Mathematical Model of Output Intensity

November 25, 2020

1 Visual representation of the system

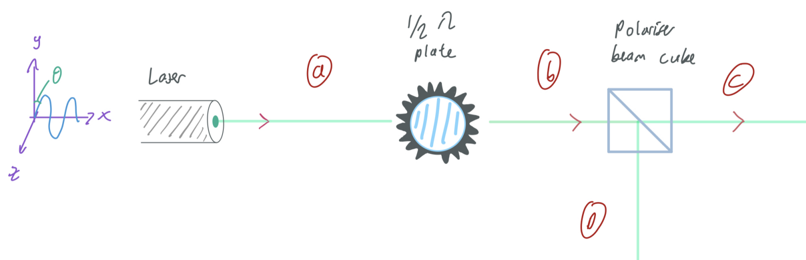

Figure 1: Path of laser

Figure above shows the path of the laser and the optical components which modulates it. A mathematical model of Fig.1 must be made and Incorporated into the control code so output can be tuned accordingly.

Laser Specs, CPS532-C2:

1. Output linearly polarised in the E_y direction
2. Beam diameter, 3.5mm
3. Gaussian beam
4. 532 nm, frequency changes with temprature
5. 0.9 mW Power
6. Impedance of free space, $\eta = 376.7303\Omega$

Half Wave Plate Specs, WPH05ME-532:

1. reflectance 0.25% @ 532nm
2. Transmittance 98.04996% @ 532nm

Polarising Beam Cube Specs, PBS121:

1. P-pol transmittance 95.54% @ 532nm
2. S-pol transmittance 0.04348% @ 532nm (ignore too small)
3. S-pol reflectance $>$ 99.50% @ 532nm

1.1 Jones Matrix

A Jones Matrix for each system component has to be made. Jones matrices make it easier to look at the system as a whole as it captures both axis at the same time. It can also be thought as a transfer function.

1.1.1 Polarising Beam Cube

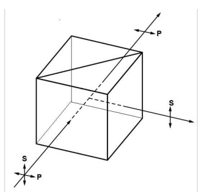

Figure 2: P=$\hat{y}$, S=$\hat{z}$

This cube allows $\hat{y}$ waves to be transmitted through to C, while reflecting $\hat{z}$ to path D. Thus, we can imagine the cube to have 2 polarisers, for path C in Fig.1 the fast axis is in the $\hat{y}$ direction. While for path D fast axis is in $\hat{z}$.

Suppose Beam Cube from Fig.1 has a fast axis in the ŷ direction. An electromagnetic wave vector $\vec{E}$ can be split into it's components. $\begin{pmatrix} E_y \\ E_z \end{pmatrix}$, axis defined in Fig.1. Now lets create its Jones Matrix. If a purely horizontally polarised EM wave passes through it, $\vec{E_x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \hat{z}$ the output would be the nil $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \hat{z}$ as incident wave is perpendicular to its axis. If a purely vertically polarised wave passes it, $\vec{E_x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \hat{y}$ output would also be $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \hat{y}$ as incident wave and fast axis align. We can combine both the output matrix together to form the Jones matrix at Fig.1 C. J = $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

The above can be repeated to find the Jones matrix for the polariser in the $\hat{z}$ direction.

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \hat{y} \longrightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad , \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \hat{z} \longrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$J = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{1}$$

1.1.2 1/2 Wave Plate

Half wave plate works as a retarder, slowing down each of the axis independently. This is caused by bifringence where the crystal has 2 different refractive indexes for each of its axis. We call refractive index in the fast axis n_f and slow axis n_s. In Fig.1 the fast axis of the wave plate **originally** lies in the $\hat{y}$ direction.

A linearly polarised wave travelling in the $\hat{x}$ direction can be given the equation (2). After passing the wave plate it emerges with modified terms (3) as a phase difference is introduced.

$$\vec{E_x} = \hat{y} E_o e^{j(\omega t - kx)} \tag{2}$$

$$\begin{aligned} \vec{E_x} &= \hat{y} E_o e^{j(\omega t - k_o n_f L)} \\ &= \hat{z} E_o e^{-jk_o n_f L} e^{j(\omega t - kz)} \end{aligned} \tag{3}$$

$e^{-jk_o n_f L}$ *is the phase difference*

k_o=wave number, n_f=fast axis index, L=length of waveplate, E_o=incident wave amplitude, ω=Angular frequency, t=time

Like the polarising beam cube, we intend to find the Jones matrix. So we input waves in $\hat{y}$ and $\hat{z}$ to find its response.

$$\begin{pmatrix}1\\0\end{pmatrix}\hat{y} \longrightarrow \begin{pmatrix}e^{-jk_o n_f L}\\0\end{pmatrix} \quad , \quad \begin{pmatrix}0\\1\end{pmatrix}\hat{z} \longrightarrow \begin{pmatrix}0\\e^{-jk_o n_s L}\end{pmatrix}$$

We combine both to make (4) which is the phase **relative** to incident.

$$\begin{pmatrix}e^{-jk_o n_f L} & 0\\0 & e^{-jk_o n_s L}\end{pmatrix} \tag{4}$$

Multiply (4) by $e^{jk_o n_f L}$

$$\begin{pmatrix}1 & 0\\0 & e^{-jk_o (n_f - n_s) L}\end{pmatrix} \tag{5}$$

$$\Delta n = n_f - n_s \ \textit{and}\ L = \frac{\lambda_o}{2\Delta n}$$

Therefore its Jones matrix is:

$$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} \tag{6}$$

1.2 1/2 Wave Plate Rotating Jones Matrix

Jones matrix only works for a fixed theta, since our system requires the half wave plate to rotate we have to modify (6) by multiplying a rotation matrix term $R(\theta)$ to find J_{new}.

$$\begin{gathered} J_{new} = R(\theta) J_{current} R(\theta)^{-1} \\ R(\theta) = \begin{pmatrix}\cos(\theta) & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix} \end{gathered} \tag{7}$$

By substituting (6) and finding the inverse of $R(\theta)$ we get (8) which was calculated in matlab, complex!

$$J_{new} = \begin{pmatrix} \cos\left(\frac{\pi\,\theta}{180}\right)^2 - \sin\left(\frac{\pi\,\theta}{180}\right)^2 & 2\,\cos\left(\frac{\pi\,\theta}{180}\right)\,\sin\left(\frac{\pi\,\theta}{180}\right) \\ 2\,\cos\left(\frac{\pi\,\theta}{180}\right)\,\sin\left(\frac{\pi\,\theta}{180}\right) & \sin\left(\frac{\pi\,\theta}{180}\right)^2 - \cos\left(\frac{\pi\,\theta}{180}\right)^2 \end{pmatrix} \tag{8}$$

2 Combining it together

Order of multiplication is important, we start by multiplying leftwards components in Fig.1. Remember in this case the polarisers fast axis is in the $\hat{y}$ direction.

$$E_{output} = J_{cube} * J_{waveplate} * E_{input} \tag{9}$$

2.1 Output at Fig.1 C

(9) gives us the output amplitude as a function of θ, E_y and E_z. Notice there is no E_z term, remember the polariser is aligned with the $\hat{y}$ axis so it totally blocks any $\hat{z}$ component.

$$\vec{E}_{c,out} = \begin{pmatrix} \mathrm{Ey}\,\left(\cos\left(\frac{\pi\,\theta}{180}\right)^2 - \sin\left(\frac{\pi\,\theta}{180}\right)^2\right) + 2\,\mathrm{Ez}\,\cos\left(\frac{\pi\,\theta}{180}\right)\,\sin\left(\frac{\pi\,\theta}{180}\right) \\ 0 \end{pmatrix} \tag{10}$$

2.2 Output at Fig.1 D

Sin and Cos contained within equations of this paper are calculated in radians (matlab default). So θ which is in degrees has to be multiplied by $\frac{\pi}{180}$//

Remember in this case the polariser's fast axis is in the $\hat{z}$ direction. Notice that there is no E_y component as the polarising beam cube only reflects $\hat{z}$ So using (9) we get

$$\vec{E}_{d,out} = \begin{pmatrix} 0 \\ 2\,\mathrm{Ey}\,\cos\left(\frac{\pi\,\theta}{180}\right)\,\sin\left(\frac{\pi\,\theta}{180}\right) - \mathrm{Ez}\,\left(\cos\left(\frac{\pi\,\theta}{180}\right)^2 - \sin\left(\frac{\pi\,\theta}{180}\right)^2\right) \end{pmatrix} \tag{11}$$

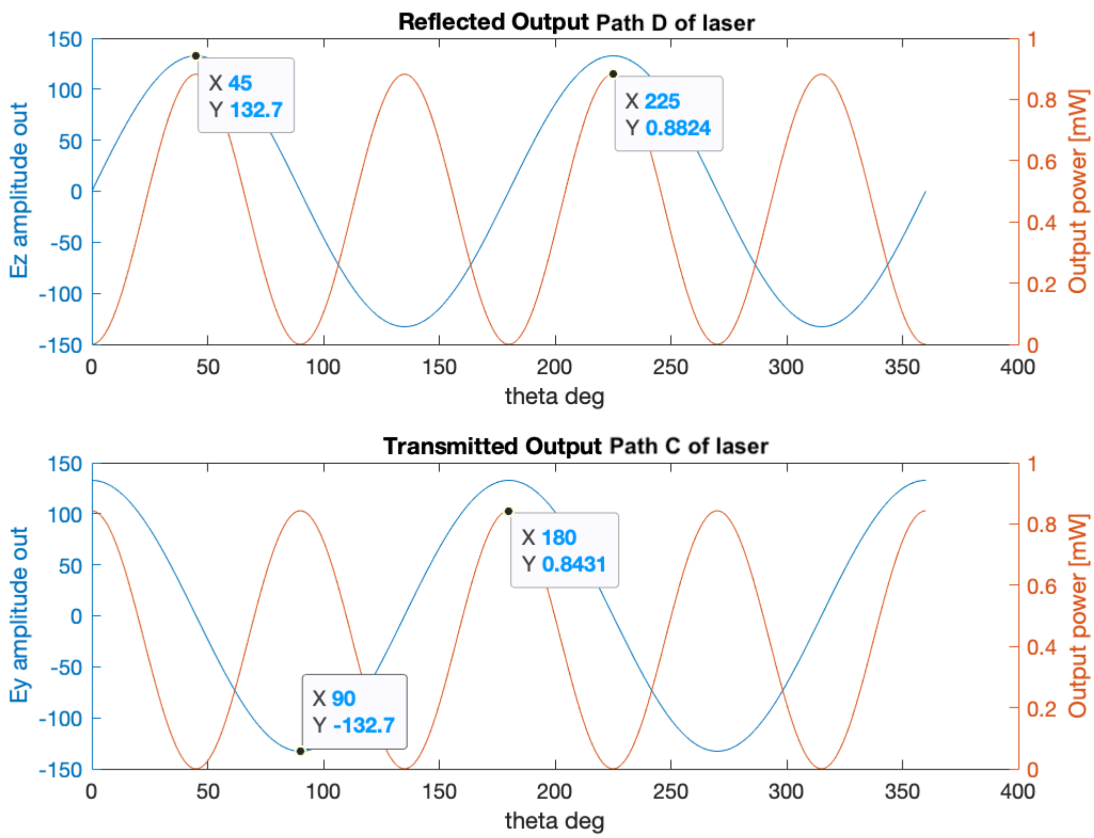

Figure 3: Output for Path C and D

Theta is the angle of rotation of the half wave plate relative to its fast axis pointing towards $\hat{y}$. We can theoretically sustain full range 0 to Max power with a rotation range of $\pi/2$. After including the transmission coefficients, the maximum power would be less at a max of 0.8431mW. $E_y$andE_z are phase shifted from each other by $\pi/4$.

Power of E_y also has a factor of 0.9554 multiplied to it relative to E_z (checked by isolating regions with the same shape and divided those regions, constant factor found). Both graphs are essentially the same by adding a phase shift and a factor to one of the outputs (checked by plotting with matlab).

Might use DTFT to add time shift, $Power_{E_y} = 0.9554 Power_{E_z} +$ *phase shift,* $\pi/2$ factor is just the P-Pol transmittance.

2.3 Equation Relating both Power at C & D

Total intensity can be found by relating eqn (10) and (11):

$$\begin{aligned} I_{total} &= C_t \lambda_t \frac{E_c^2}{2\eta} + \lambda_t \frac{E_d^2}{2\eta} \\ &= \frac{\lambda_t}{2\eta}(C_t E_c^2 + E_d^2) \end{aligned} \tag{12}$$

c_t is cube transmittance, λ_t is waveplate transmittance.
Total power can be found by multiplying total intensity with area:

$$\begin{aligned} P_{total} &= I_{total} * BeamArea \\ P_c + P_d &= A\frac{\lambda_t}{2\eta}(C_t E_c^2 + E_d^2) \end{aligned} \tag{13}$$

Expanding the above, we see that the final equation relies on 3 variables, P_c, P_d and θ in degrees.

$$P_c + P_d = AE_y^2 \frac{\lambda_t}{2\eta}(C_t(cos^2(\frac{\pi\theta}{180}) - sin^2(\frac{\pi\theta}{180}) + 4cos^2(\frac{\pi\theta}{180})sin^2(\frac{\pi\theta}{180})) \tag{14}$$

The terms after the = sign represent the fluctuating power output as a sum of both split beams. This is caused by the difference in transmittivity for P and S pol in the beam cube.

The equation (14) essentially deducts the sensor power from the total power (with regards to θ) to find power in path C.

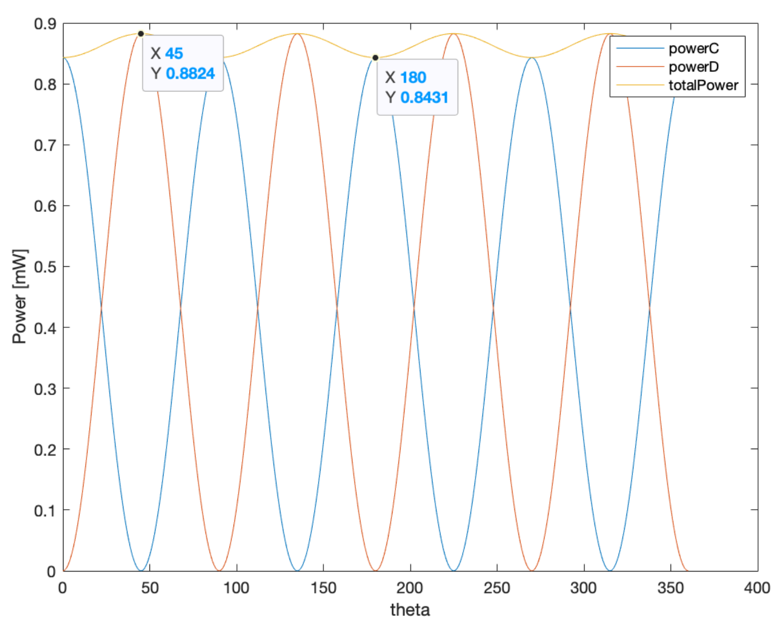

Figure 4: Power vs theta

UNFORTUNATELY (14) CONTAINS LASER VARIABLES FROM THE ORIGINAL LASER INPUT WHICH IS THEORETICALLY CONSTANT. THIS DOES NOT TAKE INTO ACCOUNT THE FLUCTUATIONS OF THE LASER ITSELF. INVERSE FOR θ HARD TO FIND as shown in the long equation (15) below

$$-\frac{\ln\left(\left(\frac{16384000\sqrt{(500000\,T\,Z-4405109641\,A\,L)\,(137438953472\,T\,Z-1210867317976915\,A\,C\,L)}-4294967296000000\,T\,Z+18919801843389300736\,A\,L+18}{18919801843389300736\,A\,L-18919801843389296875\,A\,C\,L}\right.\right.}{\pi} \tag{15}$$

3 POSSIBLE BREAKTHROUGH

Looking at figure 3. There is a possible connection between both output power except the obvious phase shifts.

By dividing powerC/powerD I get the figure below. This does not change with the change with different laser intensity. The transmittance of the optical elements cause it to change as explained below.

$$intensity = \frac{E^2}{2\eta}$$
$$power = intensity * area$$

Since we're dividing Power C by Power D, and beam area is assumed to be constant, the area cancels out. This gives us the individual intensities for both path C and D. Don't forget to multiply by transmittance of half wave plate λ_t and cube C_t.

$$\begin{aligned} I_D &= \lambda_t \frac{E_z^2}{2\eta} \\ I_C &= \lambda_t C_t \frac{E_y^2}{2\eta} \\ Ratio,\ \frac{P_C}{P_D} = \frac{I_C}{I_D} &= C_t \frac{E_y^2}{E_z^2} \end{aligned} \tag{16}$$

We get the ratio which can be seen in Fig. 5. Ratio changes with theta and at multiples of 0, 45 deg the ratio will be 0. So this means at multiples of 0 and 90 deg power D will be 0. This is the same for power C at multiples of 45 deg. This can be confirmed by referring to Fig. 3

Ratio scaled based on cube transmittance C_t and not half wave plate as they cancel out. If the cube reflected tramsmittance is significant, te scale would instead be, $\frac{C_t}{Ref.Trans.}$

We now can relate the Power C and Power D without the original laser input. This can be done by checking the motor angle and referring to the ratio dataset to calculate the corresponding power. The datapoints can be found in github to be used! `https://github.com/aliceDuhem/LaserClosedLoop`

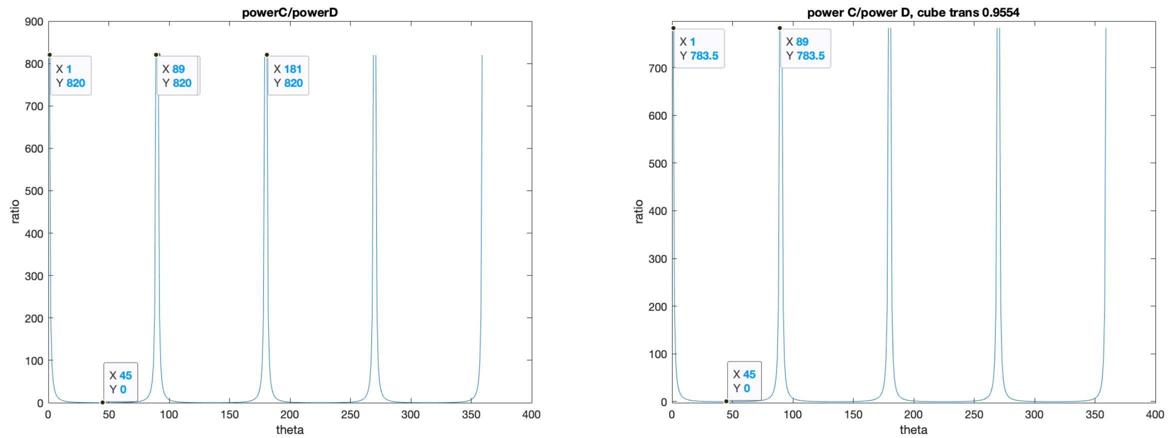

Figure 5: PowerD:PowerC

4 Comparison with Malus Law Ignore

Malus Law states that output intensity, I_o, is proportional to $\cos^2\theta$, when a wave passes a linear polariser.

$$I_o = I_{in}\cos^2\theta \tag{17}$$

In Fig.6 we can see that Jones has an angular velocity twice that of Malus. Transmission coefficients have been incorperated into the calculations for the graph. Both have the same peak, thus we are able to model **ALL** the above with (18) instead, but keep in mind to add the transmission coefficients.

$$I_o = I_{in}\cos^2 2\theta \tag{18}$$

I_{in} contains the original intensity, cube P-pol transmittance, wave plate transmittance.

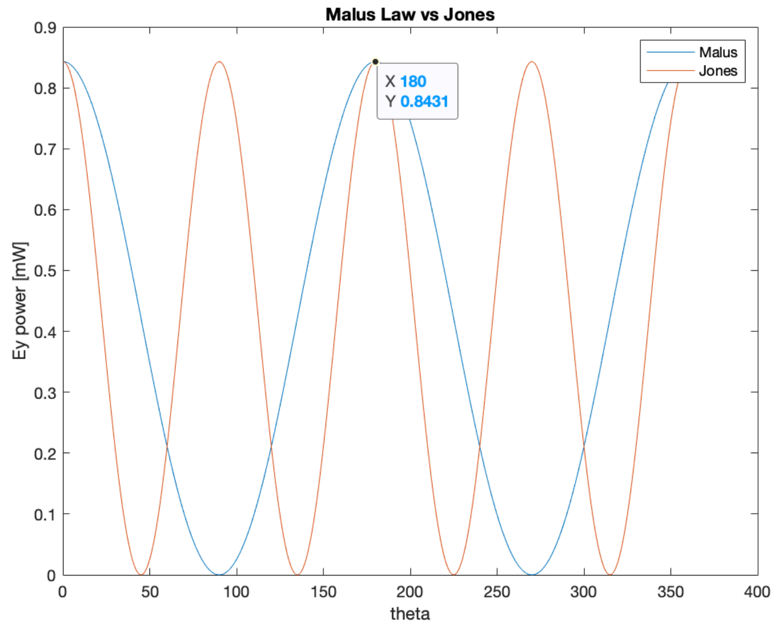

Figure 6: Malus VS Jones in Path C

Appendix A MATLAB Code

```
clc
clear all
close all


% polarizing beam cube errors and specs (reflects s-pol)
% p-pol transmittance 95.54% @ 532nm
% s-pol transmittance 0.04348% @ 532nm
% total R_s > 99.50%
cubeP_trans = 0.9554;
cubeS_trans = 0.0004348; %this not needed as the max difference in power is 3.83e-4 mW

% half wave plate errors and specs
% reflectance 0.25% @ 532nm
% transmittamnce 98.04996% @ 532nm

halfWave_trans = 0.9804996; % incldue as mean sqr err between original and transmitted Ey
↪  = 0.8482

% laser spec and error
% 0.8mW < power < 1mW, typical @ 0.9mW
% frequency changes with temprature


%% Find transmitted output Ey

% use jones matrix to find 'transfer function' of whole system
```

```
syms theta Ey Ez;

r = [cosd(theta) -sind(theta);sind(theta) cosd(theta)];
rInv = inv(r);

JonesHalf = [1 0;0 -1];
JonesPol=[1 0 ; 0 0];
JonesHalfRot = r*JonesHalf*rInv;
initial =[Ey; Ez];  % only adjust Ey becsaue laser linearly pol.
                    % keep Ez as 0, laser pol in Y direc only.
                    % adjust in bottom {} brackets

TotalTrans = JonesPol*JonesHalfRot  * initial;

thetaAxis =[0:1:360];     % adjust division accodring to servo motor

for i=1:size(thetaAxis,2)

solT =  subs (TotalTrans, {theta, Ey,Ez} ,{thetaAxis(i),132.742,0});

outEy(i) = double(solT(1));
outEz(i) = double(solT(2));
end

% plot theta against Ey proportion
% Ez remains 0 as 2nd polariser cuts it off
figure(1)
subplot(2,1,2)

yyaxis left
plot(thetaAxis,outEy)
xlabel('theta')
ylabel('Ey amplitude out')
title('Transmitted Output Ey of laser')

hold on
intensity = cubeP_trans .* halfWave_trans .* outEy.^2 ./ (2.*376.7303);
%include transmittance above
power = intensity .* (pi.*(3.5e-3.^2));

yyaxis right
plot(thetaAxis,power.*1000)
xlabel('theta deg')
ylabel('Output power [mW]')


%% test with malus law instead of jones

intensity1 = 0.9e-3 / (pi*(3.5e-3^2));
intensity1Out = cubeP_trans .* halfWave_trans .* intensity1 .* cosd(thetaAxis).^2;


figure
```

```
plot(thetaAxis,intensity1Out.*1000.* (pi.*(3.5e-3.^2)),thetaAxis, power.*1000)
legend('Malus','Jones')
ylabel('Ey power [mW]')
xlabel('theta')
title('Malus Law vs Jones')

% intensity - intensity1Out gave a mean squared error of 1.5903e-10, which
% is neglegable. So i suggest we ignore it.
% this shows that my equations are very close to malus law


%% reflected output Ez to detector

JonesPolRef=[0 0 ; 0 1];
JonesHalfRot = r*JonesHalf*rInv;
TotalRef = JonesPolRef * JonesHalfRot * initial;

for i=1:size(thetaAxis,2)

solT =  subs (TotalRef, {theta, Ey,Ez} ,{thetaAxis(i),132.742,0});

outRefEy(i) = double(solT(1));
outRefEz(i) = double(solT(2));
end

figure(1)
subplot(2,1,1)
yyaxis left
plot(thetaAxis,outRefEz)
xlabel('theta')
ylabel('Ez amplitude out')
title('Reflected Output Ez of laser')
hold on

intensityRef = halfWave_trans.* outRefEz.^2 ./ (2.*376.7303);
powerRef = intensityRef .* (pi.*(3.5e-3.^2));

% figure
yyaxis right
plot(thetaAxis,powerRef.*1000)
xlabel('theta deg')
ylabel('Output power [mW]')
% title('Power vs theta')

%% Ez and Ey between half wave and cube

between = JonesHalfRot * initial;

for i=1:size(thetaAxis,2)

solBet =  subs (between, {theta, Ey,Ez} ,{thetaAxis(i),132.742,0});

BetEy(i) = double(solBet(1));
BetEz(i) = double(solBet(2));
```

```
end

figure
hold on
plot(thetaAxis, sqrt(halfWave_trans).* BetEy, thetaAxis, sqrt(halfWave_trans).*BetEz)
plot(thetaAxis, BetEy, thetaAxis, BetEz)
% legend('Ey','Ez')

%% plot path C including s-pol
% cannot use equaton outEY as that contains final transform
% can ignore as difference too small 1e-4 mW in power

FinalEz = BetEz.*sqrt(halfWave_trans).*sqrt(cubeS_trans);
FinalEy = BetEy.*sqrt(halfWave_trans).*sqrt(cubeP_trans);
% outEy = outEy .* sqrt(halfWave_trans);

Etot = sqrt(FinalEz.^2 + FinalEy.^2);


intensityNew = Etot.^2 ./ (2.*376.7303);
powerNew = intensityNew .*(pi.*(3.5e-3.^2));
figure
plot(thetaAxis, powerNew*1000)
```