Hamiltonian decompositions of the wreath product of two hamiltonian decomposable directed graphs

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Terminology

Definition

A **2-factor** of a (directed) graph G is a 2-regular spanning subgraph of G.

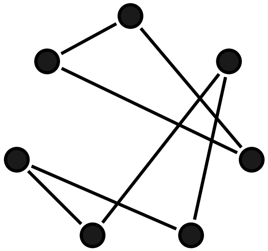


Figure: A 2-factor of K_7.

Terminology

Definition

A 2-**factorization** of (directed) graph G is a decomposition of G into 2-factors.

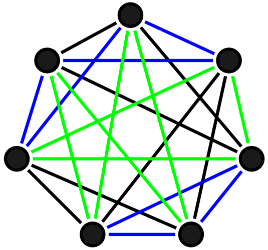


Figure: A 2-factorization of K_7.

Hamiltonian decomposable

Definition

A graph (directed graph) is **hamiltonian decomposable** if it admits a decomposition into (directed) hamiltonian cycles.

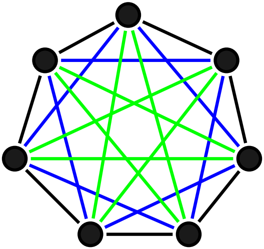
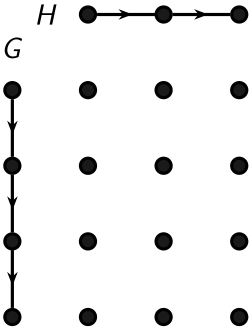


Figure: A hamiltonian decomposition of K_7.

Wreath product

Definition

The **wreath product** of G and H, denoted $G \wr H$, is a digraph on vertex set $V(G) \times V(H)$, where $((x, y), (u, v)) \in A(G \wr H)$ if and only if...



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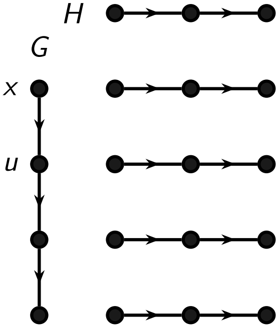

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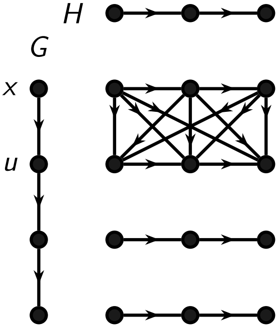

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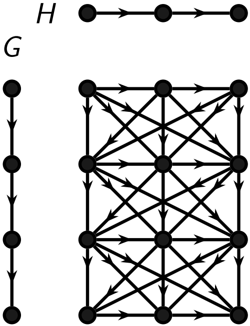

Main problem

Question: Given two hamiltonian decomposable (directed) graphs G and H, is $G \wr H$ also hamiltonian decomposable?

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Theorem (Baranyai and Szás, 1981)

If G and H are hamiltonian decomposable graphs, then $G \wr H$ is also hamiltonian decomposable.

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If G and H are hamiltonian decomposable graphs, then $G \wr H$ is also hamiltonian decomposable.

Theorem (Ng, 1998)

If G and H are hamiltonian decomposable digraphs, $|V(G)|$ is odd, and $|V(H)| > 2$, then $G \wr H$ is also hamiltonian decomposable.

Main question refined

Question: Given two hamiltonian decomposable digraphs graphs G and H, such that $|V(G)|$ is even, is $G \wr H$ also hamiltonian decomposable?

Reduction

Proposition (Ng, 1998)

Let G and H be hamiltonian decomposable directed graphs such that $|V(G)| = n$ and $|V(H)| = m$. If

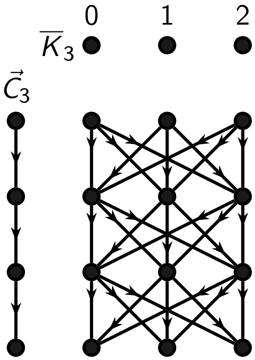
1. $\vec{C}_n \wr H$ *is hamiltonian decomposable,*
2. *and* $\vec{C}_n \wr \overline{K}_m$ *are hamiltonian decomposable,*

then $G \wr H$ *is hamiltonian decomposable.*

The directed graph $\vec{C}_n \wr \overline{K}_m$

Lemma (Ng, 1998)

If $m \geqslant 3$, then $\vec{C}_n \wr \overline{K}_m$ is hamiltonian decomposable.



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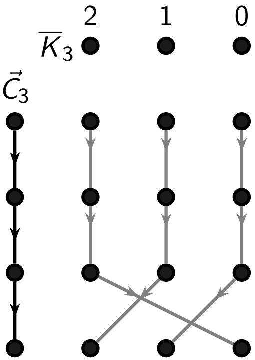

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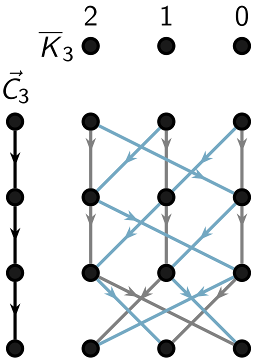

$$F_0 = (id, id, (0,1,2))$$
$$F_1 = ((0,1,2), (0,1,2), (0,2,1))$$

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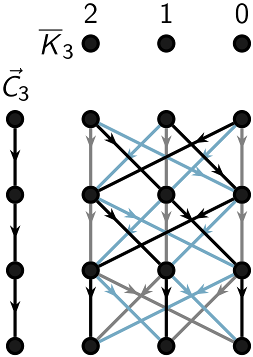

$$
\begin{aligned}
F_0 &= (id, id, (0,1,2)) \\
F_1 &= ((0,1,2),(0,1,2),(0,2,1)) \\
F_2 &= ((0,2,1),(0,2,1), id)
\end{aligned}
$$

2-factorization of $\vec{C}_n \wr \overline{K}_m$

Each 2-factorization of $\vec{C}_n \wr \overline{K}_m$ can be described as a set of m n-tuples of permutations from S_m:

$$\mathcal{F} = \left\{ \begin{array}{cccc} (\mu_{(0,0)}, & \mu_{(0,1)}, & \dots, & \mu_{(0,n-1)}); \\ (\mu_{(1,0)}, & \mu_{(1,1)}, & \dots, & \mu_{(1,n-1)}); \\ & \vdots & & \\ (\mu_{(m-1,0)}, & \mu_{(m-1,1)}, & \dots, & \mu_{(m-1,n-1)}). \end{array} \right\}$$

Decomposition families

Definition

Let $S = \{\sigma_0, \sigma_1, \ldots, \sigma_{m-1}\}$ be a set of m permutations of S_m acting on $\mathbb{Z}_m$. The set S is a **regular permutation set of order** m if $j^{\sigma_{k_1}} \neq j^{\sigma_{k_2}}$ for all $j \in \mathbb{Z}_m$ and $k_1, k_2 \in \mathbb{Z}_m$ such that $k_1 \neq k_2$.

Example:

$$\mathcal{F} = \left\{ \begin{array}{lll} (id, & id, & (0,1,2)) \\ ((0,1,2), & (0,1,2), & (0,2,1)) \\ ((0,2,1), & (0,2,1), & id) \end{array} \right\}$$

Hamiltonian n-tuple

Definition

Let $\mu_0, \mu_1, \ldots, \mu_{n-1} \in S_m$. The n-tuple $(\mu_0, \mu_1, \ldots, \mu_{n-1})$ is a **hamiltonian n-tuple** if

$$\tau_i = \mu_0 \mu_1 \ldots \mu_{n-1}$$

is a permutation with a single cycle.

Example: Let $n = 3$:

$F_0 = (id, id, (0, 1, 2)) \Rightarrow \tau_0 = (0, 1, 2)$.

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Example: Let $n = 3$:

$$F_0 = (id, id, (0,1,2)) \Rightarrow \tau_0 = (0,1,2);$$
$$F_1 = ((0,1,2),(0,1,2),(0,2,1)) \Rightarrow \tau_1 = (0,1,2)(0,1,2)(0,2,1) = (0,1,2).$$

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$$\begin{aligned} F_0 &= (id, id, (0,1,2)) \Rightarrow \tau_0 = (0,1,2); \\ F_1 &= ((0,1,2),(0,1,2),(0,2,1)) \Rightarrow \tau_1 = (0,1,2)(0,1,2)(0,2,1) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\quad = (0,1,2); \\ F_2 &= ((0,2,1),(0,2,1), id) \Rightarrow \tau_2 = (0,2,1)(0,2,1) = (0,1,2). \end{aligned}$$

In summary

The digraph $\vec{C}_n \wr \overline{K}_m$ is hamiltonian decomposable if we have

$$\left.\begin{array}{llll}(\mu_{0,0}, & \mu_{0,1}, & \ldots, & \mu_{0,n-1}) \\ (\mu_{1,0}, & \mu_{1,1}, & \ldots, & \mu_{1,n-1}) \\ \vdots & \vdots & \vdots & \vdots \\ (\mu_{m-1,0}, & \mu_{m-1,1}, & \ldots, & \mu_{m-1,n-1})\end{array}\right\} m \text{ hamiltonian } n\text{-tuples}$$

where $\{\mu_{(0,i)}, \mu_{(1,i)}, \ldots \mu_{(m-1,i)}\}$ is a decomposition family of order m for each $i \in \mathbb{Z}_n$.

Hamiltonian decomposition of $\vec{C}_n \wr H$

We will take a similar approach for the digraph $\vec{C}_n \wr H$:

$$\left.\begin{array}{llll}(\mu_{0,0}, & \mu_{0,1}, & \ldots, & \mu_{0,n-1}) \\ (\mu_{1,0}, & \mu_{1,1}, & \ldots, & \mu_{1,n-1}) \\ \vdots & \vdots & \vdots & \vdots \\ (\mu_{m-1,0}, & \mu_{m-1,1}, & \ldots, & \mu_{m-1,n-1}).\end{array}\right\} m\ n\text{-tuples}$$

where $\{\mu_{(0,i)}, \mu_{(1,i)}, \ldots \mu_{(m-1,i)}\}$ is a decomposition family of order m for each $i \in \mathbb{Z}_n$.

Truncation of a permutation

Definition

Let $\mu \in S_m$ be such that $(m-1)^\mu \neq m-1$. The **truncation** of μ, denoted $\hat{\mu}$, is the permutation

$$\hat{\mu} = \mu\,(m-1, (m-1)^\mu).$$

Example: $\mu = (0, 1, 2, 3, 4, 5, 6, 7) \in S_8$

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Truncated hamiltonian n-tuple

Definition

Let $\mu_0, \mu_1, \ldots, \mu_{n-1} \in S_m$. The n-tuple $(\mu_0, \mu_1, \ldots, \mu_{n-1})$ is a **truncated hamiltonian n-tuple** if

$$\sigma_i = \hat{\mu}_0 \hat{\mu}_1 \ldots \hat{\mu}_{n-1}$$

is a permutation with exactly two cycles in its disjoint cycle notation.

Example: $((0,2),(0,2),(0,1,2))$, where $(0,2),(0,1,2) \in S_3$

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Example: $((0,2),(0,2),(0,1,2))$

$$\sigma = id \cdot id \cdot (0,1)(2)$$

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Example:$((0,2),(0,2),(0,1,2))$

$$\sigma = id \ id \,(0,1)(2)$$
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General Approach

Let H be a digraph on m vertices that admits a decomposition into c directed hamiltonian cycles ($1 \leqslant c \leqslant m-2$). The digraph $\vec{C}_n \wr H$ is hamiltonian decomposable if there exist m n-tuples of permutations such that:

$$\left.\begin{array}{llll}
(\mu_{0,0}, & \mu_{0,1}, & \ldots, & \mu_{0,n-1}) \\
(\mu_{1,0}, & \mu_{1,1}, & \ldots, & \mu_{1,n-1}) \\
 & \vdots & & \\
(\mu_{c-1,0}, & \mu_{c-1,1}, & \ldots, & \mu_{c-1,n-1})
\end{array}\right\} c \text{ truncated hamiltonian } n\text{-tuples}$$

$$\left.\begin{array}{llll}
(\mu_{c,0}, & \mu_{c,1}, & \ldots, & \mu_{c,n-1}) \\
(\mu_{c+1,0}, & \mu_{c+1,1}, & \ldots, & \mu_{c+1,n-1}) \\
 & \vdots & & \\
(\mu_{m-1,0}, & \mu_{m-1,1}, & \ldots, & \mu_{m-1,n-1})
\end{array}\right\} m-c \text{ hamiltonian } n\text{-tuples}$$

One more reduction step

Proposition

Let n be an even integer and c be an integer such that $0 \leqslant c \leqslant m-2$. If $\vec{C}_2 \wr \overline{K}_m$ admits a c-twined 2-factorization than so does $\vec{C}_n \wr \overline{K}_m$.

Summary: It suffices to consider the digraph $\vec{C}_2 \wr \overline{K}_m$.

Consequences

Let H be a digraph on m vertices that admits a decomposition into c directed hamiltonian cycles ($0 \leqslant c \leqslant m-2$). The digraph $\vec{C}_2 \wr H$ is hamiltonian decomposable if there exist m pairs of permutations such that:

$$\left.\begin{array}{c}(\mu_0, \tau_0)\\(\mu_1, \tau_1)\\\vdots\\(\mu_{c-1}, \tau_{c-1})\end{array}\right\} c \text{ truncated hamiltonian pairs}$$

$$\left.\begin{array}{c}(\mu_c, \tau_c)\\(\mu_{c+1}, \tau_{c+1})\\\vdots\\(\mu_{m-1}, \tau_{m-1})\end{array}\right\} m-c \text{ hamiltonian pairs}$$

Solution for the case for $m = 13$ and $c = 2$

If H is a digraph on $m = 13$ vertices that admits a decomposition into 2 hamiltonian cycles, then we aim to construct a set of 13 pairs of permutations.

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If H is a digraph on $m = 13$ vertices that admits a decomposition into 2 hamiltonian cycles, then we aim to construct a set of 13 pairs of permutations.

Step 1: Construct two decomposition families.

The decomposition family $\mathcal{F}_{13}$

$$\mathcal{F}_{13} = \left\{ \begin{array}{l}
\sigma_1 = (0,1,12,2,3,4,5,6,7,8,9,10,11); \\
\sigma_2 = (0,2,4,6,12,8,10)(1,3,5,7,9,11); \\
\sigma_3 = (0,12,3,6,9)(1,4,7,10)(2,5,8,11); \\
\sigma_4 = (0,4,8)(1,5,12,9)(2,6,10)(3,7,11); \\
\sigma_5 = (0,5,10,3,8,1,6,11,12,4,9,2,7); \\
\sigma_6 = (0,6)(1,7)(2,8)(3,9)(4,12,10)(5,11); \\
\sigma_7 = (0,7,2,9,4,11,6,1,8,3,10,12,5); \\
\sigma_8 = (0,8,4)(1,9,5)(2,10,6)(3,12,11,7); \\
\sigma_9 = (0,9,12,6,3)(1,10,7,4)(2,11,8,5); \\
\sigma_{10} = (0,10,8,6,4,2,12)(1,11,9,7,5,3); \\
\sigma_{11} = (0,11,10,9,8,12,7,6,5,4,3,2,1); \\
\sigma_{12} = (0,3,11,4,10,5,9,6,8,7,12,1,2); \\
\sigma_0 = id.
\end{array} \right\}$$

Solution for the case for $m = 13$ and $c = 2$

If H is a digraph on $m = 13$ vertices that admits a decomposition into 2 hamiltonian cycles, then we aim to construct a set of 13 pairs of permutations.

Step 1: Construct two decomposition families.

Step 2: Construct a set of 13 pairs of permutations from $\mathcal{F}_{13} \times \mathcal{F}_{13}$.

Hamiltonian array of $\mathcal{F}_{13} \times \mathcal{F}_{13}$

	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}	σ_{11}	σ_{12}	σ_0
σ_1													
σ_2													
σ_3													
σ_4													
σ_5													
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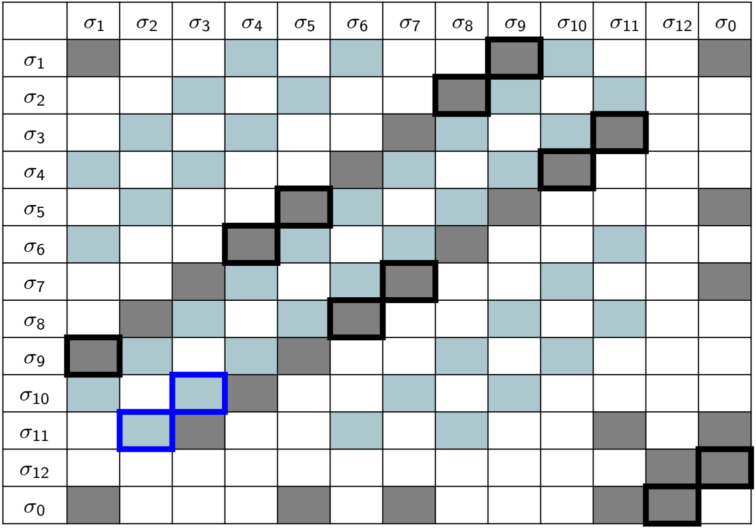
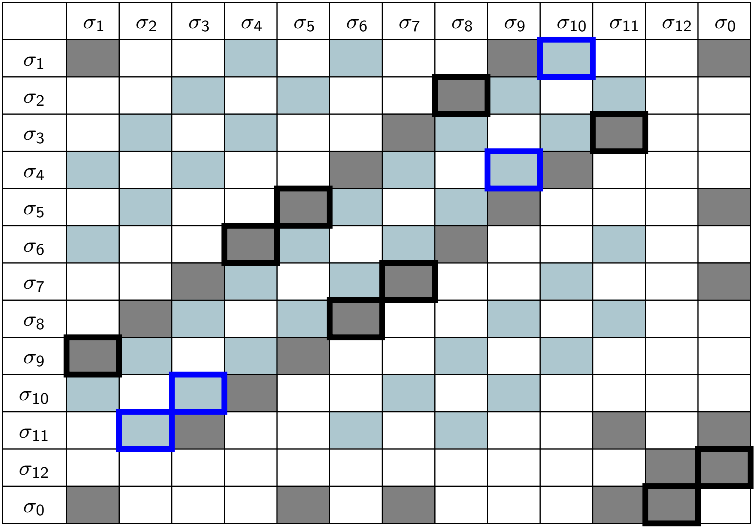
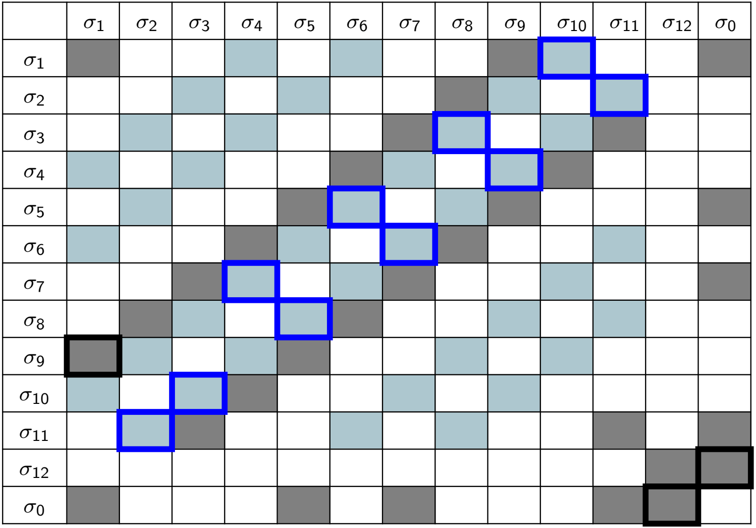


Solution for $m = 13$ and $c = 4$



Solution for $m = 13$ and $c = 10$



Summary of results

Theorem

Let G and H be hamiltonian decomposable directed graphs such that $|V(H)| > 3$ and $|V(G)|$ is even. Then $G \wr H$ is hamiltonian decomposable except possibly when

1. *G is a directed cycle,*
2. *$|V(H)|$ is even,* **and**
3. *H admits a decomposition into an odd number of directed hamiltonian cycles.*

Summary of results

Proposition

If $n > 2$ is even and $m > 2$ is even, then $\vec{C}_n \wr K_m^*$ and $\vec{C}_n \wr \vec{C}_m$ are hamiltonian decomposable.

Proposition

If n is even, then $\vec{C}_n \wr \vec{C}_2$ and $\vec{C}_n \wr \vec{C}_3$ are not hamiltonian decomposable.

Thank you!