# Recent advances on the directed Oberwolfach problem

Alice Lacaze-Masmonteil
University of Regina

Joint work with Daniel Horsley (Monash University)

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# The Oberwolfach problem

**The setting:** Consider a conference with 7 participants. To facilitate networking, the organizing committee decides to host 3 banquets. The banquet hall has 2 round tables that sit 4 and 3 people, respectively.

**The problem:** The organizing committee needs a set of 3 seating arrangements (one for each banquet) such that each participant is seated beside every other participants exactly once.

Is this possible?

# Graph-theoretic approach

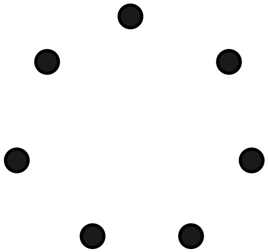

Figure: The empty graph on 7 vertices, $\overline{K}_7$.

# Graph-theoretic approach

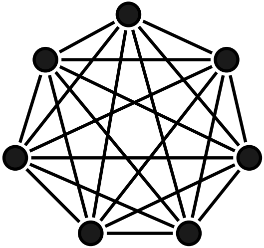

Figure: The complete graph on 7 vertices, $K_7$.

# Graph-theoretic approach

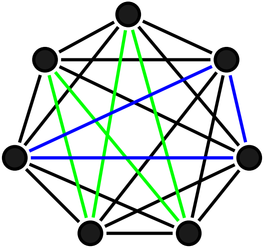

Figure: The first seating arrangement.

# Graph-theoretic approach

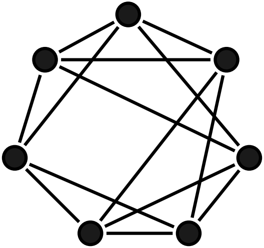

Figure: The complete $K_7$ minus 1 seating arrangement.

# Graph-theoretic approach

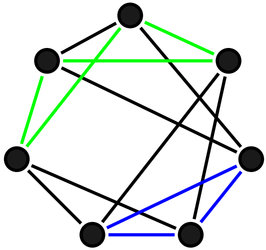

Figure: The second seating arrangement.

# Graph-theoretic approach

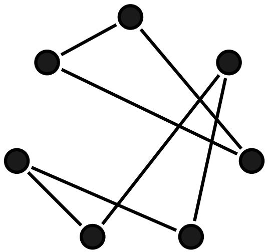

Figure: The third seating arrangement.

## The Oberwolfach problem-general case

**The setting:** Consider a conference with $n = 2k + 1$ participants. The organizing committee decides to host $k$ banquets. The banquet hall has $\alpha$ round tables that sit $m_1, m_2, \dots, m_\alpha$ participants, respectively, such that $m_1 + m_2 + \cdots + m_\alpha = n$ and each $m_i \geqslant 3$.

**The problem:** The organizing committee needs a set of $k$ seating arrangements (one for each banquet) such that each participant is seated beside every other participants exactly once.

Is this possible?

# Terminology

## Definition

A **decomposition** of a graph $G$ is a set of subgraphs $\{H_1, H_2, \ldots, H_k\}$ such that each edge of $G$ appears in exactly one subgraph. We then write $G = H_1 \oplus H_2 \oplus \cdots \oplus H_k$.

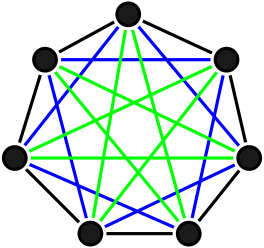

Figure: A decomposition of $K_7$ into copies of $C_7$. We see that $K_7 = C_7 \oplus C_7 \oplus C_7$.

# Terminology

## Definition

A $[m_1, m_2, \ldots, m_\alpha]$-**factor** of $G$ is a spanning subgraph of $G$ comprised to $\alpha$ disjoint cycles of lengths $m_1, m_2, \ldots, m_\alpha$.

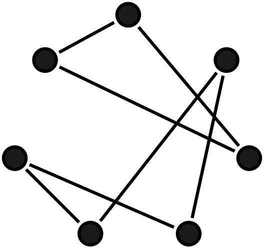

Figure: A $[3, 4]$-factor of $K_7$.

# Terminology

## Definition

A $[m_1, m_2, \ldots, m_\alpha]$-**factorization** of $G$ is a decomposition of $G$ into $[m_1, m_2, \ldots, m_\alpha]$-factors.

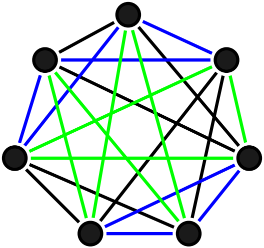

Figure: A $[3, 4]$-factorization of $K_7$.

# The graph-theoretic formulation of the OP

**Problem**

Let $n = 2k + 1$ and $m_1 + m_2 + \cdots + m_\alpha = n$. Does the graph $K_n$ admit a $[m_1, m_2, \ldots, m_\alpha]$-factorization?

# The generalized Oberwolfach problem

What if $n = 2k$?

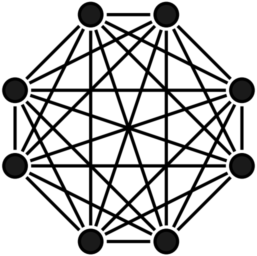

Figure: The complete graph on 8 vertices, $K_8$.

# The generalized Oberwolfach problem

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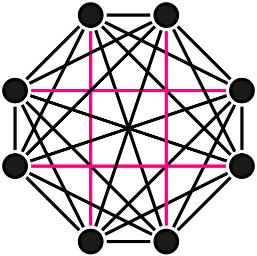

Figure: A 1-factor (perfect matching) of $K_8$ drawn in grey.

# The generalized Oberwolfach problem

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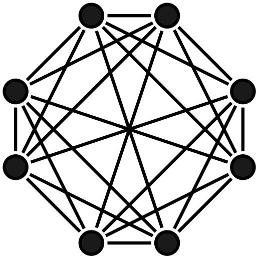

Figure: The graph $K_8 - I$.

# The graph-theoretic formulation of the OP

**Problem (OP($m_1, m_2, \ldots, m_\alpha$))**

Let $m_1 + m_2 + \cdots + m_\alpha = n$. If $n$ is odd, does the graph $K_n$ admit a $[m_1, m_2, \ldots, m_\alpha]$-factorization? If $n$ is even, does the graph $K_n - I$ admit a $[m_1, m_2, \ldots, m_\alpha]$-factorization?

If $m_1 = m_2 = \cdots = m_\alpha = m$, then we write OP($m^\alpha$).

# Hamiltonian decomposition of $K_n$

**Theorem (Walecki (1892))**

*The $OP(n)$ has a solution for all $n$.*

This is a decomposition of $K_n$ or $K_n - I$ into $C_n$ which is also known as a Hamiltonian decomposition.

# The Oberwolfach problem with tables of length $m$

**Theorem (Jiaxi (1961), Ray-Chaudhuri and Wilson (1973), Kotzig and Rosa (1974), Baker and Wilson (1977), Brouwer (1978), Rees and Stinson (1987))**

*The* $OP(3^\alpha)$ *has a solution if and only if* $\alpha \notin \{2, 4\}$

**Theorem (Walecki (1892), Alspach and Häggkvist(1985), Alspach et al. (1989), Hoffman and Schellenberg (1991))**

*If* $m \geqslant 4$, *then* $OP(m^\alpha)$ *has a solution.*

# The Oberwolfach problem with tables of varying lengths

**Theorem (Haĝgkvist (1985), Bryant and Danziger (2010))**

*The $OP(m_1, m_2, \ldots, m_\alpha)$ has a solution when $m_1, m_2, \ldots, m_\alpha$ are all even.*

**Theorem (Gvozdjak (2004) and Traetta (2013))**

*The $OP(m_1, m_2)$ has a solution if and only if $(m_1, m_2) \notin \{(3, 3), (4, 5)\}$.*

**Theorem (Traetta (2024))**

*The $OP(m_1, m_2, \ldots, m_\alpha)$ when one of $m_1, m_2, \ldots, m_\alpha$ is sufficiently greater than an explicit lower bound.*

# The Oberwolfach problem with tables of varying lengths

**Theorem (Bryant and Scharaschkin (2009))**

*The $OP(m_1, m_2, \ldots, m_\alpha)$ has a solution for infinitely many primes $n \equiv 1 \pmod{16}$.*

**Theorem (Alspach et al. (2016))**

*The $OP(m_1, m_2, \ldots, m_\alpha)$ has a solution when $n = 2p$ where $p$ is prime and $p \equiv 5 \pmod{8}$.*

## Computational results

**Theorem (P. Adams and D. Bryant (2006); A. Deza et al. (2010), F. Franek et al. (2004); F. Franek and A. Rosa.(2006); F. Salassa et al. (2021); M. Meszka (2024))**

*The* $OP(m_1, m_2, \ldots, m_\alpha)$ *has a solution for* $n \leqslant 100$ *except for* $OP(3^2), OP(3^4), OP(4,5)$*, and* $OP(3,3,5)$.

# Probabilistic approach

**Theorem (Glock et al. (2021))**

*The* $OP(m_1, m_2, \ldots, m_\alpha)$ *has a solution for* $n$ *sufficiently large.*

# The directed Oberwolfach problem

**The setting:** Consider a conference with $n$ participants. To facilitate networking, the organizing committee decides to host $n-1$ banquets. The banquet hall has $\alpha$ round tables that sit $m_1, m_2, \ldots, m_\alpha$ participants such that $m_1 + m_2 + \cdots + m_\alpha = n$.

**The problem:** The organizing committee needs a set of $n-1$ seating arrangements (one for each banquet) such that each participant is seated **to the right** of every other participants exactly once.

Is this possible?

# A simple example

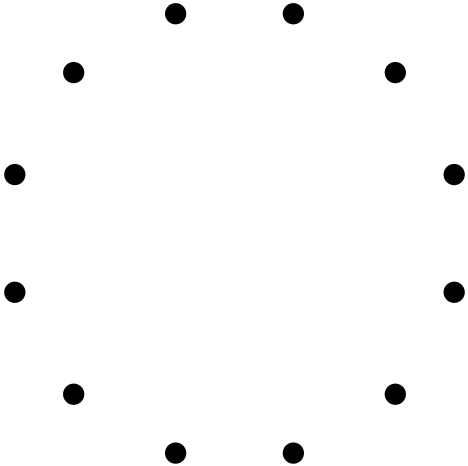

Figure: The 12 participants (one for each vertex).

# A simple example

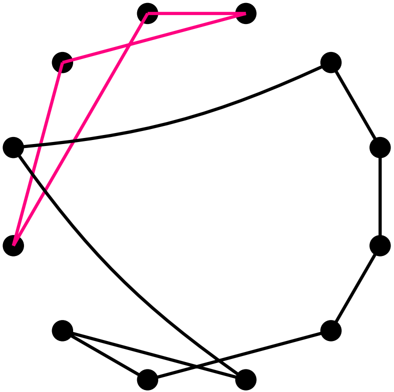

Figure: One seating arrangement with one table of length 4 and one table of length 8.

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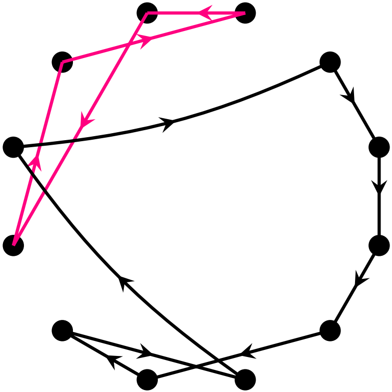

Figure: One seating arrangement with one table of length 4 and one table of length 8.

# The complete symmetric digraph

## Definition

The **complete symmetric digraph**, denoted $K_n^*$, is the digraph on $n$ vertices in which for every pair of distinct vertices $x$ and $y$, there are arcs $(x, y)$ and $(y, x)$.

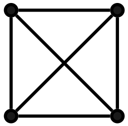

Figure: The complete graph $K_4$.

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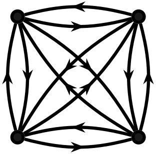

Figure: The complete symmetric digraph $K_4^*$.

# Cycle decomposition

## Definition

A $\vec{C}_m$-**factor** of digraph $G$ is a spanning subdigraph of $G$ that is the disjoint union of directed $m$-cycles.

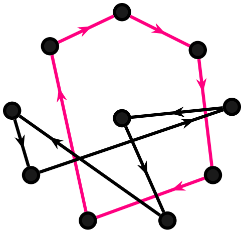

Figure: A $\vec{C}_5$-factor of $K_{10}^*$.

# Cycle decomposition

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**Definition**

A $\vec{C}_m$-**factorization** of $G$ is a decomposition of $G$ into $\vec{C}_m$-factors.

# $[m_1, m_2, \ldots, m_\alpha]$-factorization

**Definition**

A **directed** $[m_1, m_2, \ldots, m_\alpha]$**-factor** of digraph $G$ is a spanning subdigraph comprised of disjoint directed cycles of length $m_1, m_2, \ldots, m_\alpha$ .

**Definition**

A **directed** $[m_1, m_2, \ldots, m_\alpha]$**-factorization** of digraph $G$ is a decomposition of $G$ into $[m_1, m_2, \ldots, m_\alpha]$-factors.

# The graph-theoretic formulation of the directed OP

**Problem (OP$^*(m_1, m_2, \ldots, m_\alpha)$)**

Let $m_1, m_2, \ldots, m_\alpha \geqslant 2$. If $m_1 + m_2 + \cdots + m_\alpha = n$, does $K_n^*$ admit a directed $[m_1, m_2, \ldots, m_\alpha]$-factorization?

Why are we not considering two different cases based on the parity of $n$?

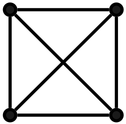

Figure: The complete graph $K_4$.

# The graph-theoretic formulation of the directed OP

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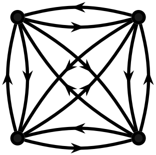

Figure: The complete symmetric digraph $K_4^*$.

## Easy consequences

**Corollary (Kadri and Šajna (2024+))**

*If* $(m_1, m_2, \ldots, m_\alpha) \notin \{(4,5),(3,3,5)\}$, *then* $OP^*(m_1, m_2, \ldots, m_\alpha$ *has a solution in each of the following cases:*

- $m_1 = m_2 = \cdots = m_t$;
- $n \leqslant 100$;
- $t = 2$.

We generally consider the case $n$ is even because, when $n$ is odd, a solution to the directed OP can be obtained by orienting a solution to the original OP.

# Directed Oberwolfach problem with tables of uniform length

**Problem (The directed Oberwolfach problem with tables of uniform length)**

To find all integers $\alpha$ and $m$ for which $K^*_{\alpha m}$ admits a $\vec{C}_m$-factorization.

Observe that $\alpha =$number of cycles in a $\vec{C}_m$-factor.

# Previous results (small $m$ or $\alpha$)

The digraph $K^*_{\alpha m}$ admits a $\vec{C}_m$-factorization when:

- $m = 3$ and $\alpha \neq 2$ (Bermond et al. (1979));
- $\alpha = 1$ and $m \notin \{4, 6\}$ (Tillson (1980));
- $m = 4$ and $\alpha \neq 1$ (Bennett and Zhang (1990); Adams and Bryant, Unpublished);
- $m = 5$ and $\alpha \geqslant 103$ (Abel et al. (2002)).

# Previous results (general $m$)

**Theorem (Burgess and Šajna, 2014)**

*If $m$ is even or $\alpha$ is odd, such that $(\alpha, m) \notin \{(1,6),(1,4)\}$, then $K^*_{\alpha m}$ admits a $\vec{C}_m$-factorization.*

We have a solution when tables are of even length or when we have an odd number of tables.

# Previous results (general $m$)

What if we have an even number of tables of odd length?

**Theorem (Burgess and Šajna, 2014)**

*Suppose that $\alpha$ is an even integer and $m \geqslant 3$ is an odd integer. If $K^*_{2m}$ admits a $\vec{C}_m$-factorization, then $K^*_{\alpha m}$ also admits a $\vec{C}_m$-factorization.*

It suffices to solve our problem when we have seating arrangements with two tables of odd length.

# Previous results (general $m$)

**Conjecture (Burgess and Šajna, 2014)**

*If $m$ is odd and $m \geqslant 5$, then $K_{2m}^*$ admits a $\vec{C}_m$-factorization.*

**Theorem (Burgess, Francetić, and Šajna, 2018)**

*If $m$ is odd and $5 \leqslant m \leqslant 49$, then $K_{2m}^*$ admits a $\vec{C}_m$-factorization.*

## New Result

**Theorem (L-M, 2024)**

*The digraph* $K_{2m}^*$ *admits a* $\vec{C}_m$*-factorization for all odd* $m \geqslant 11$.

# Tools

## Lemma (Burgess and Šajna, 2014)

*Let $\{G_1, G_2, \ldots, G_t\}$ be a decomposition of $H$ into spanning subdigraphs. If each $G_i$ admits a directed $[m_1, m_2, \ldots, m_\alpha]$-factorization, then $H$ admits a directed $[m_1, m_2, \ldots, m_\alpha]$-factorization.*

**Proof** Let $D_i$ be the $[m_1, m_2, \ldots, m_\alpha]$-factorization of $G_i$. We see that

$$\mathcal{F} = \bigcup_{i=1}^{t} D_i$$

is a $[m_1, m_2, \ldots, m_\alpha]$-factorization of $H$. $\square$

# Template

Step 1: Strategically decompose (di)graph $G$ into $t$ spanning sub(di)graphs that fall into $r$ isomorphisms classes: $H_1, H_2, \ldots, H_r$.

Step 2: Show that each isomorphism class admits the desired $[m_1, m_2, \ldots, m_\alpha]$-factorization.

## Häggkvist style constructions

**Theorem (Haǧgkvist (1985))**

*The* $OP(m_1, m_2, \ldots, m_\alpha)$ *has a solution when* $m_1, m_2, \ldots, m_\alpha$ *are all even and* $n \equiv 2 \pmod 4$*.*

# Häggkvist style constructions

**Lemma (Häggkvist (1985))**

*If $m$ is odd, $K_{2m} - I$ admits a decomposition into $\frac{m-1}{2}$ copies of $C_m \wr \overline{K}_2$.*

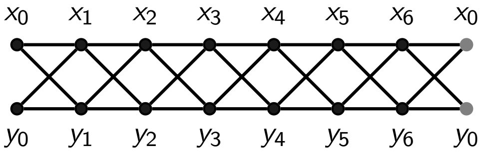


Figure: The graph $C_7 \wr \overline{K}_2$.

# Häggkvist style constructions

**Lemma (Häggkvist (1985))**

*If $m$ is odd, $K_{2m} - I$ admits a decomposition into $\frac{m-1}{2}$ copies of $C_m \wr \overline{K}_2$.*

**Proof:** We know that $K_m$ admits a decomposition into $\frac{m-1}{2}$ copies of $C_m$ when $m$ is odd.

We also know that $K_{2m} - I = K_m \wr \overline{K}_2$.

$$\begin{aligned} K_m \wr \overline{K}_2 = & \quad (C_m \oplus C_m \oplus \cdots \oplus C_m) \wr \overline{K}_2 \\ = & \quad C_m \wr \overline{K}_2 \oplus C_m \wr \overline{K}_2 \oplus \cdots \oplus C_m \wr \overline{K}_2. \end{aligned}$$

□

## Lemma (Häggkvist Lemma (1985))

*Let $m_1, m_2, \ldots, m_\alpha$ be even integers greater than 2 such that $m_1 + m_2 + \cdots + m_\alpha = 2m$. The graph $C_m \wr \overline{K}_2$ admits a $[m_1, m_2, \ldots, m_\alpha]$-factorization for all $m \geqslant 2$.*

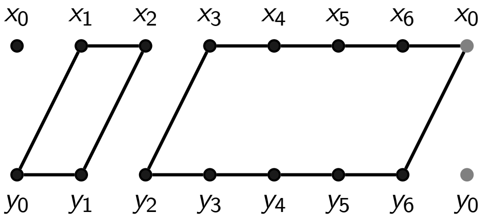


Figure: A $[4, 10]$-factor of $C_m \wr \overline{K}_2$.

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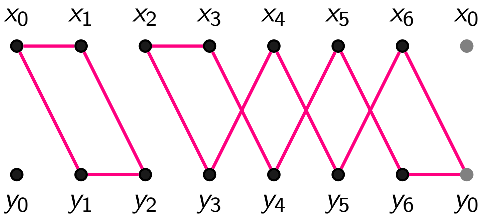


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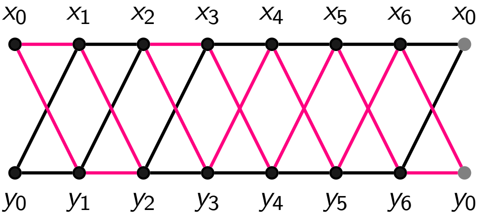


Figure: A $[4, 10]$-factorization of $C_m \wr \overline{K}_2$.

# Strategy

Step 1: Decompose $K_{2m}^*$ into $\frac{m-1}{2}$ spanning subdigraphs that fall into one of three isomorphisms classes: $G_1$, $G_2$, and $G_3$.

Step 2: Show that $G_1$, $G_2$, and $G_3$ admit a $\vec{C}_m$-factorization.

# Decomposition of $K_{2m}^*$

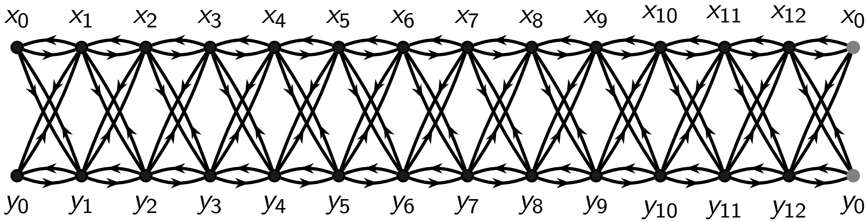


Figure: The spanning digraph $G_1 = \vec{X}(\{\pm 1\}, m) \wr \overline{K}_2$.

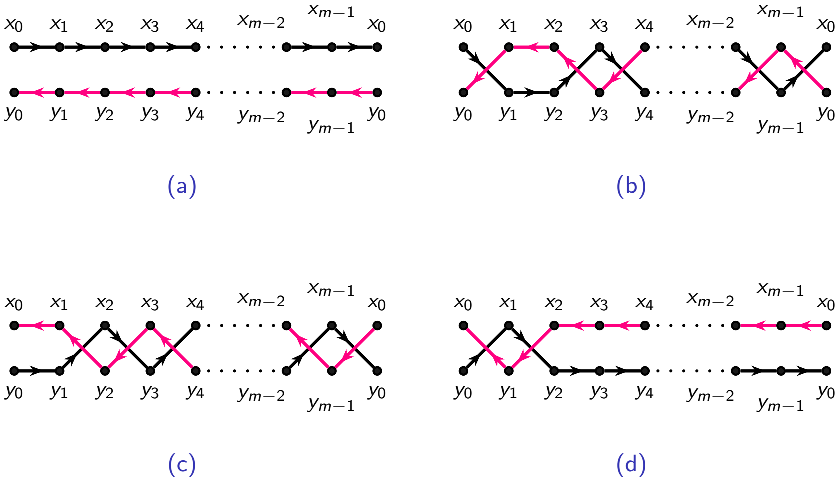


Figure: A $\vec{C}_m$-factorization of $G_1$.

# Result

**Proposition**

*Let $m \geqslant 3$ be an odd integer. The digraph $\vec{X}(\{\pm 1\}, m) \wr \overline{K}_2$ admits a $\vec{C}_m$-factorization.*

# Decomposition of $K^*_{2m}$

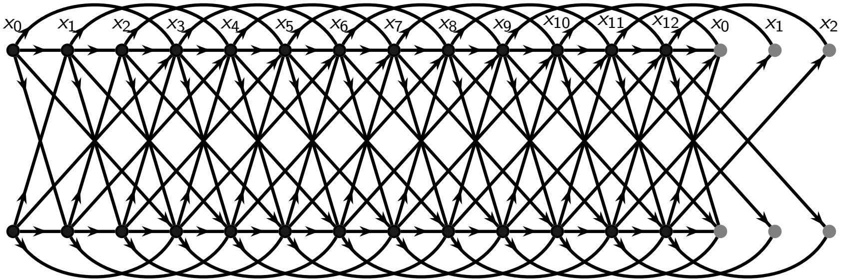


Figure: The spanning digraph $G_2 = \vec{X}(\{1,3\},13) \wr \overline{K}_2$ of $K^*_{2(13)}$.

# Key ingredients

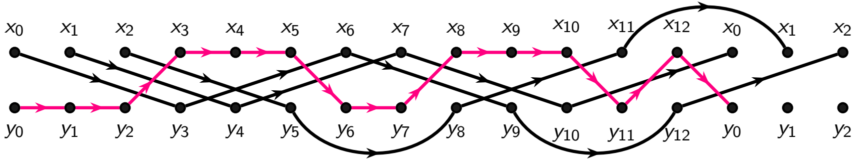


Figure: A $\vec{C}_{13}$-factorization of $\vec{X}(\{1,3\}, m)$ when $m = 13$.

# Key ingredients

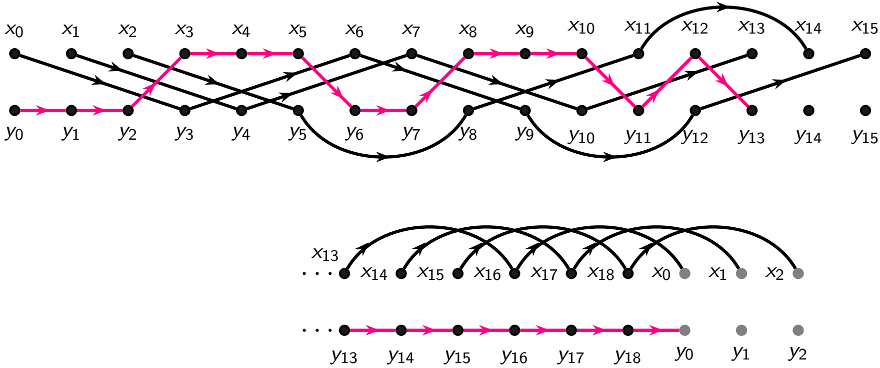


Figure: A $\vec{C}_{19}$-factor of $\vec{X}(\{1,3\},19) \wr \overline{K}_2$.

# Key ingredients

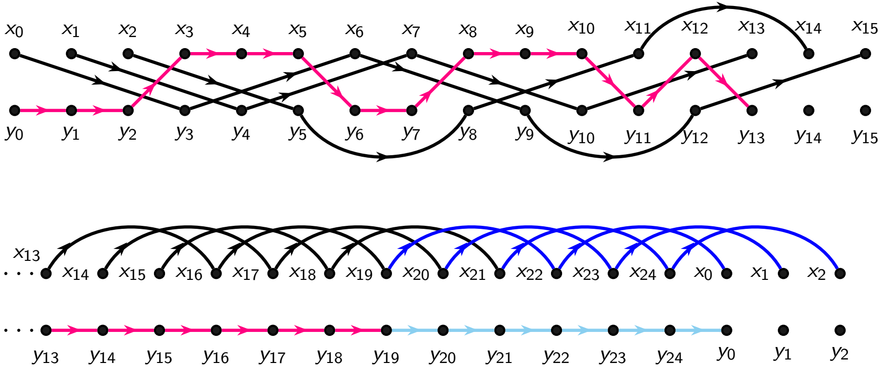


Figure: A $\vec{C}_{25}$-factor of $\vec{X}(\{1,3\},25) \wr \overline{K}_2$.

# Result

## Proposition

*Let $m \geqslant 3$ be an odd integer. The digraph $\vec{X}(\{\pm 1\}, m) \wr \overline{K}_2$ admits a $\vec{C}_m$-factorization.*

## Proposition

*Let $m \geqslant 11$ be an odd integer. The digraph $\vec{X}(\{1,3\}, m) \wr \overline{K}_2$ admits a $\vec{C}_m$-factorization if and only if $m \not\equiv 3 \pmod 6$.*

# Decomposition of $K^*_{2m}$

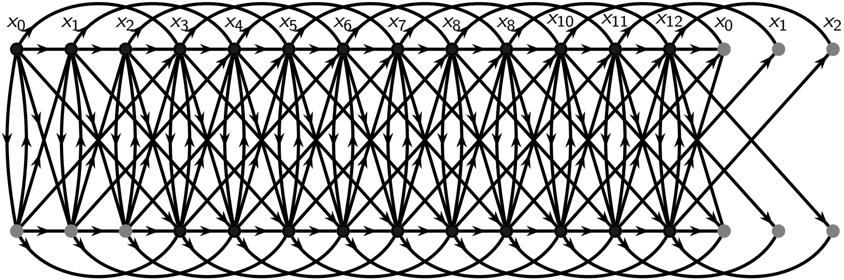


Figure: The spanning digraph $G_3 = \vec{X}(\{1,3\},13) \wr K_2^*$ of $K^*_{2(13)}$.

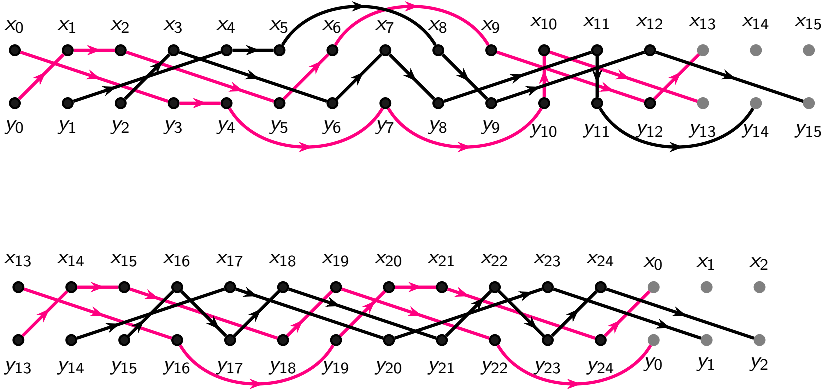


Figure: A $\vec{C}_{25}$-factor of $G_3$ when $m = 25$.

# Result

## Proposition

*Let $m \geqslant 3$ be an odd integer. The digraph $\vec{X}(\{\pm 1\}, m) \wr \overline{K}_2$ admits a $\vec{C}_m$-factorization.*

## Proposition

*Let $m \geqslant 11$ be an odd integer. The digraph $\vec{X}(\{1, 3\}, m) \wr \overline{K}_2$ admits a $\vec{C}_m$-factorization if and only if $m \not\equiv 3 \pmod 6$.*

## Proposition

*Let $m \geqslant 11$ be an odd integer such that $m \equiv 1, 5 \pmod 6$. The digraph $\vec{X}(\{1, 3\}, m) \wr K_2^*$ admits a $\vec{C}_m$-factorization.*

# Summary

## Proposition

*The digraph $K_{2m}^*$ admits a decomposition into*

1. $\frac{m-5}{2}$ *copies of* $\vec{X}(\{\pm 1\}, m) \wr \overline{K}_2$;
2. *one copy of* $\vec{X}(\{1,3\}, m) \wr \overline{K}_2$;
3. *one copy of* $\vec{X}(\{1,3\}, m) \wr K_2^*$.

## Theorem (L-M, (2024))

*If* $m \equiv 1, 5 \pmod 6$ *and* $m \geqslant 11$ *then* $K_{2m}^*$ *admits a* $\vec{C}_m$*-factorization.*

# Reduction step

**Proposition**

*If $K^*_{2m}$ admits a $\vec{C}_m$-factorization, then $K^*_{2(3^t m)}$ admits a $\vec{C}_{3^t m}$-factorization where $t$ is a positive integer.*

If $m' \equiv 3 \pmod 6$ then:

- $m' = 3^t \cdot m$ where $m \equiv 1, 5 \pmod 6$.

# Reduction step

**Proposition**

*If $K_{2m}^*$ admits a $\vec{C}_m$-factorization, then $K_{2(3^t m)}^*$ admits a $\vec{C}_{3^t m}$-factorization where $t$ is a positive integer.*

If $m' \equiv 3 \pmod 6$ then:

- $m' = 3^t \cdot m$ where $m \equiv 1, 5 \pmod 6$.

When $m \equiv 1, 5 \pmod 6$ and $m \geqslant 5$, we obtain a $\vec{C}_{m'}$-factorization of $K_{2m'}^*$ using a $\vec{C}_m$-factorization of $K_{2m}^*$.

When $m = 1$, we use a $\vec{C}_9$-factorization of $K_{18}^*$.

# Main result

**Theorem (L-M, (2024))**

*The digraph $K_{2m}^*$ admits a $\vec{C}_m$-factorization for all odd $m \geqslant 11$.*

# A complete solution

**Theorem**

*The digraph $K^*_{\alpha m}$ admits $\vec{C}_m$-factorization if and only if $(\alpha, m) \notin \{(1,6),(2,3),(1,4)\}$.*

The theorem above is a result of the work of: Bermond and Faber (1976); Bermond, Germa, and Sotteau (1979); Tillson (1980); Bennett and Zhang (1990); Adams and Bryant (Unpublished); Abel, Bennett, and Ge (2002); Burgess and Šajna (2014); Burgess, Francetić, and Šajna (2018); L-M (2024).

# The directed Oberwolfach problem with tables of varying lengths

Using a recursive approach, Kadri and Šajna (2024+) obtain several infinite families of solution to $\mathrm{OP}^*(m_1, m_2, \ldots, m_\alpha)$.

Furthermore, they establish the existence of solutions for $n \leqslant 14$ except for three already known exceptions.

**Theorem (Kadri and Šajna (2024+))**

*The $OP^*(m_1, m_2, \ldots, m_\alpha)$ has a solution for $n \leqslant 14$ except for $OP^*(4^1)$, $OP^*(6^1)$, $OP^*(3^2)$.*

# A key corollary

**Theorem (Kadri and Šajna (2024+))**

*Let $m_1 < m_2$. The $OP^*(m_1, m_2)$ has a solution except possibly when $m_1 \in \{4, 6\}$ and $m_2$ is even.*

**Idea:** Take a solution to $\mathrm{OP}^*(m_1^1)$ and construct a solution to $\mathrm{OP}^*(m_1, m_2)$.
**Problem:** $\mathrm{OP}^*(4^1)$ and $\mathrm{OP}^*(6^1)$ do not have a solution (Bermond and Faber (1976)).

# Result on two tables

**Theorem (Horsley and L-M (2024+))**

*Let $m_1 < m_2$. The $OP^*(m_1, m_2)$ has a solution when $m_1 \in \{4, 6\}$ and $m_2$ is even.*

We construct an $[m_1, m_2]$-factorization of $K_n^*$ when $m_1 + m_2 = n$, $m_1 \in \{4, 6\}$, and $m_2$ is even.

# Approach

Step 1: Decompose $K_{2m}^*$ into $\frac{m-1}{2}$ spanning subdigraphs that fall into one two isomorphisms classes: $G_1$ and $G_2$.

Step 2: Show that $G_1$ and $G_2$ both admit a $[m_1, m_2]$-factorization.

# The first class of digraphs

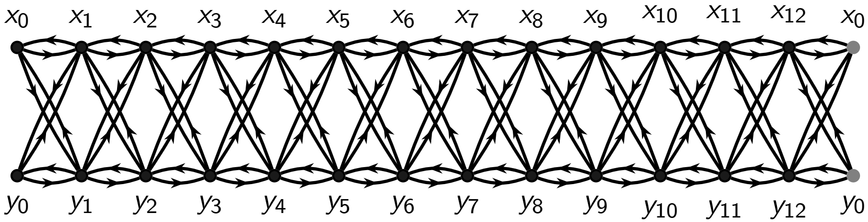


Figure: The spanning digraph $G_1 = \vec{X}(\{\pm 1\}, m) \wr \overline{K}_2$.

# The second class of digraph

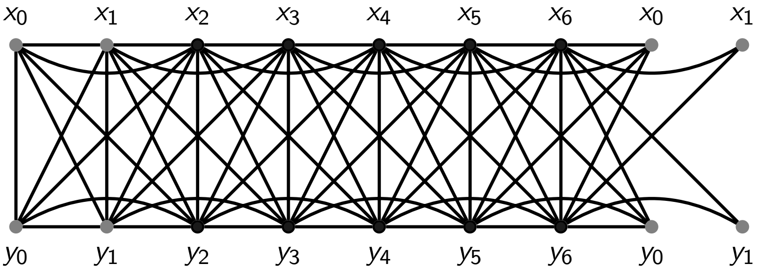


Figure: The digraph $G_2 = \vec{X}(\{\pm 1, \pm 2\}, m) \wr K_2^*$.

Each edge represents a pair of arcs, one for each direction.

# A complete solution

**Theorem (Kadri and Šajna (2024+) and Horsley and L-M (2024+))**

*Let $m_1 < m_2$. The $OP^*(m_1, m_2)$ has a solution.*

We have a complete solution to the directed Oberwolfach problem with two tables.

# Next step

**Next step:** To generalize our methods to obtain a solution to $\mathrm{OP}^*(m_1, m_2, \ldots, m_\alpha)$ for any even integers $m_1, m_2, \ldots, m_\alpha$ and $n \equiv 0, 2 \pmod{4}$.

# Thanks!