

# Lecture 6: Autoregressive Integrated Moving Average Models

Introduction to Time Series, Fall 2023

Ryan Tibshirani

Related reading: Chapters 3.1 and 3.3 Shumway and Stoffer (SS); Chapters 9.1–9.5 and 9.8–9.9 of Hyndman and Athanasopoulos (HA).

## 1 AR models

- The *autoregressive* (AR) model is one of the foundational legs of ARIMA models, which we'll cover bit by bit in this lecture. (Recall, you've already learned about AR models, which were introduced all the way back in our first lecture)
- Precisely, an AR model of order  $p \geq 0$ , denoted  $\text{AR}(p)$ , is of the form

$$x_t = \sum_{j=1}^p \phi_j x_{t-j} + w_t \quad (1)$$

where  $w_t$ ,  $t = 0, \pm 1, \pm 2, \pm 3, \dots$  is a white noise sequence. Note that we allow the time index to be negative here (we extend time back to  $-\infty$ ), which will be useful in what follows

- The coefficients  $\phi_1, \dots, \phi_p$  in (1) are fixed (nonrandom), and we assume  $\phi_p \neq 0$  (otherwise the order here would effectively be less than  $p$ ). Note that in (1), we have  $\mathbb{E}(x_t) = 0$  for all  $t$
- If we wanted to allow for a nonzero but constant mean, then we could add an intercept to the model in (1). We'll omit this for simplicity in this lecture
- A useful tool for expressing and working with AR models is the *backshift operator*: this is an operator we denote by  $B$  that takes a given time series and shifts it back in time by one index,

$$Bx_t = x_{t-1}$$

- We can extend this to powers, as in  $B^2 x_t = BBx_t = x_{t-2}$ , and so on, thus

$$B^k x_t = x_{t-k}$$

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- Returning to (1), note now that we can rewrite this as

$$x_t - \phi_1 x_{t-1} - \phi_2 x_{t-2} - \dots - \phi_p x_{t-p} = w_t$$

or in other words, using backshift notation

$$(1 - \phi_1 B - \phi_2 B^2 - \dots - \phi_p B^p)x_t = w_t \quad (2)$$

- Hence (2) is just a compact way to represent the  $\text{AR}(p)$  model (1) using the backshift operator  $B$ . Often, authors will write this model even more compactly as

$$\phi(B)x_t = w_t \quad (3)$$

where  $\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \dots - \phi_p B^p$  is called the *autoregressive operator* of order  $p$ , associated with the coefficients  $\phi_1, \dots, \phi_p$

- Figure 1 shows two simple examples of AR processes

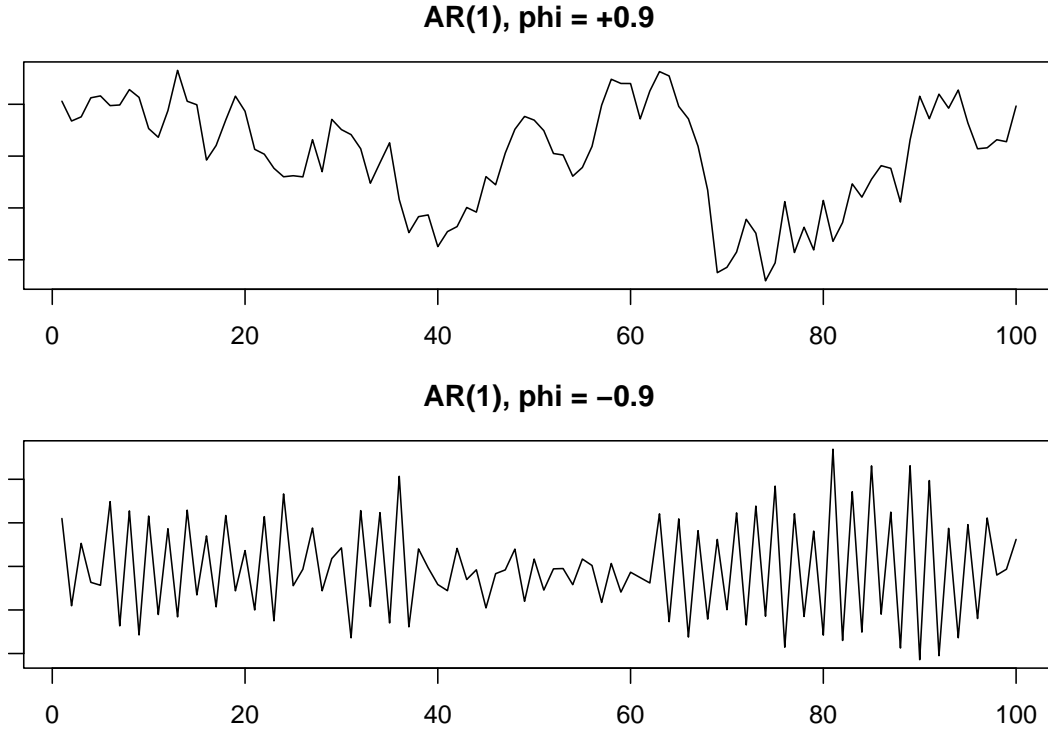


Figure 1: Two examples of AR(1) processes, with  $\phi = \pm 0.9$ .

### 1.1 AR(1): auto-covariance and stationarity

- A key question for us will be: *under what conditions does the AR model in (1), or equivalently (3), define a stationary process?*
- The answer will turn out to be fairly sophisticated, but we can glean some intuition by starting with the AR(1) case:

$$x_t = \phi x_{t-1} + w_t \quad (4)$$

for  $t = 0, \pm 1, \pm 2, \pm 3, \dots$

- Unraveling the iterations, we get

$$\begin{aligned} x_t &= \phi^2 x_{t-2} + \phi w_{t-1} + w_t \\ &= \phi^3 x_{t-3} + \phi^2 w_{t-2} + \phi w_{t-1} + w_t \\ &\vdots \\ &= \phi^k x_{t-k} + \sum_{j=0}^k \phi^j w_{t-j} \end{aligned}$$

- If  $|\phi| < 1$ , then we can send  $k \rightarrow \infty$  in the last display to get

$$x_t = \sum_{j=0}^{\infty} \phi^j w_{t-j} \quad (5)$$

This is called the *stationary representation* of the AR(1) process (4)

- Why is it called this? We can compute the auto-covariance function, writing  $\sigma^2 = \text{Var}(w_t)$  for the noise variance, as

$$\begin{aligned}
\text{Cov}(x_t, x_{t+h}) &= \text{Cov} \left( \sum_{j=0}^{\infty} \phi^j w_{t-j}, \sum_{\ell=0}^{\infty} \phi^\ell w_{t+h-\ell} \right) \\
&= \sum_{j,\ell=0}^{\infty} \phi^j \phi^\ell \text{Cov}(w_{t-j}, w_{t+h-\ell}) \\
&= \sum_{j=0}^{\infty} \phi^j \phi^{j+h} \sigma^2 \\
&= \sigma^2 \phi^h \sum_{j=0}^{\infty} \phi^{2j} \\
&= \sigma^2 \frac{\phi^h}{1 - \phi^2}
\end{aligned} \tag{6}$$

where we used the fact that  $\sum_{j=0}^{\infty} b^j = 1/(1-b)$  for  $|b| < 1$ . Since the auto-covariance in the last line only depends on  $h$ , we can see that the AR(1) process is indeed stationary

- To reiterate, the representation (5), and the auto-covariance calculation just given, would have not been possible unless  $|\phi| < 1$ . This condition is required in order for the AR(1) process to have a stationary representation. We will see later that we can generalize this to a condition that applies to an AR( $p$ ), yielding an analogous conclusion. The conclusion we will be looking for is explained next

## 1.2 Causality (no, not the usual kind)

- Now we will introduce a concept called *causality*, which generalizes what we just saw falls out of an AR(1) when  $|\phi| < 1$ . This is a slightly unfortunate bit of nomenclature that nonetheless seems to be common in the time series literature. It has really nothing to do with causality used in the broader sense in statistics. We will ... somewhat begrudgingly ... stick with the standard nomenclature in time series here
- We say that a series  $x_t$ ,  $t = 0, \pm 1, \pm 2, \pm 3, \dots$  is *causal* provided that it can be written in the form

$$x_t = \sum_{j=0}^{\infty} \psi_j w_{t-j} \tag{7}$$

for a white noise sequence  $w_t$ ,  $t = 0, \pm 1, \pm 2, \pm 3, \dots$ , and coefficients such that  $\sum_{j=0}^{\infty} |\psi_j| < \infty$

- You should think of this as a generalization of (5), where we allow for arbitrary coefficients  $\psi_0, \psi_1, \psi_2, \dots$ , subject to an absolute summability condition
- It is straightforward to check that causality actually implies stationarity: we can just compute the auto-covariance function in (7), similar to the above calculation:

$$\begin{aligned}
\text{Cov}(x_t, x_{t+h}) &= \text{Cov} \left( \sum_{j=0}^{\infty} \psi_j w_{t-j}, \sum_{\ell=0}^{\infty} \psi_\ell w_{t+h-\ell} \right) \\
&= \sum_{j,\ell=0}^{\infty} \psi_j \psi_\ell \text{Cov}(w_{t-j}, w_{t+h-\ell}) \\
&= \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+h}
\end{aligned}$$

The summability condition ensures that these calculations are well-defined and that the last display is finite. Since this only depends on  $h$ , we can see that the process is indeed stationary

- Thus, to emphasize, causality actually tells us *more* than stationary: it is stationary “plus” a representation a linear filter of past white noise variates, with summable coefficients
- Note that when  $\psi_j = \phi^j$ , the summability condition  $\sum_{j=0}^{\infty} |\psi_j| < \infty$  is true if and only if  $|\phi| < 1$ . Hence what we actually proved above for AR(1) was that it is causal if and only if  $|\phi| < 1$ . And it is this condition—for causality—that we will actually generalize for AR( $p$ ) models, and beyond

## 2 MA models

- A *moving average* (MA) model is “dual”, in a colloquial sense, to the AR model. Instead of having  $x_t$  evolve according to a linear combination of the recent past, the *errors* in the model evolve according to a linear combination of white noise
- Precisely, an MA model of order  $q \geq 0$ , denoted MA( $q$ ), is of the form

$$x_t = w_t + \sum_{j=1}^q \theta_j w_{t-j} \quad (8)$$

where  $w_t$ ,  $t = 0, \pm 1, \pm 2, \pm 3, \dots$  is a white noise sequence

- The coefficients  $\theta_1, \dots, \theta_q$  in (8) are fixed (nonrandom), and we assume  $\theta_q \neq 0$  (otherwise the order here would effectively be less than  $q$ ). Note that in (8), we have  $\mathbb{E}(x_t) = 0$  for all  $t$
- Again, we can rewrite (8), using backshift notation, as

$$x_t = (1 + \theta_1 B + \theta_2 B^2 + \dots + \theta_q B^q) w_t \quad (9)$$

- Often, authors will write (9) even more compactly as

$$x_t = \theta(B) w_t \quad (10)$$

where  $\theta(B) = 1 + \theta_1 B + \theta_2 B^2 + \dots + \theta_q B^q$  is called the *moving average operator* of order  $q$ , associated with the coefficients  $\theta_1, \dots, \theta_q$

- Figure 2 shows two simple examples of MA processes

### 2.1 Stationarity

- Unlike AR processes, an MA process (8) is stationary *for any values of the parameters*  $\theta_1, \dots, \theta_q$
- To check this, we compute the auto-covariance function using a similar calculation to those we’ve done before, writing  $\theta_0 = 1$  for convenience:

$$\begin{aligned} \text{Cov}(x_t, x_{t+h}) &= \text{Cov} \left( \sum_{j=0}^q \theta_j w_{t-j}, \sum_{\ell=0}^q \theta_\ell w_{t+h-\ell} \right) \\ &= \sum_{j,\ell=0}^q \theta_j \theta_\ell \text{Cov}(w_{t-j}, w_{t+h-\ell}) \\ &= \sigma^2 \sum_{j=0}^q \theta_j \theta_{j+h} \end{aligned} \quad (11)$$

Since this only depends on  $h$ , we can see that the process is indeed stationary

- Actually, this (similarity between this and previous calculations) brings us to emphasize the following connection: *an AR(1) model with  $|\phi| < 1$  is actually a particular infinite-order MA model*, as we saw in the stationary representation (5). But writing it as an AR(1) process is simpler
- We will see later that there is a more general connection to be made: an AR( $p$ ) process with certain constraints on its coefficients can be written as an MA( $\infty$ ) process



Figure 2: Two examples of MA(1) processes, with  $\theta = \pm 0.9$ .

## 2.2 MA(1): issues with non-uniqueness

- Consider the MA(1) model:

$$x_t = w_t + \theta w_{t-1} \quad (12)$$

for  $t = 0, \pm 1, \pm 2, \pm 3, \dots$

- According to (11), we can compute its auto-covariance simply (recalling  $\theta_0 = 1$ ) as

$$\gamma(h) = \begin{cases} (1 + \theta^2)\sigma^2 & h = 0 \\ \theta\sigma^2 & |h| = 1 \\ 0 & |h| > 1 \end{cases} \quad (13)$$

- The corresponding auto-correlation function is thus

$$\rho(h) = \begin{cases} 1 & h = 0 \\ \frac{\theta}{1 + \theta^2} & |h| = 1 \\ 0 & |h| > 1 \end{cases}$$

- If we look carefully, then we can see a problem lurking here: the auto-correlation function is unchanged if we replace  $\theta$  by  $1/\theta$
- And in fact, the auto-covariance function (13) is unchanged if we replace  $\theta$  and  $\sigma^2$  with  $1/\theta$  and  $\sigma^2\theta^2$ ; e.g., try  $\theta = 5$  and  $\sigma^2 = 1$ , and  $\theta = 1/5$  and  $\sigma^2 = 25$ , you'll find that the auto-covariance function is the same in both cases
- This is not good because it means we cannot detect the difference in an MA(1) model with parameter  $\theta$  and normal noise with variance  $\sigma^2$  from another MA(1) model with parameter  $1/\theta$  and normal noise with variance  $\sigma^2\theta^2$

- In other words, there is some *non-uniqueness of redundancy* in the parametrization—different choices of parameters will actually lead to the same behavior in the model at the end
- In the MA(1) case, the convention is to simply choose the parametrization with  $|\theta| < 1$ . Note that we can write

$$w_t = -\theta w_{t-1} + x_t$$

which is like an AR(1) process with the roles of  $x_t$  and  $w_t$  reversed. Thus by the same arguments that led to (5), when  $|\theta| < 1$ , we now have

$$w_t = \sum_{j=0}^{\infty} (-\theta)^j x_{t-j} \quad (14)$$

This is called the *invertible representation* of the MA(1) process (12)

- We will see soon that we can generalize this to a condition that applies to a general MA( $q$ ), yielding an analogous conclusion. The conclusion we will be looking for is explained next

## 2.3 Invertibility

- Before we turn to ARMA models, we define one last concept called *invertibility*, which generalizes what we just saw for MA(1) when  $|\theta| < 1$
- We say that a series  $x_t$ ,  $t = 0, \pm 1, \pm 2, \pm 3, \dots$  is *invertible* provided that it can be written in the form

$$w_t = \sum_{j=0}^{\infty} \pi_j x_{t-j} \quad (15)$$

for a white noise sequence  $w_t$ ,  $t = 0, \pm 1, \pm 2, \pm 3, \dots$ , and coefficients such that  $\sum_{j=0}^{\infty} |\pi_j| < \infty$ , where we set  $\pi_0 = 1$

- You should think of this as a generalization of (14), where we allow for arbitrary coefficients  $\pi_1, \pi_2, \dots$ , subject to an absolute summability condition
- And of course, note how invertibility (15) is kind of an opposite condition to causality (7)

## 3 ARMA models

- AR and MA models have complementary characteristics. The auto-covariance of an AR model decays away from  $h = 0$ , whereas that for an MA process has finite support—in other words, at a certain lag, variates along an MA sequence are completely uncorrelated (e.g., compare (6) and (13) for the AR(1) and MA(1) models)
- (The spectral perspective, by the way, provides another nice way of viewing these complementary characteristics. In the spectral domain, the story is somewhat flipped: the spectral density of an MA process generally decays away from  $\omega = 0$ , whereas that for an AR process can be much more locally concentrated around particular frequencies; recall our examples from the last lecture)
- Sure, there is some duplicity in representation here. We can write some AR models as infinite-order MA models; and we think of some MA models as AR models with the role of  $x_t$  and  $w_t$  reversed
- But that's OK! We can generally take the most salient features that each model represents, compactly, and combine them to get a compact representation of both sets of features, simultaneously. That is exactly what an ARMA model does
- Precisely, an ARMA model of orders  $p, q \geq 0$ , denoted ARMA( $p, q$ ), is of the form

$$x_t = \sum_{j=1}^p \phi_j x_{t-j} + \sum_{j=0}^q \theta_j w_{t-j} \quad (16)$$

where  $w_t$ ,  $t = 0, \pm 1, \pm 2, \pm 3, \dots$  is a white noise sequence

- The coefficients  $\phi_1, \dots, \phi_p, \theta_0, \dots, \theta_q$  in (16) are fixed (nonrandom), and we assume  $\phi_p, \theta_q \neq 0$ , and we set  $\theta_0 = 1$ . Note that in (16), we have  $\mathbb{E}(x_t) = 0$  for all  $t$
- As before, we can represent an ARMA model more compactly using backshift notation, rewriting (16) as

$$(1 - \phi_1 B - \phi_2 B^2 - \dots - \phi_p B^p)x_t = (1 + \theta_1 B + \theta_2 B^2 + \dots + \theta_q B^q)w_t \quad (17)$$

- Often, authors will write (17) even more compactly as

$$\phi(B)x_t = \theta(B)w_t \quad (18)$$

where  $\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \dots - \phi_p B^p$  and  $\theta(B) = 1 + \theta_1 B + \theta_2 B^2 + \dots + \theta_q B^q$  are the AR and MA operators, respectively, as before

### 3.1 Parameter redundancy

- For ARMA models, there is an issue of parameter redundancy, just like there are for MA models. If  $\eta(B)$  is any (invertible) operator, then we can transform (18) by applying  $\eta(B)$  on both sides,

$$\eta(B)\phi(B)x_t = \eta(B)\theta(B)w_t$$

which may look like a different model, but the dynamics are entirely the same

- As an example, consider white noise  $x_t = w_t$ , and multiply both sides by  $\eta(B) = 1 - B/2$ . This gives:

$$x_t = \frac{1}{2}x_{t-1} + w_t - \frac{1}{2}w_{t-1}$$

which looks like an ARMA(1,1) model, but it is nothing else than white noise!

- How do we resolve this issue? Doing so requires introducing another concept. The *AR and MA polynomials* associated with the ARMA process (16) are

$$\phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \dots - \phi_p z^p \quad (19)$$

$$\theta(z) = 1 + \theta_1 z + \theta_2 z^2 + \dots + \theta_q z^q \quad (20)$$

respectively. To be clear, these are polynomials over *complex numbers*  $z \in \mathbb{C}$ . (Note that these are just what we get by taking the AR and MA operators, and replacing the backshift operator  $B$  by a complex argument  $z$ )

- As it turns out, lots of important properties of ARMA models can be derived by placing conditions on the AR and MA polynomials. Here we'll see the first one, to deal with parameter redundancy: *when we speak of an ARMA model, we implicitly assume that the AR and MA polynomials  $\phi(z)$  and  $\theta(z)$ , in (19) and (20), have no common factors.* This rules out a case like

$$x_t = \frac{1}{2}x_{t-1} + w_t - \frac{1}{2}w_{t-1}$$

because the polynomials each have  $1 - z/2$  as a common factor. Hence we do not even refer to the above as an ARMA(1,1) model

### 3.2 Causality and invertibility

- Now we learn about two more conditions on the AR and MA polynomials that imply important general properties of the underlying ARMA process, and generalize calculations we saw earlier for AR(1) and MA(1) models

- Before we describe these, we recall the following terminology: for a polynomial  $f(z) = \sum_{j=0}^k a_j z^j$ , we say that  $z$  is a *root* of  $f$  provided  $f(z) = 0$
- And we say that a point  $z \in \mathbb{C}$  lies *outside of the unit circle* (in the complex plane  $\mathbb{C}$ ) provided that  $|z| > 1$ , where  $|z|$  is the complex modulus of  $z$  (recall  $|z|^2 = \text{Re}\{z\}^2 + \text{Im}\{z\}^2$ )
- The first property: *the ARMA process (16) has a causal representation (7) if and only if all roots of the AR polynomial (19) lie outside of the unit circle.* The coefficients  $\psi_0, \psi_1, \psi_2, \dots$  in the causal representation can be determined by solving

$$\psi(z) = \frac{\theta(z)}{\phi(z)} \iff \sum_{j=0}^{\infty} \psi_j z^j = \frac{1 + \theta_1 z + \theta_2 z^2 + \dots + \theta_q z^q}{1 - \phi_1 z - \phi_2 z^2 - \dots - \phi_p z^p}, \quad \text{for } |z| < 1$$

- The second property: *the ARMA process (16) is invertible (15) if and only if all roots of the MA polynomial (20) lie outside of the unit circle.* The coefficients  $\pi_1, \pi_2, \dots$  in the invertible representation can be determined by solving

$$\pi(z) = \frac{\phi(z)}{\theta(z)} \iff \sum_{j=0}^{\infty} \pi_j z^j = \frac{1 - \phi_1 z - \phi_2 z^2 - \dots - \phi_p z^p}{1 + \theta_1 z + \theta_2 z^2 + \dots + \theta_q z^q}, \quad \text{for } |z| < 1$$

- (As an aside, you can see that parameter redundancy issues would not affect the causal and invertible representations, as they shouldn't, because any common factors in  $\phi(z)$  and  $\theta(z)$  would cancel in their ratios which determine the coefficients in the causal and invertible expansions)
- We won't cover the proofs of these properties or go into further details about them or their importance. You can see Appendix B.2 of SS for proofs, or read more in Chapter 3.1 of SS about these properties, and read other parts of the SS book for their consequences (and significance). At a high-level, they are worth knowing because they are considered foundational results in ARMA models (just like it is worth knowing foundations for regression models, etc.)
- Also, recall that causality implies stationarity, so what we have just learned is the general result that renders an ARMA( $p, q$ ) process stationary

### 3.3 Difference equations

Diff equations  $\rightarrow$  don't cover. Auto-covariance

## 4 Partial auto-covariance

## 5 ARIMA models

Stationarity and differencing ARIMA( $p, d, q$ ) Backshift notation Seasonality extensions

### 5.1 Parameter estimation

complicated  $\rightarrow$  don't cover.

### 5.2 Regression with correlated errors

complicated  $\rightarrow$  mostly don't cover

## 6 Forecasting