

# On the directed Oberwolfach problem with tables of even lengths

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## A simple example

**The setting:** Consider a conference with 16 participants. To facilitate networking, the organizing committee decides to host 15 banquets. The banquet hall has 3 tables that seat 4, 4, and 8 participants.

**The problem:** The organizing committee needs a set of 15 seating arrangements (one for each banquet) such that each participant is seated to the right of every other participants exactly once.

Is this possible?

## Construction of a seating arrangement

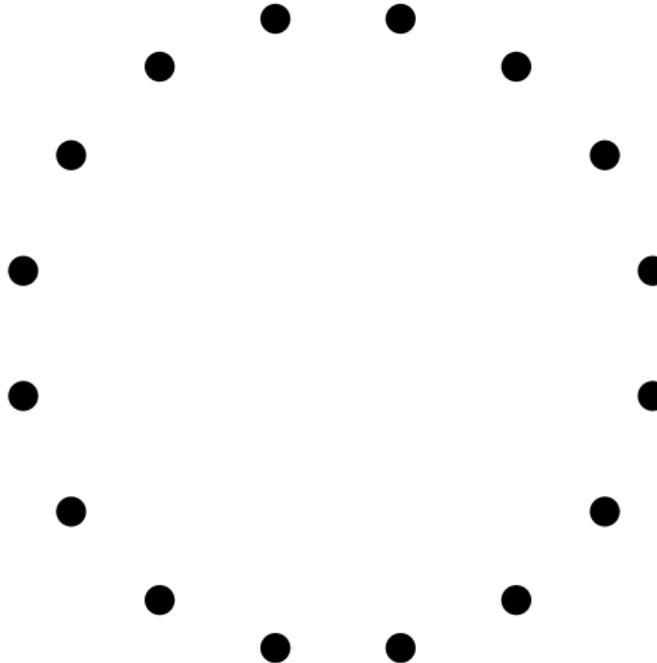


Figure: The 16 participants (one for each vertex).

## Construction of a seating arrangement

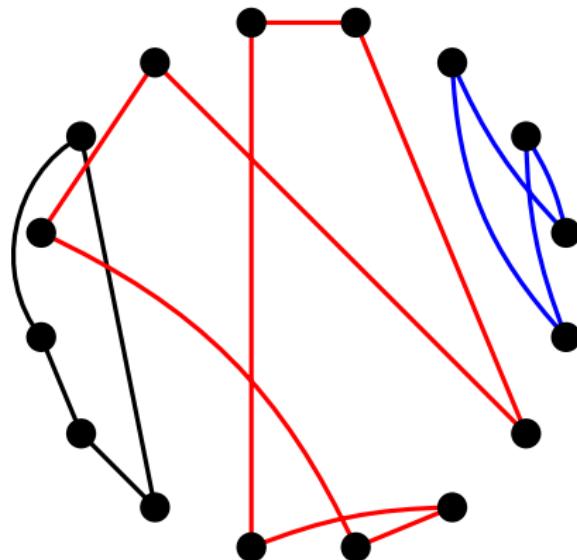


Figure: One seating arrangement with two tables of length 4 and one table of length 8.

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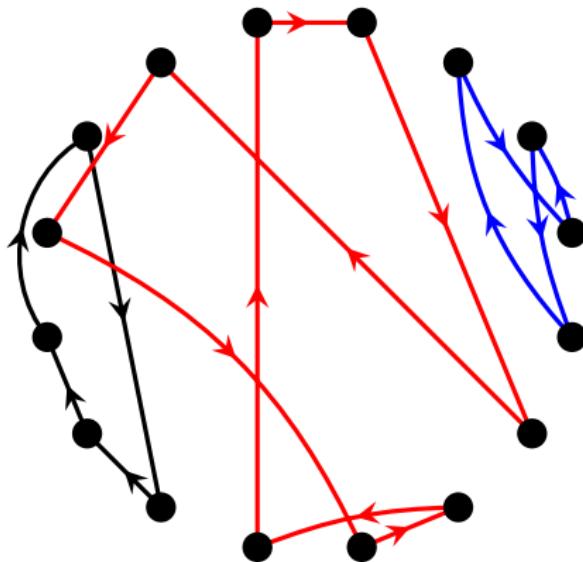


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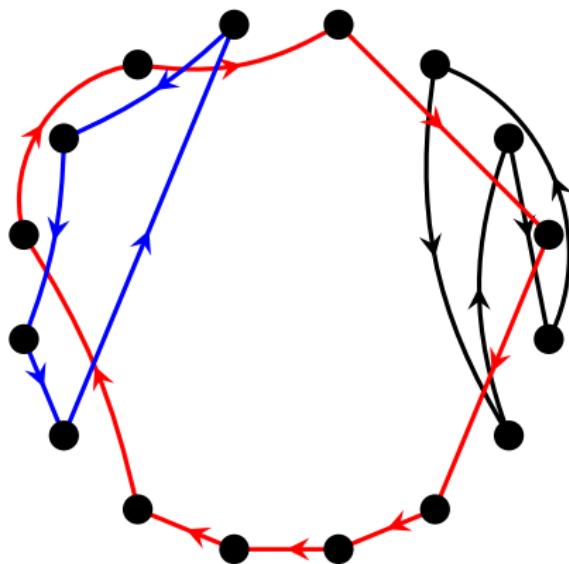


Figure: Another seating arrangement with two tables of length 4 and one table of length 8.

# The directed Oberwolfach problem

**The setting:** Consider a conference with  $n$  participants. To facilitate networking, the organizing committee decides to host  $n - 1$  banquets. The banquet hall has  $t$  round tables that sit  $m_1, m_2, \dots, m_t$  participants such that  $m_1 + m_2 + \dots + m_t = n$ .

**The problem:** The organizing committee needs a set of  $n - 1$  seating arrangements (one for each banquet) such that each participant is seated **to the right** of every other participants exactly once.

Is this possible?

# The complete symmetric digraph

## Definition

The **complete symmetric digraph**, denoted  $K_n^*$ , is the digraph on  $n$  vertices in which for every pair of distinct vertices  $x$  and  $y$ , there are arcs  $(x, y)$  and  $(y, x)$ .

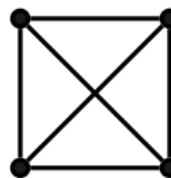


Figure: The complete graph  $K_4$ .

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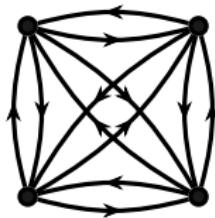


Figure: The complete symmetric digraph  $K_4^*$ .

## 2-factorization

### Definition

A **directed**  $[m_1, m_2, \dots, m_t]$ -**factor** of digraph  $G$  is a spanning subdigraph comprised of disjoint directed cycles of length  $m_1, m_2, \dots, m_t$ .

### Definition

A **directed**  $[m_1, m_2, \dots, m_t]$ -**factorization** of digraph  $G$  is a decomposition of  $G$  into  $[m_1, m_2, \dots, m_t]$ -factors.

# The graph-theoretic formulation of the directed OP

Problem  $(OP^*(m_1, m_2, \dots, m_t))$

Let  $m_1, m_2, \dots, m_t \geq 2$ . If  $m_1 + m_2 + \dots + m_\alpha = n$ , does  $K_n^*$  admit a directed  $[m_1, m_2, \dots, m_t]$ -factorization?

When all tables are of length  $m$ , we write  $OP^*(m^t)$ .

A solution to  $OP^*(m_1, m_2, \dots, m_t)$  is also a resolvable Mendelsohn design with blocks of size  $m_1, m_2, \dots, m_t$ .

## Background (free solutions!)

Corollary (Kadri and Šajna (2025))

*If the original Oberwolfach problem with  $n$  participants and tables of lengths  $3 \leq m_1 \leq m_2 \leq \dots \leq m_t$  with  $n$  being odd, then  $OP^*(m_1, m_2, \dots, m_t)$  also has a solution.*

## Background

Theorem (Bermond, Germa, and Sotteau (1979); Tillson (1980), Bennett and Zhang (1990); Adams and Bryant (Unpublished); Abel, Bennett, and Ge (2002); Burgess and Šajna (2014); Burgess, Francetić, and Šajna (2018); L-M (2024))

*The  $OP^*(m^t)$  has a solution except when  
 $(m, t) \notin \{(3, 2), (4, 1), (6, 1)\}$ .*

The directed OP has been completely resolved when all tables are of the same length.

# Background

Theorem (Zhang and Du (2005))

*The  $OP^*(3^t, s)$  has a solution when  $s \in \{4, 5\}$  for all  $t \in \mathbb{Z}^+$ .*

Theorem (Kadri and Šajna (2025))

*Let  $m_1 < m_2$ . The  $OP^*(m_1, m_2)$  has a solution except possibly when  $m_1 \in \{4, 6\}$  and  $m_2$  is even.*

Theorem (Horsley and L-M (2024+))

*Let  $m_1 < m_2$ . The  $OP^*(m_1, m_2)$  has a solution when  $m_1 \in \{4, 6\}$  and  $m_2 \geq 8$  is even.*

## Bipartite 2-factors

The original Oberwolfach problem has been completely resolved for bipartite 2-factorizations.

Theorem (Häggkvist (1985))

*The graph  $K_{2n} - I$  admits an  $[m_1, m_2, \dots, m_t]$ -factorization when each  $m_i$  is even,  $m_i \geq 4$ , and  $2n \equiv 2 \pmod{4}$ .*

Note that  $K_{2n} - I$  is the complete graph on  $2n$  vertices with a 1-factor removed.

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### Theorem (Bryant and Danziger (2011))

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Note that  $K_{2n} - I$  is the complete graph on  $2n$  vertices with a 1-factor removed.

# Our results

Theorem (Burgess, Danziger, and L-M (2025+))

*Let  $m_1, m_2, \dots, m_t$  be positive even integers and  $2n \equiv 2 \pmod{4}$ , then  $OP^*(m_1, m_2, \dots, m_t)$  has a solution except for  $OP^*(6)$ .*

# Strategy

Step 1: Decompose  $K_{2n}^*$  into  $n - 2$  spanning subdigraphs that fall into one of two isomorphisms classes  $G_1$  and  $G_2$ .

Step 2: Show that  $G_1$  and  $G_2$  both admit a  $[m_1, m_2, \dots, m_t]$ -factorization.

# First class of digraphs

Below, we show a spanning digraph of  $K_{14}^*$ .

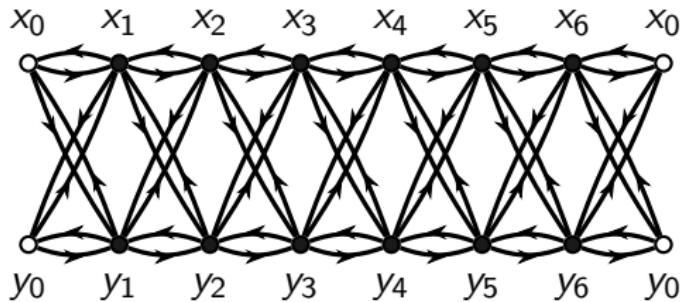


Figure: The first directed graph  $G_1 = \vec{C}_7[2]$ .

# First class of digraphs

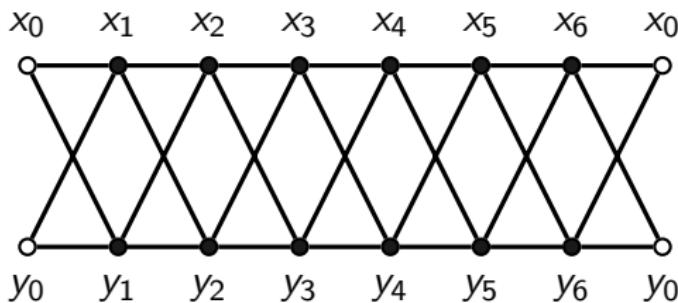


Figure: The underlying graph of  $\vec{C}_7[2]$  written  $C_7[2]$ .

# A directed version of Häggkvist Lemma

Theorem (Häggkvist Lemma (Häggkvist (1985)))

*Let  $m_1, m_2, \dots, m_t$  be even integers such that  $m_i \geq 4$ . The graph  $C_r[2]$  admits an undirected  $[m_1, m_2, \dots, m_t]$ -factorization.*

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Corollary

Let  $m_1, m_2, \dots, m_t$  be even integers such that  $m_i \geq 4$ . The graph  $\vec{C}_r[2]$  admits a  $[m_1, m_2, \dots, m_t]$ -factorization.

We still need to derive a solution that includes tables of lengths two!

# A directed version of Häggkvist Lemma

## Lemma

Let  $m_1, m_2, \dots, m_t$  be even integers such that  $m_i \geq 2$ . The graph  $\vec{C}_r[2]$  admits a  $[m_1, m_2, \dots, m_t]$ -factorization.

**Proof:** Case 1: We have  $k \geq 2$  tables of length 2.

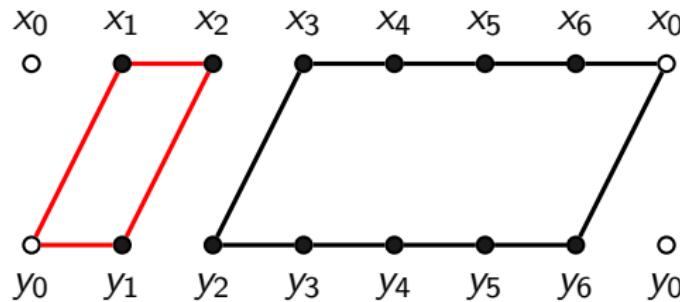


Figure: A undirected  $[4, 10]$ -factor of  $C_7[2]$ .

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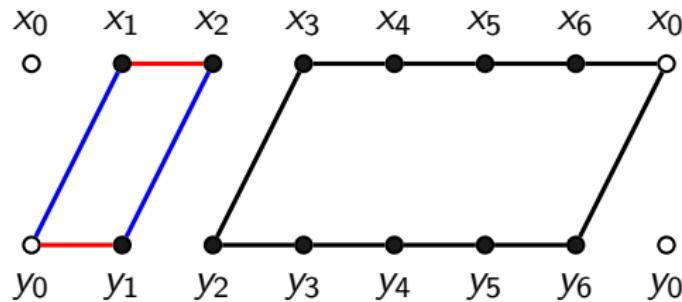


Figure: A 1-factorization of  $C_4$  in blue and grey.

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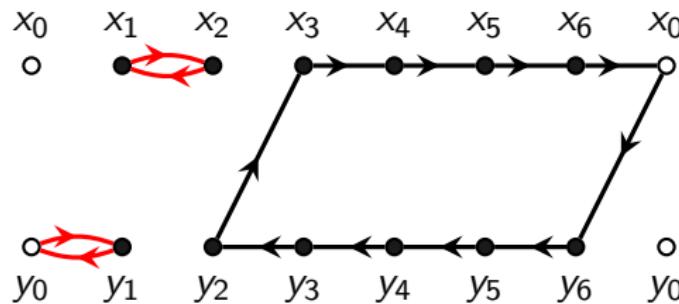


Figure: A directed  $[2, 2, 10]$  factor of  $\vec{C}_6[2]$ .

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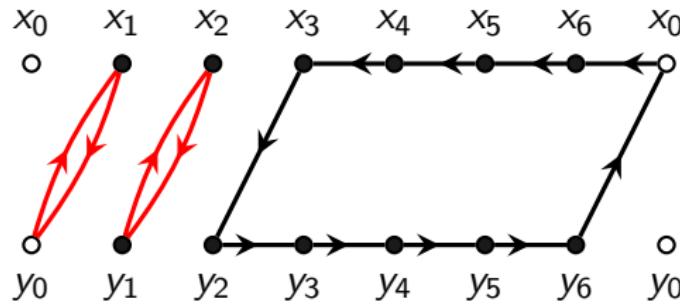


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# A directed version of Häggkvist Lemma

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**Proof:** Case 2: We have one table of length 2.

Here an explicit construction is needed. The construction follows a similar reasoning as Häggkvist's lemma.



# The second spanning subdigraph of $K_{2n}^*$

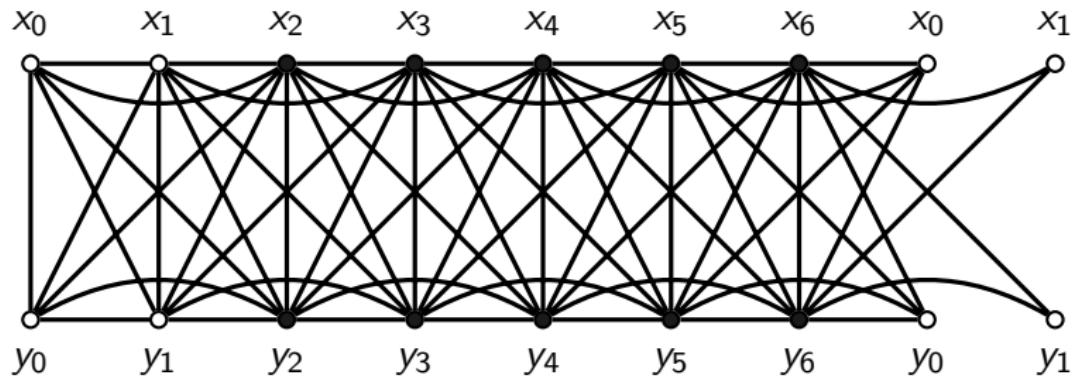
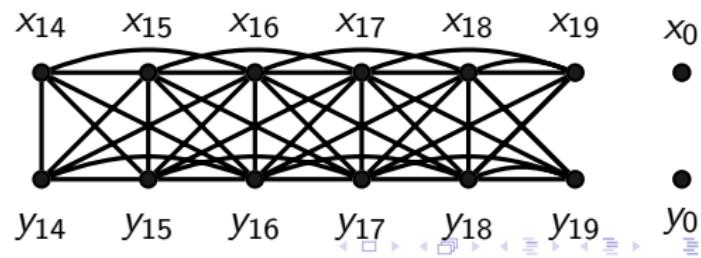
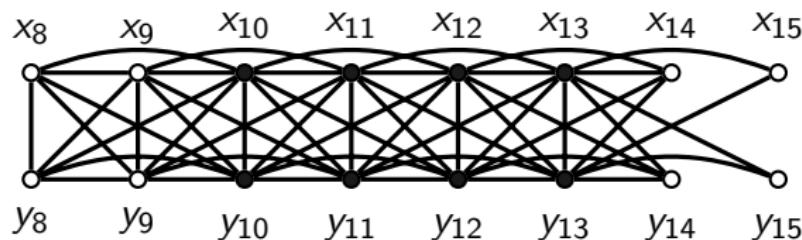
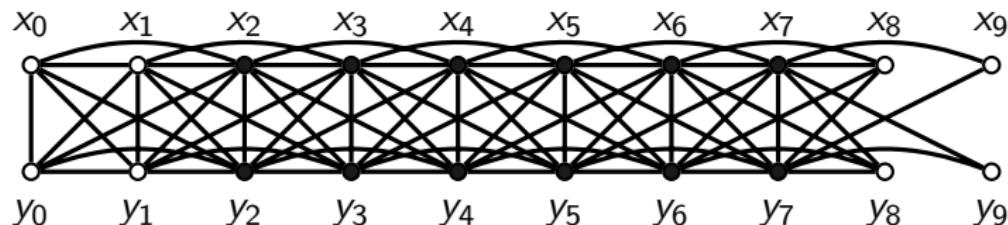


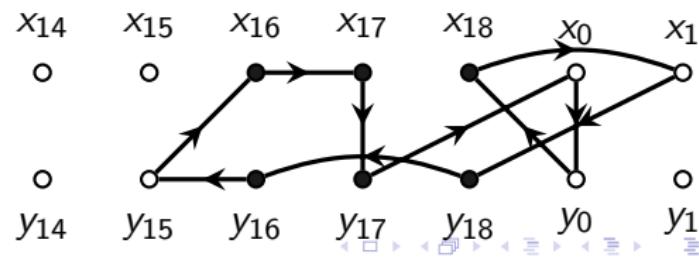
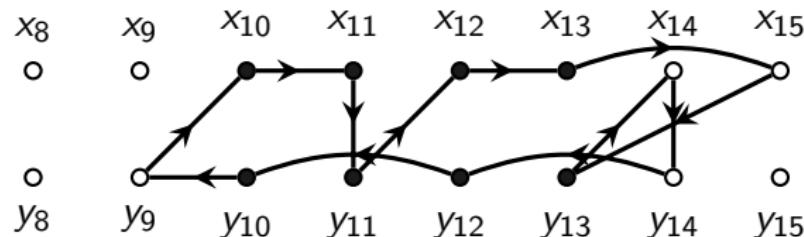
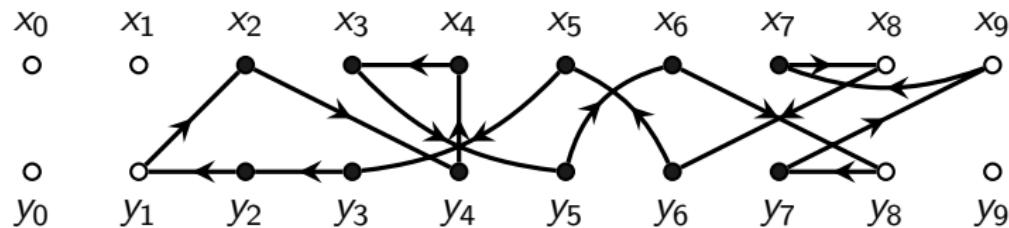
Figure: The underlying graph of  $G_2$ .

Each edge represents a pair of arcs, one for each direction. This underlying graph can be described as a wreath product (lexicographic product) of the circulant  $X(n, \{\pm 1, \pm 2\})$  with  $K_2$

# The construction of a $[16, 12, 10]$ -factor of $K_{38}^*$



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# Long tables

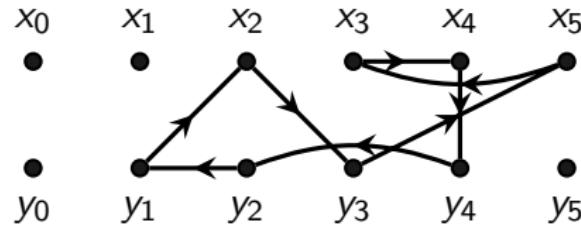
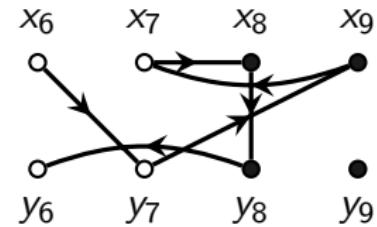
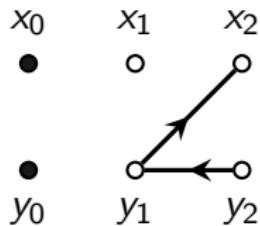
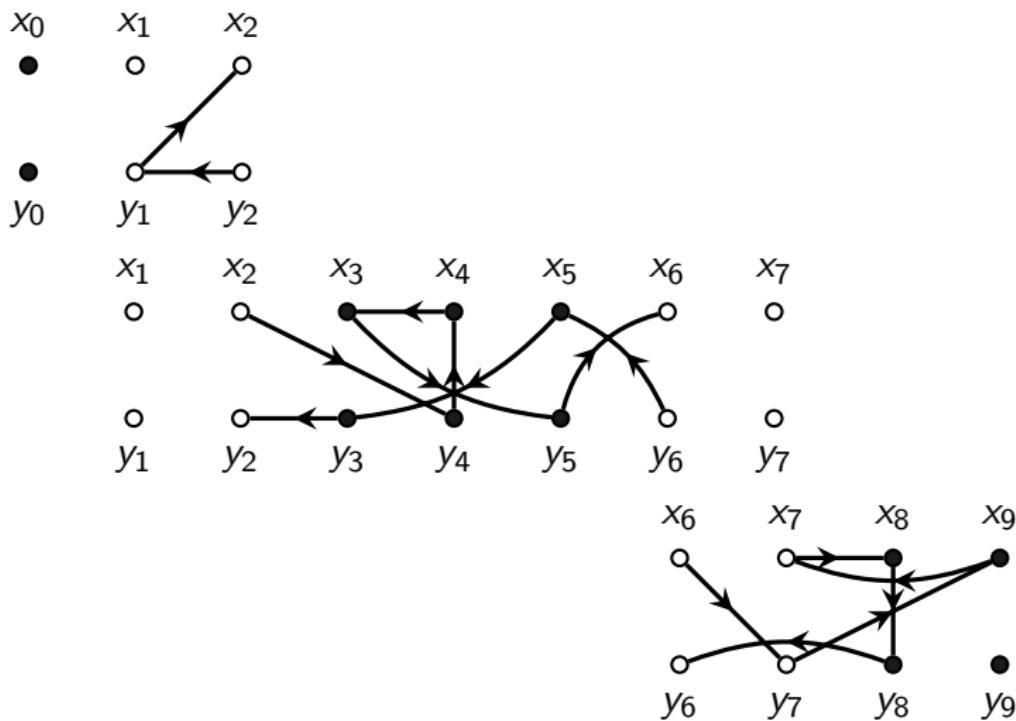


Figure: A cycle of length 8.

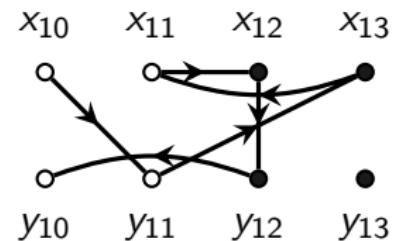
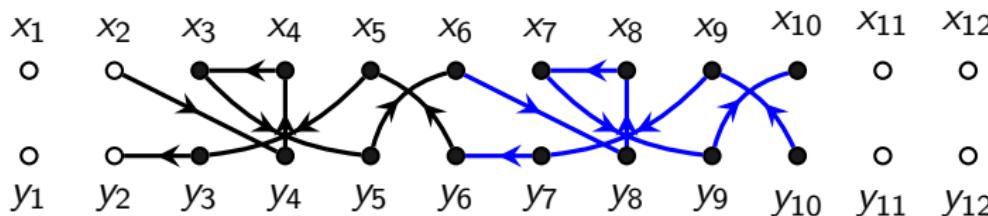
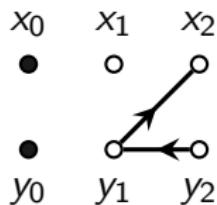
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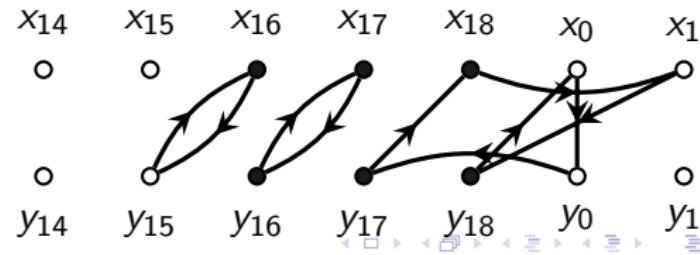
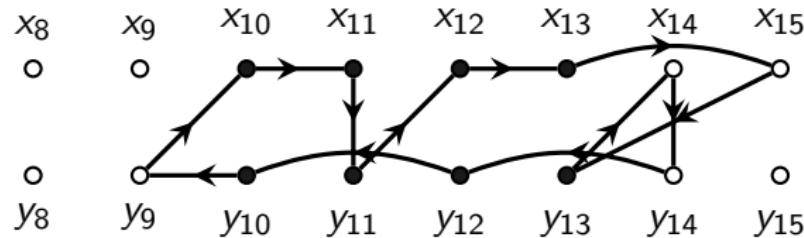
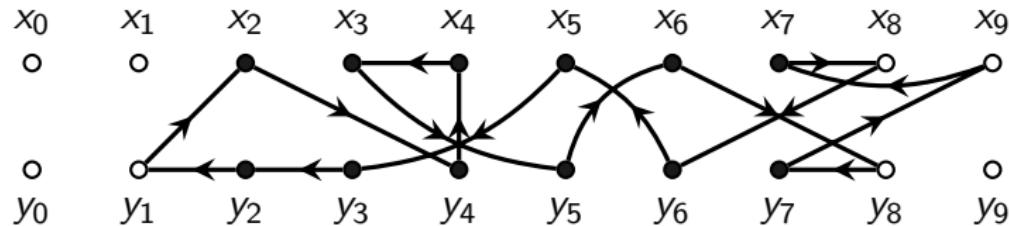
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Special cases- A [16,12,6,2,2]-factor



## Future work

**Problem:** Can we take a similar approach for the case  $2n \equiv 0 \pmod{4}$ ?

**Answer:** Yes. However, the second spanning digraph in the decomposition of  $K_{2n}^*$  is different.

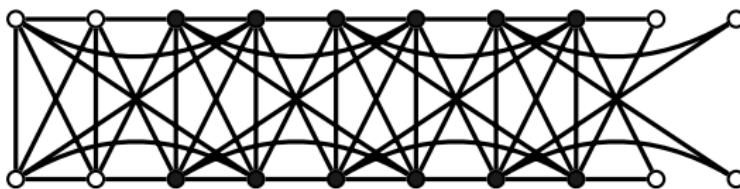


Figure: The underlying graph of  $G_2$ .

Thank you!

The 2026 Prairie Discrete Math Workshop:

- Set to take place on May 7th and 8th in Regina;
- Students and post-docs welcome! (Some travel funding may be available)