MBA Business Foundations, Quantitative Methods: Session Three

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Today

Exponential functions: review

A hypothetical:

Exponential functions: review

A hypothetical: the infinitely evil loan shark...

Exponential functions: review

A hypothetical: the infinitely evil loan shark...

Exponential functions: review

Suppose you borrow one dollar from an ordinary loan shark that charges 100% interest, n times per year. Then after the year, you will owe: $\left(1+\frac{1}{n}\right)^n$

- In particular, after 1 year:

  For $n = 1$, you will owe 2.

  For $n = 2$, you will owe 2.25

  For $n = 3$, you will owe 2.37.

  For $n = 100$, you will owe 2.704.

- The infinitely evil loan shark: $n = \infty$,

Exponential functions: review

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  For $n = 3$, you will owe 2.37.

  For $n = 100$, you will owe 2.704.
- The infinitely evil loan shark: $n = \infty$, owe e.
- That is, $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \approx 2.7182818...!!$
- Your debt to the infinitely evil loan shark is finitely bounded by e!
- Thanks Euler!

Exponential functions: review

- If you borrow A_0, after t years, you will owe,

$$A = \lim_{n \to \infty} A_0 \left[\left(1 + \frac{1}{n} \right)^n \right]^t = A_0 e^t$$

- We have now recovered the growth model for bacteria and populations that we saw last class (with k, the growth rate, equal to 1)!
- Now suppose the loan shark charges $(k \cdot 100)\%$ interest continuously.
- Reminder: A_0 is your starting point, $(k \cdot 100)\%$ is the growth rate, t is the number of periods, and $e = (1 + 1/n)^n$ as n gets arbitrarily large.
- Then in general you will owe

$$A = A_0 e^{kt}$$

- The growth model involving e appears naturally as the continuous generalization of the compounding interest formula.

Example

The parents of a newborn child want to have $\$25,000$ for the child's college education when she is 18.

At what rate of interest, compounded continuously, must $\$10,000$ be invested now to achieve this goal?

Solution

$$
\begin{aligned}
A &= A_0 e^{rt} \\
25000 &= 10000 e^{r18} \\
\ln(25000) &= \ln(10000 \cdot e^{r18}) \\
\ln(25000) &= \ln(10000) + \ln e^{r18} \\
\ln(25000) &= \ln(10000) + r18 \\
\ln(25000) - \ln(10000) &= r18 \\
\ln(\frac{25000}{10000}) &= r18 \\
r &= 0.05 = 5\%
\end{aligned}
$$

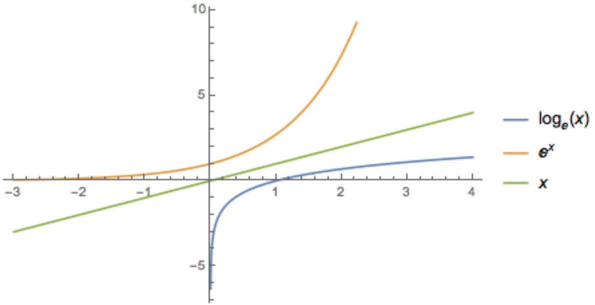
Logarithms: review

- A logarithmic function is the inverse of an exponential function:

  If $\log_b x = c$ then $b^c = x$ and $\log_b b^x = x$.
- Special case where the base b of the exponential is e.
- We call it the natural logarithm (natural log) and it has its own notation: $\log_e x = \ln x$.

Logarithms: review

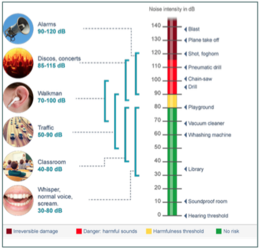
Natural log is the inverse of the exponential function of base e: $e^x = c \leftrightarrow \ln c = x$ or $\ln e^x = x$.



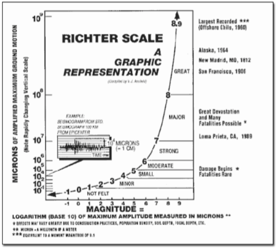
Logarithmic applications

- Logarithms are often used for measurement scales, where we need to compare orders of magnitude.
- Magnitude 6 earthquake is 10 times stronger than magnitude 5 earthquake. Magnitude 7 earthquake is $10 \times 10 = 100$ times strong than magnitude 5 earthquake.

Order in which websites are presented to you when you search on Google



Decibels for sound



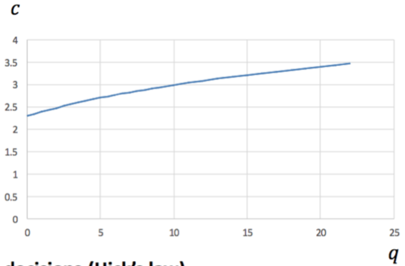
Richter scale for earthquakes

Logarithmic applications

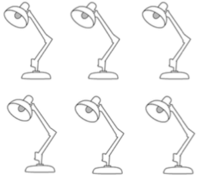
Logarithms are also present in many "laws" in economics, statistics, finance, psychology, etc.

Economics: economies of scale

Total cost as a function of quantity:
$c = \ln(10 + q)$



Psychology: how human beings make decisions (Hick's law)



$$T = b\log_2(n+1)$$

T: time to react
n: number of choices

Exponential and logarithmic equations

- How to solve equations that involve exponentials and logarithms?
- Formulas at our disposal:

$$e^x = e^y \leftrightarrow x = y$$

$$\ln x = \ln y \leftrightarrow x = y$$

$$e^{\ln x} = x$$

$$\ln(e^y) = y$$

Practice

Solve for x:

1. $e^x = p$
2. $e^{3x} = 403.43$
3. $\ln(x+1) = 2$
4. $\ln(3x-2) = 5$

– more challenging –

5. $\ln(x) + \ln(x+6) = \ln(5x+12)$
6. $\ln(10) - \ln(7-x) = \ln(x)$

Practice Solutions, 1-4

Solve for x:

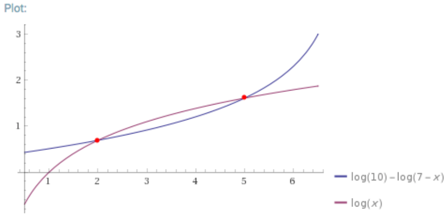
- $e^x = p \to x = \ln p$
- $e^{3x} = 403.43 \to x = 2$
- $\ln(x+1) = 2 \to x = e^2 - 1$
- $\ln(3x-2) = 5 \to x = \frac{1}{3}(e^5 + 2) \approx 50.138$

Ex 5 Solution

$$
\begin{aligned}
\ln(x) + \ln(x+6) &= \ln(5x+12) \\
\ln(x(x+6)) &= \ln(5x+12) \\
x(x+6) &= 5x+12 \\
x^2 + 6x &= 5x+12 \\
x^2 + x - 12 &= 0 \\
(x-3)(x+4) &= 0 \to x = 3
\end{aligned}
$$

Ex 6 Solution

$$
\begin{aligned}
\ln(10) - \ln(7-x) &= \ln(x) \\
\ln\left(\frac{10}{7-x}\right) &= \ln x \\
\left(\frac{10}{7-x}\right) &= x \\
10 &= x(7-x) \\
10 &= 7x - x^2 \\
x^2 - 7x + 10 &= 0 \\
(x-5)(x-2) &= 0 \\
\rightarrow x = 2, x &= 5
\end{aligned}
$$



Exponents and logarithms: changing basis

- How does one go from a^x to e^x?

$$a^x = \left(e^{\ln a}\right)^x = e^{x \ln a} = (e^x)^{\ln a}$$

- How does one go from $\log_a y$ to $\ln y$?

$$\log_a y = \frac{\ln y}{\ln a}$$

Summary of logs and exponents

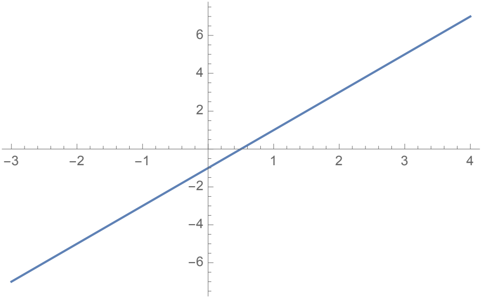
- Exponential function is a function of the type $f(x) = b^x$
- Logarithmic function is the inverse of this function, $\log_b(b^x) = x$
- Applications: for exponents, compounded growth (GDP, population growth, interest); for logarithms, order of magnitude scales, statistics, utility functions.
- Solving exponential and logarithmic equations.
- Special case of $b = e \approx 2.71828$: $f(x) = e^x$, and $f(x) = \ln x$.

Case discussion

Case 1: Motorcycle helmets with bluetooth (A): pricing bluetooth chips

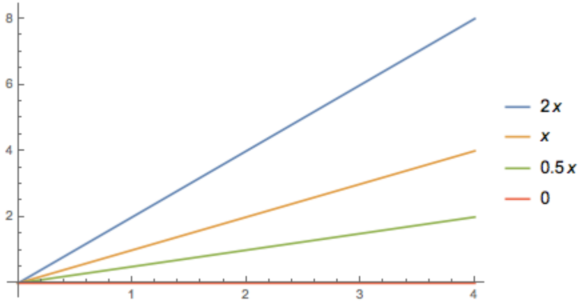
Rate of change of linear functions

Example of linear function: $f(x) = 2x - 1$



- Rate of change: $\frac{\text{change in } f(x)}{\text{change in } x}$
- If x changes from 0 to 5 what values does $f(x)$ take?
- What is the rate of change here?
- Does it depend on the points we pick? What does it correspond to?

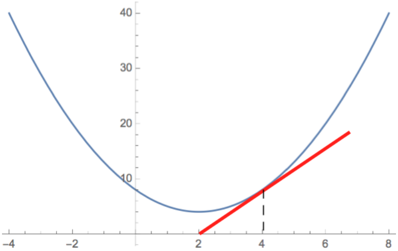
Rate of change of linear functions



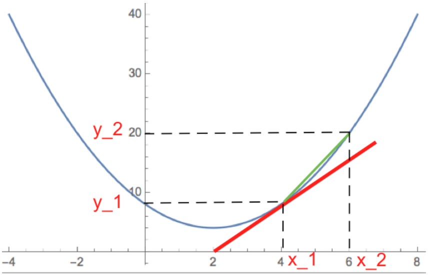
- The steeper the curve, the larger the rate of change
- Slope $= 0$, horizontal line. Slope $= \infty$, vertical line.

Rate of change of nonlinear functions

- Like linear functions, would like to find the slope of the curve.
- But the "slope" of the curve is changing along the curve.
- Thus, rate of change will be point specific (point??).
- Given by the slope of the tangent line at that "point."



Rate of change of nonlinear functions



- The slope of the tangent can be viewed as the slope of a (shrinking) chord (green) – secant line.

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

- This gives us average rate of change over a (small) interval.
- Ex: average speed (per minute).

Derivatives

- What if we want instantaneous speed (huh?)
- Find the value as the chord shrinks to 0.
- That is, as $\Delta x \to 0$ and $\Delta y \to 0$.
- Derivative of f at x:

$$f'(x) = \frac{d}{dx}f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

  where $h = \Delta x$ and $[f(x+h) - f(x)] = \Delta y$.
- Ex: use this definition to find $\frac{d}{dx}x^2$.
- Ex: use it to find $\frac{d}{dx}\log x$. Hint: $\lim_{k \to 0} \frac{\log(1+k)}{k} = 1$.

Ex: $f(x) = x^2$

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{x^2 + 2hx + h^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{2hx + h^2}{h} \\
&= \lim_{h\to 0} \frac{h(2x+h)}{h} \\
&= \lim_{h\to 0} 2x + h \\
&= 2x
\end{aligned}
$$

Ex $f(x) = \log x$

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\log(x+h) - \log(x)}{h} \\
&= \lim_{h\to 0} \frac{\log\left(\frac{x+h}{x}\right)}{h} \\
&= \lim_{h\to 0} \frac{\log\left(1 + \frac{h}{x}\right)}{h} \\
&= \lim_{h\to 0} \frac{\log\left(1 + \frac{h}{x}\right)}{\frac{h}{x}} \cdot \frac{1}{x} \\
&= 1 \cdot \frac{1}{x} = \frac{1}{x}
\end{aligned}
$$

Rules for differentiation

- $f(x) = a \to f'(x) = 0$, and $f(x) = x \to f'(x) = 1$
- $f(x) = bx \to f'(x) = b$
- $f(x) = x^n \to f'(x) = nx^{n-1}$
- $f(x) = bx^n \to f'(x) = bnx^{n-1}$
- $f(x) = x \to f'(x) = 1$
- $(f(x) + g(x))' = f'(x) + g'(x)$
- $[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$
- $f(g(x)) = f'(g(x))g'(x)$
- $[\log(x)]' = 1/x$, and $(e^x)' = e^x$

Practice

Differentiate the following functions:

- $f(x) = 7x + 5$
- $f(x) = x^2 + 3x + 4$
- $f(x) = 5x^7 + 3x^4 + x^2 + 4x + 100$
- $f(x) = e^{3x^2}$

Practice Solutions

Differentiate the following functions:

- $f(x) = 7x + 5 \to f'(x) = 7$
- $f(x) = x^2 + 3x + 4 \to f'(x) = 2x + 3$
- $f(x) = 5x^7 + 3x^4 + x^2 + 4x + 100 \to f'(x) = 35x^6 + 12x^3 + 2x + 4$
- $f(x) = e^{3x^2}$

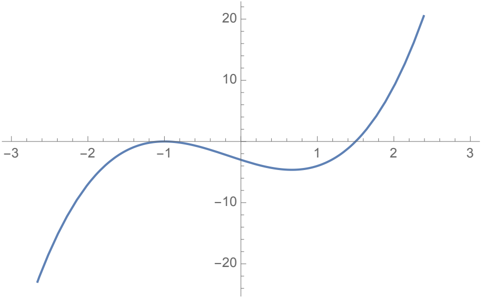
  $f(x) = e^x,\ g(x) = 3x^2 \to f'(g(x))g'(x) = e^{3x^2} \cdot 6x = 6xe^{3x^2}$

Analyzing functions with derivatives

- We are given some function $f(x)$ and want to know something about its behavior at x_1.
- Find $f'(x)$.
- Find $f'(x_1)$.
- If $f'(x_1) > 0$ the function is increasing "at that point".
- $f'(x_1) < 0$ the function is decreasing "at that point".
- if for all $x, f'(x) > 0$ the function is increasing.
- if for all $x, f'(x) < 0$ the function is decreasing.
- $f'(x_1) = 0$ the function is at a maximum or minimum (most likely).

Example

Consider the function $f(x) = 2x^3 + x^2 - 4x - 3$



- Find $f'(x)$.
- Evaluate x at $(-2, -1, 0, 1, 1.5)$.

Solution

- $f'(x) = 6x^2 + 2x - 4$
- $f'(-2) = 16$
- $f'(-1) = 0$
- $f'(0) = -4$
- $f'(1) = 4$
- $f'(1.5) = 12.5$

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