# Lecture 5: Spectral Analysis and Filtering

Introduction to Time Series, Fall 2023

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Related reading: Chapters 4.1–4.3 and 4.7–4.8 of Shumway and Stoffer (SS).

## 1 Periodic processes

- Consider a periodic process of the form

$$x_t = A \cos(2\pi\omega t + \phi) \tag{1}$$

  It will be convenient to allow the time index in the processes we study in this lecture to be *positive and negative*; hence, we write our process as $x_t$, $t = 0, \pm 1, \pm 2, \dots$

- Importantly, the quantity $\omega$ in the above definition is called *frequency* of the process; and the quantity $1/\omega$ is called the *period*. As $t$ varies from 0 to $1/\omega$, note that the process goes through one complete cycle (it ends up back where it started). See Figure 1

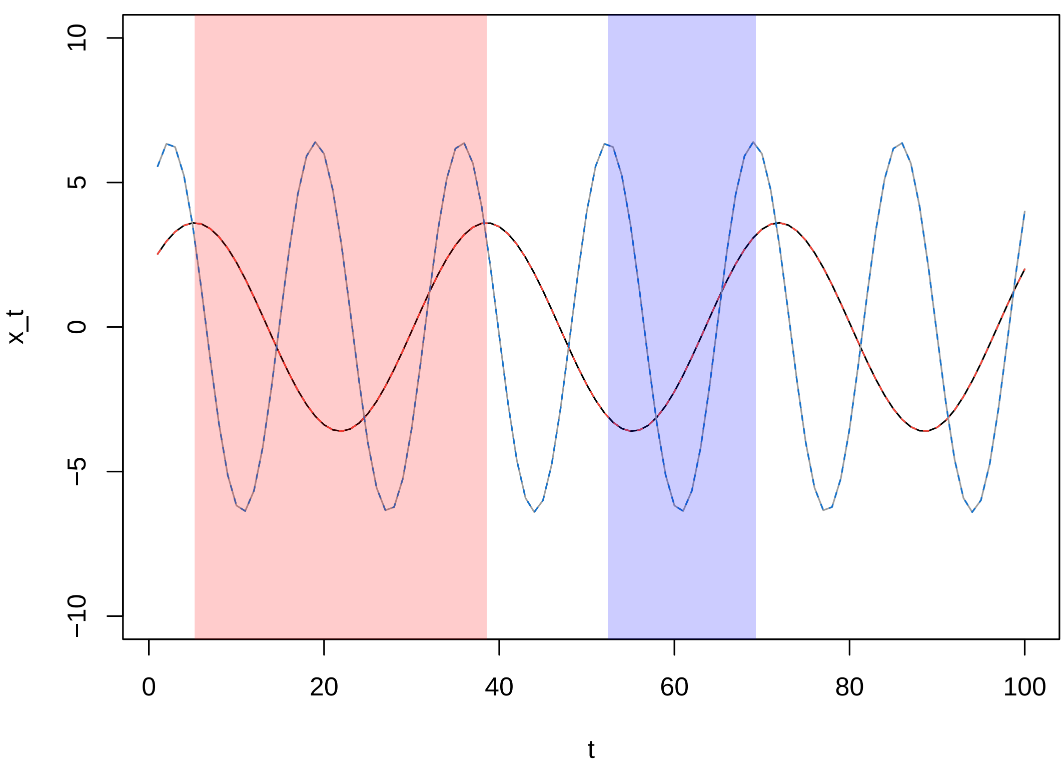


Figure 1: Two examples of cosine processes, the first (in red) having a frequency $\omega = 3/100$ and amplitude $\sqrt{2^2 + 3^2} \approx 3.6$, and the second (in blue) having a frequency $\omega = 6/100$ and amplitude $\sqrt{4^2 + 5^2} \approx 6.4$.

- The quantity $A$ is called the *amplitude* and $\phi$ the *phase* of the process. The amplitude controls how high the peaks are, and the phase determine where (along the cosine cycle) the process starts at the origin $t = 0$

- We can introduce randomness into the process (1) by allowing $A$ and $\phi$ to be random
- It will be useful to reparametrize. In general, recall the trigonometric identity (cosine compound angle formula):
$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b) \tag{2}$$
Thus, starting with (1), we can rewrite this as $x_t = A\cos(\phi)\cos(2\pi\omega t) - A\sin(\phi)\sin(2\pi\omega t)$. Simply letting $U_1 = A\cos(\phi)$, $U_2 = -A\sin(\phi)$, we can therefore write
$$x_t = U_1\cos(2\pi\omega t) + U_2\sin(2\pi\omega t) \tag{3}$$
with $U_1, U_2$ our two random variables, determining the amplitude of the cosine and sine components separately
- Note that another way of writing the relationship between $A, \phi$ and $U_1, U_2$ is (why?):
$$A = \sqrt{U_1^2 + U_2^2}, \quad \phi = \tan^{-1}(-U_2/U_1)$$
- An interesting fact (that you can try to verify as a challenge):
$$U_1, U_2 \sim N(0,1), \text{ independently } \iff A \sim \chi_2^2, \ \phi \sim \text{Unif}(-\pi, \pi), \text{ independently}$$

## 1.1 Stationarity

- If $U_1, U_2$ are uncorrelated, each with mean zero and variance $\sigma^2$, then the periodic process $x_t$, $t = 0, \pm 1, \pm 2, \ldots$ defined in (3) is stationary
- To check this: simply compute the mean function
$$\mu_t = \mathbb{E}(x_t) = 0$$
which is constant in time; and the autocovariance function
$$\begin{aligned}
\gamma(s,t) &= \text{Cov}(x_s, x_t) \\
&= \text{Cov}\Big(U_1\cos(2\pi\omega s) + U_2\sin(2\pi\omega s),\ U_1\cos(2\pi\omega t) + U_2\sin(2\pi\omega t)\Big) \\
&= \text{Cov}\Big(U_1\cos(2\pi\omega s),\ U_1\cos(2\pi\omega t)\Big) + \text{Cov}\Big(U_2\sin(2\pi\omega s),\ U_1\cos(2\pi\omega t)\Big) \\
&\quad\ + \text{Cov}\Big(U_1\cos(2\pi\omega s),\ U_2\sin(2\pi\omega t)\Big) + \text{Cov}\Big(U_2\sin(2\pi\omega s),\ U_2\sin(2\pi\omega t)\Big) \\
&= \sigma^2\cos(2\pi\omega s)\cos(2\pi\omega t) + 0 + 0 + \sigma^2\sin(2\pi\omega s)\sin(2\pi\omega t) \\
&= \sigma^2\cos(2\pi\omega(s-t))
\end{aligned}$$
which only depends on the lag $s - t$ (where in the last line we used the identity (2) once again)

## 1.2 General mixtures

- As a generalization of (3), we can also mix together a total of $p$ periodic processes, defining
$$x_t = \sum_{i=1}^{p}\Big(U_{k1}\cos(2\pi\omega_k t) + U_{k2}\sin(2\pi\omega_k t)\Big) \tag{4}$$
for $U_{k1}, U_{k2}$, $k = 1, \ldots, p$ all uncorrelated random variables with mean zero, where $U_{k1}, U_{k2}$ have variance $\sigma_k^2$
- As a generalization of the above calculation, you'll show on your homework that the process $x_t$, $t = 0, \pm 1, \pm 2, \ldots$ defined in (4) is stationary, with autocovariance function
$$\gamma(h) = \sum_{k=1}^{p}\sigma_k^2\cos(2\pi\omega_k h)$$

- Figure 2 displays a couple of mixture processes of the form (4) (with $p = 2$ and $p = 3$). Note the regular repeating nature of the mixture processes. One might wonder how we can decompose a such a mixture into its frequency components (periodic processes, each of the form (3)). This is, in fact, one of the main objectives in spectral analysis

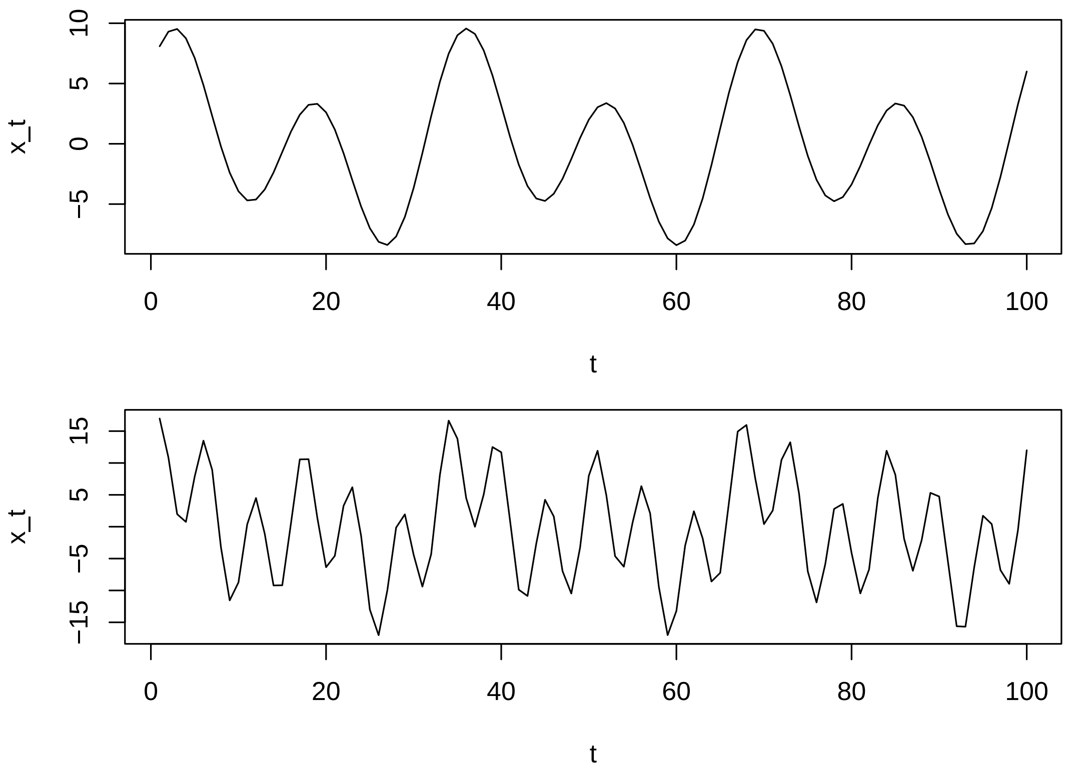


Figure 2: Mixture of periodic processes of different frequencies (and amplitudes).

- And the answer, as we'll see next, is given by something you're already quite familiar with ... regression!

## 2 Periodogram

## 3 Spectral density

## 4 Linear filtering

## 5 Lagged regression