Statistical Learning - Probability and Distributions

Probability – Meaning & Concepts

- **Probability** refers to chance or likelihood of a particular event-taking place.
- An **event** is an outcome of an experiment.
- An **experiment** is a process that is performed to understand and observe possible outcomes.
- Set of all outcomes of an experiment is called the **sample space.**

Example

- In a manufacturing unit three parts from the assembly are selected. You are observing whether they are defective or non-defective. Determine

a) The sample space.

b) The event of getting at least two defective parts.

Solution

a) Let S = Sample Space. It is pictured as under:

GGG	GGD	GDG	DGG
GDD	DGD	DDG	DDD

D = Defective

G = Non-Defective

b) Let E denote the event of getting at least two defective parts. This implies that E will contain two defectives, and three defectives. Looking at the sample space diagram above,

E ={GDD, DGD, DDG, DDD}. It is easy to see that E is a part of S and commonly called as a subset of S. Hence an event is always a subset of the sample space.

Definition of Probability

Probability of an event A is defined as the ratio of two numbers m and n. In symbols

$$P(A) = \frac{m}{n}$$

where m= number of ways that are favorable to the occurrence of A and n= the total number of outcomes of the experiment (all possible outcomes)

Please note that P (A) is always > = 0 and always < = 1.

P (A) is a pure number.

Extreme Values of Probability

The range with in which probability of an event lies can be best understood by the following diagram. The glass shows three stages- Empty, half-full, and full to explain the properties of probability.

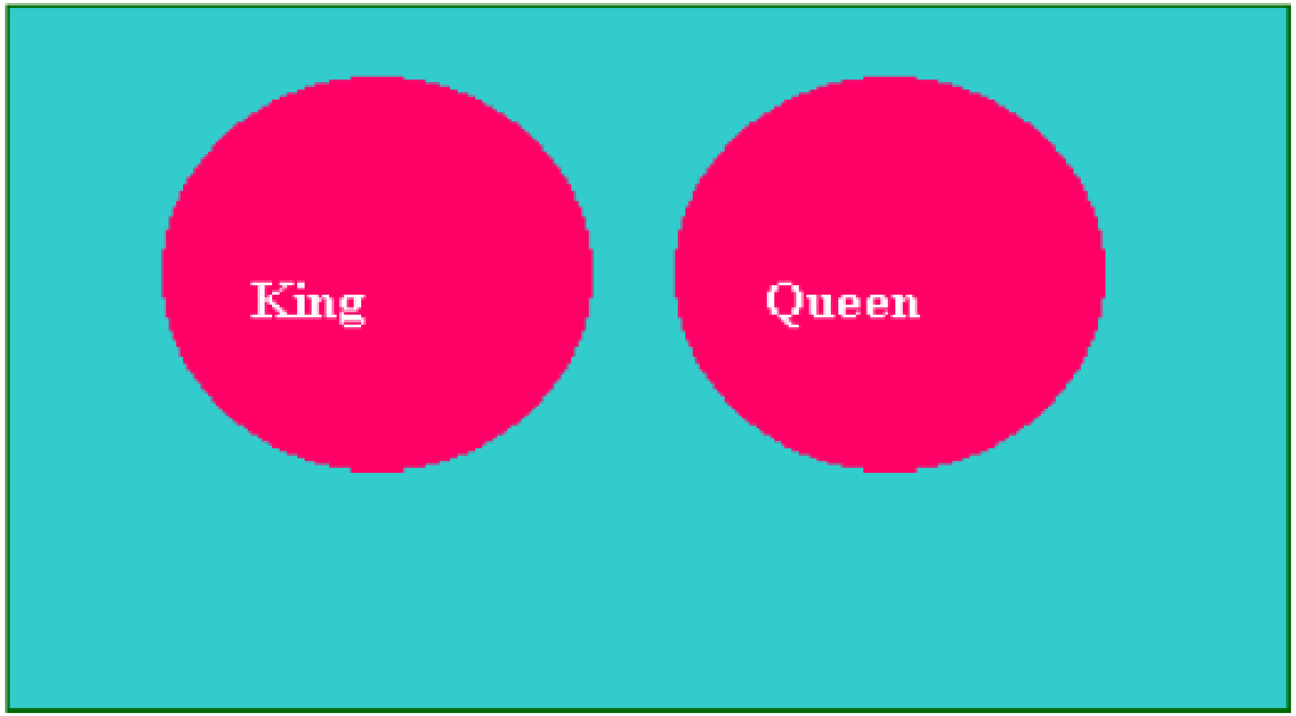
100% Chance or Certainty

(50% Chance) Equally Likely

(0% Chance) Impossibility

Mutually Exclusive Events

Two events A and B are said to be mutually exclusive if the occurrence of A precludes the occurrence of B. For example, from a well shuffled pack of cards, if you pick up one card at random and would like to know whether it is a King or a Queen. The selected card will be either a King or a Queen. It cannot be both a King and a Queen. If King occurs, Queen will not occur and Queen occurs, King will not occur.



Independent Events

- Two events A and B are said to be independent if the occurrence of A is in no way influenced by the occurrence of B. Likewise occurrence of B is in no way influenced by the occurrence of A.

Rules for Computing Probability

1) Addition Rule -Mutually Exclusive Events

$$P(A \bigcup B) = P(A) + P(B)$$

This rule says that the probability of the union of A and B is determined by adding the probability of the events A and B.

Here the symbol $A \bigcup B$ is called A union B meaning A occurs, or B occurs or both A and B simultaneously occur. When A and B are mutually exclusive, A and B cannot simultaneously occur.

greatlearning
Learning for Life

Rules for Computing Probability

2) Addition Rule –Events are not Mutually Exclusive

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

This rule says that the probability of the union of A and B is determined by adding the probability of the events A and B and then subtracting the probability of the intersection of the events A and B.

The symbol $A \cap B$ is called A intersection B meaning

both A and B simultaneously occur.

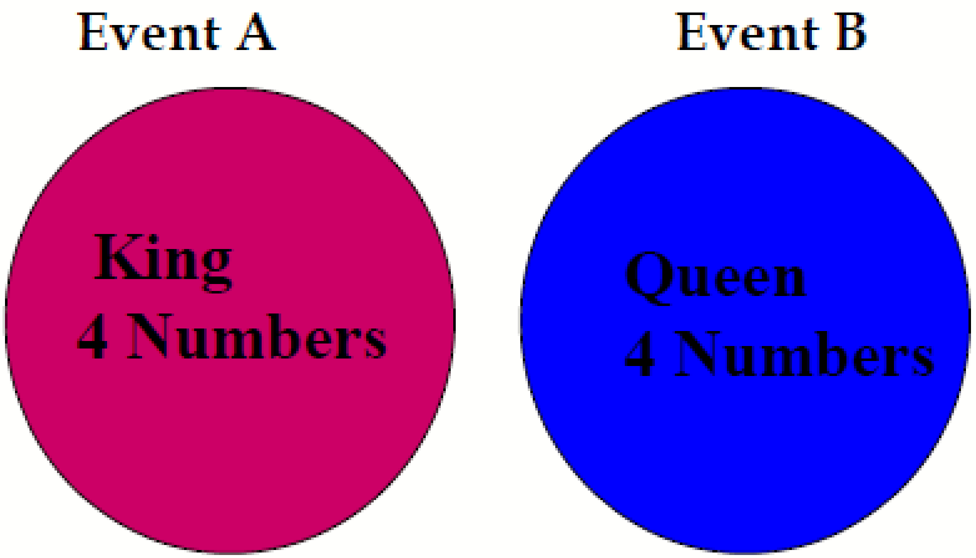
Example for Addition Rules

- From a pack of well-shuffled cards, a card is picked up at random.

1) What is the probability that the selected card is a King or a Queen?
2) What is the probability that the selected card is a King or a Diamond?

Solution to Part 1)

Look at the Diagram:



Let A = getting a King
Let B = getting a Queen

There are 4 kings and there are 4 Queens. The events are clearly mutually exclusive. Applying the formula $P(A \cup B) = P(A) + P(B)$

= 4/52 +4/52 = 8/52 =2/13

d

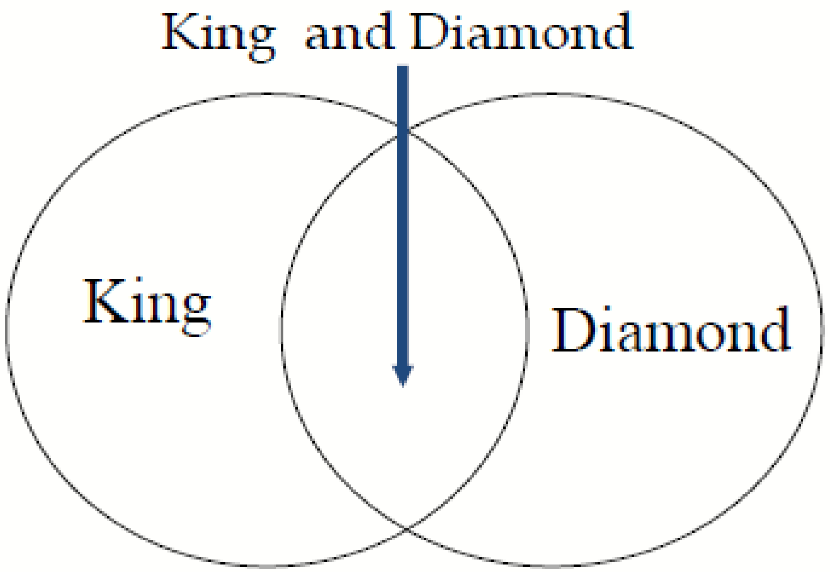
Solution to part 2)

Look at the Diagram:

There are totally 52 cards in a pack out of which 4 are Kings and 13 are Diamonds. Let A= getting a King and B= getting a Diamond. The two events here are not mutually exclusive because you can have a card, which is both a King and a Diamond called King Diamond.

$$P(K \cup D) = P(K) + P(D) - P(K \cap D)$$

$= 4/52+13/52-1/52= 16/52=4/13$



Multiplication Rule

Independent Events

$$P(A \cap B) = P(A).P(B)$$

This rule says when the two events A and B are independent, the probability of the simultaneous occurrence of A and B (also known as probability of intersection of A and B) equals the product of the probability of A and the probability of B. Of course this rule can be extended to more than two events.

Multiplication Rule

Independent Events-Example

Example:

The probability that you will get an A grade in Quantitative Methods is 0.7. The probability that you will get an A grade in Marketing is 0.5. Assuming these two courses are independent, compute the probability that you will get an A grade in both these subjects.

Solution:

Let A = getting A grade in Quantitative Methods

Let B =getting A grade in Marketing

It is given that A and B are independent.

$$P(A \cap B) = P(A).P(B) \quad = \quad 0.7.0.5 = 0.35.$$

Multiplication Rule

Events are not independent

$$P(A \cap B) = P(A).P(B/A)$$

This rule says that the probability of the intersection of the events A and B equals the product of the probability of A and the probability of B given that A has happened or known to you. This is symbolized in the second term of the above expression as P (B/ A). P(B/ A) is called the conditional probability of B given the fact that A has happened.

We can also write $P(A \cap B) = P(B).P(A/B)$ if B has already happened.

Multiplication Rule

Events are not independent-Example

From a pack of cards, 2 cards are drawn in succession one after the other. After every draw, the selected card is not replaced. What is the probability that in both the draws you will get Spades?

Solution:

Let A = getting Spade in the first draw

Let B = getting spade in the second draw.

The cards are not replaced.

This situation requires the use of conditional probability.

P(A) = 13/52 (There are 13 Spades and 52 cards in a pack)

P(B/A)=12/51(There are 12 Spades and 51 cards because the first card selected is not replaced after the first draw)

$$P(A \cap B) = P(A).P(B/A) = (13/52).(12/51)=156/2652= 1/17.$$

Marginal Probability

- Contingency table consists of rows and columns of two attributes at different levels with frequencies or numbers in each of the cells. It is a matrix of frequencies assigned to rows and columns.
- The term marginal is used to indicate that the probabilities are calculated using a contingency table (also called joint probability table).

Marginal Probability - Example

A survey involving 200 families was conducted. Information regarding family income per year and whether the family buys a car are given in the following table.

Family	Income below Rs 10 Lakhs	Income of Rs. >=10 lakhs	Total
Buyer of Car	38	42	80
Non-Buyer	82	38	120
Total	120	80	200

a) What is the probability that a randomly selected family is a buyer of the car?
b) What is the probability that a randomly selected family is both a buyer of car and belonging to income of Rs. 10 lakhs and above?
c) A family selected at random is found to be belonging to income of Rs 10 lakhs and above. What is the probability that this family is buyer of car?

greatlearning
Learning for Life

Solution

a) What is the probability that a randomly selected family is a buyer of the Car?

- 80/200 =0.40.

b) What is the probability that a randomly selected family is both a buyer of car and belonging to income of Rs. 10 lakhs and above?

- 42/200 =0.21.

c) A family selected at random is found to be belonging to income of Rs 10 lakhs and above. What is the probability that this family is buyer of car?

- 42/80 =0.525. Note this is a case of conditional probability of buyer given income is Rs. 10 lakhs and above.

Bayes' Theorem

- Bayes' Theorem is used to revise previously calculated probabilities based on new information.
- Developed by Thomas Bayes in the 18th Century.
- It is an extension of conditional probability.

greatlearning
Learning for Life

Bayes' Theorem

Given a hypothesis H and evidence E, Bayes' theorem states that the relationship between the probability of the hypothesis $P(H)$ before getting the evidence and the probability $P(H \mid E)$ of the hypothesis after getting the evidence is

$$P(H \mid E) = \frac{P(E \mid H)}{P(E)} P(H).$$

Many modern machine learning techniques rely on Bayes' theorem. For instance, spam filters use Bayesian updating to determine whether an email is real or spam, given the words in the email. Additionally, many specific techniques in statistics, such as calculating p-values or interpreting medical results, are best described in terms of how they contribute to updating hypotheses using Bayes' theorem.

Bayes' Theorem

$$P(B_i \mid A) = \frac{P(A \mid B_i)P(B_i)}{P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2) + \cdots + P(A \mid B_k)P(B_k)}$$

- where:

  B_i = i^{th} event of k mutually exclusive and collectively exhaustive events

  A = new event that might impact $P(B_i)$

Bayes' Theorem - Example

Bayesian Spam Filtering

One clever application of Bayes' Theorem is in spam filtering. We have

- Event A: The message is spam.
- Test X: The message contains certain words (X)

Plugged into a more readable formula (from Wikipedia):

$$\Pr(\text{spam}|\text{words}) = \frac{\Pr(\text{words}|\text{spam})\Pr(\text{spam})}{\Pr(\text{words})}$$

Bayes' Theorem - Example

Bayesian filtering allows us to predict the chance a message is really spam given the "test results" (the presence of certain words). Clearly, words like "viagra" have a higher chance of appearing in spam messages than in normal ones.

Spam filtering based on a blacklist is flawed — it's too restrictive and false positives are too great. But Bayesian filtering gives us a middle ground — we use *probabilities*. As we analyze the words in a message, we can compute the chance it is spam (rather than making a yes/no decision). If a message has a 99.9% chance of being spam, it probably is. As the filter gets trained with more and more messages, it updates the probabilities that certain words lead to spam messages. Advanced Bayesian filters can examine multiple words in a row, as another data point.

What is a Probability Distribution

- In precise terms, a **probability distribution** is a total listing of the various values the random variable can take along with the corresponding probability of each value. A real life example could be the pattern of distribution of the machine breakdowns in a manufacturing unit.
- The random variable in this example would be the various values the machine breakdowns could assume.
- The probability corresponding to each value of the breakdown is the relative frequency of occurrence of the breakdown.
- The probability distribution for this example is constructed by the actual breakdown pattern observed over a period of time. Statisticians use the term "observed distribution" of breakdowns.

Binomial Distribution

- The Binomial Distribution is a widely used probability distribution of a discrete random variable.
- It plays a major role in **quality control** and **quality assurance** function. Manufacturing units do use the binomial distribution for **defective** analysis.
- Reducing the number of defectives using the proportion defective control chart (p chart) is an accepted practice in manufacturing organizations.
- Binomial distribution is also being used in **service organizations** like banks, and insurance corporations to get an idea of the proportion customers who are satisfied with the service quality.

Conditions for Applying Binomial Distribution (Bernoulli Process)

- Trials are independent and random.
- There are fixed number of trials (n trials).
- There are only two outcomes of the trial designated as *success* or *failure*.
- The probability of success is uniform through out the n trials

Binomial Probability Function

Under the conditions of a Bernoulli process,

The probability of getting x successes out of n trials is indeed the definition of a Binomial Distribution. The Binomial Probability Function is given by the following expression

$$P(x) = \binom{n}{x} P^x (1 - P)^{n-x}$$

Where $P(x)$ is the probability of getting x successes in n trials

$\binom{n}{x}$ is the number of ways in which x successes can take place out of n trials

$$= \frac{n!}{x!\,(n-x)!}$$

P is the probability of success, which is the same through out the n trials.

P is the parameter of the Binomial distribution

x can take values 0, 1, 2, ..., n

Example for Binomial Distribution

A bank issues credit cards to customers under the scheme of Master Card. Based on the past data, the bank has found out that 60% of all accounts pay on time following the bill. If a sample of 7 accounts is selected at random from the current database, construct the Binomial Probability Distribution of accounts paying on time.

Mean and Standard Deviation of the Binomial Distribution

The mean μ of the Binomial Distribution is given by $\mu = E(x) = np$

The Standard Deviation σ is given by

$$\sigma = \sqrt{np(1-p)}$$

For the example problem in the previous two slides,
Mean $= 7 \times 0.6 = 4.2$.
Standard Deviation $= \sqrt{4.2(1-0.60)} = 1.30$

Poisson Distribution

- Poisson Distribution is another discrete distribution which also plays a major role in **quality control** in the context of reducing the number of defects per standard unit.
- Examples include number of defects per item, number of defects per transformer produced, number of defects per 100 m2 of cloth, etc.
- Other examples would include 1) The number of cars arriving at a highway check post per hour; 2) The number of customers visiting a bank per hour during peak business period; 3) The number of pixels in the image that are corrupted.

Poisson Probability Function

Poisson Distribution Formula

$$P(x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

where

$P(x)$ = Probability of x successes given an idea of λ

λ = Average number of successes

e = 2.71828(based on natural logarithm)

x = successes per unit which can take values 0, 1, 2, 3,…………..∞

λ is the Parameter of the Poisson Distribution.

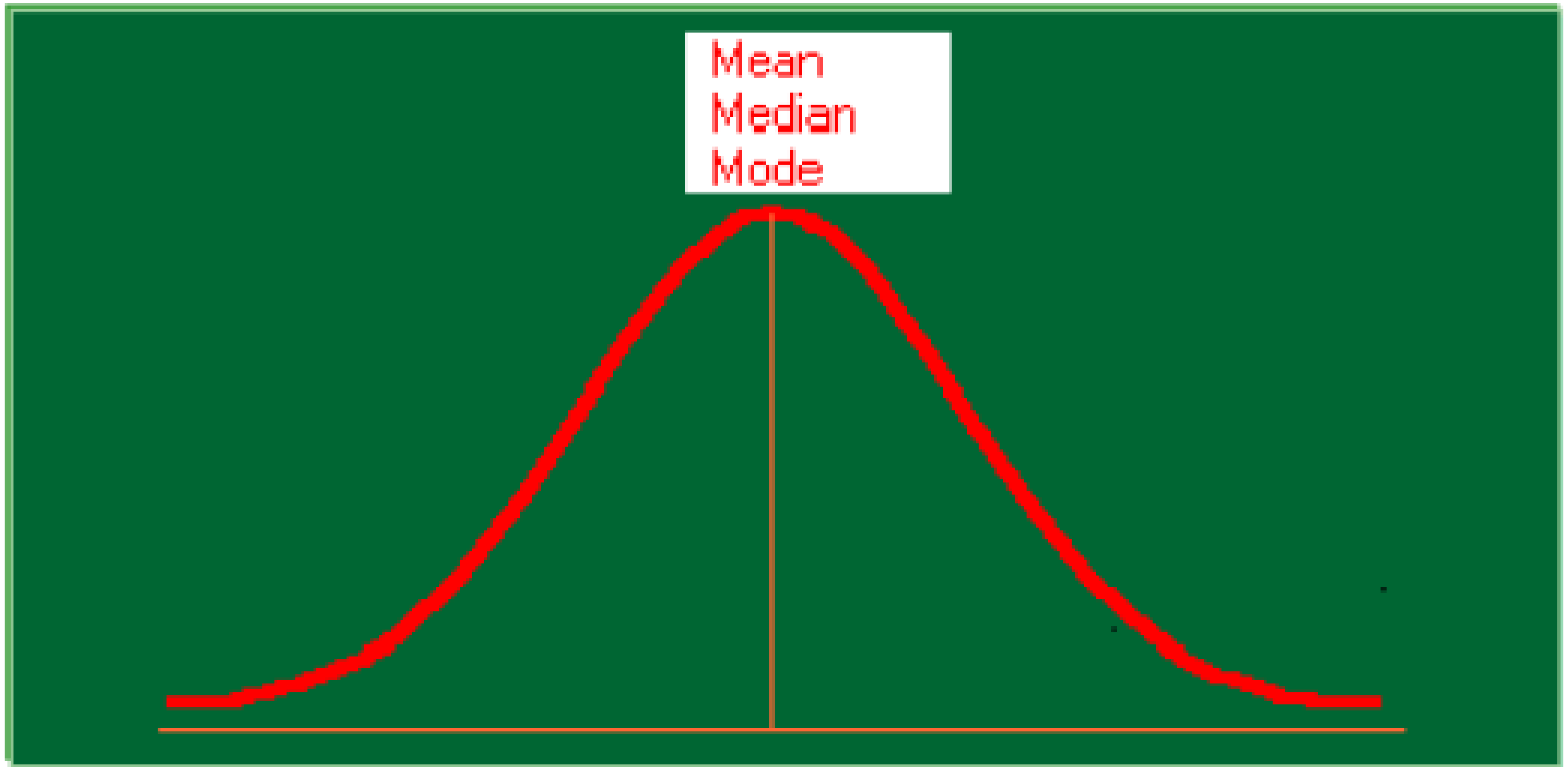
Mean of the Poisson Distribution is = λ

Standard Deviation of the Poisson Distribution is = $\sqrt{\lambda}$

Example – Poisson Distribution

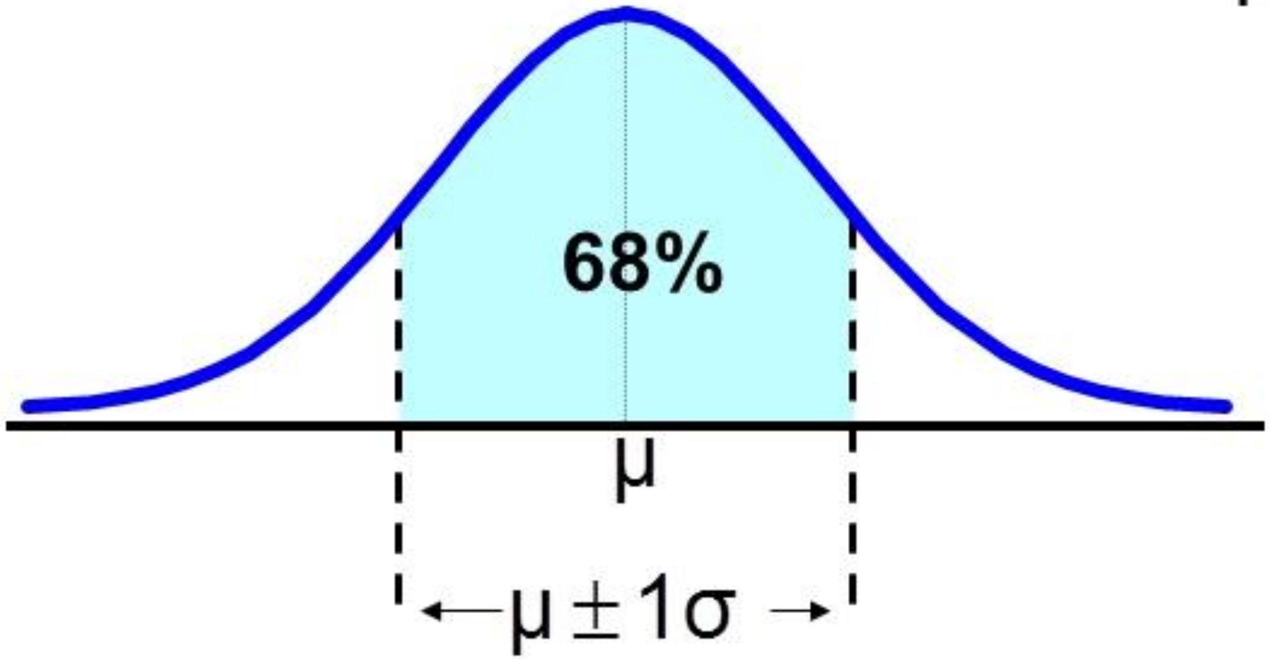
If on an average, 6 customers arrive every two minutes at a bank during the busy hours of working, a) what is the probability that exactly four customers arrive in a given minute? b) What is the probability that more than three customers will arrive in a given minute?

Normal Distribution



Normal Distribution

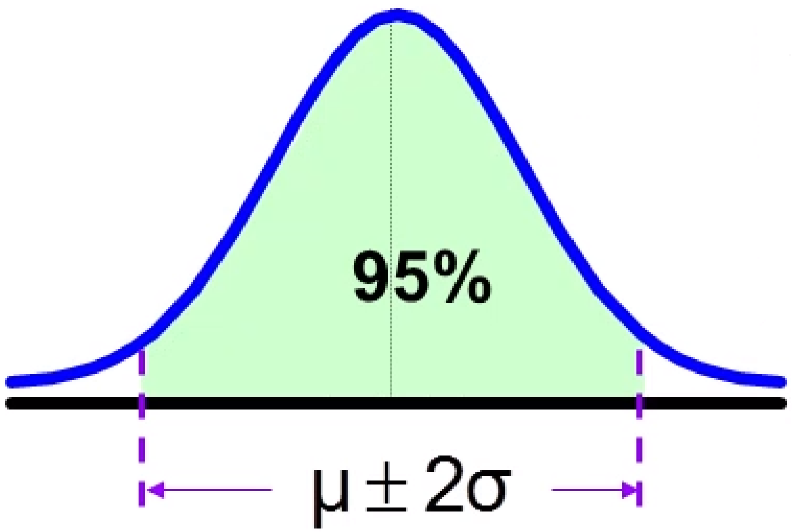
- The empirical rule approximates the variation of data in a bell-shaped distribution
- Approximately 68% of the data in a bell shaped distribution is within 1 standard deviation of the mean or $\mu \pm 1\sigma$

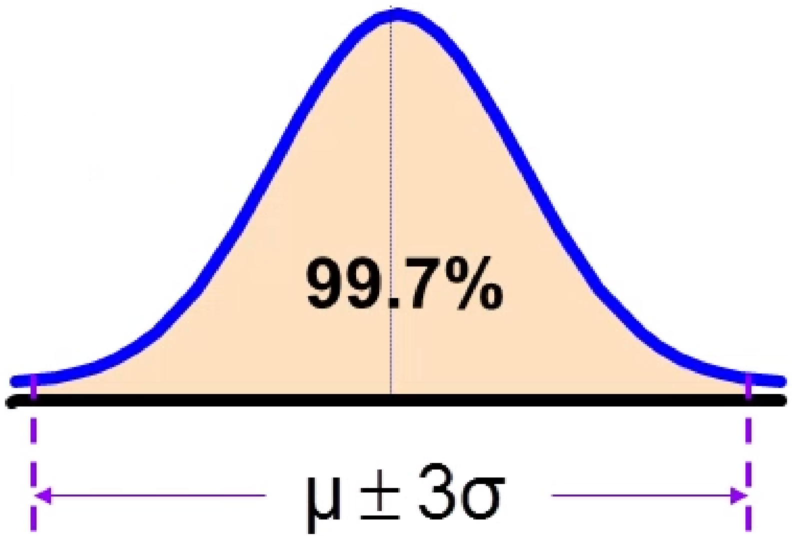


Normal Distribution

- Approximately 95% of the data in a bell-shaped distribution lies within two standard deviations of the mean, or $\mu \pm 2\sigma$

- Approximately 99.7% of the data in a bell-shaped distribution lies within three standard deviations of the mean, or $\mu \pm 3\sigma$





Properties of Normal Distribution

- The normal distribution is a continuous distribution looking like a bell. Statisticians use the expression "Bell Shaped Distribution".
- It is a beautiful distribution in which the mean, the median, and the mode are all equal to one another.
- It is symmetrical about its mean.
- If the tails of the normal distribution are extended, they will run parallel to the horizontal axis without actually touching it. (asymptotic to the x-axis)
- The normal distribution has two parameters namely the mean μ and the standard deviation σ

Normal Probability Density Function

In the usual notation, the probability density function of the normal distribution is given below:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\left[\frac{(x-\mu)^2}{2\sigma^2}\right]}$$

x is a continuous normal random variable witl the property $-\infty < x < \infty$ meaning x can take all real numbers in the interval $-\infty < x < \infty$.

Standard Normal Distribution

INSTITUTE OF MANAGEMENT, I

The Standard Normal Variable is defined as follows:

$$Z = \frac{X - \mu}{\sigma}$$

Please note that Z is a pure number independent of the unit of measurement. The random variable Z follows a normal distribution with mean=0 and standard deviation =1.

$$f(Z) = \frac{1}{\sqrt{2\pi}} e^{-\left[\frac{Z^2}{2}\right]}$$

Example Problem

The mean weight of a morning breakfast cereal pack is 0.295 kg with a standard deviation of 0.025 kg. The random variable weight of the pack follows a normal distribution.

a)What is the probability that the pack weighs less than 0.280 kg?

b)What is the probability that the pack weighs more than 0.350 kg?

c)What is the probability that the pack weighs between 0.260 kg to 0.340 kg?