On the second largest eigenvalue of certain graphs in the perfect matching association scheme

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PIMS

Discrete Mathematics Research Group at the University of Regina

Adjacency matrices of a graph

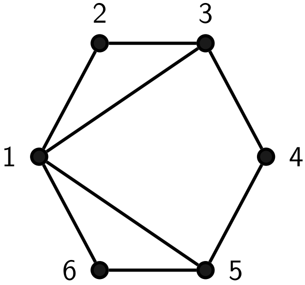
Definition

Given a graph X with vertex set $V(X)$, the **adjacency matrix of** X is a $V(X) \times V(X)$ matrix with rows and columns indexed by elements of $V(X)$. The coefficients of our matrix are defined as follows:

$$X(u, v) = \begin{cases} 1 & \text{if } u \sim v; \\ 0 & \text{if } u \perp v. \end{cases}$$

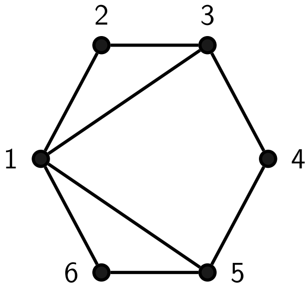
Example: Rows and columns of the matrix are indexed by the vertices of our graph.

$$\begin{array}{cc} & \begin{array}{cccccc}1&2&3&4&5&6\end{array} \\ \begin{array}{c}1\\2\\3\\4\\5\\6\end{array} & \left(\begin{array}{cccccc} &&&&& \\ &&&&& \\ &&&&& \\ &&&&& \\ &&&&& \\ &&&&& \end{array}\right)\end{array}$$



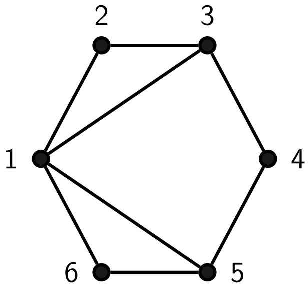
Example: $(v_1, v_2) = 1$ if and only if v_1 and v_2 are adjacent in X.

$$
\begin{array}{c c}
 & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} &
\begin{pmatrix}
  & 1 & 1 &   & 1 & 1 \\
1 &   & 1 &   &   &   \\
1 & 1 &   & 1 &   &   \\
  &   & 1 &   & 1 &   \\
1 &   &   & 1 &   & 1 \\
1 &   &   &   & 1 &
\end{pmatrix}
\end{array}
$$



Example: $(v_1, v_2) = 0$ otherwise.

$$\begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$



The spectrum of a graph

Definition

The **spectrum** of a graph G on n vertices is the spectrum of its adjacency matrix: $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$.

Definition

The **spectral gap** of a graph G is defined as $\lambda_1 - \lambda_2$.

Motivation

One of the state of the art bound on the diameter of a graph can be found in a 1991 paper of Bojan Mohar. The following is a corollary of Mohar's bound.

Corollary

Give a k-regular graph G on n-vertices with spectral gap τ, we have

$$diam(G) \leqslant 2\lceil \frac{k+\tau}{4\tau} \ln(n-1) \rceil.$$

The actual bound is given in terms of the second smallest eigenvalue of the Laplacian of G.

Association schemes

Definition

Given a set of v points, a set $\mathcal{A} = \{A_0, A_1, \ldots, A_t\}$ of $v \times v$ binary matrices is an **association scheme** if:

- $A_0 = I_v$ (the identity matrix);
- $\sum_{i=0}^{t} A_i = J$ (J is the all-one matrix);
- $A^T \in \mathcal{A}$; (A^T is the transpose)
- $A_i A_j = c_o A_0 + c_1 A_1 + \ldots + c_t A_t$, where $c_i \in \mathbb{C}$;
- $A_i A_j = A_j A_i$ (matrices commute).

The indices of the scheme are known as the **relations** of the scheme. An association scheme is **symmetric**, if $A_i = A_i^T$ for all relations.

Association schemes in lay terms

• We are given a set of v points (these points could be sets, group elements, perfect matchings....) and a set $\mathcal{A} = \{A_0, A_1, \ldots, A_t\}$ of $v \times v$ binary matrices.

• We say that v_1 is i related to v_2 if entry $(v_1, v_2) = 1$ in A_i.

• Since $\sum_{i=0}^{t} A_i = J$ (J is the all-one matrix), each pair of points is related exactly once!

Graphs in an association scheme

- For each relation i in an association scheme, we have a directed graph X_i with adjacency matrix A_i.

- The vertices of the graphs are the points of the schemes; adjacency in X_i is dictated by the relation.

- When the scheme is symmetric, then all graphs in the scheme are undirected graphs.

Perfect matching

Definition

A **matching** in a graph G is a collection of edges of G that do not have a vertex in common.

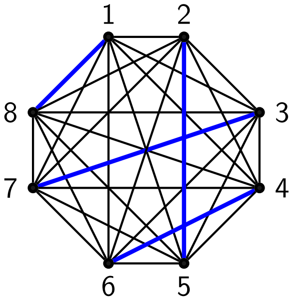


Figure: A matching of K_8 (in blue).

Perfect matching

Definition

A **perfect matching** in a graph G is a matching that covers every vertex of G.

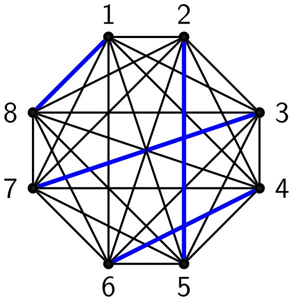


Figure: A perfect matching of K_8 (in blue).

Perfect matchings of K_{2n}

Definition

Let $M(K_{2n})$ denote the set of all perfect matchings of K_{2n}. An elementary counting argument will show that:

$$|M(K_{2n})| = (2n-1)(2n-3)\cdots(3)(1) = (2n-1)!!$$

Main goal: To construct a set of graphs, each with vertex set $M(K_{2n})$, with a relation.

Relation between two perfect matchings

We define a relation between two perfect matchings in $M(K_{2n})$.
Example: We overlap two perfect matchings of K_{2n}.

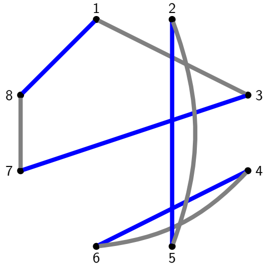


Figure: Two perfect matchings of $M(K_8)$ in grey and blue.

Relation between two perfect matchings

We define a relation between two perfect matchings in $M(K_{2n})$.
Example: This gives rise to a set of cycles of **even** lengths.

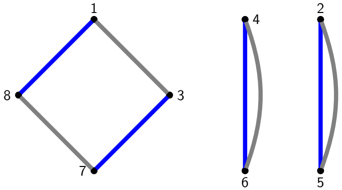


Figure: The union of these two matchings gives us 3 cycles of length 4,2, and 2 respectively.

Relation between two perfect matchings

Notation

Let $\mu \vdash n$ be a partition of n such that $\mu = [\mu_1, \mu_2, \ldots, \mu_t]$. We write $2\mu = [2\mu_1, 2\mu_2, \ldots, 2\mu_t]$ where $2\mu \vdash 2n$.

Relation between two perfect matchings

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Example: If $\mu \vdash 4$ and $\mu = [2, 1, 1]$, then $2\mu = [4, 2, 2]$, where $2\mu \vdash 8$.

Relation between two perfect matchings

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Example: If $\mu \vdash 4$ and $\mu = [2, 1, 1]$, then $2\mu = [4, 2, 2]$, where $2\mu \vdash 8$.

Observation: There exists a bijection between the set of all partitions of n and the set of even partitions of $2n$.

Relation between two perfect matchings

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Example: If $\mu \vdash 4$ and $\mu = [2, 1, 1]$, then $2\mu = [4, 2, 2]$, where $2\mu \vdash 8$.

Observation: There exists a bijection between the set of all partitions of n and the set of even partitions of $2n$.

Note: We use exponential notation to be concise. This means that

$$2\mu = [4, 2, 2] = [4, 2^2].$$

Building our graphs

Definition

Let P and Q be two perfect matchings in $M(K_{2n})$ and $\mu = [\mu_1, \mu_2, \ldots, \mu_t]$ is a partition of n. We say that P and Q are μ-related if $P \cup Q = C_{2\mu_1} \cup C_{2\mu_2} \cup \cdots \cup C_{2\mu_t}$.

Example:

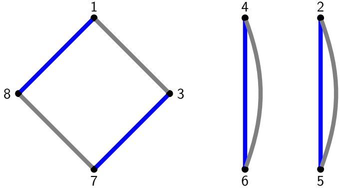

Figure: Our blue and grey perfect matching are $[2, 1^2]$-related.

Constructing a graph

Definition

The graph X_μ is a graph whose vertex set is $M(K_{2n})$. Two vertices (matchings), P and Q, are adjacent if and only if the corresponding matchings are μ-related.

Key properties:

- X_μ has $(2n-1)!!$ vertices;
- X_μ is d_μ-regular;
- X_μ is vertex transitive with automorphism group S_{2n};
- We have a graph for each even partition of $2n$.

Constructing $X_{[2,1^{n-2}]}$

Example: The graph $X_{[2,1^{n-2}]}$ is comprised of vertex set $M(K_{2n})$.

Two perfect matchings are adjacent if and only if their union gives rise to one cycle of length 4 and $n-2$ cycles of length two.

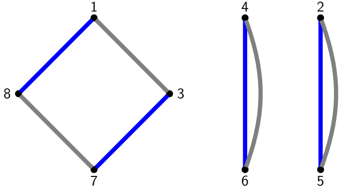

Figure: The vertices corresponding to the blue and grey perfect matching are adjacent in $X_{[2,1^2]}$.

Adjacency matrices of a perfect matching graph

Definition

Let $\lambda \vdash n$. The matrix A_λ is a $(2n-1)!! \times (2n-1)!!$ matrix with rows and columns indexed by elements of $M(K_{2n})$. The coefficients of our matrix are defined as follows:

$$X(P,Q) = \begin{cases} 1 & \text{if } P \text{ and } Q \text{ are } \mu\text{-related} \\ 0 & \text{otherwise} \end{cases}$$

The matrices A_λ is a symmetric matrix ($A_\lambda^T = A_\lambda$).

Perfect matching association schemes

Definition

The set $\mathcal{A}_{2n} = \{A_{[1^n]}, A_{[2,1^{n-2}]}, A_{[2,2,1^{n-4}]}, \ldots, A_{[n]}\}$ is known as the perfect matching association scheme.

Problem

What is the second largest eigenvalue of each graph in the perfect matching association scheme?

Observation: The set $\mathcal{A}_{2n} = \{A_{[1^n]}, A_{[2,1^{n-2}]}, A_{[2,2,1^{n-4}]}, \ldots, A_{[n]}\}$ is a set of symmetric matrices that pairwise commute.

Fact: A set of symmetric matrices that pairwise commute have the same eigenspaces.

Eigenspaces

There is an equivalent (and more technical) description of the perfect matching association scheme.

Eigenspaces \ matrices	$A_{[1^n]}$	$A_{[2,1^{n-2}]}$	$A_{[3,2,1^{n-5}]}$	$\cdots$	$A_{[n]}$
$[2n]$					
$[2n-2,2]$					
$[2n-4,4]$					
$\vdots$					
$[2^n]$					

The eigenspaces of our matrices correspond to irreducible representations of the symmetric group S_{2n} which are S_{2n}-modules. Each eigenspace is indexed by an even partition of $2n$.

Eigenvalues

Question: Given a S_{2n}-module corresponding to 2μ, what is the eigenvalue of A_λ corresponding to this eigenspace?

Eigenspaces \ matrices	$A_{[1^n]}$	$A_{[2,1^{n-2}]}$	$A_{[3,2,1^{n-5}]}$	$\cdots$	$A_{[n]}$
$[2n]$	?	?	?		?
$[2n-2,2]$	?	?	?		?
$[2n-4,4]$	?	?	?		?
$\vdots$	?	?	?		?
$[2^n]$	?	?	?		?

Eigenvalues

Eigenspaces \ matrices	$A_{[1^n]}$	$A_{[2,1^{n-2}]}$	$A_{[3,2,1^{n-5}]}$	$\cdots$	$A_{[n]}$
$[2n]$	1	?	?		?
$[2n-2,2]$	1	?	?		?
$[2n-4,4]$	1	?	?		?
$\vdots$	1	?	?		?
$[2^n]$	1	?	?		?

The first relation is the identity matrix. Its eigenvalues are well known: 1.

Eigenvalues

Eigenspaces \ matrices	$A_{[1^n]}$	$A_{[2,1^{n-2}]}$	$A_{[3,2,1^{n-5}]}$	$\cdots$	$A_{[n]}$
$[2n]$	1	✓	✓	✓	✓
$[2n-2,2]$	1	?	?		?
$[2n-4,4]$	1	?	?		?
$\vdots$	1	?	?		?
$[2^n]$	1	?	?		?

The eigenvalues of the $[2n]$-eigenspace corresponds to the degree of each graph (each graph is regular).

Eigenvalues

Theorem (MacDonald, 1979)

Let $\mu = [\mu_1^{m_1}, \ldots, \mu_k^{m_k}]$ *and* $n = \sum_{i=1}^{k} m_i \mu_i$. *Then the degree of* X_μ *is given by*

$$v_\mu = \phi_\mu^{[2n]} = \frac{2^n n!}{2^{m_1 + \cdots + m_k} \prod_i (m_i!)(\mu_i^{m_i})}.$$

Eigenvalues

Eigenspaces \ matrices	$A_{[1^n]}$	$A_{[2,1^{n-2}]}$	$A_{[3,2,1^{n-5}]}$	$\cdots$	$A_{[n]}$
$[2n]$	1	✓	✓	✓	✓
$[2n-2,2]$	1	✓	✓	✓	✓
$[2n-4,4]$	1	?	?		?
$\vdots$	1	?	?		?
$[2^n]$	1	?	?		?

MacDonald (1979) gives formulas for the eigenvalues corresponding to the $[2n-2,2]$-eigenspace.

Eigenvalues

Theorem (MacDonald, 1979)

Let μ be a partition of n and $r_1(\mu)$ denote the multiplicity of 1 in μ. Let v_u be the degree of X_μ. Then

$$\phi_{\mu}^{[2n-2,2]} = v_\mu \left(\frac{(2n-1)r_1(\mu) - n}{2n(n-1)} \right).$$

Eigenvalues

Eigenspaces \ matrices	$A_{[1^n]}$	$A_{[2,1^{n-2}]}$	$A_{[3,2,1^{n-5}]}$	$\cdots$	$A_{[n]}$
$[2n]$	1	✓	✓	✓	✓
$[2n-2,2]$	1	✓	✓	✓	✓
$[2n-4,4]$	1	✓	?		?
$\vdots$	1	✓	?		?
$[2^n]$	1	✓	?		?

Diaconis and Holmes (2002) determine all eigenvalues of $A_{[4,2,2,\ldots,2]}$.

Eigenvalues

Eigenspaces \ matrices	$A_{[1^n]}$	$A_{[2,1^{n-2}]}$	$A_{[3,2,1^{n-5}]}$	$\cdots$	$A_{[n]}$
$[2n]$	1	✓	✓	✓	✓
$[2n-2,2]$	1	✓	✓	✓	✓
$[2n-4,4]$	1	✓	?		✓
$\vdots$	1	✓	?		✓
$[2^n]$	1	✓	?		✓

MacDonald (1979) provides a formula for computing eigenvalues of $A_{[2n]}$.

An inductive algorithm

Srinivasan (2020) derived an inductive algorithm that allows us to obtain closed form formulas for the spectrum of X_μ based on content-evaluating symmetric functions.

An inductive algorithm

Example: Let $\phi^{\lambda}_{[2,1^{n-2}]}$ be the eigenvalue of $X_{[2,1^{n-2}]}$ occurring on the λ-eigenspace.

$$\phi^{\lambda}_{[2,1^{n-2}]} = \sum_{i=1}^{2n} x_i - \frac{n}{2}.$$

An inductive algorithm

Example: Let $\phi^{\lambda}_{[2,1^{n-2}]}$ be the eigenvalue of $X_{[2,1^{n-2}]}$ occurring on the λ-eigenspace.

$$\phi^{\lambda}_{[2,1^{n-2}]} = \sum_{i=1}^{2n} x_i - \frac{n}{2}.$$

x_1	x_2	x_3	x_4	x_5	x_6
x_7	x_8	x_9	x_{10}		
x_{11}	x_{12}				

(a) Assignment of $2n$ variables to the boxes of a Young tableau for $n = 6$.

0	1	2	3	4	5
-1	0	1	2		
-2	-1				

(b) Content of Young tableau corresponding to the partition $[6, 4, 2]$.

Obvious approach

An obvious approach is to construct an eigenvector w from the λ-eigenspace and evaluate $A_\mu w$.

Lemma (Godsil and Meagher, 2015)

Let $H_n = S_2 \wr S_n$ and $x_\lambda \in S_{2n}$ such that $(H_n, x_\lambda H_n)$ is a pair of cosets in the λ-orbital of H_n. Then

$$\phi_\mu^\lambda = \frac{v_\mu}{2^n(n!)} \sum_{h \in H_n} \chi^\lambda(x_\mu h).$$

This approach involves evaluating a sum of irreducible characters in a coset of H_n.

Matrix of Eigenvalues for $n = 4$

Eigenspaces \ matrices	$A_{[1^4]}$	$A_{[2,1,1]}$	$A_{[2,2]}$	$A_{[3,2,1^{n-5}]}$	$A_{[4]}$
$[8]$	1	12	12	32	48
$[6,2]$	1	5	-2	4	-8
$[4,4]$	1	2	7	-8	-2
$[4,2^2]$	1	-1	-2	-2	4
$[2^4$	1	-6	3	8	-6

We implemented Srinivasan's Maple code in Sage to obtain all eigenvalues of the perfect matching association scheme for $n \leqslant 15$.

Conjecture

Problem

On which eigenspace does the second largest eigenvalue occurs?

It is well-known that the largest eigenvalue occurs on the $[2n]$-eigenspace for each A_μ and that this eigenvalue is the degree of A_μ.

Conjecture

If μ contains at least two parts of length 1, then the second largest eigenvalue of A_μ occurs on the $[2n-2, 2]$-eigenspace.

Conjecture

Eigenspaces \ matrices	$A_{[1^n]}$	$A_{[2,1^{n-2}]}$	$A_{[3,1^{n-3}]}$	$\cdots$	$A_{[n]}$
$[2n]$	1	✓	✓	✓	✓
$[2n-2,2]$	1				
$[2n-4,2,2]$	1	✓			✓
$\vdots$	1	✓			✓
$[2^n]$	1	✓			✓

Using a computer, we can verify this conjecture for $2n \leqslant 30$.

Conjecture

Conjecture

If μ contains at least two parts of length 1, then the second highest eigenvalue of X_μ occurs on the $[2n-2, 2]$-eigenspace.

Why do we require that μ contains at least two parts of length 1?

Theorem (MacDonald, 1979)

Let μ be a partition of n and $\mathbf{r_1}(\mu)$ **denote the number of parts of size** 1 *in μ. Let v_u be the degree of X_μ. Then*

$$\phi_\mu^{[2n-2,2]} = v_\mu \left(\frac{(2n-1)r_1(\mu) - n}{2n(n-1)} \right).$$

Relations with one part of size one

$[1^5]$	$[2,1^3]$	$[2^2,1]$	$[3,1^2]$	[3,2]	[4,1]	[5]	Space	Dim.
1	20	60	80	160	240	384	[10]	1
1	11	6	26	-20	24	-48	[8,2]	35
1	6	11	-4	20	-26	-8	[6,4]	90
1	3	-10	2	-4	-8	16	$[6,2^2]$	225
1	0	5	-10	-10	10	4	$[4^2,2]$	252
1	-4	-3	2	10	6	-12	$[4,2^3]$	300
1	-10	15	20	-20	-30	24	$[2^5]$	42

Table: Eigenvalues of $\mathcal{A}_{10}$

Results

Theorem (GHLMM (2025+))

Let $\mu = [n-k, \mu']$ with $\mu' \vdash k$. If n is sufficiently large relative to k, then $\phi_\mu^{[2n-2,2]}$ is the second largest eigenvalue of X_μ in absolute value.

Theorem (GHLMM (2025+))

If

$$\mu \in \{[2, 1^{n-2}], [3, 1^{n-3}], [2^2, 1^{n-4}], [4, 1^{n-4}], [3, 2, 1^{n-5}], [5, 1^{n-5}]\}$$

then $\phi_\mu^{[2n-2,2]}$ is the second largest eigenvalue of X_μ in absolute value.

The trace trick

Theorem (GHLMM (2025+))

Let $\mu = [n-k, \mu']$ with $\mu' \vdash k$. If n is sufficiently large relative to k, then $\phi_\mu^{[2n-2,2]}$ is the second largest eigenvalue of X_μ in absolute value.

Facts:

- The trace of a matrix is the sum of its eigenvalues.
- If A is the adjacency matrix of a graph X, the trace of A^2 is twice the number of edges of X.

Goal: To use the trace of A^2 to relate the eigenvalues of A to the degree of X.

The trace trick

The degree of A_μ is v_μ, the eigenvalue of the $[2n]$-eigenspace. We see that the trace of A_μ^2 is:

$$trace(A_\mu^2) = \sum_\lambda m_\lambda (\phi_{2\mu}^\lambda)^2;$$
$$trace(A_\mu^2) = (2n-1)!!(v_\mu).$$

where the ϕ_μ^λ is the eigenvalue of A_μ occurring with multiplicity m_λ on the λ-eigenspace. This means that

$$\sum_{\lambda \vdash 2n} m_\lambda (\phi_\mu^\lambda)^2 = (2n-1)!!(v_\mu).$$

The trace trick

The equality below will allow us to bound individual terms in the left-hand sum:

$$\sum_\lambda m_\lambda(\phi_\mu^\lambda)^2 = (2n-1)!!(v_\mu).$$

Recall: ϕ_μ^λ is the eigenvalue of A_μ occurring with multiplicity m_λ on the λ-eigenspace.

The trace trick

$$v_\mu^2 + (\phi_\mu^{[2n-2,2]})^2 m_{[2n-2,2]} + \sum_{\lambda \neq [2n],[2n-2,2]} (\phi_\mu^\lambda)^2 m_\lambda = (2n-1)!!(v_\mu).$$

Recall: ϕ_μ^λ is the eigenvalue of A_μ occurring with multiplicity m_λ on the λ-eigenspace.

The trace trick

$$\sum_{\lambda \neq [2n],[2n-2,2]} (\phi_\mu^\lambda)^2 m_\lambda = ((2n-1)!!)(v_\mu) - ((\phi_\mu^{[2n-2,2]})^2 m_{[2n-2,2]} + (v_\mu)^2).$$

Recall: ϕ_μ^λ is the eigenvalue of A_μ occurring with multiplicity m_λ on the λ-eigenspace.

The trace trick

Because all terms in our sum are positive, we can remove all but one term to obtain the following inequality:

$$(\phi_\mu^\lambda)^2 m_\lambda \leqslant ((2n-1)!!)(v_\mu) - ((\phi_\mu^{[2n-2,2]})^2 m_{[2n-2,2]} + (v_\mu)^2).$$

Recall: ϕ_μ^λ is the eigenvalue of A_μ occurring with multiplicity m_λ on the λ-eigenspace such that $\lambda \neq [2n], [2n-2,2]$.

The trace trick

Recall: ϕ_μ^λ is the eigenvalue of A_μ occurring with multiplicity m_λ on the λ-eigenspace such that $\lambda \neq [2n], [2n-2,2]$.

Key Inequality:

$$(\phi_\mu^\lambda)^2 m_\lambda \leqslant \underbrace{((2n-1)!!)(v_\mu) - ((\phi_\mu^{[2n-2,2]})^2 m_{[2n-2,2]} + (v_\mu)^2)}_{\text{When } \mu \text{ and } n \text{ are fixed, this is constant}}$$

Conclusion: If the eigenvalue is ϕ_μ^λ is large, then the dimension of the λ-eigenspace cannot be large.

Example

$[1^5]$	$[2,1^3]$	$[2^2,1]$	$[3,1^2]$	[3,2]	[4,1]	[5]	Space	Dim.
1	20	60	80	160	240	384	[10]	1
1	11	6	26	-20	24	-48	[8,2]	35
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1	3	-10	2	-4	-8	16	$[6,2^2]$	225
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1	-4	-3	2	10	6	-12	$[4,2^3]$	300
1	-10	15	20	-20	-30	24	$[2^5]$	42

Table: Eigenvalues of perfect matching association scheme for $n = 5$.

The trace trick

Let's do a proof by contradiction! Suppose that

$$(\phi_\mu^{[2n-2,2]}) < (\phi_\mu^{\lambda})$$

$\lambda \neq [2n], [2n-2, 2]$.
Our previous inequality implies

$$(\phi_\mu^{[2n-2,2]})^2 m_\lambda < (\phi_\mu^{\lambda})^2 m_\lambda \leqslant ((2n-1)!!)(v_\mu) - ((\phi_\mu^{[2n-2,2]})^2 m_{[2n-2,2]} + (v_\mu)^2)$$

and

$$(\phi_\mu^{[2n-2,2]})^2 m_\lambda < ((2n-1)!!-1)(v_\mu)^2) - ((\phi_\mu^{[2n-2,2]})^2 m_{[2n-2,2]} + (v_\mu)^2)$$

The trace trick

If $\mu = [\mu_1^{\alpha_1}, \mu_2^{\alpha_2}, \ldots, \mu_t^{\alpha_t}]$, then the dimension of the eigenspace on which ϕ_μ^λ occurs is

$$m_\lambda \leq 4n^{3/2} \prod_i \alpha_i!(2\mu_i)^{\alpha_i} < 8n^{\frac{3}{2}}(n-k)(2k)!!$$

where $k = n - \mu_1$.

• Note that m_λ is the dimension of the λ-eigenspace.

• The above expression implies that, when k is small, the dimension of the λ-eigenspace is also small.

The Hook length formula

How do we compute the dimension of the eigenspaces (Young subgroups)?

• Each eigenspace is indexed by an even partition of $2n$: $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_k]$.

The Hook length formula

How do we compute the dimension of the eigenspaces (Young subgroups)?

• Each eigenspace is indexed by an even partition of $2n$: $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_k]$, with parts given in decreasing order.

• Each partition $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_k]$ corresponds to a Young Tableau.

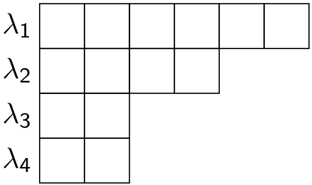


Figure: Young tableau of shape $[\lambda_1, \lambda_2, \lambda_3, \lambda_4] = [6, 4, 2, 2]$.

The Hook Length Formula

We fill each cell of our Young tableau of shape λ with a natural number according to a specific rule.

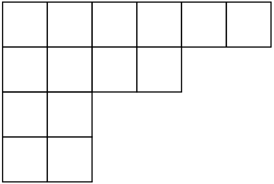


Figure: Young tableau of shape $[6, 4, 2, 2]$.

The Hook Length Formula

We fill each cell of our Young tableau of shape λ with a natural number according to a specific rule.

1. For each box, we have a hook comprised of all boxes to the right, and all boxes to the left.

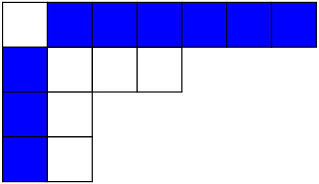


Figure: In grey is the hook at box $(1, 1)$.

The Hook Length Formula

We fill each cell of our Young tableau of shape λ with a natural number according to a specific rule.

2. We sum the number of boxes in the hook plus one.

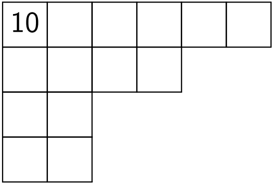


Figure: Content of box $(1, 1)$.

The Hook Length Formula

We fill each cell of our Young tableau of shape λ with a natural number according to a specific rule.

3. We repeat for each box in the tableau.

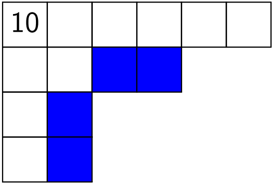


Figure: Content of box $(1, 1)$.

The Hook Length Formula

We fill each cell of our Young tableau of shape λ with a natural number according to a specific rule.

3.We repeat for each box in the tableau.

10					
	5				

Figure: Content of box $(2, 2)$.

The Hook Length Formula

We fill each cell of our Young tableau of shape λ with a natural number according to a specific rule.

3. We repeat for each box in the tableau.

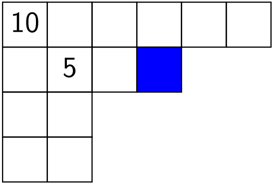


Figure: Hook at box $(2, 3)$.

The Hook Length Formula

We fill each cell of our Young tableau of shape λ with a natural number according to a specific rule.

3. We repeat for each box in the tableau.

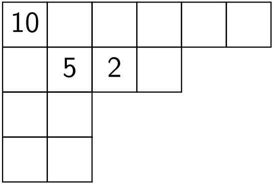


Figure: Content of box $(2, 3)$.

The Hook Length Formula

We fill each cell of our Young tableau of shape λ with a natural number according to a specific rule.

10	8	5	4	2	1
6	5	2	1		
3	2				
2	1				

Figure: Content of tableau.

Let $h_{(i,j)}$ denote the content of box (i, j) (this is the hook length at (i, j)).

The Hook Length formula

Lemma (Hook Length Formula)

Let $2\lambda \vdash 2n$ and let $H(\lambda) = \prod_{(i,j)} h_{(i,j)}$ as (i,j) runs over all $2n$ cells of λ. Then

$$m_\lambda = \frac{(2n)!}{H(\lambda)}. \tag{1}$$

Example

Formula: $m_\lambda = \frac{(2n)!}{H(\lambda)}, H(\lambda) = \prod_{(i,j)} h_{(i,j)}$.

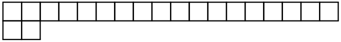
10	8	5	4	2	1
6	5	2	1		
3	2				
2	1				

$$H([6,4,2,2]) = (10)\cdot(8)\cdot(5)\cdot(4)\cdot(2)\cdot(6)\cdot(5)\cdot(2)\cdot(3)\cdot(2)\cdot(2) = 1152000$$

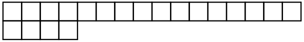
$$m_{[6,4,2,2]} = \frac{14!}{1152000} = 42042.$$

Low dimension eigenspaces

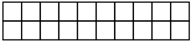
We can use the shape of a partition to gain some intuition on which eigenspaces have small dimension:



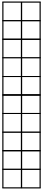
(a) Young tableau of $[18, 2]$.



(b) Young tableau of $[16, 4]$.



(c) Young tableau of $[10^2]$.



(d) Young tableau of $[2^{10}]$.

Figure: Young tableaus corresponding to λ-eigenspaces of small dimension for $2n = 20$.

Example

$[1^5]$	$[2,1^3]$	$[2^2,1]$	$[3,1^2]$	$[3,2]$	$[4,1]$	$[5]$	Space	Dim.
1	20	60	80	160	240	384	[10]	1
1	11	6	26	-20	24	-48	[8,2]	35
1	6	11	-4	20	-26	-8	[6,4]	90
1	3	-10	2	-4	-8	16	$[6,2^2]$	225
1	0	5	-10	-10	10	4	$[4^2,2]$	252
1	-4	-3	2	10	6	-12	$[4,2^3]$	300
1	-10	15	20	-20	-30	24	$[2^5]$	42

Table: Eigenvalues of $\mathcal{A}_{10}$

Eigenspaces with small dimensions

We know which eigenspaces will have small dimension!

Theorem (Meagher, Shirazi, and Stevens (2023))

The two eigenspaces of the perfect matching association scheme with dimension less than $\binom{2n}{3} - \binom{2n}{2}$ are indexed by partitions $\lambda \in \{[2n], [2n-2, 2]\}$.

Corollary: If m_λ is small, then $\lambda \in \{[2n], [2n-2, 2]\}$, a contradiction.

Conclusion

Let $\mu = [\mu_1^{\alpha_1}, \mu_2^{\alpha_2}, \ldots, \mu_t^{\alpha_t}]$ and $k = n - \mu_1$. If μ_1 is sufficiently large, then

$$m_\lambda \leq 4n^{3/2} \prod_i \alpha_i!(2\mu_i)^{\alpha_i} < 8n^{\frac{3}{2}}(n-k)(2k)!! < \binom{2n}{3} - \binom{2n}{2}.$$

Conclusion If μ_1 is sufficiently large, then $\lambda \in \{[2n], [2n-2, 2]\}$, a contradiction.

Results

Using formulas obtained from Srinivasan, we are also able to confirm our conjecture for five other matrices in the scheme.

Theorem (GHLMM (2025+))

If

$$\mu \in \{[2, 1^{n-2}], [3, 1^{n-3}], [2^2, 1^{n-4}], [4, 1^{n-4}], [3, 2, 1^{n-5}], [5, 1^{n-5}]\}$$

then $\phi_{\mu}^{[2n-2,2]}$ *is the second largest eigenvalue of* X_{μ}.

Future work

- What are the diameters of the graphs in $\mathcal{A}(M_{2n})$?
- What is the chromatic number of the graphs in $\mathcal{A}(M_{2n})$?
- Can our methods be further extended to confirm our conjecture on the second highest eigenvalue?

Thank you!

The 2026 Prairie Discrete Math Workshop:

- Set to take place on May 7th and 8th in Regina;
- Students and post-docs welcome! (Some travel funding may be available)