$$\begin{aligned} 0 &= \sum_j e_j \cdot \mathcal{R}^A_{e_j,X} \Psi \\ &= \sum_j e_j \cdot (\mathcal{R}_{e_j,X} \Psi + \frac{i}{2} \Omega(e_j, X) \Psi) \\ &= \frac{1}{2} \mathrm{Ric}(X) \cdot \Psi - \frac{i}{2} X \lrcorner \Omega \cdot \Psi. \end{aligned}$$

□

We consider Ric as a (1,1) tensor on M and for every $x \in M$ denote by $K(x)$ the image of Ric, i.e.,

$$K(x) = \{\mathrm{Ric}(X);\ X \in \mathrm{T}_x M\}$$

and by $L(x)$ the orthogonal complement of $K(x)$ in $\mathrm{T}_x M$, which by (8.3) can be written as

$$L(x) = \{X \in \mathrm{T}_x M;\ \mathrm{Ric}(X) = 0\} = \{X \in \mathrm{T}_x M;\ X \lrcorner \Omega = 0\}.$$

Since Ψ is parallel, $\mathrm{T}M \cdot \Psi$ and $i\mathrm{T}M \cdot \Psi$ are two parallel sub-bundles of ΣM. This shows that their intersection is also a parallel sub-bundle of ΣM. Let E be the inverse image of $\mathrm{T}M \cdot \Psi \cap i\mathrm{T}M \cdot \Psi$ by the isomorphism

$$F : \mathrm{T}M \longrightarrow \mathrm{T}M \cdot \Psi$$

given by

$$F(X) = X \cdot \Psi.$$

The fiber at some $x \in M$ of E can be described as

$$E_x = \{X \in \mathrm{T}_x M;\ \text{there exists } Y \in \mathrm{T}_x M \text{ such that } X \cdot \Psi = iY \cdot \Psi\}. \tag{8.6}$$

By the discussion above, E and $E^\perp$ are well-defined parallel distributions of M. Moreover, (8.3) shows that $K(x) \subset E_x$ for all x, so $E_x^\perp \subset L(x)$, i.e., the restriction of Ω to $E^\perp$ vanishes.

Assuming now that M is complete and simply connected, we use the de Rham decomposition theorem and obtain that M is isometric to a Riemannian product $M = M_1 \times M_2$, where M_1 and M_2 are arbitrary integral manifolds of E and $E^\perp$ respectively.

The Spin^c structure of M induces Spin^c structures on M_1 and M_2 with canonical line bundles whose curvature is given by the restriction of Ω to E and $E^\perp$, and (using the correspondence between parallel spinors and fixed points of the spin holonomy representation) the parallel spinor Ψ induces parallel spinors Ψ_1 and Ψ_2 on M_1 and M_2, which satisfy (8.3).

It is clear that (since the restriction of Ω to $E^\perp$ vanishes) the canonical line bundle of the Spin^c structure on M_2 is trivial and has vanishing curvature, so by Lemma 2.12, Ψ_2 is actually a parallel spinor of a *spin* structure on M_2 (see Remark 8.8 above).

On the other hand, by the very definition of E, one easily obtains that the condition

$$X \cdot \Psi = iJX \cdot \Psi, \tag{8.7}$$

defines an almost complex structure J on M_1.

Lemma 8.10. *The almost complex structure J defined by equation (8.7) is parallel, so (M_1, J) is a Kähler manifold.*

Proof. Taking the covariant derivative (on M_1) of (8.7) in an arbitrary direction Y and using (8.2) gives

$$\nabla_Y X \cdot \Psi = i\nabla_Y (JX) \cdot \Psi. \tag{8.8}$$

On the other hand, replacing X by $\nabla_Y X$ in (8.7) and subtracting from (8.8) yields

$$\nabla_Y (JX) \cdot \Psi = J\nabla_Y X \cdot \Psi,$$

so $((\nabla_Y J)(X)) \cdot \Psi = 0$, and finally $(\nabla_Y J)(X) = 0$, since Ψ never vanishes on M_1. As X and Y were arbitrary vector fields, we conclude that $\nabla J = 0$. $\square$

We finally point out that the restriction of the Spin^c structure of M to M_1 is just the canonical Spin^c structure of M_1, since (8.7) and (8.3) show that the restriction of Ω to M_1 is the Ricci form of M_1.

Remark 8.11. Of course, replacing J by $-J$ just means switching from the canonical to the anti-canonical Spin^c structure of M_1, but we solve this ambiguity by "fixing" the sign of J with the help of (8.7).

Conversely, the Riemannian product of two Spin^c manifolds carrying parallel spinors is again a Spin^c manifold carrying parallel spinors, and as we already pointed out in the first section, the canonical Spin^c structure of every Kähler manifold carries parallel spinors. Consequently, we have proved the following theorem.

Theorem 8.12. *A complete, simply connected* Spin^c *manifold M carries a parallel spinor if and only if it is isometric to the Riemannian product $M_1 \times M_2$ of a simply connected Kähler manifold and a simply connected spin manifold carrying a parallel spinor. Moreover, the* Spin^c *structure of M is the product of the canonical* Spin^c *structure of M_1 and the spin structure of M_2 (modulo the action of the gauge group $\mathcal{C}^\infty(M)$).*

There are two natural questions that one may ask.

Question 1. What is the dimension of the space of parallel spinors on M?

Question 2. How many Spin^c structures on M carry parallel spinors?

We can assume that M is irreducible, since otherwise we decompose M, endow each component with the induced Spin^c structure, and apply the argument below for each component separately.

Using Theorem 8.12, we can thus always suppose that M is either an irreducible spin manifold carrying parallel spinors, or an irreducible Kähler manifold (not Ricci flat, since these ones are already contained in the first class), endowed with the canonical Spin^c structure. We call such manifolds *of type S* (spin) and *of type K* (Kähler) respectively. Then the answers to the above questions are the following ones.

1. For manifolds of type S, the answer is given by M. Wang's classification; see [Wan89]. For manifolds of type K, we will show that the dimension of the space of parallel spinors is 1.

Suppose we have two parallel spinors Ψ_1 and Ψ_2 on M. Correspondingly we have two Kähler structures J_1 and J_2. Moreover, the Ricci forms of these Kähler structures are both equal to Ω, so $J_1 = J_2$ when restricted to the image of the Ricci tensor. On the other hand, the vectors X for which $J_1 X = J_2 X$ form a parallel distribution on M, which, by irreducibility, is either the whole of $\mathrm{T}M$ or empty.

In the first case we have $J_1 = J_2$, so both Ψ_1 and Ψ_2 are parallel sections of $\Sigma_0 M$ whose complex dimension is 1. The second case is impossible, since it would imply that the Ricci tensor vanishes.

2. This question has a meaning only for manifolds of type K. A slight modification in the above argument shows that on a manifold of type K there are no other Kähler structures except for J and $-J$.

On the other hand, we have seen that a parallel spinor with respect to some Spin^c structure induces on such a manifold M a Kähler structure whose canonical Spin^c structure is just the given one. It is now clear that we can have at most two different Spin^c structures with parallel spinors: the canonical Spin^c structures induced by J and $-J$. These are just the canonical and anti-canonical Spin^c structures on (M, J), and they both carry parallel spinors.

Summarizing, we have the following result.

Proposition 8.13. *The only* Spin^c *structures on an irreducible not Ricci-flat Kähler manifold which carry real Killing spinors are the canonical and anti-canonical ones. The dimension of the space of parallel spinors is in both cases equal to* 1.

8.2 Special holonomies and relations to warped products

This section is intended to provide a dictionary between the different geometries of the base and the total space of a special class of warped products, namely the Riemannian cones. They will be needed in the next section for the classification of complete spin or Spin^c manifolds with Killing spinors (cf. Theorem 2.41).

8.2.1 Sasakian structures

Let (M, g_M) be a Riemannian manifold with Levi-Civita covariant derivative ∇.

Definition 8.14. A *Sasakian structure* on a Riemannian manifold (M, g_M) is a Killing vector field ξ of unit norm with the property

$$\nabla^2_{X,Y}\xi = -g_M(X, Y)\xi + \eta(Y)X, \quad X, Y \in \mathrm{T}M, \tag{8.9}$$

where $\eta := g_M(\xi\cdot, \cdot)$ is the 1-form corresponding to ξ.

Taking the scalar product with ξ in (8.9) shows that the (1,1)-tensor $\varphi := -\nabla\xi$ vanishes on ξ and defines an orthogonal complex structure on $\xi^\perp$. Since ξ is a Killing vector field, its covariant derivative is related to the exterior differential of the associated 1-form η by

$$g_M(\nabla_X\xi, Y) = \frac{1}{2}d\eta(X, Y), \quad X, Y \in \mathrm{T}M. \tag{8.10}$$

Definition 8.15. The *Riemannian cone* over (M, g_M) is the manifold

$$\bar{M} = M \times \mathbb{R}^+,$$

endowed with the Riemannian metric

$$g_{\bar{M}} := r^2 g_M + dr^2.$$

The covariant derivative of the Levi-Civita connection of $g_{\bar{M}}$ will be denoted by $\bar{\nabla}$. It satisfies the warped product relations (2.32)–(2.34), where X and Y are vector fields on M, identified with their canonical extensions to $\bar{M}$.

Theorem 8.16. *The Sasakian structures on M are in one-to-one correspondence with the Kähler structures on the Riemannian cone over M preserved by the flow of the radial vector field $r\partial_r$ of $\bar{M}$.*

Proof. Suppose first that (ξ, η, φ) is a Sasakian structure on M. We define a tensor J of type (1,1) on $\bar{M}$ by

$$J(r\partial_r) = \xi, \quad J\xi = -r\partial_r, \quad JX = -\varphi(X), \quad \text{for } X \perp \partial_r, X \perp \xi.$$

The remark above shows that J is an orthogonal almost complex structure on $\bar{M}$. By (8.10), the associated 2-form $\Omega(\cdot,\cdot) := g_M(J\cdot,\cdot)$ is given by

$$\Omega = \frac{r^2}{2} d\eta + r dr \wedge \eta,$$

so $d\Omega = 0$ on $\bar{M}$.

Let N^J denote the Nijenhuis tensor of J, defined by

$$\begin{aligned} N^J(X,Y) &:= [X,Y] + J[X,JY] + J[JX,Y] - [JX,JY] \\ &= J(\mathcal{L}_X J)Y - (\mathcal{L}_{JX} J)Y. \end{aligned}$$

We claim that $\mathcal{L}_{r\partial_r} J = 0$ and $\mathcal{L}_\xi J = 0$. The first relation follows from the fact that the flow of $r\partial_r$ is given by $\varphi_t(r,x) = (e^t r, x)$ and clearly preserves J. The vector field ξ is Killing for g_M, hence also for $g_{\bar{M}}$. Moreover, $\mathcal{L}_\xi \Omega = 0$, thus proving the second relation.

Let now X and Y denote vector fields on $\bar{M}$ induced by arbitrary vector fields on M perpendicular to ξ. Using (2.33) and (2.34), we get

$$\begin{aligned} J(\bar{\nabla}_X Y) &= J(\nabla_X Y) - rJ(g_M(X,Y)\partial_r) \\ &= J(g_M(\nabla_X Y, \xi)\xi) + J(\nabla_X Y - g_M(\nabla_X Y, \xi)) - g_M(X,Y)\xi \\ &= -\, r g_M(\xi, \nabla_X Y)\partial_r - \varphi(\nabla_X Y) - g_M(X,Y)\xi, \end{aligned}$$

and

$$\bar{\nabla}_X(JY) = -\bar{\nabla}_X(\varphi(Y)) = -\nabla_X(\varphi(Y)) - rg_M(X, \nabla_Y\xi)\partial_r.$$

Since $g_M(\xi, \nabla_X Y) = -g_M(Y, \nabla_X\xi) = g_M(X, \nabla_Y\xi)$, this shows that

$$\begin{aligned}(\bar{\nabla}_X J)(Y) &= J(\bar{\nabla}_X Y) - \bar{\nabla}_X(JY) = (\nabla_X\varphi)(Y) - g_M(X, Y)\xi \\ &= g_M(X, Y)\xi - g_M(\xi, Y)X - g_M(X, Y)\xi = 0,\end{aligned}$$

so in particular $N^J(X, Y) = 0$. Combining these results we get $N^J = 0$, so $\bar{M}$ is Kähler by [KN69], Proposition 4.2, Chapter 9.

Conversely, suppose that J is a Kähler structure on $\bar{M}$ preserved by the flow of $r\partial_r$. We identify M with $\{1\} \times M \subset \bar{M}$ and denote by ξ the restriction to M of $J\partial_r$. From (2.33), $\nabla_X\xi$ equals $-JX$ when $X \in \mathrm{T}M$ and $X \perp \xi$, and

$$\nabla_\xi\xi = \bar{\nabla}_\xi\xi + \partial_r = J(\bar{\nabla}_\xi\partial_r) + \partial_r = J\xi + \partial_r = 0,$$

so ξ is a unit Killing vector field on M.

To check (8.9), we use (2.33) and (2.34):

$$\begin{aligned}-\nabla^2_{V,W}\xi &= -\nabla_V\nabla_W\xi + \nabla_{\nabla_V W}\xi \\ &= -\bar{\nabla}_V\nabla_W\xi - g_M(\nabla_W\xi, V)\partial_r + \bar{\nabla}_{\nabla_V W}\xi + g_M(\nabla_V W, \xi)\partial_r \\ &= -\bar{\nabla}_V\bar{\nabla}_W\xi - \bar{\nabla}_V(g_M(W, \xi)\partial_r) - g_M(J(\nabla_W\partial_r), V)\partial_r \\ &\quad + \bar{\nabla}_{\bar{\nabla}_V W}\xi + \bar{\nabla}_{g_M(V,W)\partial_r}\xi + g_M(\bar{\nabla}_V W, \xi)\partial_r \\ &= -J\bar{\nabla}_V W - g_M(\bar{\nabla}_V W, \xi)\partial_r - g_M(W, \bar{\nabla}_V\xi)\partial_r - g_M(W, \xi)\bar{\nabla}_V\partial_r \\ &\quad - g_M(JW, V)\partial_r + J(\bar{\nabla}_V W) + g_M(V, W)\xi + g_M(\bar{\nabla}_V W, \xi)\partial_r \\ &= -g_M(W, JV)\partial_r - g_M(W, \xi)V - g_M(JW, V)\partial_r + g_M(V, W)\xi \\ &= g_M(V, W)\xi - g_M(W, \xi)V.\end{aligned}$$

The two constructions above are clearly inverse to each other. □

8.2.2 3-Sasakian structures

Definition 8.17. A 3-*Sasakian structure* on a Riemannian manifold (M, g) is a triple $(\xi_i, \eta_i, \varphi_i)$, $i \in \{1, 2, 3\}$, of Sasakian structures such that the following relations hold:

$$[\xi_1, \xi_2] = 2\xi_3, \quad [\xi_2, \xi_3] = 2\xi_1, \quad [\xi_3, \xi_1] = 2\xi_2; \tag{8.11}$$

$$\begin{aligned}\varphi_3\varphi_2 &= -\varphi_1 + \eta_2 \otimes \eta_3, & \varphi_2\varphi_3 &= \varphi_1 + \eta_3 \otimes \eta_2; \\ \varphi_1\varphi_3 &= -\varphi_2 + \eta_3 \otimes \eta_1, & \varphi_3\varphi_1 &= \varphi_2 + \eta_1 \otimes \eta_3; \\ \varphi_2\varphi_1 &= -\varphi_3 + \eta_1 \otimes \eta_2, & \varphi_1\varphi_2 &= \varphi_3 + \eta_2 \otimes \eta_1.\end{aligned} \tag{8.12}$$

Remark 8.18. It is easy to see that (8.11) is equivalent to

$$\nabla_{\xi_i}\xi_j = \varepsilon_{ijk}\xi_k, \tag{8.13}$$

where ε_{ijk} denotes the signature of the permutation $\{ijk\}$ if i, j, k are all different, and 0 otherwise.

Definition 8.19. A *hyper-Kähler structure* on a Riemannian manifold (M, g) is a triple $\{I, J, K\}$ of Kähler structures satisfying $IJ = -JI = -K$.

Theorem 8.20. *The* 3*-Sasakian structures on* M *are in one-to-one correspondence with the hyper-Kähler structures on the Riemannian cone over* M *preserved by the flow of the radial vector field* $r\partial_r$ *of* $\bar{M}$.

Proof. Let $(\xi_i, \eta_i, \varphi_i)$, $i \in \{1, 2, 3\}$ be a 3-Sasakian structure on M. By Theorem 8.16, we obtain three Kähler structures I, J and K on $\bar{M}$. Let us check, for instance, that $IJ = -K$.

From (8.13) we have $\varphi_1(\xi_2) = \xi_3$, $\varphi_2(\xi_3) = \xi_1$, etc., so

$$\begin{aligned}
IJ(r\partial_r) &= I(\xi_2) = -\varphi_1(\xi_2) = -\xi_3 = -K(r\partial_r),\\
IJ\xi_1 &= I(-\varphi_2(\xi_1)) = I(\xi_3) = -\varphi_1(\xi_3) = \varphi_3(\xi_1) = -K(\xi_1),\\
IJ\xi_2 &= I(-r\partial_r) = -\xi_1 = \varphi_3(\xi_2) = -K(\xi_2),\\
IJ\xi_3 &= I(-\varphi_2(\xi_3)) = I(-\xi_1) = r\partial_r = -K(\xi_3),\\
IJV &= \varphi_1(\varphi_2(V)) = \varphi_3(V) + \eta_2(V)\xi_1 = \varphi_3(V) = -K(V),
\end{aligned}$$

for all vectors V orthogonal to ξ_1, ξ_2, ξ_3.

Conversely, a hyper-Kähler structure on $\bar{M}$ preserved by the flow of $r\partial_r$ induces three Sasakian structures $(\xi_i, \eta_i, \varphi_i)_{i=1,2,3}$ on M, as in Theorem 8.16. To check (8.13), we compute

$$\begin{aligned}
\nabla_{\xi_1}\xi_2 &= \nabla_{I(\partial_r)}J(\partial_r)\\
&= \bar{\nabla}_{I(\partial_r)}J(\partial_r) - g_{\bar{M}}(I(\partial_r), J(\partial_r))\partial_r\\
&= JI(\partial_r) + g_{\bar{M}}(JI(\partial_r), \partial_r)\partial_r\\
&= \xi_3 + g_{\bar{M}}(K(\partial_r), \partial_r)\partial_r\\
&= \xi_3.
\end{aligned}$$

Finally, we check, for instance, the first equation of (8.12):

$$\varphi_3(\varphi_2(\xi_1)) = -\varphi_3(JI\partial_r) = -\varphi_3(\xi_3) = 0 = (-\varphi_1 + \eta_2 \otimes \xi_3)(\xi_1), \tag{8.14}$$

etc. $\square$

8.2.3 The exceptional group G_2

Let V be the 7-dimensional Euclidean space with canonical base $\{e_1, \dots, e_7\}$ and scalar product $\langle\cdot,\cdot\rangle$ and denote by $\{\omega^1, \dots, \omega^7\}$ the canonical base of V^*. We use the notation ω^{ijk} for $\omega^i \wedge \omega^j \wedge \omega^k$. Following Bryant (see [Bry87], p. 539), we define G_2 as the subgroup of $\mathrm{GL}(V)$ preserving a suitable 3-form of $\Lambda^3 V$.

Definition 8.21. We set $\mathrm{G}_2 := \{g \in \mathrm{GL}(V) \;\; g^*(\phi) = \phi\}$, where

$$\phi := \omega^{123} + \omega^{145} + \omega^{167} + \omega^{246} + \omega^{257} + \omega^{347} + \omega^{356}. \tag{8.15}$$

We refer to [Bry87] for the proof of the following classical result.

Theorem 8.22 ([Bry87]). *The group $\mathrm{G}_2 \subset \mathrm{GL}(V)$ is a compact, connected, simple, simply connected 14-dimensional subgroup of $\mathrm{SO}(V)$. Moreover, it acts irreducibly and transitively on V and transitively on the Grassmannian of 2-planes of V.*

We define the bilinear map

$$P \colon V \times V \longrightarrow V$$

associated with ϕ via $\langle\cdot,\cdot\rangle$ as

$$\langle P(x,y), z\rangle = \phi(x,y,z), \quad x,y,z \in V;$$

P satisfies

$$P(x,y) = -\,P(y,x), \quad P(e_1,e_2) = e_3,$$

and moreover

$$\langle P(x,y), P(x,y)\rangle = \langle x,x\rangle\langle y,y\rangle - \langle x,y\rangle^2 \tag{8.16}$$

and

$$P(x, P(x,y)) = -\langle x,x\rangle y + \langle x,y\rangle x. \tag{8.17}$$

To check (8.16) and (8.17) we note that G_2 preserves ϕ by definition and the scalar product $\langle\cdot,\cdot\rangle$ by Theorem 8.22, so P is G_2-invariant. Consequently, using the last part of Theorem 8.22, we see that it is enough to prove (8.16) and (8.17) for $x = \alpha e_1$ and $y = \beta e_1 + \gamma e_2$, which is obvious.

Lemma 8.23. *Let $\psi \in \Lambda^3 V$. The following assertions are equivalent.*

(i) *For every unit vector $X \in V$, the restriction of $X \lrcorner \psi$ to $X^\perp$ defines an orthogonal complex structure on $X^\perp$ and*

$$X^* \wedge X \lrcorner \psi \wedge X \lrcorner \psi \wedge X \lrcorner \psi = 6\omega^{1234567}.$$

(ii) *There exists $g \in \mathrm{SO}(V)$ such that $\psi = g^*\phi$.*

Proof. (i) $\Longrightarrow$ (ii). We have to show that one can find an orthonormal basis of V with the property that ψ has in this basis the same form as ϕ in the basis $\{e_1, \ldots, e_7\}$.

For any unit vector X of V we denote by J_X the complex structure defined by $X \lrcorner \psi$ on $X^{\perp}$. We now choose $X = e_1$. One can find an orthonormal basis $\{v_2, \ldots, v_7\}$ of $e_1^{\perp}$ such that

$$J_{e_1} v_2 = v_3, \quad J_{e_1} v_4 = v_5, \quad J_{e_1} v_6 = v_7. \tag{8.18}$$

Let $X = v_2$. One sees immediately that $J_{v_2} e_1 = -v_3$, therefore J_{v_2} preserves the subspace spanned by v_4, v_5, v_6, v_7. By making a rotation in the plane (v_4, v_5), one can assume that $J_{v_2} v_4 \perp v_7$. Then $J_{v_2} v_4 \perp v_5$ too: if not, ψ contains the terms $e_1 \wedge v_4 \wedge v_5$ and $\alpha v_2 \wedge v_4 \wedge v_5$, $\alpha \neq 0$ (one identifies the vectors with the dual forms), therefore in any case the "coefficient" of $v_4 \wedge v_5$ in ψ would be a vector Y of norm $r = \|Y\| > 1$. By taking $X = Y/r$ in (i), we conclude that J_X maps v_4 to $(1/r)v_5$, which is impossible.

We can thus suppose $J_{v_2} v_4 = v_6$. Then

$$J_{v_2} v_5 = a v_4 + b v_6 + c v_7, \quad a^2 + b^2 + c^2 = 1,$$

so

$$-v_5 = (J_{v_2}^2) v_5 = J_{v_2}(a v_4 + b v_6 + c v_7) = a v_6 - b v_4 + c J_{v_2} v_7,$$

which implies $c^2 = 1 + a^2 + b^2$. Thus $a = b = 0$, so we obtained

$$J_{v_2} e_1 = -v_3, \quad J_{v_2} v_4 = v_6, \quad J_{v_2} v_5 = \pm v_6. \tag{8.19}$$

From (8.18), (8.19) and (i) applied to $X = e_3$ and $X = e_5$ we have

$$J_{v_4} v_3 = \pm v_7, \quad J_{v_5} v_3 = \pm v_6. \tag{8.20}$$

Using (8.18), (8.19), and (8.20) we get

$$\begin{aligned} \psi = {} & e_1 \wedge v_2 \wedge v_3 + e_1 \wedge v_4 \wedge v_5 + e_1 \wedge v_6 \wedge v_7 + v_2 \wedge v_4 \wedge v_6 \\ & \pm v_2 \wedge v_5 \wedge v_7 \pm v_3 \wedge v_4 \wedge v_7 \pm v_3 \wedge v_5 \wedge v_7 + \cdots . \end{aligned} \tag{8.21}$$

Now, each pair of subscripts between 1 and 7 is contained in exactly one of the triples of subscripts appearing in (8.21). Thus, the same argument as before shows that there is no other term in the expression of ψ given by (8.21):

$$\begin{aligned} \psi = {} & e_1 \wedge v_2 \wedge v_3 + e_1 \wedge v_4 \wedge v_5 + e_1 \wedge v_6 \wedge v_7 + v_2 \wedge v_4 \wedge v_6 \\ & \pm v_2 \wedge v_5 \wedge v_7 \pm v_3 \wedge v_4 \wedge v_7 \pm v_3 \wedge v_5 \wedge v_7 . \end{aligned}$$

Finally, to prove that the ambiguous signs are all "–", we just use the second condition of (i) with $X = v_2$, $X = v_4$ and $X = v_6$.

(ii) $\Longrightarrow$ (i) Observe first that the conditions in (i) remain true if we replace ψ by $g^*\psi$, $g \in \mathrm{SO}(V)$. It is thus enough to check (i) for ϕ. We use Theorem 8.22: (i) holds for $X = e_1$, and if X is arbitrary, we choose $g \in \mathrm{G}_2 \subset \mathrm{SO}_7$ such that $X = g(e_1)$. Then,

$$X \lrcorner \phi = (g^*)^{-1}(e_1 \lrcorner \phi) = g(e_1 \lrcorner \phi)\,, \quad X^* = (g^*)^{-1}(e_1^*) = g(e_1^*)$$

and it is clear that $X \lrcorner \phi = g(e_1 \lrcorner \phi)$ defines a complex structure on $X^\perp = g(e_1^\perp)$, and that

$$\begin{aligned} X^* \wedge X \lrcorner \psi \wedge X \lrcorner \psi \wedge X \lrcorner \psi &= g(e_1^* \wedge e_1 \lrcorner \psi \wedge e_1 \lrcorner \psi \wedge e_1 \lrcorner \psi) \\ &= g(6\omega^{1234567}) \\ &= 6\omega^{1234567}. \end{aligned}$$ □

The 3-forms satisfying the equivalent conditions of the above lemma are called *generic* 3-*forms*. For later use, we introduce the following notion

Definition 8.24. A weak G_2-structure on a 7-dimensional manifold M is given by a generic 3-form φ satisfying $\nabla\varphi = *\varphi$, with respect to the covariant derivative and the duality operator given by the metric induced by φ.

8.2.4 Nearly Kähler manifolds

Definition 8.25. A *nearly Kähler structure* on a Riemannian manifold (M, g) is an almost Hermitian structure J such that

$$(\nabla_X J)(X) = 0, \quad X \in \mathrm{T}M. \tag{8.22}$$

A nearly Kähler structure is called of *constant type* $\beta \in \mathbb{R}$ if

$$|(\nabla_X J)(Y)|^2 = \beta(|X|^2|Y|^2 - \langle X, Y\rangle^2 - \langle JX, Y\rangle^2), \quad X, Y \in \mathrm{T}M. \tag{8.23}$$

Note that (8.23) implies (8.22) and that a nearly Kähler structure of constant type β is Kähler if and only if $\beta = 0$.

From (8.22) we obtain immediately

$$(\nabla_X J)Y + (\nabla_Y J)X = 0 \quad \text{and} \quad (\nabla_X J)(JX) = 0, \tag{8.24}$$

$$(\nabla_X J)JY = (\nabla_{JX} J)Y = -\,J((\nabla_X J)Y),$$

and

$$\langle(\nabla_X J)Y, Z\rangle = -\,\langle(\nabla_X J)Z, Y\rangle. \tag{8.25}$$

The relations above show that ∇J is a skew-symmetric tensor of type (3,0)+(0,3). In particular, a 4-dimensional nearly Kähler manifold is automatically Kähler.

Proposition 8.26. *Every nearly Kähler 6-dimensional manifold M is of constant type.*

Proof. We may suppose that M is not Kähler. Let e_1, Je_1, e_2, Je_2 be mutually orthogonal unit vector fields on an open set U of M and define a positive function α and a unit vector field on U by $(\nabla_{e_1} J)e_2 = \alpha e_3$. As before, it is easy to check that $(e_1, Je_1, e_2, Je_2, e_3, Je_3)$ is an orthonormal frame on U. Moreover, we have $(\nabla_{e_2} J)e_3 \perp (e_2, Je_2, e_3, Je_3)$,

$$\langle (\nabla_{e_2} J)e_3, e_1\rangle = \langle (\nabla_{e_1} J)e_2, e_3\rangle = \alpha,$$

and

$$\langle (\nabla_{e_2} J)e_3, Je_1\rangle = \langle (\nabla_{e_1} J)e_2, Je_3\rangle = 0,$$

thus showing that

$$(\nabla_{e_2} J)e_3 = \alpha e_1.$$

Similarly,

$$(\nabla_{e_3} J)e_1 = \alpha e_2.$$

From these formulas, (8.23), with $\beta = \alpha^2$, follows directly. Note also that by polarization (8.23) is equivalent to

$$\begin{aligned}\langle (\nabla_V J)X, (\nabla_Y J)Z\rangle = \beta\{&\langle V, Y\rangle\langle X, Z\rangle - \langle V, Z\rangle\langle X, Y\rangle \\ &- \langle V, JY\rangle\langle X, JZ\rangle + \langle V, JZ\rangle\langle X, JY\rangle\}\end{aligned} \tag{8.26}$$

which concludes the proof. □

We now come to the central result of this section.

Theorem 8.27. *The nearly Kähler structures of constant type 1 on a 6-dimensional Riemannian manifold M are in one-to-one correspondence with parallel generic 3-forms on the Riemannian cone over M.*

Proof. Let φ be a parallel generic 3-form on $\bar{M}$. As before, we identify M with $M \times \{1\} \subset \bar{M}$ and using Lemma 8.23, we define an almost Hermitian structure J on M by

$$\langle X, JY\rangle = \varphi(\partial_r, X, Y).$$

Then

$$\begin{aligned}\langle (\nabla_X J)Z, Y\rangle &= \langle \nabla_X(JZ), Y\rangle - \langle J\nabla_X Z, Y\rangle \\ &= X\langle Y, JZ\rangle - \langle \nabla_X Y, JZ\rangle - \langle J\nabla_X Z, Y\rangle \\ &= X.\varphi(\partial_r, Y, Z) - \varphi(\partial_r, \nabla_X Y, Z) - \varphi(\partial_r, Y, \nabla_X Z) \\ &= X.\varphi(\partial_r, Y, Z) - \varphi(\partial_r, \bar{\nabla}_X Y, Z) - \varphi(\partial_r, Y, \bar{\nabla}_X Z) \\ &= \varphi(\bar{\nabla}_X \partial_r, Y, Z) + (\bar{\nabla}_X \varphi)(\partial_r, Y, Z) \\ &= \varphi(X, Y, Z),\end{aligned}$$

which shows that J is nearly Kähler of constant type 1.

Conversely, let J be a nearly Kähler structure of constant type 1 on M. For every $x \in M, r \in \mathbb{R}^+$, and $X, Y, Z \in \mathrm{T}_x M \subset \mathrm{T}_{(x,r)} \bar{M}$, $\partial_r \in \mathrm{T}_r \mathbb{R}^+ \subset \mathrm{T}_{(x,r)} \bar{M}$, we define

$$\varphi(\partial_r, X, Y) = -\varphi(X, \partial_r, Y) = \varphi(X, Y, \partial_r) := r^2 \langle X, JY \rangle_M,$$

$$\varphi(X, Y, Z) := r^3 \langle Y, (\nabla_X J)(Z) \rangle_M.$$

The fact that φ is skew-symmetric follows from (8.24)–(8.25). To show that φ is generic, let $\{e_1, Je_1, e_2, Je_2, e_3, Je_3\}$ be a local orthonormal frame on M such that $e_3 = (\nabla_{e_1} J)e_2$. By taking $v_1 = \partial_r$, $v_2 = e_1/r$, $v_3 = Je_1/r$, $v_4 = e_2/r$, $v_5 = Je_2/r$, $v_6 = e_3/r$, and $v_7 = Je_3/r$, we obtain a local orthonormal frame of $\bar{M}$ around (x, r). Relations (8.24)–(8.25) then show that the expression of φ in this basis is just (8.15).

Finally, we check that φ is parallel (we continue to use the notation $X, Y, \ldots$ for vector fields on M, identified with their canonical extension to $\bar{M}$):

(1) we have

$$\begin{aligned}
(\bar{\nabla}_{\partial_r} \varphi)(X, Y, Z) &= \partial_r(\varphi(X, Y, Z)) - \varphi(\bar{\nabla}_{\partial_r} X, Y, Z) \\
&\quad - \varphi(X, \bar{\nabla}_{\partial_r} Y, Z) - \varphi(X, Y, \bar{\nabla}_{\partial_r} Z) \\
&= 3r^2 \langle Y, (\nabla_X J)Z \rangle - 3r^2 \langle Y, (\nabla_X J)Z \rangle \\
&= 0;
\end{aligned}$$

(2) we have

$$\begin{aligned}
(\bar{\nabla}_{\partial_r} \varphi)(\partial_r, X, Y) &= \partial_r(\varphi(\partial_r, X, Y)) - \varphi(\partial_r, \bar{\nabla}_{\partial_r} X, Y) \\
&\quad - \varphi(\partial_r, X, \bar{\nabla}_{\partial_r} Y) \\
&= 2r \langle X, JY \rangle_M - 2r \langle X, JY \rangle_M \\
&= 0;
\end{aligned}$$

(3) we have

$$\begin{aligned}
(\bar{\nabla}_X \varphi)(\partial_r, Y, Z) &= X(\varphi(\partial_r, Y, Z)) - \varphi(\bar{\nabla}_X(\partial_r), Y, Z) \\
&\quad - \varphi(\partial_r, \bar{\nabla}_X Y, Z) - \varphi(\partial_r, Y, \bar{\nabla}_X Z) \\
&= r^2 X \langle Y, JZ \rangle_M - r^2 \langle Y, (\nabla_X J)Z \rangle_M \\
&\quad - r^2 \langle \nabla_X Y, JZ \rangle_M - r^2 \langle Y, J \nabla_X Z \rangle_M \\
&= 0.
\end{aligned}$$

□

To show that $(\bar{\nabla}_X \varphi)(Y, Z, W) = 0$, we use (8.26) and the following result of Gray (see [Gra76]).

Proposition 8.28. *For every vector fields X, Y, Z, W on a nearly Kähler manifold M,*

$$2\langle(\nabla^2_{X,Y} J)W, Z\rangle = \underset{Y,W,Z}{\mathfrak{S}} \langle(\nabla_X J)Y, J(\nabla_W J)Z\rangle.$$

We then conclude (using $\langle\cdot,\cdot\rangle$ instead of $\langle\cdot,\cdot\rangle_M$ for simplicity) that

$$\begin{aligned}\langle(\nabla^2_{X,Y} J)W, Z\rangle &= -\frac{1}{2}\underset{Y,W,Z}{\mathfrak{S}}\langle(\nabla_X J)Y, (\nabla_W J)JZ\rangle\\ &= -\frac{1}{2}\underset{Y,W,Z}{\mathfrak{S}}\{\langle X, W\rangle\langle Y, JZ\rangle - \langle X, JZ\rangle\langle Y, W\rangle\\ &\qquad + \langle X, JW\rangle\langle Y, Z\rangle - \langle X, Z\rangle\langle Y, JW\rangle\},\end{aligned}$$

and after some calculations

$$\langle(\nabla^2_{X,Y} J)W, Z\rangle = \langle X, Z\rangle\langle Y, JW\rangle + \langle X, Y\rangle\langle W, JZ\rangle + \langle X, W\rangle\langle Z, JY\rangle.$$

Consequently

$$\begin{aligned}0 &= \langle(\nabla^2_{X,Y} J)W, Z\rangle - \{\langle X, Z\rangle\langle Y, JW\rangle + \langle X, Y\rangle\langle W, JZ\rangle + \langle X, W\rangle\langle Z, JY\rangle\}\\ &= \langle\nabla_X((\nabla_Y J)W) - (\nabla_{\nabla_X Y} J)W - (\nabla_Y J)(\nabla_X W), Z\rangle\\ &\quad - \{\langle X, Z\rangle\langle Y, JW\rangle + \langle X, Y\rangle\langle W, JZ\rangle + \langle X, W\rangle\langle Z, JY\rangle\},\end{aligned}$$

and finally

$$\begin{aligned}&(\overline{\nabla}_X\varphi)(Y, Z, W)\\ &\quad = X(\varphi(Y, Z, W)) - \varphi(\nabla_X Y - r\langle X, Y\rangle\partial_r, Z, W)\\ &\qquad - \varphi(Y, \nabla_X Z - r\langle X, Z\rangle\partial_r, W) - \varphi(Y, Z, \nabla_X W - r\langle X, W\rangle\partial_r)\\ &\quad = r^3\{X\langle Z, (\nabla_Y J)W\rangle - (\langle Z, (\nabla_{\nabla_X Y} J)W\rangle - \langle X, Y\rangle\langle Z, JW\rangle)\\ &\quad - (\langle\nabla_X Z, (\nabla_Y J)W\rangle + \langle X, Z\rangle\langle Y, JW\rangle)\\ &\quad - (\langle Z, (\nabla_Y J)(\nabla_X W)\rangle - \langle X, W\rangle\langle Y, JZ\rangle)\}\\ &\quad = r^3\{\langle\nabla_X((\nabla_Y J)W) - (\nabla_{\nabla_X Y} J)W - (\nabla_Y J)(\nabla_X W), Z\rangle\\ &\qquad\qquad - \{\langle X, Z\rangle\langle Y, JW\rangle + \langle X, Y\rangle\langle W, JZ\rangle + \langle X, W\rangle\langle Z, JY\rangle\}\}\\ &\quad = 0.\end{aligned}$$

8.2.5 The group Spin_7

In this section we introduce the group Spin_7 as the stabilizer of a suitable 4-form in $\mathbb{R}^8$, and describe its relationship with the exceptional group G_2.

We view the Euclidean space $\mathbb{R}^8$ as the direct sum

$$V_+ := \mathbb{R}e_0 \oplus V,$$

where V is the 7-dimensional Euclidean space appearing in the construction of G_2. Let $\{e_0, \dots, e_7\}$ be the standard basis of V_+, let $\{\omega^0, \dots, \omega^7\}$ be the dual basis of V_+^*, and denote by $\langle \cdot, \cdot \rangle$ the standard Euclidean scalar product. We define the 4-form $\Phi \in \Lambda^4(V_+^*)$ by

$$\Phi = \omega^0 \wedge \phi + *\phi,$$

where ϕ is the standard generic 3-form on V defined by (8.15). One easily computes

$$\Phi \wedge \Phi = 14\omega^{01234567},$$

and

$$\Phi = \frac{1}{2}\alpha \wedge \alpha + \mathrm{Re}(\beta),$$

where

$$\alpha = \omega^{01} + \omega^{23} + \omega^{45} + \omega^{67},$$
$$\beta = (\omega^0 + i\omega^1) \wedge (\omega^2 + i\omega^3) \wedge (\omega^4 + i\omega^5) \wedge (\omega^6 + i\omega^7).$$

We define

$$G = \{g \in \mathrm{GL}(V_+);\ g^*(\Phi) = \Phi\}$$

and quote the following result.

Theorem 8.29 ([Bry87], p. 545). *The group G is a compact, connected, simply connected 21-dimensional Lie group preserving the Euclidean product $\langle \cdot, \cdot \rangle$. It acts irreducibly on V_+ and transitively on each Grassmannian of k-planes of V_+ $(k \le 8)$. Its center is $\mathbb{Z}/2\mathbb{Z} = \{\pm \mathrm{Id}_{V_+}\}$ and $G/(\mathbb{Z}/2\mathbb{Z}) \simeq \mathrm{SO}_7$, so G is isomorphic to* Spin_7.

For the proof, we refer to [Bry87], but we point out that the correct definition of the subgroup H introduced there is

$$H = \{h \in \mathrm{SO}_7;\ h^*(\alpha) = \alpha, h^*(\beta) = \beta\}.$$

Lemma 8.30. *Let $\Psi \in \Lambda^4(V_+^*)$. The following assertions are equivalent.*

(i) *For every unit vector $X \in V_+$, the restriction of $X \lrcorner \Psi$ to $X^\perp$ is a generic 3-form φ on $X^\perp$, and the restriction of Ψ itself to $X^\perp$ equals $*_7\varphi$, where $*_7$ denotes the Hodge duality operator on $X^\perp$ with respect to the induced Euclidean product.*

(ii) *The assertion* (i) *holds for $X = e_0$.*

(iii) *The form Ψ can be written as $\Psi = \omega^0 \wedge \varphi + *_7\varphi$, where φ is a generic 3-form on $e_0^\perp$.*

Proof. The only non-trivial point is to prove (iii) $\Longrightarrow$ (i). Let $\Psi = \omega^0 \wedge \varphi + *_7\varphi$. The fact that (i) holds for $X = e_0$ is tautological. For $X \in V_+$ arbitrary, we use Theorem 8.29 to find $g \in \mathrm{Spin}_7 \subset \mathrm{SO}_8$ with $X = g(e_0)$. Then, on the one hand, we obtain that

$$X \lrcorner \Psi = (g^*)^{-1}(e_0 \lrcorner \Psi) = g(\varphi)$$

is a generic 3-form on $X^\perp = g(e_0^\perp)$, and on the other hand

$$\Psi|_{X^\perp} = g(\Psi)|_{g(e_0^\perp)} = g(\Psi)|_{e_0^\perp} = g(*_7\varphi) = *_7 g(\varphi). \qquad \square$$

Definition 8.31. A 4-form on $\mathbb{R}^8$ satisfying the equivalent conditions of Lemma 8.30 will be called a *generic 4-form*.

Theorem 8.32. *There is a one-to-one correspondence between the weak G_2-structures on a 7-dimensional manifold M and the parallel generic 4-forms on the Riemannian cone $\bar{M}$ of M.*

Proof. Let Φ be a parallel generic 4-form on $\bar{M}$. We define a generic 3-form φ on M by

$$\varphi(X, Y, Z) = \Phi(\partial_r, X, Y, Z).$$

It is clear that $\Phi = dr \wedge \varphi + *\varphi$, so

$$\begin{aligned}
(\nabla_W \varphi)(X, Y, Z) &= W(\Phi(\partial_r, X, Y, Z)) - \Phi(\partial_r, \nabla_W X, Y, Z) \\
&\qquad - \Phi(\partial_r, X, \nabla_W Y, Z) - \Phi(\partial_r, X, Y, \nabla_W Z) \\
&= W\Phi(\partial_r, X, Y, Z) - \Phi(\partial_r, \bar{\nabla}_W X, Y, Z) \\
&\qquad - \Phi(\partial_r, X, \bar{\nabla}_W Y, Z) - \Phi(\partial_r, X, Y, \bar{\nabla}_W Z) \\
&= (\bar{\nabla}_W \Phi)(\partial_r, X, Y, Z) + \Phi(\bar{\nabla}_W \partial_r, X, Y, Z) \\
&= \Phi(W, X, Y, Z) \\
&= (*\varphi)(W, X, Y, Z),
\end{aligned}$$

thus proving that φ defines a weak G_2-structure on M.

Conversely, let φ be a generic 3-form on M satisfying $\nabla\varphi = *\varphi$. We define a 4-form Φ on $\bar{M}$ by

$$\Phi(\partial_r, X, Y, Z) = r^3\varphi(X, Y, Z), \tag{8.27}$$

$$\Phi(W, X, Y, Z) = r^4(\nabla_W\varphi)(X, Y, Z) = r^4(*\varphi)(W, X, Y, Z). \tag{8.28}$$

To avoid any possible confusion, we denote by "$*$" the duality operator on M and by "$*_{\bar{M}}$" the duality operator on the sub-bundle $E \subset \mathrm{T}^*\bar{M}$ of forms on $\bar{M}$ which do not contain dr. Consider the 3-form $\tilde{\varphi}$ on $\bar{M}$ given by

$$\tilde{\varphi}_{(x,r)}(\partial_r, X, Y) = 0,$$

$$\tilde{\varphi}_{(x,r)}(X, Y, Z) = \varphi_x(rX, rY, rZ).$$

Since the transformation $\mathrm{T}_xM \to \mathrm{T}_{(x,r)}\bar{M}$ given by $X_x \mapsto rX_{(x,r)}$ is an isometry, we obtain

$$(*_{\bar{M}}\tilde{\varphi})(W, X, Y, Z) = (*\varphi)(rW, rX, rY, rZ) = r^4(*\varphi)(W, X, Y, Z),$$

so (8.27) and (8.28) imply that

$$\Phi = dr \wedge \tilde{\varphi} + (*_{\bar{M}}\tilde{\varphi}),$$

i.e., Φ is a generic 4-form on $\bar{M}$.

It remains to check that Φ is parallel:

$$(\bar{\nabla}_{\partial_r}\Phi)(\partial_r, X, Y, Z) = \partial_r(r^3\varphi(X, Y, Z)) - 3r^2\varphi(X, Y, Z) = 0,$$
$$(\bar{\nabla}_{\partial_r}\Phi)(X, Y, Z, T) = \partial_r(r^4(*\varphi)(X, Y, Z, T)) - 4r^3(*\varphi)(X, Y, Z, T) = 0,$$

and

$$\begin{aligned}(\bar{\nabla}_W\Phi)(\partial_r, X, Y, Z) &= W(r^3\varphi(X, Y, Z)) - \Phi\Big(\frac{W}{r}, X, Y, Z\Big) - r^3\varphi(\nabla_W X, Y, Z) \\ &\quad - r^3\varphi(X, \nabla_W Y, Z) - r^3\varphi(X, Y, \nabla_W Z) \\ &= r^3(\nabla_W\varphi)(X, Y, Z) - \Phi\Big(\frac{W}{r}, X, Y, Z\Big) \\ &= 0.\end{aligned}$$

In order to show that $(\bar{\nabla}_W\Phi)(X, Y, Z, T) = 0$ for all vectors W, X, Y, Z, T of $\bar{M}$ orthogonal to ∂_r, we observe that

- the volume form of M is parallel, so $\nabla_W(*\varphi) = *(\nabla_W \varphi)$ and
- if W^* denotes the dual 1-form of W with respect to the metric $\langle\cdot,\cdot\rangle$, then
$$*(W \lrcorner (*\varphi)) = \varphi \wedge W^*$$
(by linearity, it suffices to check this for $\varphi = dx^1 \wedge dx^2 \wedge dx^3$ and $W = \partial/\partial x_3$ or $W = \partial/\partial x_4$).

Consequently,
$$\nabla_W(*\varphi) = *(\nabla_W \varphi(X,Y,Z,T)) = *(W \lrcorner (*\varphi)) = \varphi \wedge W^*,$$
which yields
$$\begin{aligned}
(\bar{\nabla}_W \Phi)&(X,Y,Z,T)\\
&= r^4(\nabla_W(*\varphi)) + r^3\varphi(Y,Z,T)\cdot r\langle W,X\rangle - r^3\varphi(X,Z,T)\cdot r\langle W,Y\rangle\\
&\quad + r^3\varphi(X,Y,T)\cdot r\langle W,Z\rangle - r^3\varphi(X,Y,Z)\cdot r\langle W,T\rangle\\
&= 0. \qquad \square
\end{aligned}$$

The results of this section are due both to Bär [Bär93] and Bryant [Bry87], as well as Gray [Gra76].

8.3 Classification of manifolds admitting real Killing spinors

Real Killing spinors appeared naturally in Chapter 5, where it was shown that they characterize the limiting case of Friedrich's inequality between the first eigenvalue of the Dirac operator and the scalar curvature of a compact spin manifold. In this section we give a complete geometrical description of all compact simply connected spin and Spin^c manifolds carrying real Killing spinors. By Lemma 2.12, it suffices to treat the Spin^c case.

All over this section, $(M^n, g, \mathcal{S}, L, A)$ will be a simply connected Spin^c manifold carrying a real Killing spinor Ψ, i.e., satisfying the equation
$$\nabla^A_X \Psi = \lambda X \cdot \Psi, \qquad X \in \mathrm{T}M,$$
for some fixed real number $\lambda \neq 0$. By rescaling the metric if necessary, we can assume, without loss of generality, that $\lambda = \pm\frac{1}{2}$. Moreover, for n even we can assume that $\lambda = \frac{1}{2}$, by taking the conjugate of Ψ if necessary.

Consider the Riemannian cone $(\bar{M}, \bar{g})$ over M and, as before, denote by ∂_r the radial vector field. Using the formulas for warped products (2.32)–(2.34), one easily computes the curvature tensor $\bar{R}$ of $\bar{M}$ (cf. [O'N66], p. 210):
$$\bar{R}(X,\partial_r)\partial_r = \bar{R}(X,Y)\partial_r = \bar{R}(X,\partial_r)Y = 0, \tag{8.29}$$
$$\bar{R}(X,Y)Z = R(X,Y)Z + g(X,Z)Y - g(Y,Z)X. \tag{8.30}$$

In particular, we have the following result.

Lemma 8.33. *The Riemannian cone $\bar{M}$ is flat if and only if M is a space form of constant sectional curvature equal to* 1.

We now quote an important result concerning the holonomy of Riemannian cone metrics, originally due to S. Gallot (see [Gal79], Proposition 3.1).

Lemma 8.34. *If M is complete, then $\bar{M}$ is irreducible or flat.*

Note that the completeness hypothesis is essential here. Indeed, it is straightforward to see that the Riemannian product of two Riemannian cones is again a Riemannian cone over a non-complete manifold (cf. [MO08], for instance). This means that there are lots of examples of non-flat Riemannian cones with reducible holonomy over non-complete manifolds.

The use of the Riemannian cone over M is the key idea for the classification of manifolds with real Killing spinors. From Lemma 2.40 and Theorem 2.41, we obtain at once the following corollary.

Corollary 8.35. *The natural* Spin^c *structure of the Riemannian cone over a* Spin^c *manifold M^n carrying real Killing spinors with Killing constant $\pm\frac{1}{2}$ admits parallel spinors. Moreover, for n odd, a Killing spinor with Killing constant $\frac{1}{2}$ generates a positive parallel half-spinor and a Killing spinor with Killing constant $-\frac{1}{2}$ generates a negative parallel half-spinor. The converse is also true.*

As in the case of parallel spinors, the gauge group $\mathcal{C}^\infty(M)$ acts freely on the pairs (A, Ψ) of connections A on L with a given curvature form and Killing spinors Ψ.

Using the corollary above together with Lemma 8.34 and Theorem 8.12, we obtain that the Riemannian cone $\bar{M}$ is either an irreducible spin manifold with parallel spinors, or a Kähler manifold with the canonical Spin^c structure, or a flat manifold. In the last case, the Spin^c structure of $\bar{M}$ is also flat, because of (8.3).

Thus, modulo the action of the gauge group we have the following result.

Theorem 8.36. *A simply connected complete* Spin^c *manifold M carries a real Killing spinor with Killing constant $\frac{1}{2}$ if and only if the associated Riemannian cone satisfies one of the two conditions below:*

(1) *$\bar{M}$ is a simply connected spin manifold carrying parallel spinors;*

(2) *$\bar{M}$ is a Kähler manifold.*

We now study these two situations in more detail. The first case is directly settled by Lemma 8.34, Lemma 8.33, and the results of the previous sections (Theorem 8.1, Theorem 8.16, Theorem 8.20, Theorem 8.27, and Theorem 8.32).

Theorem 8.37 ([Bär93]). *A simply connected complete spin manifold M^n carries a real Killing spinor if and only if it is homothetic to one of the following manifolds:*

(1) *the sphere $\mathbb{S}^n$,*

(2) *an Einstein–Sasakian manifold,*

(3) *a 3-Sasakian manifold,*

(4) *a nearly Kähler manifold of constant type 1 $(n = 6)$,*

(5) *a weak G_2-manifold $(n = 7)$.*

The second case of Theorem 8.36 is equivalent to M being Sasakian (by Theorem 8.16). The following result provides an analogue in the Sasakian setting for the canonical Spin^c structure on Kähler manifolds.

Proposition 8.38. *Every Sasakian manifold (M^{2k+1}, g, ξ) has a canonical Spin^c structure. If M is Einstein, then the auxiliary bundle of the canonical Spin^c structure is flat, so if in addition M is simply connected, then it is spin.*

Proof. The first statement follows directly from the fact that the Riemannian cone over M is Kähler, and thus carries a canonical Spin^c structure, whose restriction to M is the desired canonical Spin^c structure.

If M is Einstein, then its Einstein constant is $2k$ (see [BFGK91], p. 78), so $\bar{M}$ is Ricci flat by (8.29) and (8.30). The auxiliary bundle of the canonical Spin^c structure of $\bar{M}$, which is just the canonical bundle $K = \Lambda^{k+1,0}\bar{M}$, is thus flat, and the same is true for its restriction to M. The last statement follows from Lemma 2.12. □

One can construct the Spin^c structure more directly as follows. The frame bundle of every Sasakian manifold restricts to U_k, by considering only adapted frames, i.e., orthonormal frames of the form $\{\xi, e_1, \varphi(e_1), \dots, e_k, \varphi(e_k)\}$. Then extend this bundle of adapted frames to a Spin^c_{2k}-principal bundle using the canonical inclusion $\mathrm{U}_k \to \mathrm{Spin}^c_{2k}$ (cf. [LM89], p. 392).

Just as in the case of almost Hermitian manifolds, one can define an anti-canonical Spin^c structure for Sasakian manifolds, which has the same properties as the canonical one.

We recall that the parallel spinor of the canonical Spin^c structure of a Kähler manifold M^{2k+2} lies in $\Sigma_0 M$, so is always a positive half-spinor, and the parallel spinor of the anti-canonical Spin^c structure lies in $\Sigma_{k+1}M$, so it is positive (negative) for k odd (resp. even). Collecting these remarks together with Corollary 8.35 we obtain the following result.

Corollary 8.39. *The non-Einstein Sasakian manifolds M^{2k+1} endowed with their canonical or anti-canonical* Spin^c *structure are the only simply connected* Spin^c *manifolds admitting real Killing spinors other than the spin manifolds. For the canonical* Spin^c *structure, the dimension of the space of Killing spinors, for the Killing constant $\frac{1}{2}$, is always equal to* 1*, and there is no Killing spinor for the constant $-\frac{1}{2}$. For the anti-canonical* Spin^c *structure, the dimensions of the spaces of Killing spinors, for the Killing constants $\frac{1}{2}$ and $-\frac{1}{2}$, are* 1 *and* 0 (0 *and* 1) *for k odd* (*resp. even*).

The results in this section are due to Ch. Bär in the spin case [Bär93] and to A. Moroianu in the Spin^c case [Mor97b].

8.4 Detecting model spaces by Killing spinors

In this section we characterize the round sphere as being the only complete connected spin manifold of dimension $3, 4, 7$ or 8 admitting non-trivial Killing spinors associated with the positive and the negative first eigenvalues of the Dirac operator. Note that in dimension 4 and 8 the spectrum of the Dirac operator is symmetric around zero. The result is due to Th. Friedrich [Fri81] in dimension 4 and to Nieuwenhuizen and Warner [vNW84] in dimension 7. Here we present a uniform simple proof (see [Hij86a]) based on the use of real spinor fields and the Obata–Lichnerowicz theorem for the scalar Laplacian.

For this, recall the following definition.

Definition 8.40. A real Killing spinor Ψ is a spinor field satisfying, for any tangent vector field X, the differential equation

$$\nabla_X \Psi + \frac{\lambda_0}{n} X \cdot \Psi = 0, \tag{8.31}$$

where

$$\lambda_0 = \pm\sqrt{\frac{n}{4(n-1)}\mathrm{Scal}_0},$$

is the smallest eigenvalue of the Dirac operator. The space of such spinors is denoted by $\mathcal{KS}_\pm$.

Theorem 8.41. *Any complete connected Riemannian spin manifold of dimension* $3, 4, 7$ *or* 8 *for which* $\dim \mathcal{KS}_+ > 0$ *and* $\dim \mathcal{KS}_- > 0$ *is isometric to the round sphere.*

Proof. To show this it is sufficient to prove that the conditions of the Obata–Lichnerowicz theorem are satisfied, i.e., there exist a positive constant c and a non-trivial real function f on M^n such that

$$\mathrm{Ric} \geq c\,\mathrm{Id} \quad \text{and} \quad \Delta f = \frac{n}{n-1} c f,$$

where Δ is the scalar Laplacian.

Consider $\Psi \in \mathcal{KS}_+$ and $\overline{\Psi} \in \mathcal{KS}_-$ and let $f := \mathrm{Re}\langle \Psi, \overline{\Psi} \rangle$. With the help of (8.31) it is straightforward to see that

$$\Delta f = \frac{\mathrm{Scal}}{n-1} f.$$

Hence, since the manifold is Einstein, it is sufficient to take $c = \frac{\mathrm{Scal}}{n}$ and to prove that $f \not\equiv 0$.

For each integer n, denote

$$d_n := \dim_{\mathbb{R}} \Sigma,$$

where Σ is an irreducible $\mathbb{R}$-module of Cl_n and

$$d_n^{\mathbb{C}} := \dim_{\mathbb{C}} \Sigma^{\mathbb{C}},$$

where $\Sigma^{\mathbb{C}}$ is an irreducible $\mathbb{C}$-module of Cl_n (hence for $\mathbb{C}\mathrm{l}_n := \mathrm{Cl}_n \otimes_{\mathbb{R}} \mathbb{C}$). From Chapter 1, see [LM89], we have Table 5.

Table 5

n	1	2	3	4
Cl_n	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{H}(2)$
d_n	2	4	4	8
$\mathbb{C}\mathrm{l}_n$	$\mathbb{C} \oplus \mathbb{C}$	$\mathbb{C}(2)$	$\mathbb{C}(2) \oplus \mathbb{C}(2)$	$\mathbb{C}(4)$
$d_n^{\mathbb{C}}$	1	2	2	4

n	5	6	7	8
Cl_n	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{R}(16)$
d_n	8	8	8	16
$\mathbb{C}\mathrm{l}_n$	$\mathbb{C}(4) \oplus \mathbb{C}(4)$	$\mathbb{C}(8)$	$\mathbb{C}(8) \oplus \mathbb{C}(8)$	$\mathbb{C}(16)$
$d_n^{\mathbb{C}}$	4	8	8	16

From this table we can read that for $n = 1, 2, 3, 4, 5$, the bundle of real spinors is the same as the bundle of complex spinors (considered as real vector bundles) and for $n = 6, 7, 8$, the bundle of complex spinors is the complexification of the bundle of real spinors. In both cases, the real and the complex Dirac operators have the same spectrum. We denote by ΣM the real spinor bundle.

We first consider the case $n = 3$ or 7. For $\Psi \in \mathcal{KS}_+$ define the linear injective map

$$\begin{aligned} \mathbf{\Psi}\colon \mathrm{T}M &\longrightarrow \Sigma M, \\ X &\longmapsto X \cdot \Psi, \end{aligned}$$

which is an isomorphism onto its image $\mathbf{\Psi}(\mathrm{T}M)$, a subspace in ΣM of codimension 1, for $n = 3$ or 7.

Since Ψ is orthogonal to $\mathbf{\Psi}(\mathrm{T}M)$, the function $f := \operatorname{Re}\langle \Psi, \overline{\Psi}\rangle$ has no zero on M.

For $n = 4$ or 8, we consider the decomposition of the real spinor bundle ΣM into positive and negative spinors, i.e., $\Sigma M = \Sigma^+ M \oplus \Sigma^- M$ and $\Psi = \Psi^+ + \Psi^-$. We note that the linear injective map

$$\begin{aligned} \mathbf{\Psi}^+\colon \mathrm{T}M &\longrightarrow \Sigma^- M, \\ X &\longmapsto X \cdot \Psi^+, \end{aligned}$$

is an isomorphism since in these dimensions, $\dim_{\mathbb{R}} \mathrm{T}M = \dim_{\mathbb{R}} \Sigma M$. Note that in this case, if $\Psi \in \mathcal{KS}_+$, then $\overline{\Psi} := \Psi^+ - \Psi^- \in \mathcal{KS}_-$. We then conclude by noting that, for any tangent vector field X,

$$X(f) = -4\frac{\lambda_0}{n}(X \cdot \Psi^+, \Psi^-) \not\equiv 0. \qquad \square$$

8.5 Generalized Killing spinors

If $\mathcal{Z}$ is a Riemannian spin manifold, any oriented hypersurface $M \subset \mathcal{Z}$ inherits a spin structure, and by Lemma 2.38, the restriction to M of the complex spinor bundle $\Sigma\mathcal{Z}$ if n is even (resp. $\Sigma^+\mathcal{Z}$ if n is odd) is canonically isomorphic to the complex spinor bundle ΣM. Moreover, Theorem 2.39 shows that if W denotes the Weingarten tensor of M, the spin covariant derivatives $\nabla^{\mathcal{Z}}$ on $\Sigma\mathcal{Z}$ and ∇ on ΣM are related by

$$(\nabla^{\mathcal{Z}}_X \Psi)|_M = \nabla_X(\Psi|_M) - \frac{1}{2}W(X)\cdot(\Psi|_M), \quad X \in \Gamma(\mathrm{T}M),$$

for all spinors (resp. half-spinors for n odd) Ψ on $\mathcal{Z}$. This motivates the following definition.

Definition 8.42. A *generalized Killing spinor* on a spin manifold (M, g) is a non-zero spinor $\psi \in \Gamma(\Sigma M)$ satisfying for all vector fields X on M the equation

$$\nabla_X \psi = A(X) \cdot \psi \, , \tag{8.32}$$

where $A \in \Gamma(\mathrm{End}^+(\mathrm{T}M))$ is some field of symmetric endomorphisms called the *stress-energy* (or *energy-momentum*) *tensor* of ψ.

We thus see that if Ψ is a parallel spinor on $\mathcal{Z}$, its restriction ψ to any hypersurface M is a generalized Killing spinor on M. It is natural to ask whether the converse holds.

Question. If ψ is a generalized Killing spinor on M^n, does there exist an isometric embedding of M into a spin manifold $(\mathcal{Z}^{n+1}, g^{\mathcal{Z}})$ carrying a parallel spinor Ψ whose restriction to M is ψ?

This question is the Cauchy problem for metrics with parallel spinors asked in [BGM05].

The answer is known to be positive in several special cases: if the stress-energy tensor A of ψ is the identity [Bär93], if A is parallel [Mor03], and if A is a Codazzi tensor [BGM05]. Even earlier, Friedrich [Fri98] had worked out the case $n = 2$, which is also covered by [BGM05], Theorem 8.1, since on surfaces the stress-energy tensor of a generalized Killing spinor is automatically a Codazzi tensor. Some related embedding results were also obtained by Kim [Kim02], Lawn and Roth [LR10], and Morel [Mor05]. The common feature of these cases is that one can actually construct in an explicit way the "ambient" metric $g^{\mathcal{Z}}$ as a generalized cylinder metric on the product $(-\varepsilon, \varepsilon) \times M$ (see also [BG92]).

Our aim in this section is to show that the same holds more generally, under the sole additional assumption that (M, g) and A are analytic.

Theorem 8.43 ([AMM13]). *Let ψ be a spinor field on an analytic spin manifold (M^n, g), and A an analytic field of symmetric endomorphisms of* $\mathrm{T}M$*. Assume that ψ is a generalized Killing spinor with respect to A, i.e., it satisfies* (8.32)*. Then there exists a unique metric $g^{\mathcal{Z}}$ of the form $g^{\mathcal{Z}} = dt^2 + g_t$ with $g_0 = g$ on a sufficiently small neighborhood $\mathcal{Z}$ of $\{0\} \times M$ inside $\mathbb{R} \times M$, such that $(\mathcal{Z}, g^{\mathcal{Z}})$, endowed with the spin structure induced from M, carries a parallel spinor Ψ whose restriction to M is ψ.*

In particular, the solution $g^{\mathcal{Z}}$ must be Ricci-flat. Einstein manifolds are analytic, but of course hypersurfaces can lose this structure, so our hypothesis is restrictive. Einstein metrics with smooth initial data can be constructed for small time as metrics of constant sectional curvature when the second fundamental form is a Codazzi tensor; see [BGM05], Theorem 8.1. In particular, in dimensions $n = 1$ and $n = 2$

Theorem 8.43 remains valid in the smooth category since the tensor A associated to a generalized Killing spinor is automatically a Codazzi tensor.

Note that several particular instances of Theorem 8.43 have been proved in recent years, based on the characterization of generalized Killing spinors in terms of exterior forms in low dimensions. Indeed, in dimensions 5, 6 and 7, generalized Killing spinors are equivalent to so-called *hypo*, *half-flat* and *co-calibrated* G_2 structures, respectively. Hitchin proved that the cases $n = 6$ and $n = 7$ can be solved up to the local existence of a certain gradient flow [Hit01]. Later on, Conti and Salamon [CS06] and [CS07] treated the cases $n = 5$, $n = 6$, and $n = 7$ in the analytical setting, cf. also [Con11] for further developments.

8.6 The Cauchy problem for Einstein metrics

Let $(\mathcal{Z}, g^{\mathcal{Z}})$ be an oriented Riemannian manifold of dimension $n+1$, and M an oriented hypersurface with induced Riemannian metric

$$g := g^{\mathcal{Z}}|_M.$$

We start by fixing some notations. Denote by $\nabla^{\mathcal{Z}}$ and ∇ the Levi-Civita covariant derivatives on $(\mathcal{Z}, g^{\mathcal{Z}})$ and (M, g), by ν the unit normal vector field along M compatible with the orientations, and by $W \in \Gamma(\mathrm{End}^+(\mathrm{T}M))$ the Weingarten tensor, defined by

$$\nabla^{\mathcal{Z}}_X \nu = -W(X), \quad X \in \mathrm{T}M. \tag{8.33}$$

Using the normal geodesics issued from M, the metric on $\mathcal{Z}$ can be expressed in a neighborhood $\mathcal{Z}_0$ of M as $g^{\mathcal{Z}} = dt^2 + g_t$, where t is the distance function to M and g_t is a family of Riemannian metrics on M with $g_0 = g$ (cf. [BGM05]). The vector field ν extends to $\mathcal{Z}_0$ as $\nu = \partial/\partial t$ and (8.33) defines a symmetric endomorphism on $\mathcal{Z}_0$ which can be viewed as a family W_t of endomorphisms of M, symmetric with respect to g_t, and satisfying (cf. [BGM05], equation (4.1))

$$g_t(W_t(X), Y) = -\tfrac{1}{2}\dot{g}_t(X, Y), \quad X, Y \in \mathrm{T}M.$$

By [BGM05], equations (4.5)–(4.8), the Ricci tensor and the scalar curvature of $\mathcal{Z}$ satisfy for every vectors $X, Y \in \mathrm{T}M$ the relations

$$\mathrm{Ric}^{\mathcal{Z}}(\nu, \nu) = \mathrm{tr}(W_t^2) - \frac{1}{2}\mathrm{tr}_{g_t}(\ddot{g}_t), \tag{8.34}$$

$$\mathrm{Ric}^{\mathcal{Z}}(\nu, X) = d\,\mathrm{tr}(W_t)(X) + \delta^{g_t}(W)(X), \tag{8.35}$$

$$\begin{aligned} \mathrm{Ric}^{\mathcal{Z}}(X, Y) = \mathrm{Ric}^{g_t}(X, Y) &+ 2g_t(W_t X, W_t Y) \\ &+ \frac{1}{2}\mathrm{tr}(W_t)\dot{g}_t(X, Y) - \frac{1}{2}\ddot{g}_t(X, Y), \end{aligned} \tag{8.36}$$

and

$$\mathrm{Scal}^{\mathcal{Z}} = \mathrm{Scal}^{g_t} + 3\,\mathrm{tr}(W_t^2) - \mathrm{tr}^2(W_t) - \mathrm{tr}_{g_t}(\ddot{g}_t), \tag{8.37}$$

where in (8.35) the divergence operator

$$\delta^g : \mathrm{End}(\mathrm{T}M) \longrightarrow \mathrm{T}^*M$$

is defined in a local g-orthonormal basis $\{e_i\}$ of $\mathrm{T}M$ by

$$\delta^g(A)(X) = -\sum_{i=1}^{n} g((\nabla_{e_i} A)(e_i), X).$$

Using (8.34) and (8.37) we get

$$-2\mathrm{Ric}^{\mathcal{Z}}(\nu,\nu) + \mathrm{Scal}^{\mathcal{Z}} = \mathrm{Scal}^{g_t} + \mathrm{tr}(W_t^2) - \mathrm{tr}^2(W_t). \tag{8.38}$$

Assume now that the metric $g^{\mathcal{Z}}$ is Einstein with scalar curvature $(n+1)\lambda$, i.e., $\mathrm{Ric}^{\mathcal{Z}} = \lambda g^{\mathcal{Z}}$. Evaluating (8.35) and (8.38) at $t = 0$ yields

$$d\,\mathrm{tr}(W) + \delta^g W = 0, \tag{8.39}$$

and

$$\mathrm{Scal}^g + \mathrm{tr}(W^2) - \mathrm{tr}^2(W) = (n-1)\lambda. \tag{8.40}$$

If

$$g_t : \mathrm{End}(\mathrm{T}M) \longrightarrow \mathrm{T}^*M \otimes \mathrm{T}^*M$$

is the isomorphism defined by

$$g_t(A)(X,Y) := g_t(A(X),Y)$$

and

$$g_t^{-1} : \mathrm{T}^*M \otimes \mathrm{T}^*M \longrightarrow \mathrm{End}(\mathrm{T}M)$$

denotes its inverse, then taking (8.34) into account, (8.36) reads

$$\ddot{g}_t = 2\mathrm{Ric}^{g_t} + \dot{g}_t(g_t^{-1}(\dot{g}_t)\cdot,\cdot) - \mathrm{tr}(g_t^{-1}(\dot{g}_t))\dot{g}_t - 2\lambda g_t.$$

Using the Cauchy–Kowalewskaya theorem (see, e.g., [Die75]), Koiso proved the following existence and unique continuation result for Einstein metrics starting from an analytic metric and an analytic stress-energy tensor satisfying the above constraints.

Theorem 8.44 ([Koi81]). *Let (M^n, g) be an analytic Riemannian manifold and let W be an analytic field of symmetric endomorphisms on M satisfying (8.39) and (8.40). Then for $\varepsilon > 0$, there exists a unique analytic germ near $\{0\} \times M$ of an Einstein metric $g^{\mathcal{Z}}$ with scalar curvature $(n+1)\lambda$ of the form $g^{\mathcal{Z}} = dt^2 + g_t$ on $\mathcal{Z} := \mathbb{R} \times M$ with $g_0 = g$, whose Weingarten tensor at $t = 0$ is W.*

Corollary 8.45. *Assume that (M^n, g) is an analytic spin manifold carrying a non-trivial generalized Killing spinor ψ with analytic stress-energy tensor A. Then in a neighborhood of $\{0\} \times M$ in $\mathcal{Z} := \mathbb{R} \times M$ there exists a unique Ricci-flat metric $g^{\mathcal{Z}}$ of the form $g^{\mathcal{Z}} = dt^2 + g_t$ whose Weingarten tensor at $t = 0$ is $W = 2A$.*

Proof. We just need to check that the constraints (8.39) and (8.40) are a consequence of (8.32). In order to simplify the computations, we will drop the reference to the metric g and denote respectively by ∇, R, Ric and Scal the Levi-Civita covariant derivative, curvature tensor, Ricci tensor and scalar curvature of (M, g). As usual, $\{e_i\}$ will denote a local g-orthonormal basis of $\mathrm{T}M$.

We will use the following classical formula in Clifford calculus which says that the Clifford contraction of a symmetric tensor A depends only on its trace:

$$\sum_{i=1}^{n} e_i \cdot A(e_i) = -\operatorname{tr}(A). \tag{8.41}$$

Now, let ψ be a non-trivial generalized Killing spinor satisfying (8.32) and denote $W := 2A$. Being parallel with respect to a modified connection on ΣM, ψ is nowhere vanishing (and actually of constant norm).

Taking a further covariant derivative in (8.32) and skew-symmetrizing yields

$$\begin{aligned}\mathcal{R}_{X,Y}\psi = {}& \frac{1}{4}(W(Y) \cdot W(X) - W(X) \cdot W(Y)) \cdot \psi \\ & + \frac{1}{2}((\nabla_X W)(Y) - (\nabla_Y W)(X)) \cdot \psi\end{aligned}$$

for all $X, Y \in \mathrm{T}M$. In this formula we set $Y = e_i$, take the Clifford product with e_i and sum over i. From (8.41) and the classical formula for the Ricci tensor (2.3) we get

$$\begin{aligned}\operatorname{Ric}(X) \cdot \psi = {}& -\frac{1}{2}\sum_{i=1}^{n} e_i \cdot (W(e_i) \cdot W(X) - W(X) \cdot W(e_i)) \cdot \psi \\ & - \sum_{i=1}^{n} e_i \cdot ((\nabla_X W)(e_i) - (\nabla_{e_i} W)(X)) \cdot \psi \\ = {}& \frac{1}{2}\operatorname{tr}(W) W(X) \cdot \psi \\ & + \frac{1}{2}\sum_{i=1}^{n}(-W(X) \cdot e_i - 2g(W(X), e_i)) \cdot W(e_i) \cdot \psi \\ & + X(\operatorname{tr}(W))\psi + \sum_{i=1}^{n} e_i \cdot (\nabla_{e_i} W)(X) \cdot \psi,\end{aligned}$$

whence

$$\begin{aligned}\mathrm{Ric}(X)\cdot\psi &= \mathrm{tr}(W)W(X)\cdot\psi - W^2(X)\cdot\psi + X(\mathrm{tr}(W))\psi \\ &\quad + \sum_{i=1}^{n} e_i\cdot(\nabla_{e_i}W)(X)\cdot\psi.\end{aligned} \tag{8.42}$$

We set $X = e_j$ in (8.42), take the Clifford product with e_j, and sum over j. Using (8.41) again we obtain

$$\begin{aligned}-\mathrm{Scal}\psi &= -\mathrm{tr}^2(W)\psi + \mathrm{tr}(W^2)\psi + \nabla(\mathrm{tr}(W))\cdot\psi \\ &\quad + \sum_{i,j=1}^{n} e_j\cdot e_i\cdot(\nabla_{e_i}W)(e_j)\cdot\psi \\ &= -\mathrm{tr}^2(W)\psi + \mathrm{tr}(W^2)\psi + d\,\mathrm{tr}(W)\cdot\psi \\ &\quad + \sum_{i,j=1}^{n}(-e_i\cdot e_j - 2\delta_{ij})\cdot(\nabla_{e_i}W)(e_j)\cdot\psi \\ &= -\mathrm{tr}^2(W)\psi + \mathrm{tr}(W^2)\psi + 2d\,\mathrm{tr}(W)\cdot\psi + 2\delta W\cdot\psi,\end{aligned}$$

which implies simultaneously (8.39) and (8.40) (indeed, if $f\psi = X\cdot\psi$ for some real f and vector X, then $-|X|^2\psi = X\cdot X\cdot\psi = X\cdot(f\psi) = f^2\psi$, so both f and X vanish). □

Theorem 8.46. *Let $(\mathcal{Z}, g^{\mathcal{Z}})$ be a Ricci-flat spin manifold with Levi-Civita connection $\nabla^{\mathcal{Z}}$ and let $M \subset \mathcal{Z}$ be any oriented analytic hypersurface. Assume there exists some spinor $\psi \in \Gamma(\Sigma\mathcal{Z}|_M)$ which is parallel along M:*

$$\nabla^{\mathcal{Z}}_X\psi = 0, \qquad X \in \mathrm{T}M \subset \mathrm{T}\mathcal{Z}. \tag{8.43}$$

Assume moreover that the map $\pi_1(M) \to \pi_1(\mathcal{Z})$ induced by the inclusion is onto. Then there exists a parallel spinor $\Psi \in \Gamma(\Sigma\mathcal{Z})$ such that $\Psi|_M = \psi$.

Proof. Any Ricci-flat manifold is analytic, cf. [Bes87], p. 145, thus the analyticity of M makes sense.

Let ν denote the unit normal vector field along M. Every $x \in M$ has an open neighborhood V in M such that the exponential map $(-\varepsilon,\varepsilon)\times V \to \mathcal{Z}$, $(t,y) \mapsto \exp_y(t\nu)$, is well defined for some $\varepsilon > 0$. Its differential at $(0,x)$ being the identity, one can assume, by shrinking V and choosing a smaller ε if necessary, that it maps $(-\varepsilon,\varepsilon)\times V$ diffeomorphically onto some open neighborhood U of x in $\mathcal{Z}$. We extend the spinor ψ to a spinor Ψ on U by parallel transport along the normal geodesics $\exp_y(t\nu)$ for every fixed y. It remains to prove that Ψ is parallel on U in horizontal directions.

Let $\{e_i\}$ be a local orthonormal basis along M. We extend it on U by parallel transport along the normal geodesics, and note that $\{e_i, \nu\}$ is a local orthonormal basis on U. More generally, every vector field X along V gives rise to a unique horizontal vector field on U also denoted X, satisfying $\nabla_\nu X = 0$. For every such vector field we get

$$\nabla^{\mathcal{Z}}_\nu(\nabla^{\mathcal{Z}}_X \Psi) = \mathcal{R}^{\mathcal{Z}}(\nu, X)\Psi + \nabla^{\mathcal{Z}}_{[\nu,X]}\Psi = \mathcal{R}^{\mathcal{Z}}(\nu, X)\Psi + \nabla^{\mathcal{Z}}_{W(X)}\Psi, \tag{8.44}$$

where in this section $\mathcal{R}^{\mathcal{Z}}(\nu, X)\Psi$ stands for $\mathcal{R}^{\mathcal{Z}}_{\nu,X}\Psi$.

Since $\mathcal{Z}$ is Ricci-flat, (2.3) applied to the local orthonormal basis $\{e_i, \nu\}$ of $\mathcal{Z}$ yields

$$0 = \tfrac{1}{2}\mathrm{Ric}^{\mathcal{Z}}(X)\cdot\Psi = \sum_{i=1}^{n} e_i \cdot \mathcal{R}^{\mathcal{Z}}(e_i, X)\Psi + \nu\cdot\mathcal{R}^{\mathcal{Z}}(\nu, X)\Psi. \tag{8.45}$$

We take the Clifford product by ν in this relation, differentiate again with respect to ν, and use the second Bianchi identity to obtain

$$\begin{aligned}
\nabla^{\mathcal{Z}}_\nu(\mathcal{R}^{\mathcal{Z}}(\nu, X)\Psi) &= \nabla^{\mathcal{Z}}_\nu\Big(\nu\cdot\sum_{i=1}^{n} e_i\cdot\mathcal{R}^{\mathcal{Z}}(e_i, X)\Psi\Big)\\
&= \nu\cdot\sum_{i=1}^{n} e_i\cdot(\nabla^{\mathcal{Z}}_\nu\mathcal{R}^{\mathcal{Z}})(e_i, X)\Psi\\
&= \nu\cdot\sum_{i=1}^{n} e_i\cdot((\nabla^{\mathcal{Z}}_{e_i}\mathcal{R}^{\mathcal{Z}})(\nu, X)\Psi + (\nabla^{\mathcal{Z}}_X\mathcal{R}^{\mathcal{Z}})(e_i, \nu)\Psi),
\end{aligned}$$

whence

$$\begin{aligned}
&\nabla^{\mathcal{Z}}_\nu(\mathcal{R}^{\mathcal{Z}}(\nu, X)\Psi)\\
&\quad = \nu\cdot\sum_{i=1}^{n} e_i\cdot(\nabla^{\mathcal{Z}}_{e_i}(\mathcal{R}^{\mathcal{Z}}(\nu, X)\Psi) + \mathcal{R}^{\mathcal{Z}}(W(e_i), X)\Psi - \mathcal{R}^{\mathcal{Z}}(\nu, \nabla^{\mathcal{Z}}_{e_i}X)\Psi\\
&\qquad\quad - \mathcal{R}^{\mathcal{Z}}(\nu, X)\nabla^{\mathcal{Z}}_{e_i}\Psi + \nabla^{\mathcal{Z}}_X(\mathcal{R}^{\mathcal{Z}}(e_i, \nu)\Psi) - \mathcal{R}^{\mathcal{Z}}(\nabla^{\mathcal{Z}}_X e_i, \nu)\Psi\\
&\qquad\quad + \mathcal{R}^{\mathcal{Z}}(e_i, W(X))\Psi - \mathcal{R}^{\mathcal{Z}}(e_i, \nu)\nabla^{\mathcal{Z}}_X\Psi).
\end{aligned} \tag{8.46}$$

Let $\nu^\perp$ denote the distribution orthogonal to ν on U and consider the sections

$$B, C \in \Gamma((\nu^\perp)^* \otimes \Sigma U) \quad\text{and}\quad D \in \Gamma(\Lambda^2(\nu^\perp)^* \otimes \Sigma U)$$

defined for all $X, Y \in \nu^\perp$ by

$$B(X) := \nabla^{\mathcal{Z}}_X \Psi,$$
$$C(X) := \mathcal{R}^{\mathcal{Z}}(\nu, X)\Psi,$$
$$D(X,Y) := \mathcal{R}^{\mathcal{Z}}(X,Y)\Psi.$$

We have noted that the metric $g^{\mathcal{Z}}$ is analytic since it is Ricci-flat. From the assumptions that M is analytic and that ψ is parallel along M it follows that Ψ, and thus the tensors B, C, and D, are analytic.

Equations (8.44) and (8.46) read in our new notation

$$(\nabla^{\mathcal{Z}}_\nu B)(X) = C(X) + B(W(X)),$$

and

$$\begin{aligned}(\nabla^{\mathcal{Z}}_\nu C)(X) = \nu \cdot \sum_{i=1}^n e_i \cdot ((\nabla^{\mathcal{Z}}_{e_i} C)(X) + D(W(e_i), X) - \mathcal{R}^{\mathcal{Z}}(\nu, X) B(e_i) \\ - (\nabla^{\mathcal{Z}}_X C)(e_i) + D(e_i, W(X)) - \mathcal{R}^{\mathcal{Z}}(e_i, \nu) B(X)).\end{aligned} \tag{8.47}$$

Moreover, the second Bianchi identity yields

$$\begin{aligned}(\nabla^{\mathcal{Z}}_\nu D)(X,Y) =\ & (\nabla^{\mathcal{Z}}_\nu \mathcal{R}^{\mathcal{Z}})(X,Y)\Psi \\ =\ & (\nabla^{\mathcal{Z}}_X \mathcal{R}^{\mathcal{Z}})(\nu, Y)\Psi + (\nabla^{\mathcal{Z}}_Y \mathcal{R}^{\mathcal{Z}})(X, \nu)\Psi \\ =\ & \nabla^{\mathcal{Z}}_X (R^{\mathcal{Z}}(\nu, Y)\Psi) - \mathcal{R}^{\mathcal{Z}}(\nabla^{\mathcal{Z}}_X \nu, Y)\Psi \\ & - \mathcal{R}^{\mathcal{Z}}(\nu, \nabla^{\mathcal{Z}}_X Y)\Psi - \mathcal{R}^{\mathcal{Z}}(\nu, Y)\nabla^{\mathcal{Z}}_X \Psi \\ & - \nabla^{\mathcal{Z}}_Y (\mathcal{R}^{\mathcal{Z}}(\nu, X)\Psi) + \mathcal{R}^{\mathcal{Z}}(\nabla^{\mathcal{Z}}_Y \nu, X)\Psi \\ & + \mathcal{R}^{\mathcal{Z}}(\nu, \nabla^{\mathcal{Z}}_Y X)\Psi + \mathcal{R}^{\mathcal{Z}}(\nu, X)\nabla^{\mathcal{Z}}_Y \Psi \\ =\ & (\nabla^{\mathcal{Z}}_X C)(Y) + D(W(X), Y) - \mathcal{R}^{\mathcal{Z}}(\nu, Y)\nabla^{\mathcal{Z}}_X \Psi \\ & - (\nabla^{\mathcal{Z}}_Y C)(X) + D(X, W(Y)) + \mathcal{R}^{\mathcal{Z}}(\nu, X)\nabla^{\mathcal{Z}}_Y \Psi,\end{aligned}$$

thus showing that

$$\begin{aligned}(\nabla^{\mathcal{Z}}_\nu D)(X,Y) =& (\nabla^{\mathcal{Z}}_X C)(Y) + D(W(X), Y) - \mathcal{R}^{\mathcal{Z}}(\nu, Y)(B(X)) \\ & - (\nabla^{\mathcal{Z}}_Y C)(X) + D(X, W(Y)) + \mathcal{R}^{\mathcal{Z}}(\nu, X)(B(Y)).\end{aligned} \tag{8.48}$$

Hypothesis (8.43) is equivalent to $B = 0$ for $t = 0$. Differentiating this again in the direction of M and skew-symmetrizing yields $D = 0$ for $t = 0$. Finally, (8.45) shows that $C = 0$ for $t = 0$. We thus see that the section $S := (B, C, D)$ vanishes along the hypersurface $\{0\} \times V$ of U.

The system (8.47)–(8.48) is a linear PDE for S and the hypersurfaces $t =$ *constant* are clearly non-characteristic. The Cauchy–Kowalewskaya theorem shows that S vanishes everywhere on U. In particular, $B = 0$ on U, thus proving our claim.

Now we prove that there exists a parallel spinor $\Psi \in \Gamma(\Sigma\mathcal{Z})$ such that $\Psi|_M = \psi$. Take any $x \in M$ and an open neighborhood U like in Theorem 8.46 on which a parallel spinor Ψ extending ψ is defined. The spin holonomy group $\widetilde{\mathrm{Hol}}(U, x)$ thus preserves Ψ_x. Since any Ricci-flat metric is analytic, the restricted spin holonomy group $\widetilde{\mathrm{Hol}}_0(\mathcal{Z}, x)$ is equal to $\widetilde{\mathrm{Hol}}_0(U, x)$ for every $x \in \mathcal{Z}$ and for every open neighborhood U of x. By the local extension result proved above, $\widetilde{\mathrm{Hol}}_0(U, x)$ acts trivially on Ψ_x, thus showing that Ψ_x can be extended (by parallel transport along every curve in $\tilde{\mathcal{Z}}$ starting from x) to a parallel spinor $\tilde{\Psi}$ on the universal cover $\tilde{\mathcal{Z}}$ of $\mathcal{Z}$. The group of deck transformations acts trivially on $\tilde{\Psi}$ since every element in $\pi_1(\mathcal{Z}, x)$ can be represented by a curve in M (here we use the surjectivity hypothesis) and Ψ was assumed to be parallel along M. Thus $\tilde{\Psi}$ descends to $\mathcal{Z}$ as a parallel spinor. This completes the proof of Theorem 8.46. $\square$

Theorem 8.43 is now a direct consequence of Corollary 8.45 and Theorem 8.46.

Chapter 9

Special spinors on conformal manifolds

In this chapter we continue the study of conformal spin manifolds initiated in Chapter 2. The first section is devoted to the conformal analogue of the Schrödinger–Lichnerowicz formula, which is derived in the framework of conformal geometry (i.e., without making use of Riemannian metrics). We then give in the next section an application of this formula to the problem of the existence of parallel spinors on conformal manifolds. Finally, the Hijazi inequality (see Chapter 5) is proved in the last section by using methods of conformal spin geometry. The notations are those of Section 2.2.3.

9.1 The conformal Schrödinger–Lichnerowicz formula

Let (M^n, c) be a conformal spin manifold and let D be a Weyl structure on M. Recall that the Clifford product (denoted by γ) of 1-forms on weighted spinors lowers the conformal weight by one. We define the *conformal Dirac operator* $\mathcal{D}^{(k)}$ (or simply $\mathcal{D}$ when there is no risk of confusion), mapping sections of $\Sigma^{(k)}M$ to sections of $\Sigma^{(k-1)}M$, as the composition $\mathcal{D}^{(k)} := \gamma \circ D$,

$$\Gamma(\Sigma^{(k)}M) \xrightarrow{D} \Gamma(\mathrm{T}^*M \otimes \Sigma^{(k)}M) \xrightarrow{\gamma} \Gamma(\Sigma^{(k-1)}M).$$

The conformal invariance of the Dirac operator, first observed by Hitchin [Hit74], can be stated in the following way:

Theorem 9.1. *The Dirac operator $\mathcal{D}^{(k)}$ is independent of D for $k = \frac{1-n}{2}$. In particular, for every Riemannian metric g in the conformal class c, it coincides with the usual Dirac operator after the identification, defined in Section 2.2.3, of the weighted spinor bundles with the g-spinor bundle.*

Proof. Let g be a (local) metric in the conformal class c. It is sufficient to observe that, according to Theorem 2.21, the conformal Dirac operator $\mathcal{D}^{(k)}$ written in this metric satisfies

$$\mathcal{D}^{(k)}\Phi = \mathcal{D}\Phi + \Big(k + \frac{n-1}{2}\Big)\theta \cdot \Phi. \qquad \square$$

We now prove the following conformal analogue of Corollary 2.8.

Proposition 9.2. *Let Φ be a spinor of weight k on a conformal manifold M. Then, for every local conformal frame $\{e_i\}$ and vector X on M,*

$$\sum_{i=1}^{n} e_i^* \cdot \mathcal{R}^D_{X,e_i}\Phi = -\frac{1}{2}\mathrm{Ric}^D_{\mathrm{s}}(X)\cdot\Phi - \frac{1}{2}X\cdot F\cdot\Phi + \Big(k + \frac{n-4}{4}\Big)(X \lrcorner F)\cdot\Phi, \quad (9.1)$$

where $\mathrm{Ric}^D_{\mathrm{s}}$ denotes the symmetric part of the Ricci tensor and F is the Faraday form of D.

Proof. We fix a local conformal basis $\{e_i\}$, i.e., satisfying $c(e_i, e_j) = l^2\delta_{ij}$ for some section l of $\mathcal{L}$. Then its dual basis can be written as $e_i^* = e_i l^{-2}$ (via the identification by c of T^*M with $\mathrm{T}M \otimes \mathcal{L}^{-2}$).

We start by noting that (with the notations of Section 2.2.3)

$$\mathcal{R}^{D,(k)}_{X,Y}\Phi = \frac{1}{2}R^D_{\mathrm{a}}(X,Y)\cdot\Phi + kF(X,Y)\Phi, \quad (9.2)$$

the proof being similar to that of Theorem 2.7. Then, using the "Bianchi-type" relation (2.10), Lemma 2.16, and the Clifford relations

$$e_j^* \cdot e_k^* \cdot e_i^* = e_i \cdot e_j^* \cdot e_k^* - 2l^{-2}e_j^*\delta_{ik} + 2l^{-2}e_k^*\delta_{ij}$$

similarly to (2.4), we get

$$\begin{aligned}
&\sum_i e_i^* \cdot R^D_{\mathrm{a}}(X,e_i)\cdot\Phi \\
&\quad= \frac{1}{2}\sum_{i,j,k} c(R^D_a(X,e_i)e_j, e_k)e_i^*\cdot e_j^*\cdot e_k^*\cdot\Phi \\
&\quad= \frac{1}{6}\underset{i,j,k}{\mathfrak{S}}\sum_{i,j,k} c(R^D_{\mathrm{a}}(X,e_i)e_j, e_k)e_i^*\cdot e_j^*\cdot e_k^*\cdot\Phi - \sum_i \mathrm{Ric}^D(X,e_i)e_i^*\cdot\Phi \\
&\quad= \frac{1}{6}\underset{i,j,k}{\mathfrak{S}}\sum_{i,j,k} F(e_j,e_i)c(X,e_k)e_i^*\cdot e_j^*\cdot e_k^*\cdot\Phi \\
&\qquad\qquad\qquad - \mathrm{Ric}^D_{\mathrm{s}}(X)\Phi + \frac{n-2}{2}(X \lrcorner F)\cdot\Phi \\
&\quad= -X\cdot F\cdot\Phi - \mathrm{Ric}^D_{\mathrm{s}}(X)\Phi + \frac{n-4}{2}(X \lrcorner F)\cdot\Phi.
\end{aligned}$$

This relation, together with (9.2), proves the desired formula. $\square$

We now introduce the *conformal spin Laplacian* Δ^D mapping sections of $\Sigma^{(k)}M$ to sections of $\Sigma^{(k-2)}M$, as the composition $\Delta^D := -(c \otimes \mathrm{Id}) \circ D \circ D$,

$$\Gamma(\Sigma^{(k)}M) \xrightarrow{D} \Gamma(\mathrm{T}^*M \otimes \Sigma^{(k)}M) \xrightarrow{D} \Gamma(\mathrm{T}^*M \otimes \mathrm{T}^*M \otimes \Sigma^{(k)}M) \xrightarrow{-c\otimes \mathrm{Id}} \Gamma(\Sigma^{(k-2)}M).$$

We end this section by proving the following theorem.

Theorem 9.3 (Conformal Schrödinger–Lichnerowicz formula). *The square of the conformal Dirac operator and the conformal spin Laplacian, both acting on the spinor bundle of weight k, are related by*

$$\mathcal{D}^{(k-1)} \circ \mathcal{D}^{(k)}\Phi = \Delta^D\Phi + \frac{\mathrm{Scal}^D}{4}\Phi + \Big(k + \frac{n-2}{2}\Big)F \cdot \Phi,$$

where Scal^D *is the* ($\mathcal{L}^{(-2)}$*-valued*) *conformal scalar curvature of D.*

Proof. Consider a local conformal frame $\{e_i\}$ as before. For a local section Φ of $\Sigma^{(k)}M$ we have

$$\begin{aligned}
\Delta^D\Phi &= -\sum_{i,j} c \otimes \mathrm{Id}(e_i^* \otimes D_{e_i}(e_j^* \otimes D_{e_j}\Phi)) \\
&= -\sum_{i,j} c \otimes \mathrm{Id}(e_i^* \otimes e_j^* \otimes D_{e_i}D_{e_j}\Phi + e_i^* \otimes D_{e_i}e_j^* \otimes D_{e_j}\Phi) \\
&= l^{-2}\sum_i (D_{D_{e_i}e_i}\Phi - D_{e_i}D_{e_i}\Phi).
\end{aligned}$$

On the other hand, using the Clifford relations we compute

$$\begin{aligned}
\mathcal{D}^{(k-1)} \circ \mathcal{D}^{(k)}\Phi &= \sum_{i,j} e_i^* \cdot D_{e_i}(e_j^* \cdot D_{e_j}\Phi) \\
&= \sum_{i,j} (e_i^* \cdot D_{e_i}e_j^* \cdot D_{e_j}\Phi + e_i^* \cdot e_j^* \cdot D_{e_i}D_{e_j}\Phi) \\
&= -\sum_{i,j} e_i^* \cdot e_j^* \cdot D_{D_{e_i}e_j}\Phi - l^{-2}\sum_i D_{e_i}D_{e_i}\Phi \\
&\quad + \frac{1}{2}\sum_{i,j} e_i^* \cdot e_j^* \cdot (D_{e_i}D_{e_j}\Phi - D_{e_j}D_{e_i}\Phi) \\
&= \Delta^D\Phi + \frac{1}{2}\sum_{i,j} e_i^* \cdot e_j^* \cdot \mathcal{R}^{D,(k)}_{e_i,e_j}\Phi,
\end{aligned}$$

where $\mathcal{R}^{D,(k)}$ denotes the curvature tensor of the connection D acting on spinors of weight k. We thus have

$$\mathcal{D}^{(k-1)} \circ \mathcal{D}^{(k)} \Phi = \Delta^D \Phi + \frac{1}{2} \sum_{i,j} e_i^* \cdot e_j^* \cdot \mathcal{R}^{D,(k)}_{e_i,e_j} \Phi,$$

which together with (9.1) yields the desired result. □

The conformal Schrödinger–Lichnerowicz formula is a result due to V. Buchholz [Buc98], P. Gauduchon [Gau95a], and A. Moroianu [Mor98a].

9.2 Parallel spinors with respect to Weyl structures

Consider the following question.

Problem 9.4. Classify the triples $((M, c), D, k)$, where M is an n-dimensional spin manifold with conformal structure c, D is a Weyl structure on M and $k \in \mathbb{R}$, which admit D-parallel spinors of weight k.

We will give a partial answer to this question in the non-compact case, and solve it completely for M compact.

The first remark is that, if (M, g) is a Riemannian spin manifold with parallel spinors, then $((M, [g]), \nabla^g, k)$ is a solution to our problem for every k, by Theorem 2.21. Such solutions are called *Riemannian*. The Weyl structure of every Riemannian solution is exact and conversely, a solution with exact Weyl structure is Riemannian. Indeed, if D is exact, we can choose a metric g in the conformal class c such that $D = \nabla^g$, and the D-parallel spinor can be identified with a parallel spinor on (M, g), again by Theorem 2.21.

The second remark is that every solution $((M, c), D, k)$ of our problem with $k \neq 0$ is Riemannian. Indeed, if Φ is a D-parallel spinor of weight $k \neq 0$, then $\|\Phi\|^{\frac{1}{k}}$ is a D-parallel section of $\mathcal{L}$, so, by definition, D is exact. Here $\|\cdot\|$ denotes the norm associated with the $\mathcal{L}^{2k}$-valued Hermitian product on $\Sigma^{(k)}M$.

Accordingly, from now on we will only consider D-parallel spinors of weight 0. In particular, every such spinor has constant (scalar-valued) norm.

9.2.1 Parallel conformal spinors on Riemann surfaces

It happens quite often in conformal geometry that the 2-dimensional case has to be treated separately. In this section we consider our problem on spin Riemann surfaces (M, c).

Recall that a conformal structure in dimension 2 is equivalent to a complex structure. Up to a change of orientation, we may suppose that the parallel spinor lies in $\Sigma^+ M$. Let D be a Weyl structure on M, let g be a metric in the conformal class c and let θ be the Lee form of D with respect to g (identified by g with a vector field).

Lemma 9.5. *A spinor field $\Psi \in \Gamma(\Sigma^+ M)$ is D-parallel if and only if it satisfies*

$$\nabla^g_X \Psi = \frac{i}{2} g(X, J\theta)\Psi, \quad X \in \Gamma(\mathrm{T}M).$$

Proof. For every $\Psi \in \Gamma(\Sigma^+ M)$ we have $JX \cdot X \cdot \Psi = i|X|^2\Psi$, so

$$\begin{aligned}[X \cdot \theta + \theta(X)] \cdot \Psi &= \Big[g\Big(X, \frac{\theta}{|\theta|}\Big)\frac{\theta}{|\theta|} \cdot \theta + g\Big(X, \frac{J\theta}{|\theta|}\Big)\frac{J\theta}{|\theta|} \cdot \theta + g(X, \theta)\Big] \cdot \Psi \\ &= i g(X, J\theta)\Psi.\end{aligned}$$

The statement now follows immediately from Theorem 2.21. □

This lemma implies that every spinor Ψ of constant norm on a Riemann surface is parallel with respect to some Weyl structure. Indeed, we can write

$$\nabla^g_X \Psi = \alpha(X)\Psi$$

for some purely imaginary-valued 1-form α and consider the Weyl structure D whose Lee form θ satisfies

$$\frac{i}{2} J\theta = \alpha.$$

However, this does not answer our problem, since we need to determine whether a given Weyl structure D admits D-parallel spinors. We have already seen that a necessary condition is the triviality of the spinor bundle $\Sigma^+ M$. Let Φ be a unit section of $\Sigma^+ M$ and define a 1-form α by

$$\nabla^g_X \Phi = i\alpha(X)\Phi.$$

Here g is a fixed metric on M and we denote by θ the Lee form of D with respect to g. Every spinor of constant norm can be written as $f\Phi$ for some map $f : M \to \mathbb{S}^1$. By Lemma 9.5, it is clear that $f\Phi$ is D-parallel if and only if

$$\frac{1}{2} J\theta - \alpha = \frac{-i\,df}{f}.$$

Then the answer to our problem is given by the following lemma.

Lemma 9.6. *A* 1*-form on a manifold M can be written as $\frac{idf}{f}$ for some map $f : M \to \mathrm{U}_1$, if and only if it is closed and has integral cohomology class.*

Proof. Let $\mathcal{A}$ be the sheaf of germs of U_1-valued functions and let $\mathcal{F}$ be the sheaf of germs of closed 1-forms on M. The exact sequence of sheaves

$$0 \longrightarrow \mathrm{U}_1 \longrightarrow \mathcal{A} \xrightarrow{f \mapsto \frac{idf}{f}} \mathcal{F} \longrightarrow 0$$

induces a long exact sequence of cohomology groups

$$\cdots \longrightarrow H^0(M, \mathcal{A}) \longrightarrow H^0(M, \mathcal{F}) \xrightarrow{\delta} H^1(M, \mathrm{U}_1) \longrightarrow \cdots .$$

This exact sequence shows that a closed 1-form ω can be written as $\frac{idf}{f}$ for some $f : M \to \mathrm{U}_1$ if and only if $\delta(\omega) = 0$. On the other hand, we also have a long exact sequence

$$\cdots \longrightarrow H^1(M, \mathbb{Z}) \longrightarrow H^1(M, \mathbb{R}) \xrightarrow{\exp} H^1(M, \mathrm{U}_1) \longrightarrow \cdots$$

and, moreover, if $[\cdot]$ denotes the natural mapping $H^0(M, \mathcal{F}) \to H^1(M, \mathbb{R})$, then $\exp \circ [\cdot] = \delta$ (by naturality). Thus $\delta(\omega) = 0$ if and only if $\exp([\omega]) = 0$, which, by the last exact sequence, is equivalent to $[\omega] \in H^1(M, \mathbb{Z})$. □

We thus, have proved the following theorem.

Theorem 9.7. *Let (M, c) be a spin Riemann surface with a Weyl structure D, let g be a metric in the conformal class c, and let θ be the Lee form of D with respect to g. Then $\Sigma^+ M$ carries a D-parallel spinor field if and only if it is trivial (as complex line bundle) and $\frac{1}{2} J\theta - \alpha$ is a closed integral form, where $i\alpha$ denotes the connection form of the Levi-Civita spin connection written in some arbitrary gauge of $\Sigma^+ M$.*

9.2.2 The non-compact case

Consider now the n-dimensional case, with $n > 2$.

Theorem 9.8. *Let (M^n, c) $(n > 2)$ be a conformal spin manifold with Weyl structure D. If M carries a D-parallel spinor field Ψ and $n \neq 4$, then D is closed.*

Proof. The conformal Schrödinger–Lichnerowicz formula implies that

$$\mathrm{Scal}^D \Psi = -2(n-2) F \cdot \Psi. \tag{9.3}$$

By taking here the scalar product with Ψ, we obtain

$$\mathrm{Scal}^D ||\Psi||^2 = -2(n-2)\langle F \cdot \Psi, \Psi \rangle.$$

As the left term is real and the right term is purely imaginary, they must vanish, so in particular $\mathrm{Scal}^D = 0$. As $n > 2$, (9.3) shows that

$$F \cdot \Psi = 0. \tag{9.4}$$

This equation, together with (9.1) gives

$$2\mathrm{Ric}_s^D = (n-4)F, \tag{9.5}$$

so, for $n \neq 4$, the Faraday form must vanish. □

This theorem says that – in every dimension other than 2 and 4 – the solutions to Problem 9.4 are locally Riemannian (i.e., every D-parallel spinor is locally ∇^g-parallel for some local metric g). In the previous section we have seen that this theorem does not hold in dimension 2, and the example below shows that it also fails for $n = 4$. We first emphasize the special relations between forms and spinors which hold on 4-dimensional manifolds.

Recall that for n even, every weighted spinor bundle decomposes under the action of the complex volume element $\omega^{\mathbb{C}} \in \Lambda^n M \otimes \mathcal{L}^n$ as $\Sigma M = \Sigma^+ M \oplus \Sigma^- M$, where

$$\Sigma^{\pm} M = \{\Psi;\ \omega^{\mathbb{C}} \cdot \Psi = \pm\Psi\}. \tag{9.6}$$

This decomposition is preserved by every Weyl structure D since the volume element is D-parallel. As usual, the projections of a spinor Ψ on $\Sigma^{\pm} M$ are denoted by $\Psi^{\pm}$. Then (9.6) shows that changing the orientation of M is equivalent to switching from $\Sigma^+ M$ to $\Sigma^- M$. Thus, if Ψ is a D-parallel spinor, we can assume that $\Psi \in \Gamma(\Sigma^+ M)$.

Now let M be 4-dimensional. In this case, the conformal Hodge operator $*$ acts involutively on $\Lambda^2 M \otimes \mathcal{L}^2$. On the other hand, $\Lambda^2 M \otimes \mathcal{L}^2$ is canonically isomorphic to the bundle $\mathrm{A}(M)$ of skew-symmetric endomorphisms of the tangent bundle. Hence we obtain an orthogonal decomposition $\mathrm{A}(M) = \mathrm{A}^+(M) \oplus \mathrm{A}^-(M)$, where $\mathrm{A}^+(M)$ (resp. $\mathrm{A}^-(M)$) denotes the eigenspace of $*$ corresponding to the eigenvalue 1 (resp. -1). The elements of $\mathrm{A}^+(M)$ (resp. $\mathrm{A}^-(M)$) are called self-dual (resp. anti-self-dual) endomorphisms. Using the isomorphism between $\mathrm{A}(M)$ and $\Lambda^2 M \otimes \mathcal{L}^2$, we see that the skew-symmetric endomorphisms act by Clifford product on spinors and preserve their weight.

Lemma 9.9. (i) *If $A \in \mathrm{A}^-(M)$ is a fixed anti-self-dual endomorphism, then the restriction to $\Sigma^+ M$ of the Clifford product by A is the zero endomorphism.*

(ii) *The endomorphism $u \colon \mathrm{A}^+(M) \to \Sigma^+ M$ defined by $A \mapsto A \cdot \Psi$ is injective for every fixed spinor $\Psi \in \Sigma^+ M$.*

Proof. (i) The Clifford product by A is a skew-Hermitian endomorphism of $\Sigma^+ M$. By (1.15), the weightless volume form vol satisfies $\mathrm{vol} = -\omega^{\mathbb{C}}$. If A is anti-self-dual we thus have

$$\begin{aligned} A \cdot A \cdot \Psi &= (A \wedge A) \cdot \Psi - \langle A, A\rangle \Psi \\ &= -(A \wedge *A) \cdot \Psi - \langle A, A\rangle \Psi \\ &= -\langle A, A\rangle \mathrm{vol} \cdot \Psi - \langle A, A\rangle \Psi \\ &= 0, \end{aligned}$$

for all $\Psi \in \Sigma^+ M$. The statement follows from the fact that a skew-Hermitian endomorphism whose square vanishes must itself vanish.

(ii) If $A \in \mathrm{Ker}(u)$, then

$$\begin{aligned} 0 &= A \cdot A \cdot \Psi \\ &= (A \wedge A) \cdot \Psi - \langle A, A\rangle \Psi \\ &= (A \wedge *A) \cdot \Psi - \langle A, A\rangle \Psi \\ &= \langle A, A\rangle \mathrm{vol} \cdot \Psi - \langle A, A\rangle \Psi \\ &= -2\langle A, A\rangle \Psi, \end{aligned}$$

so $A = 0$. □

Now, by Theorem 2.6, the spinor bundle of a 4-dimensional manifold carries a quaternionic structure, i.e., a parallel complex anti-linear endomorphism j commuting with the Clifford product by real vectors and satisfying $\mathrm{j}^2 = -1$. In particular, j preserves the decomposition $\Sigma M = \Sigma^+ M \oplus \Sigma^- M$.

Moreover, the uniqueness of the Hermitian product on spinors shows that there exists some positive real constant α such that

$$\overline{\langle \mathrm{j}\Psi, \mathrm{j}\Phi\rangle} = \alpha \langle \Psi, \Phi\rangle, \quad \Psi, \Phi \in \Gamma(\Sigma M).$$

Applying this twice, we get $\alpha^2 = 1$. Thus j satisfies

$$\langle \mathrm{j}\Psi, \mathrm{j}\Phi\rangle = \overline{\langle \Psi, \Phi\rangle}. \tag{9.7}$$

Lemma 9.10. *There is a canonical isomorphism between the complexification of the bundle of self-dual endomorphisms and the second symmetric power of* $\Sigma^+ M$.

Proof. Indeed, let A_i, $i = 1, 2, 3$ be a local orthonormal basis of $\mathrm{A}^+(M)$. The endomorphism

$$F : \Sigma^+ M \otimes \Sigma^+ M \longrightarrow \mathrm{A}^+(M) \otimes \mathbb{C}$$

defined by

$$F(\Psi \otimes \Phi) = \sum_i A_i \langle A_i \cdot \Psi, \mathrm{j}\Phi \rangle$$

is symmetric by (9.7), so it descends to an endomorphism

$$F_{\mathrm{s}} : \mathrm{Sym}^2(\Sigma^+ M) \longmapsto \mathrm{A}^+(M) \otimes \mathbb{C}.$$

Using Lemma 9.9 and a local basis of $\mathrm{Sym}^2(\Sigma^+ M)$ we easily obtain that F_{s} is injective, thus an isomorphism (for dimensional reasons). □

Definition 9.11. Let J be an almost complex structure on a conformal manifold (M, c) with Weyl structure D. The 1-form θ_J^D given by

$$\theta_J^D(X) = \mathrm{tr}\{V \mapsto J(D_V J)X\}$$

is called the Lee form of D with respect to J.

Lemma 9.12. *On a conformal manifold with almost complex structure J, there exists exactly one Weyl structure D whose Lee form with respect to J vanishes.*

Proof. Choose a metric g in the conformal class c, and let D be an arbitrary Weyl structure whose Lee form with respect to g is θ. An easy calculation using formula (2.8) yields

$$\theta_J^D = -J\delta^g \Omega + (n-2)J\theta,$$

where $\Omega(X, Y) := g(JX, Y)$. Thus $\theta_J^D = 0$ if and only if $\theta = \frac{1}{n-2}\delta^g \Omega$. □

The connection D is sometimes called the *canonical Weyl structure* associated to (M, c, J). Note that if J is D-parallel for some Weyl structure D, then D has to be the canonical Weyl structure and moreover J has to be integrable, being parallel with respect to a torsion-free connection. Conversely, in dimension 4, we have the following

Proposition 9.13. *A Hermitian structure on a conformal* 4*-dimensional manifold is parallel with respect to its canonical Weyl structure.*

Proof. Let D be the canonical Weyl structure of J and let $\Omega(X, Y) := c(JX, Y)$ be the associated ($\mathcal{L}^2$-valued) Kähler form. It is straightforward to check that in dimension 4 one has $d^D \Omega = \delta^D \Omega \wedge \Omega$. In our case, this implies that $d^D \Omega = 0$. The fact that J is D-parallel is then a direct consequence of the conformal analogue of [KN69], Proposition 4.2, Chapter IX. □

Definition 9.14. A *hyper-Hermitian structure* on a conformal manifold (M, c) is given by three complex structures J_1, J_2, J_3 which are compatible with c and satisfy the quaternionic relations.

Lemma 9.15. *If J_1, J_2, J_3 define a hyper-Hermitian structure, then for each Weyl structure D, the Lee forms $\theta^D_{J_i}$ are all equal.*

Proof. It is well known that an almost complex structure J is integrable if and only if $JD_X J = D_{JX} J$, no matter what Weyl structure D is used. We then compute, using a local conformal basis $\{e_i\}$

$$\begin{aligned}\sum_{i=1}^{n}(D_{e_i}J_1)J_2e_i &= \sum_{i=1}^{n}(D_{J_1e_i}J_1)J_2J_1e_i\\ &= -\sum_{i=1}^{n}(J_1D_{e_i}J_1)J_1J_2e_i\\ &= -\sum_{i=1}^{n}(D_{e_i}J_1)J_2e_i,\end{aligned}$$

so

$$\sum_{i=1}^{n}(D_{e_i}J_1)J_2e_i = 0.$$

This yields

$$\begin{aligned}\theta^D_{J_3} = \sum_{i=1}^{n} J_3(D_{e_i}J_3)e_i &= \sum_{i=1}^{n} J_3\{(D_{e_i}J_1)J_2e_i + J_1(D_{e_i}J_2)e_i\}\\ &= \sum_{i=1}^{n} J_3J_1(D_{e_i}J_2)e_i\\ &= \theta^D_{J_2}.\end{aligned}$$

□

From Proposition 9.13 and Lemma 9.15 we obtain the following corollary.

Corollary 9.16. *On a 4-dimensional hyper-Hermitian conformal manifold M there exists a unique Weyl structure with respect to which the three Hermitian structures are parallel.*

Theorem 9.17. *Let M be a 4-dimensional conformal spin manifold. If M carries a Weyl structure D and a D-parallel spinor, then it is hyper-Hermitian. The converse is also true if M is simply connected.*

Proof. Let Ψ be a D-parallel spinor, and suppose as before that it lies in $\Gamma(\Sigma^+ M)$. Then $\{\Psi, \mathrm{j}\Psi\}$ defines a parallelization of $\Sigma^+ M$, so, by Lemma 9.10, one can find three D-parallel orthonormal sections J_1, J_2, J_3 of $\mathrm{A}^+(M)$. It is easy to check that J_i are almost Hermitian structures satisfying the quaternion relations. Their integrability is obvious, since they are parallel with respect to a torsion-free connection.

Conversely, suppose that M has a hyper-Hermitian structure J_1, J_2, J_3. Corollary 9.16 implies that J_i are all D-parallel for some Weyl structure D on M. This shows that $\mathrm{A}^+(M)$ is D-flat, thus $\mathrm{Sym}^2(\Sigma^+ M)$ is D-flat, too. Note that for every non-zero spinor Ψ, the symmetric product with Ψ maps $\Sigma^+ M$ *injectively* into $\mathrm{Sym}^2(\Sigma^+ M)$. Let $\Psi \odot \Psi$ be a section of $\mathrm{Sym}^2(\Sigma^+ M)$. Then

$$0 = \mathcal{R}^D_{X,Y}(\Psi \odot \Psi) = 2(\mathcal{R}^D_{X,Y}\Psi) \odot \Psi,$$

so by the above remark, $\Sigma^+ M$ is itself D-flat. If M is simply connected, one can thus find a D-parallel spinor. $\square$

Example 9.18 ([Apo96]). Let $u \colon U \subset \mathbb{C}^2 \to \mathbb{C}^*$ be a smooth function and consider the Riemannian metric on U given by

$$g_u = dz_1 d\bar{z}_1 + |u|^2 dz_2 d\bar{z}_2.$$

The relations

$$\Omega_1 + i\Omega_2 = \bar{u} dz_1 \wedge d\bar{z}_2$$

and

$$\Omega_3 = i(dz_1 \wedge d\bar{z}_1 + |u|^2 dz_2 \wedge d\bar{z}_2)$$

define the Kähler forms of three almost Hermitian structures J_1, J_2, J_3 on (U, g_u).

A straightforward computation shows that the Lee forms of J_1 and J_2 are both equal to

$$\tau = \frac{1}{u}\frac{\partial u}{\partial z_1} dz_1 + \frac{1}{\bar{u}}\frac{\partial \bar{u}}{\partial \bar{z}_1} d\bar{z}_1 + \frac{1}{\bar{u}}\frac{\partial \bar{u}}{\partial z_2} dz_2 + \frac{1}{u}\frac{\partial u}{\partial \bar{z}_2} d\bar{z}_2,$$

while the Lee form of J_3 is

$$\theta = 2\frac{\partial}{\partial z_1} \ln |u| dz_1 + 2\frac{\partial}{\partial \bar{z}_1} \ln |u| d\bar{z}_1.$$

Moreover, J_1, J_2, J_3 are integrable if and only if $\tau = \theta$, which is equivalent to u being holomorphic.

Thus, if u is holomorphic, $\{J_1, J_2, J_3\}$ define a hyper-Hermitian structure on (U, g_u). In that case, the Weyl structure defined by each of $\{J_1, J_2, J_3\}$ is closed if and only if $d\theta = 0$, which is equivalent to

$$u^2 \frac{\partial^2 u}{\partial z_1 \partial z_2} = \frac{\partial u}{\partial z_1}\frac{\partial u}{\partial z_2}. \tag{9.8}$$

By Theorem 9.17, we deduce that if U is simply connected and u is a non-zero holomorphic function on U which *does not* satisfy (9.8), then (U, g_u) carries parallel spinors with respect to the non-closed Weyl structure defined above.

9.2.3 The compact case

In this section we will provide a complete solution to Problem 9.4 under the assumption that M is compact. The first step is to note that, in the compact case, Theorem 9.8 also holds for $n = 4$. Indeed, if we express equation (9.4) in a fixed metric, Lemma 9.9 implies that the self-dual part of the Faraday form F of D vanishes. Since F is exact, the compactness of M easily implies $F = 0$. This proves

Corollary 9.19. *Let (M^n, c), $n > 2$, be a compact conformal spin manifold with Weyl structure D, carrying a D-parallel spinor Ψ. Then D is closed.*

Actually, much more can be said on such manifolds.

Theorem 9.20. *A conformal manifold (M, c) with Weyl structure D carries a D-parallel spinor if and only if there exists a metric g in the conformal class c (which turns out to be unique up to a constant rescaling) such that one of the two statements below holds:*

- *$D = \nabla^g$ and (M, g) is a spin manifold with parallel spinors;*
- *the universal cover of (M, g) is isometric to $N \times \mathbb{R}$, where N is a Riemannian manifold with Killing spinors, and D is the Weyl structure on M whose Lee form with respect to g is dt.*

Proof. The main ingredient of the proof is the use of the *standard* or *Gauduchon metric* associated with every Weyl structure (in dimension greater than 2). Recall the following theorem.

Theorem 9.21 ([Gau84]). *Let (M, c) be a compact conformal manifold of dimension $n > 2$ and let D be a Weyl structure on M. Then c contains a unique metric g_0 (up to homothety) such that the Lee form of D with respect to g_0 is g_0-coclosed. This metric is called the standard metric associated to D.*

Now, Corollary 9.19 together with (9.5) show that D is a Weyl–Einstein structure (see Definition 2.17). Let g be the standard metric of D, and denote by θ the Lee form of D with respect to g. A fundamental observation by P. Tod is that the Lee form of a closed Weyl–Einstein structure on a compact manifold is parallel with respect to the standard metric (cf. [Gau95b]). We distinguish 2 cases.

If the Lee form θ vanishes, then $D = \nabla^g$ and the D-parallel spinor of $\Sigma^{(0)}(M, c)$ induces a parallel spinor on $\Sigma(M, g)$.

The second case is when θ is a non-zero parallel form on (M, g). By rescaling the metric, we can suppose that θ has unit norm. Let $\widetilde{M}$ be the universal cover of M, which inherits by pull-back the Weyl structure D, the D-parallel spinor and the metric g from M. By the de Rham decomposition theorem, $(\widetilde{M}, g)$ is isometric to $N \times \mathbb{R}$ for some Riemannian (spin) manifold N. With respect to this identification, we have of course $\theta = dt$. Let Ψ be the D-parallel spinor field, expressed in the metric g. By Theorem 2.21, it satisfies

$$\nabla^g_X \Psi = \frac{1}{2} X \cdot \theta \cdot \Psi + \frac{1}{2} \theta(X) \Psi.$$

As before, by changing the orientation of M if necessary, we can suppose that, for n even, Ψ is a non-zero section of $\Sigma^+ M$. Then Theorem 2.38 shows that (for every n), by restriction Ψ induces a well-defined spinor on N, which, by (2.27) and (2.28), is a Killing spinor with Killing constant $\frac{1}{2}$. Conversely, the same equations show that a manifold satisfying one of the two cases above has a Weyl structure with parallel spinors. This proves the theorem. □

9.3 A conformal proof of the Hijazi inequality

In this section we give a proof of the Hijazi inequality which makes essential use of conformal geometry.

Theorem 9.22. *Let (M^n, g) be a compact Riemannian spin manifold of dimension $n \geq 3$. Then, the first eigenvalue λ_1 of the Dirac operator on the spinor bundle satisfies*

$$\lambda_1^2 \geq \frac{n}{4(n-1)} \mu_1,$$

where μ_1 is the first eigenvalue of the scalar curvature operator L_g defined by

$$\mathrm{L}_g = 4 \frac{n-1}{n-2} \Delta_g + \mathrm{Scal}_g.$$

Proof. Consider an arbitrary conformal change of the metric given by $\bar{g} = f^{-2} g$. We have defined in Section 2.2.3 a family of isomorphisms

$$\Phi^{(k)} \colon \Sigma^g M \longrightarrow \Sigma^{\bar{g}} M,$$

which satisfy (2.13).

Every choice of a Weyl structure on $(M, [g])$ induces Dirac and Penrose operators, $\mathcal{D}^g$ and $\mathcal{P}^g$, acting on $\Sigma^g M$, as well as *weighted Dirac and Penrose operators*, $\mathcal{D}$ and $\mathcal{P}$, which act on the weighted spinor bundles and lower the weights by 1.

Moreover, for $k = -(n-1)/2$ (resp. $k = 1/2$) the weighted Dirac operator (resp. Penrose operator) is conformally invariant (i.e., does not depend on the choice of the Weyl structure), by Theorem 9.1. As a consequence, this yields

$$\mathcal{D}^g = (\Phi^{(-\frac{1}{2}(1+n))})^{-1} \circ \mathcal{D}^{\bar{g}} \circ \Phi^{(\frac{1}{2}(1-n))},$$

and

$$(\mathcal{P}^g)_X = (\Phi^{(\frac{1}{2})})^{-1} \circ (\mathcal{P}^{\bar{g}})_X \circ \Phi^{(\frac{1}{2})}.$$

The Schrödinger–Lichnerowicz formula relating the Dirac and Penrose operators associated to a metric g can be written as

$$\frac{n-1}{n}(\mathcal{D}^g)^*(\mathcal{D}^g)\psi - \frac{1}{4}\mathrm{Scal}_g\psi = (\mathcal{P}^g)^*(\mathcal{P}^g)\psi.$$

By integration over M with respect to the volume element ν_g we get

$$\int_M \Big(\frac{n-1}{n}|\mathcal{D}^g\psi|^2 - \frac{1}{4}\mathrm{Scal}_g|\psi|^2\Big)\nu_g = \int_M |\mathcal{P}^g(\psi)|_g^2\nu_g.$$

This formula also holds for the metric $\bar{g} = f^{-2}g$ and the spinor $\overline{\psi} = \Phi(\psi)$, where Φ is the isomorphism $\Phi^{(\frac{1-n}{2})}$ above. Hence

$$\int_M \Big(\frac{n-1}{n}|\mathcal{D}^{\bar{g}}\overline{\psi}|^2 - \frac{1}{4}\mathrm{Scal}_{\bar{g}}|\overline{\psi}|^2\Big)\nu_{\bar{g}} = \int_M |\mathcal{P}^{\bar{g}}(\overline{\psi})|_{\bar{g}}^2\nu_{\bar{g}}, \tag{9.9}$$

On the other hand, we obviously have

$$\nu_{\bar{g}} = f^{-n}\nu_g, \quad |\mathcal{D}^{\bar{g}}\overline{\psi}|_{\bar{g}}^2 = f^{n+1}|\mathcal{D}^g\psi|_g^2, \quad |\mathcal{P}^{\bar{g}}\overline{\psi}|_{\bar{g}}^2 = f|\mathcal{P}^g(f^{\frac{n}{2}}\psi)|_g^2,$$

and moreover, by 5.19,

$$\mathrm{Scal}_{\bar{g}} = h^{-\frac{n+2}{n-2}}\mathrm{L}_g h,$$

where $h = f^{-\frac{n-2}{2}}$. Plugging these formulas into (9.9) yields

$$\begin{aligned}\int_M \Big(\frac{n-1}{n}|\mathcal{D}^g\psi|^2 - \frac{1}{4}h^{-1}(\mathrm{L}_g h)|\psi|^2\Big)h^{-\frac{2}{n-2}}\nu_g \\ = \int_M f^{1-n}|\mathcal{P}^g(f^{\frac{n}{2}}\psi)|_g^2\nu_g.\end{aligned} \tag{9.10}$$

Let us now choose a spinor ψ and a function h such that

$$\mathcal{D}^g\psi = \lambda_1\psi \quad \text{and} \quad \mathrm{L}_g h = \mu_1 h.$$

By the maximum principle, h is everywhere strictly positive, so (9.10) yields

$$\lambda_1^2 \geq \frac{n}{4(n-1)}\mu_1$$

as required. □

Chapter 10

Special spinors on Kähler manifolds

It was pointed out in Section 5.6 that Friedrich's inequality (see [Fri80]) is not sharp on Kähler manifolds, for which better estimates hold (see Kirchberg's inequalities, Theorem 6.1). In this chapter we show that Kirchberg's inequalities are sharp, we characterize their limiting cases and we give an application in complex contact geometry.

The first section is devoted to twistor theory, which will play a central role in the sequel. The next section treats the limiting case of Kirchberg's inequality in odd complex dimensions, following closely [Mor95]. An important application of this result – the characterization of positive Kähler–Einstein contact manifolds – is given in Section 10.3, making use of Spin^c geometry.

The last section in this chapter deals with the limiting case of Kirchberg's inequality in even complex dimensions. This problem was first studied by A. Lichnerowicz [Lic90] and the complete solution was obtained in [Mor97a] and [Mor99].

10.1 An introduction to the twistor correspondence

The twistor theory of self-dual Einstein 4-manifolds was introduced by R. Penrose [PM73] and [Pen76] and further studied by Atiyah, Hitchin, and Singer in their seminal paper [AHS78]. A few years later, this theory was extended both by S. Salamon [Sal82] and L. Bérard Bergery [Ber79] to quaternion-Kähler manifolds and turned out to be closely related with the Ishihara–Konishi theory of 3-Sasakian bundles over positive quaternion-Kähler manifolds.

The aim of this section is to briefly recall the interplay of these theories. For further details the reader is referred to the surveys [BG99] and [Sal99].

10.1.1 Quaternion-Kähler manifolds

Quaternion-Kähler Manifolds were introduced in Chapter 7. For the convenience of the reader we briefly review the basic definitions and notations.

Let $\mathrm{GL}_n(\mathbb{H})$ be the algebra of $n \times n$ matrices with quaternionic entries, acting on the left on $\mathbb{H}^n$. Using the canonical identification $\mathbb{H}^n \cong \mathbb{R}^{4n}$, the algebra $\mathrm{GL}_n(\mathbb{H})$

becomes a sub-algebra of $\mathrm{GL}_{4n}(\mathbb{R})$. Let Sp_n be the subgroup of SO_{4n} defined by

$$\mathrm{Sp}_n = \mathrm{SO}_{4n} \cap \mathrm{GL}_n(\mathbb{H})$$

and consider the embedding of the group Sp_1 of unit quaternions into SO_{4n} given by

$$q \in \mathrm{Sp}_1 \longrightarrow A(q) \in \mathrm{SO}_{4n}, \quad A(q)v := vq^{-1}, \quad v \in \mathbb{R}^{4n}.$$

We define $\mathrm{Sp}_1 \cdot \mathrm{Sp}_n$ to be the group generated by these two subgroups of SO_{4n}. It is easy to check that $\mathrm{Sp}_1 \cdot \mathrm{Sp}_n \cong \mathrm{Sp}_1 \times_{\mathbb{Z}/2\mathbb{Z}} \mathrm{Sp}_n$. Note that $\mathrm{Sp}_1 \cdot \mathrm{Sp}_1 \cong \mathrm{SO}_4$. From now on we assume $n \geq 2$.

Definition 10.1. A *quaternion-Kähler manifold* is a $4n$-dimensional Riemannian manifold M $(n \geq 2)$ satisfying one of the following equivalent conditions:

- The holonomy of M is contained in $\mathrm{Sp}_1 \cdot \mathrm{Sp}_n$.
- There exists a parallel 3-dimensional sub-bundle $\mathcal{Q}M$ of $\mathrm{End}(M)$, called the *quaternionic bundle*, locally spanned by three endomorphisms I, J, and K satisfying the quaternion relations.

The equivalence of the two conditions follows from the fact that the vector subspace of $\mathrm{End}(\mathbb{H}^n)$ given by right multiplication with imaginary quaternions is left invariant by the adjoint action of $\mathrm{Sp}_1 \cdot \mathrm{Sp}_n$.

Note that the local endomorphisms I, J, and K do not extend, in general, to the whole manifold, unless the holonomy of M is contained in Sp_n.

Theorem 10.2 ([Ber66]). *Every quaternion-Kähler manifold is Einstein.*

Proof. The argument goes roughly as follows. If $G \subset \mathrm{SO}_n$ is the holonomy group of a Riemannian manifold, then its curvature tensor at every point belongs to the space $R(\mathfrak{g})$ of algebraic curvature tensors, defined as the kernel of the Bianchi map

$$b\colon \mathrm{Sym}^2(\mathfrak{g}) \longrightarrow \Lambda^4(\mathbb{R}^n), \quad b(\omega \odot \omega) = \omega \wedge \omega.$$

The theorem then follows from the fact that

$$R(\mathfrak{sp}_1\mathfrak{sp}_n) = R(\mathfrak{sp}_n) \oplus \mathbb{R},$$

where the one-dimensional summand $\mathbb{R}$ is generated by the curvature tensor of the quaternionic projective space $\mathbb{H}\mathrm{P}^n$. □

Depending on the value of the Einstein constant, one can characterize more precisely quaternion-Kähler manifolds in terms of holonomy: if the Einstein constant vanishes, then the manifold is *hyper-Kähler* and its restricted holonomy is a subgroup of Sp_n. If the Einstein constant is non-zero, then the manifold is irreducible, and its restricted holonomy group is a subgroup of $\mathrm{Sp}_1 \cdot \mathrm{Sp}_n$ not contained in Sp_n.

A quaternion-Kähler manifold is called *positive* if it has positive scalar curvature.

10.1.2 3-Sasakian structures

We now focus our attention on 3-Sasakian structures (which were introduced in Section 8.2.2). If $(S, g, \xi_1, \xi_2, \xi_3)$ is a 3-Sasakian manifold, the distribution $\mathcal{V}$ spanned by ξ_1, ξ_2, and ξ_3 is integrable by (8.11), and totally geodesic by (8.13). This last equation shows, moreover, that the integral leaves of $\mathcal{V}$ have constant sectional curvature 1. If S is compact, every maximal integral leaf is therefore isometric to some lens space $\mathbb{S}^3/\Gamma$. This induces, in particular, an isometric $\mathbb{S}^3$-action on every compact 3-Sasakian manifold.

Recall that the action of a Lie group G on a smooth manifold is called *regular* if the quotient M/G is smooth. A compact 3-Sasakian manifold is called *regular* if the $\mathbb{S}^3$-action defined above is regular.

There is a close relation between regular 3-Sasakian and positive quaternion-Kähler manifolds, provided by the following theorem.

Theorem 10.3 ([Ish73], [Ish74], [IK72], and [Kon75]). *The quotient of a* regular 3-*Sasakian manifold S by the above-defined isometric $\mathbb{S}^3$ action is a positive quaternion-Kähler manifold. Conversely, if M is a positive quaternion-Kähler manifold, the* SO_3*-principal bundle S associated with the quaternionic bundle $\mathcal{Q}M$* (*cf. Definition* 10.1) *carries a* 3-*Sasakian metric which turns the bundle projection $S \to M$ into a Riemannian submersion with totally geodesic fibers.*

Proof. If S is a 3-Sasakian manifold, the quotient $M := S/\mathbb{S}^3$ is exactly the space of leaves of the distribution $\mathcal{V}$. In the regular case, the pull-back of the tangent bundle of M by the submersion $\pi\colon S \to M$ can be identified with $\mathcal{V}^\perp$. Every tensor on $\mathcal{V}^\perp$ "constant along $\mathcal{V}$" (i.e., whose Lie derivative with respect to ξ_i vanishes) induces by projection a tensor on M.

In particular, the restriction of the metric of S to $\mathcal{V}^\perp$ induces a metric on M making π a Riemannian submersion. The restrictions to $\mathcal{V}^\perp$ of the endomorphisms φ_i introduced in Definition 8.14 satisfy the quaternion relations, but do not project to M. Nevertheless, every local section σ of the fibration $S \to M$ induces a local quaternionic structure $\{I^\sigma, J^\sigma, K^\sigma\}$ on M by

$$I^\sigma_x := \pi_*((\varphi_1)_{\sigma(x)}), \qquad J^\sigma_x := \pi_*((\varphi_2)_{\sigma(x)}), \qquad K^\sigma_x := \pi_*((\varphi_3)_{\sigma(x)}).$$

It remains to check (using the properties of 3-Sasakian structures and the O'Neill formulas for Riemannian submersions) that M is Einstein with positive scalar curvature and that the sub-bundle of $\mathrm{End}(\mathrm{T}M)$ spanned by I^σ, J^σ and K^σ does not depend on σ and is parallel with respect to the Levi-Civita connection of M. The converse is similar. □

The above result shows that the study of (regular) 3-Sasakian manifolds is equivalent to that of positive quaternion-Kähler manifolds. For later use, let us mention the following lemma.

Lemma 10.4. *A compact* 3*-Sasakian manifold is regular if and only if at least one of the Sasakian structures is regular.*

Proof. Assume that $(S, g, \xi_1, \xi_2, \xi_3)$ is regular and let $\pi: S \to M$ denote the projection of S onto its quotient by the $\mathbb{S}^3$ action. The manifold M is quaternion-Kähler by Theorem 10.3. Let Z denote the sphere bundle of the quaternionic bundle $\mathcal{Q}M$ of M. We claim that the map $f_1: S \to Z$ given by $x \mapsto \pi_*((\varphi_1)_x)$ is a submersion whose fibers are exactly the integral curves of ξ_1. This follows easily from the fact that $\xi_1(f_1) = \pi_*(\mathcal{L}_{\xi_1}\varphi_1) = 0$ (showing that f_1 is constant along the flow of ξ_1) and from the fact that

$$(a\xi_2 + b\xi_3)(f_1) = \pi_*(\mathcal{L}_{a\xi_2+b\xi_3}\varphi_1) = \pi_*(-2a\varphi_3 + 2b\varphi_2),$$

is non-zero, whenever $(a, b) \neq (0, 0)$. For the converse statement, see [Tan71]. □

10.1.3 The twistor space of a quaternion-Kähler manifold

The manifold Z introduced in the proof of Lemma 10.4 turns out to play a central role in the theory of quaternion-Kähler manifolds.

Definition 10.5 ([Sal82] and [Ber79]). The unit sphere bundle Z of the quaternionic bundle $\mathcal{Q}M$ over a quaternion-Kähler manifold M is called the *twistor space* of M. The corresponding fibration $\pi: Z \to M$ is called the *twistor fibration*.

Theorem 10.6 ([Sal82] and [Ber79]). *Let M be a quaternion-Kähler manifold and Z its twistor space.*

(i) *Z admits a natural complex structure such that the fibers of the twistor fibration are compact complex curves of genus* 0.

(ii) *If M has non-zero scalar curvature, Z carries a natural complex contact structure* (*cf. Definition* 10.9 *below*).

(iii) *If M has positive scalar curvature, Z carries a natural Kähler–Einstein metric such that the twistor fibration becomes a Riemannian submersion with totally geodesic fibers.*

We will not give the proof of these results here (the interested reader is referred to the original articles or to [Sal99] for details).

As a consequence of the classification, in the next section, of manifolds with Kählerian Killing spinors, we will obtain the converse of Theorem 10.6, stating that every compact Kähler–Einstein manifold with positive scalar curvature admitting a complex contact structure is the twistor space of some positive quaternion-Kähler manifold (cf. Theorem 10.11 below and [Mor98b]).

10.2 Kählerian Killing spinors

In this section we study Kählerian Killing spinors on spin and Spin^c Kähler manifolds. Unless otherwise stated, we use the notations of Chapter 6.

Definition 6.13 has the following Spin^c counterpart.

Definition 10.7. Let (M^{4k+2}, g, J) be a Kähler manifold of odd complex dimension $m = 2k+1$ endowed with a Spin^c structure with Spin^c connection ∇^A. A Spin^c *Kählerian Killing spinor* is a section Ψ of the spinor bundle of M satisfying

$$\nabla^A_X \Psi = \frac{1}{2} X \cdot \Psi + \frac{i\epsilon}{2} JX \cdot \bar{\Psi}, \quad X \in \mathrm{T}M, \tag{10.1}$$

where $\epsilon := (-1)^k$.

In order to state the main result, we recall that a *Hodge manifold* is a compact Kähler manifold with the property that the cohomology class of the Kähler form is a real multiple of an integral class.

Theorem 10.8 ([Mor98b]). *Let (M^{4k+2}, g, J) be a simply connected Hodge manifold carrying a non-zero* Spin^c *Kählerian Killing spinor $\Psi \in \Gamma(\Sigma_k M \oplus \Sigma_{k+1} M)$. Then*

(i) *if k is even, M is isometric to $\mathbb{C}\mathrm{P}^{2k+1}$;*

(ii) *if k is odd, either M is isometric to $\mathbb{C}\mathrm{P}^{2k+1}$, or the* Spin^c *structure on M is actually a spin structure and M is isometric to the twistor space over a positive quaternion-Kähler manifold.*

Proof. The proof goes roughly as follows. Using the theory of projectable spinors for Riemannian submersions from Section 2.4.3, one gets a real Killing spinor on an $\mathbb{S}^1$-bundle S over M endowed with some natural metric and Spin^c structure induced from M. This spinor induces a parallel spinor on the Riemannian cone $\bar{S}$ over S, lying in the kernel of the Clifford multiplication with the Kähler form of $\bar{S}$. Using the classification of Spin^c manifolds carrying parallel spinors (Theorem 8.12) and the irreducibility of $\bar{S}$, we must have on the one hand that the Spin^c structure is a spin structure, and then that $\bar{S}$ is hyper-Kähler, which easily translates into the geometrical characterization for M stated in the theorem.

Let Ω be Kähler form of M, defined as usual by $\Omega(X, Y) := g(JX, Y)$. Since M is Hodge, there exists some $s \in \mathbb{R}^*$ such that $s[\Omega] \in H^2(M, \mathbb{Z})$. We fix $s > 0$ of least absolute value with this property. The isomorphism $H^2(M, \mathbb{Z}) \simeq H^1(M, \mathbb{S}^1)$ (cf. Proposition 3.2) guarantees the existence of some $\mathbb{S}^1$-principal bundle $\pi: S \to M$ whose first Chern class satisfies $c_1(S) = s\epsilon[\Omega]$. We claim that

S is simply connected. To see this, consider like in [BFGK91], Example 1, p. 84, the Thom–Gysin exact sequence for the $\mathbb{S}^1$-fibration π:

$$\cdots \longrightarrow H^{2m-2}(M;\mathbb{Z}) \xrightarrow{\cup c_1(S)} H^{2m}(M;\mathbb{Z}) \\ \xrightarrow{\pi^*} H^{2m}(S;\mathbb{Z}) \longrightarrow H^{2m-1}(M;\mathbb{Z}) \longrightarrow \cdots .$$

The last arrow is just the cap product with the homology class of a fibre. By assumption M is simply connected, so $H^{2m-1}(M;\mathbb{Z}) = H_1(M;\mathbb{Z}) = 0$. Moreover, the strong Poincaré duality implies that the first arrow in the above exact sequence is onto (since $c_1(S)$ has no integral quotient). This shows that $H^{2m}(S;\mathbb{Z}) = H_1(S;\mathbb{Z}) = 0$. In other words, the center of $\pi_1(S)$ is trivial. On the other hand, the exact homotopy sequence of the $\mathbb{S}^1$-fibration $S \to M$ shows that $\pi_1(S)$ is a quotient of $\mathbb{Z}$, so in particular it is commutative. This proves our claim.

By construction, the $\mathbb{S}^1$-bundle S carries a connection whose curvature form is $2\pi i s\epsilon\Omega$. With the notations from Section 2.4.3 we then have $F = s\epsilon\Omega$ and $T = s\epsilon J$. Theorem 2.45 says that for every positive number ℓ, there exists a Riemannian metric on S such that $\pi\colon S \to M$ becomes a Riemannian submersion, with totally geodesic fibres of length ℓ, and such that the pull-back to S of the $\mathrm{Spin^c}$ structure on M defines a $\mathrm{Spin^c}$ structure with connection $\overline{\nabla}^A$ satisfying the following relations:

$$X^* \cdot (\pi^*\Psi) = \pi^*(X \cdot \Psi), \quad X \in \mathrm{T}M, \tag{10.2}$$

$$V \cdot (\pi^*\Psi) = i\pi^*(\bar{\Psi}), \tag{10.3}$$

$$\overline{\nabla}^A_{X^*}\pi^*\Psi = \pi^*\Big(\nabla^A_X\Psi - \frac{i\ell}{4}TX \cdot \bar{\Psi}\Big), \tag{10.4}$$

$$\overline{\nabla}^A_V\pi^*\Psi = \frac{\ell}{4}\pi^*(F \cdot \Psi). \tag{10.5}$$

If we choose $\ell = \frac{2}{s}$, then (10.1), (10.2), and (10.4) imply that

$$\overline{\nabla}^A_{X^*}\pi^*\Psi = \frac{1}{2}X^* \cdot \pi^*\Psi.$$

On the other hand, since Ψ is a section of $\Sigma_k M \oplus \Sigma_{k+1} M$, we have

$$\Omega \cdot \Psi = i\epsilon\bar{\Psi}, \tag{10.6}$$

so from (6.3), (10.3), and (10.5) we get

$$\overline{\nabla}^A_V\pi^*\Psi = \frac{\ell}{4}\pi^*(F \cdot \Psi) = \frac{1}{2s}\pi^*(s\epsilon\Omega \cdot \Psi)\frac{i}{2}\pi^*(\bar{\Psi}) = \frac{1}{2}V \cdot \pi^*\Psi,$$

hence $\pi^*\Psi$ is a Spin^c Killing spinor on S. Moreover, if ξ denotes the 1-form dual to the vertical Killing vector field V, then Proposition 2.43 shows that

$$d\xi = \ell F = 2\epsilon\pi^*\Omega.$$

From (10.6) and (10.3) we get

$$\pi^*\Omega \cdot \pi^*\Psi = \epsilon V \cdot \pi^*\Psi. \tag{10.7}$$

Now consider the Riemannian cone $\overline{S} := \mathbb{R}_+^* \times S$ with the metric $dt^2 + t^2 g^S$ and with the pull-back Spin^c structure from S. Theorem 2.41 shows that the Spin^c Killing spinor $\pi^*\Psi$ on S induces a parallel Spin^c spinor Φ on $\Sigma^{-\epsilon}\overline{S}$. The Kähler form $\overline{\Omega}$ of $\overline{S}$ is given by the formula $\overline{\Omega} = t^2 d\xi - 2t\,dt \wedge \xi$. We claim that $\overline{\Omega}\cdot\Phi = 0$. It is enough to show this relation along the submanifold $S = \{1\} \times S$ of $\overline{S}$, since both $\overline{\Omega}$ and Φ are parallel on $\overline{S}$. By restricting to S and using (2.30), we get

$$\overline{\Omega}\cdot\Phi|_S = d\xi\cdot\pi^*\Psi + 2\xi \wedge dt \cdot \Phi|_S = 2\epsilon\pi^*\Omega\cdot\pi^*\Psi - 2\epsilon V\cdot\pi^*\Psi \overset{(10.7)}{=} 0.$$

This shows that $\overline{S}$ is a Kähler manifold which carries a parallel spinor whose Clifford product with the Kähler form vanishes.

Assume from now on that M is not isometric to $\mathbb{C}\mathrm{P}^{2k+1}$. Then S is not isometric to the round sphere, and $\overline{S}$ is not flat. Lemma 8.34 thus implies that $\overline{S}$ is irreducible.

From Theorem 8.12 we then deduce that either the Spin^c structure of $\overline{S}$ is actually a spin structure, or there exists a Kähler structure I on $\overline{S}$ such that

$$X\cdot\Phi = iI(X)\cdot\Phi, \qquad X \in \mathrm{T}\overline{S}, \tag{10.8}$$

and the Spin^c structure of $\overline{S}$ is the canonical Spin^c structure induced by I (these two cases do not exclude each other).

Case I. If the Spin^c structure of $\overline{S}$ is a spin structure, Theorem 8.1 shows that the holonomy of $\overline{S}$ is either SU_{2k+2} or Sp_{k+1}. Recall that by (6.7) the spinor bundle of the Ricci-flat Kähler manifold $\overline{S}$ can be written as

$$\Sigma\overline{S} = \Lambda^{0,0} \oplus \Lambda^{0,1} \oplus \cdots \oplus \Lambda^{0,2k+2},$$

and the Clifford action of the Kähler form $\overline{\Omega}$ on each factor $\Lambda^{0,p}$ is the scalar $i(2k+2-p)$. Since $\overline{\Omega}\cdot\Phi = 0$, Φ is a section of $\Lambda^{0,k+1}$. Now, it is easy to see that SU_{2k+2} has no fixed point in $\Lambda^{0,k+1}$, and Sp_{k+1} has a fixed point in $\Lambda^{0,k+1}$ if and only if k is odd.

We thus obtain that k is odd and $\overline{S}$ is hyper-Kähler. In order to complete the classification, it remains to interpret this latter condition in terms of the original manifold M. From Theorem 8.20 we obtain directly that S has to be 3-Sasakian. Moreover, if S is not isometric to the round sphere, the vector field V defines one of the Sasakian structures. To see this, we consider the Kähler structure induced by V on $\overline{S}$. If this structure does not belong to the 2-sphere of Kähler structures defined by the hyper-Kähler structure, then this would provide a further reduction of the holonomy of $\overline{S}$, a contradiction. Thus M is the quotient of the 3-Sasakian manifold S by the flow of one of the Sasakian vector fields. By Lemma 10.4 and Theorem 10.3, M is the twistor space of a positive quaternion-Kähler manifold.

Case II. If $\overline{S}$ carries a Kähler structure I satisfying (10.8) (in addition to the original Kähler structure, say $\bar{J}$, defined by $\bar{\Omega}$), we will show that $\overline{S}$ has to be reducible, and thus obtain a contradiction.

Taking the Clifford product with $\bar{\Omega}$ in (10.8) and using (6.3) yields

$$\bar{J}X \cdot \Phi = i\bar{J}I(X) \cdot \Phi, \quad X \in \mathrm{T}\overline{S}, \tag{10.9}$$

so replacing X by $\bar{J}X$ in (10.8) and using (10.9) shows that $I\bar{J} = \bar{J}I$. Now it is clear that $I \neq \pm\bar{J}$ since $\bar{\Omega} \cdot \Phi = 0$ and, by (10.8), $\Omega_I \cdot \Phi = 2i(k+1)\Phi$, where Ω_I denotes the Kähler form of I.

On the other hand, $I\bar{J}$ is a symmetric parallel involution of $\mathrm{T}\overline{S}$, so the decomposition $\mathrm{T}\overline{S} = \mathrm{T}^+S \oplus \mathrm{T}^-S$, where $\mathrm{T}^{\pm}S := \{X;\ I\bar{J}X = \pm X\}$ gives a holonomy reduction of $\overline{S}$. This finishes the proof of the theorem. □

10.3 Complex contact structures on positive Kähler–Einstein manifolds

As an application of Theorem 10.8, we will describe in this section the classification of positive Kähler–Einstein contact manifolds obtained by C. LeBrun [LeB95] and by A. Moroianu and U. Semmelmann [KS95], [MS96], and [Mor98b].

Definition 10.9 ([Kob59]). Let M^{2m} be a complex manifold of complex dimension $m = 2k+1$. A *complex contact structure* is a family $\mathcal{C} = \{(U_i, \omega_i)\}$ satisfying the following conditions:

(i) $\{U_i\}$ is an open covering of M.

(ii) ω_i is a holomorphic 1-form on U_i.

(iii) $\omega_i \wedge (\partial\omega_i)^k \in \Gamma(\Lambda^{m,0}M)$ is different from zero at every point of U_i.

(iv) $\omega_i = f_{ij}\omega_j$ on $U_i \cap U_j$, where f_{ij} is a holomorphic function on $U_i \cap U_j$.

Let $\mathcal{C} = \{(U_i, \omega_i)\}$ be a complex contact structure. Then there exists an associated holomorphic line sub-bundle $L_{\mathcal{C}} \subset \Lambda^{1,0}M$, with transition functions $\{f_{ij}^{-1}\}$ and local sections ω_i. It is easy to see that

$$\{Z \in \mathrm{T}^{1,0}M \colon \omega(Z) = 0, \text{ for all } \omega \in L_{\mathcal{C}}\}$$

is a codimension 1 maximally non-integrable holomorphic sub-bundle of $\mathrm{T}^{1,0}M$, and conversely, every such bundle defines a complex contact structure. From condition (iii) immediately follows the isomorphism $L_{\mathcal{C}}^{k+1} \cong K$, where $K = \Lambda^{m,0}M$ denotes the canonical bundle of M.

Let M be a compact Kähler–Einstein manifold of odd complex dimension $m = 2k + 1$ with positive scalar curvature, admitting a complex contact structure $\mathcal{C}$. By rescaling the metric on M if necessary, we can suppose that the scalar curvature of M is equal to $2m(2m + 2)$, and thus the Ricci form ρ and the Kähler form Ω are related by $\rho = (2m + 2)\Omega$. We will denote by l the integer part of $\frac{k}{2}$, and let $k = 2l + \delta$, where δ is 0 or 1. Our main goal is to construct, starting from the complex contact structure $\mathcal{C}$, a Kählerian Killing spinor on M. This can only be done for k odd. When k is even we will actually construct a $\mathrm{Spin}^{\mathrm{c}}$ Kählerian Killing spinor for a certain $\mathrm{Spin}^{\mathrm{c}}$ structure on M, determined by $\mathcal{C}$.

Theorem 10.10. *Let $\mathcal{C}$ be a complex contact structure on a Kähler–Einstein manifold (M, g, J) of odd complex dimension $m = 2k + 1$.*

(i) *If k is odd, then M is spin and carries a Kählerian Killing spinor $\Psi \in \Gamma(\Sigma_k M \oplus \Sigma_{k+1} M)$.*

(ii) *If k is even, then the* $\mathrm{Spin}^{\mathrm{c}}$ *structure on M with auxiliary bundle $L_{\mathcal{C}}$ carries a Kählerian Killing spinor $\Psi \in \Gamma(\Sigma_{k+1} M \oplus \Sigma_{k+2} M)$.*

Proof. The collection $(U_i, \omega_i \wedge (\partial\omega_i)^l)$ defines a holomorphic line bundle $L_l \subset \Lambda^{2l+1,0}M$, and from the definition of $\mathcal{C}$ we easily obtain

$$L_l \cong L_{\mathcal{C}}^{l+1}. \tag{10.10}$$

We now fix some $(U, \omega) \in \mathcal{C}$ and define a local section $\Psi_{\mathcal{C}}$ of $\Lambda^{0,2l+1}M \otimes L_{\mathcal{C}}^{l+1}$ by

$$\Psi_{\mathcal{C}}|_U := |\xi_\tau|^{-2}\bar{\tau} \otimes \xi_\tau,$$

where $\tau := \omega \wedge (\partial\omega)^l$ and ξ_τ is the element corresponding to τ through the isomorphism (10.10). The fact that $\Psi_{\mathcal{C}}$ does not depend on the element $(U, \omega) \in \mathcal{C}$ shows that it actually defines a global section $\Psi_{\mathcal{C}}$ of $\Lambda^{0,2l+1}M \otimes L_{\mathcal{C}}^{l+1}$.

We now recall that $\Lambda^{0,*}M$ is just the spinor bundle associated to the canonical $\mathrm{Spin}^{\mathrm{c}}$ structure on M, whose auxiliary line bundle is K^{-1}, so that $\Lambda^{0,*}M \otimes L_{\mathcal{C}}^{l+1}$ is actually the spinor bundle associated to the $\mathrm{Spin}^{\mathrm{c}}$ structure on M with auxiliary bundle

$$L = K^{-1} \otimes L_{\mathcal{C}}^{2l+2} \cong L_{\mathcal{C}}^{-k-1} \otimes L_{\mathcal{C}}^{2l+2} \cong L_{\mathcal{C}}^{1-\delta}.$$

The section $\Psi_{\mathcal{C}}$ is thus a usual spinor for k odd, and a Spin^c spinor for k even, and is always a section of $\Sigma_{2l+1}M$. In particular,

$$\Omega \cdot \Psi_{\mathcal{C}} = i(m - 4l - 2)\Psi_{\mathcal{C}} = i(2\delta - 1)\Psi_{\mathcal{C}}. \tag{10.11}$$

We claim that the spinor field $\Psi_{\mathcal{C}}$ satisfies $\nabla_Z \Psi_{\mathcal{C}} = 0$, for all $Z \in \Gamma(\mathrm{T}^{1,0}M)$ (this shows, in particular, that $\mathcal{D}_-\Psi_{\mathcal{C}} = 0$), and

$$\mathcal{D}^2\Psi_{\mathcal{C}} = \mathcal{D}_-\mathcal{D}_+\Psi_{\mathcal{C}} = \frac{l+1}{k+1}\Big(\frac{1}{2}\,\mathrm{Scal}\,\Psi_{\mathcal{C}} - i\rho \cdot \Psi_{\mathcal{C}}\Big), \tag{10.12}$$

where as before Scal denotes the scalar curvature of M.

This claim actually follows directly from [KS95], Proposition 5, keeping in mind that we treat simultaneously the cases k even and odd, so the coefficients $1/2$ in formulas (8) and (9) of [KS95] have to be replaced by $\frac{l+1}{k+1}$.

Using (10.11), (10.12), and the fact that

$$\rho = \frac{1}{2m}\,\mathrm{Scal}\,\Omega = (2m+2)\Omega,$$

we obtain that the spinor field $\Psi_{\mathcal{C}}$ is an eigenspinor of $\mathcal{D}^2$ with respect to the eigenvalue $16(l+1)(l+\delta)$.

Note that

$$\frac{m+1}{4m}\mathrm{Scal} = (m+1)^2.$$

Thus, for $\delta = 1$ (i.e., for k odd) we get $m+1 = 4l+4$, so we are in the limiting case of Kirchberg's inequality on compact Kähler manifolds of odd complex dimension. By Theorem 6.14, the spinor field $(4l+4)\Psi_{\mathcal{C}} + \mathcal{D}\Psi_{\mathcal{C}}$ is a Kählerian Killing spinor on M.

The case where k is even is more delicate, as there are no Spin^c analogues of Kirchberg's inequalities. One thus needs an *ad hoc* argument to prove that the Spin^c spinor $(4l+4)\Psi_{\mathcal{C}} + \mathcal{D}\Psi_{\mathcal{C}}$ is a Kählerian Killing spinor. We refer the reader to [Mor98b] for the proof of this fact. □

The main application of Theorem 10.10 is the converse to Theorem 10.6.

Theorem 10.11. *Every compact Kähler–Einstein manifold of positive scalar curvature and odd complex dimension $2k+1$ carrying a complex contact structure is the twistor space over a positive quaternion-Kähler manifold.*

Proof. For k odd, the result follows from Theorem 10.10 (i) together with Theorem 10.8.

Assume now that $k = 2l$ is even. Theorem 10.10 (ii) provides a Spin^c structure with auxiliary line bundle $L_{\mathcal{C}}$ and a Kählerian Killing spinor

$$\Psi \in \Gamma(\Sigma_{2l+1}M \oplus \Sigma_{2l+2}M).$$

The rest of the proof parallels that of Theorem 10.8, with a notable exception. One uses again projectable spinors for the submersion $S \to M$ (where S denotes the principal $\mathbb{S}^1$-bundle associated to L_C), except that in the present situation we endow S with the spin structure inherited from the Spin^c structure of M, rather than with the pull-back Spin^c structure. A correspondingly modified version of Theorem 2.45 shows that Ψ induces a Killing spinor on S, and then a parallel spinor on its Riemannian cone $\overline{S}$, eventually showing that $\overline{S}$ is hyper-Kähler. Details can be found in [Mor98b]. □

10.4 The limiting case of Kirchberg's inequalities

Recall Kirchberg's inequalities (Theorem 6.1): on a compact Kähler spin manifold (M^{2m}, g, J) with positive scalar curvature Scal, any eigenvalue λ of the Dirac operator satisfies

$$\lambda^2 \geq \frac{m+1}{4m}\mathrm{Scal}_0, \quad \text{if } m \text{ is odd,}$$

and

$$\lambda^2 \geq \frac{m}{4(m-1)}\mathrm{Scal}_0, \quad \text{if } m \text{ is even,}$$

where Scal_0 denotes the infimum of the scalar curvature on M.

Compact Kähler spin manifolds satisfying the limiting case in one of the above inequalities are called *limiting manifolds*. The aim of this section is to give their complete description.

If the complex dimension m is odd, Theorem 6.14 and Theorem 10.8 yield at once

Theorem 10.12 ([Mor95]). *Limiting manifolds of complex dimension $4l-1$ are twistor spaces over positive quaternion-Kähler manifolds. The only limiting manifold of complex dimension $4l+1$ is $\mathbb{C}\mathrm{P}^{4l+1}$.*

Suppose now that m is even. The case $m = 2$ was completely solved by Th. Friedrich [Fri93], who proved that M is isometric to $\mathbb{S}^2 \times \mathbb{T}^2$. We assume from now on that $m > 2$.

Theorem 10.13. *Let M^{2m} be a limiting manifold of even complex dimension $m = 2l$, $l \geq 2$, and let $\widetilde{M}$ be its universal cover. Then:*

- *if l is odd, $\widetilde{M}$ is isometric to the Riemannian product $\mathbb{C}\mathrm{P}^{2l-1} \times \mathbb{R}^2$;*
- *if l is even, $\widetilde{M}$ is isometric to the Riemannian product $N \times \mathbb{R}^2$, where N is the twistor space of some quaternion-Kähler manifold of positive scalar curvature.*

Proof. We give here a sketch of the proof. See [Mor97a] and [Mor99] for the details. The proof is in several steps.

Step 1. Like in the odd-dimensional case, limiting manifolds can be characterized by the existence of special spinor fields (see Theorem 6.15).

Applying the Dirac operator to (6.35) and using Corollary 2.8 yields

$$\nabla_X \mathcal{D}\Psi = -\frac{1}{4}(\mathrm{Ric}(X) + iJ\mathrm{Ric}(X)) \cdot \Psi. \tag{10.13}$$

If the manifold M were Einstein, (6.35) and (10.13) would imply that $\lambda\Psi + \mathcal{D}\Psi$ is a Kählerian Killing spinor. The point is that M *cannot* be Einstein (see Step 2 below).

Step 2. It is thus necessary to study the Ricci tensor of M. A tricky argument (cf. [Mor97a], Theorem 3.1) yields that the Ricci tensor of a limiting manifold of even complex dimension has only two eigenvalues: the eigenvalue 0 with multiplicity 2, and the eigenvalue $\mathrm{Scal}/(2m-2)$ with multiplicity $2m-2$.

Step 3. The tangent bundle of M thus splits into a J-invariant orthogonal direct sum $\mathrm{T}M = \mathcal{E} \oplus \mathcal{F}$ (where $\mathcal{E}$ and $\mathcal{F}$ are the eigenbundles of $\mathrm{T}M$ corresponding to the eigenvalues 0 and $\kappa := \mathrm{Scal}/(2m-2)$ of Ric, respectively). Moreover, the distributions $\mathcal{E}$ and $\mathcal{F}$ are integrable (this follows from the fact that the Ricci form is harmonic).

Let N denote a maximal leaf of the distribution $\mathcal{F}$. The main point here is that the Ricci tensor of N is greater than the restriction to N of the Ricci tensor of M, and is thus positive. Since N is complete, Myers' theorem implies that N is compact. Moreover, N being Kähler with positive defined Ricci tensor, a classical theorem of Kobayashi (see [Kob61], Theorem A) shows that N is simply connected.

Step 4. We now consider the restriction Φ^N of $\Phi := \Psi + \frac{2}{\sqrt{2m\kappa}}\mathcal{D}\Psi$ to N. In order to understand the nature of Φ^N, we recall Hitchin's representation of spinors (6.7):

$$\Sigma M \simeq (\mathcal{K}^M)^{\frac{1}{2}} \otimes \Lambda^{0,*}M.$$

The isomorphisms of complex vector bundles

$$\mathcal{K}^M \simeq \Lambda^m(\mathrm{T}^{0,1}M) \simeq \Lambda^m(\mathcal{E}^{0,1} \oplus \mathcal{F}^{0,1}) \simeq \mathcal{E}^{0,1} \otimes \Lambda^{m-1}(\mathcal{F}^{0,1})$$

yields

$$\mathcal{K}^M|_N \simeq \mathcal{E}^{0,1}|_N \otimes \mathcal{K}^N.$$

Similarly,

$$\Lambda^{0,*}M|_N \simeq \Lambda^{0,*}N \oplus (\mathcal{E}^{1,0}|_N \otimes \Lambda^{0,*}N),$$

and thus

$$\Sigma M|_N \simeq ((\mathcal{K}^M)^{\frac{1}{2}}|_N \otimes \Lambda^{0,*}N) \oplus ((\mathcal{K}^M)^{\frac{1}{2}}|_N \otimes (\mathcal{E}^{1,0}|_N \otimes \Lambda^{0,*}N)).$$

One can check that Φ^N is a section of

$$(\mathcal{K}^M)^{\frac{1}{2}}|_N \otimes (\mathcal{E}^{1,0}|_N \otimes \Lambda^{0,*}N),$$

and by the above, this is just the spinor bundle of some Spin^c structure on N with associated line bundle $\mathcal{E}^{1,0}|_N$. In fact, we can write locally

$$\begin{aligned}(\mathcal{K}^M)^{\frac{1}{2}}|_N \otimes (\mathcal{E}^{1,0}|_N \otimes \Lambda^{0,*}N) &\simeq ((\mathcal{E}^{1,0})|_N)^{\frac{1}{2}} \otimes ((\mathcal{K}^N)^{\frac{1}{2}} \otimes \Lambda^{0,*}N)\\ &\simeq ((\mathcal{E}^{1,0})|_N)^{\frac{1}{2}} \otimes \Sigma N,\end{aligned}$$

but, of course, neither $((\mathcal{E}^{1,0})|_N)^{\frac{1}{2}}$, nor ΣN need to exist globally on N.

Step 5. We compute the covariant derivative of Φ^N as Spin^c spinor on N and show that it coincides with the restriction to N of its covariant derivative on M. Thus Φ^N is a *Kählerian Killing* Spin^c *spinor* on N. Moreover, N is a Hodge manifold: if we denote by i the inclusion $N \to M$ and by ρ the Ricci form of M, then $\kappa\Omega^N = i^*\rho$, which implies $\kappa[\Omega^N] = i^*(2\pi c_1(M))$, and thus $[\Omega^N]$ is a real multiple of $i^*(c_1(M)) \in H^2(N;\mathbb{Z})$.

Step 6. We then apply Theorem 10.8 and deduce that the Spin^c structure on N has actually to be a *spin* structure (i.e., $\mathcal{E}^{1,0}|_N$ is a flat bundle on N). This implies that $\mathcal{E}$ and $\mathcal{F}$ are parallel distributions at each point of N, so they are parallel on M because the N's foliate M.

This immediately shows that the universal cover $\tilde{M}$ of a limiting manifold M of even complex dimension is isometric to the Riemannian product $N \times \mathbb{R}^2$, where N is a limiting manifold of odd complex dimension.

By taking into account the classification of limiting manifolds of odd complex dimension (Theorem 10.12), we obtain the claimed result. □

Remark 10.14. Note that once Step 2 is achieved, there is an alternative way to prove that if the Ricci tensor of a compact Kähler manifold has only two eigenvalues, both constant, then the two eigendistributions of the Ricci tensor are parallel. This was done in [ADM11] using the Weitzenböck formula applied to the curvature operator.

Chapter 11

Special spinors on quaternion-Kähler manifolds

In this chapter we examine the problem of the classification of spin compact quaternion-Kähler manifolds with positive scalar curvature admitting an eigenspinor for the minimal eigenvalue

$$\lambda = \pm\sqrt{\frac{m+3}{m+2}\frac{\mathrm{Scal}}{4}}$$

of the Dirac operator (cf. Chapter 7, recall that such manifolds have constant scalar curvature). As in the Riemannian and Kählerian settings, such eigenspinors will be called "quaternion-Kähler Killing spinors."

However, the only positive spin quaternion-Kähler manifolds with quaternion-Kähler Killing spinors are the quaternionic projective spaces. This was proved by W. Kramer, U. Semmelmann, and G. Weingart in [KSW98a], following the approach used in [Mor95] for the Kählerian case: any quaternion-Kähler Killing spinor on a positive spin quaternion-Kähler manifold M^{4m} can be lifted to a Killing spinor on the canonical 3-Sasakian SO_3-principal bundle S associated with the quaternion-Kähler structure, cf. Theorem 10.3. Now by the general cone construction (cf. 2.4.2), this induces a parallel spinor on the Riemannian cone over S. Moreover, this cone is a hyper-Kähler manifold (cf. 8.2.2). Hence, using results of M. Wang (cf. [Wan89]), one concludes that M^{4m} is isometric to the quaternionic projective space $\mathbb{H}\mathrm{P}^m$.

This method of pull-back to the 3-Sasakian SO_3-principal bundle $S \to M$ was already used by D. V. Alekseevsky and S. Marchiafava in [AM95] to prove the limit case of the following result: on any positive compact quaternion-Kähler manifold, the first non-zero eigenvalue of the Laplace operator Δ is greater or equal to

$$\frac{m+1}{m+2}\frac{\mathrm{Scal}}{2m},$$

and equality is attained if and only if the manifold is isometric to the quaternionic projective space.

In the following, we give the proof of [KSW98a], but with a simpler formalism, using the results of the first proof of the eigenvalue estimate given in Chapter 7 together with the formulas for the Dirac operator on the canonical 3-Sasakian SO_3-principal bundle S given in [Mor96].

We will use notations and results of Chapter 7.

11.1 The canonical 3-Sasakian SO_3-principal bundle

According to the survey [BG99], the existence of a 3-Sasakian structure on the canonical SO_3-principal bundle S over a quaternion-Kähler manifold was first studied by Konishi [Kon75].

The canonical SO_3-principal bundle $\pi\colon S \to M$ is the principal frame bundle associated with the vector bundle $\mathcal{Q}M$. Alternatively, considering the $Sp_1 . Sp_m$ reduction P of the principal bundle $P_{SO_{4m}}M$, it can be defined as the SO_3-associated bundle

$$S = P \times_{Sp_1 . Sp_m} SO_3,$$

where $Sp_1 . Sp_m$ acts on SO_3 via the covering $Sp_1 \to SO_3$ induced by the adjoint representation on the space of imaginary quaternions $\mathrm{Im}(\mathbb{H})$; cf. (7.4).

The Levi-Civita principal connection on P defines canonically a horizontal distribution $\mathrm{T}^{\mathrm{H}}S$, which complements the vertical distribution $\mathrm{T}^{\mathrm{V}}S$:

$$\mathrm{T}S = \mathrm{T}^{\mathrm{V}}S \oplus \mathrm{T}^{\mathrm{H}}S.$$

Note that $\mathrm{T}^{\mathrm{H}}S$ indeed gives rise to a principal connection θ on S, and that the covariant derivative on the associated vector bundle $\mathcal{Q}M$, induced by this principal connection, is just the extension of ∇ to $\mathcal{Q}M$.

According to a general result of Vilms [Vil70], given any $Sp_1 . Sp_m$-invariant (indeed a SO_3-invariant) metric g_{SO_3} on SO_3, there exists a unique Riemannian metric on S such that $\pi\colon S \to M$ is a Riemannian submersion with totally geodesic fibers isometric to (SO_3, g_{SO_3}), and horizontal distribution $\mathrm{T}^{\mathrm{H}}S$.

To define such a metric on S, we renormalize the metric on M in such a way that

$$\mathrm{Scal}_M = 16m(m+2),$$

(as for the quaternionic projective space with its canonical metric; the reason for this particular choice will be explained by the result below) and consider the bi-invariant metric on SO_3 induced by $(-1/8)$ times the Killing form of $\mathfrak{sp}_1$, in such a way that $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ becomes an orthonormal basis of $\mathfrak{so}_3 \simeq \mathrm{Im}(\mathbb{H})$.

Let V_1, V_2, V_3 be the fundamental vector fields on S induced by the vectors

$$\boldsymbol{i}_1 := \boldsymbol{i}, \quad \boldsymbol{i}_2 := \boldsymbol{j}, \quad \boldsymbol{i}_3 := \boldsymbol{k},$$

of the standard basis of $\mathfrak{sp}_1$. Those three vector fields define a 3-Sasakian structure on S (cf. Section 8.2.2 or the survey [BG99]). Let (J_1^S, J_2^S, J_3^S) be the tensor fields on S defined by

$$J_\alpha^S = -\nabla V_\alpha, \quad \alpha = 1, 2, 3,$$

(the same symbol is used to denote the covariant derivatives on M and S).

Then the quaternion-Kähler structure on M is related to the 3-Sasakian structure on S by the following relations.

Lemma 11.1. *Let* $s\colon x \mapsto s(x) = (J^x_\alpha)$ *be a local section of* $\pi\colon S \to M$. *Let* X *be a vector field on* M *and denote by* X^* *its horizontal lift. Then*

$$\pi_*(J^S_\alpha(X^*_{s(x)})) = J^x_\alpha(X_x),$$

and

$$J^S_\alpha(X^*_{s(x)}) \in \mathrm{T}^{\mathrm{H}}_{s(x)}S.$$

In other words, $J^S_\alpha(X^*_{s(x)})$ *is the horizontal lift of* $J^x_\alpha(X_x)$.

Proof. Let $x \in M$. Let $\{e_1, \ldots, e_{4m}\}$ be a local orthonormal frame of $\mathrm{T}M$, and for any $i = 1, \ldots, 4m$, let e^*_i be the horizontal lift of e_i. First, one gets at the point $s(x)$,

$$\begin{aligned}
\langle J^S_\alpha(e^*_i), e^*_j\rangle &= \frac{1}{2}\langle J^S_\alpha(e^*_i), e^*_j\rangle - \frac{1}{2}\langle e^*_i, J^S_\alpha(e^*_j)\rangle \\
&= -\frac{1}{2}\langle \nabla_{e^*_i} V_\alpha, e^*_j\rangle + \frac{1}{2}\langle e^*_i, \nabla_{e^*_j} V_\alpha\rangle \\
&= \frac{1}{2}\langle V_\alpha, \nabla_{e^*_i} e^*_j\rangle - \frac{1}{2}\langle \nabla_{e^*_j} e^*_i, V_\alpha\rangle \\
&= \frac{1}{2}\langle [e^*_i, e^*_j], V_\alpha\rangle \\
&= \frac{1}{2}\langle [e^*_i, e^*_j] - [e_i, e_j]^*, V_\alpha\rangle,
\end{aligned}$$

where $[e_i, e_j]^*$ is the horizontal lift of $[e_i, e_j]$.

Now recall that any section J of $\mathcal{Q}M$, defined on a neighborhood of x, may be identified with the $\mathrm{Sp}_1 . \mathrm{Sp}_m$-equivariant function $f_J\colon S \to \mathrm{Im}(\mathbb{H})$ that maps any $s \in S_x$ into $s^{-1}\big(Jx\big)$, viewing s as an isometry $\mathrm{Im}(\mathbb{H}) \to \mathcal{Q}_x M$. Under this identification, it is well known that $\nabla_{e_i} J$ corresponds to $e^*_i \cdot f_J$. It then follows easily that the same identification maps $R_{\mathcal{Q}M}(e_i, e_j)J$ into $\big([e^*_i, e^*_j] - [e_i, e_j]^*\big) \cdot f_J$.

We apply this to one of the three sections given by the choice of the section

$$s\colon x \longmapsto s(x) = (J^x_\alpha),$$

where each $s(x)$ may also be viewed as the isometry $\mathrm{Im}(\mathbb{H}) \to \mathcal{Q}_x M$ which maps the basis $(\boldsymbol{i}_\alpha)_{\alpha=1,2,3}$ into the basis $(J^x_\alpha)_{\alpha=1,2,3}$. On the one hand, since $\mathcal{Q}M$ is a subbundle of the bundle $\mathrm{End}(\mathrm{T}M)$, its curvature $R_{\mathcal{Q}M}$ is defined by

$$R_{\mathcal{Q}M}(X, Y) \cdot J = [R(X, Y), J], \quad X, Y \in \Gamma(\mathrm{T}M), J \in \Gamma(\mathcal{Q}M).$$

Hence, using the basic equality (7.23), one has that

$$R_{QM}(X,Y)\cdot J_\alpha^x = 4\sum_{\beta,\gamma}\varepsilon_{\alpha\beta\gamma}^{123}\Omega_\beta^x(X,Y)J_\gamma^x. \tag{11.1}$$

On the other hand

$$([e_i^*,e_j^*]-[e_i,e_j]^*)\cdot f_{J_\alpha} = \sum_\beta \langle [e_i^*,e_j^*]-[e_i,e_j]^*, V_\beta\rangle V_\beta\cdot f_{J_\alpha}.$$

But at the point $s(x)$

$$\begin{aligned} V_\beta\cdot f_{J_\alpha} &= \frac{d}{dt} f_{J_\alpha}(s(x)\cdot \exp(t\boldsymbol{i}_\beta))|_{t=0} \\ &= \frac{d}{dt}\exp(-t\boldsymbol{i}_\beta) f_{J_\alpha}(s(x))\exp(t\boldsymbol{i}_\beta)|_{t=0} \\ &= -\boldsymbol{i}_\beta\boldsymbol{i}_\alpha + \boldsymbol{i}_\alpha\boldsymbol{i}_\beta \end{aligned}$$

(since $f_{J_\alpha}(s(x)) = \boldsymbol{i}_\alpha$)

$$= 2\varepsilon_{\alpha\beta\gamma}^{123}\boldsymbol{i}_\gamma,$$

whence

$$([e_i^*,e_j^*]-[e_i,e_j]^*)\cdot f_{J_\alpha} = 2\sum_{\beta,\gamma}\varepsilon_{\alpha\beta\gamma}^{123}\langle [e_i^*,e_j^*]-[e_i,e_j]^*, V_\beta\rangle \boldsymbol{i}_\gamma. \tag{11.2}$$

Using the above identification, one deduces from equations (11.1) and (11.2) that

$$\langle [e_i^*,e_j^*]-[e_i,e_j]^*, V_\alpha\rangle = 2\Omega_\alpha^x(e_i,e_j),$$

hence

$$\langle J_\alpha^S(e_i^*), e_j^*\rangle = \frac{1}{2}\langle [e_i^*,e_j^*]-[e_i,e_j]^*, V_\alpha\rangle = \Omega_\alpha^x(e_i,e_j) = \langle J_\alpha^x(e_i), e_j\rangle.$$

Therefore,

$$\pi_*(J_\alpha^S(e_i^*)) = \sum_j \langle J_\alpha^S(e_i^*), e_j^*\rangle \pi_*(e_j^*) = \sum_j \langle J_\alpha^x(e_i), e_j\rangle e_j = J_\alpha^x(e_i).$$

For the second result, let V be a vertical vector field. $\nabla_{V_\alpha}V$ is vertical because the fibers of $\pi\colon S\to M$ are totally geodesic, so

$$0 = \langle \nabla_{V_\alpha}V, e_i^*\rangle = -\langle V, \nabla_{V_\alpha}e_i^*\rangle = \langle V, J_\alpha^S(e_i^*)\rangle,$$

i.e., $J_\alpha^S(e_i^*)$ is horizontal. □

Finally, we mention that S with its 3-Sasakian structure is an Einstein space (cf. [BG99]), hence its scalar curvature Scal_S is constant. This constant can be easily computed using O'Neill's formulas (cf. for instance 9.37 and 9.63 in [Bes87]). One finds

$$\mathrm{Scal}_S = (4m+2)(4m+3).$$

11.2 The Dirac operator acting on projectable spinors

Here we review results of the Chapter "Spineurs G-invariants sur une G-fibration principale" in [Mor96].

First S admits a spin structure. The decomposition $\mathrm{T}S = \mathrm{T}^{\mathrm{V}}S \oplus \mathrm{T}^{\mathrm{H}}S$ allows to reduce the bundle $P_{\mathrm{SO}_{4m+3}}S$ of orthonormal frames over S by considering local orthonormal frames of $\mathrm{T}S$ of the form $\{e_1^*, \ldots, e_{4m}^*, V_1, V_2, V_3\}$, where $\{e_1^*, \ldots, e_{4m}^*\}$ is the horizontal lift of an orthonormal local frame $\{e_1, \ldots, e_{4m}\}$ of $\mathrm{T}M$.

Proposition 11.2. *Let $P_{\mathrm{SO}_{4m+3}}S$ be the bundle of orthonormal frames over S. There is a bundle isomorphism*

$$P_{\mathrm{SO}_{4m+3}}S \simeq \pi^*(P_{\mathrm{SO}_{4m}}M) \times_{\mathrm{SO}_{4m}} \mathrm{SO}_{4m+3},$$

considering the group SO_{4m} *as a subgroup of* SO_{4m+3}.

Let $\sigma(x) = \{e_1(x), \ldots, e_{4m}(x)\}$ be an orthonormal frame at $x = \pi(s)$. Then, denoting by $\sigma^*(s)$ the orthonormal frame

$$\{e_1^*(s), \ldots, e_{4m}^*(s), V_1(s), V_2(s), V_3(s)\}, \tag{11.3}$$

any other orthonormal frame at s has the form $\sigma^*(s) \circ g$, where $g \in \mathrm{SO}_{4m+3}$, the frame $\sigma^*(s)$ being viewed here as the isomorphism that maps the standard basis of $\mathbb{R}^{4m+3}$ into the orthonormal frame (11.3). Clearly, the map that sends $\sigma^*(s) \circ g$ into the equivalence class $[(s, \sigma(x)), g]$ does not depend on the particular choice of $\sigma(x)$, and gives the above isomorphism.

One then obtains a spin structure on S by considering the bundle

$$P_{\mathrm{Spin}_{4m+3}}S := \pi^*(P_{\mathrm{Spin}_{4m}}M) \times_{\mathrm{Spin}_{4m}} \mathrm{Spin}_{4m+3},$$

identifying the group Spin_{4m} with a subgroup of Spin_{4m+3}. On the other hand, the complex Clifford algebra $\mathbb{C}\mathrm{l}_{4m+3}$ being canonically isomorphic to the graded tensor product $\mathbb{C}\mathrm{l}_{4m} \mathbin{\widehat{\otimes}} \mathbb{C}\mathrm{l}_3$, (cf. Proposition 1.12), the spinor space Σ_{4m+3} may be identified with $\Sigma_{4m} \otimes \Sigma_3$, endowed with its structure of $\mathbb{C}\mathrm{l}_{4m} \mathbin{\widehat{\otimes}} \mathbb{C}\mathrm{l}_3$-module (cf. [ABS64]): any $a \otimes b \in \mathbb{C}\mathrm{l}_{4m} \mathbin{\widehat{\otimes}} \mathbb{C}\mathrm{l}_3$ acts on a decomposable tensor of the form $\psi \otimes \varphi$ by

$$(a \otimes b) \cdot (\psi \otimes \varphi) := (-1)^{\deg(b)\deg(\psi)} (a \cdot \psi \otimes b \cdot \varphi),$$

where the structure of graded module of Σ_{4m} is given by the decomposition $\Sigma_{4m} = \Sigma_{4m}^+ \oplus \Sigma_{4m}^-$ into even and odd spinors (cf. Proposition 1.32).

This allows the following identification.

Proposition 11.3. *Let $\Sigma_{4m+3}S$ be the spinor bundle of S. Then*

$$\Sigma_{4m+3}S \simeq \pi^*(\Sigma M) \otimes \Sigma_3 S,$$

where $\Sigma_3 S$ denotes the trivial bundle on S with fibre Σ_3.

Proof. Let $s \in S$. Any element Ψ_s of the fiber over s of $\Sigma_{4m+3}S$ is represented by a quadruple

$$\Big(s, \tilde{\sigma}_{\pi(s)}, u, \sum_i \psi_i \otimes \varphi_i\Big),$$

where $\tilde{\sigma}_{\pi(s)}$ is an element of the fiber of $P_{\mathrm{Spin}_{4m}} M$ over $\pi(s)$, $u \in \mathrm{Spin}_{4m+3}$, and

$$\sum_i \psi_i \otimes \varphi_i \in \Sigma_{4m+3} \simeq \Sigma_{4m} \otimes \Sigma_3.$$

Any other quadruple $(s, \tilde{\sigma}'_{\pi(s)}, u', \sum_j \psi'_j \otimes \varphi'_j)$, represents Ψ_s if and only if there exist $u_0 \in \mathrm{Spin}_{4m}$ and $u_1 \in \mathrm{Spin}_{4m+3}$ such that $\tilde{\sigma}'_{\pi(s)} = \tilde{\sigma}_{\pi(s)} u_0$, $u' = u_0^{-1} u u_1$, and $u_1 \cdot \sum_j \psi'_j \otimes \varphi'_j = \sum_i \psi_i \otimes \varphi_i$, where u_1 is viewed as an element of $\mathbb{C}\mathrm{l}_{4m} \hat{\otimes} \mathbb{C}\mathrm{l}_3$. Viewing u and u' also as elements of $\mathbb{C}\mathrm{l}_{4m} \hat{\otimes} \mathbb{C}\mathrm{l}_3$, set

$$\sum_a \phi_a \otimes \omega_a = u \cdot \sum_i \psi_i \otimes \varphi_i \quad \text{and} \quad \sum_b \phi'_b \otimes \omega'_b = u' \cdot \sum_j \psi'_j \otimes \varphi'_j.$$

Since $u_0 \in \mathrm{Spin}_{4m}$, one has

$$\sum_b (u_0 \cdot \phi'_b) \otimes \omega'_b = u_0 \cdot \sum_b \phi'_b \otimes \omega'_b = \sum_a \phi_a \otimes \omega_a,$$

hence we can define

$$F(\Psi_s) := \sum_a (s, [\tilde{\sigma}_{\pi(s)}, \phi_a]) \otimes (s, \omega_a),$$

since the right-hand side in the above equation does not depend on the quadruple representing Ψ_s.

The map

$$\Psi_s \longmapsto F(\Psi_s)$$

is then a bundle morphism

$$F\colon \Sigma_{4m+3}S \longrightarrow \pi^*(\Sigma M) \otimes \Sigma_3 S.$$

This is actually a bundle isomorphism: it is easy to verify that the inverse morphism maps a decomposable element of the form

$$(s, [\tilde{\sigma}_{\pi(s)}, \psi]) \otimes (s, \varphi)$$

into the equivalence class of the quadruple

$$(s, \tilde{\sigma}_{\pi(s)}, 1, \psi \otimes \varphi). \qquad \square$$

This identification allows us to introduce the following particular spinor fields on S.

Definition 11.4. Any couple (Ψ, φ) with $\Psi \in \Gamma(\Sigma M)$ and $\varphi \in \Sigma_3$ defines a spinor field on S called *projectable* and denoted by $\Psi \otimes \varphi$.

Proposition 11.5. *Let X^* be the horizontal lift of a vector field X on M, and V be the fundamental vertical field on S induced by a vector $v \in \mathfrak{so}_3 \simeq \mathbb{R}^3$. Their Clifford multiplication on a projectable spinor $\Psi \otimes \varphi$ is given by*

$$X^* \cdot (\Psi \otimes \varphi) = (X \cdot \Psi) \otimes \varphi$$

and

$$V \cdot (\Psi \otimes \varphi) = \overline{\Psi} \otimes (v \cdot \varphi),$$

where $\overline{\Psi}$ is the image of Ψ by the conjugation map (cf. Definition 1.34), $\overline{\Psi} = \omega^{\mathbb{C}}_{4m} \cdot \Psi$.

Proof. Let $s \in S$ and $x = \pi(s)$. Let $\sigma = \{e_1, \ldots, e_{4m}\}$ be a local section of the bundle $P_{\mathrm{SO}_{4m}} S$ defined in a neighborhood of x. Let $\tilde{\sigma}$ be a section of $P_{\mathrm{Spin}_{4m}} M$ which projects onto σ. Denote also by $\tilde{\sigma}$ the section of $P_{\mathrm{Spin}_{4m+3}} S$ given by

$$s \longmapsto [(s, \tilde{\sigma}(\pi(s)), 1)].$$

This section projects onto the local section of $P_{\mathrm{SO}_{4m+3}} S$, $s \mapsto \sigma^*(s)$.

Finally, let $\psi \in \Sigma_{4m}$ be defined by $\Psi(x) = [\tilde{\sigma}(x), \psi]$. Viewing $\sigma(x)$ (resp. $\sigma^*(s)$) as the isometry that sends the standard basis of $\mathbb{R}^{4m}$ (resp. $\mathbb{R}^{4m+3}$) into $\{e_1(x), \ldots, e_{4m}(x)\}$ (resp. $\{e_1^*(s), \ldots, e_{4m}^*(s), V_1(s), V_2(s), V_3(s)\}$), one has at the point s

$$\begin{aligned}
e_i^* \cdot (\Psi \otimes \varphi) &\simeq [\tilde{\sigma}(s), \sigma^*(s)^{-1}(e_i^*(s)) \cdot \psi \otimes \varphi] \\
&= [\tilde{\sigma}(s), \sigma(x)^{-1}(e_i(x)) \cdot \psi \otimes \varphi] \\
&\simeq [\tilde{\sigma}(x), \sigma(x)^{-1}(e_i(x)) \cdot \psi] \otimes \varphi \\
&= (e_i \cdot \Psi) \otimes \varphi.
\end{aligned}$$

On the other hand, considering the decomposition $\psi = \psi_+ + \psi_-$ induced by the splitting $\Sigma_{4m} = \Sigma_{4m}^+ \oplus \Sigma_{4m}^-$ into even and odd spinors, for any $\alpha = 1, 2, 3$, one gets

$$\begin{aligned}
V_\alpha \cdot (\Psi \otimes \varphi) &\simeq (-1)^{\deg \psi} [\tilde{\sigma}(s), \psi \otimes \sigma^*(s)^{-1}(V_\alpha(s)) \cdot \varphi] \\
&= [\tilde{\sigma}(s), (\psi_+ - \psi_-) \otimes \boldsymbol{i}_\alpha \cdot \varphi] \\
&\simeq [\tilde{\sigma}(x), \psi_+ - \psi_-] \otimes (\boldsymbol{i}_\alpha \cdot \varphi) \\
&= \overline{\Psi} \otimes (\boldsymbol{i}_\alpha \cdot \varphi).
\end{aligned}$$

□

Proposition 11.6 ([Mor96]). *Let s be a local section of $\pi\colon S \to M$ and $x \in M$. Denote by $\{e_1, \dots, e_{4m}\}$ a local orthonormal frame of $\mathrm{T}M$ such that $(\nabla e_i)_x = 0$, and for any $i = 1, \dots, 4m$, let e_i^* be the horizontal lift of e_i. Introduce the 2-forms Ω_α^S on S, locally defined by*

$$\Omega_\alpha^S = \frac{1}{2}\sum_i e_i^* \wedge J_\alpha^S(e_i^*).$$

Then for any projectable spinor $\Psi \otimes \varphi$ on S one has

$$\nabla_{e_i^*}(\Psi \otimes \varphi) = (\nabla_{e_i}\Psi) \otimes \varphi + \frac{1}{2}\sum_\alpha J_\alpha^S(e_i^*) \cdot V_\alpha \cdot (\Psi \otimes \varphi), \tag{11.4}$$

$$\nabla_{V_\alpha}(\Psi \otimes \varphi) = -\frac{1}{2}\Omega_\alpha^S \cdot (\Psi \otimes \varphi) - \frac{1}{2}(-1)^m V_\alpha \cdot (\bar{\Psi} \otimes \varphi), \tag{11.5}$$

Hence,

$$\mathcal{D}^S(\Psi \otimes \varphi) = (\mathcal{D}\Psi) \otimes \varphi + \frac{1}{2}\sum_\alpha \Omega_\alpha^S \cdot V_\alpha \cdot (\Psi \otimes \varphi) + \frac{3}{2}(-1)^m(\bar{\Psi} \otimes \varphi).$$

Proof. Consider the local section $\sigma = \{e_1^*, \dots, e_{4m}^*, V_1, V_2, V_3\}$ of $P_{\mathrm{SO}_{4m+3}}S$. Let $\tilde{\sigma}$ be a section of $P_{\mathrm{Spin}_{4m}}M$ which projects onto $\{e_1, \dots, e_{4m}\}$. It induces a section of $P_{\mathrm{Spin}_{4m+3}}S$, also denoted by $\tilde{\sigma}$, which projects onto σ. Now, the spinor field Ψ is locally defined by $\Psi = [\tilde{\sigma}, \psi]$, where $\psi\colon U \subset M \to \Sigma_{4m}$, hence $\Psi \otimes \varphi$ is locally defined by $[\tilde{\sigma}, (\psi \circ \pi) \otimes \varphi]$. Therefore,

$$\begin{aligned}\nabla_{e_i^*}(\Psi \otimes \varphi) &= \frac{1}{2}\sum_{j<k}\langle \nabla_{e_i^*} e_j^*, e_k^*\rangle e_j^* \cdot e_k^* \cdot (\Psi \otimes \varphi) + [\tilde{\sigma}, e_i^*(\psi \circ \pi) \otimes \varphi] \\ &\quad + \frac{1}{2}\sum_{j,\alpha}\langle \nabla_{e_i^*} e_j^*, V_\alpha\rangle e_j^* \cdot V_\alpha \cdot (\Psi \otimes \varphi).\end{aligned}$$

But since $\pi\colon S \to M$ is a Riemannian submersion, and since $[e_i^*, e_j^*] - [e_i, e_j]^*$ is a vertical vector, it follows that

$$\begin{aligned}\langle \nabla_{e_i^*} e_j^*, e_k^*\rangle &= \frac{1}{2}(-\langle e_i^*, [e_j^*, e_k^*]\rangle + \langle e_j^*, [e_k^*, e_i^*]\rangle + \langle e_k^*, [e_i^*, e_j^*]\rangle) \\ &= \frac{1}{2}(-\langle e_i^*, [e_j, e_k]^*\rangle + \langle e_j^*, [e_k, e_i]^*\rangle + \langle e_k^*, [e_i, e_j]^*\rangle) \\ &= \frac{1}{2}(-\langle e_i, [e_j, e_k]\rangle + \langle e_j, [e_k, e_i]\rangle + \langle e_k, [e_i, e_j]\rangle) \\ &= \langle \nabla_{e_i} e_j, e_k\rangle.\end{aligned}$$

One the other hand,

$$\langle \nabla_{e_i^*} e_j^*, V_\alpha \rangle = -\langle e_j^*, \nabla_{e_i^*} V_\alpha \rangle = \langle e_j^*, J_\alpha^S(e_i^*) \rangle,$$

hence,

$$\begin{aligned}\nabla_{e_i^*}(\Psi \otimes \varphi) &= \Big(\frac{1}{2}\sum_{j<k} \langle \nabla_{e_i} e_j, e_k \rangle e_j \cdot e_k \cdot \Psi\Big) \otimes \varphi + [\tilde{\sigma}, e_i(\psi)] \otimes \varphi \\ &\quad + \frac{1}{2}\sum_j \langle J_\alpha^S(e_i^*), e_j^* \rangle e_j^* \cdot V_\alpha \cdot (\Psi \otimes \varphi) \\ &= (\nabla_{e_i}\Psi) \otimes \varphi + \frac{1}{2}\sum_\alpha J_\alpha^S(e_i^*) \cdot V_\alpha \cdot (\Psi \otimes \varphi),\end{aligned}$$

i.e., (11.4). In the same way,

$$\begin{aligned}\nabla_{V_\alpha}(\Psi \otimes \varphi) &= \frac{1}{2}\sum_{j<k} \langle \nabla_{V_\alpha} e_j^*, e_k^* \rangle e_j^* \cdot e_k^* \cdot (\Psi \otimes \varphi) + [\tilde{\sigma}, V_\alpha(\psi \circ \pi) \otimes \varphi] \\ &\quad + \frac{1}{2}\sum_{\beta<\gamma} \langle \nabla_{V_\alpha} V_\beta, V_\gamma \rangle V_\beta \cdot V_\gamma \cdot (\Psi \otimes \varphi).\end{aligned}$$

But since the fibres of the Riemannian submersion $\pi\colon S \to M$ are totally geodesic, $\nabla_{V_\alpha} V_\beta$ is a vertical vector and

$$\langle \nabla_{V_\alpha} V_\beta, V_\gamma \rangle = \frac{1}{2}(-\langle V_\alpha, [V_\beta, V_\gamma] \rangle + \langle V_\beta, [V_\gamma, V_\alpha] \rangle + \langle V_\gamma, [V_\alpha, V_\beta] \rangle) = \varepsilon_{\alpha\beta\gamma}^{123},$$

hence

$$\nabla_{V_\alpha} V_\beta = \sum_\gamma \varepsilon_{\alpha\beta\gamma}^{123} V_\gamma. \tag{11.6}$$

Consequently,

$$\sum_{\beta<\gamma} \langle \nabla_{V_\alpha} V_\beta, V_\gamma \rangle V_\beta \cdot V_\gamma \cdot (\Psi \otimes \varphi) = \frac{1}{2}\sum_{\beta,\gamma} \varepsilon_{\alpha\beta\gamma}^{123} V_\beta \cdot V_\gamma \cdot (\Psi \otimes \varphi).$$

Now, by considering an equivalent representation if necessary, we may always assume that the volume element in $\mathbb{C}l_3$ acts on Σ_3 in such a way that

$$V_1 \cdot V_2 \cdot V_3 \cdot (\Psi \otimes \varphi) = (-1)^m \overline{\Psi} \otimes \varphi$$

(cf. Proposition 1.32), therefore

$$\begin{aligned}(-1)^m V_\alpha \cdot (\overline{\Psi} \otimes \varphi) &= V_\alpha \cdot (V_1 \cdot V_2 \cdot V_3) \cdot (\Psi \otimes \varphi) \\ &= \frac{1}{2} V_\alpha \cdot \sum_{\beta,\gamma} \varepsilon^{123}_{\alpha\beta\gamma} V_\alpha \cdot V_\beta \cdot V_\gamma \cdot (\Psi \otimes \varphi) \\ &= -\frac{1}{2} \sum_{\beta,\gamma} \varepsilon^{123}_{\alpha\beta\gamma} V_\beta \cdot V_\gamma \cdot (\Psi \otimes \varphi)\end{aligned} \tag{11.7}$$

and finally

$$\sum_{\beta<\gamma} \langle \nabla_{V_\alpha} V_\beta, V_\gamma \rangle V_\beta \cdot V_\gamma \cdot (\Psi \otimes \varphi) = (-1)^{m+1} V_\alpha \cdot \overline{\Psi} \otimes \varphi.$$

On the other hand, the right invariance of horizontal vectors implies $[V_\alpha, e_i^*] = 0$, hence, since the connection is torsion free,

$$\nabla_{V_\alpha} e_i^* = \nabla_{e_i^*} V_\alpha = -J^S_\alpha(e_i^*), \tag{11.8}$$

and

$$\begin{aligned}\nabla_{V_\alpha}(\Psi \otimes \varphi) = &-\frac{1}{4} \sum_j e_j^* \cdot J^S_\alpha(e_j^*) \cdot (\Psi \otimes \varphi) \\ &- \frac{1}{2}(-1)^m V_\alpha \cdot \overline{\Psi} \otimes \varphi,\end{aligned}$$

which yields (11.5). The third result is easily obtained from (11.4) and (11.5). □

11.3 Characterization of the limiting case

Observe first that the volume element $\omega^{\mathbb{C}}_{4m}$ commutes with the fundamental 4-form Ω, hence, by the Schur lemma, the restriction of $\omega^{\mathbb{C}}_{4m}$ to $\Sigma_r M$ is either Id or −Id, since any fiber $\Sigma_r M$ is an irreducible $\mathrm{Sp}_1 \cdot \mathrm{Sp}_m$-space equivalent to $\Sigma_{4m,r}$. Considering for instance a maximal vector in $\Sigma_{4m,r}$, it is easy to verify that, for m even, $\omega^{\mathbb{C}}_{4m|\Sigma_{4m,r}} = \mathrm{Id}$ if r is even, and $\omega^{\mathbb{C}}_{4m|\Sigma_{4m,r}} = -\mathrm{Id}$ if r is odd, whereas the contrary holds when m is odd.

The limiting case in the main estimate provides two spinor fields

$$\Psi_0 \in \Gamma(\Sigma_0 M) \quad \text{and} \quad \Psi_1 = \mathcal{D}\Psi_0 \in \Gamma(\Sigma_1 M),$$

(cf. Section 7.4). Proceeding heuristically, we begin by considering the action of the Dirac operator $\mathcal{D}^S$ on the two projectable spinors $\Psi_0 \otimes \varphi$ and $\Psi_1 \otimes \varphi$, φ being a fixed non-zero vector in Σ_3. First, since by Lemma 7.5,

$$\Omega^S_\alpha \cdot (\Psi_0 \otimes \varphi) = (\Omega_\alpha \cdot \Psi_0) \otimes \varphi = 0,$$

and by the above remark $\overline{\Psi_0} = (-1)^m \Psi_0$, one gets the following lemma.

Lemma 11.7.
$$\mathcal{D}^S(\Psi_0 \otimes \varphi) = \Psi_1 \otimes \varphi + \frac{3}{2}(\Psi_0 \otimes \varphi).$$

Knowing that $\overline{\Psi_1} = (-1)^{m+1}\Psi_1$, one also gets the following lemma.

Lemma 11.8.
$$\mathcal{D}^S(\Psi_1 \otimes \varphi) = 4m(m+3)\Psi_0 \otimes \varphi + \frac{1}{2}\sum_\alpha \Omega^S_\alpha \cdot V_\alpha \cdot (\Psi_1 \otimes \varphi) - \frac{3}{2}(\Psi_1 \otimes \varphi).$$

We now compute
$$\mathcal{D}^S\Big(\sum_\alpha \Omega^S_\alpha \cdot V_\alpha \cdot (\Psi_1 \otimes \varphi)\Big).$$

For that, we need the following formulas.

Lemma 11.9. *Assume that the local frame $\{e_i\}$ is such that $\nabla_{e_i} e_j(x) = 0$. Then at the point $s(x)$,*
$$\nabla_{e_j^*} J^S_\alpha(e_i^*) = \delta_{ij} V_\alpha - \sum_{\beta,\gamma} \varepsilon^{123}_{\alpha\beta\gamma}\langle J^S_\beta(e_i^*), e_j^*\rangle V_\gamma, \tag{11.9}$$
and
$$\nabla_{V_\beta} J^S_\alpha(e_i^*) = \delta_{\alpha\beta} e_i^* - \sum_\gamma \varepsilon^{123}_{\alpha\beta\gamma} J^S_\gamma(e_i^*). \tag{11.10}$$

Proof. Since each V_α is a Sasakian structure (cf. Definition 8.14), using the arguments of the above proof, one has
$$\begin{aligned}
\nabla_{e_j^*} J^S_\alpha(e_i^*) &= -\nabla_{e_j^*}(\nabla_{e_i^*} V_\alpha) \\
&= -\nabla^2_{e_j^*, e_i^*} V_\alpha - \nabla_{\nabla_{e_j^*} e_i^*} V_\alpha \\
&= \delta_{ij} V_\alpha - \sum_k \langle \nabla_{e_j^*} e_i^*, e_k^*\rangle \nabla_{e_k^*} V_\alpha - \sum_\beta \langle \nabla_{e_j^*} e_i^*, V_\beta\rangle \nabla_{V_\beta} V_\alpha \\
&= \delta_{ij} V_\alpha + \sum_{\beta,\gamma} \varepsilon^{123}_{\alpha\beta\gamma}\langle e_i^*, J^S_\beta(e_j^*)\rangle V_\gamma.
\end{aligned}$$

On the other hand,
$$\nabla_{V_\beta} J^S_\alpha(e_i^*) = -\nabla_{V_\beta}(\nabla_{e_i^*} V_\alpha) = -\nabla^2_{V_\beta, e_i^*} V_\alpha - \nabla_{\nabla_{V_\beta} e_i^*} V_\alpha.$$

But $\nabla^2_{V_\beta, e_i^*} V_\alpha = 0$, by (8.9), and
$$\nabla_{\nabla_{V_\beta} e_i^*} V_\alpha = \nabla_{\nabla_{e_i^*} V_\beta} V_\alpha = J^S_\alpha(J^S_\beta(e_i^*)),$$

by (11.8), hence the second result. $\square$

From the relations obtained above one deduces

Lemma 11.10. *Assume that the local frame $\{e_i\}$ is such that $\nabla_{e_i} e_j(x) = 0$. Then at the point $s(x)$*

$$\nabla_{e_j^*} \Omega_\alpha^S = e_j^* \cdot V_\alpha + \sum_{\beta,\gamma} \varepsilon_{\alpha\beta\gamma}^{123} J_\beta^S(e_j^*) \cdot V_\gamma, \tag{11.11}$$

and

$$\nabla_{V_\beta} \Omega_\alpha^S = 0. \tag{11.12}$$

Proof. As already noted,

$$\begin{aligned}
\nabla_{e_j^*} e_i^* &= \sum_k \langle \nabla_{e_j^*} e_i^*, e_k^* \rangle e_k^* + \sum_\beta \langle \nabla_{e_j^*} e_i^*, V_\beta \rangle V_\beta \\
&= \sum_\beta \langle e_i^*, J_\beta^S(e_j^*) \rangle V_\beta,
\end{aligned}$$

whence

$$\begin{aligned}
\nabla_{e_j^*} \Omega_\alpha^S &= \frac{1}{2} \sum_i \nabla_{e_j^*} e_i^* \cdot J_\alpha^S(e_i^*) + \frac{1}{2} \sum_i e_i^* \cdot \nabla_{e_j^*} J_\alpha^S(e_i^*) \\
&= \frac{1}{2} \Big(\sum_{\beta,i} \langle e_i^*, J_\beta^S(e_j^*) \rangle V_\beta \cdot J_\alpha^S(e_i^*) + e_j^* \cdot V_\alpha \\
&\qquad + \sum_{i,\beta,\gamma} \varepsilon_{\alpha\beta\gamma}^{123} \langle e_i^*, J_\beta^S(e_j^*) \rangle e_i^* \cdot V_\gamma \Big) \\
&= \frac{1}{2} \Big(\sum_\beta V_\beta \cdot J_\alpha^S(J_\beta^S(e_j^*)) + e_j^* \cdot V_\alpha + \sum_{\beta,\gamma} \varepsilon_{\alpha\beta\gamma}^{123} J_\beta^S(e_j^*) \cdot V_\gamma \Big) \\
&= e_j^* \cdot V_\alpha + \sum_{\beta,\gamma} \varepsilon_{\alpha\beta\gamma}^{123} J_\beta^S(e_j^*) \cdot V_\gamma.
\end{aligned}$$

On the other hand, using (11.8), one gets

$$\begin{aligned}
\nabla_{V_\beta} \Omega_\alpha^S &= \frac{1}{2} \sum_i \nabla_{V_\beta} e_i^* \cdot J_\alpha^S(e_i^*) + \frac{1}{2} \sum_i e_i^* \cdot \nabla_{V_\beta} J_\alpha^S(e_i^*) \\
&= -\frac{1}{2} \sum_i J_\beta^S(e_i^*) \cdot J_\alpha^S(e_i^*) - 2m\delta_{\alpha\beta} - \sum_\gamma \varepsilon_{\alpha\beta\gamma}^{123} \Omega_\gamma^S.
\end{aligned}$$

The tensorial form of the first term in the right-hand side of the above equation shows that we can assume that $\{e_i\}$ is an adapted local orthonormal frame (i.e., such that any e_i has the form $\pm J_\beta e_j$). Then, by Lemma 11.1, any e_i^* has then the form $\pm J_\beta(e_j^*)$, therefore

$$\nabla_{V_\beta}\Omega_\alpha^S = \frac{1}{2}\sum_i e_i^* \cdot J_\alpha^S(J_\beta^S(e_i^*)) - 2m\delta_{\alpha\beta} - \sum_\gamma \varepsilon_{\alpha\beta\gamma}^{123}\Omega_\gamma^S = 0. \qquad \square$$

Lemma 11.11. *We have*

$$\mathcal{D}^S\Big(\sum_\alpha \Omega_\alpha^S \cdot V_\alpha \cdot (\Psi_1 \otimes \varphi)\Big)$$
$$= 6\Psi_1 \otimes \varphi + \frac{9}{2}\sum_\alpha \Omega_\alpha^S \cdot V_\alpha \cdot (\Psi_1 \otimes \varphi) + 2\sum_\alpha V_\alpha \cdot \mathcal{D}_\alpha^S(\Psi_1 \otimes \varphi),$$

where $\mathcal{D}_\alpha^S$ is the operator locally defined by

$$\mathcal{D}_\alpha^S(\Psi \otimes \varphi) = \sum_i J_\alpha^S(e_i^*) \cdot (\nabla_{e_i}\Psi \otimes \varphi).$$

Proof. With the notations of the proof of Proposition 11.6, using (11.4), (11.9), and (11.11) one gets

$$\begin{aligned}\nabla_{e_i^*}\Big(\sum_\alpha \Omega_\alpha^S \cdot V_\alpha \cdot (\Psi_1 \otimes \varphi)\Big) = &-3e_i^* \cdot (\Psi_1 \otimes \varphi)\\ &+ \sum_{\alpha,\beta,\gamma} \varepsilon_{\alpha\beta\gamma}^{123} J_\beta^S(e_i^*) \cdot V_\gamma \cdot V_\alpha \cdot (\Psi_1 \otimes \varphi)\\ &- \sum_\alpha \Omega_\alpha^S \cdot J_\alpha^S(e_i^*) \cdot (\Psi_1 \otimes \varphi)\\ &+ \sum_\alpha \Omega_\alpha^S \cdot V_\alpha \cdot (\nabla_{e_i}\Psi_1 \otimes \varphi)\\ &- \frac{1}{2}\sum_{\alpha,\beta} \Omega_\alpha^S \cdot J_\beta^S(e_i^*) \cdot V_\alpha \cdot V_\beta \cdot (\Psi_1 \otimes \varphi).\end{aligned}$$

Now, by (11.7), for any $\alpha, \beta \in \{1, 2, 3\}$ and any projectable spinor $\Psi \otimes \varphi$,

$$\begin{aligned}\sum_\gamma \varepsilon_{\alpha\beta\gamma}^{123} V_\gamma \cdot (\Psi \otimes \varphi) &= -\frac{1}{2}(-1)^m \sum_{\gamma,\lambda,\mu} \varepsilon_{\alpha\beta\gamma}^{123}\varepsilon_{\gamma\lambda\mu}^{123} V_\lambda \cdot V_\mu \cdot (\bar{\Psi} \otimes \varphi)\\ &= -\frac{1}{2}(-1)^m \sum_\gamma (\varepsilon_{\alpha\beta\gamma}^{123})^2 (V_\alpha \cdot V_\beta - V_\beta \cdot V_\alpha) \cdot (\bar{\Psi} \otimes \varphi)\\ &= (-1)^m(\delta_{\alpha\beta} - 1)V_\alpha \cdot V_\beta \cdot (\bar{\Psi} \otimes \varphi).\end{aligned} \tag{11.13}$$

Hence since $\overline{\Psi_1} = (-1)^{m+1}\Psi_1$, one gets, using again (11.7),

$$\begin{aligned}
\nabla_{e_i^*}\Big(\sum_\alpha \Omega_\alpha^S \cdot V_\alpha \cdot (\Psi_1 \otimes \varphi)\Big) = -3e_i^* \cdot (\Psi_1 \otimes \varphi) \\
+ 2\sum_\beta J_\beta^S(e_i^*) \cdot V_\beta \cdot (\Psi_1 \otimes \varphi) \\
- \frac{1}{2}\sum_\alpha \Omega_\alpha^S \cdot J_\alpha^S(e_i^*) \cdot (\Psi_1 \otimes \varphi) \\
+ \sum_\alpha \Omega_\alpha^S \cdot V_\alpha \cdot (\nabla_{e_i}\Psi_1 \otimes \varphi) \\
- \frac{1}{2}\sum_{\alpha,\beta,\gamma} \varepsilon_{\alpha\beta\gamma}^{123}\Omega_\alpha^S \cdot J_\beta^S(e_i^*) \cdot V_\gamma \cdot (\Psi_1 \otimes \varphi).
\end{aligned}$$

Using the straightforward formula $e_i^* \cdot \Omega_\alpha^S = \Omega_\alpha^S \cdot e_i^* - 2J_\alpha^S(e_i^*)$, and noting that, by Lemma 11.1

$$\begin{aligned}
\sum_\alpha (\Omega_\alpha^S \cdot \Omega_\alpha^S) \cdot (\Psi_1 \otimes \varphi) &= \sum_\alpha (\Omega_\alpha \cdot \Omega_\alpha \cdot \Psi_1) \otimes \varphi \\
&= -12\Psi_1 \otimes \varphi,
\end{aligned} \tag{11.14}$$

$$\Omega_\alpha^S \cdot (\Psi_0 \otimes \psi) = 0$$

(cf. the proof of Lemma 11.7), and, by (7.10),

$$\sum_{\alpha,\beta,\gamma} \varepsilon_{\alpha\beta\gamma}^{123}\Omega_\alpha^S \cdot \Omega_\beta^S = 4\Omega_\gamma^S, \tag{11.15}$$

it follows

$$\begin{aligned}
&\sum_i e_i^* \cdot \nabla_{e_i^*}\Big(\sum_\alpha \Omega_\alpha^S \cdot V_\alpha \cdot (\Psi_1 \otimes \varphi)\Big) \\
&\quad = 12(\Psi_1 \otimes \varphi) + 2\sum_\alpha V_\alpha \cdot \mathcal{D}_\alpha^S(\Psi_1 \otimes \varphi) \\
&\qquad + \sum_{\alpha,\beta,\gamma} \varepsilon_{\alpha\beta\gamma}^{123} J_\alpha^S(e_i^*) \cdot J_\beta^S(e_i^*) \cdot V_\gamma \cdot (\Psi_1 \otimes \varphi).
\end{aligned}$$

In view of the tensorial form of the last term in the right-hand side of the above equation, we may suppose that $\{e_i\}$ is an adapted local orthonormal frame (i.e., such that any e_i has the form $\pm J_\alpha e_j$). Using once again Lemma 11.1, any e_i^* has then the form $\pm J_\alpha(e_j^*)$, and we get

$$\begin{aligned}
&\sum_{\alpha,\beta,\gamma} \varepsilon^{123}_{\alpha\beta\gamma} J^S_\alpha(e_i^*)\cdot J^S_\beta(e_i^*)\cdot V_\gamma\cdot(\Psi_1\otimes\varphi)\\
&\qquad = -\sum_{\alpha,\beta,\gamma} \varepsilon^{123}_{\alpha\beta\gamma} e_i^*\cdot J_\beta(J^S_\alpha(e_i^*))\cdot V_\gamma\cdot(\Psi_1\otimes\varphi)\\
&\qquad = 2\sum_{\alpha,\beta,\gamma,\lambda} \varepsilon^{123}_{\alpha\beta\gamma}\varepsilon^{123}_{\alpha\beta\lambda}\Omega^S_\lambda\cdot V_\gamma\cdot(\Psi_1\otimes\varphi)\\
&\qquad = 4\sum_\alpha \Omega^S_\alpha\cdot V_\alpha\cdot(\Psi_1\otimes\varphi).
\end{aligned}$$

So finally

$$\begin{aligned}
&\sum_i e_i^*\cdot\nabla_{e_i^*}\Big(\sum_\alpha \Omega^S_\alpha\cdot V_\alpha\cdot(\Psi_1\otimes\varphi)\Big)\\
&\qquad = 12(\Psi_1\otimes\varphi) + 2\sum_\alpha V_\alpha\cdot\mathcal{D}^S_\alpha(\Psi_1\otimes\varphi)\\
&\qquad\qquad + 4\sum_\alpha \Omega^S_\alpha\cdot V_\alpha\cdot(\Psi_1\otimes\varphi).
\end{aligned}$$

On the other hand, using (11.5), (11.6), (11.7), and (11.12), the expression

$$E := \sum_\beta V_\beta\cdot\nabla_{V_\beta}\Big(\sum_\alpha \Omega^S_\alpha\cdot V_\alpha\cdot(\Psi_1\otimes\varphi)\Big)$$

reads

$$\begin{aligned}
E &= -\sum_{\alpha,\beta,\gamma} \varepsilon^{123}_{\alpha\beta\gamma}\Omega^S_\alpha\cdot V_\beta\cdot V_\gamma\cdot(\Psi_1\otimes\varphi)\\
&\quad -\frac{1}{2}\sum_{\alpha,\beta} \Omega^S_\alpha\cdot\Omega^S_\beta\cdot V_\beta\cdot V_\alpha\cdot(\Psi_1\otimes\varphi)\\
&\quad +\frac{1}{2}\sum_\alpha \Omega^S_\alpha\cdot V_\alpha\cdot(\Psi_1\otimes\varphi)\\
&= -2\sum_\alpha \Omega^S_\alpha\cdot V_\alpha\cdot(\Psi_1\otimes\varphi) + \frac{1}{2}\sum_\alpha \Omega^S_\alpha\cdot\Omega^S_\alpha\cdot(\Psi_1\otimes\varphi)\\
&\quad +\frac{1}{2}\sum_{\alpha,\beta,\gamma} \varepsilon^{123}_{\alpha\beta\gamma}\Omega^S_\alpha\cdot\Omega^S_\beta\cdot V_\gamma\cdot(\Psi_1\otimes\varphi) + \frac{1}{2}\sum_\alpha \Omega^S_\alpha\cdot V_\alpha\cdot(\Psi_1\otimes\varphi)\\
&= -6(\Psi_1\otimes\varphi) + \frac{1}{2}\sum_\alpha \Omega^S_\alpha\cdot V_\alpha\cdot(\Psi_1\otimes\varphi),
\end{aligned}$$

again by making use of (11.14) and (11.15). $\square$

We now have

Lemma 11.12.

$$\mathcal{D}^S\Big(\sum_\alpha V_\alpha\cdot\mathcal{D}^S_\alpha(\Psi_1\otimes\varphi)\Big)=2(m-1)(m+4)\sum_\alpha\Omega^S_\alpha\cdot V_\alpha\cdot(\Psi_1\otimes\varphi)$$
$$-\frac{13}{2}\sum_\alpha V_\alpha\cdot\mathcal{D}^S_\alpha(\Psi_1\otimes\varphi).$$

Proof. Using (11.9) and (11.13), one can write the term

$$F:=\sum_i e_i^*\cdot\nabla_{e_i^*}\Big(\sum_\alpha V_\alpha\cdot\mathcal{D}^S_\alpha(\Psi_1\otimes\varphi)\Big)$$

as

$$\begin{aligned}F=&-\sum_\alpha\Omega^S_\alpha\cdot\mathcal{D}^S_\alpha(\Psi_1\otimes\varphi)+\sum_{\alpha,\beta,\gamma}\varepsilon^{123}_{\alpha\beta\gamma}V_\alpha\cdot V_\beta\cdot\mathcal{D}^S_\gamma(\Psi_1\otimes\varphi)\\&-\sum_{\alpha,i,j}V_\alpha\cdot e_i^*\cdot J^S_\alpha(e_j^*)\cdot(\nabla_{e_i}\nabla_{e_j}\Psi_1\otimes\varphi)\\&+\frac{1}{2}\sum_{\alpha,\beta,\gamma,i,j}\varepsilon^{123}_{\alpha\beta\gamma}e_i^*\cdot J^S_\alpha(e_j^*)\cdot J^S_\beta(e_i^*)\cdot V_\gamma\cdot(\nabla_{e_j}\Psi_1\otimes\varphi).\end{aligned}$$

Now, from Lemma 11.1 and Lemma 7.10,

$$\sum_\alpha\Omega^S_\alpha\cdot\mathcal{D}^S_\alpha(\Psi_1\otimes\varphi)=0.$$

From (11.7), using once again Lemma 11.1 and the fact that $\mathcal{D}_\alpha\Psi_1$ is a local section of $\Sigma_0M\oplus\Sigma_2M$, one gets

$$\sum_{\alpha,\beta,\gamma}\varepsilon^{123}_{\alpha\beta\gamma}V_\alpha\cdot V_\beta\cdot\mathcal{D}^S_\gamma(\Psi_1\otimes\varphi)=-2\sum_\alpha V_\alpha\cdot\mathcal{D}^S_\alpha(\Psi_1\otimes\varphi).\tag{11.16}$$

By Lemma (11.1),

$$\sum_{\alpha,i,j}V_\alpha\cdot e_i^*\cdot J^S_\alpha(e_j^*)\cdot(\nabla_{e_i}\nabla_{e_j}\Psi_1\otimes\varphi)=\sum_\alpha V_\alpha\cdot\Big(\sum_{i,j}e_i\cdot J_\alpha e_j\cdot\nabla_{e_i}\nabla_{e_j}\Psi_1\Big)\otimes\varphi,$$

but we saw in (7.33) that

$$\mathcal{D}^2(\imath\Psi_1)=4(m+4)(m-1)\imath\Psi_1,\tag{11.17}$$

where $\imath\Psi_1$ is locally defined by $\imath\Psi_1=\sum_\alpha J_\alpha\otimes\Omega_\alpha\cdot\Psi_1$.

But since $\mathcal{D}(\imath\Psi_1) = -2\sum_\alpha J_\alpha \otimes \mathcal{D}_\alpha \Psi_1$ (cf. (7.31)), equality (11.17) gives (at the point x)

$$\sum_{i,j} e_i \cdot J_\alpha e_j \cdot \nabla_{e_i} \nabla_{e_j} \Psi_1 = -2(m+4)(m-1)\Omega_\alpha \cdot \Psi_1,$$

whence

$$\sum_{\alpha,i,j} V_\alpha \cdot e_i^* \cdot J_\alpha^S(e_j^*) \cdot (\nabla_{e_i} \nabla_{e_j} \Psi_1 \otimes \varphi) = -2(m+4)(m-1) \sum_\alpha V_\alpha \cdot \Omega_\alpha^S \cdot (\Psi_1 \otimes \varphi).$$

Finally, using that for $\alpha \neq \beta$

$$\begin{aligned}\sum_i e_i^* \cdot J_\alpha^S(e_j^*) \cdot J_\beta^S(e_i^*) &= -2\sum_i \langle J_\alpha^S(e_j^*), J_\beta^S(e_i^*)\rangle e_i^* - 2\Omega_\beta^S \cdot J_\alpha^S(e_j^*) \\ &= -2\sum_\gamma \varepsilon_{\alpha\beta\gamma}^{123} J_\gamma^S(e_j^*) - 2\Omega_\beta^S \cdot J_\alpha^S(e_j^*),\end{aligned}$$

one gets

$$\begin{aligned}&\frac{1}{2} \sum_{\alpha,\beta,\gamma,i,j} \varepsilon_{\alpha\beta\gamma}^{123} e_i^* \cdot J_\alpha^S(e_j^*) \cdot J_\beta^S(e_i^*) \cdot V_\gamma \cdot (\nabla_{e_j} \Psi_1 \otimes \varphi) \\ &\quad = 2\sum_\alpha V_\alpha \cdot \mathcal{D}_\alpha^S \cdot (\Psi_1 \otimes \varphi) - \sum_{\alpha,\beta,\gamma} \varepsilon_{\alpha\beta\gamma}^{123} V_\alpha \cdot \Omega_\beta^S \cdot \mathcal{D}_\gamma^S(\Psi_1 \otimes \varphi).\end{aligned}$$

Now using Lemma 7.10 and Lemma 11.1, the last term in the right-hand side of the above equation is $-8\sum_\alpha V_\alpha \cdot \mathcal{D}_\alpha^S \cdot (\Psi_1 \otimes \varphi)$, and so

$$\begin{aligned}&\sum_i e_i^* \cdot \nabla_{e_i^*}\Big(\sum_\alpha V_\alpha \cdot \mathcal{D}_\alpha^S(\Psi_1 \otimes \varphi)\Big) \\ &\quad = 2(m-1)(m+4)\sum_\alpha V_\alpha \cdot \Omega_\alpha^S \cdot (\Psi_1 \otimes \varphi) - 8\sum_\alpha V_\alpha \cdot \mathcal{D}_\alpha^S(\Psi_1 \otimes \varphi).\end{aligned}$$

On the other hand, using (11.5), (11.6), (11.10), and the definition of $\mathcal{D}_\alpha^S$, one gets

$$\begin{aligned}\nabla_{V_\beta}\Big(\sum_\alpha V_\alpha \cdot \mathcal{D}_\alpha^S(\Psi_1 \otimes \varphi)\Big) = &-\sum_{\alpha,\gamma} \varepsilon_{\alpha\beta\gamma}^{123} V_\alpha \cdot \mathcal{D}_\gamma^S(\Psi_1 \otimes \varphi) \\ &-\frac{1}{2}\sum_\alpha V_\alpha \cdot \Omega_\beta^S \cdot \mathcal{D}_\alpha^S(\Psi_1 \otimes \varphi) \\ &-\frac{1}{2}\sum_\alpha V_\alpha \cdot V_\beta \cdot \mathcal{D}_\alpha^S(\Psi_1 \otimes \varphi).\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{\beta} V_\beta \cdot \nabla_{V_\beta}\Big(\sum_{\alpha} V_\alpha \cdot \mathcal{D}^S_\alpha(\Psi_1 \otimes \varphi)\Big) &= \sum_{\alpha,\beta,\gamma} \varepsilon^{123}_{\alpha\beta\gamma} V_\alpha \cdot V_\beta \cdot \mathcal{D}^S_\gamma(\Psi_1 \otimes \varphi)\\ &\quad - \frac{1}{2}\sum_{\alpha,\beta} V_\beta \cdot V_\alpha \cdot \Omega^S_\beta \cdot \mathcal{D}^S_\alpha(\Psi_1 \otimes \varphi)\\ &\quad - \frac{1}{2}\sum_{\alpha,\beta} V_\beta \cdot V_\alpha \cdot V_\beta \cdot \mathcal{D}^S_\alpha(\Psi_1 \otimes \varphi).\end{aligned}$$

Now, we already noted in (11.16) that

$$\sum_{\alpha,\beta,\gamma} \varepsilon^{123}_{\alpha\beta\gamma} V_\alpha \cdot V_\beta \cdot \mathcal{D}^S_\gamma(\Psi_1 \otimes \varphi) = -2\sum_{\alpha} V_\alpha \cdot \mathcal{D}^S_\alpha(\Psi_1 \otimes \varphi).$$

On the other hand, by (11.13), since $\mathcal{D}_\alpha \Psi_1$ is a local section of $\Sigma_0 M \oplus \Sigma_2 M$ one has

$$\begin{aligned}-\frac{1}{2}\sum_{\alpha,\beta} V_\beta \cdot V_\alpha \cdot \Omega^S_\beta \cdot \mathcal{D}^S_\alpha(\Psi_1 \otimes \varphi) &= \frac{1}{2}\sum_{\alpha} \Omega^S_\alpha \cdot \mathcal{D}^S_\alpha(\Psi_1 \otimes \varphi)\\ &\quad + \frac{1}{2}\sum_{\alpha} V_\alpha \cdot \Big(\sum_{\beta,\gamma} \varepsilon^{123}_{\alpha\beta\gamma} \Omega^S_\beta \cdot \mathcal{D}^S_\gamma(\Psi_1 \otimes \varphi)\Big)\\ &= 4\sum_{\alpha} V_\alpha \cdot \mathcal{D}^S_\alpha(\Psi_1 \otimes \varphi),\end{aligned}$$

since, as we already noted,

$$\sum_{\alpha} \Omega^S_\alpha \cdot \mathcal{D}^S_\alpha(\Psi_1 \otimes \varphi) = 0$$

and

$$\sum_{\beta,\gamma} \varepsilon^{123}_{\alpha\beta\gamma} \Omega^S_\beta \cdot \mathcal{D}^S_\gamma(\Psi_1 \otimes \varphi) = 8\sum_{\alpha} \mathcal{D}^S_\alpha(\Psi_1 \otimes \varphi).$$

Finally, since

$$-\frac{1}{2}\sum_{\alpha,\beta} V_\beta \cdot V_\alpha \cdot V_\beta \cdot \mathcal{D}^S_\alpha(\Psi_1 \otimes \varphi) = -\frac{1}{2}\sum_{\alpha} V_\alpha \cdot \mathcal{D}^S_\alpha(\Psi_1 \otimes \varphi),$$

one obtains

$$\sum_{\beta} V_\beta \cdot \nabla_{V_\beta}\Big(\sum_{\alpha} V_\alpha \cdot \mathcal{D}^S_\alpha(\Psi_1 \otimes \varphi)\Big) = \frac{3}{2}\sum_{\alpha} V_\alpha \cdot \mathcal{D}^S_\alpha(\Psi_1 \otimes \varphi). \qquad \square$$

We may now state the main result. It is a straightforward computation using Lemmas 11.7, 11.8, 11.11, and 11.12.

Proposition 11.13. *Let Φ be the spinor field on S defined by*

$$\Phi = 2(m+3)(\Psi_0 \otimes \varphi) + \Psi_1 \otimes \varphi - \frac{1}{2}\sum_\alpha V_\alpha \cdot \Omega^S_\alpha \cdot (\Psi_1 \otimes \varphi)$$
$$- \frac{1}{2(m+4)} \sum_\alpha V_\alpha \cdot \mathcal{D}^S_\alpha (\Psi_1 \otimes \varphi).$$

Then

$$\mathcal{D}^S \Phi = \frac{4m+3}{2}\Phi = \sqrt{\frac{4m+3}{4m+2}\frac{\mathrm{Scal}_S}{4}}\,\Phi,$$

i.e., Φ is a Killing spinor on S.

Remark 11.14. The heuristic construction of the above Killing spinor on S seems rather "miraculous". By reviewing the basic argument in [KSW98a], we will explain why it is not. We will see that the above spinor fields,

$$\sum_\alpha V_\alpha \cdot \Omega^S_\alpha \cdot (\Psi_1 \otimes \varphi) \quad \text{and} \quad \sum_\alpha V_\alpha \cdot \mathcal{D}^S_\alpha (\Psi_1 \otimes \varphi),$$

are indeed related to the two sections of the twisted bundle $\mathcal{Q}M \otimes \Sigma M$, $\imath\Psi_1$ and $\mathcal{D}(\imath\Psi_1)$, which appear in the study of the limiting case of the estimate in Section 7.4.

The approach developed in [KSW98a] is based on the following remark. On the one hand, the limiting case of the estimate brings into play two spinor fields belonging, respectively, to $\Gamma(\Sigma_0 M)$ and $\Gamma(\Sigma_1 M)$, together with a section of a vector bundle $P_{\Lambda^{m-2}_\circ E}$, with fiber isomorphic to $\Lambda^{m-2}_\circ E$. On the other hand, the determination of the first eigenvalue of the Dirac operator on $(\mathbb{H}\mathrm{P}^m, \mathrm{can})$ (cf. Section 15.4), brings into play the irreducible fundamental representation of the group Sp_{m+1} in the space $\Lambda^m_\circ E_{m+1}$, where

$$E_{m+1} := (\mathbb{C} + \boldsymbol{j}\,\mathbb{C})^{m+1} \simeq \mathbb{H}^{m+1}.$$

But the decomposition of this representation under the action of $\mathrm{Sp}_1 \times \mathrm{Sp}_m$ (viewed as a subgroup of Sp_{m+1}), is given by

$$\begin{aligned} \Lambda^m_\circ E_{m+1} &= \Lambda^m_\circ E \oplus (H \otimes \Lambda^{m-1}_\circ E) \oplus \Lambda^{m-2}_\circ E \\ &\simeq \Sigma_{4m,0} \oplus \qquad \Sigma_{4m,1} \qquad \oplus \Lambda^{m-2}_\circ E, \end{aligned} \tag{11.18}$$

(cf. for instance the branching rules given in [Tsu81]).

In [KSW98a]), the authors give an explicit description of the inclusion

$$\Lambda^m_\circ E \oplus (H \otimes \Lambda^{m-1}_\circ E) \oplus \Lambda^{m-2}_\circ E \hookrightarrow \Lambda^m_\circ E_{m+1},$$

and then use it to construct a section of a bundle with fibre isomorphic to $\Lambda^m_\circ E_{m+1}$, which appears to be a spinor field on S verifying the Killing equation.

In order to understand the relation between $\Lambda^m_\circ E_{m+1}$ and the spinor representation $\Sigma_{4(m+1)}$, recall that the restriction of the homomorphism (cf. (7.2))

$$\mathrm{Sp}_1 \times \mathrm{Sp}_{m+1} \longrightarrow \mathrm{SO}_{4(m+1)}, \quad (q, A) \longmapsto (x \in \mathbb{H}^{m+1} \longmapsto Ax\bar{q})$$

to the subgroup Sp_{m+1}, gives the standard inclusion $\mathrm{Sp}_{m+1} \subset \mathrm{SO}_{4(m+1)}$.

Now, the spinor space $\Sigma_{4(m+1)}$ decomposes under the action of $\mathrm{Sp}_1 \times \mathrm{Sp}_{m+1}$ as (cf. (7.6))

$$\Sigma_{4(m+1)} = \bigoplus_{r=0}^{m+1} \Sigma_{4(m+1),r},$$

where

$$\Sigma_{4(m+1),r} \simeq \mathrm{S}^r H \otimes \Lambda^{m+1-r}_\circ E_{m+1}.$$

Hence, since Sp_{m+1} acts trivially on $\mathrm{S}^r H$, each $\Sigma_{4(m+1),r}$ decomposes under the action of Sp_{m+1} into $r+1$ copies of the representation $\Lambda^{m+1-r}_\circ E_{m+1}$. In particular,

(a) $\Sigma_{4(m+1),m+1}$ decomposes into $(m+2)$ trivial representations;

(b) $\Sigma_{4(m+1),1}$ decomposes into two copies of $\Lambda^m_\circ E_{m+1}$.

Hence, the fundamental representation of Sp_{m+1} in the space $\Lambda^m_\circ E_{m+1}$ is contained (with multiplicity 2) in $\Sigma_{4(m+1),1}$. Note moreover that, as it was noted before, $\Sigma_{4m+4,1}$ is contained in either $\Sigma^+_{4(m+1)}$ or $\Sigma^-_{4(m+1)}$, hence the fundamental representation of Sp_{m+1} in the space $\Lambda^m_\circ E_{m+1}$ can be seen as contained in the spinor representation $\Sigma_{4m+3} \simeq \Sigma^+_{4(m+1)} \simeq \Sigma^-_{4(m+1)}$.

Now from the definition of the spin structure of S given above, together with the description of the spin structure of M given in Section 7.2, it follows that the spinor bundle $\Sigma_{4m+3}S$ is indeed an $\mathrm{Sp}_1 \times \mathrm{Sp}_m$-bundle. Hence, by (11.18), one concludes that the bundles $\pi^*(\Sigma_0 M)$, $\pi^*(\Sigma_1 M)$, $\pi^*(V_{1,1}M)$, and $\pi^*(V_{0,2}M)$, where $V_{1,1}M$ and $V_{0,2}M$ are the subbundles of $\mathcal{Q}M \otimes \Sigma_2 M$, with fiber respectively isomorphic to $H \otimes \Lambda^{m-1}_\circ E$ and $\Lambda^{m-2}_\circ E$, can be embedded into the spinor bundle $\Sigma_{4m+3}S \simeq \pi^*(\Sigma M) \otimes \Sigma_3 S$.

By considering explicit embeddings, it can be proved that, up to a scalar multiple, the spinor fields Ψ_0, Ψ_1 correspond respectively to the projectable spinors $\Psi_0 \otimes \varphi$ and $\Psi_1 \otimes \varphi$, whereas the two sections $\imath\Psi_1$ and $\mathcal{D}(\imath\Psi_1)$ of the twisted bundle $\mathcal{Q}M \otimes \Sigma M$ correspond respectively to

$$\sum_\alpha V_\alpha \cdot \Omega^S_\alpha \cdot (\Psi_1 \otimes \varphi) \quad \text{and} \quad \sum_\alpha V_\alpha \cdot \mathcal{D}^S_\alpha (\Psi_1 \otimes \varphi).$$

11.4 Conclusion

We may now conclude as in [KSW98a]. The Killing spinor Φ induces a parallel spinor on the Riemannian cone $\overline{S}$ over S. Moreover, $\overline{S}$ is a hyper-Kähler manifold (cf. Theorem 8.20). Now, due to a result of Wang [Wan89], on an hyper-Kähler manifold of dimension $4(m+1)$ there are exactly $m+2$ linearly independent parallel spinors, corresponding to the $m+2$ trivial representations induced by the decomposition of $\Sigma_{4(m+1),m+1}$ under the group Sp_{m+1} (cf. Remark 11.14).

By the general description of spinors on Riemannian cones, cf. Section 2.4.2, together with Remark 11.14 above, the parallel spinor induced by Φ is indeed a section of $\Sigma_{4(m+1),1}\overline{S}$. This forces the holonomy to be further reduced. But from the Berger–Simons theorem (cf. for instance Theorem 10.90 in [Bes87]), this is only possible if $\overline{S}$ is reducible or locally symmetric.

In the first case, $\overline{S}$ has to be flat, by a result of Gallot [Gal79]. This is also the case if $\overline{S}$ is locally symmetric, because any hyper-Kähler manifold is Ricci-flat and any homogeneous Ricci-flat manifold is flat (cf. for instance Theorem 7.61 in [Bes87]). Therefore $\overline{S}$ is flat. Since S is simply connected (cf. [BG99], Proposition 4.4.1), this implies that S is isometric to the standard sphere $\mathbb{S}^{4m+3}$. Hence, by a result of Ishihara [Ish73], M is isometric to the quaternionic projective space.

Part IV

Dirac spectra of model spaces

In the study of the spectrum of the Dirac operator, it is natural and useful to consider examples, and the archetypal examples in Riemannian Geometry are the *symmetric spaces*. For a symmetric space, the spectrum of the Dirac operator can be (theoretically) computed by classical harmonic analysis methods.

The aim of this part is to present a method for the determination of the spectrum of the Dirac operator on a spin, compact, simply connected and irreducible symmetric space, and to give the explicit computation of the spectrum in those three cases:

(1) spheres $(\mathbb{S}^n, \mathrm{can})$, $n \geq 2$;

(2) complex projective spaces $(\mathbb{C}\mathrm{P}^m, \mathrm{can})$, $m = 2q + 1$, $q \geq 0$;

(3) quaternionic projective spaces $(\mathbb{H}\mathrm{P}^m, \mathrm{can})$, $m \geq 1$.

Note that the holonomy of those manifolds is, respectively, SO_n, U_{2q+1} and $\mathrm{Sp}_1 \,.\, \mathrm{Sp}_m$, so they are standard examples of, respectively, Riemannian, Kähler and quaternion-Kähler manifolds.

Chapter 12

A brief survey on representation theory of compact groups

In this chapter we sketch the theory of finite-dimensional complex representations of a compact and connected Lie group G, that is Lie groups homomorphisms

$$\rho: G \longrightarrow \mathrm{GL}(V),$$

where V is a finite-dimensional $\mathbb{C}$-vector space.

We assume known the basic terminology on representation theory. Our aim is to give a brief survey on the construction of the theory in order to consider the following problems:

- Describe all irreducible representations (up to equivalence) of G.
- Give the decomposition of an arbitrary representation of G into irreducible representations.

Results and detailed proofs can be found, for example, in [Ada69], [BtD85], [Die75], [FH91], [GW98], [Kna01], [Hum72], and [Žel73].

12.1 Reduction of the problem to the study of irreducible representations

Because of the compactness of the group, to describe all finite-dimensional representations of G it is enough to know its irreducible representations (up to equivalence). This is due to the following fundamental result.

Theorem 12.1. *Every finite-dimensional representation of G is unitary, and hence completely reducible.*

Proof. Choose a Hermitian product on the Lie algebra $\mathfrak{g} = T_e G$ and construct a right-invariant Riemannian metric on G by right translations:

$$R_g: G \longrightarrow G, \quad R_g x := xg.$$

This provides us with a right-invariant volume form dg. We normalize the metric so that $\mathrm{vol}(G) = \int_G dg = 1$. Let (ρ, V) be a finite-dimensional representation. Let $\langle \cdot, \cdot \rangle$ be any Hermitian product on V and let $\langle \cdot, \cdot \rangle_G$ be the Hermitian product on V defined by

$$\langle v_1, v_2 \rangle_G = \int_G \langle \rho(g) v_1, \rho(g) v_2 \rangle dg, \quad (v_1, v_2) \in V \times V.$$

By the right-invariance of dg, the representation ρ is unitary for this Hermitian product.

Suppose that V is not irreducible. Then it admits a non-trivial G-invariant subspace U. But the orthogonal complement $U^\perp$ for the Hermitian product $\langle \cdot, \cdot \rangle_G$, is also a G-invariant subspace of V, so V decomposes into the sum of G-spaces $V = U \oplus U^\perp$. If either U or $U^\perp$ is not irreducible, V can be further decomposed. The process must eventually stop because V is finite dimensional. $\square$

This theorem applies, in particular, to the adjoint representation of G, yielding an Ad_G-invariant Hermitian product on $\mathfrak{g}$. The right-invariant metric on G defined by this product is then bi-invariant, and its volume form dg is thus the Haar measure on G. Note also that the averaging construction continues to hold when V is an infinite-dimensional Hilbert space.

A simple but crucial fact in representation theory is the Schur lemma.

Lemma 12.2. *Let ρ, ρ' be irreducible unitary representations of G and $A\colon V \to V'$ a linear map commuting with the G-actions. If V and V' are not isomorphic as representation spaces, then A must be* 0. *If $\rho = \rho'$ and* $\dim(V) < \infty$, *then A must be a constant multiple of the identity.*

Proof. The kernel of A and the orthogonal complement of its image are both representation spaces of G. Since the two representations are assumed irreducible, it follows that A is either 0 or an isomorphism of G-spaces, proving the first claim. If $\rho = \rho'$, $V = V'$, the endomorphism A must have an eigenspace for some eigenvalue λ. Since A commutes with the G action, this eigenspace is a sub-representation, so it must be the whole of V. $\square$

For any unitary representation (ρ, V) (where V is a complex Hilbert space of possibly infinite dimension) and any orthonormal basis $\{e_i\}$ of V, the matrix of $\rho(g)$ is unitary for any $g \in G$. Let

$$m_{ij}^\rho \colon G \longrightarrow \mathbb{C}, \quad m_{ij}^\rho(g) := \langle \rho(g) e_j, e_i \rangle,$$

be the (smooth) functions defined by considering the coefficients of this matrix. These functions are called *matrix coefficients* of the representation ρ. With respect to the usual Hermitian scalar product $(\cdot, \cdot)$ on $L^2(G)$, the matrix coefficients verify the following orthogonality properties.

Theorem 12.3. *Let ρ and ρ' be two irreducible representations of G (possibly of infinite dimension). Their matrix coefficients satisfy the following properties:*

(i) $(m^{\rho}_{ij}, m^{\rho'}_{k',l'}) = 0$, *if ρ and ρ' are not equivalent.*

(i) *If $\rho = \rho'$ and* $\dim(V) = n_\rho < \infty$, *then* $(m^{\rho}_{ij}, m^{\rho}_{kl}) = \frac{1}{n_\rho}\delta_{ik}\delta_{jl}$.

Proof. Let (ρ, V) and (ρ', V') be two inequivalent irreducible representations of G. The Schur lemma implies that the space of G-invariant homomorphisms $V \to V'$,

$$\mathrm{Hom}_G(V, V') := \{A \in \mathrm{Hom}(V, V');\ \rho'(g) \circ A = A \circ \rho(g),\ \text{for all } g \in G\},$$

is trivial because any G-invariant homomorphism $V \to V'$ has to be an isomorphism.

Now to each $A \in \mathrm{Hom}(V, V') \simeq V^* \otimes V'$ can be associated, using the averaging technique of the above proof, the G-invariant endomorphism

$$\widehat{A} = \int_G \rho'(g^{-1}) \circ A \circ \rho(g) dg.$$

But since $\mathrm{Hom}_G(V, V') = \{0\}$, any such $\widehat{A}$ has to be zero. Applying this to the basis $\{e_i^* \otimes e_j'\}$ of $V^* \otimes V'$, where $\{e_i\}$ (resp. $\{e_j'\}$) is a basis of V (resp. V'), in which the matrix of $\rho(g)$ (resp. $\rho'(g)$) is unitary for any $g \in G$, one gets the first set of orthogonality relations.

On the other hand, by applying the averaging technique to $A \in \mathrm{Hom}(V, V)$, since by the Schur lemma any vector in $\mathrm{Hom}_G(V, V)$ is a scalar multiple of identity, one gets

$$\widehat{A} = \int_G \rho(g^{-1}) \circ A \circ \rho(g) dg = \lambda \mathrm{Id},$$

where $\lambda \in \mathbb{C}$. Note that

$$\mathrm{tr}\, \widehat{A} = \mathrm{tr}\, A = \lambda \dim V.$$

Applying this to the basis $\{e_i^* \otimes e_j\}$ of $V^* \otimes V$, one gets the second set of orthogonality relations. □

Instead of matrix coefficients which involve the choice of a basis, representations can be uniquely characterized, up to equivalence, by a single function on G.

Definition 12.4. Let ρ be a representation of G. The (smooth) function

$$\chi_\rho \colon G \longrightarrow \mathbb{C}, \quad g \longmapsto \chi_\rho(g) := \mathrm{tr}(\rho(g))$$

is called the *character* of the representation ρ. Characters of irreducible representations are called *irreducible characters*.

The following results are easily deduced from the definition.

Proposition 12.5. *Let ρ and ρ' be two representations of G.*

(i) *If ρ and ρ' are equivalent, then $\chi_\rho = \chi_{\rho'}$, so χ_ρ depends only on the equivalence class of ρ.*

(ii) *Characters are* class functions, *i.e., they verify*

$$\chi_\rho(hgh^{-1}) = \chi_\rho(g), \quad g, h \in G,$$

(iii) $\chi_{\rho \oplus \rho'} = \chi_\rho + \chi_{\rho'}$.

(iv) $\chi_{\rho \otimes \rho'} = \chi_\rho \chi_{\rho'}$.

(v) *If ρ is unitary, $\chi_{\rho^*} = \overline{\chi_\rho}$.*

By Theorem 12.3 we get the following proposition.

Proposition 12.6. *Let χ_ρ and $\chi_{\rho'}$ be two irreducible characters. Then*

$$(\chi_\rho, \chi_{\rho'}) = \begin{cases} 0 & \textit{if } \rho \textit{ and } \rho' \textit{ are not equivalent;} \\ 1 & \textit{if } \rho = \rho'. \end{cases}$$

This implies that irreducible characters are linearly independent and

Corollary 12.7. *If two irreducible characters satisfy $\chi_\rho = \chi_{\rho'}$, then ρ and ρ' are equivalent.*

Corollary 12.8. *The character χ_ρ of every representation ρ satisfies $(\chi_\rho, \chi_\rho) \in \mathbb{Z}$, $(\chi_\rho, \chi_\rho) \geq 1$, with equality if and only if ρ is irreducible.*

Proof. Let $\rho = \bigoplus_i n_i \rho_i$ be the decomposition of ρ into irreducible components, where $n_i \in \mathbb{N}$ denotes the multiplicity of ρ_i in ρ. Then $(\chi_\rho, \chi_\rho) = \sum_i n_i^2$, hence the result. □

Proposition 12.9. *Let G and H be compact Lie groups. Let ρ_G, ρ'_G and ρ_H, ρ'_H be irreducible representations of G, respectively of H. Then the tensor products $\rho_G \otimes \rho_H$ and $\rho'_G \otimes \rho'_H$ are irreducible representations of $G \times H$. Moreover we have $\rho_G \otimes \rho_H \sim \rho'_G \otimes \rho'_H$ if and only if $\rho_G \sim \rho'_G$ and $\rho_H \sim \rho'_H$.*

Conversely, every irreducible representation of $G \times H$ is equivalent to a unique tensor product of this form.

Proof. From Proposition 12.5 (iv) and Proposition 12.6, one has

$$
\begin{aligned}
(\chi_{\rho_G\otimes\rho_H}, \chi_{\rho'_G\otimes\rho'_H}) &= \int_{G\times H} \overline{\chi_{\rho'_G}(g)\chi_{\rho'_H}(h)}\chi_{\rho_G}(g)\chi_{\rho_H}(h)dgdh \\
&= \int_G \overline{\chi_{\rho'_G}(g)}\chi_{\rho_G}(g)dg \cdot \int_H \overline{\chi_{\rho'_H}(h)}\chi_{\rho_G}(h)dh \\
&= (\chi_{\rho_G}, \chi_{\rho'_G})(\chi_{\rho_H}, \chi_{\rho'_H}).
\end{aligned}
$$

Hence, by Corollary 12.8, $\rho_G \otimes \rho_H$ is an irreducible $(G \times H)$-representation and the first part of the proposition follows.

Let (ρ, V) be an irreducible $(G \times H)$-representation. Identifying H with the closed subgroup $\{e\} \times H$ of $G \times H$, the space V can be decomposed into the direct sum $V = \bigoplus_i V_i$, where the V_i are the irreducible H-subspaces of V.

Now the space $\mathrm{Hom}_H(V_i, V)$ of H-invariant homomorphisms $V_i \to V$ is a G-space. Indeed, for any $g \in G$, for any $A \in \mathrm{Hom}_H(V_i, V)$, set

$$g \cdot A := \rho(g, e) \circ A.$$

It is easy to see that $(g \cdot A) \in \mathrm{Hom}_H(V_i, V)$. Thus the space $\mathrm{Hom}_H(V_i, V)$ can be decomposed into the direct sum $\mathrm{Hom}_H(V_i, V) = \bigoplus_j W_{ij}$, where the W_{ij} are the irreducible G-subspaces of $\mathrm{Hom}_H(V_i, V)$. Consider then the endomorphism

$$
\begin{aligned}
F_{ij}\colon W_{ij} \otimes V_i &\longrightarrow \quad V \\
(A \otimes v) &\longmapsto A(v).
\end{aligned}
$$

It is easy to check that $F_{ij} \in \mathrm{Hom}_{G\times H}(W_{ij} \otimes V_i, V)$. Hence, since $W_{ij} \otimes V_i$ and V are irreducible $(G \times H)$-spaces, the Schur lemma implies that F_{ij} is either zero or an isomorphism.

Now $\mathrm{Hom}_H(V_i, V)$ contains the canonical injection $A\colon V_i \hookrightarrow V$. According to the decomposition $\mathrm{Hom}_H(V_i, V) = \bigoplus_j W_{ij}$, A splits as $A = \sum_j A_j$. Let v_i be a non-zero vector in V_i. Necessarily one of the vectors $A_j(v_i) = F_{ij}(A_j, v_i)$ has to be non-zero, thus the corresponding F_{ij} is an isomorphism, and V is equivalent to $W_{ij} \otimes V_i$. □

Let R_G be the set of equivalence classes of irreducible finite-dimensional complex representations of G, and let Φ_G be the vector space generated by the matrix coefficients m^ρ_{ij} of all the $\rho \in R_G$. By choosing different bases in V, the matrix coefficients change by linear combinations, so Φ_G does not depend on these choices.

By Theorem 12.1, Φ_G is a subalgebra of the algebra $\mathcal{C}^\infty(G)$ of smooth functions on G (indeed, the product of two matrix coefficients $m^\rho_{ij}, m^{\rho'}_{k'l'}$ is a matrix coefficient of the tensor product $\rho \otimes \rho'$, which factors as the direct sum of irreducible representations).

Theorem 12.10 (Peter–Weyl). *Let G be a compact Lie group.*

(1) *The functions $(\frac{1}{n_\rho} m_{ij}^\rho)$, with $\rho \in R_G$, form a Hilbert basis of the Hilbert space $L^2(G)$.*

(2) *Every continuous function on G is uniformly approximable by functions from Φ_G.*

Proof. We first give the proof of the second statement of the theorem under the assumption that G has a faithful representation (such as classical matrix groups or spin groups), because in this case the result can be simply deduced from the Stone–Weierstraß theorem.

First, the constant function $f_0(g) \equiv 1$ is the matrix coefficient of the trivial representation on the space $\mathbb{C}$, hence constant functions belong to Φ_G. For any $m_{ij}^\rho \in \Phi_G$, $(\rho, V) \in R_G$, the function $\overline{m_{ij}^\rho}$ is also in Φ_G, because it is a matrix coefficient of the dual representation (ρ, V^*), hence Φ_G is closed under complex conjugation. Finally, the existence of a faithful representation ensures that Φ_G separates points of G. Thus, from the Stone–Weierstraß theorem, Φ_G is dense in $\mathcal{C}^0(G)$ for the supremum norm topology.

Let us now turn to the general case. There exists another algebra structure on $\mathcal{C}^\infty(G)$, with respect to the *convolution product*

$$(f * f')(x) := \int_G f(xy) f'(y^{-1}) dy = \int_G f(y) f'(y^{-1}x) dy.$$

The algebra $\mathcal{C}^\infty(G)$ is endowed with the usual L^2 inner product with respect to the bi-invariant metric on G of volume 1,

$$(f, f') = \int_G f(x) \overline{f'(x)} dx.$$

Finally, there is a $G \times G$ action on $\mathcal{C}^\infty(G)$, called the *regular representation*, derived from the (mutually commuting) left and right actions of G on itself:

$$(g, g') f := L_g R_{g'} f, \quad (L_g f)(x) = f(g^{-1}x), \quad (R_g f)(x) = f(xg).$$

This action preserves the L^2 inner product, and is compatible with the convolution

$$L_g R_{g'}(f * f') = (L_g f) * (R_{g'} f').$$

Thus there is an induced Hermitian action of $G \times G$ on the norm-completion $L^2(G)$ of $\mathcal{C}^\infty(G)$. Note also that the Laplacian Δ on G commutes with the $G \times G$ action. Since the eigenspaces of the Laplacian are finite-dimensional and span $L^2(G)$ by Corollary 4.37, $G \times G$ preserves these finite-dimensional eigenspaces and hence its action on $L^2(G)$ splits into finite-dimensional representations.

Let $\rho: G \to \mathrm{Aut}(V)$ be an irreducible unitary representation. Then the algebra $\mathrm{End}(V)$ is also a Hermitian representation of $G \times G$:

$$(g, g')A = \rho(g)A\rho(g'^{-1}), \quad \langle A, B\rangle = \dim(V)\,\mathrm{tr}(AB^*).$$

By Proposition 12.9, this representation is irreducible (since $\mathrm{End}(V) = V \otimes V^*$). We define a map from $\mathrm{End}(V)$ to $\mathcal{C}^\infty(G)$ by

$$\iota_V: \mathrm{End}(V) \longrightarrow \mathcal{C}^\infty(G), \quad A \longmapsto F_A, \quad F_A(x) = n_V\,\mathrm{tr}(\rho(x^{-1})A).$$

where $n_V = \dim(V)$. We also define

$$p_V: \mathcal{C}^\infty(G) \longrightarrow \mathrm{End}(V), \quad f \longmapsto A_f = \int_G f(x)\rho(x)dx.$$

Directly from the definitions, these two maps commute with the $G \times G$ actions.

Lemma 12.11. *Let ρ^V, $\rho^{V'}$ be irreducible representations of G. For all $A \in \mathrm{End}(V)$ one has $p_{V'} \circ \iota_V A = A$ if $\rho^V = \rho^{V'}$, and zero if ρ^V and $\rho^{V'}$ are inequivalent.*

Proof. We write

$$p_{V'} \circ \iota_V: \mathrm{End}(V) \longrightarrow \mathrm{End}(V'),$$

$$p_{V'}(\iota_V(A)) = n_V \int_G \mathrm{tr}(\rho(x^{-1})A)\rho^{V'}(x)dx \in \mathrm{End}(V').$$

By Schur lemma and Proposition 12.9, this $(G \times G)$-equivariant map vanishes if V is not equivalent to V', while if $V = V'$ it is a constant multiple of the identity transformation of $\mathrm{End}(V)$, i.e., there exists $\lambda_V \in \mathbb{C}$ such that $p_V(\iota_V(A)) = \lambda_V A$ for all A. To compute the constant λ_V we apply $p_V \circ \iota_V$ to the identity I_V and compute its trace:

$$\mathrm{tr}(p_V(\iota_V(I_V))) = n_V \int_G \mathrm{tr}(\rho(x^{-1}))\,\mathrm{tr}(\rho(x))dx = n_V \|\chi_\rho\|^2_{L^2} = n_V,$$

where in the last equality we have used Corollary 12.8. Therefore, $\lambda_V = 1$. □

It is easy to see that p_V and ι_V are compatible with the algebra structures (the composition of linear maps, respectively the convolution product). Moreover, we claim that ι_V preserves the inner products, more precisely, for two representations ρ^V, $\rho^{V'}$ we have

$$\begin{aligned}(F_A^V, F_B^{V'}) =& n_V n_{V'} \int_G \mathrm{tr}(\rho^V(x^{-1})A)\overline{\mathrm{tr}(\rho^{V'}(x^{-1})B)}dx \\ &= n_V n_{V'}\,\mathrm{tr}\left[B^* \int_G \mathrm{tr}(\rho^V(x^{-1})A)\rho^{V'}(x)dx\right] \\ &= n_{V'}\,\mathrm{tr}(B^* p_{V'}\iota_V(A)).\end{aligned}$$

By Lemma 12.11, this is 0 if V is not equivalent to V', respectively $n_V \operatorname{tr}(B^*A)$ if $V = V'$, thus proving the claim.

It follows that ι_V embeds $\operatorname{End}(V)$ into $\mathcal{C}^\infty(G)$ as an irreducible representation $\mathcal{A}_V$ of $G \times G$. Since the L^2 orthogonal projection on every eigenspace of Δ commutes with the $G \times G$ action, it follows that $\iota_V(\operatorname{End}(V))$ lives inside some eigenspace of the Laplacian.

The complex conjugates of the matrix coefficients of the representation V are in fact, by definition, a linear basis of $\mathcal{A}_V$. Thus for inequivalent representations V, V', the spaces $\mathcal{A}_V, \mathcal{A}_{V'}$ are orthogonal in $L^2(G)$. Clearly, $\bigoplus_{V \in R(G)} \mathcal{A}_V$ is just the space of matrix coefficients Φ_G.

We will show that $\bigoplus_{V \in R(G)} \mathcal{A}_V$ is dense in $L^2(G)$. Otherwise, there would exist some orthogonal complement of it inside some eigenspace of Δ. Call $\mathcal{B}$ this finite-dimensional Hilbert space. It consists of smooth functions, L^2-orthogonal to the matrix coefficients of every finite-dimensional representation of G. It is moreover a $G \times G$ representation space, hence also closed under the left regular action of G, i.e., if $f \in \mathcal{B}$, then for every $x \in G$, the map $L_x f$ sending g to $f(x^{-1}g)$ also belongs to $\mathcal{B}$. Let $f, f' \in \mathcal{B}$ be of norm 1, living in some irreducible component $\mathcal{B}_0$ of $\mathcal{B}$ with respect to the right action of G on $\mathcal{C}^\infty(G)$. Then

$$(R_g f, f') = \int_G f(xg)\overline{f'(x)}dx = \left(\int_G \overline{f'(x)} L_{x^{-1}}(f)dx\right)(g).$$

As a function of g, this integral belongs to the complete linear space $\mathcal{B}$. We reach now a contradiction: on the one hand this function is a matrix coefficient for the irreducible component $\mathcal{B}_0$ of the right regular representation, on the other hand, since it belongs to $\mathcal{B}$, it is orthogonal to every irreducible matrix coefficient. Thus all the matrix coefficients of the representation $\mathcal{B}_0$ vanish, which contradicts the character identity of Proposition 12.6.

For the second claim of the theorem, by Stone–Weierstraß, every continuous function on G can be uniformly approximated by smooth functions. The decomposition of a smooth function into eigenmodes (eigenfunctions of the Laplacian) converges in every Sobolev norm. By the Sobolev embedding theorem

$$H^{\frac{n}{2}+\epsilon}(G) \hookrightarrow \mathcal{C}^0(G);$$

such a convergence holds in $\mathcal{C}^0$ (i.e., uniform) norm. □

For a more "classical" proof in the general case, see [BtD85], Theorem (3.1), p. 134.

Corollary 12.12. *R_G is a countable set.*

Note that for a general compact Lie group, the existence of a faithful representation is deduced from the Peter–Weyl theorem.

Theorem 12.13. *Every compact Lie group G has a finite-dimensional faithful representation.*

Proof. Consider the eigenspaces E_λ of the Laplacian inside $L^2(G)$, which are finite-dimensional representation spaces for the left regular action of G, and at the same time the spaces of matrix coefficients of these representations:

$$E_\lambda = \mathrm{End}(V_\lambda^1) \oplus \cdots \oplus \mathrm{End}(V_\lambda^{k(\lambda)}).$$

Let

$$L_N^2 := \bigoplus_{\lambda \leq N} E_\lambda,$$

and set

$$V_N := \bigoplus_{\substack{\lambda \leq N, \\ j \leq k(\lambda)}} V_\lambda^j$$

the corresponding representation. The kernel of the representation V_N is the set of $g \in G$ such that all the matrix coefficients of V_N take the same values at g and at $1 \in G$, or equivalently the algebra of functions V_N does not separate 1 and g. The set of such g is clearly a closed subgroup, and becomes smaller as N increases. If the intersection of these nested Lie groups contains some point g other than 1, it would mean that L_N^2 does not separate 1 and g for any N, hence by the Peter–Weyl theorem $\mathcal{C}^0(G)$ would not separate these two points, which is absurd. Hence the intersection is $\{1\}$. Now note that a decreasing sequence of compact Lie groups is stationary after a finite number of steps. Thus there exists N such that V_N is faithful. □

For another proof see, for instance, [BtD85], Theorem (4.1), p. 136.

Remark 12.14. Using the Peter–Weyl theorem, we can also prove that any continuous irreducible representation of a compact Lie group on a Hilbert space is finite dimensional. Indeed, by Proposition 12.3, the matrix coefficients of an infinite-dimensional representation must be L^2-orthogonal to every matrix coefficient of any finite-dimensional representation, thus by Peter–Weyl they must vanish, which implies the absurd conclusion that the representation itself is 0. See also [BtD85], Corollary (5.8), p. 141, or [Žel73], Lemma 7, p. 77.

The existence of a faithful representation allows us to describe any irreducible representation as acting on a space of tensors.

Theorem 12.15. *Let ρ_1 be a finite-dimensional faithful representation of G. Every irreducible representation of G is contained in a tensor product of the form*

$$\rho_{m,n} := \underbrace{\rho_1 \otimes \cdots \otimes \rho_1}_{m} \otimes \underbrace{\overline{\rho_1} \otimes \cdots \otimes \overline{\rho_1}}_{n}, \quad m, n \in \mathbb{N},$$

($\rho_{0,0}$ *being the trivial representation on the space* $\mathbb{C}$).

Proof. Suppose there exists an irreducible representation ρ which does not satisfy the above condition. Let m_{ij}^{ρ} be the matrix coefficients of ρ. From Theorem 12.1 and Theorem 12.3, any m_{ij}^{ρ} should be orthogonal to any matrix coefficient of the representations $\rho_{m,n}$, $m, n \in \mathbb{N}$. But it is not hard to see that the matrix coefficients of the representations $\rho_{m,n}$, $m, n \in \mathbb{N}$, define a subalgebra of $\mathcal{C}^0(G)$ which satisfies all the hypothesis of the Stone–Weierstraß theorem. Thus, all the m_{ij}^{ρ} should be equal to 0, a contradiction. □

Corollary 12.16. *Let H be a closed subgroup of G (hence a compact Lie subgroup of G). Every irreducible representation of H is contained in the restriction to H of a representation of G.*

Proof. Let ρ_1 be a faithful representation of G. Then $\rho_1|_H$ is a faithful representation of H. By Theorem 12.15, any irreducible representation of H is contained in a tensor product of $\rho_1|_H$ and its conjugate, which is clearly the restriction of a representation of G. □

Theorem 12.10 admits the

Corollary 12.17. (i) *Every continuous class function*[1] *on G is uniformly approximable by a linear combination of irreducible characters.*

(ii) *The irreducible characters χ_ρ, $\rho \in R_G$, form a Hilbert basis of the center of $L^2(G)$ with respect to the convolution product.*

The previous results allow one to endow the $\mathbb{Z}$-module $\mathbb{Z}^{R_G}$ generated by R_G with a commutative ring structure given by identifying an equivalence class with its character.

Classically, $\mathbb{Z}^{R_G}$ is called *ring of equivalence classes of representations of G*. Note that a linear combination of representations in R_G with integer coefficients is the class of a representation only if all the integers are non-negative. Accordingly, the elements of $\mathbb{Z}^{R_G}$ are also called *virtual representations*.

The following result is straightforward.

[1]The definition of "class function" was given in Proposition 12.5 (ii).

Proposition 12.18. *Let G be a compact and connected Lie group and let*

$$\pi : G \longrightarrow H$$

be an onto homomorphism of Lie groups (thus H is compact and connected too). Any irreducible representation of H is obtained as a factor through the quotient of an irreducible representation ρ of G with the property that for all $g \in \operatorname{Ker} \pi$, we have $\rho(g) = \mathrm{Id}$.

12.2 Reduction of the problem to the study of irreducible representations of a maximal torus

Definition 12.19. A *torus* of G is a closed, connected, and Abelian subgroup of G. A torus of G is said to be *maximal* if there is no other torus of G containing it strictly.

Note that every closed, connected, and Abelian subgroup of G is a compact connected Abelian Lie group, hence is isomorphic to a torus T^n, hence the terminology.

Theorem 12.20. *The Lie group G is the union of its maximal tori.*

Proof. Let $\mathfrak{g}$ be the Lie algebra of G. The proof uses the fact that the exponential map $\exp : \mathfrak{g} \to G$ is onto. Indeed, the compact group G posses a bi-invariant metric (see the remark after Theorem 12.1) for which the (Riemannian) exponential map Exp_e at the point e coincides with the group exponential exp (cf. for instance Example 2.90 in [GHL87]). But since G is compact and connected, it results from the Hopf–Rinow theorem that Exp_e is onto, hence exp is onto too.

Let $g \in G$. Since exp is onto, there exists $X \in \mathfrak{g}$ such that $\exp(X) = g$. Thus g belongs to the one-parameter subgroup $\{\exp(tX), t \in \mathbb{R}\}$. The Lie algebra of this subgroup being Abelian, is contained in a maximal Abelian subalgebra $\mathfrak{t}$ of $\mathfrak{g}$.

Now let T be the connected Abelian Lie subgroup of G whose Lie algebra is $\mathfrak{t}$. This group is necessarily closed: otherwise, its closure $\overline{T}$ would be a connected compact and Abelian Lie subgroup of G whose Lie algebra $\mathfrak{t}'$ satisfies $\mathfrak{t} \subsetneq \mathfrak{t}'$, contradicting the maximality of $\mathfrak{t}$. Hence, T is a maximal torus such that $g \in T$. □

Theorem 12.21. *Let T be a maximal torus of G. For any $g \in G$, there exists $s \in G$ such that $sgs^{-1} \in T$.*

Proof. First recall that since T is isomorphic to a torus $\mathbb{T}^n$, it has a generator, that is an element u such that the group $\{u^k, k \in \mathbb{Z}\}$ is dense in T (cf. for instance [BtD85], Theorem (4.13), p. 38). Take $g_0 \in G$. Since the exponential map of G is onto (cf. the proof of Theorem 12.20), there exist a $X_0 \in \mathfrak{g}$ and a Z in the Lie algebra $\mathfrak{t}$ of T such that $\exp(X_0) = g_0$ and $\exp(Z) = u$. Note that the one-parameter subgroup $\{\exp(tZ); t \in \mathbb{R}\}$ is dense in T.

Let $\langle \cdot, \cdot \rangle$ be an $\mathrm{Ad}(G)$-invariant scalar product on $\mathfrak{g}$ (such a scalar product on the finite-dimensional representation space $\mathfrak{g}$ exists as a consequence of Theorem 12.1 since G is compact), and let $\| \cdot \|$ be the norm associated with it. Now G being compact, there exists an $s \in G$ such that the function on G defined by

$$g \longmapsto \| \mathrm{Ad}(g) \cdot X_0 - Z \|^2$$

admits a minimum at the point s. Therefore, the tangent map at s of this function is zero, whence

$$\frac{d}{dt}\{\| \mathrm{Ad}(\exp(tX)s) \cdot X_0 - Z \|^2\}_{t=0} = 0, \quad X \in \mathfrak{g}.$$

Setting $Y_0 := \mathrm{Ad}(s) \cdot X_0$, one gets

$$0 = 2\langle [X, Y_0], Y_0 - Z \rangle.$$

But by the $\mathrm{Ad}(G)$-invariance of the scalar product, this implies that

$$\langle X, [Y_0, Z] \rangle = 0, \quad X \in \mathfrak{g},$$

hence

$$\mathrm{ad}(Z) \cdot Y_0 = 0.$$

This in turn yields

$$\mathrm{Ad}_{\exp(tZ)}\, Y_0 = \mathrm{e}^{t\,\mathrm{ad}(Z)} \cdot Y_0 = Y_0, \quad t \in \mathbb{R}.$$

Hence, since the one-parameter subgroup $\{\exp(tZ), t \in \mathbb{R}\}$ is dense in T,

$$\mathrm{Ad}(t) \cdot Y_0 = Y_0, \quad t \in T$$

and so

$$[X, Y_0] = 0, \quad X \in \mathfrak{t}.$$

But since $\mathfrak{t}$ is a maximal Abelian subalgebra of $\mathfrak{g}$, this forces Y_0 to be in $\mathfrak{t}$ (otherwise $\mathfrak{t} \oplus \langle Y_0 \rangle$ would be an Abelian subalgebra of $\mathfrak{g}$ containing $\mathfrak{t}$ strictly). Thus

$$s g_0 s^{-1} = \exp(\mathrm{Ad}(s) \cdot X_0) = \exp(Y_0) \in T. \qquad \square$$

Corollary 12.22. *Any two maximal tori of G are conjugate. Thus all maximal tori of G have same dimension, called the* rank *of G.*

Proof. Let T_1 and T_2 be two maximal tori. Let t_1 be a generator of T_1. By Theorem 12.21, there exists $s \in G$ such that $s t_1 s^{-1} \in T_2$. Since t_1 is a generator of T_1, this implies that $s T_1 s^{-1} \subset T_2$. But $s T_1 s^{-1}$ is a maximal torus, hence $s T_1 s^{-1} = T_2$. $\square$

Corollary 12.23. *Every irreducible representation of G is uniquely defined, up to equivalence, by the restriction of its character to a maximal torus T.*

Proof. For $\rho \in R_G$, let χ_ρ be its character. By Theorem 12.21, for any $g \in G$, there exists $s \in G$ such that $sgs^{-1} \in T$, hence

$$\chi_\rho(g) = \chi_\rho(sgs^{-1}) = \chi_\rho|_T(sgs^{-1}). \qquad \square$$

This last result is interesting since irreducible representations of tori are explicitly known. Let T be a torus of G, and let $\mathfrak{t}$ be its Lie algebra. The exponential map $\exp\colon \mathfrak{t} \to T$ is an onto homomorphism of Lie groups. Furthermore, its kernel

$$\Gamma_T := \exp^{-1}(e),$$

is a lattice of $\mathfrak{t}$.

Theorem 12.24. *Let T be a torus of G and $\mathfrak{t}$ its Lie algebra.*

(i) *The representation space of every irreducible representation of T is one-dimensional.*

(ii) *The irreducible characters of T are the Lie groups homomorphisms*

$$\chi\colon T \longrightarrow \mathrm{U}_1$$

such that

$$\chi(\exp X) = \mathrm{e}^{2i\pi\varphi(X)}, \quad X \in \mathfrak{t},$$

where $\varphi \in \mathfrak{t}^*$ has the property that for all $X \in \Gamma_T$, we have $\varphi(X) \in \mathbb{Z}$.

Proof. Let $\rho\colon T \to \mathrm{GL}(V)$ be an irreducible representation of T. Since T is Abelian,

$$\rho(t) = \rho(s) \circ \rho(t) \circ \rho(s^{-1}), \quad t, s \in T.$$

Hence, for any $t \in T$, $\rho(t)$ is a T-invariant $\mathbb{C}$-automorphism of V. The Schur lemma then implies that

$$\rho(t) = \lambda(t)\mathrm{Id}_V$$

with $\lambda(t) \in \mathbb{C}$, $t \in T$. So every one-dimensional subspace of V is T-invariant, and since V is irreducible, V has to be one-dimensional.

Furthermore, since ρ is unitary, one has for any $t \in T$, $\lambda(t)\overline{\lambda(t)} = 1$, hence $\lambda(t) \in \mathrm{U}_1$. Now the character χ of ρ is the Lie group homomorphism

$$T \longrightarrow \mathrm{U}_1, \quad t \longmapsto \lambda(t).$$

Considering the tangent map of χ at the point e, $\chi_*\colon \mathfrak{t} \to \mathfrak{u}_1 = i\mathbb{R}$, one gets

$$\chi(\exp X) = \mathrm{e}^{\chi_*(X)}, \quad X \in \mathfrak{t}.$$

Let $\varphi = \frac{1}{2i\pi}\chi_*$. Note that $\varphi \in \mathfrak{t}^*$ and satisfies $\varphi(X) \in \mathbb{Z}$, for any $X \in \Gamma_T$. Conversely, if $\varphi \in \mathfrak{t}^*$ satisfies this condition, then the map $2i\pi\varphi$ factors into an irreducible character of T. $\square$

The linear forms $\varphi \in \mathfrak{t}^*$ that verify the condition of the theorem form a lattice Γ_T^* of $\mathfrak{t}^*$. This lattice is called *dual of the lattice* Γ_T.

Definition 12.25. The set $P_T(G) = 2i\pi\Gamma_T^*$ is a lattice of the complexified Lie algebra $\mathfrak{t}_{\mathbb{C}}^*$ of $\mathfrak{t}^*$, called the *weight lattice* of G relative to the maximal torus T. The elements of $P_T(G)$ are called *weights*.

Remark 12.26. Note that elements of $i\,\mathfrak{t}^*$ are often also called weights. In this case, if there is some ambiguity, weights of $P_T(G)$ will be said to be *integral weights*.

From now on, we fix a maximal torus T of G and denote the corresponding weight lattice by $P(G)$. Let χ be the irreducible character of an element $\rho \in R_G$. From the results above it follows that

$$\chi|_T = \sum_{\beta \in P(G)} n(\beta)\mathrm{e}^{\beta},$$

where $n(\beta) \in \mathbb{N}$, which is zero except for a finite number of values of β, is the multiplicity in $\rho|_T$ of the representation

$$\mathrm{e}^{\beta}\colon T \longrightarrow \mathrm{U}_1, \quad t \longmapsto \mathrm{e}^{\beta(X)},$$

X being any element in $\exp^{-1}(t)$. If $n(\beta)$ is a positive integer, β is said to be contained in χ with multiplicity $n(\beta)$.

Remark 12.27. Note that, conversely, Corollary 12.16 implies that any weight $\beta \in P(G)$ is contained in the character of some irreducible representation of G.

Now as $\chi|_T$ is the restriction to T of a class function, one has

$$\chi|_T(gtg^{-1}) = \chi|_T(t), \quad g \in G, \quad gTg^{-1} \subset T.$$

(Note that the condition $gTg^{-1} \subset T$ indeed implies $gTg^{-1} = T$, since gTg^{-1} is a maximal torus of G). Hence $\chi|_T$ is invariant under the action of the normalizer $N(T)$ of T,

$$N(T) := \{g \in G;\ gTg^{-1} = T\},$$

which acts on T by inner automorphisms

$$\mathcal{I}_g\colon T \longrightarrow T, \quad t \longmapsto gtg^{-1}.$$

Since $\mathcal{I}_g = \mathrm{Id}$ if $g \in T$, this action factors into a smooth action of the group $N(T)/T$ on T, given by

$$w \cdot t = \mathcal{I}_g(t),$$

where $g \in N(T)$ is a representative of $w \in N(T)/T$.

Theorem 12.28. *The group*

$$\mathfrak{W}_G := N(T)/T$$

is a finite group, called the Weyl group *of* G.

Indeed it would be more correct to call it the Weyl group of G corresponding to T, but this does not matter since from Corollary 12.22, the Weyl groups corresponding to two maximal tori are isomorphic.

Proof. The group $N(T)$ is a closed group of G, hence a Lie subgroup of G. For any $g \in N(T)$, $\mathrm{Ad}(g)$ is an automorphism of the Lie algebra $\mathfrak{t}$ of T. But T being isomorphic to a torus $T^n \simeq \mathbb{R}^n/\mathbb{Z}^n$, any automorphism of $\mathfrak{t}$ can be identified with an isomorphism of $\mathbb{R}^n$ which factors through the quotient, hence an element of $\mathrm{GL}(n, \mathbb{Z})$.

Thus, the set $\mathrm{Aut}(\mathfrak{t}) \simeq \mathrm{GL}(n, \mathbb{Z})$ being discrete, the image of the component of identity $N(T)_e$ of $N(T)$ by the homomorphism

$$\mathrm{Ad}\,|_{N(T)} \colon N(T) \longrightarrow \mathrm{Aut}(\mathfrak{t}),$$

reduces to the identity of $\mathrm{Aut}(\mathfrak{t})$. Hence for any X in the Lie algebra of $N(T)$, $\mathrm{ad}(X) \cdot \mathfrak{t} = \{0\}$, so $X \in \mathfrak{t}$ (otherwise $\mathfrak{t} \oplus \langle X \rangle$ would be an Abelian subalgebra of $\mathfrak{g}$ containing $\mathfrak{t}$ stictly, contradicting the maximality).

From this, one deduces that the Lie algebra of $N(T)$ is equal to $\mathfrak{t}$, hence $N(T)_e = T$, since those two connected subgroups of G have the same Lie algebra. Now $N(T)$, being closed in G, is compact, so $\mathfrak{W}_G = N(T)/N(T)_e$ is compact and discrete, hence finite. □

Now considering the differential at e of the action of $N(T)$ on T, we get an action of $\mathfrak{W}_G$ on $\mathfrak{t}$ given by

$$\mathfrak{w} \cdot X := \mathrm{Ad}(g)(X), \quad \mathfrak{w} \in \mathfrak{W}_G,$$

where $g \in N(T)$ is a representative of $\mathfrak{w} \in N(T)/T$. We consider the natural extension of this action to the complexified Lie algebra $\mathfrak{t}_{\mathbb{C}}$ of $\mathfrak{t}$, and define a left action of $\mathfrak{W}_G$ on $\mathfrak{t}^*_{\mathbb{C}}$ by

$$(\mathfrak{w} \cdot \beta)(X) := \beta(\mathfrak{w}^{-1} \cdot X), \quad \mathfrak{w} \in \mathfrak{W}_G, \beta \in \mathfrak{t}^*_{\mathbb{C}}, X \in \mathfrak{t}_{\mathbb{C}}.$$

Lemma 12.29. *The weight lattice $P(G)$ is invariant under the action of the Weyl group $\mathfrak{W}_G$.*

Proof. Let $\mathfrak{w} \in \mathfrak{W}_G$, $\beta \in P(G)$, and $X \in \Gamma_T$. One has

$$(\mathfrak{w} \cdot \beta)(X) = \beta(\mathfrak{w}^{-1} \cdot X) = \beta(\mathrm{Ad}(g^{-1}) \cdot X),$$

where $g \in N(T)$ is some representative of $\mathfrak{w}$. Now, since

$$\exp(\mathrm{Ad}(g^{-1}) \cdot X) = g^{-1}(\exp X)g = e,$$

we have $\mathrm{Ad}(g^{-1})(X) \in \Gamma_T$, hence $\beta(\mathrm{Ad}(g^{-1}) \cdot X) \in 2i\pi\mathbb{Z}$. Thus $\mathfrak{w} \cdot \beta \in P(G)$. $\square$

Consider now the decomposition of the restriction to T of an irreducible character χ of G,

$$\chi|_T = \sum_{\beta \in P(G)} n(\beta)\mathrm{e}^{\beta}.$$

Since χ is invariant under the action of the Weyl group, for any $\mathfrak{w} \in \mathfrak{W}_G$ and any $t \in T$, considering some element X in $\exp^{-1}(t)$, one deduces from the preceding lemma that

$$\begin{aligned}
\chi|_T(t) &= \chi|_T(\mathfrak{w} \cdot t) \\
&= \sum_{\beta \in P(G)} n(\beta)\mathrm{e}^{\beta}(\mathfrak{w} \cdot t) \\
&= \sum_{\beta \in P(G)} n(\beta)\mathrm{e}^{\beta(\mathfrak{w} \cdot X)} \\
&= \sum_{\beta \in P(G)} n(\beta)\mathrm{e}^{(\mathfrak{w}^{-1} \cdot \beta)(X)} \\
&= \sum_{\beta \in P(G)} n(\mathfrak{w} \cdot \beta)\mathrm{e}^{\beta}(t).
\end{aligned}$$

Irreducible characters being linearly independent, we get $n(\mathfrak{w} \cdot \beta) = n(\beta)$, for any $\mathfrak{w} \in \mathfrak{W}_G$ and any $\beta \in P(G)$. Hence considering the set $P(G)/\mathfrak{W}_G$ of orbits of the group $\mathfrak{W}_G$ in $P(G)$, one can write

$$\chi|_T = \sum_{\Pi \in P(G)/\mathfrak{W}_G} n(\Pi) S(\Pi),$$

where $S(\Pi) = \sum_{\beta \in \Pi} \mathrm{e}^{\beta}$ and $n(\Pi) = n(\beta)$, β being some element in Π.

We may thus conclude the following consequence of the above considerations:

Proposition 12.30. *Let* $\mathbb{Z}^{R(T)}_{\mathfrak{W}_G}$ *be the subring of* $\mathbb{Z}^{R(T)}$ *formed by the sums*

$$\sum_{\beta \in P(G)} n(\beta) e^{\beta},$$

where $n(\beta) \in \mathbb{Z}$ *is such that* $n(\beta) = 0$*, except for a finite number of values of* β*, and* $n(\mathfrak{w} \cdot \beta) = n(\beta)$*, for any* $\mathfrak{w} \in \mathfrak{W}_G$*.*

Restriction to T *induces an injective homomorphism from the representation ring* $\mathbb{Z}^{R(G)}$ *of* G *to* $\mathbb{Z}^{R(T)}_{\mathfrak{W}_G}$*.*

The next step in the theory is then to characterize which elements in $\mathbb{Z}^{R(T)}_{\mathfrak{W}_G}$ are restrictions to T of irreducible characters of G. Here come into play two fundamental objects in the theory: the adjoint representation of G, and the representation ring of SU_2.

12.3 Characterization of irreducible representations by means of dominant weights

12.3.1 Restriction to semi-simple simply connected groups

From now on, the Lie group G will be assumed to be semi-simple and simply connected. This assumption can be made thanks to the following result on the structure of compact and connected groups.

Definition 12.31. A Lie group is said to be *semi-simple* if the Killing form of its Lie algebra is non-degenerate.

Theorem 12.32. *Any compact connected Lie group* G *has a finite covering isomorphic to a product* $\mathbb{T}^n \times K$*, where* $\mathbb{T}^n$ *is a torus and* K *a compact semi-simple and simply connected Lie group.*

For a proof see for instance [Die75], (21.6.9), p. 40. It is proved that the universal covering $\widetilde{G}$ of G is isomorphic to a product of the form $\mathbb{R}^n \times K$, where K is a compact semi-simple and simply connected Lie group. Now, coverings of G are of the form $\widetilde{G}/D$, where D is a discrete subgroup of the center $\mathbb{R}^n \times Z$ of $\widetilde{G}$, Z being the (finite) center of K. Thus considering $D = \mathbb{Z}^n \times \{e\}$, one gets a finite covering of G isomorphic to $\mathbb{T}^n \times K$.

Remark 12.33. So we only consider compact semi-simple simply connected Lie groups. Indeed, with the notations of the above theorem, if the irreducible representations of K are known, then irreducible representations of the product $\mathbb{T}^n \times K$ are known by Theorems 12.9 and 12.24. Then, the irreducible representations of G can be deduced from Proposition 12.18.

12.3.2 Roots and their properties

Let $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$ be the adjoint representation of G, and $\mathrm{Ad}_{\mathbb{C}}\colon G \to \mathrm{GL}_{\mathbb{C}}(\mathfrak{g}_{\mathbb{C}})$ its extension to the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$, $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i\,\mathfrak{g}$. Denote by T a maximal torus of G. According to the results of the previous section, from the decomposition into irreducible parts of the representation $\mathrm{Ad}_{\mathbb{C}}|_T$ it follows that there exists a family of distinct weights $\theta_1, \dots, \theta_p$ in $P(G)$, such that

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_0 \oplus \Big(\bigoplus_{i=1}^{p} \mathfrak{g}_{\theta_i} \Big), \tag{12.1}$$

where

$$\mathfrak{g}_0 = \{Y \in \mathfrak{g}_{\mathbb{C}};\ \text{for all } X \in \mathfrak{t}, [X, Y] = 0\},$$

and

$$\mathfrak{g}_{\theta_i} = \{Y \in \mathfrak{g}_{\mathbb{C}};\ \text{for all } X \in \mathfrak{t}, [X, Y] = \theta_i(X)Y\}.$$

Of course, $\mathfrak{t}_{\mathbb{C}} \subset \mathfrak{g}_0$, but in fact one has

Lemma 12.34. $\mathfrak{g}_0 = \mathfrak{t}_{\mathbb{C}}$ *and this property characterizes the maximal tori of* G.

Proof. Let $Z = X + iY$, $X, Y \in \mathfrak{g}$, be an element of $\mathfrak{g}_0$. Since $[U, Z] = 0$ for any $U \in \mathfrak{t}$, one deduces that, for any $U \in \mathfrak{t}$, $[U, X] = 0$ and $[U, Y] = 0$, hence X and Y belong to the centralizer of $\mathfrak{t}$, which is equal to $\mathfrak{t}$, since T is a maximal torus. □

Definition 12.35. The weights θ_i, $1 \le i \le p$, are called *roots* of G.

Lemma 12.36. *The Weyl group of* G *permutes the roots.*

Proof. Let $\mathfrak{w} \in \mathfrak{W}_G$ and let $g \in N(T)$ be a representative of $\mathfrak{w}$. For any $t \in T$, one has

$$\mathrm{Ad}_{\mathbb{C}}(\mathfrak{w} \cdot t) = \mathrm{Ad}_{\mathbb{C}}(gtg^{-1}) = \mathrm{Ad}_{\mathbb{C}}(g)\,\mathrm{Ad}_{\mathbb{C}}(t)\,\mathrm{Ad}_{\mathbb{C}}(g^{-1}).$$

Therefore, the two representations $t \to \mathrm{Ad}_{\mathbb{C}}(t)$ and $t \to \mathrm{Ad}_{\mathbb{C}}(\mathfrak{w} \cdot t)$ are equivalent, hence have the same weights. Furthermore, considering the extension of the action of $\mathfrak{W}_G$ to $\mathfrak{g}_{\mathbb{C}}$, for any $X \in \mathfrak{t}$ and $X_i \in \mathfrak{g}_{\theta_i}$, one has

$$\begin{aligned}
[X, \mathfrak{w} \cdot X_i] &= [X, \mathrm{Ad}_{\mathbb{C}}(g) \cdot X_i] \\
&= \mathrm{Ad}_{\mathbb{C}}(g)([\mathrm{Ad}_{\mathbb{C}}(g^{-1}) \cdot X, X_i]) \\
&= \mathrm{Ad}_{\mathbb{C}}(g)(\theta_i(\mathfrak{w}^{-1} \cdot X) X_i) \\
&= ((\mathfrak{w} \cdot \theta_i)(X))\mathfrak{w} \cdot X_i,
\end{aligned}$$

and so

$$\mathfrak{w} \cdot \mathfrak{g}_{\theta_i} = \mathfrak{g}_{\mathfrak{w} \cdot \theta_i}.$$

□

Lemma 12.37. *If θ_i is a root of G, so is $(-\theta_i)$, and one has* $\overline{\mathfrak{g}_{\theta_i}} = \mathfrak{g}_{-\theta_i}$.

Proof. For any root θ_i, any $X_i \in \mathfrak{g}_{\theta_i}$, and any $X \in \mathfrak{t}$, one has

$$[X, X_i] = \theta_i(X)X_i,$$

hence

$$\overline{[X, X_i]} = [X, \overline{X_i}] = -\theta_i(X)\overline{X_i},$$

since $\theta_i \in 2i\pi\mathfrak{t}^*$. So for any root θ_i, $(-\theta_i)$ is a root and $\overline{\mathfrak{g}_{\theta_i}} \subset \mathfrak{g}_{-\theta_i}$, as claimed. □

Now let θ_j be a root of G and let X_j be a non-trivial vector of $\mathfrak{g}_{\theta_j}$. Consider the two vectors

$$Y_j = X_j + \overline{X_j}, \quad Z_j = i(X_j - \overline{X_j}),$$

of $\mathfrak{g}$. As X_j ranges over a basis (over $\mathbb{C}$) of $\mathfrak{g}_{\theta_j}$, they form a basis (over $\mathbb{R}$) of $\mathfrak{g} \cap \left(\mathfrak{g}_{\theta_j} \oplus \mathfrak{g}_{-\theta_j}\right)$.

Let $\varphi_j \in \mathfrak{t}^*$, defined by $\theta_j = 2i\pi\varphi_j$. For any $X \in \mathfrak{t}$,

$$[X, Y_j] = 2\pi\varphi_j(X)Z_j, \quad [X, Z_j] = -2\pi\varphi_j(X)Y_j. \tag{12.2}$$

Lemma 12.38. *For any root θ_j, let $H_j := [Y_j, Z_j]$. Then $H_j \in \mathfrak{t}$ and $\varphi_j(H_j) > 0$.*

Furthermore, $\{H_j, Y_j, Z_j\}$ is a basis of a subalgebra of $\mathfrak{g}$, isomorphic to the Lie algebra $\mathfrak{su}_2$.

Proof. From relations (12.2), one has for any $X \in \mathfrak{t}$ that

$$[X, H_j] = [X, [Y_j, Z_j]] = [Y_j, [X, Z_j]] - [Z_j, [X, Y_j]] = 0,$$

hence H_j belongs to the centralizer of $\mathfrak{t}$, which is equal to $\mathfrak{t}$. Let $\langle\cdot,\cdot\rangle$ be an $\mathrm{Ad}(G)$-invariant scalar product on $\mathfrak{g}$ (for instance, the Killing form of $\mathfrak{g}$ sign-changed); then for any $X \in \mathfrak{t}$,

$$\langle X, H_j\rangle = \langle X, [Y_j, Z_j]\rangle = \langle [X, Y_j], Z_j\rangle = 2\pi\varphi_j(X)\| Z_j \|^2.$$

First note that this equality implies, by choosing $X \in \mathfrak{t}$ such that $\varphi_j(X) \neq 0$, that H_j is non-zero. Now applying this equality to $X = H_j$ gives $\varphi_j(H_j) > 0$. Finally, from relations (12.2) it follows

$$[H_j, Y_j] = 2\pi\varphi_j(H_j)Z_j \quad \text{and} \quad [H_j, Z_j] = -2\pi\varphi_j(H_j)Y_j,$$

so letting

$$\boldsymbol{H}_j = \frac{2}{a_j}H_j, \quad \boldsymbol{Y}_j = \frac{2}{\sqrt{a_j}}Y_j, \quad \boldsymbol{Z}_j = \frac{2}{\sqrt{a_j}}Z_j,$$

with

$$a_j = 2\pi\varphi_j(H_j) > 0,$$

one gets

$$[\boldsymbol{H}_j, \boldsymbol{Y}_j] = 2\boldsymbol{Z}_j, \quad [\boldsymbol{H}_j, \boldsymbol{Z}_j] = -2\boldsymbol{Y}_j, \quad [\boldsymbol{Y}_j, \boldsymbol{Z}_j] = 2\boldsymbol{H}_j.$$

These are the same multiplication rules as for the Lie algebra $\mathfrak{su}_2$. □

We now consider the scalar product on $\mathfrak{g}$ given by the Killing form of $\mathfrak{g}$ sign-changed. (In fact, one could use any $\mathrm{Ad}(G)$-invariant scalar product on $\mathfrak{g}$). This scalar product being $\mathrm{Ad}(G)$-invariant, its restriction $\langle\cdot,\cdot\rangle$ to $\mathfrak{t}$ is invariant under the action of the Weyl group $\mathfrak{W}_G$. We consider the canonical isomorphism $\sharp$ between $\mathfrak{t}$ and its dual $\mathfrak{t}^*$ given by

$$\sharp\colon X \in \mathfrak{t} \longrightarrow X^\sharp \in \mathfrak{t}^*, \quad X^\sharp\colon Y \longmapsto X^\sharp(Y) = \langle X, Y\rangle.$$

We endow $\mathfrak{t}^*$ with the scalar product $\langle,\rangle$ which makes $\sharp$ an isometry:

$$\langle X^\sharp, Y^\sharp\rangle := \langle X, Y\rangle, \quad X, Y \in \mathfrak{t}.$$

This scalar product is invariant under the action of $\mathfrak{W}_G$. In fact, for any $\mathfrak{w} \in \mathfrak{W}_G$ and $X, Y \in \mathfrak{t}$, one has $\mathfrak{w}\cdot X^\sharp = (\mathfrak{w}\cdot X)^\sharp$, so

$$\langle \mathfrak{w}\cdot X^\sharp, \mathfrak{w}\cdot Y^\sharp\rangle = \langle (\mathfrak{w}\cdot X)^\sharp, (\mathfrak{w}\cdot Y)^\sharp\rangle = \langle \mathfrak{w}\cdot X, \mathfrak{w}\cdot Y\rangle = \langle X, Y\rangle = \langle X^\sharp, Y^\sharp\rangle.$$

Finally, we consider the natural extension of this scalar product on $i\,\mathfrak{t}^*$, given by

$$\langle i\beta, i\beta'\rangle := \langle \beta, \beta'\rangle, \quad \beta, \beta' \in \mathfrak{t}^*.$$

Using Lemma 12.38 and the explicit description of irreducible representations of SU_2, one gets the following "fundamental" properties of roots.

Theorem 12.39. (i) *Roots span* $i\,\mathfrak{t}^*$.

(ii) *For any pair of roots* (θ_i, θ_j), $\theta_j - 2\frac{\langle\theta_i,\theta_j\rangle}{\langle\theta_i,\theta_i\rangle}\theta_i$ *is a root.* (*In other words, for any root* θ_i, *the set* Φ *of roots is invariant under reflections wrt the hyperplane perpendicular to* θ_i).

(iii) *For any pair of roots* (θ_i, θ_j), *one has* $2\frac{\langle\theta_i,\theta_j\rangle}{\langle\theta_i,\theta_i\rangle} \in \mathbb{Z}$.

(iv) *If* θ_i *is a root, then* $\pm\theta_i$ *are the only scalar multiples of* θ_i *which are also roots.*

As it was mentioned before, the proof of this result is based on the description of irreducible representations of SU_2, so we first consider this group.

12.3.3 Irreducible representations of the group SU_2

Since SU_2 is simply connected, the map $\rho \mapsto \rho_*$ is a one-to-one correspondence between finite-dimensional complex representations of SU_2 and finite-dimensional complex representations of its Lie algebra $\mathfrak{su}_2$.

The map $\rho_* \mapsto \rho_*^{\mathbb{C}}$ (extension to the complexified Lie algebra of $\mathfrak{su}_2$) is a one-to-one correspondence between finite-dimensional complex representations of $\mathfrak{su}_2$ and finite-dimensional complex representations of $\mathfrak{sl}_{2,\mathbb{C}}$, which is the complexified Lie algebra of $\mathfrak{su}_2$. (The inverse map is simply $\rho_*^{\mathbb{C}} \mapsto \rho_*^{\mathbb{C}}|_{\mathfrak{su}_2}$.)

Hence the map $\rho \mapsto \rho_*^{\mathbb{C}}$ gives a one-to-one correspondence between irreducible representations of SU_2 and irreducible finite-dimensional complex representations of $\mathfrak{sl}_{2,\mathbb{C}}$.

Notations 12.40. For any representation (ρ, V) of SU_2, and for any $X \in \mathfrak{sl}_{2,\mathbb{C}}$ and $v \in V$, we will simply write $X \cdot v$ instead of $\rho_*^{\mathbb{C}}(X) \cdot v$.

Let

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

be the standard basis of $\mathfrak{sl}_{2,\mathbb{C}}$. One has

$$[H, X^+] = 2X^+, \quad [H, X^-] = -2X^-, \quad [X^+, X^-] = H. \tag{12.3}$$

Theorem 12.41. *For any integer $m \geq 0$, there exists an irreducible representation of SU_2 in a complex vector space of dimension $m+1$, having a basis $\{e_0, e_1, \ldots, e_m\}$ such that*

$$H \cdot e_k = (m-2)e_k, \quad X^+ \cdot e_k = (m-k+1)e_{k-1}, \quad X^- \cdot e_k = (k+1)e_{k+1}, \tag{12.4}$$

setting $e_{-1} = e_{m+1} = 0$.

Every (complex) irreducible representation of SU_2 is equivalent to one of the indicated representations.

Proof. First if (ρ, V) and (ρ', V') are two representations of SU_2 such that both V and V' have a basis $\{e_k\}$, resp. $\{e'_k\}$, $k = 0, \ldots, m$, verifying (12.4), then they are equivalent via the $\mathbb{C}$-isomorphism $V \to V'$ that sends the basis $\{e_k\}$ into the basis $\{e'_k\}$.

Now let $\rho\colon \mathrm{SU}_2 \to \mathrm{GL}(V)$ be a (finite-dimensional complex) irreducible representation. Consider the Lie subgroup of SU_2 given by

$$T = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}; \theta \in \mathbb{R} \right\};$$

its Lie algebra is $\mathbb{R}(iH)$.

By (12.3), T is a maximal torus of SU_2 (see (12.1) and Lemma 12.34). The decomposition of $\rho|_T$ into irreducible parts induces the decomposition of V into weight spaces:

$$V = \bigoplus_{\lambda \in \Lambda} V_\lambda,$$

where Λ is a finite subset of $\mathbb{C}$ and

$$V_\lambda = \{v \in V; H \cdot v = \lambda v\}.$$

Take $v_\lambda \in V_\lambda$. Then by (12.3), one has

$$X^+ \cdot v_\lambda \in V_{\lambda+2} \quad \text{and} \quad X^- \cdot v_\lambda \in V_{\lambda-2}.$$

Since V is finite dimensional and since the sum $V = \bigoplus_{\lambda\in\Lambda} V_\lambda$ is direct, there exists a $\lambda_0 \in \Lambda$ such that $V_{\lambda_0} \neq \{0\}$ and $V_{\lambda_0+2} = \{0\}$.

Consider a non-zero vector $v_0 \in V_{\lambda_0}$. It satisfies $X^+ \cdot v_0 = 0$. Such a weight vector is said to be a "maximal vector" of the representation. Set

$$e_{-1} := 0, \quad e_j := \left(\frac{1}{j!}\right)(X^-)^j \cdot v_0, \quad j \in \mathbb{N}.$$

It is easy to check that

(a) $H \cdot e_j = (\lambda_0 - 2j)e_i$;

(b) $X^- \cdot e_j = (j+1)e_{j+1}$;

(c) $X^+ \cdot e_j = (\lambda_0 - j + 1)e_{j-1}$.

From (a), one deduces that the non-zero e_j are linearly independent. But V is finite dimensional. Hence, there exists a smallest integer m for which $e_m \neq 0$, and $e_{m+j} = 0$ for all $j > 0$. It is easy to see from relations (a)–(c) that the subspace of V with basis $\{e_0, e_1, \dots, e_m\}$ is a $\mathfrak{sl}_{2,\mathbb{C}}$-submodule of V. But V is an irreducible $\mathfrak{sl}_{2,\mathbb{C}}$-module, hence this subspace has to be V. Finally, taking $j = m+1$ in (c) gives $\lambda_0 = m$. Note that weights are integers, that m is the "highest" one and that, up to a scalar, V admits a unique maximal vector, namely e_0.

Conversely, for any $m \in \mathbb{N}$, relations (12.4), as it was noted before, define a representation of $\mathfrak{sl}_{2,\mathbb{C}}$ on a complex space L_m of dimension $m+1$, with basis $\{e_0, \dots, e_m\}$. Let U be a non-trivial invariant $\mathfrak{sl}_{2,\mathbb{C}}$-submodule of L_m and denote by $U = \bigoplus_{\lambda\in\Lambda} U_\lambda$ its decomposition into weight spaces under the action of H. By the first formula in (12.4), one has necessarily $\Lambda \subset \{-m, -m+2, \dots, m-2, m\}$, hence one of the e_j belongs to U. But the two last formulas in (12.4) imply that all the vectors of the basis $\{e_j\}$ belong to U. Hence $U = L_m$, so L_m is irreducible. $\square$

Remarks 12.42. (i) Let (V, ρ) be a representation of SU_2. Decomposing ρ into irreducible components, one deduces from (12.4) that the weights of ρ are integers (and an integer occurs as a weight as many times as its negative).

Furthermore, since weights of an irreducible representation are all simultaneously even integers or odd integers, the number of summands in the decomposition of ρ is $\dim V_0 + \dim V_1$, where V_0 (resp. V_1) is the weight space corresponding to the weight 0 (resp. 1).

(ii) Let $\rho\colon \mathrm{SU}_2 \to \mathrm{GL}_{\mathbb{C}}(V)$, $V = \mathbb{C}^2$, be the usual (faithful) representation of SU_2. Recall Theorem 12.15: irreducible representations of SU_2 are contained in tensor products of ρ and its conjugate.

For any $m \in \mathbb{N}$, let $\mathcal{S}^m(V)$ be the space of symmetric tensors of order m (note that it has dimension $(m+1)$). This is (up to equivalence) the irreducible representation with highest weight m since the basis $\{e_0, e_1, \ldots, e_m\}$ of $\mathcal{S}^m(V)$ given by

$$e_k = \frac{1}{k!(m-k)!} {\epsilon_1}^{m-k} \vee {\epsilon_2}^{k},$$

where $\{\epsilon_1, \epsilon_2\}$ is the standard basis of $\mathbb{C}^2$, satisfies (12.4).

12.3.4 Proof of the fundamental properties of roots

Proof of Theorem 12.39.

Step 1. First, we show that roots (extended to $\mathfrak{t}^*_{\mathbb{C}}$) span $\mathfrak{t}^*_{\mathbb{C}}$. Assuming this is not the case, there exists $X \in \mathfrak{t}_{\mathbb{C}}$ such that $\theta_j(X) = 0$ for any root θ_j. Then $[X, Y] = 0$ for any $Y \in \mathfrak{g}_{\mathbb{C}}$, hence X belongs to the center of $\mathfrak{g}_{\mathbb{C}}$. But G is supposed to be semi-simple, hence $\mathfrak{g}$ and then $\mathfrak{g}_{\mathbb{C}}$ are semi-simple, so the center of $\mathfrak{g}_{\mathbb{C}}$ is trivial: contradiction.

Step 2. Now recall that for any root θ_j, the vectors $\boldsymbol{H}_j$, $\boldsymbol{Y}_j$, $\boldsymbol{Z}_j$ introduced in Lemma 12.38 define a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{su}_2$. Let h_j, x_j^+, x_j^- be the following vectors of $\mathfrak{g}_{\mathbb{C}}$:

$$h_j = -i\,\boldsymbol{H}_j, \quad x_j^+ = -\frac{1}{2}(\boldsymbol{Z}_j + i\,\boldsymbol{Y}_j), \quad x_j^- = \frac{1}{2}(\boldsymbol{Z}_j - i\,\boldsymbol{Y}_j).$$

It is easy to verify that

$$h_j \in i\mathfrak{t}, \quad x_j^+ \in \mathfrak{g}_{\theta_j}, \quad x_j^- \in \mathfrak{g}_{-\theta_j},$$

$$\theta_j(h_j) = 2,$$

$$[h_j, x_j^+] = 2x_j^+, \quad [h_j, x_j^-] = -2x_j^-, \quad [x_j^+, x_j^-] = h_j,$$

so

$$\mathfrak{s}_j := \langle h_j, x_j^+, x_j^- \rangle,$$

is a Lie subalgebra of $\mathfrak{g}_{\mathbb{C}}$ isomorphic to $\mathfrak{sl}_{2,\mathbb{C}}$. Now, for any pair of roots (θ_j, θ_k), one has

$$[h_j, Y] = \theta_k(h_j)Y, \quad Y \in \mathfrak{g}_{\theta_k},$$

so $\theta_k(h_j)$ is a weight of the representation $\mathrm{ad}_{\mathbb{C}}$ restricted to $\mathfrak{s}_j \simeq \mathfrak{sl}_{2,\mathbb{C}}$. Hence, by Remark 12.42 i), $\theta_k(h_j) \in \mathbb{Z}$.

Step 3. Using the Jacobi identity it is easy to verify that

$$[\mathfrak{g}_{\theta_j}, \mathfrak{g}_{\theta_k}] \subset \mathfrak{g}_{\theta_j+\theta_k} \quad \text{if } \theta_j + \theta_k \text{is a root,} \tag{12.5}$$

and $[\mathfrak{g}_{\theta_j}, \mathfrak{g}_{\theta_k}] = \{0\}$ otherwise. For any root θ_j, let

$$\mathfrak{M}_j := \mathfrak{t}_{\mathbb{C}} \oplus \Big(\bigoplus_{c \in C} \mathfrak{g}_{c\theta_j} \Big),$$

where C is the set of scalars c such that $c\theta_j$ is a root. By (12.5), $\mathfrak{M}_j$ is a $\mathfrak{s}_j$-invariant subspace of $\mathfrak{g}_{\mathbb{C}}$ (for the representation $\mathrm{ad}_{\mathbb{C}}|_{\mathfrak{s}_j}$). By Step 2, its weights under the action of h_j are 0 and $c\theta_j(h_j) = 2c$. So by Remark 12.42 i), c has to be an integral multiple of $\frac{1}{2}$.

Now note that the hyperplane $\mathrm{Ker}\,\theta_j$ (which complements $\langle h_j \rangle$ in $\mathfrak{t}_{\mathbb{C}}$) and the subspace $\mathfrak{s}_j = \langle h_j, x_j^+, x_j^- \rangle$ are $\mathfrak{s}_j$-invariant subspaces of $\mathfrak{M}_j$, and they exhaust the occurrences of the weight 0 for the action of h_j on $\mathfrak{M}_j$. Observe that $\mathfrak{s}_j$ acts trivially on $\mathrm{Ker}\,\theta_j$ and that $\mathfrak{s}_j$ is an irreducible $\mathfrak{s}_j$-module (by formulas (12.4)).

By Remark 12.42, one sees that the only irreducible $\mathfrak{s}_j$-modules with even highest weight contained in $\mathfrak{M}_j$ are (with the notations of the proof of Theorem 12.41) L_0 (with multiplicity $\dim \mathfrak{t}_{\mathbb{C}} - 1$) and L_2 (with multiplicity 1), so the only even weights of h_j on $\mathfrak{M}_j$ are 0 and ± 2. Hence one deduces that

for any root θ_j, $2\theta_j$ is not a root.

Consequently,

for any root θ_j, $\frac{1}{2}\theta_j$ cannot be a root.

From this last result, one sees that 1 cannot occur as a weight of h_j on $\mathfrak{M}_j$. From Remark 12.42, one then concludes that

$$\mathfrak{M}_j = \mathrm{Ker}\,\theta_j \oplus \mathfrak{s}_j,$$

which establishes the property (iv) of roots. Moreover, this implies:

for any root θ_j, $\dim \mathfrak{g}_{\theta_j} = 1$.

Step 4. For any root θ_j, let σ_j be the map

$$\sigma_j : \mathfrak{t}^*_{\mathbb{C}} \longrightarrow \mathfrak{t}^*_{\mathbb{C}}, \quad \lambda \longmapsto \lambda - \lambda(h_j)\theta_j .$$

Since $\theta_j(h_j) = 2$, σ_j is an involution, distinct from the identity (since $\sigma_j(\theta_j) = -\theta_j$). We claim that it leaves invariant the set of roots Φ. To see this, consider, for any $i \neq j$, the space

$$\mathfrak{N}_{ji} := \bigoplus_{k \in K} \mathfrak{g}_{\theta_j + k\theta_i},$$

where K is the set of integers k such that $\theta_j + k\theta_i$ is a root. By (12.5), $\mathfrak{N}_{ji}$ is an $\mathfrak{s}_i$-invariant subspace of $\mathfrak{g}_{\mathbb{C}}$ (for the representation $\mathrm{ad}_{\mathbb{C}}|_{\mathfrak{s}_i}$). Its weights under the action of h_i are $\theta_j(h_i) + 2k$. Moreover, by Step 3, the corresponding weight spaces $\mathfrak{g}_{\theta_j + k\theta_i}$ are one-dimensional, and by Step 1, $\theta_j(h_i) \in \mathbb{Z}$.

Now since the weights $\theta_j(h_i) + 2k$ are simultaneously all even or odd integers and since 0 or 1 (but not both) can occur only once, as weights of this form, one deduces from Remark 12.42 that $\mathfrak{N}_{ji}$ is irreducible. Its highest (resp. lowest) weight must be $\theta_j(h_i) + 2p$, (resp. $\theta_j(h_i) - 2q$), where p (resp. q) is the largest integer $k \in K$ for which $\theta_j + k\theta_i$ (resp $\theta_j - k\theta_i$) is a root.

Moreover, since the other weights have the form $\theta_j(h_i) + 2k$, $-q \leq k \leq p$ (cf. (12.4)), $\theta_j + k\theta_i$ is a root for any k such that $-q \leq k \leq p$. Now, relations (12.4) imply

$$\theta_j(h_i) - 2q = -(\theta_j(h_i) + 2p),$$

hence, $\theta_j(h_i) = q - p$. So, $\sigma_i(\theta_j) = \theta_j + (p-q)\theta_i$ is a root, since $-q \leq p-q \leq p$.

Let F be the $\mathbb{R}$-vector subspace of $i\,\mathfrak{t}^*$ spanned by the roots θ_i, endowed with the induced scalar product $\langle \cdot, \cdot \rangle$. Since $\theta_j(h_i) \in \mathbb{Z}$, any σ_i leaves F invariant. Now consider $\beta \in F$ such that $\langle \beta, \theta_i \rangle = 0$. By the definition of the scalar product,

$$\langle \beta, \theta_i \rangle = -i\,\theta_i(X_\beta),$$

where X_β is the vector in $\mathfrak{t}$ corresponding to β via the canonical isomorphism

$$\mathfrak{t} \longrightarrow i\,\mathfrak{t}^*, \; X \longmapsto i X^\sharp .$$

So $X_\beta \in \operatorname{Ker}\theta_i$. Using the $\mathrm{Ad}(G)$-invariance of the scalar product, one gets

$$\begin{aligned}
\langle X_\beta, \boldsymbol{H}_i \rangle &= \frac{1}{2}\langle X_\beta, [\boldsymbol{Y}_i, \boldsymbol{Z}_i] \rangle \\
&= \frac{1}{2}\langle [X_\beta, \boldsymbol{Y}_i], \boldsymbol{Z}_i \rangle \\
&= -\frac{i}{2}\theta_i(X_\beta)\|\boldsymbol{Z}_i\|^2 \\
&= 0.
\end{aligned}$$

Consequently,

$$\beta(h_i) = -i\beta(\boldsymbol{H}_i) = \langle X_\beta, \boldsymbol{H}_i \rangle = 0.$$

Hence, $\sigma_i(\beta) = \beta$, for any $\beta \in \theta_i^\perp$. Thus $\sigma_i|_F$ is the reflection across the hyperplane $\theta_i^\perp$. This implies that

$$\theta_j(h_i) = 2\frac{\langle \theta_i, \theta_j \rangle}{\langle \theta_i, \theta_i \rangle},$$

which establishes properties (ii) and (iii).

Step 5. Since the θ_j span $\mathfrak{t}_{\mathbb{C}}^*$ (cf. Step 1), there exists a basis of $\mathfrak{t}_{\mathbb{C}}^*$ of the form $\{\theta_1, \theta_2, \ldots, \theta_l\}$, so any $\beta \in i\mathfrak{t}^*$ can be written uniquely as $\beta = \sum_{j=1}^{l} c_j \theta_j$, where $c_j \in \mathbb{C}$. Besides, for any $k = 1, \ldots, l$, one has $\beta(h_k) = \sum_{j=1}^{l} c_j \theta_j(h_k)$.

This is a system of l equations with l unknowns c_j and with real coefficients ($\beta(h_k) \in \mathbb{R}$, $\theta_j(h_k) \in \mathbb{Z}$). The matrix corresponding to this system is non-degenerate (from the result before, it is, up to a scalar, the matrix of the $\mathbb{C}$-linear extension of the scalar product $\langle \cdot, \cdot \rangle$ of $i\mathfrak{t}^*$ to $\mathfrak{t}_{\mathbb{C}}^*$, in the basis (θ_j)), so by Cramer's formulas $c_j \in \mathbb{R}$. Thus $F = i\mathfrak{t}^*$, hence (i). □

Definition 12.43. Let $(E, \langle \cdot, \cdot \rangle)$ be an Euclidean space. Any finite subset Φ of E whose elements θ_i verify properties (i)–(iv) of Theorem 12.39 is called a *root system* of E.

Definition 12.44. Let Φ be a root system of an Euclidean space $(E, \langle \cdot, \cdot \rangle)$. A subset Δ of Φ is called a *basis* of Φ if it satisfies the two following properties:

(i) Δ is a basis of E;

(ii) every $\theta \in \Phi$ can be written as $\theta = \sum n_i \theta_i$, where $\theta_i \in \Delta$ and n_i are integers, all of the same sign.

Roots belonging to a basis are called *simple roots*.

Theorem 12.45. *Every root system has a basis.*

Definitions 12.46. A vector γ in E is said to be *regular* if $\gamma \in E \setminus \left(\bigcup_{\theta \in \Phi} \theta^\perp\right)$, *singular* otherwise. Any connected component of $E \setminus \left(\bigcup_{\theta \in \Phi} \theta^\perp\right)$ is called a *Weyl chamber*.

Idea of the proof of Theorem 12.45. Let $\gamma \in E$ be regular and let

$$\Phi_\gamma^+ = \{\theta \in \Phi; \langle \theta, \gamma \rangle > 0\} \quad \text{and} \quad \Phi_\gamma^- = \{\theta \in \Phi; \langle \theta, \gamma \rangle < 0\}.$$

One has $\Phi = \Phi_\gamma^+ \cup \Phi_\gamma^-$. An element θ in Φ_γ^+ is said to be decomposable if it can be written as $\theta = \theta_1 + \theta_2$ with $\theta_1, \theta_2 \in \Phi_\gamma^+$. Otherwise, θ is said to be indecomposable.

It is then shown (cf. for instance [Hum72], p. 48), that the set $\Delta(\gamma)$ of indecomposable elements in Φ_γ^+ is a basis and that any basis of Φ has this form. For this reason, roots belonging to a basis are called *simple* roots. □

Definition 12.47. Let $\Delta = \{\theta_1, \theta_2, \ldots, \theta_l\}$ be a basis of a root system Φ in an Euclidean space $(E, \langle\cdot,\cdot\rangle)$. Any root $\theta \in \Phi$ such that $\theta = \sum_{i=1}^l n_i \theta_i$, where $n_i \in \mathbb{N}$, is called a *positive root*. The set of positive roots is denoted by Φ^+.

Now we get back to “our” root system Φ defined by the choice of a maximal torus T in G. By the previous results, it has a basis $\Delta = \{\theta_1, \theta_2, \ldots, \theta_l\}$, $l = \dim(i\,\mathfrak{t}^*)$.

Remark 12.48. From the proof of Theorem 12.39 (Step 2), for any simple root θ_i there exists $h_i \in i\,\mathfrak{t}$, $x_i^+ \in \mathfrak{g}_{\theta_i}$ and $x_i^- \in \mathfrak{g}_{-\theta_i}$ such that $\operatorname{span}\{h_i, x_i^+, x_i^-\}$ is a subalgebra of $\mathfrak{g}_{\mathbb{C}}$ isomorphic to $\mathfrak{sl}_{2,\mathbb{C}}$. In fact, those vectors $h_1, \ldots, h_l$, form a basis of $\mathfrak{t}_{\mathbb{C}}$.

Indeed, let $c_1, \ldots, c_l \in \mathbb{C}$ such that $\sum_{i=1}^l c_i h_i = 0$. Applying each simple root θ_i to this equation, one obtains a system of l equations in l unknowns c_i whose matrix $\left(\left(\theta_i(h_j) = 2\frac{\langle \theta_i, \theta_j \rangle}{\langle \theta_j, \theta_j \rangle}\right)\right)$ is non-singular, cf. the proof of Theorem 12.39 (Step 5).

Hence all the c_i are 0. Thus, the h_i, $1 \le i \le l$, are linearly independent and they form a basis, since $l = \operatorname{Card} \Delta = \dim(i\,\mathfrak{t}^*) = \dim_{\mathbb{C}}(\mathfrak{t}_{\mathbb{C}})$.

The existence of simple roots allows to give a simple characterization of the Weyl group.

Theorem 12.49. *The Weyl group $\mathfrak{W}_G$ is spanned by reflections σ_i across the hyperplanes $\theta_i^\perp$, where $\theta_i \in \Delta$.*

Proof. We first prove that any such reflection is an element of $\mathfrak{W}_G$. Let θ_i be a simple root and consider $\operatorname{span}\{\mathbf{H}_i, \mathbf{Y}_i, \mathbf{Z}_i\}$, the corresponding subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{su}_2$, introduced in Lemma 12.38. Let $g_i = \exp(\frac{\pi}{2}\mathbf{Y}_i)$. We claim that g_i defines an element $\mathfrak{w}_i \in \mathfrak{W}_G$ whose action on roots is the reflection σ_i across the hyperplane $\theta_i^\perp$. One has $\operatorname{Ad}(g_i) = \exp(\frac{\pi}{2}\operatorname{ad}(\mathbf{Y}_i))$. Now

$$\operatorname{ad}(\mathbf{Y}_i) \cdot X = -[X, \mathbf{Y}_i] = -i\theta_i(X)\mathbf{Z}_i = 0, \quad X \in \operatorname{Ker}\theta_i,$$

so

$$\operatorname{Ad}(g_i) \cdot X = X, \quad X \in \operatorname{Ker}\theta_i.$$

On the other hand, it is easy to verify that, for all $k \in \mathbb{N}$,

$$(\operatorname{ad} \boldsymbol{Y}_i)^{2k} \cdot \boldsymbol{H}_i = (-1)^k 2^{2k} \boldsymbol{H}_i,$$

and

$$(\operatorname{ad} \boldsymbol{Y}_i)^{2k+1} \cdot \boldsymbol{H}_i = (-1)^{k+1} 2^{2k+1} \boldsymbol{H}_i.$$

Therefore,

$$\begin{aligned}
\operatorname{Ad}(g_i) \cdot \boldsymbol{H}_i &= \exp\left(\frac{\pi}{2} \operatorname{ad}(\boldsymbol{Y}_i)\right) \cdot \boldsymbol{H}_i \\
&= \sum_{k \geq 0} \frac{(\pi/2)^k}{k!} (\operatorname{ad} \boldsymbol{Y}_i)^k \cdot \boldsymbol{H}_i \\
&= (\cos \pi) \boldsymbol{H}_i - (\sin \pi) \boldsymbol{Z}_i \\
&= -\boldsymbol{H}_i.
\end{aligned}$$

Thus, since $\operatorname{Ad}(g_i)$ leaves invariant the space $\mathfrak{t} = \operatorname{Ker} \theta_i \oplus \langle \boldsymbol{H}_i \rangle$, one concludes that $g_i \in N(T)$.

Furthermore, since $\boldsymbol{H}_i$ is orthogonal to $\operatorname{Ker} \theta_i$ (cf. Step 4 in the proof of Theorem 12.39), the action on $\mathfrak{t}$ of the element $\mathfrak{w}_i \in \mathfrak{W}_G$ defined by g_i is the reflection across $\boldsymbol{H}_i^{\perp}$. Now, by the results of Step 4 in the proof of Theorem 12.39, $\operatorname{Ker} \theta_i$ corresponds to $\theta_i^{\perp}$ under the canonical isomorphism $\mathfrak{t} \to i\mathfrak{t}^*$, $X \mapsto iX^{\sharp}$. Hence the action of $\mathfrak{w}_i$ on $i\mathfrak{t}^*$ is the reflection across $\theta_i^{\perp}$. So the group $\mathfrak{W}$ spanned by the σ_i corresponding to simple roots satisfies $\mathfrak{W} \subset \mathfrak{W}_G$.

Conversely, let $\mathfrak{w} \in \mathfrak{W}_G$. Let γ be a regular element in $i\mathfrak{t}^*$ such that Δ is the set $\Delta(\gamma)$ of indecomposable elements in Φ_{γ}^{+}, cf. the proof of Theorem 12.45. Since $\mathfrak{w}$ permutes the roots, cf. Lemma 12.36, it is easy to see that $\Delta' := \mathfrak{w} \cdot \Delta$ is the basis of Φ defined by the set $\Delta(\mathfrak{w} \cdot \gamma)$ of indecomposable elements in $\Phi_{\mathfrak{w}\cdot\gamma}^{+}$.

We first prove that there exists $\sigma \in \mathfrak{W}$ such that $\sigma \cdot \Delta' = \Delta$. To see this, observe that for any σ_i corresponding to a simple root, one has

$$\sigma_i(\theta) \in \Phi^{+}, \quad \theta \in \Phi^{+}, \theta \neq \theta_i.$$

Indeed, by property (ii) of Theorem 12.39, $\sigma_i(\theta) = \theta - 2\frac{\langle \theta, \theta_i \rangle}{\langle \theta_i, \theta_i \rangle}\theta_i$ is a root, and furthermore a positive root, since it has to belong to either Φ^{+} or Φ^{-}.

From this last result, one deduces easily that the half-sum of the positive roots,

$$\delta := \frac{1}{2} \sum_{\theta \in \Phi^{+}} \theta,$$

is such that for any simple root θ_i, $\sigma_i(\delta) = \delta - \theta_i$. Now, since the group $\mathfrak{W} \subset \mathfrak{W}_G$ is finite, there exists $\sigma \in \mathfrak{W}$ such that $\langle \sigma(\mathfrak{w} \cdot \gamma), \delta \rangle$ is as big as possible.

By the definition of σ, one gets for any simple root θ_i,

$$\begin{aligned}\langle \sigma(\mathfrak{w}\cdot\gamma), \delta\rangle &\geq \langle \sigma_i\sigma(\mathfrak{w}\cdot\gamma), \delta\rangle \\ &= \langle \sigma(\mathfrak{w}\cdot\gamma), \sigma_i(\delta)\rangle \\ &= \langle \sigma(\mathfrak{w}\cdot\gamma), \delta\rangle - \langle \sigma(\mathfrak{w}\cdot\gamma), \theta_i\rangle.\end{aligned}$$

Hence

$$\langle \sigma(\mathfrak{w}\cdot\gamma), \theta_i\rangle \geq 0, \quad \theta_i \in \Delta.$$

But $\langle \sigma(\mathfrak{w}\cdot\gamma), \theta_i\rangle = \langle \mathfrak{w}\cdot\gamma, \sigma^{-1}(\theta_i)\rangle \neq 0$, since $\mathfrak{w}\cdot\gamma$ is regular. Therefore,

$$\langle \sigma(\mathfrak{w}\cdot\gamma), \theta_i\rangle > 0, \quad \theta_i \in \Delta.$$

This implies $\Phi^+_{\sigma(\mathfrak{w}\cdot\gamma)} = \Phi^+_\gamma(=: \Phi^+)$, hence $\Delta(\sigma(\mathfrak{w}\cdot\gamma) = \Delta(\gamma)(=: \Delta)$. So $\sigma\cdot\Delta' = \Delta$ as required.

We now prove that $\sigma\mathfrak{w}$ is the neutral element $\mathfrak{e}$ of $\mathfrak{W}$. Let $\mathfrak{w}' := \sigma\mathfrak{w}$, and denote by g' a representative of $\mathfrak{w}'$ and by X_γ the vector in $\mathfrak{t}$ corresponding to γ under the canonical isomorphism $\mathfrak{t} \to i\mathfrak{t}^*$, $X \mapsto iX^\sharp$. For any simple root θ_j, one has

$$-i\theta_j(X_\gamma) = \langle \theta_j, \gamma\rangle > 0,$$

and since $\mathfrak{w}'\cdot\Delta = \Delta$, this implies

$$-i\theta_j(\mathfrak{w}'\cdot X_\gamma) = \langle \mathfrak{w}'^{-1}\cdot\theta_j, \gamma\rangle > 0.$$

Consider $X = \sum_{k=0}^{m-1} \mathfrak{w}'^k \cdot X_\gamma$, with m the order of $\mathfrak{w}'$ in $\mathfrak{W}_G$. By definition, $\mathfrak{w}'\cdot X = X$. This implies that g' belongs to the group

$$\mathcal{Z}_G(X) := \{g \in G; \mathrm{Ad}(g)\cdot X = X\}.$$

It can be shown that this compact subgroup of G, which contains $\mathcal{T}$, is actually connected, hence its Lie algebra is

$$\mathcal{B}(X) := \{Y \in \mathfrak{g}; [Y, X] = 0\}.$$

Now let $Y \in \mathcal{B}(X)$. Thanks to decomposition (12.1) and Lemma 12.34, Y can be uniquely written as

$$Y = Y' + \sum_{\theta\in\Phi} Y_\theta, \quad Y' \in \mathfrak{t}_{\mathbb{C}}, Y_\theta \in \mathfrak{g}_\theta.$$

Then

$$[X, Y] = 0 = \sum_{\theta\in\Phi} \theta(X) Y_\theta.$$

But we noted before that $-i\theta_j(X) > 0$, for any simple root θ_j. Now, since any root belongs either to Φ^+ or Φ^-, this implies that $\theta(X) \neq 0$, for any root θ. Hence all the Y_θ are 0, and then $Y = Y' \in \mathfrak{g} \cap \mathfrak{t}_{\mathbb{C}} = \mathfrak{t}$.

This proves the inclusion $\mathcal{B}(X) \subset \mathfrak{t}$, which implies $\mathcal{B}(X) = \mathfrak{t}$. We then get $\mathcal{Z}_G(X) = \mathcal{T}$, since these two connected subgroups of G have same Lie algebras.

Finally $g' \in \mathcal{T}$, so $\mathfrak{w}' = \sigma\mathfrak{w} = \mathfrak{e}$ and $\mathfrak{w} = \sigma^{-1} \in W$. □

The following result is a first step in the characterization of weights of G by means of simple roots.

Theorem 12.50. *Let β be a weight* (*in the "integral" sense*) *of G, that is $\beta \in P(G)$. Then*

$$2\frac{\langle \beta, \theta_i \rangle}{\langle \theta_i, \theta_i \rangle} \in \mathbb{Z}, \qquad \theta_i \in \Delta.$$

Proof. Any $\beta \in P(G)$ can be written as $\beta = \sum_{\theta_i \in \Delta} c_i \theta_i$. Therefore,

$$2\frac{\langle \beta, \theta_j \rangle}{\langle \theta_j, \theta_j \rangle} = \sum_i c_i \frac{2\langle \theta_i, \theta_j \rangle}{\langle \theta_j, \theta_j \rangle}.$$

Let θ_j be a simple root, and $\mathfrak{s}_j = \operatorname{span}\{h_j, x_j^+, x_j^-\}$ the Lie subalgebra of $\mathfrak{g}_{\mathbb{C}}$ is isomorphic to $\mathfrak{sl}_{2,\mathbb{C}}$ introduced in the proof of Theorem 12.39. From the proof of this same theorem, we have

$$\beta(h_j) = \sum_i c_i \theta_i(h_j) = \sum_i c_i \frac{2\langle \theta_i, \theta_j \rangle}{\langle \theta_j, \theta_j \rangle}.$$

Hence

$$2\frac{\langle \beta, \theta_j \rangle}{\langle \theta_j, \theta_j \rangle} = \beta(h_j).$$

But β is contained in the character of an irreducible representation ρ of G (cf. Remark 12.27), hence $\beta(h_j)$ is a weight of the representation $\rho_*^{\mathbb{C}}$ restricted to $\mathfrak{s}_j$. By Remark 12.42 i), one has $\beta(h_j) \in \mathbb{Z}$. □

12.3.5 Dominant weights

Proceeding further in this direction, we are going to characterize irreducible representations of G in terms of certain weights, called dominant weights. Since G is supposed to be simply connected, irreducible representations of G are in one-to-one correspondence with complex finite-dimensional representations of its Lie algebra $\mathfrak{g}$.

This allows to derive the study of irreducible representations of G from that of irreducible complex finite-dimensional representations of its Lie algebra $\mathfrak{g}$. We adopt this “algebraic” point of view in the following, using a method which may be seen as a generalization of the method we used in the study of the representations of SU_2.

We begin by introducing a (partial) order $\leq$ in $i\,\mathfrak{t}^*$.

Definition 12.51. For $\beta, \beta' \in \mathfrak{t}^*$, we will say that

$$\beta' \leq \beta \quad \text{if } \beta - \beta' = \sum_{\theta_i \in \Delta} c_i \theta_i, \text{ where the } c_i\text{'s belong to } \mathbb{R}^+.$$

Since G is simply connected, the map $\rho \mapsto \rho_*^{\mathbb{C}}$ is a one-to-one correspondence (as it was remarked in the case $G = \mathrm{SU}_2$), between complex finite-dimensional representations of G and complex finite-dimensional representations of the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$. Indeed, considering the enveloping algebra $\mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$ of $\mathfrak{g}_{\mathbb{C}}$, the map $\rho \mapsto \mathfrak{U}\left(\rho_*^{\mathbb{C}}\right)$ is a one-to-one correspondence between complex finite-dimensional representations of G and complex finite-dimensional representations of $\mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$.

Moreover, this one-to-one correspondence maps irreducible representations of G into irreducible finite-dimensional complex representation of $\mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$. Recall that if $\{X_i, i = 1, \dots, N\}$, is a basis of $\mathfrak{g}_{\mathbb{C}}$, then $\mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$ can be seen as the algebra spanned by (formal) monomials of the form

$$X_{i_1}^{k_1} X_{i_2}^{k_2} \cdots X_{i_p}^{k_p}, \quad 1 \leq i_1 < i_2 < \cdots < i_p \leq N, \quad k_i \in \mathbb{N},$$

with the (associative) multiplication induced by the relations

$$X_i X_j - X_j X_i = [X_i, X_j].$$

If $\rho\colon G \to \mathrm{GL}_{\mathbb{C}}(V)$ is a representation, a monomial of the form $X_{i_1}^{k_1} X_{i_2}^{k_2} \cdots X_{i_p}^{k_p}$ acts on V by the endomorphism

$$(\rho_*^{\mathbb{C}}(X_{i_1}))^{k_1} \circ \cdots \circ (\rho_*^{\mathbb{C}}(X_{i_p}))^{k_p}.$$

Indeed, it seems “natural” to consider such an endomorphism in the study of the representation, however it is not in general an element of $\rho_*^{\mathbb{C}}(\mathfrak{g}_{\mathbb{C}})$, a reason to introduce this trick of enveloping algebra. Moreover, note that there is a fine description of irreducible G-representation spaces in terms of cyclic modules on $\mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$, see Theorem 12.58 below.

Notations 12.52. For any representation (ρ, V) of G, $X \in \mathfrak{g}_{\mathbb{C}}$ and $v \in V$, we write $X \cdot v$ instead of $\rho_*^{\mathbb{C}}(X) \cdot v$.

Lemma 12.53. *If (ρ, V) is an irreducible representation of G and β a weight of ρ, we denote by*

$$V_\beta = \{v \in V;\ \text{for all } X \in \mathfrak{t}, \rho_*(X) \cdot v = \beta(X)v\},$$

the corresponding weight space. For any root θ_i, one has

1. *either $\beta + \theta_i$ is not a weight and then $\mathfrak{g}_{\theta_i} \cdot V_\beta = \{0\}$ or*
2. *$\beta + \theta_i$ is a weight and then $\mathfrak{g}_{\theta_i} \cdot V_\beta \subset V_{\beta+\theta_i}$.*

Proof. Let $X \in \mathfrak{t}$, $X_i \in \mathfrak{g}_{\theta_i}$. For any $v \in V$,

$$X \cdot (X_i \cdot v) = X_i \cdot (X \cdot v) + [X, X_i] \cdot v = (\beta(X) + \theta_i(X))(X_i \cdot v).$$

Hence the result. □

Let ρ be an irreducible representation of G. Since it has only a finite number of weights, there exists a weight λ such that $\lambda + \theta_i$ is *not* a weight for any *positive root* θ_i. From the above lemma it follows that

$$X_i \cdot v = 0, \quad \theta_i \in \Phi^+, X_i \in \mathfrak{g}_{\theta_i}, v \in V_\lambda.$$

Definition 12.54. A weight vector of a representation $\rho\colon G \to \mathrm{GL}_{\mathbb{C}}(V)$ is said to be a *maximal vector*, if it is annihilated by all the $\mathfrak{g}_{\theta_i}$, where $\theta_i \in \Phi^+$.

Remark 12.55. The above argument shows that every representation (irreducible or not) has a maximal vector. Thus if a representation has, up to a scalar multiple, only one maximal vector, it is irreducible.

Remark 12.56. By (12.5) and the fact that any positive root can be written as a sum of simple roots $\theta_1 + \cdots + \theta_k$ in such a way that each partial sum $\theta_1 + \cdots + \theta_i$ is a root (cf. for instance Lemma 10.2A in [Hum72]), one concludes that for any positive root θ, any element in $\mathfrak{g}_\theta$ belongs to the Lie subalgebra of $\mathfrak{g}_{\mathbb{C}}$ generated by the $\mathfrak{g}_{\theta_i}$, $i = 1, \ldots, l$, corresponding to simple roots.

Hence, in order for a vector of a G-space to be maximal it suffices that it is annihilated by all the $\mathfrak{g}_{\theta_i}$ corresponding to simple roots.

Theorem 12.57. *Denote by $\rho\colon G \to \mathrm{GL}_{\mathbb{C}}(V)$ a representation of G (not necessarily irreducible), and by v^+ a maximal vector of ρ with weight λ. Let V^+ be the cyclic module $\mathfrak{U}(\mathfrak{g}_{\mathbb{C}}) \cdot v^+$. Then*

(i) *every weight μ of V^+ has the form $\mu = \lambda - \sum_{\theta_i \in \Delta} n_i \theta_i$, where $n_i \in \mathbb{N}$, hence $\mu \leq \lambda$;*

(ii) $\dim V_\lambda = 1$, *that is, the weight λ has multiplicity 1 in V^+;*

(iii) *V^+ is an irreducible G-space.*

Proof. Let $\theta_1, \ldots, \theta_n$ be the positive roots of G. From the proof of Theorem 12.39 (Step 2) it follows that for any positive root θ_i there exist $h_i \in i\mathfrak{t}$, $x_{\theta_i} \in \mathfrak{g}_{\theta_i}$ and $x_{-\theta_i} \in \mathfrak{g}_{-\theta_i}$ such that $\operatorname{span}\{h_i, x_{\theta_i}, x_{-\theta_i}\}$ is a subalgebra of $\mathfrak{g}_{\mathbb{C}}$ isomorphic to $\mathfrak{sl}_{2,\mathbb{C}}$.

From decomposition (12.1), Lemma 12.34, and Remark 12.48, using moreover the fact that $\dim \mathfrak{g}_\theta = 1$, for any root θ, (cf. the proof of Theorem 12.39, Step 3), one deduces that the h_i, $i = 1, \ldots, l$ (corresponding to simple roots) and the x_{θ_i}, $x_{-\theta_i}$, $i = 1, \ldots, n$, form a basis of $\mathfrak{g}_{\mathbb{C}}$. Hence the enveloping algebra $\mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$ has a basis of the form

$$x_{-\theta_1}^{a_1} \cdots x_{-\theta_n}^{a_n} h_1^{c_1} \cdots h_l^{c_l} x_{\theta_1}^{b_1} \cdots x_{\theta_n}^{b_n}, \tag{12.6}$$

where the a_i, b_i, and c_i belong to $\mathbb{N}$.

Now, since v^+ is a maximal vector, the action of a basis vector of $\mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$ of the form (12.6) on v^+ gives 0, if one of the integer b_i is nonzero. Furthermore, since v^+ is a weight vector, the action of a basis vector of $\mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$ of the form $h_1^{c_1} \cdots h_l^{c_l}$ on v^+ gives a scalar multiple of v^+. Hence V^+ is spanned by vectors of the form

$$x_{-\theta_1}^{a_1} \cdots x_{-\theta_n}^{a_n} \cdot v^+.$$

Now by the proof of Lemma 12.53, if such a vector is nonzero, then it is a weight vector for the weight $\mu = \lambda - \sum_{i=1}^{n} a_i \theta_i$. Expressing each positive root θ_i in terms of simple roots, one gets (i).

Moreover, note that up to a scalar, the only vector of the above form of weight λ is v^+, hence (ii). Now suppose V^+ is not irreducible; then it decomposes into the sum of two G-invariant subspaces: $V^+ = V_1^+ \oplus V_2^+$. According to this decomposition, one gets $v^+ = v_1^+ + v_2^+$. If both v_1^+ and v_2^+ are nonzero, then they are two linearly independent (maximal) vectors for the weight λ, contradicting (ii). Hence one of v_1^+ and v_2^+ has to be zero. If, for instance, $v_2^+ = 0$, then $v^+ = v_1^+$, so

$$V^+ = \mathfrak{U}(\mathfrak{g}_{\mathbb{C}}) \cdot v_1^+ \subset V_1^+,$$

hence $V_1^+ = V^+$ and $V_2^+ = \{0\}$. Thus, V^+ is irreducible. $\square$

Theorem 12.58. *If $\rho\colon G \to \mathrm{GL}_{\mathbb{C}}(V)$ is an irreducible representation, then V has, up to a scalar multiple, only one maximal vector v^+, and $V = V^+ := \mathfrak{U}(\mathfrak{g}_{\mathbb{C}}) \cdot v^+$.*

Proof. Let v^+ be a maximal weight vector of V. Since V is irreducible, $V^+ = V$. Uniqueness of v^+ results from Theorem 12.57 (ii). $\square$

Definition 12.59. Let $\rho\colon G \to \mathrm{GL}_{\mathbb{C}}(V)$ be an irreducible representation of G. Let v^+ be the maximal vector of ρ and λ the corresponding weight. This weight is said to be the *dominant weight* of the representation ρ.

This dominant weight λ has the following property.

Theorem 12.60. *Let λ be the dominant weight of an irreducible representation of G. For any simple root θ_i,*

$$2\frac{\langle \lambda, \theta_i \rangle}{\langle \theta_i, \theta_i \rangle} \in \mathbb{N}.$$

Proof. Let θ_i be a simple root and $\operatorname{span}\{h_i, x_{\theta_i}, x_{-\theta_i}\}$ the corresponding Lie subalgebra of $\mathfrak{g}_{\mathbb{C}}$ (it was introduced in the proof of Theorem 12.57). If v^+ is the maximal vector of ρ, then by definition, $x_{\theta_i} \cdot v^+ = 0$, since $x_{\theta_i} \in \mathfrak{g}_{\theta_i}$.

Moreover, $2\frac{\langle \lambda, \theta_i \rangle}{\langle \theta_i, \theta_i \rangle} = \lambda(h_i)$, see the proof of Theorem 12.50. Therefore,

$$h_i \cdot v^+ = \lambda(h_i) v^+ \quad \text{and} \quad x_{\theta_i} \cdot v^+ = 0,$$

so by Theorem 12.41 and its proof, $\lambda(h_i)$ must belong to $\mathbb{N}$. □

Theorem 12.61. *Let $\lambda \in i\mathfrak{t}^*$. If for any simple root θ_j, the real number $2\frac{\langle \lambda, \theta_j \rangle}{\langle \theta_j, \theta_j \rangle}$ is a non negative integer, then λ is the dominant weight of an irreducible representation of G. Hence the map which assigns to each equivalence class of irreducible representations of G its dominant weight is a one-to-one correspondence onto the set*

$$\Lambda^+ = \left\{ \lambda \in i\mathfrak{t}^* \colon \textit{for all } \theta_j \in \Delta, 2\frac{\langle \lambda, \theta_j \rangle}{\langle \theta_j, \theta_j \rangle} \in \mathbb{N} \right\}.$$

Elements of Λ^+ are called dominant weights.

In particular, the vectors $\beta_k \in i\mathfrak{t}^*$, $k = 1, \dots, l$, defined by $2\frac{\langle \beta_k, \theta_j \rangle}{\langle \theta_j, \theta_j \rangle} = \delta_{jk}$, for any $\theta_j \in \Delta$, are dominant weights of irreducible representations of G.

Definition 12.62. The irreducible representations ρ_k, $k = 1, \dots, l$, of G with corresponding dominant weights β_k defined by the condition

$$2\frac{\langle \beta_k, \theta_j \rangle}{\langle \theta_j, \theta_j \rangle} = \delta_{jk}, \quad \theta_j \in \Delta,$$

are called *fundamental representations* of G. Their corresponding dominant weights β_j are called *fundamental weights*.

Note that, by definition, fundamental weights define a basis $\{\beta_1, \dots, \beta_l\}$ of $i\mathfrak{t}^*$. Thus any $\beta \in i\mathfrak{t}^*$ can be written as

$$\beta = \sum_{j=1}^{l} c_j \beta_j, \quad c_j \in \mathbb{R}.$$

But for each $\theta_j \in \Delta$, we have $2\frac{\langle \beta, \theta_j \rangle}{\langle \theta_j, \theta_j \rangle} = c_j$, hence

$$\beta = \sum_{j=1}^{l} 2\frac{\langle \beta, \theta_j \rangle}{\langle \theta_j, \theta_j \rangle} \beta_j.$$

Applying this to a weight β, we conclude from Theorem 12.50 that

Proposition 12.63. *Fundamental weights define a basis $\{\beta_1, \beta_2, \ldots, \beta_l\}$ of the weight lattice $P(G)$. Moreover, a vector $\beta = \sum_{j=1}^{n} m_j \beta_j \in i\mathfrak{t}^*$ is a dominant weight if and only if the m_j's are non-negative integers.*

For the proof of Theorem 12.61 see [Hum72], Sections 20.3 and 21.2, or [Die75], pp. 116–120. Indeed, instead of this rather abstract proof, we prefer to show how to construct an irreducible representation with dominant weight $\beta = \sum_{j=1}^{l} m_j \beta_j$, $m_j \in \mathbb{N}$, assuming that the fundamental representations are explicitly known.

Remark 12.64. Suppose that fundamental representations $\rho_1, \rho_2, \ldots, \rho_l$ are explicitly known.

Let $\beta_1, \beta_2, \ldots, \beta_l$, be the corresponding fundamental weights, $V_1, V_2, \ldots, V_l$ the corresponding representation spaces, and $v_1, v_2, \ldots, v_l$ the corresponding maximal vectors. Let $\beta \in i\mathfrak{t}^*$ given by

$$\beta = \sum_{j=1}^{l} m_j \beta_j,$$

where $m_j \in \mathbb{N}$. In the space

$$W = \mathcal{S}^{m_1} V_1 \otimes \mathcal{S}^{m_2} V_2 \otimes \cdots \otimes \mathcal{S}^{m_l} V_l,$$

where for any j, $\mathcal{S}^{m_j} V_j$ denotes the space of symmetric tensors of order m_j over V_j, consider the vector

$$v = v_1^{m_1} \otimes v_2^{m_2} \otimes \cdots \otimes v_l^{m_l}.$$

By construction, this is a maximal vector of W corresponding to the weight β. So by Theorem 12.57, the cyclic module $V = \mathfrak{U}(\mathfrak{g}_{\mathbb{C}}) \cdot v$ is an irreducible G-space of W with dominant weight β.

In the following we give the explicit description of the fundamental representations of the groups SU_n, Spin_n and Sp_n. Hence this remark can be seen as a "proof" of Theorem 12.61 in those cases (quotation marks because the existence of fundamental representations is indeed asserted by Theorem 12.61!).

12.3.6 The Weyl formulas

By now, we are able to answer the two questions pointed out in the introduction of this chapter. However, a more precise answer can be given to the question that follows Proposition 12.30: which elements in $\mathbb{Z}_{\mathfrak{W}_G}^{R(T)}$ are restrictions to T of irreducible characters of G?

This answer (which is a more explicit formulation of the one-to-one correspondence established in Theorem 12.61) is given by the Weyl character formula. This formula is obtained using an "analytic" approach (which is Weyl's original approach, cf. [Wey46]), dealing more with the groups themselves rather than their Lie algebras. In spite of its importance (it plays for instance a central role in the references [Ada69] and [BtD85]), we only give this formula since only one of its corollaries, the Weyl dimension formula, cf. Theorem 12.67 below, will be useful for the problems we are dealing with.

Theorem 12.65 (Weyl character formula). *Let β be a dominant weight and χ_ρ the character of the corresponding irreducible representation of G. Then the restriction to T of χ_ρ satisfies*

$$\Delta_G \chi_\rho|_T = \sum_{\mathfrak{w} \in \mathfrak{W}_G} (\det \mathfrak{w}) \mathrm{e}^{\mathfrak{w}\cdot(\beta+\delta)}, \tag{12.7}$$

where δ is the half-sum of the positive roots and Δ_G is the function on T defined by

$$\Delta_G(t) = \prod_{\theta \in \Phi+} \left(\mathrm{e}^{\frac{1}{2}\theta(X)} - \mathrm{e}^{-\frac{1}{2}\theta(X)} \right) = \mathrm{e}^{\delta(X)} \prod_{\theta \in \Phi+} (1 - \mathrm{e}^{-\theta(X)}), \tag{12.8}$$

X being any element in $\exp^{-1}(t)$.

Remark 12.66. First, note that since

$$\delta = \beta_1 + \cdots + \beta_l,$$

where the β_i are the fundamental weights of G, δ is an integral weight of G (cf. the comments following Theorem 12.61), hence the function Δ_G is well defined on T. Now observe that $\Delta_G(t) \neq 0$ if and only if

$$\theta(X) \notin 2i\pi\mathbb{Z}, \quad X \in \exp^{-1}(t), \theta \in \Phi.$$

An element $t \in T$ satisfying this condition is said to be *regular*. Otherwise it is said to be *singular*.

Thus the value of $\chi_\rho|_T$ at regular elements can be obtained by multiplying both sides of equation (12.7) by Δ_G^{-1}. Now if t is a singular element, then $\Delta_G(t) = 0$, so the right-hand side of equation (12.7) has also to be zero at the point t. Let us check this. Indeed, suppose that there exists $X \in \exp^{-1}(t)$ such that $\theta_i(X) \in 2i\pi\mathbb{Z}$ for some root θ_i and let $\sigma_i \in \mathfrak{W}_G$ be defined by the reflection across $\theta_i^\perp$, cf. Theorem 12.49.

For any integral weight β, one has

$$\sum_{\mathfrak{w}\in\mathfrak{W}_G} (\det \mathfrak{w}) \mathrm{e}^{\mathfrak{w}\cdot\beta(X)} = \frac{1}{2} \sum_{\mathfrak{w}\in\mathfrak{W}_G} (\det \mathfrak{w}) \mathrm{e}^{\mathfrak{w}\cdot\beta(X)} + \frac{1}{2} \sum_{\mathfrak{w}\in\mathfrak{W}_G} (\det \sigma_i \mathfrak{w}) \mathrm{e}^{\sigma_i \mathfrak{w}\cdot\beta(X)},$$

where X is any element in $\exp^{-1}(t)$. But from Theorem 12.50 and its proof, since $2\frac{\langle \mathfrak{w}\cdot\beta, \theta_i\rangle}{\langle \theta_i, \theta_i\rangle}\theta_i(X) \in 2i\pi\mathbb{Z}$, one gets $\mathrm{e}^{\sigma_i \mathfrak{w}\cdot\beta(X)} = \mathrm{e}^{\mathfrak{w}\cdot\beta(X)}$. Hence, since $\det \sigma_i \mathfrak{w} = -\det \mathfrak{w}$, the right-hand side of equation (12.7) vanishes at t.

Thus the value of $\chi_\rho|_T$ at singular elements cannot be obtained directly from equation (12.7). In fact, the set T_{reg} of regular elements is dense in T (since any generator of T has to be regular), so the value of $\chi_\rho|_T$ at singular elements can be obtained from the value at regular elements by a continuous extension argument.

For a proof of Theorem 12.65 following Weyl's original approach, see [Ada69] Chapter 6, [BtD85] Chapter VI, [Die75] Chapter 15, or for classical groups [GW98] (Section 7.4). Indeed there is also a (rather abstract) algebraic proof of this result, cf. for instance [Hum72] (Section 24.3), [Žel73] (Paragraph 123), or for classical groups [GW98] (Section 7.3).

Note that, taking $\beta = 0$ (which is the dominant weight of the trivial representation) in formula (12.7) yields

$$\Delta_G = \sum_{\mathfrak{w}\in\mathfrak{W}_G} (\det \mathfrak{w}) \mathrm{e}^{\mathfrak{w}\cdot\delta}. \tag{12.9}$$

From the Weyl character formula one obtains a useful formula that gives the dimension of the irreducible representation corresponding to a dominant weight in terms of the dominant weight itself.

Theorem 12.67 (Weyl dimension formula). *Let β be the dominant weight of an irreducible representation ρ of G. The dimension n_ρ of the representation is given by*

$$n_\rho = \frac{\prod_{\theta\in\Phi^+} \langle \beta+\delta, \theta\rangle}{\prod_{\theta\in\Phi^+} \langle \delta, \theta\rangle},$$

where δ is the half-sum of the positive roots.

Proof. By definition, $n_\rho = \chi_\rho(e)$, but, as it was noted in the above remark, $\chi_\rho(e)$ cannot be computed directly from (12.7) since e is singular. For this, we first compute χ_ρ at the point $\exp(sX_\delta)$, where X_δ is the vector in $\mathfrak{t}$ corresponding to δ under the canonical isomorphism

$$\mathfrak{t} \longrightarrow i\mathfrak{t}^*, \quad X \longmapsto iX^\sharp, \tag{12.10}$$

and then let s tend to 0. Note that, by definition, for any integral weight β,

$$-i\beta(X_\delta) = \langle \beta, \delta \rangle.$$

Note moreover that since δ is the sum of the fundamental weights, one has

$$\langle \theta_i, \delta \rangle > 0, \quad \theta_i \in \Delta,$$

so

$$\theta(X_\delta) \neq 0, \quad \theta \in \Phi,$$

hence $\exp(sX_\delta)$ is regular for s sufficiently small $s \neq 0$. Now, for any integral weight β and for any $\mathfrak{w} \in \mathfrak{W}_G$, one gets from the $\mathfrak{W}_G$-invariance of the scalar product that

$$(\mathfrak{w} \cdot \beta)(X_\delta) = i \langle \mathfrak{w} \cdot \beta, \delta \rangle = i \langle \beta, \mathfrak{w}^{-1}\delta \rangle = (\mathfrak{w}^{-1} \cdot \delta)(X_\beta),$$

where $X_\beta \in \mathfrak{t}$ corresponds to β under the canonical isomorphism 12.10. Hence

$$\begin{aligned} \sum_{\mathfrak{w} \in \mathfrak{W}_G} (\det \mathfrak{w}) \mathrm{e}^{(\mathfrak{w} \cdot \beta)(sX_\delta)} &= \sum_{\mathfrak{w} \in \mathfrak{W}_G} (\det \mathfrak{w}) \mathrm{e}^{(\mathfrak{w}^{-1} \cdot \delta)(sX_\beta)} \\ &= \sum_{\mathfrak{w} \in \mathfrak{W}_G} (\det \mathfrak{w}) \mathrm{e}^{(\mathfrak{w} \cdot \delta)(sX_\beta)}. \end{aligned}$$

But from (12.9), this is the value of Δ_G at the point $\exp(sX_\beta)$. Using the first expression of Δ_G in (12.8), one gets

$$\Delta_G(\exp(sX_\beta)) = \prod_{\theta \in \Phi^+} \left(\mathrm{e}^{\frac{1}{2} i \langle \beta, \theta \rangle s} - \mathrm{e}^{-\frac{1}{2} i \langle \beta, \theta \rangle s}\right).$$

As s tends to 0, this expression is equivalent to

$$\prod_{\theta \in \Phi^+} i \langle \beta, \theta \rangle s.$$

Thus when s tends to 0, $\chi_\rho(\exp(sX_\delta))$ is equivalent to

$$\frac{\displaystyle\prod_{\theta \in \Phi^+} i \langle \beta + \delta, \theta \rangle s}{\displaystyle\prod_{\theta \in \Phi^+} i \langle \delta, \theta \rangle s},$$

which completes the proof. □

12.4 Application: irreducible representations of the classical groups SU_n, Spin_n, and Sp_n

12.4.1 Irreducible representations of the groups SU_n and U_n, $n \geq 3$

Irreducible representations of the group SU_n. Let T_0 be the torus of U_n, i.e.,

$$T_0 = \{\text{diag. matrix}(\mathrm{e}^{i\beta_1}, \ldots, \mathrm{e}^{i\beta_n});\ \beta_1, \ldots, \beta_n \in \mathbb{R}\}. \tag{12.11}$$

Its Lie algebra is

$$\mathfrak{t}_0 = \{\text{diag. matrix } (i\beta_1, \ldots, i\beta_n);\ \beta_1, \ldots, \beta_n \in \mathbb{R}\}.$$

We *denote* by $\{y_1, \ldots, y_n\}$ the basis of $\mathfrak{t}_0{}^*$ given by

$$y_k[\text{diag. matrix}(i\beta_1, \ldots, i\beta_n)] = \beta_k.$$

We consider the torus T of SU_n defined by $T = T_0 \cap \mathrm{SU}_n$. Its Lie algebra is

$$\mathfrak{t} = \Big\{\text{diag. matrix}(i\beta_1, \ldots, i\beta_n);\ \beta_1, \ldots, \beta_n \in \mathbb{R}, \sum_{k=1}^{n} \beta_k = 0\Big\}.$$

We *denote* by $(x_1, \ldots, x_{n-1})$ the basis of $\mathfrak{t}^*$ given by

$$x_k[\text{diag. matrix}(i\beta_1, \ldots, i\beta_n)] = \beta_k.$$

Let $\{\hat{x}_k\}_{k=1,\ldots,n-1}$ be the basis of $i\,\mathfrak{t}^*$ defined by $\hat{x}_k = i x_k$. A vector $\mu \in i\,\mathfrak{t}^*$ such that $\mu = \sum_{k=1}^{n-1} \lambda_k \hat{x}_k$ is *denoted* by

$$\mu = (\lambda_1, \ldots, \lambda_{n-1}). \tag{12.12}$$

Since the Lie algebra $\mathfrak{u}_n$ of the group U_n splits into

$$\mathfrak{u}_n = \mathfrak{su}_n \oplus \mathfrak{c},$$

$\mathfrak{c} = i\,\mathbb{R}\mathrm{Id}_n$ being the Lie algebra of the center of U_n, each vector $\mu \in i\,\mathfrak{t}^*$ can be identified with a vector in $i\,\mathfrak{t}_0^*$ which is identically zero on $\mathfrak{c}$. Thus, we can also express μ in the basis $\{\hat{y}_k \equiv i y_k\}_{k=1,\ldots,n}$, of $i\,\mathfrak{t}_0^*$:

$$\mu = \sum_{k=1}^{n} \mu_k \hat{y}_k \quad \text{and} \quad \sum_{k=1}^{n} \mu_k = 0.$$

In this case we *denote* μ by

$$\mu = [\mu_1, \ldots, \mu_n]. \tag{12.13}$$

The correspondence between the two expressions (12.12) and (12.13), is given by

$$\lambda_i = \mu_i - \mu_n \iff \begin{cases} \mu_i = \lambda_i + \lambda_0, & i = 1, \dots, n-1, \\ \mu_n = \lambda_0 := -\frac{1}{n} \sum_{i=1}^{n-1} \lambda_i. \end{cases} \tag{12.14}$$

We consider the SU_n-invariant scalar product on $\mathfrak{t}$ given by

$$\langle X, Y \rangle = -\mathrm{Re}[\mathrm{Tr}(XY)] = -\frac{1}{2n}\mathrm{B}(X, Y), \quad X, Y \in \mathfrak{t},$$

where B is the Killing form of $\mathfrak{su}_n$. Its extension to $i\,\mathfrak{t}^*$ is given by

$$\langle \mu, \mu' \rangle = \sum_{i=1}^{n} \mu_i \mu_i', \quad \mu = [\mu_1, \dots, \mu_n], \mu' = [\mu_1', \dots, \mu_n'] \in i\,\mathfrak{t}^*. \tag{12.15}$$

The decomposition of $\mathfrak{su}_{n,\mathbb{C}} \simeq \mathfrak{sl}_{n,\mathbb{C}}$ into root spaces (see (12.1)) is easily obtained. One gets

$$\mathfrak{su}_{n,\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \Big(\bigoplus_{\substack{1 \le i,j \le n \\ i \ne j}} \mathfrak{g}_{ij} \Big),$$

where $\mathfrak{g}_{ij} = \mathbb{C}E_{ij}$ and $\{E_{ij}\}$ is the usual basis of $\mathrm{M}_n(\mathbb{C})$. The corresponding roots θ_{ij} are given by

$$\begin{aligned} \theta_{ij} &= \hat{x}_i - \hat{x}_j, & 1 \le i \ne j \le n-1, \\ \theta_{in} &= -\theta_{ni} = \hat{x}_1 + \cdots + 2\hat{x}_i + \cdots + \hat{x}_{n-1}, & 1 \le i \le n-1. \end{aligned}$$

Using Lemma 12.34, we first note that T is a maximal torus of SU_n. Therefore, the $n-1$ roots

$$\theta_i = \hat{x}_i - \hat{x}_{i+1}, \quad 1 \le i \le n-2, \theta_{n-1} = \hat{x}_1 + \cdots + 2\hat{x}_{n-1},$$

form a basis Δ of the root system Φ corresponding to this maximal torus T. Indeed, we have

$$\begin{aligned} \theta_{ij} &= -\theta_{ji} = \theta_i + \cdots + \theta_{j-1}, & 1 \le i < j \le n-1, \\ \theta_{in} &= -\theta_{ni} = \theta_i + \cdots + \theta_{n-2} + \theta_{n-1}, & 1 \le i \le n-1. \end{aligned}$$

Note that

$$\begin{aligned} \theta_i &= (0, \dots, 0, \underbrace{1}_{i}, \underbrace{-1}_{i+1}, 0, \dots, 0) \\ &= [0, \dots, 0, \underbrace{1}_{i}, \underbrace{-1}_{i+1}, 0, \dots, 0], \quad 1 \le i \le n-2, \end{aligned}$$

and

$$\theta_{n-1} = (1, \dots, 1, 2) = [0, \dots, 0, 1, -1].$$

Thus from (12.15), the reflection across $\theta_i^{\perp}$ is given by

$$\mu = [\mu_1, \dots, \mu_n] \longmapsto [\mu_1, \dots, \mu_{i+1}, \mu_i, \dots, \mu_n].$$

From Theorem 12.49 we conclude that the Weyl group $\mathfrak{W}_{SU_n}$ acts on $i\mathfrak{t}^*$ by

$$\mu = [\mu_1, \dots, \mu_n] \longmapsto [\mu_{\sigma(1)}, \dots, \mu_{\sigma(n)}], \quad \sigma \in \mathfrak{S}_n.$$

Hence,

$$\mathfrak{W}_{SU_n} \simeq \mathfrak{S}_n.$$

Furthermore, we have

$$\begin{aligned} \langle \theta_i, \theta_j \rangle &= 0, && 1 \le i, j \le n-1, |i-j| \ne 0, 1, \\ \langle \theta_i, \theta_i \rangle &= 2, && 1 \le i \le n-1, \\ \langle \theta_i, \theta_{i+1} \rangle &= -1, && 1 \le i \le n-2. \end{aligned}$$

We then conclude from Theorem 12.61 the following result.

Proposition 12.68. *A vector $\beta = (k_1, \dots, k_{n-1}) = [p_1, \dots, p_n] \in i\mathfrak{t}^*$ is a dominant weight of* SU_n *if and only if*

$$k_i \in \mathbb{N},\ i = 1, \dots, n-1, \quad \textit{and} \quad k_1 \ge k_2 \ge \cdots \ge k_{n-1},$$

which is equivalent to

$$p_i - p_{i+1} \in \mathbb{N}, \quad i = 1, \dots, n-1.$$

It is easy to see using (12.14) and (12.15) that the fundamental weights of SU_n are given by

$$\beta_i = \hat{x}_1 + \hat{x}_2 + \cdots + \hat{x}_i, \quad i = 1, 2, \dots, n-1.$$

Let ρ be the standard representation of SU_n on $\mathbb{C}^n$. Its restriction to the maximal torus T splits into the direct sum of n irreducible representations in the spaces $\mathbb{C}e_i$, where $\{e_i\}$ is the usual basis of $\mathbb{C}^n$, with corresponding weight $\hat{x}_i$.

Since only one of those weights is dominant, namely $\hat{x}_1$, the representation ρ is irreducible with dominant weight $\hat{x}_1 = \beta_1$, thus ρ is the fundamental representation ρ_1 corresponding to the fundamental weight β_1.

Note also that there is only one maximal vector annihilated by all the matrices $E_{i,i+1}$, $1 \le i \le n-1$, corresponding to simple roots, namely e_1, so we may also conclude from Remark 12.55 that ρ is irreducible.

We now consider the fundamental representations ρ_j, $j = 2, \ldots, n-1$. From Theorem 12.15 we know that each one of them is contained in some tensor product of the faithful representation ρ and its conjugate.

The simplest example of such a tensor product is the representation $\otimes^2 \rho$ whose space $\mathcal{T}^2\mathbb{C}^n$ is the space of tensors of order 2 over $\mathbb{C}^n$. This space splits into $\mathcal{T}^2\mathbb{C}^n = \Lambda^2\mathbb{C}^n \oplus \mathcal{S}^2\mathbb{C}^n$, where $\Lambda^2\mathbb{C}^n$ (resp. $\mathcal{S}^2\mathbb{C}^n$) is the space of anti-symmetric (resp. symmetric) tensors of order 2 over $\mathbb{C}^n$.

Each of those spaces is SU_n-invariant. Considering the one of lowest dimension, we get the representation

$$\Lambda^2\rho\colon \mathrm{SU}_n \longrightarrow \mathrm{GL}_{\mathbb{C}}(\Lambda^2(\mathbb{C}^n)).$$

Its restriction to the maximal torus T splits into the direct sum of $\binom{n}{2}$ irreducible representations in the spaces $\mathbb{C}(e_i \wedge e_j)$, $1 \leq i < j \leq n$, with corresponding weight $\hat{x}_i + \hat{x}_j$.

Only one of those weights is a dominant weight, namely $\hat{x}_1 + \hat{x}_2 = \beta_2$, so $\Lambda^2\rho$ is irreducible with dominant weight β_2. Hence $\Lambda^2\rho$ is the fundamental representation ρ_2 corresponding to the fundamental weight β_2.

Considering now each representation

$$\Lambda^j\rho\colon \mathrm{SU}_n \longrightarrow \mathrm{GL}_{\mathbb{C}}(\Lambda^j(\mathbb{C}^n)), \quad j = 3, \ldots, n-1,$$

and determining its weights under the action of the maximal torus as above, one sees that only one of those weights is a dominant weight, namely

$$\hat{x}_1 + \cdots + \hat{x}_j = \beta_j,$$

hence, $\Lambda^j\rho$ is irreducible with dominant weight β_j. Thus for any $j = 2, \ldots, n-1$, $\Lambda^j\rho$ is the fundamental representation corresponding to the fundamental weight β_j.

Note that, with the help of Theorem 12.67, we can already know the dimension $n_j = \binom{n}{j}$ of ρ_j, an indication to look at the representations $\Lambda^j\rho$.

Irreducible representations of the group U_n. Consider the homomorphism of Lie groups

$$\pi\colon \mathbb{T}^1 \times \mathrm{SU}_n \longrightarrow \mathrm{U}_n, \quad (\mathrm{e}^{i\theta}, A) \longmapsto \mathrm{e}^{i\theta}A.$$

Note that π is a finite covering with kernel isomorphic to the group of n^{th}-roots of unity. So we can apply Proposition 12.18: irreducible representations of U_n are irreducible representations of $\mathbb{T}^1 \times \mathrm{SU}_n$ which factor through the quotient. Thus, let us consider an irreducible representation $\tilde{\rho}$ of $\mathbb{T}^1 \times \mathrm{SU}_n$.

By Theorem 12.9, the representation $\tilde{\rho}$ is the tensor product of an irreducible representation χ of $\mathbb{T}^1$ and an irreducible representation ρ of SU_n. The representation χ has the following form (see Theorem 12.24):

$$\chi\colon e^{i\theta} \longmapsto e^{im\theta}, \quad m \in \mathbb{Z}.$$

On the other hand, the representation ρ has a dominant weight of the form

$$(k_1, \ldots, k_{n-1}),$$

where the k_i are non-negative integers satisfying

$$k_1 \geq k_2 \geq \cdots \geq k_{n-1}$$

(see Proposition 12.68). By Remark 12.64 and the description of fundamental representations, the vector space corresponding to ρ is the cyclic module $\mathfrak{U}(\mathfrak{su}_{n,\mathbb{C}}) \cdot v$, where

$$\begin{aligned} v = e_1^{k_1-k_2} &\otimes (e_1 \wedge e_2)^{k_2-k_3} \otimes \cdots \otimes (e_1 \wedge \cdots \wedge e_{n-2})^{k_{n-2}-k_{n-1}} \\ &\otimes (e_1 \wedge \cdots \wedge e_{n-1})^{k_{n-1}}. \end{aligned}$$

The kernel of π is the group spanned by the elements

$$(e^{\frac{2ik\pi}{n}}, \text{diag. matrix}(e^{-\frac{2ik\pi}{n}}, \ldots, e^{-\frac{2ik\pi}{n}})), \quad k = 0, 1, \ldots, n-1.$$

Therefore, a necessary and sufficient condition for $\tilde{\rho}$ to factor through the quotient is

$$(e^{\frac{2ik\pi}{n}})^m \rho[\text{diag. matrix}(e^{-\frac{2ik\pi}{n}}, \ldots, e^{-\frac{2ik\pi}{n}})] \cdot v = v, \quad k = 0, 1, \ldots, n-1.$$

This is equivalent to

$$e^{\frac{2ik\pi}{n}(m-\sum_{i=1}^{n-1} k_i)} = 1, \quad k = 0, 1, \ldots, n-1.$$

Hence, $\tilde{\rho}$ factors through the quotient if and only if

$$m - \sum_{i=1}^{n-1} k_i \text{ is a multiple of } n.$$

In this case, one can write

$$\tilde{\rho}(g) = (\det g)^{m/n} \rho((\det g)^{-1/n} g), \quad g \in \mathrm{U}_n,$$

since the right-hand side in this equality does not depend on the choice of the n^{th}-root of $\det g$.

12.4.2 Irreducible representations of the groups Spin_n and SO_n, $n \geq 3$

For any $n \geq 3$, we denote by $\{e_i\}_{1\leq i\leq n}$ the standard basis of $\mathbb{R}^n$. We identify U_n with a subgroup of SO_{2n}, also denoted by U_n, using the injective homomorphism which maps a matrix in $\mathrm{M}_n(\mathbb{C})$ with coefficients $z_{jk} = x_{jk} + iy_{jk}$ into the matrix in $\mathrm{M}_{2n}(\mathbb{R})$

$$\begin{pmatrix} [Z_{11}] & \dots & [Z_{1n}] \\ \vdots & \ddots & \vdots \\ [Z_{n1}] & \dots & [Z_{nn}] \end{pmatrix}$$

formed by the (2×2)-block matrices

$$Z_{jk} = \begin{pmatrix} x_{jk} & -y_{jk} \\ y_{jk} & x_{jk} \end{pmatrix}.$$

We also identify SO_{2n} with a subgroup of SO_{2n+1}, also denoted by SO_{2n}, using the injective homomorphism

$$\mathrm{M}_{2n}(\mathbb{R}) \ni A \longmapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{M}_{2n+1}(\mathbb{R}).$$

Let T be the torus of Spin_n,

$$T = \Big\{ \prod_{k=1}^{[n/2]} (\cos\beta_k + \sin\beta_k e_{2k-1} \cdot e_{2k});\ \beta_k \in \mathbb{R} \Big\}.$$

Its image $\xi(T)$ under the group homomorphism $\xi\colon \mathrm{Spin}_n \to \mathrm{SO}_n$ is the image of the maximal torus of U_n, see (12.11), under the above injective homomorphism. The Lie algebra of T is

$$\mathfrak{t} = \Big\{ \sum_{k=1}^{[n/2]} \beta_k e_{2k-1} \cdot e_{2k};\ \beta_k \in \mathbb{R} \Big\}.$$

We *denote* by $(x_1, \dots, x_{[n/2]})$ the basis of $\mathfrak{t}^*$ given by

$$x_k\Big(\sum_{k=1}^{[n/2]} \beta_k e_{2k-1} \cdot e_{2k} \Big) = \beta_k.$$

We consider the Spin_n-invariant scalar product on $\mathfrak{t}$ defined by

$$\langle X, Y\rangle = -(1/2)\,\mathrm{Tr}[\xi_*(X)\xi_*(Y)] = -\frac{1}{2(n-2)}\mathrm{B}(X, Y), \quad X, Y \in \mathfrak{t},$$

where B is the Killing form of $\mathfrak{spin}_n$. We have

$$\langle e_{2k-1}\cdot e_{2k}, e_{2l-1}\cdot e_{2l}\rangle = 4\delta_{kl}.$$

The canonical isomorphism $\mathfrak{t}\to\mathfrak{t}^*$ is given by $e_{2k-1}\cdot e_{2k}\mapsto 4x_k$. So we have

$$\langle x_k, x_l\rangle = (1/4)\delta_{kl}.$$

Thus, in order to have an orthonormal basis of $(i\,\mathfrak{t}^*, \langle\cdot,\cdot\rangle)$, we consider

$$\hat{x}_k := 2ix_k, \quad k = 1, 2, \ldots, [n/2].$$

A vector $\mu \in i\,\mathfrak{t}^*$ such that

$$\mu = \sum_{k=1}^{[n/2]} \mu_k \hat{x}_k,$$

in the basis $\{\hat{x}_k\}$, is *denoted* by

$$\mu = (\mu_1, \ldots, \mu_{[n/2]}).$$

So

$$\langle \mu, \mu'\rangle = \sum_{k=1}^{[n/2]} \mu_k\mu'_k, \quad \mu = (\mu_1, \ldots, \mu_{[n/2]}), \mu' = (\mu'_1, \ldots, \mu'_{[n/2]}) \in i\,\mathfrak{t}^*. \tag{12.16}$$

Irreducible representations of the group $\mathbf{Spin}_{2m}$, $m \geq 2$. Let (u_k, v_k), $k = 1, 2, \ldots, m$, be the Witt basis of $\mathbb{C}^{2m}$ defined by

$$u_k = \frac{1}{2}(e_{2k-1} - ie_{2k}) \quad \text{and} \quad v_k = \frac{1}{2}(e_{2k-1} + ie_{2k}).$$

One has

$$(u_i, u_j) = (v_i, v_j) = 0,$$

$$(u_i, v_j) = \frac{1}{2}\delta_{ij}, \quad 1 \leq i, j \leq m,$$

where $(\cdot,\cdot)$ is the $\mathbb{C}$-linear extension to $\mathbb{C}^{2m}$ of the standard scalar product on $\mathbb{R}^{2m}$.

After some computations one gets the decomposition of the complexified Lie algebra $\mathfrak{spin}_{2m,\mathbb{C}}$ into root spaces (see (12.1)):

$$\mathfrak{spin}_{2m,\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \Big(\bigoplus_{1\le i<j\le m} (\mathfrak{g}_{\theta_{ij}^+} \oplus \mathfrak{g}_{-\theta_{ij}^+} \oplus \mathfrak{g}_{\theta_{ij}^-} \oplus \mathfrak{g}_{-\theta_{ij}^-}) \Big),$$

where

$$\mathfrak{g}_{\theta_{ij}^+} = \mathbb{C}u_i \cdot u_j, \quad \mathfrak{g}_{-\theta_{ij}^+} = \mathbb{C}v_i \cdot v_j, \quad \mathfrak{g}_{\theta_{ij}^-} = \mathbb{C}u_i \cdot v_j, \quad \mathfrak{g}_{-\theta_{ij}^-} = \mathbb{C}v_i \cdot u_j,$$

the corresponding roots being

$$\pm\theta_{ij}^+ = \pm(\hat{x}_i + \hat{x}_j), \quad \pm\theta_{ij}^- = \pm(\hat{x}_i - \hat{x}_j), \quad 1 \le i < j \le m.$$

By Lemma 12.34, T is a maximal torus of Spin_{2m}. Therefore, we conclude that the m roots

$$\begin{aligned} \theta_i &= \hat{x}_i - \hat{x}_{i+1}, \quad 1 \le i \le m-1, \\ \theta_m &= \hat{x}_{m-1} + \hat{x}_m, \end{aligned}$$

form a basis Δ of the root system Φ corresponding to this maximal torus T. Indeed we have

$$\theta_{ij}^- = \theta_i + \theta_{i+1} + \cdots + \theta_{j-1}$$

and

$$\theta_{ij}^+ = \theta_i + \cdots + \theta_{j-1} + 2\theta_j + 2\theta_{j+1} + \cdots + 2\theta_{m-2} + \theta_{m-1} + \theta_m.$$

Note that

$$\begin{aligned} \theta_i &= (0, \ldots, 0, \underbrace{1}_{i}, \underbrace{-1}_{i+1}, 0, \ldots, 0), \quad 1 \le i \le m-1, \\ \theta_m &= (0, \ldots, 0, 1, 1). \end{aligned}$$

Thus, by (12.16), the reflection across $\theta_i^\perp$ is given by

$$\mu = (\mu_1, \ldots, \mu_m) \longmapsto (\mu_1, \ldots, \mu_{i+1}, \mu_i, \ldots, \mu_m),$$

and the reflection across $\theta_m^\perp$ is given by

$$\mu = (\mu_1, \ldots, \mu_m) \longmapsto (\mu_1, \ldots, -\mu_m, -\mu_{m-1}).$$

So from Theorem 12.49 we conclude that the Weyl group $\mathfrak{W}_{\mathrm{Spin}_{2m}}$ acts on $i\mathfrak{t}^*$ by

$$\mu = (\mu_1, \ldots, \mu_m) \longmapsto (\epsilon_1 \mu_{\sigma(1)}, \ldots, \epsilon_m \mu_{\sigma(m)}), \quad \sigma \in \mathfrak{S}_m,$$

where $\epsilon_i = \pm 1$, $\epsilon_1 \epsilon_2 \ldots \epsilon_m = 1$. Hence $\mathrm{Card}(\mathfrak{W}_{\mathrm{Spin}_{2m}}) = 2^{m-1}(m!)$.

Furthermore, we have

$$\langle \theta_i, \theta_{i+1} \rangle = -1, \quad 1 \le i \le m-2, \tag{12.17}$$

$$\langle \theta_i, \theta_i \rangle = 2, \quad 1 \le i \le m, \tag{12.18}$$

$$\langle \theta_i, \theta_j \rangle = 0, \quad 1 \le i \le m-3,\ j \ge i+2, \tag{12.19}$$

$$\langle \theta_{m-2}, \theta_m \rangle = -1, \tag{12.20}$$

$$\langle \theta_{m-1}, \theta_m \rangle = 0. \tag{12.21}$$

We then conclude from Theorem 12.61

Proposition 12.69. *A vector $\beta = (k_1, k_2, \ldots, k_m)$ of $i\mathfrak{t}^*$ is a dominant weight if and only if*

$$k_1 \ge k_2 \ge \cdots \ge k_{m-1} \ge |k_m|,$$

and the k_i are all simultaneously integers or half-integers.

By the above results, the (fundamental) weights of the fundamental representations ρ_i of Spin_{2m} are given by

$$\beta_i = \hat{x}_1 + \hat{x}_2 + \cdots + \hat{x}_i, \quad i = 1, 2, \ldots, m-2,$$

$$\beta_{m-1} = (1/2)(\hat{x}_1 + \hat{x}_2 + \cdots + \hat{x}_{m-1} - \hat{x}_m),$$

$$\beta_m = (1/2)(\hat{x}_1 + \hat{x}_2 + \cdots + \hat{x}_{m-1} + \hat{x}_m).$$

We already know that the spin representation of Spin_{2m} splits into two irreducible representations (see Definition 1.33), so we first have a look at this representation.

Consider the complex spinor representation

$$\sigma_{2m} \colon \mathrm{Spin}(2m) \longrightarrow \mathrm{GL}_{\mathbb{C}}(\Sigma_{2m}). \tag{12.22}$$

As it was mentioned at the end of Chapter 1, the space Σ_{2m} can be defined in the following way. Let

$$\mathrm{w} := v_1 \cdot v_2 \cdots v_m (\in \mathbb{C}\mathrm{l}_{2m}) \quad \text{and} \quad \Sigma_{2m} := \mathbb{C}\mathrm{l}_{2m} \cdot \mathrm{w}.$$

For any subset $I \subset \{1, 2, \ldots, m\}$, let

$$u_I \cdot \mathrm{w} := u_{i_1} \cdot u_{i_2} \cdots u_{i_k} \cdot \mathrm{w},$$

where $i_1 < i_2 < \cdots < i_k$ are the elements of I. The vectors $u_I \cdot \mathrm{w}$ form a basis of Σ_{2m}, so $\dim \Sigma_{2m} = 2^m$. Hence, the homomorphism $\sigma_{2m}\colon \mathbb{C}\mathrm{l}_{2m} \to \mathrm{Hom}_{\mathbb{C}}(\Sigma_{2m})$, defined for all $\varphi \in \mathbb{C}\mathrm{l}_{2m}$ by

$$\sigma_{2m}(\varphi)\colon \Sigma_{2m} \longrightarrow \Sigma_{2m}, \quad \psi \cdot \mathrm{w} \longmapsto \varphi \cdot \psi \cdot \mathrm{w},$$

is an irreducible representation of $\mathbb{C}\mathrm{l}_{2m}$. Hence $\sigma_{2m}|_{\mathrm{Spin}_{2m}}$ is the complex spin representation of Spin_{2m}. Observe that for any $j = 1, \dots, m$,

$$\sigma_{2m}(e_{2j-1} \cdot e_{2j}) \cdot (u_I \cdot \mathrm{w}) = -i u_I \cdot \mathrm{w}, \quad \text{if } j \notin I,$$

and

$$\sigma_{2m}(e_{2j-1} \cdot e_{2j}) \cdot (u_I \cdot \mathrm{w}) = i u_I \cdot \mathrm{w}, \qquad \text{if } j \in I.$$

Thus the restriction of σ_{2m} to the maximal torus T splits into the direct sum of 2^m irreducible representations in the spaces

$$\mathbb{C} u_I \cdot \mathrm{w},$$

with weights

$$\frac{1}{2}(\epsilon_1, \dots, \epsilon_m), \quad \epsilon_i = \pm 1.$$

Note that only two of those weights are dominant, namely

$$\frac{1}{2}(1, \dots, 1, -1) = \beta_{m-1}$$

and

$$\frac{1}{2}(1, \dots, 1, 1) = \beta_m,$$

and that they correspond respectively to the two maximal vectors

$$u_1 \cdot u_2 \cdots u_{m-1} \cdot \mathrm{w} \quad \text{and} \quad u_1 \cdot u_2 \cdots u_m \cdot \mathrm{w},$$

annihilated by the root vectors $u_i \cdot v_{i+1}$, $1 \le i \le m-1$, $u_{m-1} \cdot u_m$, corresponding to simple roots. So we recover the fact that the spin representation splits into two irreducible representations. The corresponding spaces are the cyclic modules

$$\mathfrak{U}(\mathfrak{spin}_{2m,\mathbb{C}}) \cdot (u_1 \cdot u_2 \cdots u_{m-1} \cdot \mathrm{w}) \quad \text{and} \quad \mathfrak{U}(\mathfrak{spin}_{2m,\mathbb{C}}) \cdot (u_1 \cdot u_2 \cdots u_m \cdot \mathrm{w}).$$

Consider the complex volume element $\omega := i^m e_1 \cdots e_{2m}$ in $\mathbb{C}\mathrm{l}_{2m}$. Since ω commutes with the action of Spin_{2m} and since

$$\omega \cdot (u_1 \cdot u_2 \cdots u_{m-1} \cdot \mathrm{w}) = (-1)^{m+1}(u_1 \cdot u_2 \cdots u_{m-1} \cdot \mathrm{w})$$

and

$$\omega \cdot (u_1 \cdot u_2 \cdots u_m \cdot \mathrm{w}) = (-1)^m (u_1 \cdot u_2 \cdots u_m \cdot \mathrm{w}),$$

we recover the decomposition of Σ_{2m} into negative and positive spinor spaces: if m is even, the fundamental representation ρ_{m-1} is σ_{2m}^- and the fundamental representation ρ_m is σ_{2m}^+; the contrary if m is odd.

On the other hand, Spin_{2m} has a natural complex representation ρ on the space $\mathbb{C}^{2m}$, obtained by considering the composition

$$\xi \colon \mathrm{Spin}_{2m} \xrightarrow{\xi} \mathrm{SO}_{2m} \longrightarrow \mathrm{SO}_{2m,\mathbb{C}} \, .$$

Its restriction to the maximal torus T splits into the direct sum of $2m$ irreducible representations in the spaces $\mathbb{C}u_i$, $\mathbb{C}v_i$, $i = 1, \ldots, m$, with respective weights $\hat{x}_i$ and $-\hat{x}_i$.

Since only one of these weights is dominant, namely $\hat{x}_1$, ρ is irreducible with dominant weight $\hat{x}_1 = \beta_1$, thus the fundamental representation ρ_1 is ρ. (Note also that the only maximal vector is u_1).

Inspired by the description of the fundamental representations of SU_n, we now consider the representations

$$\Lambda^k \rho \colon \mathrm{Spin}_{2m} \longrightarrow \mathrm{GL}_{\mathbb{C}}(\Lambda^k(\mathbb{C}^{2m})), \quad 2 \le k \le m-2.$$

Note that, with the help of Theorem 12.67, we already know the dimension $n_k = \binom{2m}{k}$ of ρ_k, an indication to consider $\Lambda^k \rho$.

Indeed those representations are irreducible. The "classical" proof of this result is given in Theorem 1.26. It is easy to see that the vector $u_1 \wedge \cdots \wedge u_k$ is a maximal vector with weight β_k, hence $\Lambda^k \rho$ is the fundamental representation ρ_k.

We are going to retrieve the result by proving that the representation $\Lambda^k \rho$ has only one maximal vector.

We introduce the following basis of $\Lambda^k(\mathbb{C}^{2m})$: for any triple (A, B, C) of disjoint subsets of $\{1, \ldots, m\}$ such that

$$\mathrm{Card}\, A + \mathrm{Card}\, B + 2\, \mathrm{Card}\, C = k,$$

we set

$$w_{A,B,C} = u_{a_1} \wedge \cdots \wedge u_{a_p} \wedge v_{b_1} \wedge \cdots \wedge v_{b_q} \wedge u_{c_1} \wedge v_{c_1} \wedge \cdots \wedge u_{c_r} \wedge v_{c_r}, \tag{12.23}$$

where $a_1 < \cdots < a_p$ (resp. $b_1 < \cdots < b_q$, $c_1 < \cdots < c_r$) are the elements of A (resp. B, C).

Under the action of the maximal torus T, $\Lambda^k \rho$ splits into the direct sum of $\binom{2m}{k}$ irreducible representations in the spaces $\mathbb{C}w_{A,B,C}$, with corresponding weights

$$\hat{x}_{a_1} + \cdots + \hat{x}_{a_p} - \hat{x}_{b_1} - \cdots - \hat{x}_{b_q} \, .$$

By Proposition 12.69, the dominant weights are

$$\hat{x}_1 + \cdots + \hat{x}_{k-2l}, \quad l = 0, \ldots, [k/2] \quad (0 \text{ if } k \text{ is even and } l = k/2),$$

with corresponding weight vectors

$$w_{A,\emptyset,C}, \quad A = \{1, \ldots, k-2l\} \quad (A = \emptyset \text{ if } k \text{ is even and } l = k/2),$$

and C being a subset of $\{k-2l+1, \ldots, m\}$ with l elements. But there is only one maximal vector, namely

$$w_{A,\emptyset,\emptyset} = u_1 \wedge \cdots \wedge u_k.$$

Indeed, for any $l = 1, \ldots, [k/2]$, any vector in the weight space corresponding to $\hat{x}_1 + \cdots + \hat{x}_{k-2l}$ (0 if k is even and $l = k/2$) is of the form

$$v = \sum_C \lambda_C w_{A,\emptyset,C}, \quad \lambda_C \in \mathbb{C},$$

where $A = \{1, \ldots, k-2l\}$ ($A = \emptyset$ if k is even and $l = k/2$), the sum being taken over the distinct subsets C of $\{k-2l+1, \ldots, m\}$ with l elements.

Now let $c, d \in \{k-2l+1, \ldots, m\}$ such that $c < d$ (note that the condition $k \leq m-2$ implies the existence of such a couple (c, d)). Recall that $u_c \cdot u_d$ is a root vector for the positive root $\hat{x}_c + \hat{x}_d$. It is easily checked that for any given subset C of $\{k-2l+1, \ldots, m\}$ with l elements, one has (perhaps up to a sign which has no importance for the proof)

$$(\Lambda^k \rho)_*^{\mathbb{C}}(u_c \cdot u_d) \cdot w_{A,\emptyset,C} = 0,$$

if $c \notin C$ and $d \notin C$, or $c \in C$ and $d \in C$,

$$(\Lambda^k \rho)_*^{\mathbb{C}}(u_c \cdot u_d) \cdot w_{A,\emptyset,C} = w_{A\cup\{c,d\},\emptyset,C\setminus\{c\}},$$

if $c \in C$ and $d \notin C$, and

$$(\Lambda^k \rho)_*^{\mathbb{C}}(u_c \cdot u_d) \cdot w_{A,\emptyset,C} = w_{A\cup\{c,d\},\emptyset,C\setminus\{d\}},$$

if $c \notin C$ and $d \in C$. Therefore,

$$(\Lambda^k \rho)_*^{\mathbb{C}}(u_c \cdot u_d) \cdot v = \sum_{C'} (\lambda_{C'\cup\{c\}} + \lambda_{C'\cup\{d\}}) w_{A\cup\{c,d\},\emptyset,C'}, \tag{12.24}$$

where the sum is taken over the distinct subsets C' of $\{k-2l+1, \ldots, m\}\setminus\{c, d\}$ with $l-1$ elements.

Consider now the root vector $u_c \cdot v_d$ corresponding to the positive root $\hat{x}_c - \hat{x}_d$. Then for any given subset C of $\{k-2l+1, \ldots, m\}$ with l elements (perhaps up to a sign which has no importance for the proof) one has

$$(\Lambda^k \rho)_*^{\mathbb{C}}(u_c \cdot v_d) \cdot w_{A,\emptyset,C} = 0,$$

if $c \notin C$ and $d \notin C$, or $c \in C$ and $d \in C$,

$$(\Lambda^k \rho)_*^{\mathbb{C}}(u_c \cdot v_d) \cdot w_{A,\emptyset,C} = w_{A \cup \{c\},\{d\},C \setminus \{c\}},$$

if $c \in C$ and $d \notin C$,

$$(\Lambda^k \rho)_*^{\mathbb{C}}(u_c \cdot v_d) \cdot w_{A,\emptyset,C} = -w_{A \cup \{c\},\{d\},C \setminus \{d\}},$$

if $c \notin C$ and $d \in C$. Consequently,

$$(\Lambda^k \rho)_*^{\mathbb{C}}(u_c \cdot u_d) \cdot v = \sum_{C'} (\lambda_{C' \cup \{c\}} - \lambda_{C' \cup \{d\}}) w_{A \cup \{c\},\{d\},C'}, \tag{12.25}$$

where the sum is taken over the distinct subsets C' of $\{k-2l+1, \ldots, m\} \setminus \{c, d\}$ with $l-1$ elements.

From (12.24) and (12.25), one sees that if v is annihilated by the root vectors corresponding to the positive roots, then necessarily $\lambda_{C' \cup \{c\}} = \lambda_{C' \cup \{d\}} = 0$, for any subset C' of $\{k-2l+1, \ldots, m\} \setminus \{c, d\}$ with $l-1$ elements. As c and d are arbitrary, this implies that $\lambda_C = 0$ for any subset C of $\{k-2l+1, \ldots, m\}$ with l elements. Hence $v = 0$, so there is no maximal vector corresponding to

$$\hat{x}_1 + \cdots + \hat{x}_{k-2l}, \quad l = 1, \ldots, [k/2].$$

Then, since $\Lambda^k \rho$ has only one maximal vector, namely

$$w_{A,\emptyset,\emptyset} = u_1 \wedge \cdots \wedge u_k,$$

it is irreducible. This maximal vector corresponds to the weight

$$\hat{x}_1 + \cdots + \hat{x}_k = \beta_k,$$

and so $\Lambda^k \rho$ is the fundamental representation ρ_k.

Irreducible representations of the group $\mathbf{SO}_{2m}$. Considering the two-fold covering $\xi\colon \mathrm{Spin}_{2m} \to \mathrm{SO}_{2m}$, one deduces from Proposition 12.18, that irreducible representations of SO_{2m} correspond to irreducible representations of Spin_{2m} which factor through the quotient.

Let ρ be an irreducible representation of Spin_{2m} with dominant weight $\beta = (k_1, k_2, \dots, k_m)$. From Remark 12.64 and the description of fundamental representations, it follows that the vector space corresponding to ρ is the cyclic module $\mathfrak{U}(\mathfrak{spin}_{2m,\mathbb{C}}) \cdot v$, where

$$v = u_1{}^{k_1-k_2} \otimes \cdots \otimes (u_1 \wedge \cdots \wedge u_{m-2})^{k_{m-2}-k_{m-1}} \\ \otimes (u_1 \cdot u_2 \cdots u_{m-1} \cdot \mathrm{w})^{k_{m-1}-k_m} \otimes (u_1 \cdot u_2 \cdots u_m \cdot \mathrm{w})^{k_{m-1}+k_m}.$$

A necessary and sufficient condition for ρ to factor through the quotient is that $\rho(-1) \cdot v = v$. But

$$\rho(-1) \cdot v = (-1)^{2k_{m-1}} v.$$

Hence ρ factors through the quotient if and only if k_{m-1} is an integer. Therefore, the irreducible representations of SO_{2m} are in one-to-one correspondence with the dominant weights $(k_1, k_2, \dots, k_m)$ of Spin_{2m} such that all the k_i are integers.

Irreducible representations of the groups $\mathbf{Spin}_{2m+1}$ and $\mathbf{SO}_{2m+1}$, $\boldsymbol{m \geq 1}$. The decomposition of the complexified Lie algebra $\mathfrak{spin}_{2m+1,\mathbb{C}}$ into root spaces, see equation (12.1), is given by

$$\mathfrak{spin}_{2m+1,\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \Big(\bigoplus_{1 \leq i < j \leq m} (\mathfrak{g}_{\theta_{ij}^+} \oplus \mathfrak{g}_{-\theta_{ij}^+} \oplus \mathfrak{g}_{\theta_{ij}^-} \oplus \mathfrak{g}_{-\theta_{ij}^-}) \Big) \\ \oplus \Big(\bigoplus_{i=1}^{m} (\mathfrak{g}_{\hat{x}_i} \oplus \mathfrak{g}_{-\hat{x}_i}) \Big),$$

where

$$\mathfrak{g}_{\theta_{ij}^+} = \mathbb{C} u_i \cdot u_j, \quad \mathfrak{g}_{-\theta_{ij}^+} = \mathbb{C} v_i \cdot v_j, \quad \mathfrak{g}_{\theta_{ij}^-} = \mathbb{C} u_i \cdot v_j, \quad \mathfrak{g}_{-\theta_{ij}^-} = \mathbb{C} v_i \cdot u_j,$$

$$\mathfrak{g}_{\hat{x}_i} = \mathbb{C} u_i \cdot e_{2m+1}, \quad \mathfrak{g}_{-\hat{x}_i} = \mathbb{C} v_i \cdot e_{2m+1},$$

the corresponding roots being

$$\pm\theta_{ij}^+ = \pm(\hat{x}_i + \hat{x}_j), \pm\theta_{ij}^- = \pm(\hat{x}_i - \hat{x}_j), \quad 1 \leq i < j \leq m, \pm\hat{x}_i, 1 \leq i \leq m.$$

Hence T is a maximal torus of Spin_{2m+1}. Furthermore, the m roots

$$\theta_i = \hat{x}_i - \hat{x}_{i+1}, \quad 1 \leq i \leq m-1, \quad \theta_m = \hat{x}_m,$$

form a basis Δ of the root system Φ corresponding to this maximal torus T, since we have

$$\theta_{ij}^- = \theta_i + \theta_{i+1} + \cdots + \theta_{j-1},$$
$$\theta_{ij}^+ = \theta_i + \cdots + \theta_{j-1} + 2\theta_j + 2\theta_{j+1} + \cdots + 2\theta_m,$$
$$\hat{x}_i = \theta_i + \theta_{i+1} + \cdots + \theta_m.$$

Note that

$$\theta_i = (0, \ldots, 0, \underbrace{1}_{i}, \underbrace{-1}_{i+1}, 0, \ldots, 0), \quad 1 \leq i \leq m-1,$$
$$\theta_m = (0, \ldots, 0, 0, 1).$$

Thus by (12.16), the reflection across $\theta_i^\perp$ is given by

$$\mu = (\mu_1, \ldots, \mu_m) \longmapsto (\mu_1, \ldots, \mu_{i+1}, \mu_i, \ldots, \mu_m),$$

and the reflection across $\theta_m^\perp$ is given by

$$\mu = (\mu_1, \ldots, \mu_m) \longmapsto (\mu_1, \ldots, \mu_{m-1}, -\mu_m).$$

Using Theorem 12.49, we conclude that the Weyl group $\mathfrak{W}_{\mathrm{Spin}_{2m+1}}$ acts on $i\mathfrak{t}^*$ by

$$\mu = (\mu_1, \ldots, \mu_m) \longmapsto (\epsilon_1\mu_{\sigma(1)}, \ldots, \epsilon_m\mu_{\sigma(m)}), \quad \sigma \in \mathfrak{S}_m, \epsilon_i = \pm 1.$$

Hence, $\mathrm{Card}(\mathfrak{W}_{\mathrm{Spin}_{2m+1}}) = 2^m(m!)$. Furthermore we have

$$\langle\theta_i, \theta_{i+1}\rangle = -1, \quad 1 \leq i \leq m-1,$$
$$\langle\theta_i, \theta_i\rangle = 2, \quad 1 \leq i \leq m-1,$$
$$\langle\theta_i, \theta_j\rangle = 0, \quad 1 \leq i \leq m-2, j \geq i+2,$$
$$\langle\theta_m, \theta_m\rangle = 1.$$

We then conclude from Theorem 12.61

Proposition 12.70. *A vector $\beta = (k_1, k_2, \ldots, k_m)$ of $i\mathfrak{t}^*$ is a dominant weight if and only if*

$$k_1 \geq k_2 \geq \cdots \geq k_{m-1} \geq k_m \geq 0,$$

and the k_i are all simultaneously integers or half-integers.

The fundamental weights are

$$\beta_i = \hat{x}_1 + \hat{x}_2 + \cdots + \hat{x}_i, \quad i = 1, 2, \ldots, m-1,$$

$$\beta_m = \frac{1}{2}(\hat{x}_1 + \hat{x}_2 + \cdots + \hat{x}_{m-1} + \hat{x}_m).$$

Recall that the complex spin representation σ_{2m+1} can be defined as follows. There exists an isomorphism between $\mathbb{C}\mathrm{l}_{2m+1}$ and $\mathbb{C}\mathrm{l}^0_{2m+2}$, given by

$$\begin{aligned} e_{i_1} \cdots e_{i_k} &\longmapsto e_{i_1} \cdots e_{i_k}, && \text{if } k \text{ is even,} \\ e_{i_1} \cdots e_{i_k} &\longmapsto e_{i_1} \cdots e_{i_k} \cdot e_{2m+2}, && \text{if } k \text{ is odd.} \end{aligned}$$

Under this isomorphism, one obtains, using the representations σ^+_{2m+2} and σ^-_{2m+2} of $\mathbb{C}\mathrm{l}_{2m+2}$, two inequivalent irreducible representations of $\mathbb{C}\mathrm{l}_{2m+1}$.

The spin representation σ_{2m+1} is obtained by restricting to Spin_{2m+1} any of those two representations (see Definition 1.33). With the notations used on page 351, one chooses for instance Σ_{2m+1} to be the space generated by the vectors

$$u_I \cdot \mathrm{w}_{m+1} \in \Sigma_{2m+2}, \quad I \subset \{1, 2, \ldots, m+1\}, \ \mathrm{card} I \text{ even},$$

with

$$\mathrm{w}_{m+1} := v_1 \cdot v_2 \cdots v_{m+1}.$$

Since for any $j = 1, \ldots, m$, if $j \notin I$,

$$\begin{aligned} \sigma_{2m+1}(e_{2j-1} \cdot e_{2j}) \cdot (u_I \cdot \mathrm{w}_{m+1}) &= \sigma_{2m+2}(e_{2j-1} \cdot e_{2j}) \cdot (u_I \cdot \mathrm{w}_{m+1}) \\ &= -i u_I \cdot \mathrm{w}_{m+1} \end{aligned}$$

and, if $j \in I$,

$$\begin{aligned} \sigma_{2m+1}(e_{2j-1} \cdot e_{2j}) \cdot (u_I \cdot \mathrm{w}_{m+1}) &= \sigma_{2m+2}(e_{2j-1} \cdot e_{2j}) \cdot (u_I \cdot \mathrm{w}_{m+1}) \\ &= i u_I \cdot \mathrm{w}_{m+1}, \end{aligned}$$

the restriction of σ_{2m+1} to the maximal torus T splits into the direct sum of 2^m irreducible representations in the spaces

$$\mathbb{C} u_I \cdot \mathrm{w}_{m+1}, \quad \mathrm{card} I \text{ even, with weights } \frac{1}{2}(\epsilon_1, \ldots, \epsilon_m), \epsilon_i = \pm 1.$$

Since only one of these weights is dominant, namely

$$\frac{1}{2}(1, \ldots, 1) = \beta_m,$$

we recover that σ_{2m+1} is irreducible. Hence, σ_{2m+1} is the fundamental representation corresponding to the weight β_m.

With the same arguments used in the study of the fundamental representation of Spin_{2m}, it can be proved that the fundamental representation corresponding to β_k, $k = 1, \ldots, m-1$, is

$$\Lambda^k \rho \colon \mathrm{Spin}_{2m+1} \longrightarrow \mathrm{GL}_{\mathbb{C}}(\Lambda^k(\mathbb{C}^{2m+1})),$$

where ρ is the natural representation of Spin_{2m+1}:

$$\mathrm{Spin}_{2m+1} \longrightarrow \mathrm{SO}_{2m+1} \longrightarrow \mathrm{SO}_{2m+1,\mathbb{C}} \hookrightarrow \mathrm{GL}_{\mathbb{C}}(\mathbb{C}^{2m+1}).$$

(With the same notations as on page 353, consider the basis of $\Lambda^k\mathbb{C}^{2m+1}$ obtained by taking the vectors $w_{A,B,C}$ of the basis of $\Lambda^k(\mathbb{C}^{2m})$, together with the vectors $w_{A,B,C} \wedge e_{2m+1}$, such that $\operatorname{card} A + \operatorname{card} B + 2 \operatorname{card} C = k-1$).

Finally, one can show that the irreducible representations of SO_{2m+1} are in one–to–one correspondence with the dominant weights $(k_1, k_2, \ldots, k_m)$ of Spin_{2m+1} such that all the k_i are integers, using the same arguments as in the description of the irreducible representations of the group SO_{2m}.

An application to the decomposition of $\mathbb{C}^n \otimes \Sigma_n$ into irreducible modules. This decomposition (which may be obtained with a more elementary method) leads to the notion of the twistor operator (see Section 2.3.3). We consider the case n even, $n = 2m$ (the case n odd can be studied using analogous arguments), and use notations of page 351.

Under the action of the maximal torus T, the vector space $\mathbb{C}^{2m} \otimes \Sigma_{2m}$ splits into the direct sum of $m2^{m+1}$ irreducible representations

$$\mathbb{C}u_i \otimes u_I \cdot \mathrm{w} \quad \text{and} \quad \mathbb{C}v_i \otimes u_I \cdot \mathrm{w}, \quad i = 1, \ldots, m,$$

with corresponding weights

$$\frac{1}{2}(\epsilon_1, \ldots, \epsilon_m),$$

where $\epsilon_i = \pm 1$, except one of them which can be equal to ± 3. Thus, the dominant weights occurring in this decomposition are

- $\frac{1}{2}(3, 1, \ldots, 1)$, corresponding to the weight-vector

$$u_1 \otimes u_1 \cdot u_2 \cdots u_m \cdot \mathrm{w},$$

- $\frac{1}{2}(3, 1, \ldots, 1, -1)$, corresponding to the weight-vector

$$u_1 \otimes u_1 \cdot u_2 \cdots u_{m-1} \cdot \mathrm{w},$$

- $\frac{1}{2}(1, \ldots, 1)$, corresponding to the weight vectors

$$u_i \otimes u_1 \cdot u_2 \cdots \widehat{u_i} \cdots u_m \cdot \mathrm{w}, \quad i = 1, \ldots, m,$$

the notation $\widehat{u_i}$ meaning that the term u_i is omitted, and

- $\frac{1}{2}(1, \ldots, 1, -1)$, corresponding to the weight vectors

$$u_i \otimes u_1 \cdot u_2 \cdots \widehat{u_i} \cdots u_{m-1} \cdot \mathrm{w}, \quad i = 1, \ldots, m-1,$$

  and

$$v_m \otimes u_1 \cdot u_2 \cdots u_m \cdot \mathrm{w}.$$

It is easy to check that the two weight vectors

$$u_1 \otimes u_1 \cdot u_2 \cdots u_m \cdot \mathrm{w} \quad \text{and} \quad u_1 \otimes u_1 \cdot u_2 \cdots u_{m-1} \cdot \mathrm{w},$$

are maximal vectors. Hence the two modules

$$V_{(\frac{3}{2},\frac{1}{2},\ldots,\frac{1}{2})} := \mathfrak{U}(\mathfrak{spin}_{2m,\mathbb{C}}) \cdot (u_1 \otimes u_1 \cdot u_2 \cdots u_m \cdot \mathrm{w})$$

and

$$V_{(\frac{3}{2},\frac{1}{2},\ldots,\frac{1}{2},-\frac{1}{2})} := \mathfrak{U}(\mathfrak{spin}_{2m,\mathbb{C}}) \cdot (u_1 \otimes u_1 \cdot u_2 \cdots u_{m-1} \cdot \mathrm{w}),$$

appear with multiplicity one in the decomposition.

Let v be a weight vector corresponding to $\frac{1}{2}(1, \ldots, 1)$:

$$v = \sum_{k=1}^{m} \lambda_k u_k \otimes u_1 \cdot u_2 \cdots \widehat{u_k} \cdots u_m \cdot \mathrm{w}, \quad \lambda_k \in \mathbb{C}.$$

For any $j = 2, \ldots, m$, one has

$$\rho_*^{\mathbb{C}}(u_1 \cdot v_j) \cdot v = ((-1)^{j-1}\lambda_1 - \lambda_j) u_1 \otimes u_1 \cdot u_2 \cdots \widehat{u_j} \cdots u_m \cdot \mathrm{w}.$$

Thus, in order for v to be a maximal vector, one must have, for any $j = 2, \ldots, m$, $\lambda_j = (-1)^{j-1}\lambda_1$. So, in order for v to be a maximal vector, it has to be a scalar multiple of the vector

$$\sum_{k=1}^{m} (-1)^{k+1} u_k \otimes u_1 \cdot u_2 \cdots \widehat{u_k} \cdots u_m \cdot \mathrm{w}.$$

Now, it is easy to verify that the above vector is a maximal vector. So, up to a scalar, there exists one and only one maximal vector corresponding to $\frac{1}{2}(1, \ldots, 1)$. Hence there is a module $V_{(\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2})}$, isomorphic to Σ_{2m}^{+}, in the decomposition and it has multiplicity 1.

Similarly, it can be proved that, up to a scalar, there exists one and only one maximal vector corresponding to $\frac{1}{2}(1, \ldots, 1, -1)$, namely

$$(-1)^m v_m \otimes u_1 \cdot u_2 \cdots u_m \cdot \mathrm{w} + \sum_{k=1}^{m-1} (-1)^{k+1} u_k \otimes u_1 \cdot u_2 \cdots \widehat{u_k} \cdots u_{m-1} \cdot \mathrm{w}.$$

Hence, there is a module $V_{(\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2},-\frac{1}{2})}$, isomorphic to Σ_{2m}^{-}, in the decomposition and it has multiplicity 1. Thus, since there are four and only four linearly independent maximal vectors, corresponding to the four weights above, we get the decomposition of $\mathbb{C}^{2m} \otimes \Sigma_{2m}$ into irreducible modules:

$$\mathbb{C}^{2m} \otimes \Sigma_{2m} = V_{(\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2})} \oplus V_{(\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2},-\frac{1}{2})} \oplus V_{(\frac{3}{2},\frac{1}{2},\ldots,\frac{1}{2})} \oplus V_{(\frac{3}{2},\frac{1}{2},\ldots,\frac{1}{2},-\frac{1}{2})}.$$

Let γ be the Clifford multiplication. Since γ is a Spin_{2m}-equivariant operator, it follows from the Schur lemma that γ restricted to any irreducible component of $\mathbb{C}^{2m} \otimes \Sigma_{2m}$ is either 0 or an isomorphism. Now, it is easy to verify that

$$\gamma(u_1 \otimes u_1 \cdot u_2 \cdots u_m \cdot \mathrm{w}) = \gamma(u_1 \otimes u_1 \cdot u_2 \cdots u_{m-1} \cdot \mathrm{w}) = 0,$$

$$\gamma\Big(\sum_{k=1}^{m} (-1)^{k+1} u_k \otimes u_1 \cdot u_2 \cdots \widehat{u_k} \cdots u_m \cdot \mathrm{w}\Big) = m u_1 \cdot u_2 \cdots u_m \cdot \mathrm{w},$$

and

$$\gamma\Big((-1)^m v_m \otimes u_1 \cdot u_2 \cdots u_m \cdot \mathrm{w} + \sum_{k=1}^{m-1} (-1)^{k+1} u_k \otimes u_1 \cdot u_2 \cdots \widehat{u_k} \cdots u_{m-1} \cdot \mathrm{w}\Big)$$
$$= m u_1 \cdot u_2 \cdots u_{m-1} \cdot \mathrm{w}.$$

From this, one deduces that γ restricted to $V_{(\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2})}$ is an isomorphism onto Σ_{2m}^{+}, γ restricted to $V_{(\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2},-\frac{1}{2})}$ is an isomorphism onto Σ_{2m}^{-}, and

$$\operatorname{Ker}\gamma = V_{(\frac{3}{2},\frac{1}{2},\ldots,\frac{1}{2})} \oplus V_{(\frac{3}{2},\frac{1}{2},\ldots,\frac{1}{2},-\frac{1}{2})}.$$

12.4.3 Irreducible representations of the group Sp_n

We consider $\mathbb{H}^n$ as a right vector space over $\mathbb{H}$. The canonical $\mathbb{R}$-basis of $\mathbb{H}$ is denoted by $(\mathbf{1}, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ and $\mathbb{H}$ is identified with $\mathbb{C} \oplus \boldsymbol{j}\mathbb{C}$ via the map

$$a\mathbf{1} + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k} \longmapsto (a + ib) \oplus \boldsymbol{j}(c - id).$$

We consider the matrix representation of the group

$$\mathrm{Sp}_n = \{A \in \mathrm{M}_n(\mathbb{H}); {}^t\bar{A}A = I\},$$

and introduce the following basis of the Lie algebra $\mathfrak{sp}_n$:

$$\begin{aligned}
M_{rs} &= E_{rs} - E_{sr}, && 1 \le r < s \le n,\\
N_{rs} &= (E_{rs} + E_{sr})\boldsymbol{i}, && 1 \le r \le s \le n,\\
P_{rs} &= (E_{rs} + E_{sr})\boldsymbol{j}, && 1 \le r \le s \le n,\\
Q_{rs} &= (E_{rs} + E_{sr})\boldsymbol{k}, && 1 \le r \le s \le n,
\end{aligned}$$

where $\{E_{rs}\}_{r,s=1,\ldots,n}$ is the canonical basis of $\mathrm{M}_n(\mathbb{H})$. Let T be the torus of Sp_n:

$$T = \{\text{diag. matrix } (\mathrm{e}^{\boldsymbol{i}\beta_1}, \ldots, \mathrm{e}^{\boldsymbol{i}\beta_n}); \beta_1, \beta_2, \ldots, \beta_n \in \mathbb{R}\}.$$

Its Lie algebra is

$$\mathfrak{t} = \{\text{diag. matrix}(\boldsymbol{i}\beta_1, \ldots, \boldsymbol{i}\beta_n); \beta_1, \beta_2, \ldots, \beta_n \in \mathbb{R}\}.$$

We denote by $\{x_1, \ldots, x_n\}$ the basis of $\mathfrak{t}^*$ given by

$$x_k[\text{diag. matrix}(\boldsymbol{i}\beta_1, \ldots, \boldsymbol{i}\beta_n)] = \beta_k.$$

A vector $\mu \in i\mathfrak{t}^*$ such that $\mu = \sum_{k=1}^n \mu_k \hat{x}_k$, in the basis $\{\hat{x}_k \equiv i x_k\}_{k=1,\ldots,n}$, is denoted by

$$\mu = (\mu_1, \ldots, \mu_n). \tag{12.26}$$

We consider the Sp_n-invariant scalar product on $\mathfrak{t}$ given by

$$\langle X, Y\rangle = -\mathrm{Re}[\mathrm{Tr}(XY)] = -\frac{1}{4(n+1)}\mathrm{B}(X, Y), \quad X, Y \in \mathfrak{t},$$

where B is the Killing form of $\mathfrak{sp}_n$. Its extension to $i\mathfrak{t}^*$ is given by

$$\langle \mu, \mu'\rangle = \sum_{k=1}^n \mu_k \mu'_k, \quad \mu = (\mu_1, \ldots, \mu_n), \mu' = (\mu'_1, \ldots, \mu'_n) \in i\mathfrak{t}^*. \tag{12.27}$$

Some computations lead to the decomposition of the complexified Lie algebra $\mathfrak{sp}_{n,\mathbb{C}}$ into root spaces; see (12.1):

$$\mathfrak{sp}_{n,\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \Big(\bigoplus_{1\le r\le s\le m} (\mathfrak{g}_{\theta^+_{rs}} \oplus \mathfrak{g}_{-\theta^+_{rs}})\Big) \oplus \Big(\bigoplus_{1\le r< s\le m} (\mathfrak{g}_{\theta^-_{rs}} \oplus \mathfrak{g}_{-\theta^-_{rs}})\Big),$$

where

$$\mathfrak{g}_{\theta^+_{rs}} = \mathbb{C}(P_{rs} - iQ_{rs}), \quad \mathfrak{g}_{-\theta^+_{rs}} = \mathbb{C}(P_{rs} + iQ_{rs}), \quad 1 \le r \le s \le n,$$
$$\mathfrak{g}_{\theta^-_{rs}} = \mathbb{C}(M_{rs} - iN_{rs}), \quad \mathfrak{g}_{-\theta^-_{rs}} = \mathbb{C}(M_{rs} + iN_{rs}), \quad 1 \le r < s \le n,$$

the corresponding roots being

$$\pm\theta^+_{rs} = \pm(\hat{x}_r + \hat{x}_s), \quad 1 \le r \le s \le n,$$
$$\pm\theta^-_{rs} = \pm(\hat{x}_r - \hat{x}_s), \quad 1 \le r < s \le n.$$

By Lemma 12.34, T is a maximal torus of Sp_n. Therefore, we get that the n roots

$$\begin{aligned}\theta_i &= \hat{x}_i - \hat{x}_{i+1}, \quad 1 \leq i \leq n-1,\\ \theta_n &= 2\hat{x}_n,\end{aligned}$$

form a basis Δ of the root system Φ corresponding to this maximal torus T. Indeed, we have

$$\begin{aligned}\theta_{ij}^- &= \theta_i + \theta_{i+1} + \cdots + \theta_{j-1},\\ \theta_{ij}^+ &= \theta_i + \cdots + \theta_{j-1} + 2\theta_j + 2\theta_{j+1} + \cdots + 2\theta_{n-1} + \theta_n, \quad 1 \leq i < j \leq n,\\ \theta_{ii}^+ &= 2\theta_i + \cdots + 2\theta_{n-1} + \theta_n, \quad 1 \leq i \leq n-1.\end{aligned}$$

Note that

$$\theta_i = (0, \ldots, 0, \underbrace{1}_{i}, \underbrace{-1}_{i+1}, 0, \ldots, 0), \quad 1 \leq i \leq n-1,$$

and

$$\theta_n = (0, \ldots, 0, 0, 2).$$

Thus, by (12.27), the reflection across $\theta_i^\perp$ is given by

$$\mu = (\mu_1, \ldots, \mu_n) \longmapsto (\mu_1, \ldots, \mu_{i+1}, \mu_i, \ldots, \mu_n),$$

and the reflection across $\theta_n^\perp$ is given by

$$\mu = (\mu_1, \ldots, \mu_n) \longmapsto (\mu_1, \ldots, \mu_{n-1}, -\mu_n).$$

So from Theorem 12.49, we conclude that the Weyl group $\mathfrak{W}_{\mathrm{Sp}_n}$ acts on $i\,\mathfrak{t}^*$ by

$$\mu = (\mu_1, \ldots, \mu_n) \longmapsto (\epsilon_1\mu_{\sigma(1)}, \ldots, \epsilon_n\mu_{\sigma(n)}), \quad \sigma \in \mathfrak{S}_n, \epsilon_i = \pm 1.$$

Hence $\mathrm{Card}(\mathfrak{W}_{\mathrm{Sp}_n}) = 2^n(n!)$. Furthermore we have

$$\begin{aligned}\langle \theta_i, \theta_{i+1} \rangle &= -1, \quad 1 \leq i \leq n-2,\\ \langle \theta_{n-1}, \theta_n \rangle &= -2,\\ \langle \theta_i, \theta_i \rangle &= 2, \qquad 1 \leq i \leq n-1,\\ \langle \theta_n, \theta_n \rangle &= 4,\\ \langle \theta_i, \theta_j \rangle &= 0, \qquad 1 \leq i \leq n-1, |i-j| \neq 0, 1.\end{aligned}$$

We then conclude by Theorem 12.61.

Proposition 12.71. *A vector $\beta = (k_1, \dots, k_n)$ of $i\mathfrak{t}^*$ is a dominant weight if and only if*

$$k_1 \geq \cdots \geq k_n \geq 0,$$

and all the k_i are integers.

From the above results, the (fundamental) weights of the fundamental representations ρ_i of Sp_n are given by

$$\beta_i = \hat{x}_1 + \cdots + \hat{x}_i, \quad i = 1, 2, \dots, n.$$

In order to describe the fundamental representations, we identify $\mathbb{H}^n$ with $\mathbb{C}^{2n}$, by the isomorphism $(\mathbb{C} \oplus \boldsymbol{j}\,\mathbb{C})^n \to \mathbb{C}^{2n}$, given by

$$(\xi_1 + \boldsymbol{j}\,\zeta_1, \dots, \xi_n + \boldsymbol{j}\,\zeta_n) \longmapsto (\xi_1, \dots, \xi_n, \zeta_1, \dots, \zeta_n).$$

Under this identification, we get an injective $\mathbb{R}$-homomorphism

$$\Phi\colon \mathrm{End}_{\mathbb{H}}(\mathbb{H}^n) \longrightarrow \mathrm{End}_{\mathbb{C}}(\mathbb{C}^{2n}),$$

which in terms of matrices is given by

$$A \longmapsto \Phi(A) = \begin{pmatrix} U & -\bar{V} \\ V & \bar{U} \end{pmatrix},$$

where $A = U + jV \in \mathrm{M}_n(\mathbb{H})$, with $U, V \in \mathrm{M}_n(\mathbb{C})$. The image of the group Sp_n under Φ is the group $\mathrm{Sp}_{2n,\mathbb{C}} \cap \mathrm{U}_{2n}$, with

$$\mathrm{Sp}_{2n,\mathbb{C}} := \{A \in \mathrm{M}_{2n}(\mathbb{C});\ {}^t\!AJA = J\}, \tag{12.28}$$

where

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

Thus Sp_n has a natural complex (faithful) representation $\rho\colon \mathrm{Sp}_n \to \mathrm{GL}_{\mathbb{C}}(\mathbb{C}^{2n})$.

Denoting the canonical basis of $\mathbb{C}^{2n}$ by $\{e_1, \dots, e_n, e_{-1}, \dots, e_{-n}\}$, under the action of the maximal torus T the representation ρ splits into the sum of $2n$ irreducible representations in the spaces

$$\mathbb{C}e_i, \quad \mathbb{C}e_{-i}, \quad i = 1, \dots, n,$$

with corresponding weights $\pm\hat{x}_i$. Since only one of those weights is dominant, namely $\hat{x}_1$, ρ is irreducible with dominant weight $\hat{x}_1 = \beta_1$, hence ρ is the fundamental representation ρ_1. With all the previous examples in mind, we now consider the representation

$$\Lambda^2\rho\colon \mathrm{Sp}_n \longrightarrow \mathrm{GL}_{\mathbb{C}}(\Lambda^2(\mathbb{C}^{2n})).$$

Under the action of the maximal torus, it splits into the direct sum of $\binom{2n}{2}$ irreducible representations in the spaces

$$\mathbb{C}e_i \wedge e_j, \quad \mathbb{C}e_{-i} \wedge e_{-j}, \quad 1 \le i < j \le n,$$
$$\mathbb{C}e_i \wedge e_{-j}, \qquad\qquad 1 \le i, j \le n,$$

with corresponding weights

$$\pm(\hat{x}_i + \hat{x}_j), \quad 1 \le i < j \le n,$$
$$\pm(\hat{x}_i - \hat{x}_j), \quad 1 \le i \le j \le n.$$

Thus the dominant weights occurring in the decomposition are

- $(1, 1, 0, \dots, 0)$, with corresponding weight vector $e_1 \wedge e_2$,
- $(0, \dots, 0)$, with corresponding weight vectors $e_i \wedge e_{-i}$, $i = 1, \dots, n$.

It is easy to see that the weight vector $e_1 \wedge e_2$ is maximal.

Let v be a weight vector corresponding to $(0, \dots, 0)$,

$$v = \sum_{i=1}^{n} \lambda_i e_i \wedge e_{-i}, \quad \lambda_i \in \mathbb{C}.$$

Now note that the root vector $M_{r,r+1} - iN_{r,r+1}$ corresponding to the simple roots $\theta_r = \hat{x}_r - \hat{x}_{r+1}$, $r = 1, \dots, n-1$, acts on $\mathbb{C}^{2n}$ by the map

$$x = (x_1, \dots, x_n, x_{-1}, \dots, x_{-n}) \longmapsto 2x_{r+1}e_r - 2x_{-r}e_{-(r+1)},$$

whereas the root vector $P_{nn} - iQ_{nn}$ corresponding to the simple root $\theta_n = 2\hat{x}_n$ acts on $\mathbb{C}^{2n}$ by the map

$$x = (x_1, \dots, x_n, x_{-1}, \dots, x_{-n}) \longmapsto -4x_{-n}e_n.$$

So, one has

$$(\Lambda^2\rho)_*(M_{r,r+1} - iN_{r,r+1}) \cdot v = 2(\lambda_{r+1} - \lambda_r)e_r \wedge e_{-(r+1)}, \quad r = 1, \dots, n-1,$$
$$(\Lambda^2\rho)_*(P_{nn} - iQ_{nn}) \cdot v = 0.$$

Thus, in order for v to be a maximal vector, one must have $\lambda_{r+1} = \lambda_r$, for any $r = 1, \dots, n-1$.

Hence, up to a scalar, there exists exactly one maximal vector corresponding to $(0, \dots, 0)$, namely the symplectic 2-form

$$\omega = \sum_{i=1}^{n} e_i \wedge e_{-i}.$$

Therefore, the representation $\Lambda^2\rho$ splits into two irreducible components: the first one with weight $\hat{x}_1 + \hat{x}_2 = \beta_2$, in the space

$$\Lambda^2_\circ \mathbb{C}^{2n} := \mathfrak{U}(\mathfrak{sp}_{n,\mathbb{C}}) \cdot (e_1 \wedge e_2),$$

is the fundamental representation ρ_2, whereas the second is the trivial representation.

Note that the dimension of ρ_2 is $\binom{2n}{2}-1$. To get a description of the fundamental representations ρ_k, $k = 3, \dots, n$, observe that the vector

$$e_1 \wedge \cdots \wedge e_k \in \Lambda^k(\mathbb{C}^{2n})$$

is a maximal vector with corresponding weight β_k. Hence the vector space of the fundamental representation is the cyclic module

$$\Lambda^k_\circ \mathbb{C}^{2n} := \mathfrak{U}(\mathfrak{sp}_{n,\mathbb{C}}) \cdot (e_1 \wedge \cdots \wedge e_k).$$

As in the case $k = 2$, the representations $\Lambda^k\rho$, $k = 3, \dots, n$ are not irreducible.

Proposition 12.72. *The representations $\Lambda^k\rho$, $k = 2, \dots, n$ decompose under the action of the symplectic group as*

$$\Lambda^k(\mathbb{C}^{2n}) = \Lambda^k_\circ \mathbb{C}^{2n} \oplus (\Lambda^{k-2}_\circ \mathbb{C}^{2n} \wedge \omega) \oplus (\Lambda^{k-4}_\circ \mathbb{C}^{2n} \wedge \omega^2) \oplus \cdots .$$

This decomposition is often called “Lepage decomposition” following [Lep46]. For a proof, see [Bou75], p. 202. Note that

$$\Lambda^k(\mathbb{C}^{2n}) = \Lambda^k_\circ \mathbb{C}^{2n} \oplus (\Lambda^{k-2}(\mathbb{C}^{2n}) \wedge \omega),$$

hence the dimension of the fundamental representation ρ_k is

$$n_k = \binom{2n}{k} - \binom{2n}{k-2}.$$

As we did in the examples above, we will prove here the decomposition by determining explicitly the maximal vectors.

Consider the following basis of $\Lambda^k(\mathbb{C}^{2n})$: for any triple (A, B, C) of disjoint subsets of $\{1, \dots, n\}$ such that

$$\mathrm{Card} A + \mathrm{Card} B + 2\mathrm{Card} C = k,$$

we set

$$w_{A,B,C} = e_{a_1} \wedge \cdots \wedge e_{a_p} \wedge e_{-b_1} \wedge \cdots \wedge e_{-b_q} \wedge e_{c_1} \wedge e_{-c_1} \wedge \cdots \wedge e_{c_r} \wedge e_{-c_r},$$

where $a_1 < \cdots < a_p$ (resp. $b_1 < \cdots < b_q$, $c_1 < \cdots < c_r$) are the elements of A (resp. B, C).

Under the action of the maximal torus T, the representation $\Lambda^k \rho$ splits into the direct sum of $\binom{2n}{k}$ irreducible representations in the spaces $\mathbb{C} w_{A,B,C}$, with corresponding weights

$$\hat{x}_{a_1} + \cdots + \hat{x}_{a_p} - \hat{x}_{b_1} - \cdots - \hat{x}_{b_q}.$$

By Proposition 12.71, the dominant weights are

$$\hat{x}_1 + \cdots + \hat{x}_{k-2l}, \quad l = 0, \ldots, [k/2] \quad (0 \text{ if } k \text{ is even and } l = k/2),$$

with corresponding weight vectors

$$\begin{aligned} w_{A,\emptyset,C}, \quad & A = \{1, \ldots, k-2l\} \; (A = \emptyset, \text{if } k \text{ is even and } l = k/2), \\ & C \subset \{k-2l+1, \ldots, n\}, \operatorname{Card} C = l. \end{aligned}$$

For any given $l = 0, \ldots, [k/2]$, let us consider the existence of a maximal vector with weight

$$\hat{x}_1 + \cdots + \hat{x}_{k-2l} \quad (0 \text{ if } k \text{ is even and } l = k/2).$$

Any such vector v has the form

$$\begin{aligned} v = \sum_C \lambda_C w_{A,\emptyset,C}, \quad & \lambda_C \in \mathbb{C}, \\ & A = \{1, \ldots, k-2l\} \; (A = \emptyset, \text{if } k \text{ is even and } l = k/2), \end{aligned}$$

where the sum is taken over the distinct subsets C of $\{k-2l+1, \ldots, n\}$ with l elements. One has (possibly up to a sign which has no importance for the proof)

$$(\Lambda^2 \rho)_*(M_{r,r+1} - i N_{r,r+1}) \cdot v = 0,$$

for $1 \leq r \leq k-2l$,

$$\begin{aligned} (\Lambda^2 \rho)_*(M_{r,r+1} - i N_{r,r+1}) \cdot v = & -2 \sum_{C, r \in C, (r+1) \notin C} \lambda_C w_{A \cup \{r\}, \{r+1\}, C \setminus \{r\}} \\ & + 2 \sum_{C, r \notin C, (r+1) \in C} \lambda_C w_{A \cup \{r\}, \{r+1\}, C \setminus \{r+1\}}, \end{aligned}$$

for $k-2l+1 \leq r \leq n-1$, and

$$(\Lambda^2 \rho)_*(P_{nn} - i Q_{nn}) \cdot v = 0.$$

Thus, in order for v to be a maximal vector, one must have $\lambda_{C' \cup \{r\}} = \lambda_{C' \cup \{r+1\}}$ for any subset C' of $\{k-2l+1, \ldots, n\} \setminus \{r, r+1\}$ with $l-1$ elements, $r = k-2l+1, \ldots, n-1$. Hence, for any subset C of $\{k-2l+1, \ldots, n\}$ with l elements, one has

$$\lambda_C = \lambda_{\{k-2l+1, \ldots, k-l\}}.$$

Indeed, consider such a subset $C = \{c_1, \ldots, c_l\}$ ($c_1 < \cdots < c_l$), and let j be the lowest integer such that $c_j \neq k - 2l + j$. Then

$$\{k - 2l + j, k - 2l + j + 1, \ldots, c_j - 1\} \cap C = \emptyset.$$

Setting $C' = C \setminus \{c_j\}$, we get from the previous result, considering $r = k - 2l + j$, $k - 2l + j + 1, \ldots, c_j - 1$,

$$\lambda_C = \lambda_{C' \cup \{c_j\}} = \lambda_{C' \cup \{c_j - 1\}} = \cdots = \lambda_{C' \cup \{k-2l+j\}}.$$

Considering now the set $C' \cup \{k - 2l + j\}$ instead of C, one deduces by induction that $\lambda_C = \lambda_{\{k-2l+1,\ldots,k-l\}}$.

Hence there is, up to a scalar, exactly one maximal vector for the weight $\hat{x}_1 + \cdots + \hat{x}_{k-2l}$, which can be written as

$$e_1 \wedge \cdots \wedge e_{k-2l} \wedge \omega^l.$$

The corresponding cyclic module is

$$\mathfrak{U}(\mathfrak{sp}_{n,\mathbb{C}}) \cdot (e_1 \wedge \cdots \wedge e_{k-2l} \wedge \omega^l).$$

But the symplectic form ω is invariant under the action of Sp_n, thus

$$\begin{aligned}\mathfrak{U}(\mathfrak{sp}_{n,\mathbb{C}}) \cdot (e_1 \wedge \cdots \wedge e_{k-2l} \wedge \omega^l) &= (\mathfrak{U}(\mathfrak{sp}_{n,\mathbb{C}}) \cdot (e_1 \wedge \cdots \wedge e_{k-2l})) \wedge \omega^l \\ &= \Lambda_\circ^{k-2l} \mathbb{C}^{2n} \wedge \omega^l.\end{aligned}$$

Therefore, since corresponding to the weights $\hat{x}_1 + \cdots + \hat{x}_{k-2l}$, $l = 0, \ldots, [k/2]$, there are exactly $[k/2] + 1$ linearly independent maximal vectors, one gets the following decomposition of $\Lambda^k(\mathbb{C}^{2n})$ into irreducible modules:

$$\Lambda^k(\mathbb{C}^{2n}) = \Lambda_\circ^k \mathbb{C}^{2n} \oplus (\Lambda_\circ^{k-2} \mathbb{C}^{2n} \wedge \omega) \oplus (\Lambda_\circ^{k-4} \mathbb{C}^{2n} \wedge \omega^2) \oplus \cdots .$$

Chapter 13

Symmetric space structure of model spaces

A classical reference for an introduction to symmetric spaces is [Bes87], Chapter 7. By definition, a Riemannian manifold (M, g_M) is said to be a symmetric space if for any p in M, there exists an isometry s_p of (M, g_M) such that $s_p(p) = p$ and $ds_p = -\mathrm{Id}_{\mathrm{T}_p M}$.

Theorem 13.1 (É. Cartan). *A Riemannian manifold (M, g_M) is a symmetric space if and only if there exists a triple (G, H, σ) satisfying the following conditions.*

(i) *G is a connected Lie group, H is a compact subgroup of G, and σ is an involutive automorphism of G such that*

$$G_e^\sigma \subset H \subset G^\sigma,$$

where G^σ is the group $\{g \in G;\ \sigma(g) = g\}$ and G_e^σ is the connected component of the identity in G^σ.

(ii) *There exists a G-invariant metric $g_{G/H}$ on G/H such that $(G/H, g_{G/H})$ is isometric to (M, g_M).*

Notations 13.2. In the following, any symmetric space (M, g_M) will be identified with the corresponding homogeneous space $(G/H, g_{G/H})$ by means of the isometry $\imath\colon (G/H, g_{G/H}) \to (M, g_M)$.

The neutral element in G will be denoted by e, the equivalence class in G/H of any $g \in G$ by $[g]$. The same notation L_g will be used to denote the left action of $g \in G$ on G and the induced action on the quotient G/H.

Remark 13.3. Note that the group G acts transitively on M by the isometries $\imath \circ L_g \circ \imath^{-1}$, $g \in G$, and that H is the isotropy subgroup corresponding to the point $p := \imath([e])$.

Note also that since M is G-homogeneous, all the geometric objects on M which are invariant under isometries are completely determined by their value at p.

Theorem 13.4 ([Hel78]). *If in the above theorem the symmetric space (M, g_M) is supposed to be compact and simply connected, then there exists a triple (G, H, σ) such that G is compact and simply connected, and $H = G_e^\sigma$.*

From now on, we consider a compact and simply connected symmetric space (M, g_M), together with a triple (G, H, σ) satisfying the conditions of Theorem 13.4.

Let $\mathfrak{g}$ (resp. $\mathfrak{h}$) be the Lie algebra of G (resp. H). The tangent map of ι at the point $[e]$, gives an isometry

$$(\mathrm{T}_{[e]}(G/H)) \simeq (\mathfrak{g}/\mathfrak{h}, g_{G/H,[e]}) \xrightarrow{\mathrm{T}_{[e]}\iota} (\mathrm{T}_p M, g_{M,p}).$$

Note that under the identification $\mathrm{T}_{[e]}(G/H) \simeq \mathfrak{g}/\mathfrak{h}$, the action of $h \in H$ on $\mathfrak{g}/\mathfrak{h}$ is given by $\mathrm{Ad}_G(h)$, since for any $h \in H$ and any $X \in \mathfrak{g}$,

$$\begin{aligned}\frac{d}{dt}\{L_h([\exp(tX)])\}\big|_{t=0} &= \frac{d}{dt}\{[h\exp(tX)h^{-1}]\}\big|_{t=0}\\ &= \frac{d}{dt}\{[\exp(t\,\mathrm{Ad}_G(h)\cdot X)]\}\big|_{t=0}.\end{aligned}$$

Now the structure of symmetric space of (M, g_M) provides an $\mathrm{Ad}_G(H)$-invariant subspace $\mathfrak{p}$ of $\mathfrak{g}$ which complements $\mathfrak{h}$ in $\mathfrak{g}$. Indeed, let σ_* be the tangent map at e of the involutive automorphism σ. It is a Lie algebra automorphism of $\mathfrak{g}$ such that $\sigma_*^2 = \mathrm{Id}_{\mathfrak{g}}$. Since $H = G_e^\sigma$, one has

$$\mathfrak{h} = \{X \in \mathfrak{g}; \sigma_*(X) = X\},$$

thus the decomposition of $\mathfrak{g}$ into eigenspaces is

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p},$$

with

$$\mathfrak{p} = \{X \in \mathfrak{g};\ \sigma_*(X) = -X\}. \tag{13.1}$$

It follows easily from the condition $H = G_e^\sigma$ that $\mathfrak{p}$ is an $\mathrm{Ad}_G(H)$-invariant subspace of $\mathfrak{g}$: for any $X \in \mathfrak{p}$ and any $h \in H$

$$\begin{aligned}\sigma_*(\mathrm{Ad}_G(h)(X)) &= \frac{d}{dt}\big(\sigma\big(h\exp(tX)h^{-1}\big)\big)\big|_{t=0}\\ &= \frac{d}{dt}\big(h\sigma(\exp(tX))h^{-1}\big)\big|_{t=0}\\ &= \frac{d}{dt}\big(h\exp(t\sigma_*(X))h^{-1}\big)\big|_{t=0}\\ &= \frac{d}{dt}\big(h\exp(-tX)h^{-1}\big)\big|_{t=0}\\ &= -\mathrm{Ad}_G(h)(X).\end{aligned}$$

Furthermore, we have the following result.

Lemma 13.5. *The complementing Lie algebras* $\mathfrak{h}$ *and* $\mathfrak{p}$ *satisfy the commutation relations*

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{h}.$$

Proof. For any X and Y in $\mathfrak{p}$, one has

$$\sigma_* \cdot [X, Y] = [\sigma_* \cdot X, \sigma_* \cdot Y] = [X, Y],$$

hence the last inclusion. □

We consider on $\mathfrak{p}$ the $\mathrm{Ad}_G(H)$-invariant scalar product $(\cdot, \cdot)$ which makes the isomorphism

$$\Phi \colon \mathfrak{p} \longrightarrow \mathrm{T}_{[e]}(G/H), \quad X \longmapsto \frac{d}{dt}\{[\exp(tX)]\}\Big|_{t=0}, \tag{13.2}$$

an isometry. Thus $\mathrm{Ad}_G(H)$ is a compact and connected (by the condition $H = G^{\sigma}_e$) subgroup of $\mathrm{SO}(\mathfrak{p})$.

We fix, once for all, an orthonormal basis $\{X_i\}$, $1 \le i \le n$, of $\mathfrak{p}$. The choice of such a basis allows one to identify $\mathrm{SO}(\mathfrak{p})$ with SO_n, so in what follows $\mathrm{Ad}_G(H)$ is considered as a subgroup of SO_n.

Definition 13.6. The symmetric space G/H is said to be irreducible if the representation $\mathrm{Ad}_G \colon H \to \mathrm{GL}(\mathfrak{p}) \simeq \mathrm{GL}_n$ is irreducible.

From now on, we assume that our symmetric space (M, g_M) is irreducible. Note that, by the de Rham decomposition theorem (see [Bes87], p. 194, for instance), this is a rather weak condition. In fact, any compact simply connected symmetric space is the Riemannian product of a finite number of simply connected compact irreducible symmetric spaces.

Lemma 13.7. *Any* $\mathrm{Ad}_G(H)$*-invariant symmetric bilinear form on* $\mathfrak{p}$ *is proportional to* $(\cdot, \cdot)$*. In particular, all the* $\mathrm{Ad}_G(H)$*-invariant scalar products on* $\mathfrak{p}$ *are proportional.*

Proof. The scalar product $(\cdot, \cdot)$ allows one to identify any $\mathrm{Ad}_G(H)$-invariant symmetric bilinear form on $\mathfrak{p}$ with an $\mathrm{Ad}_G(H)$-invariant symmetric endomorphism of $\mathfrak{p}$. This endomorphism is diagonalizable in some orthonormal basis of $\mathfrak{p}$. But since it is $\mathrm{Ad}_G(H)$-invariant, any eigenspace is invariant under the action of $\mathrm{Ad}_G(H)$, hence is equal to $\mathfrak{p}$ because $\mathfrak{p}$ is irreducible. Therefore, the endomorphism is a scalar multiple of the identity. □

Note that since the group G is compact and simply connected, G is semi-simple (cf. Theorem 12.32), so the restriction to $\mathfrak{p} \times \mathfrak{p}$ of the Killing form B of $\mathfrak{g}$ sign-changed provides an $\mathrm{Ad}_G(H)$-invariant scalar product on $\mathfrak{p}$. From the above lemma, this scalar product is proportional to $(\cdot, \cdot)$. Thus we can renormalize the metric $g_{G/H}$ on G/H (hence the metric g_M on M) in such a way that

$$(\cdot, \cdot) = -\mathrm{B}|_{\mathfrak{p}}. \tag{13.3}$$

Note also that the Ricci tensor at p, viewed as a symmetric bilinear form on $\mathfrak{p}$, is proportional to $(\cdot, \cdot)$. Hence, because the Ricci tensor is totally determined by its value at p (see the above remark), (M, g_M) is an Einstein manifold.

Finally, we mention the following result about the holonomy of symmetric spaces

Theorem 13.8. *The holonomy group* Hol *of any irreducible simply connected symmetric space G/H is the image of the group H under the homomorphism*

$$H \longrightarrow \mathrm{SO}(\mathfrak{p}) \simeq \mathrm{SO}_n, \quad h \longmapsto \mathrm{Ad}_G(h)|\mathfrak{p}.$$

For a proof, see [Bes87], Proposition 10.79.

13.1 Symmetric space structure of spheres

Classically, $\mathbb{S}^n$ is the symmetric space $\mathrm{SO}_{n+1} / \mathrm{SO}_n$ (see for instance [KN69]). Since SO_{n+1} is not simply connected, in order to obtain a triple verifying the conditions of Theorem 13.4, we consider the two-fold covering $\xi \colon \mathrm{Spin}_{n+1} \to \mathrm{SO}_{n+1}$.

Then $G = \mathrm{Spin}_{n+1}$ acts transitively on $\mathbb{S}^n$ by the isometry $\xi(g)$, for any $g \in G$. Let $\{e_0, \dots, e_n\}$ be the standard basis of $\mathbb{R}^{n+1}$. We identify Spin_n with the subgroup H of Spin_{n+1}, also denoted by Spin_n, defined by

$$H = \{v_1 \cdots v_{2k};\ v_i \in \mathrm{span}\{e_1, \dots, e_n\},\ \| v_i \| = 1,\ 1 \le i \le 2k\}.$$

It is easy to see that H is the isotropy subgroup of the point $p = e_0$, for the transitive action of Spin_{n+1} on $\mathbb{S}^n$. Hence the map

$$\imath \colon \mathrm{Spin}_{n+1}/\mathrm{Spin}_n \longrightarrow \mathbb{S}^n, \quad [g] \longmapsto \xi(g)(p),$$

is a diffeomorphism and it becomes an isometry if $\mathrm{Spin}_{n+1}/\mathrm{Spin}_n$ is endowed with the metric $\imath^*\mathrm{can}$ (note that this metric is G-invariant, since G acts on $\mathbb{S}^n$ by isometries).

Now consider the involutive automorphism of Spin_{n+1}

$$\sigma\colon \mathrm{Spin}_{n+1} \longrightarrow \mathrm{Spin}_{n+1}, \quad \sigma(\psi) := e_0 \cdot \psi \cdot {e_0}^{-1} = -e_0 \cdot \psi \cdot e_0.$$

Observe that for any $\psi \in H$, $\psi \cdot e_0 = e_0 \cdot \psi$, hence $H \subset G^\sigma$, and then $H \subset G_e^\sigma$, since H is connected. But it is easy to see that $\mathfrak{h}$ is the eigenspace of σ_* corresponding to the eigenvalue 1. Indeed, consider the basis $\{e_i \cdot e_j\}$, $0 \leq i < j \leq n$, of $\mathfrak{g}$. One easily gets

$$\sigma_*(e_i \cdot e_j) \ = e_i \cdot e_j, \quad 1 \leq i < j \leq n, \tag{13.4a}$$

$$\sigma_*(e_0 \cdot e_i) = -e_0 \cdot e_i, \quad 1 \leq i \leq n. \tag{13.4b}$$

Hence X in $\mathfrak{g}$ verifies $\sigma_*(X) = X$ if and only if $X \in \mathfrak{h}$.

Thus the two connected subgroups of G, H and G_e^σ, have the same Lie algebra, hence $H = G_e^\sigma$. So the triple $(G = \mathrm{Spin}_{n+1}, H = \mathrm{Spin}_n, \sigma)$ verifies the conditions of Theorem 13.4. By (13.4), it is easy to see that the eigenspace $\mathfrak{p}$ of σ_* corresponding to the eigenvalue -1 has $\{e_0 \cdot e_i\}_{1 \leq i \leq n}$ as basis. Under the isometry

$$\Phi\colon \mathfrak{p} \longrightarrow \mathrm{T}_{e_0}\mathbb{S}^n \simeq \langle e_0 \rangle^\perp, \quad X \longmapsto \frac{d}{dt}\{\xi(\exp(tX)) \cdot e_0\}|_{t=0} = \xi_*(X)(e_0),$$

the vector $e_0 \cdot e_i$ is mapped onto $2e_i$, hence setting

$$X_i := \frac{1}{2} e_0 \cdot e_i, \quad 1 \leq i \leq n,$$

we get an orthonormal basis (X_i) of $\mathfrak{p}$.

Under the identifications $H \simeq \mathrm{Spin}_n$ and $\mathrm{SO}(\mathfrak{p}) \simeq \mathrm{SO}_n$ (the last being given by the choice of the above basis), the homomorphism $\mathrm{Ad}_G|_H$ is nothing else but the covering $\xi\colon \mathrm{Spin}_n \to \mathrm{SO}_n$. Hence the representation $\mathrm{Ad}_G\colon H \to \mathrm{GL}(\mathfrak{p}) \simeq \mathrm{GL}_n$ is irreducible, since it is the standard representation of the group Spin_n on $\mathbb{R}^n$, see Section 1.2.2.2. By applying Theorem 13.8, this in particular implies

Remark 13.9. For any $n \geq 2$, $\mathrm{Hol}(\mathbb{S}^n, \mathrm{can}) = \mathrm{SO}_n$.

The Killing form B of the group Spin_{n+1} is given by

$$\mathrm{B}(X, Y) = (n-1)\,\mathrm{tr}(\xi_*(X) \cdot \xi_*(Y)), \quad X, Y \in \mathfrak{g}.$$

By Lemma 13.7, its restriction to $\mathfrak{p}$ sign-changed has to be proportional to the scalar product $(\cdot,\cdot)$ on $\mathfrak{p}$. Considering any vector of the above orthonormal basis $\{X_i\}$, one gets

$$-\mathrm{B}(X_i, X_i) = 2(n-1),$$

so in order to have $(\cdot,\cdot) = -\mathrm{B}|_{\mathfrak{p}}$, we have to normalize the canonical metric on the sphere by considering the metric $2(n-1)\mathrm{can}$.

13.2 Symmetric space structure of the complex projective space

We denote by $\{e_0, \dots, e_m\}$, the canonical basis of $\mathbb{C}^{m+1}$. Recall that the canonical metric on complex projective space is defined by requiring the Hopf fibration $\mathbb{S}^{2m+1} \to \mathbb{C}\mathrm{P}^m$ to be a Riemannian submersion, cf. [BGM71]. Let G be the group

$$\mathrm{SU}_{m+1} = \{A \in \mathrm{M}_{m+1}(\mathbb{C});\ {}^t\bar{A}A = \mathrm{Id}, \det_{\mathbb{C}} A = 1\}.$$

The group G acts transitively on the sphere $\mathbb{S}^{2m+1}$, and since $\mathrm{SU}_{m+1} \subset \mathrm{SO}_{2m+2}$, it acts by isometries. This action factors through the quotient, hence the group SU_{m+1} acts transitively on $\mathbb{C}\mathrm{P}^m$, and by definition of the canonical metric, it acts by isometries. The isotropy subgroup H of the point $p := [e_0]$ for this transitive action of SU_{m+1} on $\mathbb{C}\mathrm{P}^m$ is given by

$$S(\mathrm{U}_1 \times \mathrm{U}_m) := \left\{A \in \mathrm{SU}_{m+1};\ A = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & B \end{pmatrix}, \theta \in \mathbb{R}, B \in \mathrm{U}_m\right\}.$$

Hence the map

$$\iota\colon \mathrm{SU}_{m+1} / S(\mathrm{U}_1 \times \mathrm{U}_m) \longrightarrow \mathbb{C}\mathrm{P}^m, \quad [g] \longmapsto g(p) = [g(e_0)],$$

is a diffeomorphism and it becomes an isometry if $\mathrm{SU}_{m+1} / S(\mathrm{U}_1 \times \mathrm{U}_m)$ is endowed with the G-invariant metric $g_{G/H} = \iota^*\mathrm{can}$. Now consider the map

$$\sigma\colon \mathrm{SU}_{m+1} \longrightarrow \mathrm{SU}_{m+1}, \quad \sigma(A) = SAS^{-1},$$

where S is the $(m+1) \times (m+1)$-matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & -I_m \end{pmatrix}.$$

It is easy to see that σ is an involutive automorphism of SU_{m+1} such that $G^\sigma = S(\mathrm{U}_1 \times \mathrm{U}_m)$, hence $G_e^\sigma = S(\mathrm{U}_1 \times \mathrm{U}_m)$, since $S(\mathrm{U}_1 \times \mathrm{U}_m)$ is connected. Hence, the triple $(G = \mathrm{SU}_{m+1}, H = S(\mathrm{U}_1 \times \mathrm{U}_m), \sigma)$ verifies the conditions of Theorem 13.4.

The eigenspace $\mathfrak{p}$ of σ_* corresponding to the eigenvalue -1 is precisely

$$\mathfrak{p} = \left\{X_x = \begin{pmatrix} 0 & -{}^t\bar{x} \\ x & 0 \end{pmatrix};\ x \in \mathrm{M}_{m,1}(\mathbb{C}) \simeq \mathbb{C}^m\right\}.$$

Since $\mathfrak{p}$ is endowed with the scalar product $(\cdot,\cdot)$ that makes the isomorphism Φ an isometry, it is easy to see from the definition of the canonical metric on $\mathbb{C}\mathrm{P}^m$ that

$$(X_x, X_{x'}) = \mathrm{Re}\langle x, x'\rangle, \quad x, x' \in \mathbb{C}^m, \tag{13.5}$$

where $\langle\cdot,\cdot\rangle$ is the usual Hermitian product on $\mathbb{C}^m$. Hence the map

$$\mathbb{C}^m \simeq \mathbb{R}^{2m} \longrightarrow \mathfrak{p}, \quad x \longmapsto X_x,$$

is an isometry.

Identifying $\mathfrak{p}$ with $\mathbb{C}^m$ by this isometry, the adjoint representation of the group H on $\mathfrak{p}$ is

$$\mathrm{Ad}_G \begin{pmatrix} \mathrm{e}^{i\theta} & 0 \\ 0 & B \end{pmatrix} \cdot x = \mathrm{e}^{-i\theta} Bx. \tag{13.6}$$

This representation is irreducible since its restriction to the subgroup

$$\left\{ A \in H;\ A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & B \end{smallmatrix}\right), B \in \mathrm{SU}_m \right\},$$

is the standard representation of the group SU_m, see Paragraph 12.4.1. So the symmetric space $(\mathrm{SU}_{m+1}/S(\mathrm{U}_1 \times \mathrm{U}_m), g_{G/H})$ is irreducible.

Remark 13.10. The image of the group H under the homomorphism

$$H \longrightarrow \mathrm{SO}(\mathfrak{p}) \simeq \mathrm{SO}_{2m}, \quad h \longmapsto \mathrm{Ad}_G(h)|\mathfrak{p},$$

is the group U_m, hence from Theorem 13.8 it follows that $\mathrm{Hol}(\mathbb{C}\mathrm{P}^m, \mathrm{can}) = \mathrm{U}_m$.

The Killing form B of the group SU_{m+1} is given by

$$\mathrm{B}(X,Y) = 2(m+1)\mathrm{Re}(\mathrm{Tr}(XY)), \quad X, Y \in \mathfrak{g}.$$

From Lemma 13.7, its restriction to $\mathfrak{p}$ sign-changed has to be proportional to the scalar product $(\cdot,\cdot)$ on $\mathfrak{p}$. By (13.5), for any $x \in \mathbb{C}^m$, one has

$$-\mathrm{B}(X_x, X_x) = 4(m+1)\|x\|^2 = 4(m+1)(X_x, X_x),$$

so in order to have $(\cdot,\cdot) = -\mathrm{B}|_{\mathfrak{p}}$, we need to normalize the canonical metric on the complex projective space by considering $g_{G/H} = 4(m+1)\mathrm{can}$.

13.3 Symmetric space structure of the quaternionic projective space

We consider $\mathbb{H}^{m+1}$ as a right vector space over $\mathbb{H}$ and denote by $\{e_0,\dots,e_m\}$, the canonical basis of $\mathbb{H}^{m+1}$. The quaternionic projective space $\mathbb{H}\mathrm{P}^m$ is the set of equivalence classes of $(m+1)$-tuples of quaternions $x = (q_0,\dots,q_m)$, two such $(m+1)$-tuples x and x' being equivalent if there exists a non-zero quaternion q such that $x' = xq$.

The canonical metric on $\mathbb{HP}^m$ is defined by requiring the Hopf fibration

$$\mathbb{S}^{4m+3} \longrightarrow \mathbb{HP}^m$$

to be a Riemannian submersion, cf. [BGM71]. Let G be the group

$$\mathrm{Sp}_{m+1} = \{A \in \mathrm{M}_{m+1}(\mathbb{H});\ {}^t\bar{A}A = I\}.$$

The group G acts transitively on the sphere $\mathbb{S}^{4m+3} \subset \mathbb{H}^{m+1}$, and it acts by isometries since $\mathrm{Sp}_{m+1} \subset \mathrm{SO}_{4m+4}$. This action factors through the quotient, hence the group Sp_{m+1} acts transitively on $\mathbb{HP}^m$, and by definition of the canonical metric, it acts by isometries. The isotropy subgroup H at the point $p := [e_0]$ for this transitive action of Sp_{m+1} on $\mathbb{HP}^m$ is the group identified with $\mathrm{Sp}_1 \times \mathrm{Sp}_m$:

$$H = \left\{A \in \mathrm{Sp}_{m+1};\ A = \begin{pmatrix} q & 0 \\ 0 & B \end{pmatrix}, q \in \mathrm{Sp}_1, B \in \mathrm{Sp}_m\right\}.$$

Hence the map

$$\iota\colon \mathrm{Sp}_{m+1} / \mathrm{Sp}_1 \times \mathrm{Sp}_m \longrightarrow \mathbb{HP}^m, \quad [g] \longmapsto g(p) = [g(e_0)],$$

is a diffeomorphism and it becomes an isometry if $\mathrm{Sp}_{m+1} / \mathrm{Sp}_1 \times \mathrm{Sp}_m$ is endowed with the G-invariant metric $g_{G/H} = \iota^*\mathrm{can}$.

Now consider the map

$$\sigma\colon \mathrm{Sp}_{m+1} \longrightarrow \mathrm{Sp}_{m+1}, \quad \sigma(A) := SAS^{-1},$$

where S is the $(m+1) \times (m+1)$ matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & -I_m \end{pmatrix}.$$

It is straightforward to see that σ is an involutive automorphism of Sp_{m+1} such that $G^\sigma = \mathrm{Sp}_1 \times \mathrm{Sp}_m$, hence $G_e^\sigma = \mathrm{Sp}_1 \times \mathrm{Sp}_m$, since $\mathrm{Sp}_1 \times \mathrm{Sp}_m$ is connected. So the triple $(G = \mathrm{Sp}_{m+1}, H = \mathrm{Sp}_1 \times \mathrm{Sp}_m, \sigma)$ satisfies the conditions of Theorem 13.4.

The eigenspace $\mathfrak{p}$ of σ_* corresponding to the eigenvalue -1 is

$$\mathfrak{p} = \left\{X_x = \begin{pmatrix} 0 & -{}^t\bar{x} \\ x & 0 \end{pmatrix};\ x \in \mathrm{M}_{m,1}(\mathbb{H}) \simeq \mathbb{H}^m\right\}.$$

Since $\mathfrak{p}$ is endowed with the scalar product $(\cdot,\cdot)$ which makes the isomorphism Φ an isometry, from the definition of the canonical metric on $\mathbb{HP}^m$ it follows that

$$(X_x, X_{x'}) = \mathrm{Re}\langle x, x'\rangle, \quad x, x' \in \mathbb{H}^m, \tag{13.7}$$

where $\langle\cdot,\cdot\rangle$ is the symplectic product on $\mathbb{H}^m$. Hence the map

$$\mathbb{H}^m \simeq \mathbb{R}^{4m} \longrightarrow \mathfrak{p}, \quad x \longmapsto X_x,$$

is an isometry. Identifying $\mathfrak{p}$ with $\mathbb{H}^m$ by this isometry, the adjoint representation of the group H on $\mathfrak{p}$ is then

$$\mathrm{Ad}_G \begin{pmatrix} q & 0 \\ 0 & B \end{pmatrix} \cdot x = Bx\bar{q}.$$

This representation is irreducible since its restriction to the subgroup

$$\left\{ A \in H;\ A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & B \end{smallmatrix}\right), B \in \mathrm{Sp}_m \right\},$$

is the standard representation of the group Sp_m, see Section 12.4.3. We conclude that the symmetric space $(\mathrm{Sp}_{m+1} \,/\, \mathrm{Sp}_1 \times \mathrm{Sp}_m, g_{G/H})$ is irreducible.

Remark 13.11. The image of the group H under the homomorphism

$$H \longrightarrow \mathrm{SO}(\mathfrak{p}) \simeq \mathrm{SO}_{4m}, \quad h \longmapsto \mathrm{Ad}_G(h)|_{\mathfrak{p}},$$

is the group $\mathrm{Sp}_1 \cdot \mathrm{Sp}_m$ defined in (7.2) (this could be indeed a definition of this group), hence from Theorem 13.8, $\mathrm{Hol}(\mathbb{H}\mathrm{P}^m, \mathrm{can}) = \mathrm{Sp}_1 \cdot \mathrm{Sp}_m$.

The Killing form B of the group Sp_{m+1} is given by

$$\mathrm{B}(X, Y) = 4(m+2)\mathrm{Re}(\mathrm{Tr}(XY)), X, Y \in \mathfrak{g}.$$

By Lemma 13.7, its restriction to $\mathfrak{p}$ sign-changed has to be proportional to the scalar product $(\cdot, \cdot)$ on $\mathfrak{p}$. By (13.7), one has for any $x \in \mathbb{H}^m$,

$$-\mathrm{B}(X_x, X_x) = 8(m+2)\langle x, x\rangle = 8(m+2)(X_x, X_x),$$

so in order to have $(\cdot, \cdot) = -\mathrm{B}|_{\mathfrak{p}}$, we have to normalize the canonical metric on the quaternionic projective space by considering $g_{G/H} = 8(m+2)\mathrm{can}$.

Chapter 14

Riemannian geometry of model spaces

The results of this chapter hold for any symmetric space, but we continue to consider a compact, simply connected, n-dimensional irreducible symmetric space $(M = G/H, g_{G/H})$ as in the previous chapter, with corresponding triple (G, H, σ) verifying the conditions of Theorem 13.4. As before, we fix once for all an orthonormal basis $\{X_i\}$, $1 \leq i \leq n$, of $\mathfrak{p}$, which allows us to identify $\mathfrak{p}$ with $\mathbb{R}^n$, and we denote by Φ the isometry

$$\Phi\colon \mathfrak{p} \longrightarrow \mathrm{T}_{[e]}G/H, \quad X \longmapsto \frac{d}{dt}\left([\exp(tX)]\right)\Big|_{t=0}.$$

In the following, for simplicity we denote by α the homomorphism

$$H \longrightarrow \mathrm{SO}_n, \quad h \longmapsto \mathrm{Ad}_G(h)|_{\mathfrak{p}}.$$

The manifold $M = G/H$ has a G-invariant orientation. Indeed, consider the n-form

$$\Phi(X_1)^* \wedge \cdots \wedge \Phi(X_n)^*,$$

on $\mathrm{T}_{[e]}G/H$, and extend it to any point $[g]$ of G/H by applying the diffeomorphism L_g given by the left action of some representative g of $[g]$ on G/H. Note that, this definition does not depend on the choice of the representative since $\alpha(H) \subset \mathrm{SO}_n$.

Proposition 14.1. *Let $P_{\mathrm{SO}_n}M$ be the bundle of positive orthonormal frames of M. Let $\pi\colon G \to G/H$ be the canonical principal bundle over G/H with structural group H. Consider the associated* SO_n*-principal bundle given by $G \times_\alpha \mathrm{SO}_n$. Then the principal bundles $P_{\mathrm{SO}_n}M$ and $G \times_\alpha \mathrm{SO}_n$ are isomorphic.*

Proof. Let $[g]$ be some element in G/H and let $b_{[g]}$ be a positive orthonormal frame at $[g]$, that is, an isometry $\mathbb{R}^n \to \mathrm{T}_{[g]}(G/H)$ preserving the orientations. Let g be a representative of $[g]$, and denote by ${L_g}_*$ the tangent map at the point $p := [e]$ of L_g. Consider the map $u_g\colon \mathbb{R}^n \to \mathbb{R}^n \simeq \mathfrak{p}$, defined by

$$u_g = \Phi^{-1} \circ {L_g}_*^{-1} \circ b_{[g]}. \tag{14.1}$$

Since u_g is an isometry preserving the orientation of $\mathbb{R}^n$, u_g belongs to SO_n. Furthermore for any h in H, it is easy to verify that $u_{gh} = \alpha(h^{-1})u_g$. Hence the element $[g, u_g]$ in the fibre of $G \times_\alpha \mathrm{SO}_n$ at $[g]$ depends only on the equivalence class $[g]$ of g. The map

$$P_{\mathrm{SO}_n} M \longrightarrow G \times_\alpha \mathrm{SO}_n, \quad b_{[g]} \longmapsto [g, u_g],$$

gives the claimed SO_n-isomorphism between $P_{\mathrm{SO}_n} M$ and $G \times_\alpha \mathrm{SO}_n$. (The inverse map is given by $[g, u] \mapsto L_{g*} \circ \Phi \circ u$, where $(g, u) \in G \times \mathrm{SO}_n$ is a representative of $[g, u]$). □

Consider the vector bundle associated with the principal bundle $\pi : G \to G/H$ by the linear representation $\alpha : H \to \mathrm{SO}_n \subset \mathrm{GL}_n$, that is $G \times_\alpha \mathbb{R}^n$.

Proposition 14.2. *The vector bundles* $\mathrm{T}M$ *and* $G \times_\alpha \mathbb{R}^n$ *are isomorphic.*

Proof. The tangent bundle $\mathrm{T}M$ is the vector bundle associated to $P_{\mathrm{SO}_n} M$ by the standard linear representation ρ of SO_n, that is $P_{\mathrm{SO}_n} M \times_\rho \mathbb{R}^n$. Let $[g]$ be an element in G/H and $[b_{[g]}, x]$ an element in the fibre of $P_{\mathrm{SO}_n} M \times_\rho \mathbb{R}^n$ over $[g]$, where $b_{[g]}$ is a positive orthonormal frame at $[g]$ and $x \in \mathbb{R}^n$. It is easy to see that the element $[g, u_g(x)]$ of the fibre of $G \times_\alpha \mathbb{R}^n$ at $[g]$, where u_g is the isometry of $\mathbb{R}^n$ defined in (14.1), depends only on the equivalence class $[b_{[g]}, x]$, and that the map

$$P_{\mathrm{SO}_n} M \times_\rho \mathbb{R}^n \longrightarrow G \times_\alpha \mathbb{R}^n, \quad [b_{[g]}, x] \longmapsto [g, u_g(x)],$$

is the claimed isomorphism between the two vectors bundles. (The inverse map is given by $[g, x] \mapsto [L_{g*} \circ \Phi, x]$, where $(g, x) \in G \times \mathbb{R}^n$ is a representative of $[g, x]$.) □

Corollary 14.3. *Let* $\mathcal{C}_H^\infty(G, \mathbb{R}^n)$ *be the space of* H*-equivariant smooth functions* $G \to \mathbb{R}^n$*, that is, the set of functions* $f : G \to \mathbb{R}^n$ *satisfying the condition*

$$f(gh) = \alpha(h^{-1}) \cdot f(g), \quad g \in G, h \in H.$$

The space $\Gamma(\mathrm{T}M)$ *of smooth sections of the tangent bundle of* M *is isomorphic to* $\mathcal{C}_H^\infty(G, \mathbb{R}^n)$.

Proof. According to Proposition 14.2, a smooth vector field on M, that is, a smooth section of the bundle $\mathrm{T}M$ can be seen as a section of the vector bundle $G \times_\alpha \mathbb{R}^n$. The proof is a classical result of the theory of associated bundles. Let f be a function in $\mathcal{C}_H^\infty(G, \mathbb{R}^n)$. Take $[g] \in G/H$ and denote by g a representative of the class $[g]$. Since f is H-equivariant, the element $[g, f(g)]$ of the fibre of $G \times_\alpha \mathbb{R}^n$ at $[g]$ depends only on the equivalence class $[g]$ of g, and the map

$$X_f : G/H \longrightarrow G \times_\alpha \mathbb{R}^n, \quad [g] \longmapsto [g, f(g)],$$

is a smooth section of the bundle $G \times_\alpha \mathbb{R}^n$.

Conversely, let X be a smooth section of the bundle $G \times_\alpha \mathbb{R}^n$. For any g in G, there exists a unique x_g in $\mathbb{R}^n$ such that $X([g]) = [g, x_g]$. Note that for any $h \in H$, $x_{gh} = \alpha(h^{-1}) \cdot x_g$. Hence the map

$$f_X : G \longrightarrow \mathbb{R}^n, \quad g \longmapsto x_g, \tag{14.2}$$

is H-equivariant. Using local trivializations of the bundle, it follows that f_X is smooth. The map $X \mapsto f_X$ is the claimed isomorphism with inverse $f \mapsto X_f$. Note that for any tangent vector field X, with the identification $\mathbb{R}^n \simeq \mathfrak{p}$, the relation between X and the H-equivariant function f_X is given by

$$X_{[g]} = L_{g*} \circ \Phi(f_X(g)), \quad g \in G \tag{14.3}$$

which concludes the proof. □

Remark 14.4. Similarly, (p,q)-tensor fields on M (resp. p-forms on M), can be identified with H-equivariant functions $G \to \mathrm{T}^{(p,q)}\mathbb{R}^n$ (resp. $G \to \Lambda^p \mathbb{R}^n$).

14.1 The Levi-Civita connection

Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ be the canonical decomposition of $\mathfrak{g}$ induced by the structure of symmetric space of M. Since $\mathfrak{p}$ is an $\mathrm{Ad}_G(H)$-invariant subspace of $\mathfrak{g}$, the G-invariant horizontal distribution $\mathcal{H}$ on G given by

$$\mathcal{H}_g := \mathrm{T}_e L_g(\mathfrak{p}), \quad g \in G,$$

is a principal connection on the canonical bundle $\pi : G \to G/H$. Let ω be the corresponding connection 1-form. Recall that, by definition, denoting by $\mathrm{pr}_{\mathfrak{h}}$ the projection $\mathfrak{g} \to \mathfrak{h}$, for any vector field X on G one has

$$\omega(X)(g) = \mathrm{pr}_{\mathfrak{h}}(\mathrm{T}_e L_g{}^{-1}(X_g)).$$

It is a classical result in the theory of connections that this principal connection ω induces a linear connection ∇ on the associated bundle $G \times_\alpha \mathbb{R}^n \simeq \mathrm{T}M$, connection which is defined as follows. Let $[g]$ be an element of G/H and denote by g a representative of the class $[g]$. If Y is a section of the bundle $G \times_\alpha \mathbb{R}^n \simeq \mathrm{T}M$, then by Corollary 14.3 we can identify Y with the H-equivariant function $f_Y : G \to \mathbb{R}^n$, defined by $Y_{[g]} = [g, f_Y(g)]$ (see (14.2)). If X is a vector field on M, we denote by X^H the horizontal lift of X. Then

$$(\nabla_X Y)([g]) = [g, (X^H . f_Y)(g)].$$

Note that if σ is a local section of the principal bundle $G \to G/H$, then

$$X^H . f_Y(\sigma([g])) = (X.\sigma^* f_Y)_{[g]} - \alpha_*(\sigma^*\omega(X)) \cdot f_Y(\sigma([g])).$$

This implies that the linear connection ∇ is metric, since $\alpha : H \to \mathrm{SO}_n$.

Proposition 14.5. *The connection ∇ is G-invariant, that is, for any g in G and for any tangent vector fields X and Y,*

$$L_{g^*}(\nabla_X Y) = \nabla_{L_{g^*}X} L_{g^*}Y.$$

Proof. The G-invariance of the principal connection $\mathcal{H}$ implies that for any $g_0 \in G$,

$$(L_{g_0^*}X)^H = L_{g_0^*}(X^H).$$

On the other hand, using (14.3), we get for any tangent vector field Z that

$$Z_{[g]} = \frac{d}{dt}\{[g\exp(tf_Z(g))]\}\big|_{t=0},$$

hence, for any smooth function $f\colon G/H \to \mathbb{R}$,

$$\begin{aligned}\big(L_{g_0^*}Z\big)_{[g]}.f &= Z_{[g_0^{-1}g]}(f\circ L_{g_0})\\ &= \frac{d}{dt}\{f([g\exp(tf_Z(g_0^{-1}g))])\}\big|_{t=0}.\end{aligned}$$

Therefore,

$$\begin{aligned}(L_{g_0^*}Z)_{[g]} &= \frac{d}{dt}\{[g\exp(tf_{L_{g_0^*}Z}(g))]\}\big|_{t=0}\\ &= \frac{d}{dt}\{[g\exp\big(tf_Z(g_0^{-1}g)\big)]\}\big|_{t=0}.\end{aligned}$$

Using the "uniqueness" of integral curves, and differentiating at $t=0$, this implies that for any tangent vector field Z

$$f_{L_{g_0^*}Z}(g) = f_Z(g_0^{-1}g), \quad g \in G.$$

So finally

$$\begin{aligned}\nabla_{L_{g_0^*}X}L_{g_0^*}Y[g] &= [g, L_{g_0^*}(X^H)\cdot f_{L_{g_0^*}Y}(g)]\\ &= [g, X^H_{g_0^{-1}g}\cdot f_{L_{g_0^*}Y}\circ L_{g_0}]\\ &= [g, X^H_{g_0^{-1}g}\cdot f_Y]\\ &= [g, f_{\nabla_X Y}(g_0^{-1}g)]\\ &= L_{g_0^*}(\nabla_X Y)[g].\end{aligned}$$

□

Definition 14.6. Let $\sigma\colon U \subset M \to G$ be a local section of the canonical bundle $(G, \pi, G/H)$ and b be the local frame $U \subset M \to P_{\mathrm{SO}_n} M$ defined by

$$b_{[g]} = [\sigma([g]), e].$$

Recall that, by Proposition 14.1,

$$b([g]) = (\overline{X}_1([g]), \dots, \overline{X}_n([g])),$$

where

$$\overline{X}_i([g]) = L_{\sigma([g])*} \circ \Phi(X_i).$$

We will say for short that such a local frame is *induced* by the local section σ.

Proposition 14.7. *Let $[g_0]$ be the class of $g_0 \in G$ and let $\sigma\colon U \subset M \to G$ be a local section of the canonical bundle $(G, \pi, G/H)$ such that $\sigma([g_0]) = g_0$. In the local frame $(\overline{X}_1([g]), \dots, \overline{X}_n([g]))$ induced by the section σ, one has*

$$\nabla_{\overline{X}_i} \overline{X}_j([g_0]) = [g_0, (\widetilde{X}_i)_{g_0} \cdot f_{\overline{X}_j}], \tag{14.4}$$

where $\widetilde{X}_i$ are the left-invariant vector fields on G defined by the $X_i \in \mathfrak{p}$. Furthermore,

$$(\widetilde{X}_i)_{g_0} \cdot f_{\overline{X}_j} = \frac{d}{dt}\{\alpha(\exp(-tX_i)g_0^{-1}\sigma([g_0 \exp(tX_i)])) \cdot X_j\}\big|_{t=0}. \tag{14.5}$$

Proof. First, it is easy to see that the horizontal lift of the vector field $\overline{X}_i$ is defined by

$$\overline{X}_i^H(g) = \widetilde{X}_i(\sigma([g])),$$

hence (14.4), since $\overline{X}_i^H(g_0) = \widetilde{X}_i(g_0)$.

On the other hand, by (14.3), the H-equivariant function $f_{\overline{X}_i}\colon \pi^{-1}(U) \to \mathbb{R}^n$ corresponding to the local vector field $\overline{X}_i$ verifies

$$\overline{X}_i([g]) = L_{\sigma([g])*} \circ \Phi(f_{\overline{X}_i}(\sigma([g]))),$$

whence

$$f_{\overline{X}_i}(\sigma([g])) = X_i.$$

Noting that $\sigma([g])^{-1} g \in H$, we get

$$\begin{aligned} f_{\overline{X}_i}(g) &= f_{\overline{X}_i}(\sigma([g])\sigma([g])^{-1}g) \\ &= \alpha(g^{-1}\sigma([g]))(X_i), \end{aligned} \tag{14.6}$$

which implies (14.5). □

Proposition 14.8. *The linear connection ∇ is the Levi-Civita connection on $M = G/H$.*

Proof. We already noted that ∇ is a metric connection, so we only have to show that ∇ is torsion free. Since M is G-homogeneous and ∇ is G-invariant, we only have to verify this property at the point $p := [e]$. Let $(\overline{X}_i)$ be the orthonormal basis of $\mathrm{T}_p(G/H)$ given by

$$\overline{X}_i = \Phi(X_i), \tag{14.7}$$

where $\{X_i\}$ is the orthonormal basis of $\mathfrak{p}$ we already fixed.

Let us compute the value of $\mathrm{T}_p(\overline{X}_i, \overline{X}_j)$. For this we introduce a local orthonormal frame $b\colon U \subset M \to P_{\mathrm{SO}_n} M$, defined on an open neighborhood U of p, such that $b(p) = (\overline{X}_1, \ldots, \overline{X}_n)$. Since T is a tensor, the value of $\mathrm{T}_p(\overline{X}_i, \overline{X}_j)$ does not depend on the choice of b, so we are free to consider the local frame induced by the following local section σ of the canonical bundle $(G, \pi, G/H)$. Since the map

$$\mathfrak{p} \longrightarrow G/H, \quad \sum_{i=1}^{n} x_i X_i \longmapsto [\exp(x_1 X_1) \ldots \exp(x_n X_n)],$$

is a local diffeomorphism from a neighborhood of 0 in $\mathfrak{p}$ onto a neighborhood U of $[e]$ in G/H, we can consider the local section $\sigma\colon U \to G$ given by

$$\sigma([\exp(x_1 X_1) \ldots \exp(x_n X_n)]) = \exp(x_1 X_1) \ldots \exp(x_n X_n). \tag{14.8}$$

We first prove that for the local frame $\{\overline{X}_1, \ldots, \overline{X}_n\}$ induced by σ,

$$(\nabla_{\overline{X}_i} \overline{X}_j)(p) = 0.$$

Indeed, using (14.4) and (14.5), one gets since $\sigma([\exp(tX_i)]) = \exp(tX_i)$,

$$\begin{aligned}
(\overline{X}_i^H \cdot f_{\overline{X}_j})(e) &= \frac{d}{dt}\{f_{\overline{X}_j}(\exp(tX_i))\}\big|_{t=0} \\
&= \frac{d}{dt}\{\alpha(\exp(-tX_i)\sigma([\exp(tX_i)]))(X_j)\}\big|_{t=0} \\
&= 0.
\end{aligned}$$

This implies that $(\nabla_{\overline{X}_i} \overline{X}_j)(p) = (\nabla_{\overline{X}_j} \overline{X}_i)(p) = 0$. Note also that we may conclude $(\nabla g)_p = 0$, hence $\nabla g = 0$ using homogeneity, so ∇ is a metric connection.

We now compute $[\overline{X}_i, \overline{X}_j](p)$. Since the local flows $\varphi_{i,t}$ of the vector fields $\overline{X}_i$ are defined by $\varphi_{i,t}([g]) = [\sigma([g])\exp(tX_i)]$, we get by definition of σ,

$$\begin{aligned}
[\overline{X}_i, \overline{X}_j](p) &= -\frac{d}{dt}\frac{d}{ds}\{\varphi_{i,t}\circ\varphi_{j,s}\circ\varphi_{i,-t}([e])\}\Big|_{\substack{t=0\\s=0}},\\
&= -\frac{d}{dt}\frac{d}{ds}\{[\exp(-tX_i)\exp(sX_j)\exp(tX_i)]\}\Big|_{\substack{t=0\\s=0}},\\
&= -\frac{d}{dt}\frac{d}{ds}\{[\exp(s\,\mathrm{Ad}_{\exp(-tX_i)}\cdot X_j)]\}\Big|_{\substack{t=0\\s=0}},\\
&= -\frac{d}{dt}\frac{d}{ds}\{[\exp(s(\mathrm{Ad}_{\exp(-tX_i)}\cdot X_j)_{\mathfrak{p}})]\}\Big|_{\substack{t=0\\s=0}},\\
&= -\Phi\Big(\frac{d}{dt}\{(\mathrm{Ad}_{\exp(-tX_i)}\cdot X_j)_{\mathfrak{p}}\}\big|_{t=0}\Big),\\
&= -\Phi\Big(\Big(\frac{d}{dt}\{\mathrm{Ad}_{\exp(-tX_i)}\cdot X_j\}\big|_{t=0}\Big)_{\mathfrak{p}}\Big),\\
&= \Phi([X_i, X_j]_{\mathfrak{p}}).
\end{aligned}$$

So finally

$$\mathrm{T}_p(\overline{X}_i, \overline{X}_j) = -[\overline{X}_i, \overline{X}_j](p) = -\Phi([X_i, X_j]_{\mathfrak{p}}) = 0,$$

since $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{h}$, by Lemma 13.5. □

We now examine the curvature of M. Since M is G-homogeneous, the curvature R is completely determined by its value at $p = [e]$.

Proposition 14.9. *Let $\{\overline{X}_i\}$ be the orthonormal basis of $\mathrm{T}_p(G/H)$ defined in (14.7). One has*

$$R_p(\overline{X}_i, \overline{X}_j)\overline{X}_k = -\Phi([[X_i, X_j], X_k]).$$

Proof. Since R is a tensor, we can choose the same local frame as in the computation of the torsion at p. Since $[\overline{X}_i, \overline{X}_j](p) = 0$, we get

$$R_p(\overline{X}_i, \overline{X}_j)\overline{X}_k = (\nabla_{\overline{X}_i}\nabla_{\overline{X}_j} - \nabla_{\overline{X}_j}\nabla_{\overline{X}_i})(\overline{X}_k)(p).$$

Now, with the notations of the above proof, we have by (14.4)

$$(\nabla_{\overline{X}_i}\nabla_{\overline{X}_j})(\overline{X}_k)(p) = [e, (\widetilde{X}_i)_e\cdot(\overline{X}_j^H\cdot f_{\overline{X}_k})].$$

But, by definition of the local section σ, see (14.8), we have

$$\begin{aligned}
(\widetilde{X}_i)_e\cdot(\overline{X}_j^H.f_{\overline{X}_k}) &= \frac{d}{dt}\{(\overline{X}_j^H)_{\exp(tX_i)}.f_{\overline{X}_k}\}\big|_{t=0},\\
&= \frac{d}{dt}\{(\widetilde{X}_j)_{\exp(tX_i)}.f_{\overline{X}_k}\}\big|_{t=0},\\
&= (\widetilde{X}_i)_e\cdot(\widetilde{X}_j.f_{\overline{X}_k}).
\end{aligned}$$

Hence

$$R_p(X_i, X_j)X_k = [e, [\widetilde{X}_i, \widetilde{X}_j]_e \cdot f_{\overline{X}_k}].$$

Now since $[\widetilde{X}_i, \widetilde{X}_j](e) = \widetilde{[X_i, X_j]}(e)$, and $[X_i, X_j] \in \mathfrak{h}$, we get by (14.6)

$$\begin{aligned}
[\widetilde{X}_i, \widetilde{X}_j]_e \cdot f_{\overline{X}_k} &= \frac{d}{dt}\{\alpha(\exp(-t[X_i, X_j]))(X_k)\}\Big|_{t=0}, \\
&= \frac{d}{dt}\{\mathrm{Ad}_G(\exp(-t[X_i, X_j]))(X_k)\}\Big|_{t=0}, \\
&= -[[X_i, X_j], X_k].
\end{aligned}$$

The proof is complete. $\square$

Using this proposition, we compute the value at p of the Ricci tensor.

Proposition 14.10. *The Ricci tensor is given by*

$$\mathrm{Ric}_p(\overline{X}_i, \overline{X}_j) = -\frac{1}{2}\mathrm{B}(X_i, X_j),$$

where B is the Killing form of $\mathfrak{g}$.

Proof. We follow [Bes87]. Let $(\cdot,\cdot)_{\mathfrak{g}}$ be any $\mathrm{Ad}_G(H)$-invariant extension of the scalar product $(\cdot,\cdot)$ to $\mathfrak{g}$ such that $(\mathfrak{h},\mathfrak{p})_{\mathfrak{g}} = 0$. For instance, consider the Killing form sign-changed on $\mathfrak{h}$, $-\mathrm{B}_{\mathfrak{h}}$, and define

$$(\cdot,\cdot)_{\mathfrak{g}} = \begin{cases} (\cdot,\cdot) & \text{on} \quad \mathfrak{p}\times\mathfrak{p}, \\ -\mathrm{B}_{\mathfrak{h}} & \text{on} \quad \mathfrak{h}\times\mathfrak{h}, \\ 0 & \text{otherwise}. \end{cases}$$

Let $\{E_a\}$ be a basis of $\mathfrak{h}$ such that $\{X_i, E_a\}$ is an orthonormal basis of $(\mathfrak{g}, (\cdot,\cdot)_{\mathfrak{g}})$. Then

$$\begin{aligned}
\mathrm{B}(X_i, X_j) &= \mathrm{tr}(\mathrm{ad}(X_i)\circ \mathrm{ad}(X_j)), \\
&= \sum_k ([X_i, [X_j, X_k]], X_k)_{\mathfrak{g}} + \sum_a ([X_i, [X_j, E_a]], E_a)_{\mathfrak{g}}.
\end{aligned}$$

From the condition $[\mathfrak{p},\mathfrak{p}]\subset\mathfrak{h}$, we get

$$\begin{aligned}\sum_k([X_i,[X_j,X_k]],X_k)_{\mathfrak{g}} &= \sum_{k,a}([X_j,X_k],E_a)_{\mathfrak{g}}([X_i,E_a],X_k)_{\mathfrak{g}}\\ &= \sum_a([X_j,\sum_k([X_i,E_a],X_k)_{\mathfrak{g}}X_k],E_a)_{\mathfrak{g}}\\ &= \sum_a\big(\big[X_j,[X_i,E_a]\big],E_a\big)_{\mathfrak{g}}\\ &= \sum_a([X_i,[X_j,E_a]],E_a)_{\mathfrak{g}},\end{aligned}$$

where the last equality is obtained using the Jacobi identity and the $\mathrm{Ad}_G(H)$-invariance of $(\cdot,\cdot)$. Hence,

$$\mathrm{B}(X_i,X_j)=2\sum_k([X_i,[X_j,X_k]],X_k)_{\mathfrak{g}}.$$

From Proposition 14.9, finally we obtain

$$\begin{aligned}\mathrm{B}(X_i,X_j) &= 2\sum_k([X_i,[X_j,X_k]],X_k)_{\mathfrak{g}}\\ &= 2\sum_k([[X_k,X_j],X_i],X_k)_{\mathfrak{g}}\\ &= -2\sum_k(R_p(\overline{X}_k,\overline{X}_j)\overline{X}_i,\overline{X}_k)\\ &= -2\mathrm{Ric}_p(\overline{X}_j,\overline{X}_i)\\ &= -2\mathrm{Ric}_p(\overline{X}_i,\overline{X}_j).\end{aligned}$$ □

Remark 14.11. Recall that for the symmetric spaces we consider, the scalar product $(\cdot,\cdot)$ on $\mathfrak{p}$ is (eventually after a renormalization) $-\mathrm{B}|_{\mathfrak{p}}$, (see (13.3)). Hence we recover the fact that those symmetric spaces are Einstein manifolds. Moreover, the scalar curvature Scal is given by

$$\mathrm{Scal}=\frac{1}{2}\dim M. \tag{14.9}$$

14.2 Spin structures

Theorem 14.12 ([CG88]). *Let $M = G/H$ be a compact, simply connected irreducible symmetric space, with corresponding triple (G, H, σ) satisfying the conditions of Theorem 13.4. Then, M has a spin structure (necessarily unique, since M is simply connected) if and only if the homomorphism*

$$\alpha\colon H \longrightarrow \mathrm{SO}_n, \quad h \longmapsto \mathrm{Ad}_G(h)|_{\mathfrak{p}},$$

lifts to a homomorphism $\tilde{\alpha}\colon H \to \mathrm{Spin}_n$ such that the diagram

$$\begin{tikzcd} & \mathrm{Spin}_n \arrow[d, "\xi"] \\ H \arrow[ur, "\tilde{\alpha}"] \arrow[r, "\alpha"'] & \mathrm{SO}_n \end{tikzcd}$$

commutes. In this case, the spin structure is G-invariant.

Proof. The condition is sufficient. Suppose α lifts to $\tilde{\alpha}\colon H \to \mathrm{Spin}_n$. Let $P_{\mathrm{Spin}_n}M$ be the principal bundle over M with structural group Spin_n, associated with the principal bundle $(G, \pi, G/H)$ by the homomorphism $\tilde{\alpha}$, that is

$$P_{\mathrm{Spin}_n}M = G \times_{\tilde{\alpha}} \mathrm{Spin}_n,$$

and let ξ_M be the map

$$\begin{aligned} \xi_M\colon P_{\mathrm{Spin}_n}M = G \times_{\tilde{\alpha}} \mathrm{Spin}_n &\longrightarrow G \times_{\alpha} \mathrm{SO}_n \simeq P_{\mathrm{SO}_n}M, \\ [g, u] &\longmapsto [g, \xi(u)]. \end{aligned}$$

It is easy to see that $(P_{\mathrm{Spin}_n}M, \xi_M)$ is a spin structure on M. Furthermore, this spin structure is G-invariant since the left action of the group G on $P_{\mathrm{Spin}_n}M$ given by

$$g_0 \cdot [g, u] = [g_0 g, u],$$

and the right action of the group Spin_n on $P_{\mathrm{Spin}_n}M$ given by

$$[g, u] \cdot u_0 = [g, u u_0],$$

clearly commute.

The condition is necessary. Here is a sketch of the proof. Suppose there exists a spin structure $(P_{\mathrm{Spin}_n}M, \xi_M)$ on $M = G/H$. First, as G is assumed to be simply connected, the monodromy principle allows us to lift the action of G on $P_{\mathrm{SO}_n}M$ to a G-action on $P_{\mathrm{Spin}_n}M$, which commutes with the right action of Spin_n on $P_{\mathrm{Spin}_n}M$.

The induced action of H on $P_{\mathrm{Spin}_n}M$ stabilizes the fibre $(P_{\mathrm{Spin}_n}M)_{[e]}$ of $P_{\mathrm{Spin}_n}M$ at the point $[e]$. Let $\tilde{b}_{[e]}$ be a fixed element in $(P_{\mathrm{Spin}_n}M)_{[e]}$. For any $h \in H$, denote by $h \cdot \tilde{b}_{[e]}$ the action of h on $\tilde{b}_{[e]}$. Then define $\tilde{\alpha}(h) \in \mathrm{Spin}_n$ by the relation

$$h \cdot \tilde{b}_{[e]} = \tilde{b}_{[e]} \cdot \tilde{\alpha}(h).$$

It can be checked that $\tilde{\alpha}$ is a homomorphism $H \to \mathrm{Spin}_n$ which is a lift of α, and that (see [CG88]):

$$P_{\mathrm{Spin}_n}M \simeq G \times_{\tilde{\alpha}} \mathrm{Spin}_n. \qquad \square$$

Examples 14.13. i) For $n \geq 2$ the sphere $(\mathbb{S}^n, \mathrm{can})$ has a unique spin structure. Recall that, up to the identification $H \simeq \mathrm{Spin}_n$, the homomorphism α is the two-fold covering ξ, so α lifts trivially to $\tilde{\alpha} = \mathrm{Id}$. Note that since Spin_n is identified with H, the spin bundle $P_{\mathrm{Spin}_n}\mathbb{S}^n$ is isomorphic to the canonical bundle

$$(\mathrm{Spin}_{n+1}, \pi, \mathrm{Spin}_{n+1}/\mathrm{Spin}_n)$$

via the map

$$P_{\mathrm{Spin}_n}\mathbb{S}^n := \mathrm{Spin}_{n+1} \times_H \mathrm{Spin}_n \longrightarrow \mathrm{Spin}_{n+1}, \quad [g,h] \longmapsto gh.$$

ii) The quaternionic projective space $(\mathbb{H}\mathrm{P}^m, \mathrm{can})$ has a unique spin structure. Recall that H is isomorphic to the group $\mathrm{Sp}_1 \times \mathrm{Sp}_m$, hence is simply connected. It then follows from the classical monodromy principle that the homomorphism $\alpha\colon H \to \mathrm{SO}_n$ lifts to a unique homomorphism $\tilde{\alpha}\colon H \to \mathrm{Spin}_n$.

iii) The complex projective space $(\mathbb{C}\mathrm{P}^m, \mathrm{can})$ has a (unique) spin structure if and only if m is odd. Indeed, by the monodromy principle, α has a lift if and only if $\alpha_*\,(\Pi_1(H))$ is trivial. Note that the map

$$\mathbb{R} \times \mathrm{SU}_m \longrightarrow H = S(\mathrm{U}_1 \times \mathrm{U}_m), \quad (\theta, B) \longmapsto \begin{pmatrix} \mathrm{e}^{i\theta} & 0 \\ 0 & \mathrm{e}^{-\frac{i\theta}{m}} B \end{pmatrix},$$

is a covering of H with kernel isomorphic to $\mathbb{Z}$. Since $\mathbb{R} \times \mathrm{SU}_m$ is simply connected, one has $\Pi_1(H) \simeq \mathbb{Z}$ and any non-trivial element of $\Pi_1(H)$ has the form

$$\gamma_k\colon [0,1] \longrightarrow H, \quad t \longmapsto \gamma_k(t) = \begin{pmatrix} \mathrm{e}^{2ik\pi t} & 0 & 0 \\ 0 & \mathrm{e}^{-2ik\pi t} & 0 \\ 0 & 0 & I_{m-1} \end{pmatrix}, k \in \mathbb{Z}^*.$$

Consider the natural basis $\{e_i\}_{1\le i\le m}$ of $\mathbb{C}^m$ and denote by $\{\epsilon_i\}_{1\le i\le 2m}$ the basis of $\mathbb{R}^{2m}$ defined by $\epsilon_{2k-1} = e_k$, $\epsilon_{2k} = i e_k$, $1 \le k \le m$. From (13.6), denoting by $u_{\epsilon_i\epsilon_j}(\theta)$ the rotation with angle θ in the plane generated by ϵ_i and ϵ_j, it is easy to see that

$$\alpha(\gamma_k(t)) = u_{\epsilon_1\epsilon_2}(-4k\pi t)\circ u_{\epsilon_3\epsilon_4}(-2k\pi t)\circ\cdots\circ u_{\epsilon_{2m-1}\epsilon_{2m}}(-2k\pi t).$$

Now $\alpha_*(\Pi_1(H))$ is trivial if and only if the lift $\widetilde{\gamma_k}$ starting at 1 in Spin_{2m} of any $\alpha(\gamma_k)$ satisfies $\widetilde{\gamma_k}(1) = 1$. But

$$\widetilde{\gamma_k}(t) = (\cos(2k\pi t) - \sin(2k\pi t)\epsilon_1\cdot\epsilon_2)\prod_{j=2}^{m}(\cos(k\pi t) - \sin(k\pi t)\epsilon_{2j-1}\cdot\epsilon_{2j}),$$

so $\widetilde{\gamma_k}(1) = (-1)^{k(m-1)}$. Hence $\widetilde{\gamma_k}(1) = 1$, for any $k\in\mathbb{Z}^*$ if and only if m is odd.

14.3 Spinor bundles on symmetric spaces

Proposition 14.14. *Let $\rho_n\colon \mathrm{Spin}_n \to \mathrm{GL}_{\mathbb{C}}(\Sigma_n)$, be the spinor representation. Consider the vector bundle associated to $(G,\pi,G/H)$ by the representation*

$$\widetilde{\rho_n} := \rho_n\circ\tilde{\alpha},$$

that is $G\times_{\widetilde{\rho_n}}\Sigma_n$. Then the spinor bundle ΣM is isomorphic to $G\times_{\widetilde{\rho_n}}\Sigma_n$.

Proof. The proof is analogous to that of Proposition 14.2. By definition, the spinor bundle ΣM is the associated bundle $P_{\mathrm{Spin}_n}M\times_{\rho_n}\Sigma_n$. Let $[g]$ be an element in G/H and $[\tilde{b}_{[g]},\psi]$ an element in the fibre of $P_{\mathrm{Spin}_n}M\times_{\rho_n}\Sigma_n$ over $[g]$, where $\tilde{b}_{[g]}$ is an element of the fibre of $P_{\mathrm{Spin}_n}M$ over $[g]$ and $\psi\in\Sigma_n$. But $\tilde{b}_{[g]}$ has the form $[g,u]$, where g is a representative of $[g]$ and $u\in\mathrm{Spin}_n$. Now, it is straightforward that the element $[g,\rho_n(u)\cdot\psi]$ of the fibre of $G\times_{\widetilde{\rho_n}}\Sigma_n$ at $[g]$ depends only on the equivalence class $[[g,u],\psi]$, and that the map

$$P_{\mathrm{Spin}_n}M\times_{\rho_n}\Sigma_n\longrightarrow G\times_{\widetilde{\rho_n}}\Sigma_n,\quad [[g,u],\psi]\longmapsto[g,\rho_n(u)\cdot\psi],$$

is the claimed isomorphism between the two vector bundles. (The inverse map is given by $[g,\psi]\mapsto[[g,e],\psi]$, $g\in G$, and $\psi\in\Sigma_n$). $\square$

Corollary 14.15. *Let $\mathcal{C}^\infty_H(G,\Sigma_n)$ be the space of H-equivariant smooth functions $G\to\Sigma_n$, that is, the set of functions $f\colon G\to\Sigma_n$ satisfying the condition*

$$f(gh) = \widetilde{\rho_n}(h^{-1})\cdot f(g),\quad g\in G, h\in H.$$

The space $\Gamma(\Sigma M)$ of smooth sections of the bundle ΣM is isomorphic to the space $\mathcal{C}^\infty_H(G,\Sigma_n)$.

Proof. The proof is similar to that of Proposition 14.3. The H-equivariant function f_Ψ corresponding to a spinor field Ψ is defined by

$$\Psi([g]) = [g, f_\Psi(g)], \tag{14.10}$$

where g is some representative of $[g]$. □

Remark 14.16. Let X be a vector field on M and Ψ a spinor field on M. Identify X with the H-equivariant function $f_X : G \to \mathbb{R}^n$, see (14.2), and Ψ with the H-equivariant function f_Ψ. Then the Clifford multiplication $X \cdot \Psi$ has the expression

$$(X \cdot \Psi)([g]) = [g, f_X(g) \cdot f_\Psi(g)], \tag{14.11}$$

where g is some representative of $[g]$.

Finally, we can retrieve, in that context, the following result.

Proposition 14.17 ([Gut86]). *The* (*complex*) *spinor bundle of the sphere* $(\mathbb{S}^n, \mathrm{can})$ *is trivial.*

Proof. This is a consequence of the description of (complex) spinor representations. Identifying Spin_n with the subgroup H of Spin_{n+1}, one gets two equivalent representations of Spin_n:

- ρ_{2m} and $\rho_{2m+1}|_{\mathrm{Spin}_{2m}}$, if n is even, $n = 2m$;
- ρ_{2m+1} and $\rho^+_{2(m+1)}|_{\mathrm{Spin}_{2m+1}}$, if n is odd, $n = 2m + 1$.

Let J be the $\mathbb{C}$-isomorphism $\Sigma_{2m} \to \Sigma_{2m+1}$ (resp. $\Sigma_{2m+1} \to \Sigma^+_{2(m+1)}$), such that

$$J \circ \rho_{2m}(h) = \rho_{2m+1}|_{\mathrm{Spin}_{2m}}(h) \circ J, \quad h \in \mathrm{Spin}_{2m},$$

(resp. $J \circ \rho_{2m+1}(h) = \rho^+_{2(m+1)}|_{\mathrm{Spin}_{2m+1}}(h) \circ J$, $h \in \mathrm{Spin}_{2m+1}$). Recall (see Example 14.13 i) above) that the principal spinor bundle $P_{\mathrm{Spin}_n}\mathbb{S}^n$ is isomorphic to the canonical bundle $(\mathrm{Spin}_{n+1}, \pi, \mathbb{S}^n)$. It is then clear that the map

$$\mathrm{Spin}_{n+1} \times_{\rho_n} \Sigma_n \longrightarrow \mathbb{S}^n \times \Sigma_n$$

given by

$$[g, \psi] \longmapsto \begin{cases} (\pi(g), J^{-1} \circ \rho_{2m+1}(g) \circ J(\psi)), & \text{if } n = 2m, \\ (\pi(g), J^{-1} \circ \rho^+_{2(m+1)}(g) \circ J(\psi)), & \text{if } n = 2m+1, \end{cases}$$

is well defined and defines an isomorphism between the two vector bundles. □

14.4 The Dirac operator on symmetric spaces

The canonical principal connection ω on $\pi: G \to G/H$ (see Section 14.1) induces the covariant derivative ∇ on the associated bundle $G \times_{\widetilde{\rho_n}} \Sigma_n \simeq \Sigma M$.

Let $[g]$ be an element of G/H and g a representative of the class $[g]$. Let X be a vector field on M and Ψ be a section of the bundle $G \times_{\widetilde{\rho_n}} \Sigma_n \simeq \Sigma M$. From Corollary 14.15, we identify Ψ with the H-equivariant function $f_\Psi: G \to \Sigma_n$, such that $\Psi([g]) = [g, f_\Psi(g)]$; see (14.10). Denote by X^H the horizontal lift of the vector field X. Then, by definition

$$(\nabla_X \Psi)([g]) = [g, (X^H . f_\Psi)(g)]. \tag{14.12}$$

Proposition 14.18. *The linear connection ∇ is the spinorial Levi-Civita connection on ΣM.*

Proof. Let σ be a local section of the bundle $(G, \pi, G/H)$. It is easy to verify that

$$\begin{aligned} X^H . f_\Psi(\sigma[g]) &= (X.\sigma^* f_\Psi)_{[g]} - \tilde{\alpha}_*(\sigma^*\omega(X)) \cdot f_\Psi(\sigma[g]) \\ &= (X.\sigma^* f_\Psi)_{[g]} - \xi_*^{-1} \circ \alpha_*\big(\sigma^*\omega(X)\big) \cdot f_\Psi(\sigma[g]). \end{aligned}$$

But we saw in Section 14.1 that $\alpha_*\big(\sigma^*\omega(X)\big)$ is the local expression of the Levi-Civita connection form in the local frame bundle $(\bar{X}_1, \dots, \bar{X}_n)$ induced by the section σ, that is

$$\alpha_*(\sigma^*\omega(X)) = \frac{1}{2} \sum_{i,j} \langle \nabla_X \bar{X}_i, \bar{X}_j \rangle X_i \wedge X_j,$$

and to complete the proof it remains to apply Theorem 2.7. □

Remark 14.19. It should be pointed out that the G-invariance of the principal connection $\mathcal{H}$ implies the G-invariance of the spinorial Levi-Civita connection: for any g in G, any vector field X, and any spinor field Ψ, one has

$$L_g(\nabla_X \Psi) = \nabla_{L_{g*}X} L_g \Psi,$$

where L_g denotes the natural left action of G on $\Gamma(\Sigma M)$ given by

$$(L_{g_0}\Psi)([g]) := [g, f_\Psi({g_0}^{-1} g)].$$

Now using (14.12) and (14.11), we get the local expression of the Dirac operator.

Let $[g_0]$ be the class of a g_0 in G, and $\sigma: U \subset M \to G$ a local section of the canonical bundle $(G, \pi, G/H)$, satisfying $\sigma([g_0]) = g_0$. Denote by $\{\bar{X}_i\}$ the local orthonormal frame induced by σ,

$$\bar{X}_i([g]) = (L_{\sigma([g])})_*(X_i).$$

We already noted in Section 14.1 that the H-equivariant function

$$f_{\overline{X}_i} : \pi^{-1}(U) \longrightarrow \mathbb{R}^n$$

corresponding to the local vector field $\overline{X}_i$ is defined by

$$f_{\overline{X}_i}(g) = \alpha(g^{-1}\sigma([g]))(X_i),$$

so that $f_{\overline{X}_i}(g_0) = X_i$.

We also noted that the horizontal lift of the vector field $\overline{X}_i$ is defined by

$$\overline{X}_i^{\mathrm{H}}(g) = \widetilde{X}_i(\sigma([g])),$$

where $\widetilde{X}_i$ is the left-invariant vector field on G corresponding to $X_i \in \mathfrak{p}$, hence $\overline{X}_i^{\mathrm{H}}(g_0) = \widetilde{X}_i(g_0)$.

Now, for Ψ a spinor field on M, let f_Ψ be the corresponding H-equivariant function $G \to \Sigma_n$. One has

$$\mathcal{D}\Psi([g_0]) = \Big[g_0, \sum_{i=1}^{n} X_i \cdot (\widetilde{X}_i . f_\Psi)_{g_0}\Big],$$

Hence, we get the following result.

Proposition 14.20. *Via the identification of the space of spinor fields $\Gamma(\Sigma M)$ with the space $\mathcal{C}_H^\infty(G, \Sigma_n)$ given in Corollary 14.15, the Dirac operator $\mathcal{D}$ on $M = G/H$ is the differential operator*

$$\begin{aligned} \mathcal{D} : \mathcal{C}_H^\infty(G, \Sigma_n) &\longrightarrow \mathcal{C}_H^\infty(G, \Sigma_n), \\ \Psi &\longmapsto \mathcal{D}\Psi, \end{aligned}$$

defined by

$$\mathcal{D}\Psi(g) = \sum_{i=1}^{n} X_i \cdot (\widetilde{X}_i . \Psi)_g, \quad g \in G, \tag{14.13}$$

where $\{X_i\}$, $i = 1, \dots, n$, is an orthonormal basis of $\mathfrak{p}$, and where $\widetilde{X}_i$ is the left-invariant vector field corresponding to X_i.

Remark 14.21. Denote by L_g the natural left action of G on $\mathcal{C}_H^\infty(G, \Sigma_n)$, that is

$$(L_{g_0} \cdot \Psi)(g) := \Psi({g_0}^{-1} g), \quad g_0 \in G, \Psi \in \mathcal{C}_H^\infty(G, \Sigma_n).$$

By the G-invariance of the spinorial Levi-Civita connection, cf. Remark 14.19 (and it is clear from (14.13), since the $\widetilde{X}_i$ are left-invariant vector fields), it follows that the Dirac operator $\mathcal{D}$ is equivariant for this action, that is

$$L_g \circ \mathcal{D} = \mathcal{D} \circ L_g, \quad g \in G.$$

Chapter 15

Explicit computations of the Dirac spectrum

15.1 The general procedure

We use harmonic analysis methods to determine the spectrum of the Dirac operator on model spaces. The seminal work on the subject is due to R. Parthasarathy [Par71], who studied discrete series representations of a noncompact semisimple Lie group by considering kernels of twisted Dirac operators. Later, M. Cahen and S. Gutt started a systematic study of the Dirac operator on compact symmetric spaces, cf. [CG88] (see also [CFG89]).

In the previous chapter, we identified the space of spinor fields on G/H with the space of H-equivariant functions $\mathcal{C}^\infty_H(G, \Sigma_n)$. This is a Banach space for the sup-norm topology (the space Σ_n being endowed with the usual (Spin_n-invariant) Hermitian scalar product $\langle \cdot, \cdot \rangle$). The left action of G on $\mathcal{C}^\infty_H(G, \Sigma_n)$ induces a topological representation which is said to be *induced* by the representation

$$\widetilde{\rho_n} \colon H \longrightarrow \mathrm{GL}_{\mathbb{C}}(\Sigma_n)$$

and is classically denoted by $\mathrm{Ind}_H^G(\widetilde{\rho_n})$.

In order to obtain a continuous unitary representation in a Hilbert space, we consider the Hermitian scalar product on $\mathcal{C}^\infty_H(G, \Sigma_n)$ given by

$$(\Psi, \Psi') := \int_G \langle \Psi(g), \Psi'(g) \rangle dg,$$

where dg is the Haar measure on G. The natural completion of $\mathcal{C}^\infty_H(G, \Sigma_n)$ for this Hermitian scalar product is the Hilbert space $L^2_H(G, \Sigma_n)$ of H-equivariant L^2-functions $G \to \Sigma_n$.

The left action of G on $\mathcal{C}^\infty_H(G, \Sigma_n)$ induces a continuous unitary representation on the Hilbert space $L^2_H(G, \Sigma_n)$. Being unitary, this representation is reducible, hence can be decomposed into irreducible components which turn out to be finite-dimensional.

Definition 15.1. In the following, $\widehat{G}$ is the (countable) set of equivalence classes of unitary irreducible finite-dimensional complex representations of G. (Actually, the condition to be finite dimensional is redundant, cf. Remark 12.14). Any representative of a γ in $\widehat{G}$ is denoted by (ρ_γ, V_γ).

Let us first examine how an irreducible representation (ρ_γ, V_γ) of G may be contained in the induced representation $\mathrm{Ind}_H^G(\widetilde{\rho_n})$.

It is well known that every irreducible representation (ρ_γ, V_γ) of G is contained in the regular left representation ρ_{reg} in the space $\mathcal{C}^0(G,\mathbb{C}) \subset L^2(G,\mathbb{C})$, with multiplicity $\dim V_\gamma$. Indeed, as explained in the proof of the Peter–Weyl theorem 12.10, any non-trivial vector $F \in V_\gamma^* = \mathrm{Hom}(V_\gamma, \mathbb{C})$ defines a G-equivariant injective homomorphism $V_\gamma \to \mathcal{C}^0(G,\mathbb{C})$, defined by

$$v \longmapsto (g \longmapsto F(\rho_\gamma(g^{-1}) \cdot v)).$$

The following result can be considered as a generalization.

Proposition 15.2 (Frobenius reciprocity). *For any $\gamma \in \widehat{G}$, let $\mathrm{Hom}_H(V_\gamma, \Sigma_n)$ be the space of H-equivariant homomorphisms $V_\gamma \to \Sigma_n$, that is*

$$\mathrm{Hom}_H(V_\gamma, \Sigma_n) = \{A \in \mathrm{Hom}(V_\gamma, \Sigma_n);\ A \circ \rho_\gamma(h) = \widetilde{\rho_n}(h) \circ A, h \in H\}.$$

The representation (ρ_γ, V_γ) is contained in the representation

$$(\mathrm{Ind}_H^G(\widetilde{\rho_n}), \mathcal{C}_H^0(G, \Sigma_n))$$

if and only if $\mathrm{Hom}_H(V_\gamma, \Sigma_n) \neq \{0\}$. In this case, the multiplicity is

$$\dim(\mathrm{Hom}_H(V_\gamma, \Sigma_n)).$$

In other words, denoting by $\mathrm{Res}_H^G(\rho_\gamma)$ the restriction of (ρ_γ, V_γ) to the subgroup H, and counting multiplicities, we have

$$\mathrm{mult}(\rho_\gamma, \mathrm{Ind}_H^G(\widetilde{\rho_n})) = \mathrm{mult}(\mathrm{Res}_H^G(\rho_\gamma), \widetilde{\rho_n}).$$

Proof. Let A be a non-trivial vector in $\mathrm{Hom}_H(V_\gamma, \Sigma_n)$. For any $v \in V_\gamma$, it is easy to check that the function

$$G \longrightarrow \Sigma_n, \quad g \longmapsto A(\rho_\gamma(g^{-1}) \cdot v)$$

is H-equivariant and continuous (and even smooth). Thus there exists a homomorphism

$$F_A \colon V_\gamma \longrightarrow \mathcal{C}_H^0(G, \Sigma_n), \quad v \longmapsto (g \mapsto A\left(\rho_\gamma(g^{-1}) \cdot v\right)),$$

which turns out to be G-equivariant, since for any g in G, $F_A \circ \rho_\gamma(g) = L_g \circ F_A$. Hence, by the Schur lemma, F_A is injective (otherwise $\mathrm{Ker}\, F_A$ would be a non-trivial G-invariant subspace of V_γ). So $(\mathrm{Ind}_H^G(\widetilde{\rho_n}), \mathcal{C}_H^0(G, \Sigma_n))$ contains the irreducible G-space $F_A(V_\gamma)$.

Conversely, suppose $(\mathrm{Ind}_H^G(\widetilde{\rho_n}), C_H^0(G, \Sigma_n))$ contains a representation equivalent to (ρ_γ, V_γ): there exists a G-equivariant injective homomorphism

$$F\colon V_\gamma \hookrightarrow C_H^0(G, \Sigma_n).$$

It is easy to verify that the map

$$A\colon V_\gamma \longrightarrow \Sigma_n, \quad v \longmapsto F(v)(e)$$

is a non-trivial vector in $\mathrm{Hom}_H(V_\gamma, \Sigma_n)$ and that, with the previous notations, $F_A = F$. Thus the map $A \mapsto F_A$ defines an isomorphism between $\mathrm{Hom}_H(V_\gamma, \Sigma_n)$ and $\mathrm{Hom}_G(V_\gamma, C_H^0(G, \Sigma_n))$. It follows that the multiplicity of (ρ_γ, V_γ) in $(\mathrm{Ind}_H^G(\widetilde{\rho_n}), C_H^0(G, \Sigma_n))$ is $\dim(\mathrm{Hom}_H(V_\gamma, \Sigma_n))$. □

Corollary 15.3. *For any γ in $\widehat{G}$, the γ-isotypical part of $\mathcal{C}_H^0(G, \Sigma_n)$ is isomorphic to $V_\gamma \otimes \mathrm{Hom}_H(V_\gamma, \Sigma_n)$.*

Proof. With the notations of the previous result, the map

$$V_\gamma \otimes \mathrm{Hom}_H(V_\gamma, \Sigma_n) \longrightarrow \mathcal{C}_H^0(G, \Sigma_n), \quad v \otimes A \longmapsto F_A(v)$$

is a G-equivariant isomorphism into the "locally finite part" of $\mathcal{C}_H^0(G, \Sigma_n)$, that is the G-invariant subspace of $\mathcal{C}_H^0(G, \Sigma_n)$ spanned by all the finite-dimensional G-spaces contained in $\mathcal{C}_H^0(G, \Sigma_n)$. □

Identifying the "locally finite part" of $\mathcal{C}_H^0(G, \Sigma_n)$ with

$$\bigoplus_{\gamma \in \widehat{G}} V_\gamma \otimes \mathrm{Hom}_H(V_\gamma, \Sigma_n),$$

a generalization of the Peter–Weyl theorem (cf. [BtD85], Theorem 5.7, Chapter III), yields

Theorem 15.4. *The locally finite part*

$$\bigoplus_{\gamma \in \widehat{G}} V_\gamma \otimes \mathrm{Hom}_H(V_\gamma, \Sigma_n)$$

of $\mathcal{C}_H^0(G, \Sigma_n)$ is dense (for the sup-norm topology) in $\mathcal{C}_H^0(G, \Sigma_n)$. The unitary representation $L_H^2(G, \Sigma_n)$ splits into the Hilbert sum

$$L_H^2(G, \Sigma_n) = \bigoplus_{\gamma \in \widehat{G}} V_\gamma \otimes \mathrm{Hom}_H(V_\gamma, \Sigma_n).$$

Proof. The Peter–Weyl theorem implies, after tensoring with the finite-dimensional Hermitian space Σ_n, that the direct sum

$$\bigoplus_{\gamma\in\widehat{G}} V_\gamma \otimes \mathrm{Hom}(V_\gamma, \Sigma_n)$$

is both L^2-orthogonal (see also the orthogonality relations from Theorem 12.3) and dense inside $\mathcal{C}^0(G, \Sigma_n)$, both for the sup norm and for the L^2 norm.

Consider the linear map

$$P_H : \mathcal{C}^0(G, \Sigma_n) \longrightarrow \mathcal{C}^0(G, \Sigma_n), \quad (P_H f)(g) := \int_H \widetilde{\rho_n}(h) f(gh) dh.$$

It is immediate to see that this map takes values in the space of H-equivariant functions $\mathcal{C}^0_H(G, \Sigma_n)$, is continuous with respect to the $\mathcal{C}^0$ topology, and also with respect to the L^2 topology. Moreover, P_H acts as the identity on $\mathcal{C}^0_H(G, \Sigma_n)$, and is self-adjoint with respect to the L^2 inner product. It follows that P_H is the orthogonal projector onto $L^2_H(G, \Sigma_n)$.

Let $v \otimes A \in V_\gamma \otimes \mathrm{Hom}(V_\gamma, \Sigma_n) \hookrightarrow \mathcal{C}^0(G, \Sigma_n)$. Then from the definition,

$$P_H(v \otimes A) = v \otimes \int_H \widetilde{\rho_n}(h) \circ A \circ \rho_\gamma(h^{-1}) dh \in V_\gamma \otimes \mathrm{Hom}_H(V_\gamma, \Sigma_n).$$

Thus, if $P_{H,\gamma} : \mathrm{Hom}(V_\gamma, \Sigma_n) \to \mathrm{Hom}_H(V_\gamma, \Sigma_n)$ denotes the orthogonal projector, we have that the restriction of P_H to $V_\gamma \otimes \mathrm{Hom}(V_\gamma, \Sigma_n)$ equals $1 \otimes P_{H,\gamma}$. We have thus constructed a self-adjoint projector preserving each term of the orthogonal Hilbert sum

$$\bigoplus_{\gamma\in\widehat{G}} V_\gamma \otimes \mathrm{Hom}(V_\gamma, \Sigma_n) = L^2(G, \Sigma_n).$$

It follows that $L^2_H(G, \Sigma_n)$ (the image of this projector) decomposes into the orthogonal Hilbert sum of the images

$$\bigoplus_{\gamma\in\widehat{G}} V_\gamma \otimes \mathrm{Hom}_H(V_\gamma, \Sigma_n),$$

as claimed. □

This result may be also re-formulated as the decomposition of the Hilbert space $L^2_H(G, \Sigma_n)$, viewed as a $\mathcal{C}^0(G)$-module, by means of the idempotent elements of the convolution algebra $\mathcal{C}^0(G)$ defined by irreducible characters, cf. [BtD85], Theorem (5.10), Chapter III.

The extension of the Dirac operator to $L^2_H(G, \Sigma_n)$, being G-equivariant (cf. Remark 14.21), it leaves invariant the γ-isotypical parts $V_\gamma \otimes \mathrm{Hom}_H(V_\gamma, \Sigma_n)$ (which are in the domain of $\mathcal{D}$, since they consist of $\mathcal{C}^\infty$-functions). More precisely, we have the following result.

Proposition 15.5. *The Dirac operator leaves invariant the space*

$$V_\gamma \otimes \mathrm{Hom}_H(V_\gamma, \Sigma_n)$$

and

$$\mathcal{D}|_{V_\gamma \otimes \mathrm{Hom}_H(V_\gamma, \Sigma_n)} = \mathrm{Id} \otimes \mathcal{D}_\gamma,$$

where $\mathcal{D}_\gamma$ is the operator

$$\begin{aligned} \mathcal{D}_\gamma \colon \mathrm{Hom}_H(V_\gamma, \Sigma_n) &\longrightarrow \mathrm{Hom}_H(V_\gamma, \Sigma_n), \\ A &\longmapsto -\sum_{i=1}^n X_i \cdot (A \circ \rho_{\gamma *}(X_i)), \end{aligned}$$

$\{X_i\}$ being some orthonormal basis of $\mathfrak{p}$.

Proof. Let $v \otimes A$ be an element in $V_\gamma \otimes \mathrm{Hom}_H(V_\gamma, \Sigma_n)$. From (14.13), one has

$$\mathcal{D}(v \otimes A) = \sum_{i=1}^n X_i \cdot (\widetilde{X}_i \cdot (v \otimes A)).$$

But

$$\begin{aligned} \widetilde{X}_i \cdot (v \otimes A)(g) &= \frac{d}{dt}\{A(\rho_\gamma(\exp(-tX_i)g^{-1}) \cdot v)\}\big|_{t=0}, \\ &= A\Big(\Big(\frac{d}{dt}\{\rho_\gamma(\exp(-tX_i))\}\big|_{t=0} \circ \rho_\gamma(g^{-1})\Big) \cdot v\Big) \\ &= -(A \circ \rho_{\gamma *}(X_i))(\rho_\gamma(g^{-1}) \cdot v). \end{aligned}$$

Hence

$$\mathcal{D}(v \otimes A) = v \otimes \Big(-\sum_{i=1}^n X_i \cdot (A \circ \rho_{\gamma *}(X_i))\Big). \qquad \square$$

Note that since $\mathcal{D}$ is formally self-adjoint, the restriction of $\mathcal{D}$ to any non-trivial space $V_\gamma \otimes \mathrm{Hom}_H(V_\gamma, \Sigma_n)$ is a Hermitian operator on the finite-dimensional Hermitian space

$$(V_\gamma \otimes \mathrm{Hom}_H(V_\gamma, \Sigma_n), (\cdot,\cdot)|_{V_\gamma \otimes \mathrm{Hom}_H(V_\gamma, \Sigma_n)}),$$

hence, is diagonalizable with real eigenvalues. By Theorem 15.4 and the above result, one concludes that

$$\mathrm{Spec}(\mathcal{D}) = \bigcup_{\gamma\in\widehat{G}} \mathrm{Spec}(\mathcal{D}_{|V_\gamma\otimes\mathrm{Hom}_H(V_\gamma,\Sigma_n)}) = \bigcup_{\gamma\in\widehat{G}} \mathrm{Spec}(\mathcal{D}_\gamma).$$

(This implies the well-known property that $\mathcal{D}$ has a discrete and real spectrum). But because of the symmetric space structure, one can proceed further.

Proposition 15.6. *The spectrum of $\mathcal{D}$ is symmetric with respect to the origin. Furthermore it is completely determined by the spectrum of its square $\mathcal{D}^2$.*

Proof. Let λ be an eigenvalue of $\mathcal{D}$ and let Ψ be a non-trivial function in $\mathcal{C}^\infty_H(G,\Sigma_n)$ such that $\mathcal{D}\Psi = \lambda\Psi$. Consider the function $\sigma^*\Psi$, where σ is the involutive automorphism of G defining the symmetric structure. Since

$$\sigma(\exp(tX_i)) = \exp(t\sigma_*(X_i)) = \exp(-tX_i),$$

cf. (13.1), for any $g\in G$ one gets

$$\begin{aligned}\widetilde{X}_i(\sigma^*\Psi)(g) &= \frac{d}{dt}\{\Psi(\sigma(g\exp(tX_i)))\}\Big|_{t=0}\\ &= \frac{d}{dt}\{\Psi(\sigma(g)\sigma(\exp(tX_i)))\}\Big|_{t=0}\\ &= \frac{d}{dt}\{\Psi(\sigma(g)\exp(-tX_i))\}\Big|_{t=0}\\ &= -\widetilde{X}_i(\Psi)(\sigma(g)).\end{aligned}$$

Hence,

$$\mathcal{D}(\sigma^*\Psi)(g) = -\mathcal{D}\Psi(\sigma(g)) = -\lambda\Psi(\sigma(g)) = -\lambda\sigma^*\Psi(g),$$

so the spectrum of $\mathcal{D}$ is symmetric with respect to the origin.

The operator $\mathcal{D}^2$ is also G-equivariant, hence the spaces $V_\gamma\otimes\mathrm{Hom}_H(V_\gamma,\Sigma_n)$ is invariant, and by Proposition 15.5

$$\mathcal{D}^2|_{V_\gamma\otimes\mathrm{Hom}_H(V_\gamma,\Sigma_n)} = \mathrm{Id}\otimes\mathcal{D}^2_\gamma. \tag{15.1}$$

Thus

$$\mathrm{Spec}(\mathcal{D}^2) = \bigcup_{\gamma\in\widehat{G}} \mathrm{Spec}(\mathcal{D}^2{}_{|V_\gamma\otimes\mathrm{Hom}_H(V_\gamma,\Sigma_n)}) = \bigcup_{\gamma\in\widehat{G}} \mathrm{Spec}({\mathcal{D}_\gamma}^2).$$

As noted before, there exists an orthonormal basis of $V_\gamma\otimes\mathrm{Hom}_H(V_\gamma,\Sigma_n)$ which diagonalizes $\mathcal{D}|_{V_\gamma\otimes\mathrm{Hom}_H(V_\gamma,\Sigma_n)}$; but it also diagonalizes $\mathcal{D}^2|_{V_\gamma\otimes\mathrm{Hom}_H(V_\gamma,\Sigma_n)}$. This leads to the conclusion that

$$\mathrm{Spec}(\mathcal{D}) = \bigcup_{\gamma\in\widehat{G}} \{\pm\sqrt{\mu};\ \mu\in\mathrm{Spec}({\mathcal{D}_\gamma}^2)\}. \qquad\square$$

Now, the following formula for the square of $\mathcal{D}_\gamma$ holds. This is a particular case of the Parthasarathy formula (cf. Proposition 3.1 in [Par71]).

Proposition 15.7 (Parthasarathy formula). *For any A in* $\mathrm{Hom}_H(V_\gamma, \Sigma_n)$,

$$\mathcal{D}_\gamma^2 A = A \circ \mathcal{C}(\rho_\gamma) + \frac{\mathrm{Scal}}{8} A,$$

where $\mathcal{C}(\rho_\gamma)$ is the Casimir operator of the representation ρ_γ.

Proof. We follow S. Sulanke [Sul79]. We first recall the expression of the Casimir operator. Let $\{E_a\}$ be a basis of $\mathfrak{h}$ such that $\{X_i, E_a\}$ is an orthonormal basis of $(\mathfrak{g}, -\mathrm{B})$, B being the Killing form of $\mathfrak{g}$. The Casimir operator of a representation (ρ, V) is the endomorphism of V

$$\mathcal{C}(\rho) := -\sum_i \rho_*(X_i) \circ \rho_*(X_i) - \sum_a \rho_*(E_a) \circ \rho_*(E_a).$$

Now, for A in $\mathrm{Hom}_H(V_\gamma, \Sigma_n)$,

$$\begin{aligned}
{\mathcal{D}_\gamma}^2 A &= \sum_{i,j} X_i \cdot X_j \cdot (A \circ \rho_*(X_j) \circ \rho_*(X_i)) \\
&= \frac{1}{2} \sum_{i,j} X_i \cdot X_j \cdot (A \circ \rho_*(X_j) \circ \rho_*(X_i)) \\
&\quad + \frac{1}{2} \sum_{i,j} X_j \cdot X_i \cdot (A \circ \rho_*(X_i) \circ \rho_*(X_j)) \\
&= -\frac{1}{2} \sum_{i,j} X_i \cdot X_j \cdot (A \circ \rho_*([X_i, X_j])) \\
&\quad - \sum_i A \circ \rho_*(X_i) \circ \rho_*(X_i) \\
&= -\frac{1}{2} \sum_{i,j} X_i \cdot X_j \cdot (A \circ \rho_*([X_i, X_j])) \\
&\quad + A \circ \mathcal{C}(\rho_\gamma) \\
&\quad + \sum_a A \circ \rho_*(E_a) \circ \rho_*(E_a).
\end{aligned}$$

But

$$A \circ \rho_*([X_i, X_j]) = \widetilde{\rho_n}_*([X_i, X_j]) \circ A,$$

since $[X_i, X_j] \in \mathfrak{h}$ and A is H-equivariant. Moreover, recall that

$$\widetilde{\rho_n}_* = \rho_{n*} \circ \tilde{\alpha}_*,$$
$$\tilde{\alpha}_* = {\xi_*}^{-1} \circ \alpha_*,$$

and

$$\alpha_*(X) = \mathrm{ad}(X)|_{\mathfrak{p}}, \quad X \in \mathfrak{h}.$$

Hence,

$$\alpha_*([X_i, X_j]) = -\frac{1}{2} \sum_{k,l} \mathrm{B}([[X_i, X_j], X_k], X_l) X_k \wedge X_l$$

and

$$\tilde{\alpha}_*([X_i, X_j]) = -\frac{1}{4} \sum_{k,l} \mathrm{B}([[X_i, X_j], X_k], X_l) X_k \cdot X_l.$$

Thus

$$\sum_{i,j} X_i \cdot X_j \cdot \widetilde{\rho_n}_*([X_i, X_j]) = -\frac{1}{4} \sum_{i,j,k,l} \mathrm{B}([[X_i, X_j], X_k], X_l) X_i \cdot X_j \cdot X_k \cdot X_l.$$

On the other hand,

$$\sum_a A \circ \rho_*(E_a) \circ \rho_*(E_a) = \sum_a \widetilde{\rho_n}_*(E_a) \circ \widetilde{\rho_n}_*(E_a) \circ A.$$

But

$$\alpha_*(E_a) = -\frac{1}{2} \sum_{i,j} \mathrm{B}([E_a, X_i], X_j) X_i \wedge X_j$$

and

$$\tilde{\alpha}_*(E_a) = -\frac{1}{4} \sum_{i,j} \mathrm{B}([E_a, X_i], X_j) X_i \cdot X_j.$$

Consequently,

$$
\begin{aligned}
&\sum_a \widetilde{\rho}_{n*}(E_a) \circ \widetilde{\rho}_{n*}(E_a) \\
&\quad = \frac{1}{16} \sum_a \sum_{i,j,k,l} \mathrm{B}([E_a, X_i], X_j)\mathrm{B}([E_a, X_k], X_l) X_i \cdot X_j \cdot X_k \cdot X_l \\
&\quad = -\frac{1}{16} \sum_{i,j,k,l} \mathrm{B}\Big(\sum_a \mathrm{B}([X_i, E_a], X_j)[E_a, X_k], X_l\Big) X_i \cdot X_j \cdot X_k \cdot X_l \\
&\quad = \frac{1}{16} \sum_{i,j,k,l} \mathrm{B}\Big(\sum_a \mathrm{B}(E_a, [X_i, X_j])[E_a, X_k], X_l\Big) X_i \cdot X_j \cdot X_k \cdot X_l \\
&\quad = -\frac{1}{16} \sum_{i,j,k,l} \mathrm{B}([[X_i, X_j], X_k], X_l) X_i \cdot X_j \cdot X_k \cdot X_l,
\end{aligned}
$$

the third equality being obtained using the $\mathrm{Ad}(G)$-invariance of the Killing form. Thus

$$
\mathcal{D}_\gamma^2 A = A \circ \mathcal{C}(\rho_\gamma) + \Big(\frac{1}{16} \sum_{i,j,k,l} \mathrm{B}([[X_i, X_j], X_k], X_l) X_i \cdot X_j \cdot X_k \cdot X_l\Big) \circ A.
$$

Now, we can avoid superfluous computations by observing that according to Proposition 14.9, the term inside the parentheses in the above equality is equal to

$$
\frac{1}{16} \sum_{i,j,k,l} R_p(\overline{X}_i, \overline{X}_j, \overline{X}_k, \overline{X}_l) \overline{X}_i \cdot \overline{X}_j \cdot \overline{X}_k \cdot \overline{X}_l,
$$

where $\overline{X}_i = \Phi(X_i)$, Φ being the isometry between $(\mathfrak{p}, -\mathrm{B}|_{\mathfrak{p}})$ and $(\mathrm{T}_p M, g_{M,p})$ defined in (13.2). By the proof of the Schrödinger–Lichnerowicz formula, cf. (2.49), the above expression is equal to $\frac{\mathrm{Scal}}{8}\mathrm{Id}$. □

It is also well known that the Casimir operator of a representation (ρ, V) of G commutes with any element of $\rho(\mathfrak{g})$. Hence, by the Schur lemma, the Casimir operator of an irreducible representation (ρ_γ, V_γ) is a scalar multiple of the identity

$$
\mathcal{C}(\rho_\gamma) = c_\gamma \mathrm{Id},
$$

the constant c_γ depending only on the equivalence class of ρ_γ. Since $\mathrm{Scal} = \frac{1}{2}\dim M$, cf. (14.9), we can re-formulate the above proposition as

Proposition 15.8. *For any $A \in \mathrm{Hom}_H(V_\gamma, \Sigma_n)$, one has*

$$
\mathcal{D}_\gamma^2 A = \Big(c_\gamma + \frac{\dim M}{16}\Big) A,
$$

where c_γ is the eigenvalue of the Casimir operator of ρ_γ.

In order to conclude, we only have to determine those of the $\mathrm{Hom}_H(V_\gamma, \Sigma_n)$ which are non-trivial.

Proposition 15.9. *Let*

$$(\widetilde{\rho_n}, \Sigma_n) = \bigoplus_{i=1}^{p} (\widetilde{\rho_{n,i}}, \Sigma_{n,i})$$

be the decomposition into irreducible components of the representation

$$\widetilde{\rho_n} \colon H \to \mathrm{GL}_{n,\mathbb{C}}(\Sigma_n).$$

For any $i = 1, \ldots, p$, let

$$\mathrm{mult}(\widetilde{\rho_{n,i}}, \mathrm{Res}_G^H(\rho_\gamma))$$

be the multiplicity of the irreducible representation $\widetilde{\rho_{n,i}}$ in the restriction $\mathrm{Res}_G^H(\rho_\gamma)$ of the representation ρ_γ to H. Then

$$\dim(\mathrm{Hom}_H(V_\gamma, \Sigma_n)) = \sum_{i=1}^{p} \mathrm{mult}(\widetilde{\rho_{n,i}}, \mathrm{Res}_G^H(\rho_\gamma)).$$

Proof. Consider

$$(\mathrm{Res}_G^H(\rho_\gamma), V_\gamma) = \bigoplus_{i=1}^{q} (\mathrm{Res}_G^H(\rho_\gamma)_i, V_{\gamma,i}),$$

the irreducible decomposition of the representation $(\mathrm{Res}_G^H(\rho_\gamma), V_\gamma)$. For any $i = 1, \ldots, q$, one identifies $\mathrm{Hom}_H(V_{\gamma,i}, \Sigma_n)$ with a subspace of $\mathrm{Hom}_H(V_\gamma, \Sigma_n)$ by considering the injective homomorphism

$$\begin{aligned} \mathrm{Hom}_H(V_{\gamma,i}, \Sigma_n) &\longrightarrow \mathrm{Hom}_H(V_\gamma, \Sigma_n), \\ A_i &\longmapsto A \equiv \begin{cases} A_i & \text{on } V_{\gamma,i}, \\ 0 & \text{on } V_{\gamma,j},\ j \neq i. \end{cases} \end{aligned}$$

Under this identification, one obtains the decomposition

$$\mathrm{Hom}_H(V_\gamma, \Sigma_n) = \bigoplus_{j=1}^{q} \mathrm{Hom}_H(V_{\gamma,j}, \Sigma_n).$$

Thus,

$$\mathrm{Hom}_H(V_\gamma, \Sigma_n) = \bigoplus_{j=1}^{q} \Big(\bigoplus_{i=1}^{p} \mathrm{Hom}_H(V_{\gamma,j}, \Sigma_{n,i}) \Big),$$

and

$$\dim(\mathrm{Hom}_H(V_\gamma, \Sigma_n)) = \sum_{i=1}^{p}\Big(\sum_{j=1}^{q}\dim(\mathrm{Hom}_H(V_{\gamma,j}, \Sigma_{n,i}))\Big).$$

But by the Schur lemma, $\dim(\mathrm{Hom}_H(V_{\gamma,j}, \Sigma_{n,i})) = 1$ if $V_{\gamma,j}$ and $\Sigma_{n,i}$ are equivalent, and zero otherwise, so, in any case,

$$\dim(\mathrm{Hom}_H(V_{\gamma,j}, \Sigma_{n,i})) = \dim(\mathrm{Hom}_H(\Sigma_{n,i}, V_{\gamma,j})).$$

We conclude that

$$\sum_{j=1}^{q}\dim(\mathrm{Hom}_H(V_{\gamma,j}, \Sigma_{n,i})) = \sum_{j=1}^{q}\dim(\mathrm{Hom}_H(\Sigma_{n,i}, V_{\gamma,j})),$$

is the multiplicity of $\widetilde{\rho_{n,i}}$ in $\mathrm{Res}_G^H(\rho_\gamma)$. □

We can summarize all the previous results as

Theorem 15.10. *Let (M, g_M) be a spin compact, simply connected, n-dimensional irreducible symmetric space and (G, H, σ) the corresponding accurate triple: $M = G/H$, with G compact and simply connected, and $\sigma: G \to G$ is an involution such that H is the identity component of the subgroup of fixed elements of σ.*

Denote by

$$\Sigma_n = \bigoplus_{i=1}^{p}\Sigma_{n,i}$$

the decomposition of the spinor space Σ_n into irreducible H-modules, under the action of the group H induced by the spin structure.

The spectrum of the Dirac operator is

$$\mathrm{Spec}(\mathcal{D}) = \Big\{ \pm\sqrt{c_\gamma + \frac{n}{16}};\\ \gamma \in \widehat{G}\colon \textit{there exists } i \in \{1, \ldots, p\} \textit{such that } \mathrm{mult}(V_\gamma|_H, \Sigma_{n,i}) \geq 1\Big\},$$

where $\widehat{G}$ is the set of equivalence classes of irreducible unitary complex representations of G, c_γ the eigenvalue of the Casimir operator of any representative (ρ_γ, V_γ) of $\gamma \in \widehat{G}$, and $\mathrm{mult}(V_\gamma|_H, \Sigma_{n,i})$ is the multiplicity of the irreducible H-module $\Sigma_{n,i}$ in the H-module V_γ.

Remark 15.11. From (15.1) and Proposition 15.8, it is clear that the multiplicity of an eigenvalue of $\mathcal{D}^2$ of the form $c_\gamma + \frac{n}{16}$ is at least equal to

$$\dim(V_\gamma)\dim(\mathrm{Hom}_H(V_\gamma, \Sigma_n));$$

it may be greater, since two distinct elements γ and γ' in $\widehat{G}$ may verify $c_\gamma = c_{\gamma'}$ (cf. for instance the example of complex projective spaces below).

Thus, the spectrum can be in principle determined by means of the following steps.

(1) Determine the sets $\widehat{G}$ and $\widehat{H}$: equivalence classes of irreducible representations can be described in terms of dominant weights relative to the choice of a maximal torus.

(2) Determine the "branching rules" giving the decomposition of an irreducible G-representation into irreducible H-representations. This is a classical problem in representation theory, but far from being obvious in general: only a few results are known for classical groups.

(3) Decompose the representation

$$\widetilde{\rho_n}\colon H \longrightarrow \mathrm{GL}_{\mathbb{C}}(\Sigma_n),$$

into irreducible H-representations. This can be done by determining the maximal vectors (cf. Theorem 12.58) of the representation, or by using the following result of R. Parthasarathy: an H-dominant weight β_H occurs in the decomposition if and only if there exists an element w in the Weyl group of G such that $\beta_H = w \cdot \delta_G - \delta_H$, where δ_G (resp. δ_H) is the half-sum of the positive roots of G (resp. H). Furthermore, it occurs with multiplicity one (cf. Lemma 2.2. in [Par71]).

(4) Using Proposition (15.9), determine the γ's in $\widehat{G}$ such that $\mathrm{Hom}_H(V_\gamma, \Sigma_n) \neq \{0\}$.

(5) For those γ, compute the Casimir eigenvalue c_γ. By Freudenthal's formula, c_γ can be computed from the dominant weight β_γ:

$$c_\gamma = \langle \beta_\gamma, \beta_\gamma + 2\delta_G \rangle,$$

where $\langle\cdot,\cdot\rangle$ is the natural extension of the Killing form of $\mathfrak{g}$ sign-changed to the space of weights $i\,\mathfrak{t}^*$.

15.2 Spectrum of the Dirac operator on spheres

Using the above method, S. Sulanke computed in [Sul79] the spectrum of the Dirac operator on the sphere $(\mathbb{S}^n, \mathrm{can})$, $n \geq 2$. The two cases n even and n odd have to be considered separately.

First, let us consider the even case, $n = 2m$. Relative to the standard maximal torus, dominant weights of the group Spin_{2m+1} are of the form $(k_1, k_2, \dots, k_m)$, where the k_i are all simultaneously integers or half-integers satisfying the condition $k_1 \geq k_2 \geq \cdots \geq k_{m-1} \geq k_m \geq 0$, cf. Section 12.4.2.

Under the identification $H = \mathrm{Spin}_{2m}$, the representation $\widetilde{\rho_{2m}}$ is the spin representation (cf. Example (14.13, i)), which is known to split into two irreducible components $\rho^{\pm}_{2m}$. Using the branching rules of the group Spin_{2m+1} and its subgroup Spin_{2m}, given in [Žel73], Theorem 2, Chapter XVIII, (see also [Bra97b] and [Bra99]) one shows that the restriction to H of an irreducible representation of Spin_{2m+1} contains one of the representations $\rho^{\pm}_{2m}$ if and only if its dominant weight has the form

$$\frac{1}{2}(2k+1, 1, \dots, 1), \quad k \in \mathbb{N}.$$

In this case, each representation occurs with multiplicity one, hence

$$\dim(\mathrm{Hom}_H(V_\gamma, \Sigma_{2m})) = 2.$$

Denoting by γ_k the element of $\widehat{G}$ with dominant weight $\frac{1}{2}(2k+1, 1, \dots, 1)$, $k \in \mathbb{N}$, the eigenvalues of $\mathcal{D}^2$ are given by

$$c_{\gamma_k} + \frac{n}{16} = \frac{1}{2(2m-1)}(m+k)^2.$$

Furthermore, since $c_{\gamma_k} \neq c_{\gamma_{k'}}$ if $k \neq k'$, the multiplicity of $\frac{1}{2(2m-1)}(m+k)^2$ is $2\dim(V_{\gamma_k})$, cf. Remark 15.11, and the Weyl dimension formula (cf. Theorem 12.67) gives the value of $\dim(V_{\gamma_k})$:

$$\dim(V_{\gamma_k}) = 2^m \binom{2m+k-1}{k}.$$

Let us now consider the odd case, $n = 2m-1$, $m \geq 1$. Relative to the standard maximal torus, the dominant weights of the group Spin_{2m} are of the form $(k_1, k_2, \dots, k_m)$, where the k_i are all simultaneously integers or half-integers verifying the condition $k_1 \geq k_2 \geq \cdots \geq k_{m-1} \geq |k_m|$, cf. Section 12.4.2.

Under the identification $H = \mathrm{Spin}_{2m-1}$, the representation $\widetilde{\rho_{2m-1}}$ is the spin representation, which is irreducible. Using the branching rules of the group Spin_{2m} and its subgroup Spin_{2m-1}, given in [Žel73], Theorem 3, Chapter XVIII, one can show that the restriction to H of an irreducible representation of Spin_{2m} contains the representation ρ_{2m-1} if and only if its dominant weight has the form

$$\frac{1}{2}(2k+1, 1, \dots, 1) \quad \text{or} \quad \frac{1}{2}(2k+1, 1, \dots, -1), \quad k \in \mathbb{N}.$$

Furthermore, the multiplicity of ρ_{2m-1} is equal to one in each case. Denoting by $\gamma_k^{\pm}$ the element in $\widehat{G}$ corresponding to the dominant weight $\frac{1}{2}(2k+1, 1, \dots, \pm 1)$, the eigenvalues of $\mathcal{D}^2$ are given by

$$c_{\gamma_k^+} + \frac{n}{16} = c_{\gamma_k^-} + \frac{n}{16} = \frac{1}{4(m-1)}\Big(\frac{2m-1}{2} + k\Big)^2.$$

Since $c_{\gamma_k^{\pm}} \neq c_{\gamma_{k'}^{\pm}}$, if $k \neq k'$, the multiplicity of the eigenvalue $\frac{1}{4(m-1)}(\frac{2m-1}{2} + k)^2$ is

$$\dim(V_{\gamma_k^+}) + \dim(V_{\gamma_k^-}) = 2^m \binom{2m+k-2}{k}.$$

Thus, thanks to the symmetry of the spectrum of the Dirac operator, we deduce the following theorem.

Theorem 15.12 ([Sul79]). *The eigenvalues of the Dirac operator on* $(\mathbb{S}^n, \mathrm{can})$*, are*

$$\pm\Big(\frac{n}{2} + k\Big), \quad k \in \mathbb{N},$$

with multiplicity

$$2^{[\frac{n}{2}]}\binom{n+k-1}{k}.$$

Note that the lowest eigenvalue of $\mathcal{D}^2$ is $\frac{n^2}{4} = \frac{n}{n-1}\frac{\mathrm{Scal}}{4}$.

There are other approaches to compute the spectrum and determine multiplicities. One is due to C. Bär, cf. [Bär91] and [Bär94]. It is based on the fact that the spinor bundle $\Sigma\mathbb{S}^n$ (which is known to be trivializable, cf. Proposition 14.17), can be trivialized by Killing spinors for the constant $\mu = \frac{1}{2}$ as well as $\mu = -\frac{1}{2}$. This is shown by proving that the curvature of the modified connection defined by

$$\widetilde{\nabla}_X \Psi := \nabla_X \Psi - \mu X \cdot \Psi,$$

vanishes at every point.

Let (Ψ_j) be a trivialization of the bundle $\Sigma\mathbb{S}^n$ by Killing spinors with constant μ. From the Schrödinger–Lichnerowicz-type formula

$$(\mathcal{D}+\mu)^2 = \widetilde{\nabla}^*\widetilde{\nabla} + \frac{1}{4}(n-1)^2,$$

one deduces, considering an orthogonal L^2 eigenbasis $f_0 \equiv 1, f_1, f_2, \dots$ of the Laplace operator, that the functions $f_i\Psi_j$ form an orthogonal L^2 eigenbasis of the operator $(\mathcal{D}+\mu)^2$. Using the explicit expression of the eigenvalues and multiplicities of the Laplace operator on $\mathbb{S}^n$, the eigenvalues and multiplicities of the Dirac operator on $\mathbb{S}^n$ are then derived from those of the operator $(\mathcal{D}+\mu)^2$. See [Bär91] or [Bär94] for details.

Another approach is due to A. Trautman [Tra93] and [Tra95]. It is based on the theory of spin (pin) structures on hypersurfaces (the spin structure on the sphere $(\mathbb{S}^n, \mathrm{can})$ is inherited from the standard spin structure on $\mathbb{R}^{n+1}$) and on a formula relating the Dirac operator on $\mathbb{R}^{n+1}$ (restricted to $\mathbb{S}^n$) to the Dirac operator on $\mathbb{S}^n$, which is analogous to the formula relating the respective Laplace operators. The explicit determination of the eigenvalues, multiplicities, as well as a description of the corresponding eigenspinors is given in [Tra95].

One more approach we know is that introduced by R. Camporesi and A. Higuchi in [CH96]. It is based on a separation-of-variables technique in geodesic polar coordinates which allows to express eigenfunctions on $\mathbb{S}^n$ in terms of those on $\mathbb{S}^{n-1}$. By an induction procedure, the spectrum of the Dirac operator on $\mathbb{S}^n$ is derived from that on $\mathbb{S}^2$. We refer to [CH96] for details.

15.3 Spectrum of the Dirac operator on the complex projective space

The “Harmonic Analysis” method also provides the spectrum of the Dirac operator on the complex projective space $(\mathbb{C}\mathrm{P}^{m=2q+1}, \mathrm{can})$, $q \geq 0$. It was given by M. Cahen, A. Franc, and S. Gutt in [CFG89] and [CFG94].

The standard maximal torus T of $G = \mathrm{SU}_{m+1}$ (cf. Section 12.4.1) is a maximal common torus of G and $H = S(\mathrm{U}_1 \times \mathrm{U}_m)$. Relative to this torus, the dominant weights of G are of the form $(k_1, \dots, k_m)$ where the k_i are integers verifying the condition $k_1 \geq k_2 \geq \cdots \geq k_m \geq 0$, whereas the dominant weights of H are of the form $(\ell_1, \dots, \ell_m)$, where the ℓ_i are integers verifying the condition $\ell_2 \geq \ell_3 \geq \cdots \geq \ell_m \geq 0$ (the dominant weights of H do not have the same expression as in [CFG89] since our choice of the isotropy subgroup H is different).

The decomposition of the representation

$$\widetilde{\rho_{2m}}\colon S(\mathrm{U}_1 \times \mathrm{U}_m) \longrightarrow \mathrm{GL}_{\mathbb{C}}(\Sigma_{2m})$$

into irreducible components is given by

$$\Sigma_{2m} = \bigoplus_{k=0}^{m} \Sigma_{2m,k}, \tag{15.2}$$

where $\Sigma_{2m,k}$ is the H-subspace corresponding to the dominant weight

$$\beta_k = (q + 1 - k, \underbrace{1, \dots, 1}_{k}, 0, \dots, 0), \quad 0 \le k \le m-1,$$

$$\beta_m = (-(q+1), 0, \dots, 0).$$

In the basis of Σ_{2m} introduced in Section 12.4.2, the corresponding maximal vector is w for β_0 and $u_1 \cdots u_k \cdot$ w for β_k, $1 \le k \le m$.

Remark 15.13. Decomposition (15.2) induces the decomposition of the spinor bundle of the spin Kähler manifold $\mathbb{C}\mathrm{P}^m$ into eigenspaces under the action of the Kähler form, cf. Lemma 6.4.

Indeed, the local expression of the Kähler form is invariant under the action of the group U_m, hence, under the action of the group $S(\mathrm{U}_1 \times \mathrm{U}_m)$. Thus, viewed as an endomorphism of Σ_{2m}, it leaves invariant all the subspaces $\Sigma_{2m,k}$ and moreover, by the Schur lemma, its restriction to any of those subspaces is a scalar multiple of the identity. Computation of its action on maximal vectors shows that the eigenvalue corresponding to the subspace $\Sigma_{2m,k}$ is $i(2k-m)$.

The branching rules of the group SU_{m+1} and its subgroup $S(\mathrm{U}_1 \times \mathrm{U}_m)$ are given in [IT78]; they can also be derived from the branching rules given in [CFG89].

Using this result, one shows that an irreducible G-representation ρ_γ has the property that $\mathrm{m}(\rho_\gamma|_H, \Sigma_{2m,k}) \neq 0$, for some integer k, $0 \le k \le m$, if and only if its dominant weight β_γ has the form

$$\beta_\gamma = (2l + q - k, \underbrace{l, \dots, l}_{k-1}, l-1, \dots, l-1),$$

where $1 \le k \le m$ and l is an integer such that $l \ge \max\{1, k-q\}$, or

$$\beta_\gamma = (2l + q + 1 - k, \underbrace{l+1, \dots, l+1}_{k}, l, \dots, l),$$

where $0 \le k \le m-1$ and where l is an integer such that $l \ge \max\{0, k-q\}$. Thus from Theorem 15.10, it can be deduced

Theorem 15.14 ([CFG89, SS93, CFG94]). *The Dirac operator on* $(\mathbb{C}\mathrm{P}^m, \mathrm{can})$*, m odd, has the eigenvalues* $\pm 2\sqrt{\lambda_{l,k}}$*,* $\pm 2\sqrt{\mu_{l,k}}$*, where*

$$\begin{aligned}\lambda_{l,k} &= l^2 + \frac{1}{2}l(3m-2k-1) + \frac{1}{2}(m-k)(m-1)\\ &= \Big(l + \frac{m-1}{2}\Big)(l+m-k),\end{aligned}$$

with $k \in \{1, \dots, m\}$*, and* $l \in \mathbb{N}$ *such that* $l \geq \max\{1, k - \frac{m-1}{2}\}$*, and*

$$\begin{aligned}\mu_{l,k} &= l^2 + \frac{1}{2}l(3m-2k+1) + \frac{1}{2}(m-k)(m+1)\\ &= \Big(l + \frac{m+1}{2}\Big)(l+m-k),\end{aligned}$$

with $k \in \{0, \dots, m-1\}$*, and* $l \in \mathbb{N}$ *such that* $l \geq \max\{0, k - \frac{m-1}{2}\}$*.*

Note that for $m = 1$ one recovers the spectrum of $(\mathbb{S}^2, \frac{1}{4}\mathrm{can}) \simeq (\mathbb{C}\mathrm{P}^1, \mathrm{can})$, and that the lowest eigenvalue of $\mathcal{D}^2$ is

$$4\lambda_{1,\frac{m+1}{2}} = 4\mu_{0,\frac{m-1}{2}} = (m+1)^2 = \frac{m+1}{m}\frac{\mathrm{Scal}}{4}.$$

There are relations between the two series $\lambda_{l,k}$ and $\mu_{l,k}$, for instance

$$\lambda_{l+1,k+1} = \mu_{l,k} = \mu_{l-k+\frac{m-1}{2},m-k-1}.$$

For this reason, it seems difficult to deduce multiplicities (using Remark 15.11), since distinct elements γ and γ' in $\widehat{G}$ may have the same Casimir eigenvalue c_γ.

Using a different approach, a formula for multiplicities was obtained by B. Ammann and C. Bär (as an example) in [AB98]. Indeed, endowing the sphere $\mathbb{S}^{2m+1}$ with the Berger metric g_ℓ with parameter ℓ, the Hopf fibration

$$\mathbb{S}^1 \longrightarrow \mathbb{S}^{2m+1} \longrightarrow \mathbb{C}\mathrm{P}^m,$$

becomes a Riemannian submersion with totally geodesic fibers of length $2\pi\ell$.

It turns out that, for m odd, upon collapsing the fibers of this $\mathbb{S}^1$-principal bundle (that is, making ℓ tend to zero), eigenvalues of the Dirac operator on $(\mathbb{C}\mathrm{P}^m, \mathrm{can})$ are exactly the limits of the eigenvalues of the Dirac operator of $(\mathbb{S}^{2m+1}, g_\ell)$ that have a finite limit.

From the explicit determination of the eigenvalues and multiplicities of the Dirac operator on $(\mathbb{S}^{2m+1}, g_\ell)$ given by C. Bär in [Bär96], one obtains

Theorem 15.15 ([AB98]). *The Dirac operator on* $(\mathbb{C}\mathrm{P}^m, \mathrm{can})$*, m odd, has the eigenvalues*

$$\pm 2\sqrt{\Big(a_1 + \frac{m+1}{2}\Big)\Big(a_2 + \frac{m+1}{2}\Big)},$$

where $a_1, a_2 \in \mathbb{N}^*$, $|a_1 - a_2| \leq \frac{m-1}{2}$, with multiplicity

$$\frac{(a_1+m)!(a_2+m)!(a_1+a_2+m+1)}{a_1!a_2!\left(a_1+\frac{m+1}{2}\right)\left(a_2+\frac{m+1}{2}\right)m!\left(a_1-a_2+\frac{m-1}{2}\right)!\left(a_2-a_1+\frac{m-1}{2}\right)!}.$$

15.4 Spectrum of the Dirac operator on the quaternionic projective space

The spectrum of the Dirac operator on the quaternionic projective space $(\mathbb{H}\mathrm{P}^m, \mathrm{can})$, $m \geq 1$, was computed in [Mil92] using the above "Harmonic Analysis" method.

The standard maximal torus T of $G = \mathrm{Sp}_{m+1}$ (cf. Section 12.4.3) is a maximal common torus of G and $H = \mathrm{Sp}_1 \times \mathrm{Sp}_m$. Relative to this torus, the dominant weights of G are of the form $(k_1, \dots, k_{m+1})$, where the k_i are integers verifying the condition $k_1 \geq k_2 \geq \dots \geq k_{m+1} \geq 0$, whereas the dominant weights of H are of the form $(\ell_1, \dots, \ell_{m+1})$, where the ℓ_i are integers verifying the condition $\ell_1 \geq 0$ and $\ell_2 \geq \ell_3 \geq \dots \geq \ell_{m+1} \geq 0$. The decomposition of the representation

$$\widetilde{\rho_{4m}}\colon \mathrm{Sp}_1 \times \mathrm{Sp}_m \longmapsto \mathrm{GL}_{\mathbb{C}}(\Sigma_{4m}),$$

into irreducible components is given by

$$\Sigma_{4m} = \bigoplus_{k=0}^{m} \Sigma_{4m,k}, \tag{15.3}$$

where $\Sigma_{4m,k}$ is the H-subspace corresponding to the dominant weight

$$\beta_k = (k, 1, \dots, 1, \underbrace{0, \dots, 0}_{k}).$$

Remark 15.16. Decomposition (15.3) was previously stated, in a different context, in [BS83] and [Wan89]. As in the Kähler case, it corresponds to the decomposition of the spinor bundle of the spin quaternion-Kähler manifold $\mathbb{H}\mathrm{P}^m$ into eigenspaces under the action of the fundamental 4-form, cf. Section 7.2.

Indeed, the local expression of the fundamental 4-form is invariant under the action of the group $\mathrm{Sp}_1 \cdot \mathrm{Sp}_m$, hence, under the action of the group $\mathrm{Sp}_1 \times \mathrm{Sp}_m$. Thus, viewed as an endomorphism of Σ_{4m}, it leaves invariant all the subspaces $\Sigma_{4m,k}$ and moreover, by the Schur lemma, its restriction to any of those subspaces is a scalar multiple of the identity.

Computation of its action on maximal vectors, shows that the eigenvalue corresponding to the subspace $\Sigma_{4m,k}$ is $6m - 4k(k+2)$ (see [HM95b] for details).

The branching rules of the group Sp_{m+1} and its subgroup $\mathrm{Sp}_1 \times \mathrm{Sp}_m$ are given in [Lep71] and [Tsu81]. Using the result of [Tsu81], one shows that an irreducible G-representation ρ_γ is such that $\mathrm{m}(\rho_\gamma|_H, \Sigma_{4m,k}) \neq 0$, for some integer k, $0 \leq k \leq m$, if and only if its dominant weight β_γ has the form

$$\beta_\gamma = (p + q - 1, q, 1, \dots, 1, \underbrace{0, \dots, 0}_{p}),$$

where $1 \leq p \leq m-1$, and $q \in \mathbb{N}^*$, or

$$\beta_\gamma = (p + q + 1, q, 1, \dots, 1, \underbrace{0, \dots, 0}_{p}),$$

where $0 \leq p \leq m-1$, and $q \in \mathbb{N}$ if $p = m-1$, $q \in \mathbb{N}^*$ otherwise. Thus, from Theorem 15.10, one can deduce

Theorem 15.17 ([Mil92]). *The Dirac operator on* $(\mathbb{H}\mathrm{P}^m, \mathrm{can})$ *has the eigenvalues* $\pm\sqrt{\lambda_{p,q}}$, $\pm\sqrt{\mu_{p,q}}$, *where*

$$\begin{aligned}\lambda_{p,q} &= (2m + p + 2q)^2 - (p+2)^2 \\ &= 4(m + p + q + 1)(m + q - 1),\end{aligned}$$

where $1 \leq p \leq m-1$ *and* $q \in \mathbb{N}^*$,

$$\begin{aligned}\mu_{p,q} &= (2m + p + 2q + 1)^2 - (p-1)^2 \\ &= 4(m + p + q)(m + q + 1),\end{aligned}$$

where $1 \leq p \leq m$, *and* $q \in \mathbb{N}$ *if* $p = m$, $q \in \mathbb{N}^*$ *otherwise.*

Note that for $m = 1$, one recovers the spectrum of $(\mathbb{S}^4, \frac{1}{4}\mathrm{can}) \simeq (\mathbb{H}\mathrm{P}^1, \mathrm{can})$, and that the lowest eigenvalue of $\mathcal{D}^2$ is

$$\lambda_{1,1} = 4m(m+3) = \frac{m+3}{m+2}\frac{\mathrm{Scal}}{4}.$$

There are relations between the two series $\lambda_{l,k}$ and $\mu_{l,k}$; for instance we have $\lambda_{p,q} = \mu_{p+3,q-2}$. For this reason, it seems difficult to deduce multiplicities with this method. Up to our knowledge, there is no known formula giving multiplicities.

15.5 Other examples of spectra

There is a generalization of the above method to spin compact homogeneous spaces due to C. Bär, cf. [Bär91] and [Bär92a].

All results about spin structures and spinor bundles are mainly the same as in Chapter 14, but the Dirac operator has a more complicated expression cf. [Bär92a], Theorem 1), because of the expression of the Levi-Civita connection (cf. [KN69] Theorem 3.3, Chapter X).

All the harmonic analysis results of Chapter 15 remain obviously true, but the expression of $\mathcal{D}_\gamma$ in Proposition 15.5 has to be changed to (cf. Proposition 1, [Bär92a])

$$\begin{aligned}\mathcal{D}_\gamma(A) = &-\sum_{i=1}^{n} X_i \cdot (A \circ \rho_{\gamma *}(X_i)) \\ &+ \Big(\sum_{i=1}^{n} \beta_i X_i + \sum_{1 \leq i < j < k \leq n} \alpha_{ijk} X_i \cdot X_j \cdot X_k\Big) \cdot A,\end{aligned}$$

where

$$\alpha_{ijk} = \frac{1}{4}(\langle [X_i, X_j]_{\mathfrak{p}}, X_k\rangle + \langle [X_j, X_k]_{\mathfrak{p}}, X_i\rangle + \langle [X_k, X_i]_{\mathfrak{p}}, X_j\rangle)$$

and

$$\beta_i = \frac{1}{2}\sum_{j=1}^{n} \langle [X_j, X_i]_{\mathfrak{p}}, X_j\rangle.$$

The main difference with respect to the "symmetric" case, is that we cannot proceed further here, since there is no result analogous to Theorem 15.10. With the help of the above formula, C. Bär computed the spectrum of the Dirac operator on odd-dimensional spheres with Berger metrics [Bär96].

Actually, there are only few examples where Dirac spectra can be computed explicitly. Examples, known to us, of Dirac spectra on closed Riemannian manifolds are listed in Table 6.

Table 6

Spin manifold	**References**	**Remarks**
flat tori $\mathbb{R}^n/\Gamma$	[Fri84]	homogeneous space
sphere $(\mathbb{S}^n, \mathrm{can})$	[Sul79], [Bär94], [Tra93], [Tra95], [CH96]	irreducible symmetric space
spherical space form $\mathbb{S}^n/\Gamma$	[Bär94]	homogeneous space
sphere $\mathbb{S}^{2m+1}$ with Berger metric	[Hit74], for $m = 1$ [Bär96], for general m	homogeneous space
3-dimensional lens space $\mathbb{S}^3/(\mathbb{Z}/k\mathbb{Z})$ with the Berger metric	[Bär92a]	homogeneous space
3-dimensional (flat) Bieberbach manifolds	[Pfä00]	homogeneous space
some n-dimensional (flat) Bieberbach manifolds	[MP06]	homogeneous space
n-dimensional Heisenberg manifolds	[AB98]	homogeneous space
$\mathrm{SU}_2/\mathrm{Q}_8$, Q_8 finite group of quaternions, endowed with a three-parameter family of homogeneous metrics, and for the four different spin structures	[Gin08]	homogeneous space
$\Gamma\backslash\mathrm{PSL}_2(\mathbb{R})$ Γ co-compact Fuchsian group	[SS87]	homogeneous space
simply connected compact Lie group G with canonical metric induced by the Killing form sign-changed	[Feg87]	symmetric space
complex projective space $(\mathbb{C}\mathrm{P}^{2m+1}, \mathrm{can})$	[CFG89],[CFG94] [SS93]	Kähler irreducible symmetric space
quaternionic projective space $(\mathbb{H}\mathrm{P}^m, \mathrm{can})$	[Bun91b], for $m = 2$ [Mil92], for general m	quaternion-Kähler irreducible symmetric space
real Grassmannian $\mathrm{Gr}_2(\mathbb{R}^{2m})$ with metric induced by the Killing form sign-changed	[Str80b], for $m = 3$ [Str80a], for general m	Kähler irreducible symmetric space
real Grassmannian $\mathrm{Gr}_{2p}(\mathbb{R}^{2m})$ with metric induced by the Killing form sign-changed	[See97]	irreducible symmetric space, Kähler if and only if $p = 1$
complex Grassmannian $\mathrm{Gr}_2(\mathbb{C}^{m+2})$ (m even), with metric induced by the Killing form sign-changed	[Mil98]	Kähler and quaternion-Kähler irreducible symmetric space
$\mathrm{G}_2/\mathrm{SO}_4$, with metric induced by the Killing form sign-changed	[See99]	quaternion-Kähler irreducible symmetric space

The method described above has also been used to compute the Dirac spectrum of the compactified Minkowski space $\mathbb{S}^1 \times \mathbb{S}^3$, with the metric $r^2 \mathrm{can}_{\mathbb{S}^1} - R^2 \mathrm{can}_{\mathbb{S}^3}$, where r, R are two real parameters, cf. [OV93]. This is the only example known to us in the pseudo-Riemannian case.

As noted before, only few results are known, since even the decomposition of the spinor representation under the action of the subgroup H is far from being easy in general (see for instance the partial results obtained in [Kli07] for Grassmann manifolds).

However, the first eigenvalue may be explicitly obtained by using analogous methods.

Theorem 15.18 ([Mil05]). *Let G/H be a compact, simply connected, n-dimensional irreducible symmetric space with G compact and simply connected, endowed with the metric induced by the Killing form of G sign-changed. Assume that G and H have the same rank and that G/H has a spin structure. Let β_k, $k = 1, \dots, p$, be the H-dominant weights occurring in the decomposition into irreducible components of the spin representation under the action of H. Then the square of the first eigenvalue of the Dirac operator is*

$$2 \min_{1 \le k \le p} \|\beta_k\|^2 + n/8,$$

where $\|\cdot\|$ is the norm associated with the scalar product $\langle \cdot, \cdot \rangle$ induced by the Killing form of G sign-changed.

The proof is based on a lemma of Parthasarathy given in [Par71] and already mentioned above. The formula has been developed further, involving only data of a root system of G, in order to obtain a more efficient expression for explicit computations.

Theorem 15.19 ([Mil06]). *Under the same assumptions as in the preceding theorem, let Φ be the set of non-zero roots of G with respect to the maximal torus T, common to G and H. Let Φ_G^+ be the set of positive roots of G and Φ_K^+, the set of positive roots of K, with respect to a fixed lexicographic ordering in Φ. Let δ_G (resp. δ_K) be the half-sum of the positive roots of G (resp. K). Then the square of the first eigenvalue of the Dirac operator is given by*

$$2\|\delta_G - \delta_K\|^2 + 4 \sum_{\theta \in \Lambda} \langle \theta, \delta_K \rangle + \frac{n}{8},$$

where Λ is the set

$$\Lambda := \{\theta \in \Phi_G^+;\ \langle \theta, \delta_K \rangle < 0\},$$

the scalar product $\langle \cdot, \cdot \rangle$ on weights being induced by the Killing form of G sign-changed.

As an example, the formula is used in [Mil06] to obtain the list of the first eigenvalues of the Dirac operator for the spin compact irreducible symmetric spaces endowed with a quaternion-Kähler structure; see Table 7.

Table 7

G/K	Square of the first eigenvalue of $\mathcal{D}$
$\mathbb{H}\mathrm{P}^n = \mathrm{Sp}_{m+1}/(\mathrm{Sp}_m \times \mathrm{Sp}_1)$	$\frac{m+3}{m+2}\frac{m}{2} = \frac{m+3}{m+2}\frac{\mathrm{Scal}}{4}$
$\mathrm{Gr}_2(\mathbb{C}^{m+2}) = \mathrm{SU}_{m+2}/S(\mathrm{U}_m \times \mathrm{U}_2)$ (m even)	$\frac{m+4}{m+2}\frac{m}{2} = \frac{m+4}{m+2}\frac{\mathrm{Scal}}{4}$
$\widetilde{\mathrm{Gr}}_4(\mathbb{R}^{m+4}) = \mathrm{Spin}_{m+4}/\mathrm{Spin}_m\mathrm{Spin}_4$ (m even)	$\frac{m^2+6m-4}{m(m+2)}\frac{m}{2} = \frac{m^2+6m-4}{m(m+2)}\frac{\mathrm{Scal}}{4}$
$\mathrm{G}_2/\mathrm{SO}_4$	$\frac{3}{2} = \frac{3}{2}\frac{\mathrm{Scal}}{4}$
$\mathrm{E}_6/(\mathrm{SU}_6\mathrm{SU}_2)$	$\frac{41}{6} = \frac{41}{30}\frac{\mathrm{Scal}}{4}$
$\mathrm{E}_7/(\mathrm{Spin}_{12}\mathrm{SU}_2)$	$\frac{95}{9} = \frac{95}{72}\frac{\mathrm{Scal}}{4}$
$\mathrm{E}_8/(\mathrm{E}_7\mathrm{SU}_2)$	$\frac{269}{15} = \frac{269}{210}\frac{\mathrm{Scal}}{4}$

We end by noting that harmonic analysis methods can also be used in the non-compact case. Here again, we may restrict to simply connected non-compact irreducible symmetric spaces. Since their sectional curvature is non-positive, cf. [Hel78], they are diffeomorphic to $\mathbb{R}^n$ by the Hadamard–Cartan theorem, hence posses a unique spin structure.

The standard example of non-compact simply connected irreducible space is the real hyperbolic space

$$H^n(\mathbb{R}) = \mathrm{Spin}_{1,n}/\mathrm{Spin}_n,$$

with the metric induced by the Killing form of $\mathrm{Spin}_{1,n}$.

Its Dirac spectrum has been computed by U. Bunke [Bun91a] using harmonic analysis methods. We will not describe this method here, because harmonic analysis on non-compact homogeneous spaces is far more complicated than the compact case. But the approach is mainly the same as in the compact case. It is based on the decomposition of the unitary representation $L^2_{H=\mathrm{Spin}_n}(G = \mathrm{Spin}_{1,n}, \Sigma_n)$, but instead of the result given in Theorem 15.4, here the representation has to be decomposed into a "direct integral" of irreducible ones, because the set of equivalence classes of irreducible G-representations (which need not to be finite dimensional) does not need to be countable.

As in the compact case, the computation of the spectrum of the Dirac operator $\mathcal{D}$ is derived from the study of the restriction of its square $\mathcal{D}^2$ to the components of the above decomposition. Here again a key role is played by the Parthasarathy formula, cf. Proposition 15.7, which brings one back to study the action of the Casimir operator of $\mathrm{Spin}_{1,n}$ on the components.

We refer to [Bun91a] for details. Note however that there is an incorrect statement concerning the eigenvalue 0 (the correct statement using harmonic analysis may be found in [CP99]).

There are other approaches to compute the spectrum of the Dirac operator. Camporesi and Higuchi [CH96] obtain the spectrum by a separation-of-variables technique in geodesic polar coordinates and confirm the computation using the above harmonic analysis method.

Another approach is given by C. Bär in [Bär00], second remark in Paragraph 3: it is based on the fact that

$$\widehat{H}^n(\mathbb{R}) := (H^n(\mathbb{R}) \text{ with a point removed}),$$

is isometric to the warped product

$$\mathbb{S}^{n-1} \times (0, \infty),$$

with the warped product metric

$$ds^2(x,t) = \varrho(t)^2 \mathrm{can}_{\mathbb{S}^{n-1}}(x) + dt^2, \quad \text{where } \varrho(t) = \sinh(t).$$

It is then proved that the square of the Dirac operator $\mathcal{D}^2$ on $\widehat{H}^n(\mathbb{R})$ is unitarily equivalent to $\bigoplus_\mu L_\mu$, where the sum is taken over all eigenvalues μ of the Dirac operator on $(\mathbb{S}^{n-1}, \mathrm{can})$, and where L_μ is the operator on $L^2((0,\infty), \mathbb{C}, dt)$ given by

$$L_\mu = -\frac{d^2}{dt^2} + \frac{\mu\cosh(t) + \mu^2}{\sinh^2(t)}.$$

This leads to the computation of the spectrum of $\mathcal{D}^2$ on $\widehat{H}^n(\mathbb{R})$, and the spectrum of $\mathcal{D}^2$ on H^n is then derived using a "decomposition principle" which allows to compare the spectra on these two manifolds.

The study of the spectrum of the Dirac operator on noncompact symmetric spaces has been considerably improved by two papers, one by S. Goette and U. Semmelmann, and one by R. Camporesi and E. Pedon. Here again, we only mention those results since they require some non-trivial knowledge in "noncompact" harmonic analysis.

Indeed, unlike the compact case, the spectrum of the L^2-closed extension of the Dirac operator to $L^2(\Sigma M)$ (which is self-adjoint since any symmetric space is a complete manifold) is the union of the *point spectrum* $\mathrm{Spec}_{\mathrm{p}}(\mathcal{D})$, defined as the set of complex numbers λ for which $\mathcal{D} - \lambda\mathrm{Id}$ is not injective, and the *continuous spectrum* $\mathrm{Spec}_{\mathrm{c}}(\mathcal{D})$, defined as the set of complex numbers λ for which $\mathcal{D} - \lambda\mathrm{Id}$ has a discontinuous inverse with dense domain.

In [GS02], S. Goette and U. Semmelmann, have determined the point spectrum of the Dirac operator on all noncompact symmetric spaces. Their result can be stated as follows.

Theorem 15.20 ([GS02]). *The point spectrum of $\mathcal{D}$ on an irreducible symmetric space G/K of noncompact type is nonempty if and only if G/K is isometric to*

$$\mathrm{SU}_{p,q}\,/S(\mathrm{U}_p \times \mathrm{U}_q)$$

with $p+q$ odd, and in this case $\mathrm{Spec}_p(\mathcal{D}) = \{0\}$.

Referring to this result, R. Camporesi and E. Pedon proved in [CP01] the following theorem.

Theorem 15.21 ([CP01]). *For a noncompact Riemannian symmetric space G/K of rank one, the continuous spectrum* $\mathrm{Spec}_{\mathrm{c}}(\mathcal{D})$ *is equal to $\mathbb{R}$ except if G/K is the complex hyperbolic space $H^n(\mathbb{C})$ with n even, in which case*

$$\mathrm{Spec}_{\mathrm{c}}(\mathcal{D}) = \Big(-\infty, -\frac{1}{2}\Big] \cup \Big[\frac{1}{2}, +\infty\Big).$$

Note that the exceptional case, the complex hyperbolic space

$$H^n(\mathbb{C}) = \mathrm{SU}(1,n)/S(\mathrm{U}_1 \times \mathrm{U}_n)$$

with n even, is precisely the one for which the point spectrum is nonempty.

Let us mention here that partial results in the study of the spectrum of $H^n(\mathbb{C})$ and $H^n(\mathbb{H})$ were obtained in [Sei92] and [Bai97]. Note also that results on eigenvalues and eigenspaces of certain twisted Dirac operators over the symmetric spaces $H^{2n}(\mathbb{R})$ and $H^n(\mathbb{C})$ can be found in [GV94].

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