Behavior-Aware Queueing: The Finite-Buffer Setting with Many Strategic Servers: Technical Online Appendix

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In this technical online appendix, we provide proofs for the results related to the Erlang B and C formulae in the Electronic Companion (EC) of the manuscript titled: "Behavior-Aware Queueing: The Finite-Buffer Setting with Many Strategic Servers". For the reader's convenience, the result numbering in this file is consistent with that in the EC. Throughout, we use the notation $\rho = \frac{\lambda}{\mu}$.

EC.1. Preliminaries

LEMMA EC.1 **(Properties of Erlang B and C Formulae)**. *The following hold:*

(a) $ErlC(N,\rho)\begin{cases} <1, & \text{if } \rho<N, \\ =1, & \text{if } \rho=N, \\ >1, & \text{if } \rho>N. \end{cases}$

(b) $ErlC(N,\rho) = N\left(\frac{N-\rho}{ErlB(N,\rho)}+\rho\right)^{-1}$.

(c) $\lim_{\rho\to\infty} ErlC(N,\rho)/\rho = 1$.

(d) $\lim_{\rho\downarrow 0} ErlC(N,\rho) = 0$.

(e) $\frac{1-ErlC(N,\rho)}{N-\rho}\in(0,1)$, $\forall\rho>0$. *Moreover,* $\lim_{\rho\to 0}\frac{1-ErlC(N,\rho)}{N-\rho}=1$ *and* $\lim_{\rho\to\infty}\frac{1-ErlC(N,\rho)}{N-\rho}=1/N$.

(f) $\frac{\partial ErlB(N,\rho)}{\partial\rho} = ErlB(N,\rho)\left(\frac{N}{\rho}-(1-ErlB(N,\rho))\right)$.

Proof of Lemma EC.1

(a): When $\rho=N$, it follows that

$$ErlC(N,\rho) = \frac{\frac{N^N}{N!}\frac{N}{N-N}}{\sum_{i=0}^{N-1}\frac{N^i}{i!}+\frac{N^N}{N!}\frac{N}{N-N}} \to \frac{\frac{N^N}{N!}\frac{N}{N-N}}{\frac{N^N}{N!}\frac{N}{N-N}} = 1.$$

From Problem 2 in Whitt 2002, p.8, $ErlC(N,\rho)$ is strictly increasing in ρ, thus it is straightforward that $ErlC(N,\rho)<1$ when $\rho<N$, and $ErlC(N,\rho)>1$ when $\rho>N$.

(b): From (1.7) in Whitt (2002),

$$ErlB(N,\rho) = \frac{\rho ErlB(N-1,\rho)}{N+\rho B(N-1,\rho)},$$

which implies

$$\frac{1}{ErlB(N-1,\rho)} = \left(\frac{1}{ErlB(N,\rho)}-1\right)\frac{\rho}{N}. \tag{EC.1}$$

From (2.6) in Whitt (2002),

$$ErlC(N,\rho) = \frac{\frac{\frac{\rho}{N}ErlB(N-1,\rho)}{1-\frac{\rho}{N}}}{1+\frac{\frac{\rho}{N}ErlB(N-1,\rho)}{1-\frac{\rho}{N}}} = \frac{\frac{\rho}{N}ErlB(N-1,\rho)}{1-\frac{\rho}{N}+\frac{\rho}{N}ErlB(N-1,\rho)} = \frac{\frac{\rho}{N}}{\frac{1-\frac{\rho}{N}}{ErlB(N-1,\rho)}+\frac{\rho}{N}},$$

which, by (EC.1), evaluates to

$$\frac{\frac{\rho}{N}}{\left(1-\frac{\rho}{N}\right)\left(\frac{1}{ErlB(N,\rho)}-1\right)\frac{\rho}{N}+\frac{\rho}{N}}=\frac{1}{\left(1-\frac{\rho}{N}\right)\left(\frac{1}{ErlB(N,\rho)}-1\right)+1}=\frac{N}{\frac{N-\rho}{ErlB(N,\rho)}+\rho}.$$

Hence,

$$ErlC(N,\rho)=N\left(\frac{N-\rho}{ErlB(N,\rho)}+\rho\right)^{-1}.$$

(c): Using (b) and $\lim_{\rho\to\infty} ErlB(N,\rho)=1$ yields

$$\lim_{\rho\to\infty} ErlC(N,\rho)=\lim_{\rho\to\infty}\frac{N}{\frac{N-\rho}{ErlB(N,\rho)}+\rho}=\frac{N}{N-\rho+\rho}=1.$$

(d): Similar to (c), from (b) and $\lim_{\rho\to 0} ErlB(N,\rho)=0$, it is straightforward that

$$\lim_{\rho\to 0} ErlC(N,\rho)=\lim_{\rho\to 0}\frac{N}{\frac{N-\rho}{ErlB(N,\rho)}+\rho}=0.$$

(e): Expanding $ErlC(N,\rho)$ as finite summations yields

$$\frac{1-ErlC\,(N,\rho)}{N-\rho}=\frac{\frac{1}{N}\sum_{i=0}^{N-1}\frac{\rho^i}{i!}}{\left(1-\frac{\rho}{N}\right)\sum_{i=0}^{N-1}\frac{\rho^i}{i!}+\frac{\rho^N}{N!}}=\frac{\frac{1}{N}\sum_{i=0}^{N-1}\frac{\rho^i}{i!}}{\sum_{i=0}^{N}\frac{\rho^i}{i!}-\frac{\rho}{N}\sum_{i=0}^{N-1}\frac{\rho^i}{i!}}=\frac{\sum_{i=0}^{N-1}\frac{1}{N}\frac{\rho^i}{i!}}{\sum_{i=0}^{N-1}\left(1-\frac{i}{N}\right)\frac{\rho^i}{i!}}.$$

Since $0<\frac{1}{N}<1-\frac{i}{N}$, $\forall i<N-1$ and $\frac{1}{N}=1-\frac{i}{N}$ for $i=N-1$, it follows that $\frac{1-ErlC(N,\rho)}{N-\rho}\in(0,1)$, $\forall\rho>0$ and $\forall N\geq 2$. Furthermore, this is straightforward to observe

$$\lim_{\rho\to 0}\frac{1-ErlC(N,\rho)}{N-\rho}=\lim_{\rho\to 0}\frac{\sum_{i=0}^{N-1}\frac{1}{N}\frac{\rho^i}{i!}}{\sum_{i=0}^{N-1}\left(1-\frac{i}{N}\right)\frac{\rho^i}{i!}}=\lim_{\rho\to 0}\frac{\frac{1}{N}\frac{\rho^{N-1}}{(N-1)!}}{\left(1-\frac{N-1}{N}\right)\frac{\rho^{N-1}}{(N-1)!}}=1,\quad\text{and}$$

$$\lim_{\rho\to\infty}\frac{1-ErlC(N,\rho)}{N-\rho}=\lim_{\rho\to\infty}\frac{\frac{1}{N}\sum_{i=0}^{N-1}\frac{\rho^i}{i!}}{\left(1-\frac{\rho}{N}\right)\sum_{i=0}^{N-1}\frac{\rho^i}{i!}+\frac{\rho^N}{N!}}=\lim_{\rho\to\infty}\frac{1}{(N-\rho)+\frac{\rho^N/N!}{\frac{1}{N}\sum_{i=0}^{N-1}\frac{\rho^i}{i!}}}=\lim_{\rho\to\infty}\frac{1}{(N-\rho)+\frac{\rho^N/N!}{\frac{1}{N}\frac{\rho^{N-1}}{(N-1)!}}}=\frac{1}{N}.$$

(f): The expression for the partial derivative of $ErlB(N,\rho)$ with respect to ρ directly comes from (1.10) in Whitt (2002).

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EC.7.2. Preliminaries B: Asymptotic Properties of Erlang Formulae Under Linear Staffing

LEMMA EC.6 **(Asymptotic properties I)**. *Under the staffing rule (14),*

(a)

$$\lim_{\lambda\to\infty} ErlB\left(N^\lambda, \frac{\lambda}{\mu}\right) = \begin{cases} 0, & \text{if } a \le \mu, \\ 1-\frac{\mu}{a}, & \text{if } a > \mu, \end{cases}$$

and

$$\lim_{\lambda\to\infty} ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right) = \begin{cases} 0, & \text{if } a < \mu, \\ \infty, & \text{if } a > \mu. \end{cases}$$

(b) If $a=\mu$, then

$$\lim_{\lambda\to\infty} ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right) = \begin{cases} \infty, & \text{if } 0 < \frac{\lambda}{\mu} - N^\lambda = \omega(\sqrt{\lambda}), \\ \left(1-\frac{z\Phi^c(z)}{\phi(z)}\right)^{-1} \in (1,\infty), & \text{if } 0 < \frac{\lambda}{\mu} - N^\lambda = \Theta(\sqrt{\lambda}), \\ 1, & \text{if } |N^\lambda - \frac{\lambda}{\mu}| = o(\sqrt{\lambda}), \\ \left(1-\frac{z\Phi^c(z)}{\phi(z)}\right)^{-1} \in (0,1), & \text{if } 0 < N^\lambda - \frac{\lambda}{\mu} = \Theta(\sqrt{\lambda}), \\ 0, & \text{if } 0 < N^\lambda - \frac{\lambda}{\mu} = \omega(\sqrt{\lambda}), \end{cases}$$

where $z = \lim_{\lambda\to\infty} \frac{\frac{\lambda}{\mu}-N^\lambda}{\sqrt{N^\lambda}}$, $\Phi^c(z) = \frac{1}{\sqrt{2\pi}}\int_z^\infty e^{-\frac{t^2}{2}}dt$ *and* $\phi(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$.

(c) If $a>\mu$, then $\lim_{\lambda\to\infty} \frac{ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right)}{\lambda} = \frac{(a-\mu)^2}{a^2\mu}$. *That is,* $ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right)$ *converges to ∞ linearly fast, as $\lambda\to\infty$.*

(d) If $a>\mu$, then $\lim_{\lambda\to\infty} \frac{ErlC(N^\lambda, \frac{\lambda}{\mu})}{N^\lambda} = \frac{(a-\mu)^2}{a\mu}$.

(e) If $a<\mu$, then $\lim_{\lambda\to\infty} P(\lambda) ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right) = 0$, *where $P(\lambda)$ represents a polynomial in λ. That is,* $ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right)$ *converges to zero super-polynomially fast, as $\lambda\to\infty$.*

(f) If $a<\mu$, then $\lim_{\lambda\to\infty} \frac{\partial ErlC\left(N^\lambda,\rho\right)}{\partial\rho} = 0$, *where*

$$\frac{\partial ErlC(N,\rho)}{\partial\rho} = ErlC(N,\rho)\left(\frac{1-ErlC(N,\rho)}{N-\rho} + \frac{N-\rho}{\rho}\right). \tag{EC.2}$$

(g) If $a<\mu$, then $\lim_{\lambda\to\infty} \frac{\partial ErlC\left(N^\lambda,\rho\right)}{\partial\rho}\cdot\rho = 0$.

(h) If $a<\mu$, then $\lim_{\lambda\to\infty} \frac{\partial^2 ErlC\left(N^\lambda,\rho\right)}{\partial\rho^2}\cdot\frac{d\rho}{d\mu} = 0$, *where*

$$\frac{\partial^2 ErlC(N,\rho)}{\partial\rho^2} = ErlC(N,\rho)\left(\left(\frac{1-ErlC(N,\rho)}{N-\rho} + \frac{N-\rho}{\rho}\right)^2 + \left(\frac{1-ErlC(N,\rho)}{N-\rho}\right)^2 - \frac{N+\rho ErlC(N,\rho)}{\rho^2}\right). \tag{EC.3}$$

(i) If $a<\mu$, $\liminf_{\lambda\to\infty} k^\lambda - N^\lambda = \infty$, *then* $\lim_{\lambda\to\infty} P(\lambda)\left(\frac{\lambda}{N^\lambda\mu}\right)^{k^\lambda-N^\lambda} = 0$, *where $P(\lambda)$ represents a polynomial in λ.*

LEMMA EC.7 **(Asymptotic properties II)**. *If $N^\lambda = f(\lambda)+o(f(\lambda))$ for some function f, then*

$$\lim_{\lambda\to\infty} ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right) = \begin{cases} 0, & \text{if } f(\lambda) = \omega(\lambda), \\ \infty, & \text{if } f(\lambda) = o(\lambda). \end{cases}$$

Proof of Lemma EC.6:

(a):

Erlang B: Using the upper bound for Erlang B given in Proposition 1 in Harel (2010), when $N^\lambda \geq 2$,

$$ErlB\left(N^\lambda, \frac{\lambda}{\mu}\right) \leq \frac{-2\left(\frac{\lambda}{N^\lambda \mu}\right) - \left(N^\lambda - \frac{\lambda}{\mu}\right) + \sqrt{\left(N^\lambda - \frac{\lambda}{\mu}\right)^2 + 4\frac{\lambda}{\mu}}}{2\left(1 - \frac{1}{N^\lambda}\right)\frac{\lambda}{\mu}} = \frac{-\frac{1}{N^\lambda} - \frac{1}{2}\left(\frac{N^\lambda \mu}{\lambda} - 1\right) + \sqrt{\left(\frac{1}{2}\left(\frac{N^\lambda \mu}{\lambda} - 1\right)\right)^2 + \frac{\mu}{\lambda}}}{1 - \frac{1}{N^\lambda}},$$

Case (I): When $a \leq \mu$, the upper bound on $ErlB\left(N^\lambda, \frac{\lambda}{\mu}\right)$ converges to

$$-\frac{1}{2}\left(\frac{\mu}{a} - 1\right) + \frac{1}{2}\left(\frac{\mu}{a} - 1\right) = 0,$$

as $\lambda \to \infty$, noting that $\lim_{\lambda\to\infty} \frac{1}{N^\lambda} = 0$. Thus, it follows that

$$0 \leq \lim_{\lambda\to\infty} ErlB\left(N^\lambda, \frac{\lambda}{\mu}\right) \leq 0 \quad \Rightarrow \quad \lim_{\lambda\to\infty} ErlB\left(N^\lambda, \frac{\lambda}{\mu}\right) = 0.$$

Case (II): When $a > \mu$, the upper bound on $ErlB\left(N^\lambda, \frac{\lambda}{\mu}\right)$ converges to

$$-\frac{1}{2}\left(\frac{\mu}{a} - 1\right) + \frac{1}{2}\left(1 - \frac{\mu}{a}\right) = 1 - \frac{\mu}{a},$$

as $\lambda \to \infty$, noting that $\lim_{\lambda\to\infty} \frac{1}{N^\lambda} = 0$. Thus, it follows that

$$0 \leq \lim_{\lambda\to\infty} ErlB\left(N^\lambda, \frac{\lambda}{\mu}\right) \leq 1 - \frac{\mu}{a}. \tag{EC.4}$$

In addition, a lower bound for Erlang B given in Proposition 4 in Harel (1988) is

$$ErlB\left(N^\lambda, \frac{\lambda}{\mu}\right) \geq \left(1 - \frac{N^\lambda \mu}{\lambda}\right)^+.$$

Letting $\lambda \to \infty$ on both sides of the above inequality implies

$$\lim_{\lambda\to\infty} ErlB\left(N^\lambda, \frac{\lambda}{\mu}\right) \geq 1 - \frac{\mu}{a}. \tag{EC.5}$$

Combining the upper and lower bounds given in (EC.4) and (EC.5) implies

$$\lim_{\lambda\to\infty} ErlB\left(N^\lambda, \frac{\lambda}{\mu}\right) = 1 - \frac{\mu}{a}.$$

Hence,

$$\lim_{\lambda\to\infty} ErlB\left(N^\lambda, \frac{\lambda}{\mu}\right) = \begin{cases} 0, & \text{if } a \leq \mu, \\ 1 - \frac{\mu}{a}, & \text{if } a > \mu, \end{cases}$$

Erlang C: Using the relationship between Erlng B and C (see Lemma EC.1 (b)),

$$ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right) = \frac{N^\lambda}{\frac{N^\lambda - \frac{\lambda}{\mu}}{ErlB\left(N^\lambda, \frac{\lambda}{\mu}\right)} + \frac{\lambda}{\mu}} = \frac{ErlB\left(N^\lambda, \frac{\lambda}{\mu}\right)}{1 - \frac{\lambda}{N^\lambda \mu}\left(1 - ErlB\left(N^\lambda, \frac{\lambda}{\mu}\right)\right)}. \tag{EC.6}$$

Using the limit of $ErlB\left(N^\lambda, \frac{\lambda}{\mu}\right)$ as $\lambda \to \infty$, (EC.6) implies

$$\lim_{\lambda\to\infty} ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right) = \begin{cases} 0, & \text{if } a < \mu, \\ \infty, & \text{if } a > \mu. \end{cases}$$

Note that when $a = \mu$, the numerator and denominator of (EC.6) both tend to zero as $\lambda \to \infty$, resulting in the value of $\lim_{\lambda\to\infty} ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right)$ undetermined.

(b): When $|\mathbf{N}^\lambda - \frac{\lambda}{\mu}| = \mathbf{o}(\sqrt{\lambda})$: From Proposition 1 in Harel (2010), using the upper and lower bounds for Erlang B when $N^\lambda \geq 2$,

$$\begin{aligned} ErlB\left(N^\lambda, \frac{\lambda}{\mu}\right) \leq & \frac{-2\frac{\lambda}{N^\lambda \mu} - \left(N^\lambda - \frac{\lambda}{\mu}\right) + \sqrt{\left(N^\lambda - \frac{\lambda}{\mu}\right)^2 + 4\frac{\lambda}{\mu}}}{2\left(1 - \frac{1}{N^\lambda}\right)\frac{\lambda}{\mu}} \\ = & \frac{-\frac{2}{N^\lambda}\frac{\lambda}{N^\lambda \mu} - \left(1 - \frac{\lambda}{N^\lambda \mu}\right) + \sqrt{\left(1 - \frac{\lambda}{N^\lambda \mu}\right)^2 + \frac{4}{N^\lambda}\frac{\lambda}{N^\lambda \mu}}}{2\left(1 - \frac{1}{N^\lambda}\right)\frac{\lambda}{N^\lambda \mu}}, \end{aligned}$$

and

$$\begin{aligned} ErlB\left(N^\lambda, \frac{\lambda}{\mu}\right) \geq & \frac{-2 - 3\left(N^\lambda - \frac{\lambda}{\mu}\right) + \sqrt{\left(N^\lambda - \frac{\lambda}{\mu}\right)^2 + 4N^\lambda + 4\frac{\lambda}{\mu} + 4}}{4\frac{\lambda}{\mu}} \\ = & \frac{-\frac{2}{N^\lambda} - 3\left(1 - \frac{\lambda}{N^\lambda \mu}\right) + \sqrt{\left(1 - \frac{\lambda}{N^\lambda \mu}\right)^2 + \frac{4}{N^\lambda} + \frac{4}{N^\lambda}\frac{\lambda}{N^\lambda \mu} + \frac{4}{(N^\lambda)^2}}}{4\frac{\lambda}{N^\lambda \mu}}. \end{aligned}$$

Then, we evaluate the lower bound for $ErlB\left(N^\lambda, \frac{\lambda}{\mu}\right)$ under $N^\lambda = \frac{\lambda}{\mu} + o(\sqrt{\lambda})$. Note that the denominator tends to 4, and the numerator tends to zero, as $\lambda \to \infty$. Hence, it suffices to study the convergence rate of the numerator. Note that,

$$\begin{aligned} & \sqrt{\lambda} \cdot \left(-\frac{2}{N^\lambda} - 3\left(1 - \frac{\lambda}{N^\lambda \mu}\right) + \sqrt{\left(1 - \frac{\lambda}{N^\lambda \mu}\right)^2 + \frac{4}{N^\lambda} + \frac{4}{N^\lambda}\frac{\lambda}{N^\lambda \mu} + \frac{4}{(N^\lambda)^2}} \right) \\ = & -\frac{2\sqrt{\lambda}}{\frac{\lambda}{\mu} + o(\sqrt{\lambda})} - \frac{3\sqrt{\lambda}o(\sqrt{\lambda})}{\frac{\lambda}{\mu} + o(\sqrt{\lambda})} + \sqrt{\frac{(o(\sqrt{\lambda}))^2 + 4\left(\frac{\lambda}{\mu} + o(\sqrt{\lambda})\right) + 4\frac{\lambda}{\mu} + 4}{\left(\frac{\lambda}{\mu} + o(\sqrt{\lambda})\right)^2}}\sqrt{\lambda} \to \sqrt{\frac{\frac{8}{\mu}}{\frac{1}{\mu^2}}} = \sqrt{8\mu}. \end{aligned}$$

Thus,

$$\lim_{\lambda\to\infty} \sqrt{\lambda} \cdot ErlB\left(N^\lambda, \frac{\lambda}{\mu}\right) \geq \frac{\sqrt{8\mu}}{4} = \sqrt{\frac{\mu}{2}} > 0.$$

Then, using the relationship between Erlang B and C (see Lemma EC.1 (b)),

$$ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right) = \frac{N^\lambda}{\frac{N^\lambda - \frac{\lambda}{\mu}}{ErlB\left(N^\lambda, \frac{\lambda}{\mu}\right)} + \frac{\lambda}{\mu}} = \frac{N^\lambda}{\frac{\sqrt{\lambda}(N^\lambda - \frac{\lambda}{\mu})}{\sqrt{\lambda} \cdot ErlB\left(N^\lambda, \frac{\lambda}{\mu}\right)} + \frac{\lambda}{\mu}} = \frac{\frac{N^\lambda}{\lambda}}{\frac{\frac{1}{\sqrt{\lambda}}(N^\lambda - \frac{\lambda}{\mu})}{\sqrt{\lambda} \cdot ErlB\left(N^\lambda, \frac{\lambda}{\mu}\right)} + \frac{1}{\mu}} \to 1, \quad \text{as } \lambda \to \infty,$$

because $\frac{N^\lambda}{\lambda} \to \frac{1}{\mu}$, $\frac{1}{\sqrt{\lambda}}\left(N^\lambda - \frac{\lambda}{\mu}\right) = \pm\frac{o(\sqrt{\lambda})}{\sqrt{\lambda}} \to 0$, as $\lambda \to \infty$.

When $|\mathbf{N}^\lambda - \frac{\lambda}{\mu}| = \mathbf{\Theta}(\sqrt{\lambda})$: We adapt the proof of Theorem 14 in Jagerman (1974). When $N^\lambda = \frac{\lambda}{\mu} - z\sqrt{N^\lambda} + o\left(\sqrt{N^\lambda}\right)$ for z real, using the integral representation of Erlang B formula, i.e., $ErlB\left(N, \frac{\lambda}{\mu}\right)^{-1} = \frac{\lambda}{\mu}\int_0^\infty e^{-\frac{\lambda}{\mu}u}(1+u)^N du$, it follows that

$$\begin{aligned}
&ErlB\left(N^\lambda, N^\lambda + z\sqrt{N^\lambda} + o(\sqrt{N^\lambda})\right)^{-1}\\
=&ErlB\left(N^\lambda, N^\lambda + \left(z + \frac{o(\sqrt{N^\lambda})}{\sqrt{N^\lambda}}\right)\sqrt{N^\lambda}\right)\\
=&\left(N^\lambda + \left(z + \frac{o(\sqrt{N^\lambda})}{\sqrt{N^\lambda}}\right)\sqrt{N^\lambda}\right)\int_0^\infty e^{-\left(N^\lambda + \left(z + \frac{o(\sqrt{N^\lambda})}{\sqrt{N^\lambda}}\right)\sqrt{N^\lambda}\right)u}(1+u)^{N^\lambda} du.
\end{aligned}$$

Let $v = \sqrt{N^\lambda}u$, then the above display implies that

$$ErlB\left(N^\lambda, N^\lambda + z\sqrt{N^\lambda} + o(\sqrt{N^\lambda})\right)^{-1} = \int_0^\infty e^{-\left(\frac{1}{2}v^2 + \left(z + \frac{o(\sqrt{N^\lambda})}{\sqrt{N^\lambda}}\right)v\right)} h(v, N^\lambda) dv, \tag{EC.7}$$

where

$$\begin{aligned}
h(v, N^\lambda) =& e^{\frac{1}{2}v^2 - \sqrt{N^\lambda}v}\left(1 + \frac{v}{\sqrt{N^\lambda}}\right)^{N^\lambda}\left(N^\lambda + \left(z + \frac{o(\sqrt{N^\lambda})}{\sqrt{N^\lambda}}\right)\sqrt{N^\lambda}\right)\frac{1}{\sqrt{N^\lambda}}\\
=& e^{\frac{1}{2}v^2 - \sqrt{N^\lambda}v}\left(1 + \frac{v}{\sqrt{N^\lambda}}\right)^{N^\lambda}(\sqrt{N^\lambda} + z) + e^{\frac{1}{2}v^2 - \sqrt{N^\lambda}v}\left(1 + \frac{v}{\sqrt{N^\lambda}}\right)^{N^\lambda}\frac{o(\sqrt{N^\lambda})}{\sqrt{N^\lambda}}\\
=:& h_1(v, N^\lambda) + h_2(v, N^\lambda),
\end{aligned}$$

with

$$\begin{aligned}
h_1(v, N^\lambda) &= e^{\frac{1}{2}v^2 - \sqrt{N^\lambda}v}\left(1 + \frac{v}{\sqrt{N^\lambda}}\right)^{N^\lambda}(\sqrt{N^\lambda} + z),\\
h_2(v, N^\lambda) &= e^{\frac{1}{2}v^2 - \sqrt{N^\lambda}v}\left(1 + \frac{v}{\sqrt{N^\lambda}}\right)^{N^\lambda}\frac{o(\sqrt{N^\lambda})}{\sqrt{N^\lambda}}.
\end{aligned}$$

In order to show $ErlB\left(N^\lambda, N^\lambda + z\sqrt{N^\lambda} + o(\sqrt{N^\lambda})\right)^{-1} \sim \frac{\Phi^c(z)}{\phi(z)}\sqrt{N^\lambda}$, from (EC.7), we want to establish the following:

(i) $\lim_{\lambda\to\infty} \frac{1}{\sqrt{N^\lambda}}\int_0^\infty e^{-\left(\frac{1}{2}v^2 + \left(z + \frac{o(\sqrt{N^\lambda})}{\sqrt{N^\lambda}}\right)v\right)} h_1(v, N^\lambda) dv = \frac{\Phi^c(z)}{\phi(z)}$;

(ii) $\lim_{\lambda\to\infty} \frac{1}{\sqrt{N^\lambda}}\int_0^\infty e^{-\left(\frac{1}{2}v^2 + \left(z + \frac{o(\sqrt{N^\lambda})}{\sqrt{N^\lambda}}\right)v\right)} h_2(v, N^\lambda) dv = 0$.

For (i): From (66) and (67) in the proof of Theorem 14 in Jagerman (1974),

$$\begin{aligned}
h_1(v, N^\lambda) \sim& \sqrt{N^\lambda} + \left(\frac{1}{3}v^3 + z\right) + \left(\frac{1}{3}v^3 - \frac{1}{4}v^4 + \frac{1}{18}v^6\right)\frac{1}{\sqrt{N^\lambda}} + \dots\\
\sim& \sqrt{N^\lambda} + \sum_{j=1}^\infty \Omega_j(v^3)\mathcal{O}_j(1).
\end{aligned}$$

Thus,

$$
\begin{aligned}
&\lim_{\lambda\to\infty}\frac{1}{\sqrt{N^\lambda}}\int_0^\infty e^{-\left(\frac{1}{2}v^2+\left(z+\frac{o(\sqrt{N^\lambda})}{\sqrt{N^\lambda}}\right)v\right)}h_1(v,N^\lambda)dv\\
&=\lim_{\lambda\to\infty}\frac{1}{\sqrt{N^\lambda}}\int_0^\infty e^{-\left(\frac{1}{2}v^2+\left(z+\frac{o(\sqrt{N^\lambda})}{\sqrt{N^\lambda}}\right)v\right)}\left(\sqrt{N^\lambda}+\sum_{j=1}^\infty \Omega_j(v^3)\mathcal{O}_j(1)\right)dv\\
&=\lim_{\lambda\to\infty}\int_0^\infty e^{-\left(\frac{1}{2}v^2+\left(z+\frac{o(\sqrt{N^\lambda})}{\sqrt{N^\lambda}}\right)v\right)}dv+\lim_{\lambda\to\infty}\frac{1}{\sqrt{N^\lambda}}\int_0^\infty e^{-\left(\frac{1}{2}v^2+\left(z+\frac{o(\sqrt{N^\lambda})}{\sqrt{N^\lambda}}\right)v\right)}\left(\sum_{j=1}^\infty \Omega_j(v^3)\mathcal{O}_j(1)\right)dv\\
&=\lim_{\lambda\to\infty}\int_0^\infty e^{-\left(\frac{1}{2}v^2+\left(z+\frac{o(\sqrt{N^\lambda})}{\sqrt{N^\lambda}}\right)v\right)}dv+\sum_{j=1}^\infty\lim_{\lambda\to\infty}\frac{\mathcal{O}_j(1)}{\sqrt{N^\lambda}}\int_0^\infty e^{-\left(\frac{1}{2}v^2+\left(z+\frac{o(\sqrt{N^\lambda})}{\sqrt{N^\lambda}}\right)v\right)}\Omega_j(v^3)dv\\
&=\int_0^\infty e^{-\left(\frac{1}{2}v^2+zv\right)}dv,
\end{aligned}
$$

where the last equality follows by noting that $\frac{o(\sqrt{N^\lambda})}{\sqrt{N^\lambda}}\to 0$ as $\lambda\to\infty$ and the second integral is finite. Moreover, note that

$$
\int_0^\infty e^{-\left(\frac{1}{2}v^2+zv\right)}dv=\int_0^\infty e^{-\frac{1}{2}(v+z)^2+\frac{1}{2}z^2}dv=\frac{\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{1}{2}(v+z)^2}dv}{\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}}=\frac{\frac{1}{\sqrt{2\pi}}\int_z^\infty e^{-\frac{1}{2}v^2}dv}{\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}}=\frac{\Phi^c(z)}{\phi(z)}.
$$

Therefore, part (i) is proved.

For (ii): To adapt a similar argument as in part (i), we need to first expand $h_2(v,N^\lambda)$ in a summation form. Note that

$$
\begin{aligned}
e^{\frac{1}{2}v^2-\sqrt{N^\lambda}v}\left(1+\frac{v}{\sqrt{N^\lambda}}\right)^{N^\lambda}&=e^{\frac{1}{2}v^2-\sqrt{N^\lambda}v+N^\lambda\log\left(1+\frac{v}{\sqrt{N^\lambda}}\right)}\\
&\overset{(*)}{\sim}e^{\frac{1}{2}v^2-\sqrt{N^\lambda}v+N^\lambda\left[\frac{v}{\sqrt{N^\lambda}}-\frac{1}{2}\frac{v^2}{N^\lambda}+\frac{1}{3}\frac{v^3}{(N^\lambda)^{3/2}}-\frac{1}{4}\frac{v^4}{(N^\lambda)^2}+\dots\right]}\\
&=e^{\frac{1}{2}v^2-\sqrt{N^\lambda}v+\sqrt{N^\lambda}v-\frac{1}{2}v^2+\frac{1}{3}\frac{v^3}{\sqrt{N^\lambda}}-\frac{1}{4}\frac{v^4}{N^\lambda}+\dots}\\
&=e^{\frac{v^3}{3}\frac{1}{\sqrt{N^\lambda}}}e^{-\frac{v^4}{4}\frac{1}{N^\lambda}}\dots\\
&\overset{(**)}{\sim}\left(1+\frac{v^3}{3}\frac{1}{\sqrt{N^\lambda}}+\frac{1}{2!}\frac{v^6}{9}\frac{1}{N^\lambda}+\cdots\right)\left(1-\frac{v^4}{4}\frac{1}{N^\lambda}+\frac{1}{2!}\frac{v^8}{16}\frac{1}{(N^\lambda)^2}+\cdots\right)\cdots\\
&\sim 1+\sum_{j=2}^\infty\Omega_j(v^3)\mathcal{O}_j\left(\frac{1}{\sqrt{N^\lambda}}\right),
\end{aligned}
$$

where $(*)$ follows from the Taylor expansion of $\log(1+x)$ as $x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$, and $(**)$ follows from the Taylor expansion of e^x as $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$.

Substitution into $h_2(v,N^\lambda)$ yields

$$
h_2(v,N^\lambda)\sim\left(1+\sum_{j=2}^\infty\Omega_j(v^3)\mathcal{O}_j\left(\frac{1}{\sqrt{N^\lambda}}\right)\right)\frac{o(\sqrt{N^\lambda})}{\sqrt{N^\lambda}}.
$$

Thus,

$$
\begin{aligned}
&\lim_{\lambda\to\infty}\frac{1}{\sqrt{N^\lambda}}\int_0^\infty e^{-\left(\frac{1}{2}v^2+\left(z+\frac{o(\sqrt{N^\lambda})}{\sqrt{N^\lambda}}\right)v\right)}h_2(v,N^\lambda)dv\\
=&\lim_{\lambda\to\infty}\frac{1}{\sqrt{N^\lambda}}\int_0^\infty e^{-\left(\frac{1}{2}v^2+\left(z+\frac{o(\sqrt{N^\lambda})}{\sqrt{N^\lambda}}\right)v\right)}\left(1+\sum_{j=2}^\infty\Omega_j(v^3)\mathcal{O}_j\left(\frac{1}{\sqrt{N^\lambda}}\right)\right)\frac{o(\sqrt{N^\lambda})}{\sqrt{N^\lambda}}dv\\
=&\lim_{\lambda\to\infty}\frac{1}{\sqrt{N^\lambda}}\int_0^\infty e^{-\left(\frac{1}{2}v^2+\left(z+\frac{o(\sqrt{N^\lambda})}{\sqrt{N^\lambda}}\right)v\right)}\frac{o(\sqrt{N^\lambda})}{\sqrt{N^\lambda}}dv\\
&+\lim_{\lambda\to\infty}\frac{1}{\sqrt{N^\lambda}}\int_0^\infty e^{-\left(\frac{1}{2}v^2+\left(z+\frac{o(\sqrt{N^\lambda})}{\sqrt{N^\lambda}}\right)v\right)}\left(\sum_{j=2}^\infty\Omega_j(v^3)\mathcal{O}_j\left(\frac{1}{\sqrt{N^\lambda}}\right)\right)\frac{o(\sqrt{N^\lambda})}{\sqrt{N^\lambda}}dv\\
=&\lim_{\lambda\to\infty}\frac{1}{\sqrt{N^\lambda}}\frac{o(\sqrt{N^\lambda})}{\sqrt{N^\lambda}}\int_0^\infty e^{-\left(\frac{1}{2}v^2+\left(z+\frac{o(\sqrt{N^\lambda})}{\sqrt{N^\lambda}}\right)v\right)}dv\\
&+\sum_{j=2}^\infty\lim_{\lambda\to\infty}\frac{\mathcal{O}_j\left(\frac{1}{\sqrt{N^\lambda}}\right)}{\sqrt{N^\lambda}}\frac{o(\sqrt{N^\lambda})}{\sqrt{N^\lambda}}\int_0^\infty e^{-\left(\frac{1}{2}v^2+\left(z+\frac{o(\sqrt{N^\lambda})}{\sqrt{N^\lambda}}\right)v\right)}\Omega_j(v^3)dv\\
=&0,
\end{aligned}
$$

where the last equality follows by noting that $\frac{o(\sqrt{N^\lambda})}{\sqrt{N^\lambda}}\to 0$, $\frac{\mathcal{O}_j\left(\frac{1}{\sqrt{N^\lambda}}\right)}{\sqrt{N^\lambda}}\to 0$ as $\lambda\to\infty$, and the second integral is finite. Hence, part (ii) is proved.

Together parts (i) and (ii), we conclude that

$$
ErlB\left(N^\lambda,N^\lambda+z\sqrt{N^\lambda}+o(\sqrt{N^\lambda})\right)^{-1}\sim\frac{\Phi^c(z)}{\phi(z)}\sqrt{N^\lambda}.
$$

Then, using the relationship between Erlang B and Erlang C (see Lemma EC.1 (b)),

$$
\begin{aligned}
ErlC\left(N^\lambda,N^\lambda+z\sqrt{N^\lambda}+o(\sqrt{N^\lambda})\right)=&\frac{N^\lambda}{\frac{N^\lambda-\left(N^\lambda+z\sqrt{N^\lambda}+o(\sqrt{N^\lambda})\right)}{ErlB\left(N^\lambda,N^\lambda+z\sqrt{N^\lambda}+o(\sqrt{N^\lambda})\right)}+\left(N^\lambda+z\sqrt{N^\lambda}+o(\sqrt{N^\lambda})\right)}\\
\sim&\frac{N^\lambda}{-\left(z\sqrt{N^\lambda}+o(\sqrt{N^\lambda})\right)\frac{\Phi^c(z)}{\phi(z)}\sqrt{N^\lambda}+\left(N^\lambda+z\sqrt{N^\lambda}+o(\sqrt{N^\lambda})\right)}\\
=&\frac{1}{-\left(z+\frac{o(\sqrt{N^\lambda})}{\sqrt{N^\lambda}}\right)\frac{\Phi^c(z)}{\phi(z)}+\left(1+\frac{z}{\sqrt{N^\lambda}}+\frac{o(\sqrt{N^\lambda})}{N^\lambda}\right)}\\
\to&\frac{1}{1-\frac{z\Phi^c(z)}{\phi(z)}},\qquad\text{as }\lambda\to\infty.
\end{aligned}
$$

Hence, under $N^\lambda=\frac{\lambda}{\mu}-z\sqrt{N^\lambda}+o(\sqrt{N^\lambda})$ for z real,

$$
\lim_{\lambda\to\infty}ErlC\left(N^\lambda,\frac{\lambda}{\mu}\right)=\left(1-\frac{z\Phi^c(z)}{\phi(z)}\right)^{-1},
$$

where $z=\lim_{\lambda\to\infty}\frac{\frac{\lambda}{\mu}-N^\lambda}{\sqrt{N^\lambda}}$.

It is clear that when $o(\lambda) = +\Theta(\sqrt{\lambda})$, i.e., $z < 0$,

$$\lim_{\lambda\to\infty} ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right) = \left(1 - \frac{z\Phi^c(z)}{\phi(z)}\right)^{-1} \in (0,1).$$

On the other hand, when $o(\lambda) = -\Theta(\sqrt{\lambda})$, i.e., $z > 0$,

$$\lim_{\lambda\to\infty} ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right) = \left(1 - \frac{z\Phi^c(z)}{\phi(z)}\right)^{-1} \in (1,\infty).$$

When $\mathbf{0 < N^\lambda - \frac{\lambda}{\mu} = \omega(\sqrt{\lambda})}$: Consider any arbitrary positive function $f(\lambda) \in \omega(\sqrt{\lambda})$ and $g(\lambda) \in \Theta(\sqrt{\lambda})$. We want to derive the value of $ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right)$ under staffing rule $N^\lambda = \frac{\lambda}{\mu} + f(\lambda)$.

By definition,

(i) For any $c_1 > 0$, there exists λ_1 such $f(\lambda) \geq c_1\sqrt{\lambda}$ for all $\lambda \geq \lambda_1$;

(ii) There exist $c_2 > 0$, $c_3 > 0$ and λ_2 such that $c_2\sqrt{\lambda} \leq g(\lambda) \leq c_3\sqrt{\lambda}$ for all $\lambda \geq \lambda_2$.

Then, $f(\lambda) > g(\lambda)$, for all $\lambda \geq \max\{\lambda_1, \lambda_2\}$.

From Problem 2 in Whitt (2002), p.8., $ErlC(N,\rho)$ is decreasing in N for any fixed ρ. Thus, for all $\lambda \geq \max\{\lambda_1, \lambda_2\}$,

$$ErlC\left(\frac{\lambda}{\mu} + f(\lambda), \frac{\lambda}{\mu}\right) < ErlC\left(\frac{\lambda}{\mu} + g(\lambda), \frac{\lambda}{\mu}\right).$$

Recall that

$$\lim_{\lambda\to\infty} ErlC\left(\frac{\lambda}{\mu} + g(\lambda), \frac{\lambda}{\mu}\right) = \left(1 - \frac{z\Phi^c(z)}{\phi(z)}\right)^{-1},$$

where $z = \lim_{\lambda\to\infty} \frac{\frac{\lambda}{\mu} - N^\lambda}{\sqrt{N^\lambda}} = \lim_{\lambda\to\infty} \frac{-g(\lambda)}{\sqrt{\frac{\lambda}{\mu} + g(\lambda)}} = \lim_{\lambda\to\infty} \frac{-g(\lambda)}{\sqrt{\lambda}} < 0$. When $z = -\infty$, the above equation evaluates to 0. Hence,

$$\lim_{\lambda\to\infty} ErlC\left(\frac{\lambda}{\mu} + f(\lambda), \frac{\lambda}{\mu}\right) \leq 0.$$

By non-negativity, the above display must take value 0. Since $f(\lambda)$ is arbitrary, we conclude that when $N^\lambda = \frac{\lambda}{\mu} + \omega(\sqrt{\lambda})$,

$$\lim_{\lambda\to\infty} ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right) = 0.$$

When $\mathbf{0 < \frac{\lambda}{\mu} - N^\lambda = \omega(\sqrt{\lambda})}$: Similar to the analysis of the case when $0 < N^\lambda - \frac{\lambda}{\mu} = \omega(\sqrt{\lambda})$, for all $\lambda \geq \max\{\lambda_1, \lambda_2\}$,

$$ErlC\left(\frac{\lambda}{\mu} - f(\lambda), \frac{\lambda}{\mu}\right) > ErlC\left(\frac{\lambda}{\mu} - g(\lambda), \frac{\lambda}{\mu}\right).$$

Recall that

$$\lim_{\lambda\to\infty} ErlC\left(\frac{\lambda}{\mu} - g(\lambda), \frac{\lambda}{\mu}\right) = \left(1 - \frac{z\Phi^c(z)}{\phi(z)}\right)^{-1},$$

where $z = \lim_{\lambda\to\infty} \frac{\frac{\lambda}{\mu}-N^\lambda}{\sqrt{N^\lambda}} = \lim_{\lambda\to\infty} \frac{g(\lambda)}{\sqrt{\frac{\lambda}{\mu}-g(\lambda)}} = \lim_{\lambda\to\infty} \frac{g(\lambda)}{\sqrt{\lambda}} > 0$. When $z=\infty$, the limit in the above display evaluates to ∞ because

$$\lim_{z\to\infty} \frac{z\Phi^c(z)}{\phi(z)} = \lim_{z\to\infty} \frac{z\int_z^\infty e^{-\frac{t^2}{2}}dt}{e^{-\frac{z^2}{2}}} = 1.$$

Hence, when $N^\lambda = \frac{\lambda}{\mu} - \omega(\sqrt{\lambda})$,

$$\lim_{\lambda\to\infty} ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right) = \infty.$$

(c): Using the lower and upper bounds for Erlang C given in Propositions 3 and 4 in Harel (1988) when $\frac{\lambda}{\mu} \geq N^\lambda$ are given by

$$ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right) \geq \frac{(N^\lambda)^2}{2\frac{\lambda}{\mu}}\left[\left(\frac{\lambda}{N^\lambda\mu}-1\right)^2 + \frac{2}{N^\lambda}\frac{\lambda}{N^\lambda\mu} + \left(\frac{\lambda}{N^\lambda\mu}-1\right)\sqrt{\left(\frac{\lambda}{N^\lambda\mu}-1\right)^2+\frac{4}{N^\lambda}\frac{\lambda}{N^\lambda\mu}}\right],$$

and

$$ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right) \leq \frac{(N^\lambda)^2}{2\frac{\lambda}{\mu}}\left[\left(\frac{\lambda}{N^\lambda\mu}-1\right)^2 + \frac{2}{N^\lambda}\frac{\lambda}{N^\lambda\mu} + \left(\frac{\lambda}{N^\lambda\mu}-1\right)\sqrt{\left(\frac{\lambda}{N^\lambda\mu}-1\right)^2+\frac{4}{N^\lambda}\left(\frac{\lambda}{N^\lambda\mu}+\frac{1}{N^\lambda}+1\right)}\right].$$

When $a>\mu$, $\frac{\lambda}{\mu} > N^\lambda$ for all large enough λ. Thus, it follows that

$$\lim_{\lambda\to\infty} \frac{ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right)}{\lambda} \geq \frac{\mu}{2a^2}\left[\left(\frac{a}{\mu}-1\right)+0+\left(\frac{a}{\mu}-1\right)\sqrt{\left(\frac{a}{\mu}-1\right)^2+0}\right] = \frac{\mu}{a^2}\left(\frac{a}{\mu}-1\right)^2,$$

and

$$\lim_{\lambda\to\infty} \frac{ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right)}{\lambda} \leq \frac{\mu}{2a^2}\left[\left(\frac{a}{\mu}-1\right)^2+0+\left(\frac{a}{\mu}-1\right)\sqrt{\left(\frac{a}{\mu}-1\right)^2+0}\right] = \frac{\mu}{a^2}\left(\frac{a}{\mu}-1\right)^2,$$

noting that $\frac{\lambda}{N^\lambda\mu} \to \frac{a}{\mu}$ as $\lambda\to\infty$.

Together the above two inequalities imply

$$\lim_{\lambda\to\infty} \frac{ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right)}{\lambda} = \frac{\mu}{a^2}\left(\frac{a}{\mu}-1\right)^2 = \frac{(a-\mu)^2}{a^2\mu},$$

meaning that $ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right)$ converges to ∞ linearly fast, as $\lambda\to\infty$.

(d): From (c),

$$\frac{ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right)}{\lambda} \to \frac{(a-\mu)^2}{a^2\mu}, \quad \text{as } \lambda\to\infty.$$

This implies that

$$\frac{ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right)}{N^\lambda} = \frac{ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right)}{\lambda}\frac{\lambda}{N^\lambda} \to \frac{(a-\mu)^2}{a^2\mu}\cdot a = \frac{(a-\mu)^2}{a\mu}, \quad \text{as } \lambda\to\infty,$$

noting that $\lim_{\lambda\to\infty}\frac{\lambda}{N^\lambda} = a$ under linear staffing (14).

(e): Using the upper bound for Erlang C given in (14) in Proposition 2 of Harel (2010), when $N^\lambda \geq 2$ and $\frac{\lambda}{\mu} < N^\lambda$,

$$ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right) < \left(\frac{\lambda}{N^\lambda \mu}\right)^{\sqrt{N^\lambda}}. \tag{EC.8}$$

Under linear staffing (14), $a < \mu$ implies $\frac{\lambda}{N^\lambda \mu} < 1$ for all large enough λ, which satisfies the condition for bound (EC.8). Therefore, for any polynomial $P(\lambda)$, under (14),

$$P(\lambda) ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right) < P(\lambda)\left(\frac{\lambda}{N^\lambda \mu}\right)^{\sqrt{N^\lambda}} \to 0, \quad \text{as } \lambda \to \infty,$$

because $\frac{\rho}{N^\lambda} < 1$ for large enough λ, and exponential decay dominates polynomial growth. Therefore, by non-negativity, $\lim_{\lambda\to\infty} P(\lambda) ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right) = 0$. That is, under (12), $ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right)$ asymptotically decays to zero in a super-polynomial fashion, when $a < \mu$.

(f): Differentiating $ErlC(N,\rho)$ with respect to ρ yields

$$\begin{aligned}
\frac{\partial ErlC(N,\rho)}{\partial \rho} =& \left(\sum_{i=0}^{N-1} \frac{\rho^i}{i!} + \frac{\rho^N}{N!}\frac{N}{N-\rho}\right)^{-2} \left[\left(\frac{N\rho^{N-1}}{N!}\frac{N}{N-\rho} + \frac{\rho^N}{N!}\frac{N}{(N-\rho)^2}\right)\left(\sum_{i=0}^{N-1} \frac{\rho^i}{i!} + \frac{\rho^N}{N!}\frac{N}{N-\rho}\right) - \right. \\
&\left. \left(\frac{\rho^N}{N!}\frac{N}{N-\rho}\right)\left(\sum_{i=0}^{N-1} \frac{i\rho^{i-1}}{i!} + \frac{N\rho^{N-1}}{N!}\frac{N}{N-\rho} + \frac{\rho^N}{N!}\frac{N}{(N-\rho)^2}\right)\right] \\
=& \frac{\frac{\rho^N}{N!}\frac{N}{N-\rho}}{\sum_{i=0}^{N-1}\frac{\rho^i}{i!} + \frac{\rho^N}{N!}\frac{N}{N-\rho}} \frac{\left(\frac{N}{\rho} + \frac{1}{N-\rho}\right)\left(\sum_{i=0}^{N-1}\frac{\rho^i}{i!}\right) - \sum_{i=0}^{N-1}\frac{i\rho^{i-1}}{i!}}{\sum_{i=0}^{N-1}\frac{\rho^i}{i!} + \frac{\rho^N}{N!}\frac{N}{N-\rho}} \\
=& ErlC(N,\rho) \frac{\left(\frac{1}{N-\rho}\sum_{i=0}^{N-1}\frac{\rho^i}{i!}\right) + \left(\left(\frac{N}{\rho} - 1\right)\sum_{i=0}^{N-1}\frac{\rho^i}{i!} + \sum_{i=0}^{N-1}\frac{\rho^i}{i!} - \sum_{i=0}^{N-1}\frac{i\rho^{i-1}}{i!}\right)}{\sum_{i=0}^{N-1}\frac{\rho^i}{i!} + \frac{\rho^N}{N!}\frac{N}{N-\rho}} \\
=& ErlC(N,\rho)\left[\frac{1}{N-\rho}\frac{\sum_{i=0}^{N-1}\frac{\rho^i}{i!}}{\sum_{i=0}^{N-1}\frac{\rho^i}{i!} + \frac{\rho^N}{N!}\frac{N}{N-\rho}} + \frac{\left(\frac{N}{\rho} - 1\right)\sum_{i=0}^{N-1}\frac{\rho^i}{i!} + \frac{\rho^{N-1}}{(N-1)!}}{\sum_{i=0}^{N-1}\frac{\rho^i}{i!} + \frac{\rho^N}{N!}\frac{N}{N-\rho}}\right] \\
=& ErlC(N,\rho)\left(\frac{1 - ErlC(N,\rho)}{N-\rho} + \frac{N-\rho}{\rho}\right).
\end{aligned}$$

Thus,

$$\frac{\partial ErlC(N,\rho)}{\partial \rho} = ErlC(N,\rho)\left(\frac{1 - ErlC(N,\rho)}{N-\rho} + \frac{N-\rho}{\rho}\right), \tag{EC.9}$$

which establishes (EC.2). From (a), $ErlC(N^\lambda, \rho) \to 0$, as $\lambda \to \infty$ when $a < \mu$. From Lemma EC.1 (e), $\frac{1-ErlC(N^\lambda,\rho)}{N^\lambda-\rho} \in (0,1)$. Moreover, $\lim_{\lambda\to\infty} \frac{N^\lambda - \rho}{\rho} = \frac{\mu}{a} - 1$ under the staffing rule (14). Hence, the above equation implies

$$\lim_{\lambda\to\infty} \frac{\partial ErlC(N^\lambda, \rho)}{\partial \rho} = 0.$$

(g): From (EC.9),

$$\lim_{\lambda\to\infty} \frac{\partial ErlC(N^\lambda,\rho)}{\partial \rho}\cdot\rho = \lim_{\lambda\to\infty} ErlC(N^\lambda,\rho)\rho\left(\frac{1 - ErlC(N^\lambda,\rho)}{N^\lambda - \rho} + \frac{N^\lambda - \rho}{\rho}\right),$$

where $\frac{1-ErlC(N^\lambda,\rho)}{N^\lambda-\rho}\in(0,1)$ and $\lim_{\lambda\to\infty}\frac{N^\lambda-\rho}{\rho}=\frac{\mu}{a}-1$ under the staffing rule (14). From (e), $\lim_{\lambda\to\infty}ErlC(N^\lambda,\rho)\cdot\rho=0$, thus the above display implies

$$\lim_{\lambda\to\infty}\frac{\partial ErlC(N^\lambda,\rho)}{\partial\rho}\cdot\rho=0.$$

(h): Differentiating (EC.2) with respect to ρ yields

$$\begin{aligned}
&\frac{\partial^2 ErlC(N,\rho)}{\partial\rho^2}\\
=&\frac{\partial ErlC(N,\rho)}{\partial\rho}\left(\frac{1-ErlC(N,\rho)}{N-\rho}+\frac{N-\rho}{\rho}\right)+ErlC(N,\rho)\left(\frac{-\frac{\partial ErlC(N,\rho)}{\partial\rho}(N-\rho)+(1-ErlC(N,\rho))}{(N-\rho)^2}-\frac{N}{\rho^2}\right)\\
=&ErlC(N,\rho)\left(\frac{1-ErlC(N,\rho)}{N-\rho}+\frac{N-\rho}{\rho}\right)^2-\frac{ErlC(N,\rho)^2}{N-\rho}\left(\frac{1-ErlC(N,\rho)}{N-\rho}+\frac{N-\rho}{\rho}\right)\\
&-\frac{ErlC(N,\rho)^2-ErlC(N,\rho)}{(N-\rho)^2}-ErlC(N,\rho)\frac{N}{\rho^2}\\
=&ErlC(N,\rho)\left(\left(\frac{1-ErlC(N,\rho)}{N-\rho}+\frac{N-\rho}{\rho}\right)^2+\left(\frac{1-ErlC(N,\rho)}{N-\rho}\right)^2-\left(\frac{N+\rho ErlC(N,\rho)}{\rho^2}\right)\right),
\end{aligned}$$

which establishes (EC.3). Thus, it follows that

$$\begin{aligned}
&\lim_{\lambda\to\infty}\frac{\partial^2 ErlC(N^\lambda,\rho)}{\partial\rho^2}\cdot\rho^2\\
=&\lim_{\lambda\to\infty}ErlC(N^\lambda,\rho)\rho^2\left(\left(\frac{1-ErlC(N^\lambda,\rho)}{N^\lambda-\rho}+\frac{N^\lambda-\rho}{\rho}\right)^2+\left(\frac{1-ErlC(N^\lambda,\rho)}{N^\lambda-\rho}\right)^2-\left(\frac{N^\lambda+\rho ErlC(N^\lambda,\rho)}{\rho^2}\right)\right),
\end{aligned}$$

where $\lim_{\lambda\to\infty}ErlC(N^\lambda,\rho)\cdot\rho^2=0$ (from (e)), $\lim_{\lambda\to\infty}\frac{1-ErlC(N^\lambda,\rho)}{N^\lambda-\rho}\in[0,1]$ (from Lemma EC.1 (e)), $\lim_{\lambda\to\infty}\frac{N^\lambda-\rho}{\rho}=\frac{\mu}{a}-1$, $\lim_{\lambda\to\infty}\frac{N^\lambda}{\rho^2}=0$, and $\lim_{\lambda\to\infty}ErlC(N^\lambda,\rho)=0$ (from (a)). Hence,

$$\lim_{\lambda\to\infty}\frac{\partial^2 ErlC(N^\lambda,\rho)}{\partial\rho^2}\cdot\rho^2=0,$$

and thus

$$\lim_{\lambda\to\infty}\frac{\partial^2 ErlC(N^\lambda,\rho)}{\partial\rho^2}\cdot\frac{d\rho}{d\mu}=\lim_{\lambda\to\infty}\frac{\partial^2 ErlC(N^\lambda,\rho)}{\partial\rho^2}\cdot\left(-\frac{\lambda}{\mu^2}\right)=\lim_{\lambda\to\infty}\frac{\partial^2 ErlC(N^\lambda,\rho)}{\partial\rho^2}\cdot\left(-\frac{\rho^2}{\lambda}\right)=0.$$

(i): Consider a subsequence $\{\lambda'\}$ along which either $k^{\lambda'}/N^{\lambda'}$ diverges to ∞ or $\lim_{\lambda'\to\infty}k^{\lambda'}/N^{\lambda'}<\infty$. For ease of exposition, we simply use λ rather than λ' to denote the subsequence.

Case (I): When $\lim_{\lambda\to\infty}\frac{k^\lambda}{N^\lambda}=:b>1$,

$$\lim_{\lambda\to\infty}P(\lambda)\left(\frac{\lambda}{N^\lambda\mu}\right)^{k^\lambda-N^\lambda+1}=\lim_{\lambda\to\infty}P(\lambda)\left(\frac{\lambda}{N^\lambda\mu}\right)^{\lambda\frac{N^\lambda}{\lambda}\left(\frac{k^\lambda}{N^\lambda}-1+\frac{1}{N^\lambda}\right)}=0,$$

because $\frac{\lambda}{N^\lambda\mu}<1$ for all large enough λ, $\frac{N^\lambda}{\lambda}(\frac{k^\lambda}{N^\lambda}-1+\frac{1}{N^\lambda})\to\frac{b-1}{a}$, as $\lambda\to\infty$, and exponential decay in terms of λ dominates its polynomial growth.

Case (II): When $\lim_{\lambda\to\infty}\frac{k^\lambda}{N^\lambda} =: b = 1$, let $k^\lambda = N^\lambda + o_1(\lambda)$ with $\lim_{\lambda\to\infty}\frac{o_1(\lambda)}{\lambda} = 0$ and $\liminf_{\lambda\to\infty} o_1(\lambda) = \infty$. Note that $\lambda = P(o_1(\lambda))$ because $\lim_{\lambda\to\infty}\frac{o_1(\lambda)}{\lambda} = 0$ and $\liminf_{\lambda\to\infty} o_1(\lambda) = \infty$. Thus, $P(\lambda) = P(P(o_1(\lambda))) = P(o_1(\lambda))$. Then,

$$\lim_{\lambda\to\infty} P(\lambda)\left(\frac{\lambda}{N^\lambda\mu}\right)^{k^\lambda - N^\lambda + 1} = \lim_{\lambda\to\infty} P(\lambda)\left(\frac{\lambda}{N^\lambda\mu}\right)^{o_1(\lambda)+1} = \lim_{\lambda\to\infty} P(o_1(\lambda))\left(\frac{\lambda}{N^\lambda\mu}\right)^{o_1(\lambda)+1} = 0,$$

noting that $\frac{\lambda}{N^\lambda\mu} < 1$ for all large enough λ, $\lim_{\lambda\to\infty} o_1(\lambda) = \infty$, and exponential decay in terms of $o_1(\lambda)$ dominates its polynomial growth.

∎

Proof of Lemma EC.7:

From Problem 2 of Section 2 in Whitt (2002), p.8, (Note: Section 2 in Whitt (2002) assumes $\rho < N$. However, the proof of monotonicty in problem 2 in Section 2 holds for all $\rho > 0$), $ErlC(N,\rho)$ is decreasing in N for any fixed $\rho > 0$. Hence, for all large enough λ,

$$ErlC\left(\omega(\lambda) + o(\omega(\lambda)), \frac{\lambda}{\mu}\right) \le ErlC\left(\frac{1}{a}\lambda + o(\lambda), \frac{\lambda}{\mu}\right),\ \text{when } a < \mu,$$

and

$$ErlC\left(o(\lambda) + o(o(\lambda)), \frac{\lambda}{\mu}\right) \ge ErlC\left(\frac{1}{a}\lambda + o(\lambda), \frac{\lambda}{\mu}\right),\ \text{when } a > \mu,$$

and so the result immediately follows from the limit in Lemma EC.6 (a). ∎

EC.7.3. Proof of Lemma 5

LEMMA EC.14. *When $a=\mu$, $\lim_{\lambda\to\infty} I^\lambda(\mu_1,\mu)=0$ for all $\mu_1>0$ and $\mu>0$.*

LEMMA EC.15. *When $a=\mu$, if $\lim_{\lambda\to\infty} d_2\frac{k^\lambda-N^\lambda}{d_1}<\infty$, then the following holds for all $\mu_1>0$ and $\mu>0$: $\lim_{\lambda\to\infty}\left(\frac{k^\lambda-N^\lambda}{N^\lambda}\right)^r I^\lambda(\mu_1,\mu)=0$, for all $r\in\mathbb{N}$ if $\lim_{\lambda\to\infty} d_2\neq 0$, or for $r=1$ if $\lim_{\lambda\to\infty} d_2=0$.*

LEMMA EC.16. *When $a=\mu$, if $\lim_{\lambda\to\infty} d_2\frac{k^\lambda-N^\lambda}{d_1}\neq 0$, then $\lim_{\lambda\to\infty}\frac{\rho C}{d_2}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda} I^\lambda(\mu_1,\mu)\in\left(-\infty,-\frac{\mu_1}{\mu}\right]\cup[0,\infty)$ for all $\mu_1>0$ and $\mu>0$.*

LEMMA EC.17. *When $a=\mu$, if $\lim_{\lambda\to\infty} d_2\neq 0$, then $\lim_{\lambda\to\infty}\frac{1}{I^\lambda(\mu_1,\mu)}\left(\frac{\partial I^\lambda(\mu_1,\mu)}{\partial\mu_1}\right)^2=0$ for all $\mu_1>0$ and $\mu>0$.*

LEMMA EC.18. *When $a=\mu$, if $\left|N^\lambda-\frac{\lambda}{\mu}\right|=\mathcal{O}(\sqrt{\lambda})$, then $\lim_{\lambda\to\infty}\sqrt{\rho}\left(\frac{1-C}{N^\lambda-\rho}\right)\in(0,\infty)$ for all $\mu>0$.*

LEMMA EC.19. *When $a=\mu$, if (i) $0<N^\lambda-\frac{\lambda}{\mu}=\omega(1)$ or (ii) $0<\frac{\lambda}{\mu}-N^\lambda=\mathcal{O}(\sqrt{\lambda})$ and $0<\frac{\lambda}{\mu}-N^\lambda=\omega(1)$, then $\lim_{\lambda\to\infty}\frac{\sqrt{\rho}I^\lambda(\mu_1,\mu)}{d_2}=0$ for all $\mu_1>0$ and $\mu>0$.*

LEMMA EC.20. *When $a=\mu$, if $\left|N^\lambda-\frac{\lambda}{\mu}\right|=\mathcal{O}(1)$, then the following holds for all $\mu_1>0$ and $\mu>0$: $\lim_{\lambda\to\infty}\rho^r I^\lambda(\mu_1,\mu)=0$ for (i) $r\in\left[0,\frac{1}{2}\right)$ if $\lim_{\lambda\to\infty} d_2\left(\frac{k^\lambda-N^\lambda}{d_1}\right)=0$, or (ii) $r\in[0,1)$ if $\lim_{\lambda\to\infty} d_2\left(\frac{k^\lambda-N^\lambda}{d_1}\right)\neq 0$.*

LEMMA EC.21. *When $a=\mu$, if $\lim_{\lambda\to\infty} d_2=0$ and $\lim_{\lambda\to\infty} d_2\left(\frac{k^\lambda-N^\lambda}{d_1}\right)\neq 0$, then $\lim_{\lambda\to\infty}\frac{I^\lambda(\mu_1,\mu)}{d_2}=0$ for all $\mu_1>0$ and $\mu>0$.*

LEMMA EC.22. *When $a=\mu$, if $0<N^\lambda-\frac{\lambda}{\mu}=\omega(1)$, then $\lim_{\lambda\to\infty}\sqrt{\rho}C\left(\frac{k^\lambda-N^\lambda}{d_1}\right)^r\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda} I^\lambda(\mu_1,\mu)=0$ for all $r\in\mathbb{N}$, $\mu_1>0$ and $\mu>0$.*

LEMMA EC.23. *When $a=\mu$, if $\lim_{\lambda\to\infty} d_2>0$, $\left|N^\lambda-\frac{\lambda}{\mu}\right|=\mathcal{O}(1)$ and $\lim_{\lambda\to\infty} d_2\left(\frac{k^\lambda-N^\lambda}{d_1}\right)\neq 0$, then $\lim_{\lambda\to\infty}\left(\frac{k^\lambda-N^\lambda}{d_1}\right)^r\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}\in[0,\infty)$ for all $r\in\mathbb{N}$, $\mu_1>0$ and $\mu>0$.*

LEMMA EC.24. *When $a=\mu$, if (i) $\lim_{\lambda\to\infty} d_2\neq 0$ and (ii) $\lim_{\lambda\to\infty} d_2\left(\frac{k^\lambda-N^\lambda}{d_1}\right)=0$, then $\lim_{\lambda\to\infty}\frac{\rho C}{d_2}\left(\frac{k^\lambda-N^\lambda}{d_1}\right)^r\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}\left(I^\lambda(\mu_1,\mu)\right)^2=0$ for all $r\in\mathbb{N}$, $\mu_1>0$ and $\mu>0$.*

LEMMA EC.25. *When $a=\mu$, if $\lim_{\lambda\to\infty} d_2=0$ and $\lim_{\lambda\to\infty} d_2\left(\frac{k^\lambda-N^\lambda}{d_1}\right)=0$, then $\lim_{\lambda\to\infty}\frac{\partial I^\lambda(\mu_1,\mu)}{\partial\mu_1}=0$ for all $\mu_1>0$ and $\mu>0$.*

LEMMA EC.26. *When $a=\mu$, if $0<\frac{\lambda}{\mu}-N^\lambda=\omega(\sqrt{\lambda})$, then $\lim_{\lambda\to\infty}\frac{\rho C I^\lambda(\mu_1,\mu)}{d_2}\in\left[-\frac{\mu_1}{\mu},0\right]$ for all $\mu_1>0$ and $\mu>0$.*

Recall from (EC.76), (EC.77) and (EC.78) in the manuscript, the idle time and its first two partial derivatives are given by

$$I^\lambda(\mu_1,\mu)=\left[1+\rho\frac{\mu}{\mu_1}\left(\frac{1-C}{N^\lambda-\rho}+\left(1-\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}\right)\frac{C}{d_2}\right)\right]^{-1}, \tag{EC.10}$$

$$\mu_1 \frac{\partial I^\lambda}{\partial \mu_1} = \left(1 + \frac{1}{d_2}\frac{\mu_1}{\mu}\right) I^\lambda(1-I^\lambda) - \frac{\rho}{d_2}\frac{1-C}{N^\lambda-\rho}(I^\lambda)^2 - \frac{\rho C}{d_2}\frac{k^\lambda - N^\lambda}{d_1}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}(I^\lambda)^2, \tag{EC.11}$$

and

$$\begin{aligned}\frac{\partial^2 I^\lambda}{\partial \mu_1^2} =& -2\frac{\partial I^\lambda}{\partial \mu_1}\left[\frac{I^\lambda}{\mu_1} + \left(\frac{1}{\mu_1} - \frac{1}{I^\lambda}\frac{\partial I^\lambda}{\partial \mu_1} - \frac{I^\lambda}{\mu_1}\right)\left(1 - \frac{1}{d_2}\frac{1}{\mu}I^\lambda\left(\frac{\partial I^\lambda}{\partial \mu_1}\right)^{-1}\right)\right] \\ &+ \frac{(I^\lambda)^2}{\mu_1\mu}\frac{(k^\lambda-N^\lambda)(k^\lambda-N^\lambda+1)}{d_1}\frac{C}{d_2}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda+1} \\ =& -\frac{2}{\mu_1}I^\lambda\frac{\partial I^\lambda}{\partial \mu_1} - 2\left[\frac{1}{\mu_1}\frac{\partial I^\lambda}{\partial \mu_1} - \frac{1}{I^\lambda}\left(\frac{\partial I^\lambda}{\partial \mu_1}\right)^2 - \frac{1}{\mu_1}I^\lambda\frac{\partial I^\lambda}{\partial \mu_1} - \frac{I^\lambda}{\mu_1\mu d_2} + \frac{1}{\mu d_2}\frac{\partial I^\lambda}{\partial \mu_1} + \frac{(I^\lambda)^2}{\mu_1\mu d_2}\right] \\ &+ \frac{1}{\mu_1\mu}\left(\frac{\rho}{d_2}I^\lambda\right)\frac{(k^\lambda-N^\lambda)(k^\lambda-N^\lambda+1)}{d_1^2}C\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}I^\lambda. \end{aligned} \tag{EC.12}$$

Now, we prove Lemmas EC.14- EC.26 as follows.

Proof of Lemma EC.14: From (EC.10),

$$I^\lambda(\mu_1,\mu) = \left[1 + \rho\frac{\mu}{\mu_1}\frac{1-C}{N^\lambda-\rho} + \frac{\mu}{\mu_1}\rho\left(1-\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}\right)\frac{C}{d_2}\right]^{-1}.$$

Note that

$$\rho\frac{1-C}{N^\lambda-\rho} \overset{(*)}{=} \rho\frac{1-\frac{N^\lambda}{\frac{N-\rho}{B}+\rho}}{N^\lambda-\rho} = \frac{\rho(1-B)}{N^\lambda-\rho+\rho B} = \frac{\rho(1-B)}{o(\lambda)+\rho B} = \frac{1-B}{o(\lambda)/\rho+B} \to \lim_{\lambda\to\infty}\frac{1-B}{B} \overset{(**)}{=} \infty, \text{ as } \lambda\to\infty,$$

where $(*)$ follows by using the relationship between the Erlang B and C formulae (see Lemma EC.1 (b)), and the last equality $(**)$ follows from $\lim_{\lambda\to\infty} B = 0$ (when $a=\mu$ from Lemma EC.6 (a)). Moreover, $\lim_{\lambda\to\infty}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda} = e^{-\lim_{\lambda\to\infty} d_2\frac{k^\lambda-N^\lambda}{d_1}}$, which implies that $\lim_{\lambda\to\infty}\left(1-\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}\right)\frac{1}{d_2} \geq 0$. Hence, $\lim_{\lambda\to\infty} I^\lambda(\mu_1,\mu) = 0$, for all $\mu_1 > 0$ and $\mu > 0$. ∎

Proof of Lemma EC.15: From (EC.10),

$$\begin{aligned}&\left(\frac{k^\lambda-N^\lambda}{N^\lambda}\right)^r I^\lambda \\ =& \left[\left(\frac{N^\lambda}{k^\lambda-N^\lambda}\right)^r + \rho\frac{\mu}{\mu_1}\frac{1-C}{N^\lambda-\rho}\left(\frac{N^\lambda}{k^\lambda-N^\lambda}\right)^r + \rho\frac{\mu}{\mu_1}\left(1-\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}\right)\frac{C}{d_2}\left(\frac{N^\lambda}{k^\lambda-N^\lambda}\right)^r\right]^{-1} \\ =& \left[\left(\frac{N^\lambda}{d_2\left(k^\lambda-N^\lambda\right)}\right)^r d_2^r + \rho\frac{\mu}{\mu_1}\frac{1-C}{N^\lambda-\rho}\left(\frac{N^\lambda}{d_2\left(k^\lambda-N^\lambda\right)}\right)^r d_2^r + \rho\frac{\mu}{\mu_1}\left(1-\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}\right)C\left(\frac{N^\lambda}{d_2\left(k^\lambda-N^\lambda\right)}\right)^r d_2^{r-1}\right]^{-1},\end{aligned}$$

where $\lim_{\lambda\to\infty}\frac{N^\lambda}{d_2\left(k^\lambda-N^\lambda\right)} = \lim_{\lambda\to\infty}\left(d_2\frac{k^\lambda-N^\lambda}{d_1}\right)^{-1} < \infty$, $\lim_{\lambda\to\infty}\frac{1-C}{N^\lambda-\rho} \in [0,1]$ (from Lemma EC.1 (e)), $\lim_{\lambda\to\infty}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda} = e^{-\lim_{\lambda\to\infty} d_2\frac{k^\lambda-N^\lambda}{d_1}} \in [0,\infty)$. Then, it is clear that,

- if $d_2 \neq 0$, the above display converges to 0, as $\lambda\to\infty$, for all $r\in\mathbb{N}$, because the second term in the square bracket tends to ∞, and the first and the third terms are positive, as $\lambda\to\infty$.

- if $d_2 = 0$, then the above display converges to 0, as $\lambda \to \infty$, for $r = 1$, because the second term in the square bracket tends to ∞, as $\lambda \to \infty$.

Hence, the following hold for all $\mu_1 > 0$ and $\mu > 0$: $\lim_{\lambda\to\infty} \left(\frac{k^\lambda - N^\lambda}{N^\lambda}\right)^r I^\lambda(\mu_1,\mu) = 0$ for all $r \in \mathbb{N}$ if $\lim_{\lambda\to\infty} d_2 \neq 0$, or for $r = 1$ if $\lim_{\lambda\to\infty} d_2 = 0$. ∎

Proof of Lemma EC.16: From (EC.10)

$$\frac{\rho C}{d_2}\left(\frac{\rho}{d_1}\right)^{k^\lambda - N^\lambda} I^\lambda = \left[\frac{d_2}{\rho C}\left(\frac{d_1}{\rho}\right)^{k^\lambda - N^\lambda} + \frac{\mu}{\mu_1}\frac{d_2}{C}\frac{1-C}{N^\lambda - \rho}\left(\frac{d_1}{\rho}\right)^{k^\lambda - N^\lambda} + \frac{\mu}{\mu_1}\left(\left(\frac{d_1}{\rho}\right)^{k^\lambda - N^\lambda} - 1\right)\right]^{-1}. \tag{EC.13}$$

We discuss the following cases.

Case (I): If $0 < \frac{\lambda}{\mu} - N^\lambda = \omega(\sqrt{\lambda})$ and $o(\lambda)$. Note that $\frac{d_2}{\rho} \to 0$, $C \to \infty$ (from Lemma EC.6 (b)), $\left(\frac{d_1}{\rho}\right)^{k^\lambda - N^\lambda} \to e^{\lim_{\lambda\to\infty} d_2 \frac{k^\lambda - N^\lambda}{d_1}} \in [0,1)$ (when $\lim_{\lambda\to\infty} d_2 \frac{k^\lambda - N^\lambda}{d_1} \neq 0$), $\frac{d_2}{C}\frac{1-C}{N^\lambda - \rho} = \frac{d_2}{N^\lambda - \rho}\frac{1-C}{C} \to -1$, as $\lambda \to \infty$. Therefore, (EC.13) implies that

$$\lim_{\lambda\to\infty} \frac{\rho C}{d_2}\left(\frac{\rho}{d_1}\right)^{k^\lambda - N^\lambda} I^\lambda = \left[-\frac{\mu}{\mu_1} e^{\lim_{\lambda\to\infty} d_2 \frac{k^\lambda - N^\lambda}{d_1}} + \frac{\mu}{\mu_1}\left(e^{\lim_{\lambda\to\infty} d_2 \frac{k^\lambda - N^\lambda}{d_1}} - 1\right)\right]^{-1} = -\frac{\mu_1}{\mu}.$$

Case (II): If $0 < \frac{\lambda}{\mu} - N^\lambda = \mathcal{O}(\sqrt{\lambda})$ and $\omega(1)$. Note that $\frac{d_2}{\rho} \to 0$, $C \in [1,\infty)$ (from Lemma EC.6 (b)), $\left(\frac{d_1}{\rho}\right)^{k^\lambda - N^\lambda} \to e^{\lim_{\lambda\to\infty} d_2 \frac{k^\lambda - N^\lambda}{d_1}} \in [0,1)$ (when $\lim_{\lambda\to\infty} d_2 \frac{k^\lambda - N^\lambda}{d_1} \neq 0$), $\frac{d_2}{C}\frac{1-C}{N^\lambda - \rho} = \frac{d_2}{N^\lambda - \rho}\frac{1-C}{C} \to \frac{1}{C} - 1$, as $\lambda \to \infty$. Therefore, (EC.13) implies that

$$\begin{aligned}\lim_{\lambda\to\infty} \frac{\rho C}{d_2}\left(\frac{\rho}{d_1}\right)^{k^\lambda - N^\lambda} I^\lambda &= \left[\frac{\mu}{\mu_1}\lim_{\lambda\to\infty}\left(\frac{1}{C} - 1\right) e^{\lim_{\lambda\to\infty} d_2 \frac{k^\lambda - N^\lambda}{d_1}} + \frac{\mu}{\mu_1}\left(e^{\lim_{\lambda\to\infty} d_2 \frac{k^\lambda - N^\lambda}{d_1}} - 1\right)\right]^{-1} \\ &= \left[\frac{\mu}{\mu_1}\left(\lim_{\lambda\to\infty}\frac{1}{C}\cdot e^{\lim_{\lambda\to\infty} d_2 \frac{k^\lambda - N^\lambda}{d_1}} - 1\right)\right]^{-1} \in \left(-\infty, -\frac{\mu_1}{\mu}\right].\end{aligned}$$

Case (III): If $0 < \frac{\lambda}{\mu} - N^\lambda = \mathcal{O}(1)$. Note that $\frac{d_2}{\rho} \to 0$, $C \to 1$ (from Lemma EC.6 (b)), $\left(\frac{d_1}{\rho}\right)^{k^\lambda - N^\lambda} \to e^{\lim_{\lambda\to\infty} d_2 \frac{k^\lambda - N^\lambda}{d_1}} \in [0,1)$ (when $\lim_{\lambda\to\infty} d_2 \frac{k^\lambda - N^\lambda}{d_1} \neq 0$), $\frac{d_2}{C}\frac{1-C}{N^\lambda - \rho} = \frac{d_2}{N^\lambda - \rho}\frac{1-C}{C} \to \frac{1}{C} - 1 = 0$, as $\lambda \to \infty$. Therefore, (EC.13) implies that

$$\lim_{\lambda\to\infty} \frac{\rho C}{d_2}\left(\frac{\rho}{d_1}\right)^{k^\lambda - N^\lambda} I^\lambda = \left[\frac{\mu}{\mu_1}\left(e^{\lim_{\lambda\to\infty} d_2 \frac{k^\lambda - N^\lambda}{d_1}} - 1\right)\right]^{-1} \in \left(-\infty, -\frac{\mu_1}{\mu}\right].$$

Case (IV): If $0 < N^\lambda - \frac{\lambda}{\mu} = \mathcal{O}(1)$. Note that $\frac{d_2}{\rho} \to 0$, $C \to 1$ (from Lemma EC.6 (b)), $\left(\frac{d_1}{\rho}\right)^{k^\lambda - N^\lambda} \to e^{\lim_{\lambda\to\infty} d_2 \frac{k^\lambda - N^\lambda}{d_1}} \in (1,\infty]$ (when $\lim_{\lambda\to\infty} d_2 \frac{k^\lambda - N^\lambda}{d_1} \neq 0$), $\frac{d_2}{C}\frac{1-C}{N^\lambda - \rho} = \frac{d_2}{N^\lambda - \rho}\frac{1-C}{C} \to \frac{1}{C} - 1 = 0$, as $\lambda \to \infty$. Therefore, (EC.13) implies that

$$\lim_{\lambda\to\infty} \frac{\rho C}{d_2}\left(\frac{\rho}{d_1}\right)^{k^\lambda - N^\lambda} I^\lambda = \left[\frac{\mu}{\mu_1}\left(e^{\lim_{\lambda\to\infty} d_2 \frac{k^\lambda - N^\lambda}{d_1}} - 1\right)\right]^{-1} \in [0,\infty).$$

Case (V): If $0 < N^\lambda - \frac{\lambda}{\mu} = \mathcal{O}(\sqrt{\lambda})$ and $\omega(1)$. Note that $\frac{d_2}{\rho} \to 0$, $C \in (0,1]$ (from Lemma EC.6 (b)), $\left(\frac{d_1}{\rho}\right)^{k^\lambda - N^\lambda} \to e^{\lim_{\lambda\to\infty} d_2 \frac{k^\lambda - N^\lambda}{d_1}} \in (1,\infty]$ (when $\lim_{\lambda\to\infty} d_2 \frac{k^\lambda - N^\lambda}{d_1} \neq 0$), $\frac{d_2}{C}\frac{1-C}{N^\lambda-\rho} = \frac{d_2}{N^\lambda-\rho}\frac{1-C}{C} \to \frac{1}{C} - 1$, as $\lambda \to \infty$. Therefore, (EC.13) implies that

$$\begin{aligned}
\lim_{\lambda\to\infty} \frac{\rho C}{d_2}\left(\frac{\rho}{d_1}\right)^{k^\lambda - N^\lambda} I^\lambda &= \left[\frac{\mu}{\mu_1}\lim_{\lambda\to\infty}\left(\frac{1}{C}-1\right)e^{\lim_{\lambda\to\infty} d_2 \frac{k^\lambda - N^\lambda}{d_1}} + \frac{\mu}{\mu_1}\left(e^{\lim_{\lambda\to\infty} d_2 \frac{k^\lambda - N^\lambda}{d_1}} - 1\right)\right]^{-1} \\
&= \left[\frac{\mu}{\mu_1}\left(\lim_{\lambda\to\infty}\frac{1}{C}\cdot e^{\lim_{\lambda\to\infty} d_2 \frac{k^\lambda - N^\lambda}{d_1}} - 1\right)\right]^{-1} \in [0,\infty).
\end{aligned}$$

Case (VI): If $0 < N^\lambda - \frac{\lambda}{\mu} = \omega(\sqrt{\lambda})$ and $o(\lambda)$. Note that $\frac{d_2}{\rho} \to 0$, $C \to 0$ (from Lemma EC.6 (b)), $\left(\frac{d_1}{\rho}\right)^{k^\lambda - N^\lambda} \to e^{\lim_{\lambda\to\infty} d_2 \frac{k^\lambda - N^\lambda}{d_1}} \in (1,\infty]$ (when $\lim_{\lambda\to\infty} d_2 \frac{k^\lambda - N^\lambda}{d_1} \neq 0$), $\frac{d_2}{C}\frac{1-C}{N^\lambda-\rho} = \frac{d_2}{N^\lambda-\rho}\frac{1-C}{C} \to \infty$, as $\lambda \to \infty$. Therefore, (EC.13) implies that

$$\lim_{\lambda\to\infty} \frac{\rho C}{d_2}\left(\frac{\rho}{d_1}\right)^{k^\lambda - N^\lambda} I^\lambda \le \lim_{\lambda\to\infty}\left[\frac{\mu}{\mu_1}\frac{d_2}{C}\frac{1-C}{N^\lambda-\rho}\left(\frac{d_1}{\rho}\right)^{k^\lambda - N^\lambda}\right]^{-1} = 0,$$

where the inequality follows by noting that the first and the third terms in (EC.13) are both positive. Hence, $\lim_{\lambda\to\infty} \frac{\rho C}{d_2}\left(\frac{\rho}{d_1}\right)^{k^\lambda - N^\lambda} I^\lambda = 0$ by non-negativity.

Combining the above six cases concludes that $\lim_{\lambda\to\infty} \frac{\rho C}{d_2}\left(\frac{\rho}{d_1}\right)^{k^\lambda - N^\lambda} I^\lambda(\mu_1,\mu) \in \left(-\infty, -\frac{\mu_1}{\mu}\right] \cup [0,\infty)$ for all $\mu_1 > 0$ and $\mu > 0$. ∎

Proof of Lemma EC.17: We discuss the following cases.

Case (I): If $\lim_{\lambda\to\infty} d_2 > 0$. Since $\lim_{\lambda\to\infty} d_2 > 0$, it follows from (EC.11) that

$$\lim_{\lambda\to\infty}\frac{\partial I^\lambda}{\partial \mu_1} \le \lim_{\lambda\to\infty}\frac{1}{\mu_1}\left\{\left(1+\frac{1}{d_2}\frac{\mu_1}{\mu}\right)I^\lambda(1-I^\lambda)\right\}.$$

This implies that

$$\lim_{\lambda\to\infty}\frac{1}{I^\lambda}\left(\frac{\partial I^\lambda}{\partial \mu_1}\right)^2 \le \lim_{\lambda\to\infty}\frac{1}{\mu_1^2}\left(1+\frac{1}{d_2}\frac{\mu_1}{\mu}\right)^2 I^\lambda\left(1-I^\lambda\right)^2 = 0,$$

where the equality follows from $\lim_{\lambda\to\infty} I^\lambda(\mu_1,\mu) = 0$ for all $\mu_1 > 0$ and $\mu > 0$ by Lemma EC.14. Hence, $\lim_{\lambda\to\infty}\frac{1}{I^\lambda}\left(\frac{\partial I^\lambda}{\partial \mu_1}\right)^2 = 0$ for all $\mu_1 > 0$ and $\mu > 0$, by non-negativity.

Case (II): If $0 < \frac{\lambda}{\mu} - N^\lambda = \omega(\sqrt{\lambda})$ and $o(\lambda)$. From (EC.10) and (EC.11),

$$\begin{aligned}
&\frac{1}{I^\lambda}\left(\frac{\partial I^\lambda}{\partial \mu_1}\right)^2 = \frac{\frac{\partial I^\lambda}{\partial \mu_1}}{I^\lambda}\cdot\frac{\partial I^\lambda}{\partial \mu_1} \\
=&\frac{1}{\mu_1}\left\{\left(1+\frac{1}{d_2}\frac{\mu_1}{\mu}\right)(1-I^\lambda) - \frac{\rho}{d_2}\frac{1-C}{N^\lambda-\rho}I^\lambda - \frac{\rho C}{d_2}\frac{k^\lambda - N^\lambda}{d_1}\left(\frac{\rho}{d_1}\right)^{k^\lambda - N^\lambda} I^\lambda\right\} \\
&\cdot\frac{1}{\mu_1}\left\{\left(1+\frac{1}{d_2}\frac{\mu_1}{\mu}\right)I^\lambda(1-I^\lambda) - \frac{\rho}{d_2}\frac{1-C}{N^\lambda-\rho}(I^\lambda)^2 - \frac{\rho C}{d_2}\frac{k^\lambda - N^\lambda}{d_1}\left(\frac{\rho}{d_1}\right)^{k^\lambda - N^\lambda}(I^\lambda)^2\right\},
\end{aligned}$$

where $\frac{1}{d_2} \to 0$, $I^\lambda \to 0$ (from Lemma EC.14) and $\frac{\rho}{d_2}\frac{1-C}{N^\lambda-\rho}I^\lambda = \frac{\rho C I^\lambda}{d_2}\left(\frac{1}{C}-1\right)\frac{1}{N^\lambda-\rho} \to 0$ (from Lemmas EC.26 and EC.6 (b)), as $\lambda\to\infty$. Then, the above display implies that

$$\begin{aligned}
&\lim_{\lambda\to\infty}\frac{1}{I^\lambda}\left(\frac{\partial I^\lambda}{\partial\mu_1}\right)^2\\
&=\lim_{\lambda\to\infty}\frac{1}{\mu_1^2}\left\{\frac{\rho C}{d_2}\frac{k^\lambda-N^\lambda}{d_1}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}(I^\lambda)^2+\frac{\rho C}{d_2}\frac{k^\lambda-N^\lambda}{d_1}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}I^\lambda\cdot\frac{\rho C}{d_2}\frac{k^\lambda-N^\lambda}{d_1}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}(I^\lambda)^2\right\}\\
&=\lim_{\lambda\to\infty}\frac{1}{\mu_1^2}\left\{\left[\frac{\rho C}{d_2}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}I^\lambda\right]\left[\frac{k^\lambda-N^\lambda}{d_1}I^\lambda\right]+\left[\frac{\rho C}{d_2}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}I^\lambda\right]^2\left[\left(\frac{k^\lambda-N^\lambda}{d_1}\right)^2I^\lambda\right]\right\}=0,
\end{aligned}$$

where $\lim_{\lambda\to\infty}\frac{\rho C}{d_2}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}I^\lambda \to -\frac{\mu_1}{\mu}$ (from the proof of Lemma EC.16) and $\lim_{\lambda\to\infty}\left(\frac{k^\lambda-N^\lambda}{d_1}\right)^r I^\lambda = 0$ for all $r\in\mathbb{N}$ (from Lemma EC.15, noting that $\lim_{\lambda\to\infty} d_2\frac{k^\lambda-N^\lambda}{d_1}\le 0<\infty$ and $\lim_{\lambda\to\infty} d_2\neq 0$).

Case (III): If $0<\frac{\lambda}{\mu}-N^\lambda=\mathcal{O}(\sqrt{\lambda})$ **and** $\omega(1)$**.** From (EC.10) and (EC.11),

$$\begin{aligned}
&\frac{1}{I^\lambda}\left(\frac{\partial I^\lambda}{\partial\mu_1}\right)^2=\frac{\frac{\partial I^\lambda}{\partial\mu_1}}{I^\lambda}\cdot\frac{\partial I^\lambda}{\partial\mu_1}\\
&=\frac{1}{\mu_1}\left\{\left(1+\frac{1}{d_2}\frac{\mu_1}{\mu}\right)(1-I^\lambda)-\frac{\rho}{d_2}\frac{1-C}{N^\lambda-\rho}I^\lambda-\frac{\rho C}{d_2}\frac{k^\lambda-N^\lambda}{d_1}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}I^\lambda\right\}\\
&\quad\cdot\frac{1}{\mu_1}\left\{\left(1+\frac{1}{d_2}\frac{\mu_1}{\mu}\right)I^\lambda(1-I^\lambda)-\frac{\rho}{d_2}\frac{1-C}{N^\lambda-\rho}(I^\lambda)^2-\frac{\rho C}{d_2}\frac{k^\lambda-N^\lambda}{d_1}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}(I^\lambda)^2\right\},
\end{aligned}\tag{EC.14}$$

where $\frac{1}{d_2}\to 0$, $I^\lambda\to 0$ (from Lemma EC.14) and $\frac{\rho}{d_2}\frac{1-C}{N^\lambda-\rho}I^\lambda=\left(\frac{\sqrt{\rho}I^\lambda}{d_2}\right)\left(\sqrt{\rho}\frac{1-C}{N^\lambda-\rho}\right)\to 0$ (from Lemmas EC.18 and EC.19), as $\lambda\to\infty$.

Case (III-1): If $\lim_{\lambda\to\infty} d_2\frac{k^\lambda-N^\lambda}{d_1}\neq 0$**.** Then, (EC.14) implies that

$$\begin{aligned}
&\lim_{\lambda\to\infty}\frac{1}{I^\lambda}\left(\frac{\partial I^\lambda}{\partial\mu_1}\right)^2\\
&=\lim_{\lambda\to\infty}\frac{1}{\mu_1^2}\left\{\frac{\rho C}{d_2}\frac{k^\lambda-N^\lambda}{d_1}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}(I^\lambda)^2+\frac{\rho C}{d_2}\frac{k^\lambda-N^\lambda}{d_1}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}I^\lambda\cdot\frac{\rho C}{d_2}\frac{k^\lambda-N^\lambda}{d_1}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}(I^\lambda)^2\right\}\\
&=\lim_{\lambda\to\infty}\frac{1}{\mu_1^2}\left\{\left[\frac{\rho C}{d_2}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}I^\lambda\right]\left[\frac{k^\lambda-N^\lambda}{d_1}I^\lambda\right]+\left[\frac{\rho C}{d_2}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}I^\lambda\right]^2\left[\left(\frac{k^\lambda-N^\lambda}{d_1}\right)^2I^\lambda\right]\right\}=0,
\end{aligned}$$

where $\lim_{\lambda\to\infty}\frac{\rho C}{d_2}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}I^\lambda\in\left(-\infty,-\frac{\mu_1}{\mu}\right]$ (from the proof of Lemma EC.16) and $\lim_{\lambda\to\infty}\left(\frac{k^\lambda-N^\lambda}{d_1}\right)^r I^\lambda=0$ for all $r\in\mathbb{N}$ (from Lemma EC.15, noting that $\lim_{\lambda\to\infty} d_2\frac{k^\lambda-N^\lambda}{d_1}<0<\infty$ and $\lim_{\lambda\to\infty} d_2\neq 0$).

Case (III-2): If $\lim_{\lambda\to\infty} d_2\frac{k^\lambda-N^\lambda}{d_1}=0$**.** Then, (EC.14) implies that

$$\lim_{\lambda\to\infty}\frac{1}{I^\lambda}\left(\frac{\partial I^\lambda}{\partial\mu_1}\right)^2=\lim_{\lambda\to\infty}\frac{1}{\mu_1^2}\left\{\frac{\rho C}{d_2}\frac{k^\lambda-N^\lambda}{d_1}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}(I^\lambda)^2+\left[\left(\frac{\rho C}{d_2}\frac{k^\lambda-N^\lambda}{d_1}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}\right)^{\frac{4}{3}}(I^\lambda)^2\right]^{\frac{3}{2}}\right\}.\tag{EC.15}$$

Using (EC.10) and after algebra, we have

$$
\begin{aligned}
&\left[\frac{\rho C}{d_2}\frac{k^\lambda-N^\lambda}{d_1}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}(I^\lambda)^2\right]^{-1}\\
=&\frac{d_2^2}{\rho C}\left[\frac{1}{d_2\frac{k^\lambda-N^\lambda}{d_1}}\left(\frac{d_1}{\rho}\right)^{k^\lambda-N^\lambda}\right]+2\frac{\mu}{\mu_1}\left(\frac{1-C}{N^\lambda-\rho}\right)\frac{d_2^2}{C}\left[\frac{1}{d_2\frac{k^\lambda-N^\lambda}{d_1}}\left(\frac{d_1}{\rho}\right)^{k^\lambda-N^\lambda}\right]+2\frac{\mu}{\mu_1}d_2\left[\frac{1}{d_2\frac{k^\lambda-N^\lambda}{d_1}}\left(\left(\frac{d_1}{\rho}\right)^{k^\lambda-N^\lambda}-1\right)\right]\\
&+\left(\frac{\mu}{\mu_1}\right)^2\left(\frac{1-C}{N^\lambda-\rho}\right)^2\frac{\rho d_2^2}{C}\left[\frac{1}{d_2\frac{k^\lambda-N^\lambda}{d_1}}\left(\frac{d_1}{\rho}\right)^{k^\lambda-N^\lambda}\right]+2\left(\frac{\mu}{\mu_1}\right)^2\left(\frac{1-C}{N^\lambda-\rho}\right)\rho d_2\left[\frac{1}{d_2\frac{k^\lambda-N^\lambda}{d_1}}\left(\left(\frac{d_1}{\rho}\right)^{k^\lambda-N^\lambda}-1\right)\right]\\
&+\left(\frac{\mu}{\mu_1}\right)^2\rho C\left[\frac{1}{d_2\frac{k^\lambda-N^\lambda}{d_1}}\left(\left(\frac{d_1}{\rho}\right)^{k^\lambda-N^\lambda}+\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}-2\right)\right].
\end{aligned}
$$

Observe that the limiting value of every term in the above display, as $\lambda\to\infty$, bears an identical sign, noting that $\lim_{\lambda\to\infty}\left(\frac{d_1}{\rho}\right)^{k^\lambda-N^\lambda}-1=e^{\lim_{\lambda\to\infty}d_2\frac{k^\lambda-N^\lambda}{d_1}}-1=0$ (since $\lim_{\lambda\to\infty}d_2\frac{k^\lambda-N^\lambda}{d_1}=0$ by assumption), and $\left(\frac{d_1}{\rho}\right)^{k^\lambda-N^\lambda}+\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}\geq 2$ (using the arithmetic-geometric mean inequality, namely, $x+y\geq 2\sqrt{xy}$ for $x,y\geq 0$). In particular,

$$
\lim_{\lambda\to\infty}\left(\frac{1-C}{N^\lambda-\rho}\right)^2\frac{\rho d_2^2}{C}\left[\frac{1}{d_2\frac{k^\lambda-N^\lambda}{d_1}}\left(\frac{d_1}{\rho}\right)^{k^\lambda-N^\lambda}\right]=\lim_{\lambda\to\infty}\left(\sqrt{\rho}\frac{1-C}{N^\lambda-\rho}\right)^2\frac{d_2^2}{C}\left[\frac{1}{d_2\frac{k^\lambda-N^\lambda}{d_1}}e^{d_2\frac{k^\lambda-N^\lambda}{d_1}}\right]=\infty,
$$

where $\lim_{\lambda\to\infty}\sqrt{\rho}\frac{1-C}{N^\lambda-\rho}\in(0,\infty)$ (from Lemma EC.18), $\lim_{\lambda\to\infty}C\in[1,\infty)$ (from Lemma EC.6 (b)) and $\lim_{\lambda\to\infty}d_2\frac{k^\lambda-N^\lambda}{d_1}=0$ (by assumption). Similarly,

$$
\begin{aligned}
&\left[\left(\frac{\rho C}{d_2}\frac{k^\lambda-N^\lambda}{d_1}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}\right)^{\frac{4}{3}}(I^\lambda)^2\right]^{-1}\\
=&\left(\frac{d_2^2}{\rho C}\right)^{\frac{2}{3}}\left[\frac{1}{d_2\frac{k^\lambda-N^\lambda}{d_1}}\left(\frac{d_1}{\rho}\right)^{k^\lambda-N^\lambda}\right]^{\frac{2}{3}}+2\frac{\mu}{\mu_1}\left(\frac{1-C}{N^\lambda-\rho}\right)\left(\frac{\sqrt{\rho}d_2^2}{C}\right)^{\frac{2}{3}}\left[\frac{1}{d_2\frac{k^\lambda-N^\lambda}{d_1}}\left(\frac{d_1}{\rho}\right)^{k^\lambda-N^\lambda}\right]^{\frac{2}{3}}\\
&+2\frac{\mu}{\mu_1}(\rho C d_2)^{\frac{1}{3}}\left(\frac{1}{d_2\frac{k^\lambda-N^\lambda}{d_1}}\right)^{\frac{2}{3}}\left[\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}\right]^{\frac{1}{3}}\left(\left(\frac{d_1}{\rho}\right)^{k^\lambda-N^\lambda}-1\right)\\
&+\left(\frac{\mu}{\mu_1}\right)^2\left(\frac{1-C}{N^\lambda-\rho}\right)^2\left(\frac{\rho^2 d_2^2}{C}\right)^{\frac{2}{3}}\left[\frac{1}{d_2\frac{k^\lambda-N^\lambda}{d_1}}\left(\frac{d_1}{\rho}\right)^{k^\lambda-N^\lambda}\right]^{\frac{2}{3}}\\
&+2\left(\frac{\mu}{\mu_1}\right)^2\left(\frac{1-C}{N^\lambda-\rho}\right)(\rho^4 C d_2)^{\frac{1}{3}}\left(\frac{1}{d_2\frac{k^\lambda-N^\lambda}{d_1}}\right)^{\frac{2}{3}}\left[\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}\right]^{\frac{1}{3}}\left(\left(\frac{d_1}{\rho}\right)^{k^\lambda-N^\lambda}-1\right)\\
&+\left(\frac{\mu}{\mu_1}\right)^2\left(\frac{\rho^2C^2}{d_2}\right)^{\frac{2}{3}}\left(\frac{1}{d_2\frac{k^\lambda-N^\lambda}{d_1}}\right)^{\frac{2}{3}}\left[\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}\right]^{\frac{1}{3}}\left(\left(\frac{d_1}{\rho}\right)^{k^\lambda-N^\lambda}+\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}-2\right),
\end{aligned}
$$

where the limiting value of every term in the above display, as $\lambda\to\infty$, bears an identical sign, and, in particular, note that

$$
\lim_{\lambda\to\infty}\left(\frac{1-C}{N^\lambda-\rho}\right)^2\left(\frac{\rho^2 d_2^2}{C}\right)^{\frac{2}{3}}\left[\frac{1}{d_2\frac{k^\lambda-N^\lambda}{d_1}}\left(\frac{d_1}{\rho}\right)^{k^\lambda-N^\lambda}\right]^{\frac{2}{3}}=\lim_{\lambda\to\infty}\left(\sqrt{\rho}\frac{1-C}{N^\lambda-\rho}\right)^2\frac{\rho^{\frac{1}{3}}d_2^{\frac{4}{3}}}{C^{\frac{2}{3}}}\left[\frac{1}{d_2\frac{k^\lambda-N^\lambda}{d_1}}e^{d_2\frac{k^\lambda-N^\lambda}{d_1}}\right]^{\frac{2}{3}}=\infty.
$$

Therefore, $\lim_{\lambda\to\infty} \frac{\rho C}{d_2}\frac{k^\lambda-N^\lambda}{d_1}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}(I^\lambda)^2 = \lim_{\lambda\to\infty}\left(\frac{\rho C}{d_2}\frac{k^\lambda-N^\lambda}{d_1}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}\right)^{\frac{4}{3}}(I^\lambda)^2 = 0$. Then, substitution into (EC.15) yields $\lim_{\lambda\to\infty}\frac{1}{I^\lambda(\mu_1,\mu)}\left(\frac{\partial I^\lambda}{\partial\mu_1}\right)^2 = 0$ for all $\mu_1>0$ and $\mu>0$.

Case (IV): If $0<\frac{\lambda}{\mu}-N^\lambda=\mathcal{O}(1)$ **and** $\lim_{\lambda\to\infty} d_2<0$**.** From (EC.10) and (EC.11),

$$\begin{aligned}
&\frac{1}{I^\lambda}\left(\frac{\partial I^\lambda}{\partial\mu_1}\right)^2 = \frac{\frac{\partial I^\lambda}{\partial\mu_1}}{I^\lambda}\cdot\frac{\partial I^\lambda}{\partial\mu_1}\\
=&\frac{1}{\mu_1}\left\{\left(1+\frac{1}{d_2}\frac{\mu_1}{\mu}\right)(1-I^\lambda)-\frac{\rho}{d_2}\frac{1-C}{N^\lambda-\rho}I^\lambda-\frac{\rho C}{d_2}\frac{k^\lambda-N^\lambda}{d_1}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}I^\lambda\right\}\\
&\cdot\frac{1}{\mu_1}\left\{\left(1+\frac{1}{d_2}\frac{\mu_1}{\mu}\right)I^\lambda(1-I^\lambda)-\frac{\rho}{d_2}\frac{1-C}{N^\lambda-\rho}(I^\lambda)^2-\frac{\rho C}{d_2}\frac{k^\lambda-N^\lambda}{d_1}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}(I^\lambda)^2\right\},
\end{aligned}\tag{EC.16}$$

where $\frac{1}{d_2}\in(-\infty,0)$ and $I^\lambda\to 0$ (from Lemma EC.14), as $\lambda\to\infty$.

Case (IV-1): If $\lim_{\lambda\to\infty} d_2\frac{k^\lambda-N^\lambda}{d_1}\neq 0$**.** Then, (EC.16) implies that

$$\begin{aligned}
&\lim_{\lambda\to\infty}\frac{1}{I^\lambda}\left(\frac{\partial I^\lambda}{\partial\mu_1}\right)^2\\
=&\lim_{\lambda\to\infty}\frac{1}{\mu_1^2}\left(1+\frac{1}{d_2}\frac{\mu_1}{\mu}\right)(1-I^\lambda)\left\{-\frac{\rho}{d_2}\frac{1-C}{N^\lambda-\rho}(I^\lambda)^2-\frac{\rho C}{d_2}\frac{k^\lambda-N^\lambda}{d_1}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}(I^\lambda)^2\right\}\\
&\frac{1}{\mu_1^2}\left\{\left(\frac{\rho}{d_2}\frac{1-C}{N^\lambda-\rho}\right)^2(I^\lambda)^3+\frac{\rho}{d_2}\frac{1-C}{N^\lambda-\rho}I^\lambda\cdot\frac{\rho C}{d_2}\frac{k^\lambda-N^\lambda}{d_1}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}(I^\lambda)^2\right.\\
&\left.+\frac{\rho}{d_2}\frac{1-C}{N^\lambda-\rho}(I^\lambda)^2\cdot\frac{\rho C}{d_2}\frac{k^\lambda-N^\lambda}{d_1}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}I^\lambda+\left(\frac{\rho C}{d_2}\frac{k^\lambda-N^\lambda}{d_1}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}\right)^2(I^\lambda)^3\right\}\\
=&\lim_{\lambda\to\infty}\frac{1}{\mu_1^2}\left(1+\frac{1}{d_2}\frac{\mu_1}{\mu}\right)(1-I^\lambda)\left\{-\frac{1}{d_2}\left[\sqrt{\rho}\frac{1-C}{N^\lambda-\rho}\right]\left(\rho^{\frac{1}{4}}I^\lambda\right)^2-\left[\frac{\rho C}{d_2}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}I^\lambda\right]\left[\frac{k^\lambda-N^\lambda}{d_1}I^\lambda\right]\right\}\\
&+\frac{1}{\mu_1^2}\left\{\frac{1}{d_2^2}\left(\sqrt{\rho}\frac{1-C}{N^\lambda-\rho}\right)^2\left(\rho^{\frac{1}{3}}I^\lambda\right)^3+2\frac{1}{d_2}\left[\sqrt{\rho}\frac{1-C}{N^\lambda-\rho}\right]\left[\frac{\rho C}{d_2}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}I^\lambda\right]\left[\frac{k^\lambda-N^\lambda}{d_1}I^\lambda\right](\sqrt{\rho}I^\lambda)\right.\\
&\left.+\left[\frac{\rho C}{d_2}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}I^\lambda\right]^2\left[\left(\frac{k^\lambda-N^\lambda}{d_1}\right)^2I^\lambda\right]\right\}=0,
\end{aligned}$$

where $\lim_{\lambda\to\infty}\left(1+\frac{1}{d_2}\frac{\mu_1}{\mu}\right)(1-I^\lambda)$ is finite, $\lim_{\lambda\to\infty}\sqrt{\rho}\frac{1-C}{N^\lambda-\rho}\in(0,\infty)$ (from Lemma EC.18), $\lim_{\lambda\to\infty}\rho^r I^\lambda=0$ for any $r\in[0,1)$ (from Lemma EC.20 (ii)), $\lim_{\lambda\to\infty}\frac{\rho C}{d_2}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}I^\lambda\in(-\infty,\infty)$ (from Lemma EC.16), $\lim_{\lambda\to\infty}\left(\frac{k^\lambda-N^\lambda}{d_1}\right)^r I^\lambda=0$ for all $r\in\mathbb{N}$ (from Lemma EC.15, noting that $\lim_{\lambda\to\infty}d_2\frac{k^\lambda-N^\lambda}{d_1}<0<\infty$ and $\lim_{\lambda\to\infty}d_2\neq 0$).

Case (IV-2): If $\lim_{\lambda\to\infty} d_2\frac{k^\lambda-N^\lambda}{d_1}=0$**.** Then, (EC.16) implies that

$$\begin{aligned}
&\lim_{\lambda\to\infty}\frac{1}{I^\lambda}\left(\frac{\partial I^\lambda}{\partial\mu_1}\right)^2\\
=&\lim_{\lambda\to\infty}\frac{1}{\mu_1^2}\left(1+\frac{1}{d_2}\frac{\mu_1}{\mu}\right)(1-I^\lambda)\left\{-\frac{\rho}{d_2}\frac{1-C}{N^\lambda-\rho}(I^\lambda)^2-\frac{\rho C}{d_2}\frac{k^\lambda-N^\lambda}{d_1}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}(I^\lambda)^2\right\}\\
&\frac{1}{\mu_1^2}\left\{\left(\frac{\rho}{d_2}\frac{1-C}{N^\lambda-\rho}\right)^2(I^\lambda)^3+\frac{\rho}{d_2}\frac{1-C}{N^\lambda-\rho}I^\lambda\cdot\frac{\rho C}{d_2}\frac{k^\lambda-N^\lambda}{d_1}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}(I^\lambda)^2\right.
\end{aligned}$$

$$+\frac{\rho}{d_2}\frac{1-C}{N^\lambda-\rho}(I^\lambda)^2\cdot\frac{\rho C}{d_2}\frac{k^\lambda-N^\lambda}{d_1}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}I^\lambda+\left(\frac{\rho C}{d_2}\frac{k^\lambda-N^\lambda}{d_1}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}\right)^2(I^\lambda)^3\Bigg\}$$

$$=\lim_{\lambda\to\infty}\frac{1}{\mu_1^2}\left(1+\frac{1}{d_2}\frac{\mu_1}{\mu}\right)(1-I^\lambda)\left\{-\frac{1}{d_2}\left[\sqrt{\rho}\frac{1-C}{N^\lambda-\rho}\right]\left(\rho^{\frac14}I^\lambda\right)^2-\frac{\rho C}{d_2}\frac{k^\lambda-N^\lambda}{d_1}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}(I^\lambda)^2\right\}$$

$$+\frac{1}{\mu_1^2}\left\{\frac{1}{d_2^2}\left(\sqrt{\rho}\frac{1-C}{N^\lambda-\rho}\right)^2\left(\rho^{\frac13}I^\lambda\right)^3+2\frac{1}{d_2}\left[\sqrt{\rho}\frac{1-C}{N^\lambda-\rho}\right]\left(\rho^{\frac13}I^\lambda\right)^{\frac32}\left[\left(\frac{\rho C}{d_2}\frac{k^\lambda-N^\lambda}{d_1}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}\right)^{\frac43}(I^\lambda)^2\right]^{\frac34}\right.$$

$$\left.+\left[\left(\frac{\rho C}{d_2}\frac{k^\lambda-N^\lambda}{d_1}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}\right)^{\frac43}(I^\lambda)^2\right]^{\frac32}\right\}.$$

Note that $\lim_{\lambda\to\infty}\sqrt{\rho}\frac{1-C}{N^\lambda-\rho}\in(0,\infty)$ (from Lemma EC.18), and $\lim_{\lambda\to\infty}\rho^r I^\lambda=0$ for any $r\in[0,\frac12)$ (from Lemma EC.20 (i)). Moreover, as shown in Case (III-2), $\lim_{\lambda\to\infty}\frac{\rho C}{d_2}\frac{k^\lambda-N^\lambda}{d_1}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}(I^\lambda)^2=\lim_{\lambda\to\infty}\left(\frac{\rho C}{d_2}\frac{k^\lambda-N^\lambda}{d_1}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}\right)^{\frac43}(I^\lambda)^2=0$. Hence, $\lim_{\lambda\to\infty}\frac{1}{I^\lambda}\left(\frac{\partial I^\lambda(\mu_1,\mu)}{\partial\mu_1}\right)^2=0$ for all $\mu_1>0$ and $\mu>0$.

Combining the above four cases concludes that, if $\lim_{\lambda\to\infty}d_2\neq0$, $\lim_{\lambda\to\infty}\frac{1}{I^\lambda(\mu_1,\mu)}\left(\frac{\partial I^\lambda(\mu_1,\mu)}{\partial\mu_1}\right)^2=0$ for all $\mu_1>0$ and $\mu>0$. ∎

Proof of Lemma EC.18: Using the relationship between the Erlang B and C formulae (see Lemma EC.1 (b)),

$$\sqrt{\rho}\frac{1-C}{N^\lambda-\rho}=\sqrt{\rho}\frac{1-\frac{N^\lambda}{\frac{N-\rho}{B}+\rho}}{N^\lambda-\rho}=\frac{\sqrt{\rho}(1-B)}{N^\lambda-\rho+\rho B}=\frac{\sqrt{\rho}(1-B)}{\mathcal{O}(\sqrt{\lambda})+\rho B}=\frac{1-B}{\mathcal{O}(1)+\sqrt{\rho}B}. \tag{EC.17}$$

Note that $\lim_{\lambda\to\infty}B=0$ when $a=\mu$ (from Lemma EC.6 (a)), and $\lim_{\lambda\to\infty}\sqrt{\rho}B\in(0,\infty)$ when $\left|N^\lambda-\frac{\lambda}{\mu}\right|=\mathcal{O}(\sqrt{\lambda})$. Hence, $\lim_{\lambda\to\infty}\sqrt{\rho}\frac{1-C}{N^\lambda-\rho}\in(0,\infty)$ for all $\mu>0$. ∎

Proof of Lemma EC.19: From (EC.10),

$$\frac{\sqrt{\rho}I^\lambda}{d_2}=\left[\frac{d_2}{\sqrt{\rho}}+\frac{\mu}{\mu_1}\sqrt{\rho}d_2\left(\frac{1-C}{N^\lambda-\rho}\right)+\frac{\mu}{\mu_1}C\sqrt{\rho}\left(1-\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}\right)\right]^{-1}. \tag{EC.18}$$

We discuss the following cases.

Case (I): If $0<N^\lambda-\frac{\lambda}{\mu}=\omega(1)$**.** Then, we can rewrite (EC.18) as

$$\frac{\sqrt{\rho}I^\lambda}{d_2}=\frac{1}{\sqrt{\rho}}\left[\frac{d_2}{\rho}+\frac{\mu}{\mu_1}(1-C)\left(\frac{d_2}{N^\lambda-\rho}\right)+\frac{\mu}{\mu_1}C\left(1-\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}\right)\right]^{-1},$$

where $\frac{d_2}{\rho}\to0$, $\frac{d_2}{N^\lambda-\rho}\to1$, as $\lambda\to\infty$; moreover, $\lim_{\lambda\to\infty}C\in[0,1]$ (from Lemma EC.6 (b)), and $\lim_{\lambda\to\infty}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}=e^{-\lim_{\lambda\to\infty}d_2\frac{k^\lambda-N^\lambda}{d_1}}\in[0,1]$. Thus, the square bracket in the above display is finite, which implies that $\lim_{\lambda\to\infty}\frac{\sqrt{\rho}I^\lambda}{d_2}=0$, given that $\frac{1}{\sqrt{\rho}}\to0$, as $\lambda\to\infty$.

Case (II): If $0<\frac{\lambda}{\mu}-N^\lambda=\mathcal{O}(\sqrt{\lambda})$ and $\omega(1)$. Then, the first term of (EC.18) satisfies $\frac{d_2}{\sqrt{\rho}}\to 0$, as $\lambda\to\infty$. Moreover, $\lim_{\lambda\to\infty}\sqrt{\rho}\left(\frac{1-C}{N^\lambda-\rho}\right)\in(0,\infty)$ (from Lemma EC.18), implying that the second term of (EC.18) converges to $-\infty$ as $\lambda\to\infty$ (since $\lim_{\lambda\to\infty}d_2<0$ and $d_2=\omega(1)$). In the third term of (EC.18), $\lim_{\lambda\to\infty}C\in[1,\infty)$ (from Lemma EC.6 (b)), and $\lim_{\lambda\to\infty}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}=e^{-\lim_{\lambda\to\infty}d_2\frac{k^\lambda-N^\lambda}{d_1}}$.

Case (II-1): If $\lim_{\lambda\to\infty}d_2\frac{k^\lambda-N^\lambda}{d_1}<0$. Then,

$$\lim_{\lambda\to\infty}\sqrt{\rho}\left(1-\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}\right)=\lim_{\lambda\to\infty}\sqrt{\rho}\left(1-e^{-d_2\frac{k^\lambda-N^\lambda}{d_1}}\right)=-\infty.$$

Then, together with the common analysis under Case (II) regarding the first two terms of (EC.18), it follows that $\lim_{\lambda\to\infty}\frac{\sqrt{\rho}I^\lambda(\mu_1,\mu)}{d_2}=0$ for all $\mu_1>0$ and $\mu>0$.

Case (II-2): If $\lim_{\lambda\to\infty}d_2\frac{k^\lambda-N^\lambda}{d_1}=0$. Then,

$$\begin{aligned}\lim_{\lambda\to\infty}\sqrt{\rho}\left(1-\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}\right)&=\lim_{\lambda\to\infty}\sqrt{\rho}\cdot d_2\frac{k^\lambda-N^\lambda}{d_1}\cdot\frac{1}{d_2\frac{k^\lambda-N^\lambda}{d_1}}\left(1-e^{-d_2\frac{k^\lambda-N^\lambda}{d_1}}\right)\\&\overset{(*)}{=}\lim_{\lambda\to\infty}\sqrt{\rho}\cdot d_2\frac{k^\lambda-N^\lambda}{d_1}\le 0,\end{aligned}$$

where $(*)$ follows by L'Hopital's rule (namely, $\lim_{x\to 0}\frac{1-e^{-x}}{x}=1$), given that $\lim_{\lambda\to\infty}d_2\frac{k^\lambda-N^\lambda}{d_1}=0$ (by assumption). Then, together with the common analysis under Case (II) regarding the first two terms of (EC.18), it follows that $\lim_{\lambda\to\infty}\frac{\sqrt{\rho}I^\lambda(\mu_1,\mu)}{d_2}=0$ for all $\mu_1>0$ and $\mu>0$.

Combining the above two cases concludes that $\lim_{\lambda\to\infty}\frac{\sqrt{\rho}I^\lambda(\mu_1,\mu)}{d_2}=0$ for all $\mu_1>0$ and $\mu>0$. ∎

Proof of Lemma EC.20: From (EC.10),

$$\rho^r I^\lambda=\left[\frac{1}{\rho^r}+\frac{\mu}{\mu_1}\rho^{1-r}\frac{1-C}{N^\lambda-\rho}+\frac{\mu}{\mu_1}\rho^{1-r}\frac{C}{d_2}\left(1-\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}\right)\right]^{-1},\tag{EC.19}$$

where $\lim_{\lambda\to\infty}C=1$ (from Lemma EC.6 (b)), and $|d_2|=\mathcal{O}(1)$. We discuss the following cases.

Case (I): If $\lim_{\lambda\to\infty}d_2\frac{k^\lambda-N^\lambda}{d_1}\neq 0$. Then, note that

$$\lim_{\lambda\to\infty}\rho^{1-r}\frac{C}{d_2}\left(1-\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}\right)=\lim_{\lambda\to\infty}\rho^{1-r}\frac{C}{d_2}\left(1-e^{-d_2\frac{k^\lambda-N^\lambda}{d_1}}\right)=\infty,$$

for all $r\in[0,1)$, noting that $\lim_{\lambda\to\infty}\rho^{1-r}=\infty$, $\lim_{\lambda\to\infty}\frac{C}{d_2}\neq 0$, and $\lim_{\lambda\to\infty}\left(1-e^{-d_2\frac{k^\lambda-N^\lambda}{d_1}}\right)\neq 0$. Thus, the third term of (EC.19) converges to ∞ as $\lambda\to\infty$. Since the first two terms of (EC.19) are both positive, it follows that $\lim_{\lambda\to\infty}\rho^r I^\lambda(\mu_1,\mu)=0$ for all $r\in[0,1)$, $\mu_1>0$ and $\mu>0$.

Case (II): If $\lim_{\lambda\to\infty}d_2\frac{k^\lambda-N^\lambda}{d_1}=0$. Then, note that

$$\begin{aligned}\lim_{\lambda\to\infty}\rho^{1-r}\frac{C}{d_2}\left(1-\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}\right)&=\lim_{\lambda\to\infty}\rho^{1-r}\frac{C}{d_2}d_2\frac{k^\lambda-N^\lambda}{d_1}\frac{1}{d_2\frac{k^\lambda-N^\lambda}{d_1}}\left(1-e^{-d_2\frac{k^\lambda-N^\lambda}{d_1}}\right)\\&\overset{(**)}{=}\lim_{\lambda\to\infty}\rho^{1-r}\frac{k^\lambda-N^\lambda}{d_1},\end{aligned}$$

which is not determined, where (∗∗) follows by L'Hopital's rule (namely, $\lim_{x\to 0}\frac{1-e^{-x}}{x}=1$), given that $\lim_{\lambda\to\infty} d_2\frac{k^\lambda-N^\lambda}{d_1}=0$ (by assumption). On the other hand, note that

$$\lim_{\lambda\to\infty}\rho^{1-r}\frac{1-C}{N^\lambda-\rho}=\lim_{\lambda\to\infty}\left(\sqrt{\rho}\frac{1-C}{N^\lambda-\rho}\right)\rho^{\frac{1}{2}-r}=\infty,$$

for all $r\in[0,\frac{1}{2})$, noting that $\lim_{\lambda\to\infty}\sqrt{\rho}\frac{1-C}{N^\lambda-\rho}\in(0,\infty)$ (from Lemma EC.18) and $\lim_{\lambda\to\infty}\rho^{\frac{1}{2}-r}=\infty$. Hence, $\lim_{\lambda\to\infty}\rho^r I^\lambda(\mu_1,\mu)=0$ for all $r\in[0,\frac{1}{2})$, $\mu_1>0$ and $\mu>0$. ∎

Proof of Lemma EC.21: We discuss the following cases.

Case (I): If $\lim_{\lambda\to\infty} d_2\frac{k^\lambda-N^\lambda}{d_1}>0$. Then, from (EC.10),

$$\frac{\rho I^\lambda}{d_2}=\left[\frac{d_2}{\rho}+\frac{\mu}{\mu_1}\frac{d_2}{\sqrt{\rho}}\sqrt{\rho}\frac{1-C}{N^\lambda-\rho}+\frac{\mu}{\mu_1}C\left(1-\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}\right)\right]^{-1},$$

where $\frac{d_2}{\rho}\to 0$, $\frac{d_2}{\sqrt{\rho}}\to 0$ (since $\lim_{\lambda\to\infty} d_2=0$), as $\lambda\to\infty$; moreover, $\lim_{\lambda\to\infty}\sqrt{\rho}\frac{1-C}{N^\lambda-\rho}\in(0,\infty)$ (from Lemma EC.18), and $\lim_{\lambda\to\infty} C=1$ (from Lemma EC.6 (b)). In addition, note that $\lim_{\lambda\to\infty}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}=e^{-\lim_{\lambda\to\infty} d_2\frac{k^\lambda-N^\lambda}{d_1}}\in[0,1)$. Thus, it follows that $\lim_{\lambda\to\infty}\frac{\rho I^\lambda(\mu_1,\mu)}{d_2}\in\left[\frac{\mu_1}{\mu},\infty\right)$. Therefore, it is clear that $\lim_{\lambda\to\infty}\frac{I^\lambda(\mu_1,\mu)}{d_2}=\lim_{\lambda\to\infty}\frac{1}{\rho}\frac{\rho I^\lambda(\mu_1,\mu)}{d_2}=0$ for all $\mu_1>0$ and $\mu>0$.

Case (II): If $\lim_{\lambda\to\infty} d_2\frac{k^\lambda-N^\lambda}{d_1}<0$. Then, from (EC.10),

$$\frac{I^\lambda}{d_2}=\left[d_2+\frac{\mu}{\mu_1}d_2\rho\frac{1-C}{N^\lambda-\rho}+\frac{\mu}{\mu_1}C\rho\left(1-\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}\right)\right]^{-1},$$

where the first two terms of the above display are zero or negative, and $C\to 1$ (from Lemma EC.6 (b)), as $\lambda\to\infty$. Note that $\lim_{\lambda\to\infty}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}=e^{-\lim_{\lambda\to\infty} d_2\frac{k^\lambda-N^\lambda}{d_1}}\in(1,\infty]$, which implies that $\lim_{\lambda\to\infty}\rho\left(1-\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}\right)=-\infty$. Thus, the above display implies that $\lim_{\lambda\to\infty}\frac{I^\lambda(\mu_1,\mu)}{d_2}=0$ for all $\mu_1>0$ and $\mu>0$.

Combining the above two cases concludes that $\lim_{\lambda\to\infty}\frac{I^\lambda(\mu_1,\mu)}{d_2}=0$ for all $\mu_1>0$ and $\mu>0$. ∎

Proof of Lemma EC.22: When $N^\lambda-\frac{\lambda}{\mu}>0$, it is clear that $\lim_{\lambda\to\infty} d_2>0$. Note that

$$\lim_{\lambda\to\infty}\sqrt{\rho}C\left(\frac{k^\lambda-N^\lambda}{d_1}\right)^r\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}I^\lambda=\lim_{\lambda\to\infty}\left(\frac{\sqrt{\rho}I^\lambda}{d_2}\right)\frac{1}{d_2^{r-1}}\cdot C\cdot\left[\left(d_2\frac{k^\lambda-N^\lambda}{d_1}\right)^r e^{-d_2\frac{k^\lambda-N^\lambda}{d_1}}\right],$$

where $\lim_{\lambda\to\infty}\frac{\sqrt{\rho}I^\lambda}{d_2}=0$ (from Lemma EC.19), $\lim_{\lambda\to\infty} C\in[0,1]$ (from Lemma EC.6 (b)), $\frac{1}{d_2^{r-1}}\in[0,\infty)$ (since $\lim_{\lambda\to\infty} d_2>0$ and $d_2=\omega(1)$), and $\lim_{\lambda\to\infty}\left(d_2\frac{k^\lambda-N^\lambda}{d_1}\right)^r e^{-d_2\frac{k^\lambda-N^\lambda}{d_1}}\in[0,\infty)$ (because even if $\lim_{\lambda\to\infty} d_2\frac{k^\lambda-N^\lambda}{d_1}=\infty$, exponential decay dominates polynomial growth). Hence, the above display implies that $\lim_{\lambda\to\infty}\sqrt{\rho}C\left(\frac{k^\lambda-N^\lambda}{d_1}\right)^r\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}I^\lambda(\mu_1,\mu)=0$ for all $r\in\mathbb{N}$, $\mu_1>0$ and $\mu>0$. ∎

Proof of Lemma EC.23: We first note that $\lim_{\lambda\to\infty} d_2 \in (0,\infty)$, because $\left|N^\lambda - \frac{\lambda}{\mu}\right| = \mathcal{O}(1)$. Observe that

$$\lim_{\lambda\to\infty}\left(\frac{k^\lambda - N^\lambda}{d_1}\right)^r\left(\frac{\rho}{d_1}\right)^{k^\lambda - N^\lambda} = \lim_{\lambda\to\infty}\frac{1}{d_2^r}\left(d_2\frac{k^\lambda - N^\lambda}{d_1}\right)^r e^{-d_2\frac{k^\lambda - N^\lambda}{d_1}},$$

where $\lim_{\lambda\to\infty}\frac{1}{d_2^r} \in (0,\infty)$ (because $\lim_{\lambda\to\infty} d_2 \in (0,\infty)$ and $r\in\mathbb{N}$). We discuss the following cases.

Case (I): If $\lim_{\lambda\to\infty} d_2\frac{k^\lambda - N^\lambda}{d_1} = \infty$. Then, $\lim_{\lambda\to\infty}\left(d_2\frac{k^\lambda - N^\lambda}{d_1}\right)^r e^{-d_2\frac{k^\lambda - N^\lambda}{d_1}} \to 0$, as $\lambda\to\infty$, because exponential decay dominates polynomial growth. Hence, the above display implies that $\lim_{\lambda\to\infty}\left(\frac{k^\lambda - N^\lambda}{d_1}\right)^r\left(\frac{\rho}{d_1}\right)^{k^\lambda - N^\lambda} = 0$, for all $r\in\mathbb{N}$, $\mu_1>0$ and $\mu>0$.

Case (II): If $\lim_{\lambda\to\infty} d_2\frac{k^\lambda - N^\lambda}{d_1} \in (0,\infty)$. Then, $\lim_{\lambda\to\infty}\left(d_2\frac{k^\lambda - N^\lambda}{d_1}\right)^r e^{-d_2\frac{k^\lambda - N^\lambda}{d_1}} \in (0,\infty)$. Hence, the above display implies that $\lim_{\lambda\to\infty}\left(\frac{k^\lambda - N^\lambda}{d_1}\right)^r\left(\frac{\rho}{d_1}\right)^{k^\lambda - N^\lambda} \in (0,\infty)$, for all $r\in\mathbb{N}$, $\mu_1>0$ and $\mu>0$.

Combining the above two cases concludes that $\lim_{\lambda\to\infty}\left(\frac{k^\lambda - N^\lambda}{d_1}\right)^r\left(\frac{\rho}{d_1}\right)^{k^\lambda - N^\lambda} \in [0,\infty)$ for all $r\in\mathbb{N}$, $\mu_1>0$ and $\mu>0$. ∎

Proof of Lemma EC.24: Note that

$$\frac{\rho C}{d_2}\left(\frac{k^\lambda - N^\lambda}{d_1}\right)^r\left(\frac{\rho}{d_1}\right)^{k^\lambda - N^\lambda}(I^\lambda)^2 = \frac{\rho C}{d_2}\frac{k^\lambda - N^\lambda}{d_1}\left(\frac{\rho}{d_1}\right)^{k^\lambda - N^\lambda}(I^\lambda)^2\cdot\left(d_2\frac{k^\lambda - N^\lambda}{d_1}\right)^{r-1}\frac{1}{d_2^{r-1}}, \quad \text{(EC.20)}$$

where $\lim_{\lambda\to\infty}\left(d_2\frac{k^\lambda - N^\lambda}{d_1}\right)^{r-1} = 0$ (because $\lim_{\lambda\to\infty} d_2\frac{k^\lambda - N^\lambda}{d_1} = 0$ by assumption and $r\in\mathbb{N}$), and $\lim_{\lambda\to\infty}\frac{1}{d_2^{r-1}} \in (-\infty,\infty)$ (because $\lim_{\lambda\to\infty} d_2 \neq 0$ and $r\in\mathbb{N}$).

Recall from the proof of Case (III-2) in Lemma EC.18,

$$\begin{aligned}
&\left[\frac{\rho C}{d_2}\frac{k^\lambda - N^\lambda}{d_1}\left(\frac{\rho}{d_1}\right)^{k^\lambda - N^\lambda}(I^\lambda)^2\right]^{-1}\\
=&\frac{d_2^2}{\rho C}\left[\frac{1}{d_2\frac{k^\lambda - N^\lambda}{d_1}}\left(\frac{d_1}{\rho}\right)^{k^\lambda - N^\lambda}\right] + 2\frac{\mu}{\mu_1}\left(\frac{1-C}{N^\lambda-\rho}\right)\frac{d_2^2}{C}\left[\frac{1}{d_2\frac{k^\lambda - N^\lambda}{d_1}}\left(\frac{d_1}{\rho}\right)^{k^\lambda - N^\lambda}\right] + 2\frac{\mu}{\mu_1}d_2\left[\frac{1}{d_2\frac{k^\lambda - N^\lambda}{d_1}}\left(\left(\frac{d_1}{\rho}\right)^{k^\lambda - N^\lambda} - 1\right)\right]\\
&+\left(\frac{\mu}{\mu_1}\right)^2\left(\frac{1-C}{N^\lambda-\rho}\right)^2\frac{\rho d_2^2}{C}\left[\frac{1}{d_2\frac{k^\lambda - N^\lambda}{d_1}}\left(\frac{d_1}{\rho}\right)^{k^\lambda - N^\lambda}\right] + 2\left(\frac{\mu}{\mu_1}\right)^2\left(\frac{1-C}{N^\lambda-\rho}\right)\rho d_2\left[\frac{1}{d_2\frac{k^\lambda - N^\lambda}{d_1}}\left(\left(\frac{d_1}{\rho}\right)^{k^\lambda - N^\lambda} - 1\right)\right]\\
&+\left(\frac{\mu}{\mu_1}\right)^2\rho C\left[\frac{1}{d_2\frac{k^\lambda - N^\lambda}{d_1}}\left(\left(\frac{d_1}{\rho}\right)^{k^\lambda - N^\lambda} + \left(\frac{\rho}{d_1}\right)^{k^\lambda - N^\lambda} - 2\right)\right], \quad \text{(EC.21)}
\end{aligned}$$

where the limiting value of every term in the above display, as $\lambda\to\infty$, bears an identical sign, noting that $\lim_{\lambda\to\infty}\left(\frac{d_1}{\rho}\right)^{k^\lambda - N^\lambda} - 1 = e^{\lim_{\lambda\to\infty} d_2\frac{k^\lambda - N^\lambda}{d_1}} - 1 = 0$ (since $\lim_{\lambda\to\infty} d_2\frac{k^\lambda - N^\lambda}{d_1} = 0$ by assumption), and $\left(\frac{d_1}{\rho}\right)^{k^\lambda - N^\lambda} + \left(\frac{\rho}{d_1}\right)^{k^\lambda - N^\lambda} \geq 2$ (using the arithmetic-geometric mean inequality, namely, $x+y\geq 2\sqrt{xy}$ for $x,y\geq 0$). In particular,

$$\lim_{\lambda\to\infty}\left|\left(\frac{1-C}{N^\lambda-\rho}\right)\rho d_2\left[\frac{1}{d_2\frac{k^\lambda - N^\lambda}{d_1}}\left(\left(\frac{d_1}{\rho}\right)^{k^\lambda - N^\lambda} - 1\right)\right]\right|$$

$$= \lim_{\lambda\to\infty} \left| d_2 \left(\rho \frac{1-C}{N^\lambda-\rho}\right) \left[\frac{1}{d_2 \frac{k^\lambda-N^\lambda}{d_1}} \left(e^{d_2 \frac{k^\lambda-N^\lambda}{d_1}} - 1 \right) \right] \right|$$
$$\overset{(\dagger)}{=} \lim_{\lambda\to\infty} \left| d_2 \left(\rho \frac{1-C}{N^\lambda-\rho}\right) \right| \overset{(\ddagger)}{=} \infty,$$

where (†) follows by L'Hopital's rule (namely, $\lim_{x\to 0} \frac{e^x-1}{x}=1$), given that $\lim_{\lambda\to\infty} d_2 \frac{k^\lambda-N^\lambda}{d_1}=0$ (by assumption), and the last equality (‡) follows because $\lim_{\lambda\to\infty} d_2 \neq 0$ (by assumption), and

$$\rho\frac{1-C}{N^\lambda-\rho} \overset{(*)}{=} \rho \frac{1-\frac{N^\lambda}{\frac{N-\rho}{B}+\rho}}{N^\lambda-\rho} = \frac{\rho(1-B)}{N^\lambda-\rho+\rho B} = \frac{\rho(1-B)}{o(\lambda)+\rho B} = \frac{1-B}{o(\lambda)/\rho+B} \to \lim_{\lambda\to\infty} \frac{1-B}{B} \overset{(**)}{=} \infty, \ \text{ as } \lambda\to\infty,$$

where $(*)$ follows by using the relationship between the Erlang B and C formulae (see Lemma EC.1 (b)), and the last equality $(**)$ follows from $\lim_{\lambda\to\infty} B=0$ (when $a=\mu$ from Lemma EC.6 (a)). Thus, from (EC.21), it follows that $\lim_{\lambda\to\infty} \frac{\rho C}{d_2}\frac{k^\lambda-N^\lambda}{d_1}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}(I^\lambda)^2=0$. Then, substitution into (EC.20) yields that $\lim_{\lambda\to\infty} \frac{\rho C}{d_2}\left(\frac{k^\lambda-N^\lambda}{d_1}\right)^r\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}(I^\lambda(\mu_1,\mu))^2=0$ for all $r\in\mathbb{N}$, $\mu_1>0$ and $\mu>0$. ∎

Proof of Lemma EC.25: From (EC.10) and (EC.11),

$$\begin{aligned}
\frac{\partial I^\lambda}{\partial \mu_1} =& \frac{1}{\mu_1}\left\{ \left(1+\frac{1}{d_2}\frac{\mu_1}{\mu}\right) I^\lambda(1-I^\lambda) - \frac{\rho}{d_2}\frac{1-C}{N^\lambda-\rho}(I^\lambda)^2 - \frac{\rho C}{d_2}\frac{k^\lambda-N^\lambda}{d_1}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}(I^\lambda)^2 \right\} \\
=& \frac{1}{\mu_1} I^\lambda(1-I^\lambda) + \frac{1}{\mu_1}\frac{I^\lambda}{d_2}\left\{ \frac{\mu_1}{\mu}(1-I^\lambda) - \rho\frac{1-C}{N^\lambda-\rho}I^\lambda - \rho C\frac{k^\lambda-N^\lambda}{d_1}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda} I^\lambda \right\} \\
=& \frac{1}{\mu_1} I^\lambda(1-I^\lambda) + \frac{1}{\mu_1}\frac{I^\lambda}{d_2}\left\{ \frac{\frac{\mu_1}{\mu}\left[\rho\frac{\mu}{\mu_1}\left(\frac{1-C}{N^\lambda-\rho}+\left(1-\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}\right)\frac{C}{d_2}\right)\right] - \rho\frac{1-C}{N^\lambda-\rho} - \rho C\frac{k^\lambda-N^\lambda}{d_1}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}}{1+\rho\frac{\mu}{\mu_1}\left(\frac{1-C}{N^\lambda-\rho}+\left(1-\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}\right)\frac{C}{d_2}\right)} \right\} \\
=& \frac{1}{\mu_1} I^\lambda(1-I^\lambda) + \frac{1}{\mu_1}\frac{(I^\lambda)^2}{d_2}\left\{ \frac{\rho C}{d_2}\left[\left(1+d_2\frac{k^\lambda-N^\lambda}{d_1}\right)\frac{1-\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}}{d_2\frac{k^\lambda-N^\lambda}{d_1}}\cdot d_2\frac{k^\lambda-N^\lambda}{d_1} - d_2\frac{k^\lambda-N^\lambda}{d_1}\right]\right\} \\
\overset{(*)}{\to}& \frac{1}{\mu_1}\frac{(I^\lambda)^2}{d_2}\frac{\rho C}{d_2}\left(d_2\frac{k^\lambda-N^\lambda}{d_1}\right)^2 \\
=& \frac{C}{\mu_1}\left(\sqrt{\rho}\frac{k^\lambda-N^\lambda}{d_1}I^\lambda\right)^2,
\end{aligned} \tag{EC.22}$$

where $(*)$ follows from $\lim_{\lambda\to\infty} I^\lambda(\mu_1,\mu)=0$ for all $\mu_1>0$ and $\mu>0$ (from Lemma EC.14), and $\lim_{\lambda\to\infty}\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda} = e^{-\lim_{\lambda\to\infty} d_2\frac{k^\lambda-N^\lambda}{d_1}}$, implying that $\lim_{\lambda\to\infty}\frac{1-\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}}{d_2\frac{k^\lambda-N^\lambda}{d_1}}=1$ by L'Hopital's rule (namely, $\lim_{x\to 0}\frac{1-e^{-x}}{x}=1$), given that $\lim_{\lambda\to\infty} d_2\frac{k^\lambda-N^\lambda}{d_1}=0$ (by assumption). Note that, from (EC.10),

$$\sqrt{\rho}\frac{k^\lambda-N^\lambda}{d_1}I^\lambda = \left[\frac{d_1}{\sqrt{\rho}}\frac{1}{k^\lambda-N^\lambda} + \frac{\mu}{\mu_1}\sqrt{\rho}\frac{cd_1}{k^\lambda-N^\lambda}\frac{1-C}{N^\lambda-\rho} + \frac{\mu}{\mu_1}C\frac{\sqrt{\rho}}{d_2}\frac{d_1}{k^\lambda-N^\lambda}\left(1-\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}\right)\right]^{-1}.$$

where $\lim_{\lambda\to\infty} C = 1$ (from Lemma EC.6 (b)), and

$$\lim_{\lambda\to\infty} \frac{\sqrt{\rho}}{d_2}\frac{d_1}{k^\lambda - N^\lambda}\left(1-\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}\right) = \lim_{\lambda\to\infty} \frac{\sqrt{\rho}}{d_2}\frac{d_1}{k^\lambda - N^\lambda}\cdot d_2\frac{k^\lambda-N^\lambda}{d_1}\frac{1-\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}}{d_2\frac{k^\lambda-N^\lambda}{d_1}}$$
$$= \lim_{\lambda\to\infty}\sqrt{\rho}\frac{1-e^{-d_2\frac{k^\lambda-N^\lambda}{d_1}}}{d_2\frac{k^\lambda-N^\lambda}{d_1}} \overset{(\ddagger)}{=} \infty,$$

where the last equality (‡) follows by L'Hopital's rule (namely, $\lim_{x\to 0}\frac{1-e^{-x}}{x}=1$), given that $\lim_{\lambda\to\infty} d_2\frac{k^\lambda-N^\lambda}{d_1}=0$ (by assumption). Hence, $\lim_{\lambda\to\infty}\sqrt{\rho}\frac{k^\lambda-N^\lambda}{d_1}I^\lambda=0$. Then, substitution into (EC.22), together with $\lim_{\lambda\to\infty} C=1$ (from Lemma EC.6 (b)), implies that $\lim_{\lambda\to\infty}\frac{\partial I^\lambda(\mu_1,\mu)}{\partial\mu_1}=0$ for all $\mu_1>0$ and $\mu>0$. ∎

Proof of Lemma EC.26: From (EC.10),

$$\frac{\rho C I^\lambda}{d_2} = \left[\frac{d_2}{\rho C}+\frac{\mu}{\mu_1}\frac{d_2}{C}\frac{1-C}{N^\lambda-\rho}+\frac{\mu}{\mu_1}\left(1-\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}\right)\right]^{-1}$$
$$= \left[\frac{d_2}{\rho}\frac{1}{C}+\frac{\mu}{\mu_1}\frac{d_2}{N^\lambda-\rho}\frac{1-C}{C}+\frac{\mu}{\mu_1}\left(1-\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}\right)\right]^{-1},$$

where $\frac{d_2}{\rho}\to 0$, $C\to\infty$ (from Lemma EC.6 (b)), $\frac{d_2}{N^\lambda-\rho}\to 1$, $\frac{1-C}{C}\to -1$, and $\left(\frac{\rho}{d_1}\right)^{k^\lambda-N^\lambda}\to e^{-\lim_{\lambda\to\infty} d_2\frac{k^\lambda-N^\lambda}{d_1}}\in[1,\infty]$, as $\lambda\to\infty$. Hence, it follows that $\lim_{\lambda\to\infty}\frac{\rho C I^\lambda(\mu_1,\mu)}{d_2}\in\left[-\frac{\mu_1}{\mu},0\right]$ for all $\mu_1>0$ and $\mu>0$. ∎

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