On the directed Oberwolfach problem with tables of even lengths

Alice Lacaze-Masmonteil, University of Regina
PIMS-CNRS Postdoctoral Fellow

Joint work with Andrea Burgess and Peter Danziger

September 25th, 2025

PIMS

A simple example

The setting: Consider a conference with 16 participants. To facilitate networking, the organizing committee decides to host 15 banquets. The banquet hall has 3 tables that seat 4,4, and 8 participants.

The problem: The organizing committee needs a set of 15 seating arrangements (one for each banquet) such that each participant is seated to the right of every other participants exactly once.

Is this possible?

Construction of a seating arrangement

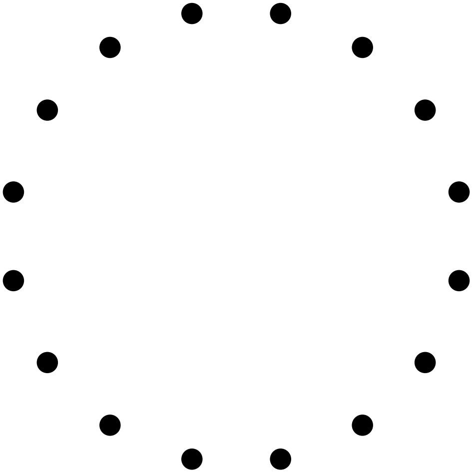


Figure: The 16 participants (one for each vertex).

Construction of a seating arrangement

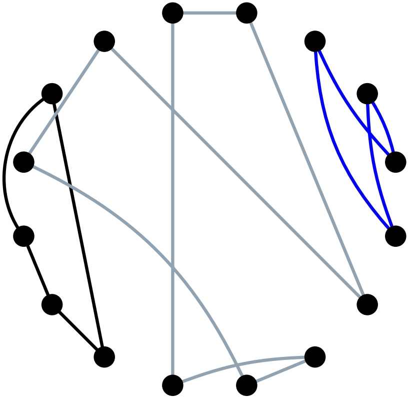


Figure: One seating arrangement with two tables of length 4 and one table of length 8.

Construction of a seating arrangement

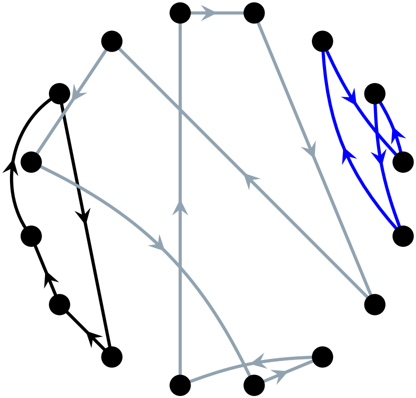


Figure: One seating arrangement with two tables of length 4 and one table of length 8.

Construction of a seating arrangement

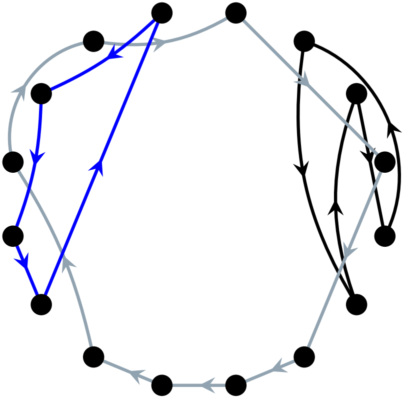


Figure: Another seating arrangement with two tables of length 4 and one table of length 8.

The directed Oberwolfach problem

The setting: Consider a conference with n participants. To facilitate networking, the organizing committee decides to host $n-1$ banquets. The banquet hall has t round tables that sit $m_1, m_2, \ldots, m_t$ participants such that $m_1 + m_2 + \cdots + m_t = n$.

The problem: The organizing committee needs a set of $n-1$ seating arrangements (one for each banquet) such that each participant is seated **to the right** of every other participants exactly once.

Is this possible?

The complete symmetric digraph

Definition

The **complete symmetric digraph**, denoted K_n^*, is the digraph on n vertices in which for every pair of distinct vertices x and y, there are arcs (x, y) and (y, x).

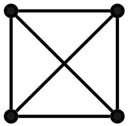


Figure: The complete graph K_4.

The complete symmetric digraph

Definition

The **complete symmetric digraph**, denoted K_n^*, is the digraph on n vertices in which for every pair of distinct vertices x and y, there are arcs (x, y) and (y, x).

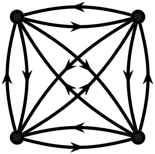


Figure: The complete symmetric digraph K_4^*.

2-factorization

Definition

A **directed** $[m_1, m_2, \ldots, m_t]$**-factor** of digraph G is a spanning subdigraph comprised of disjoint directed cycles of length $m_1, m_2, \ldots, m_t$.

Definition

A **directed** $[m_1, m_2, \ldots, m_t]$**-factorization** of digraph G is a decomposition of G into $[m_1, m_2, \ldots, m_t]$-factors.

The graph-theoretic formulation of the directed OP

Problem (OP$^*(m_1, m_2, \ldots, m_t)$)

Let $m_1, m_2, \ldots, m_t \geqslant 2$. If $m_1 + m_2 + \cdots + m_\alpha = n$, does K_n^* admit a directed $[m_1, m_2, \ldots, m_t]$-factorization?

A solution to OP$^*(m_1, m_2, \ldots, m_t)$ is also a resolvable Mendelsohn design with blocks of size $m_1, m_2, \ldots, m_t$.

Background

Theorem (Bermond, Germa, and Sotteau (1979); Tillson (1980), Bennett and Zhang (1990); Adams and Bryant (Unpublished); Abel, Bennett, and Ge (2002); Burgess and Šajna (2014); Burgess, Francetić, and Šajna (2018); L-M (2024))

The $OP^*(m^t)$ has a solution except when

$$(m,t) \notin \{(3,2),(4,1),(6,1)\}.$$

The directed OP has been completely resolved when all tables are of the same length.

Background

Theorem (Zhang and Du (2005))

The $OP^*(3^t, s)$ has a solution when $s \in \{4, 5\}$ for all $t \in \mathbb{Z}^+$.

Theorem (Kadri and Šajna (2025))

Let $m_1 < m_2$. The $OP^*(m_1, m_2)$ has a solution except possibly when $m_1 \in \{4, 6\}$ and m_2 is even.

Theorem (Horsley and L-M (2024+))

Let $m_1 < m_2$. The $OP^*(m_1, m_2)$ has a solution when $m_1 \in \{4, 6\}$ and m_2 is even.

Bipartite 2-factors

The original Oberwolfach problem has been completely resolved for bipartite 2-factorizations.

Theorem (Häggkvist (1985))

The graph K_{2n} admits an $[m_1, m_2, \ldots m_t]$-factorization when each m_i is even, $m_i \geqslant 4$, and $2n \equiv 2 \ (mod\ 4)$.

Bipartite 2-factors

The original Oberwolfach problem has been completely resolved for bipartite 2-factorizations.

Theorem (Häggkvist (1985))

The graph K_{2n} admits an $[m_1, m_2, \ldots m_t]$-factorization when each m_i is even, $m_i \geqslant 4$, and $2n \equiv 2 \pmod 4$.

Theorem (Bryant and Danziger (2011))

The graph K_{2n} admits an $[m_1, m_2, \ldots m_t]$-factorization when each m_i is even, $m_i \geqslant 4$, and $2n \equiv 0 \pmod 4$.

Our results

Theorem (Burgess, Danziger, and L-M (2025+))

Let $m_1, m_2, \ldots, m_t$ be positive even integers and $2n \equiv 2 \pmod{4}$, then $OP^*(m_1, m_2, \ldots, m_t)$ has a solution except for $OP^*(6)$.

Strategy

Step 1: Decompose K_{2n}^* into $n-2$ spanning subdigraphs that fall into one of two isomorphisms classes G_1 and G_2.

Step 2: Show that G_1 and G_2 both admit a $[m_1, m_2, \ldots, m_t]$-factorization.

First class of digraphs

Below, we show a spanning digraph of K_{14}^*.

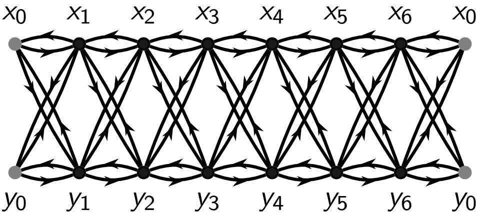


Figure: The first directed graph $G_1 = \vec{C}_7[2]$.

First class of digraphs

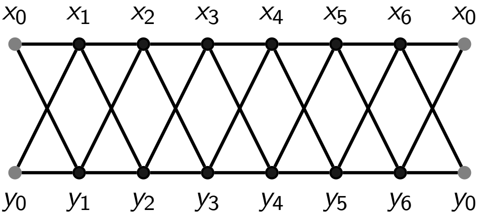


Figure: The underlying graph of $\vec{C}_7[2]$ written $C_7[2]$.

A directed version of Häggkvist Lemma

Theorem (Häggkvist Lemma (Häggkvist (1985)))

Let $m_1, m_2, \ldots, m_t$ be even integers such that $m_i \geqslant 4$. The graph $C_r[2]$ admits an undirected $[m_1, m_2, \ldots, m_t]$-factorization.

A directed version of Häggkvist Lemma

Theorem (Häggkvist Lemma (Häggkvist (1985)))

Let $m_1, m_2, \ldots, m_t$ be even integers such that $m_i \geqslant 4$. The graph $C_r[2]$ admits an undirected $[m_1, m_2, \ldots, m_t]$-factorization.

Corollary

Let $m_1, m_2, \ldots, m_t$ be even integers such that $m_i \geqslant 4$. The graph $\vec{C}_r[2]$ admits a $[m_1, m_2, \ldots, m_t]$-factorization.

We still need to derive a solution that includes tables of lengths two!

A directed version of Häggkvist Lemma

Lemma

Let $m_1, m_2, \ldots, m_t$ be even integers such that $m_i \geqslant 2$. The graph $\vec{C}_r[2]$ admits a $[m_1, m_2, \ldots, m_t]$-factorization.

Proof: Case 1: We have $k \geqslant 2$ tables of length 2.

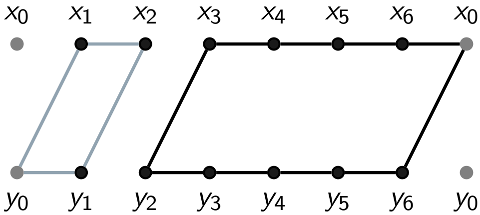


Figure: A undirected $[4, 10]$-factor of $C_7[2]$.

A directed version of Häggkvist Lemma

Lemma

Let $m_1, m_2, \ldots, m_t$ be even integers such that $m_i \geqslant 2$. The graph $\vec{C}_r[2]$ admits a $[m_1, m_2, \ldots, m_t]$-factorization.

Proof: Case 1: We have $k \geqslant 2$ tables of length 2.

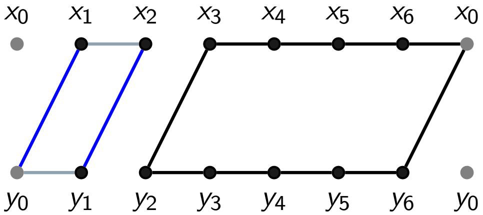


Figure: A 1-factorization of C_4 in blue and grey.

A directed version of Häggkvist Lemma

Lemma

Let $m_1, m_2, \ldots, m_t$ be even integers such that $m_i \geqslant 2$. The graph $\vec{C}_r[2]$ admits a $[m_1, m_2, \ldots, m_t]$-factorization.

Proof: Case 1: We have $k \geqslant 2$ tables of length 2.

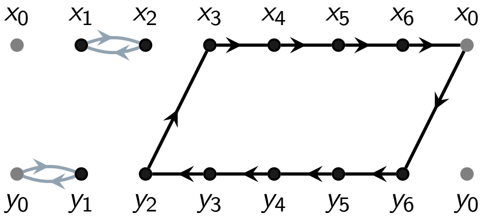


Figure: A directed $[2, 2, 10]$ factor of $\vec{C}_6[2]$.

A directed version of Häggkvist Lemma

Lemma

Let $m_1, m_2, \ldots, m_t$ be even integers such that $m_i \geqslant 2$. The graph $\vec{C}_r[2]$ admits a $[m_1, m_2, \ldots, m_t]$-factorization.

Proof: Case 1: We have $k \geqslant 2$ tables of length 2.

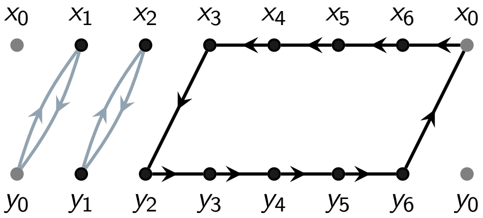


Figure: A directed $[2, 2, 10]$ factor of $\vec{C}_6[2]$.

A directed version of Häggkvist Lemma

Lemma

Let $m_1, m_2, \ldots, m_t$ be even integers such that $m_i \geqslant 2$. The graph $\vec{C}_r[2]$ admits a $[m_1, m_2, \ldots, m_t]$-factorization.

Proof: Case 2: We have one table of length 2.

Here an explicit construction is needed. The construction follows a similar reasoning as Häggkvist's lemma.

□

The second spanning subdigraph of K_{2n}^*

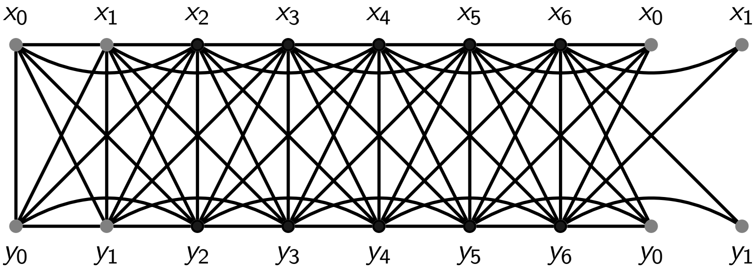
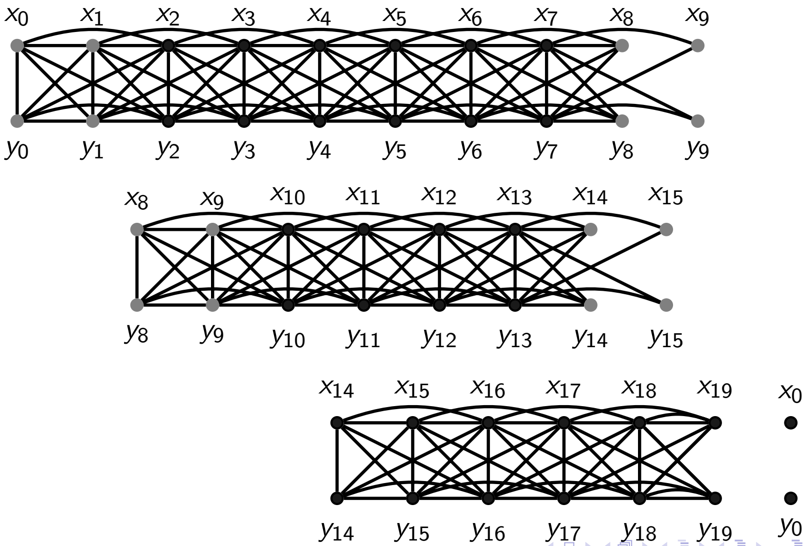


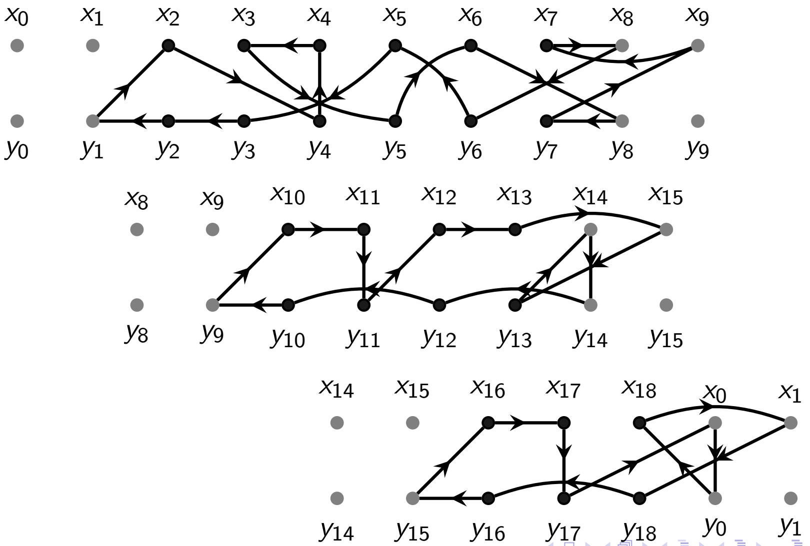
Figure: The underlying graph of G_2.

Each edge represents a pair of arcs, one for each direction. This underlying graph can be described as a wreath product (lexicographic product) of the circulant $X(n, \{\pm 1, \pm 2\})$ with K_2

The construction of a $[16, 12, 10]$-factor of K_{38}^*



The construction of a $[16, 12, 10]$-factor of K^*_{38}



Long tables

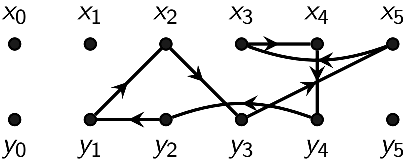
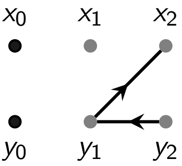
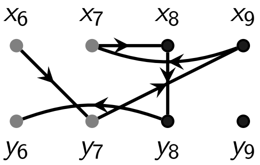


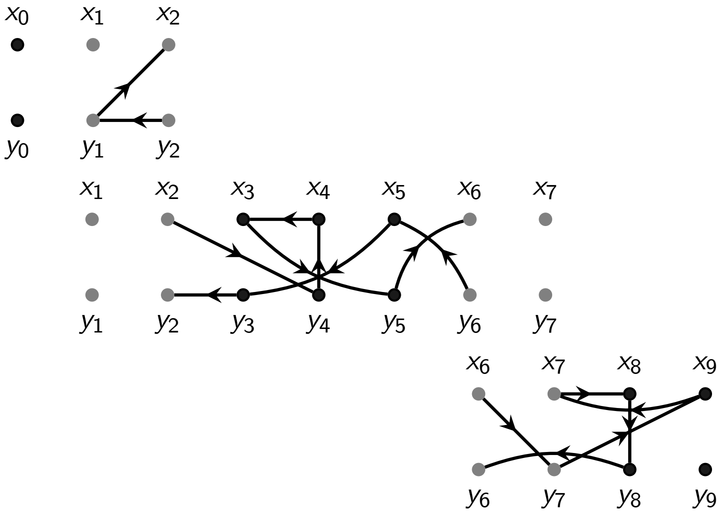
Figure: A cycle of length 8.

Long tables



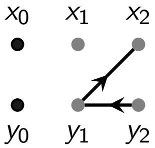


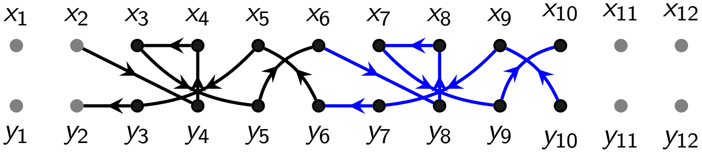
Long tables

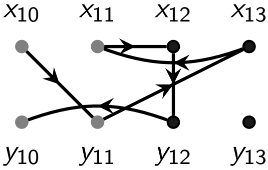


x_0
x_1
x_2
y_0
y_1
y_2
x_1
x_2
x_3
x_4
x_5
x_6
x_7
y_1
y_2
y_3
y_4
y_5
y_6
y_7
x_6
x_7
x_8
x_9
y_6
y_7
y_8
y_9

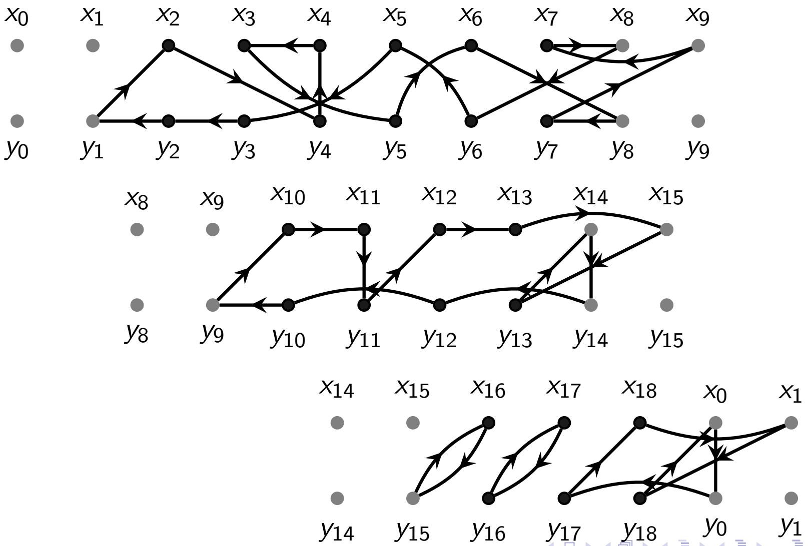
Long tables


x_0 x_1 x_2
y_0 y_1 y_2


x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9 x_{10} x_{11} x_{12}
y_1 y_2 y_3 y_4 y_5 y_6 y_7 y_8 y_9 y_{10} y_{11} y_{12}


x_{10} x_{11} x_{12} x_{13}
y_{10} y_{11} y_{12} y_{13}

Special cases- A [16,12,6,2,2]-factor



Future work

Problem: Can we take a similar approach for the case $2n \equiv 0 \pmod 4$?

Answer: Yes. However, the second spanning digraph in the decomposition of K_{2n}^* is different.

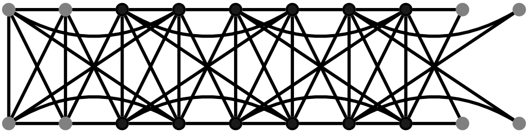


Figure: The underlying graph of G_2.

Thank you!