# Behavior-Aware Queueing:

## The Finite-Buffer Setting with Many Strategic Servers

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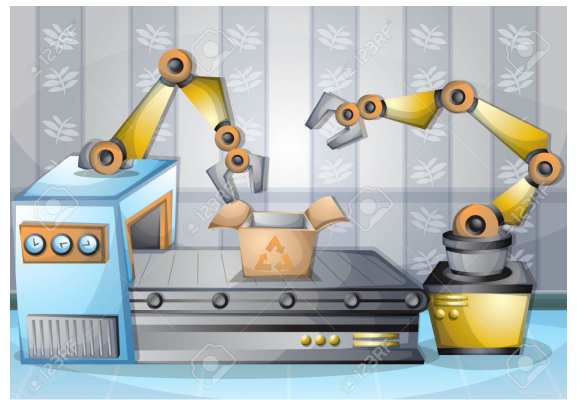
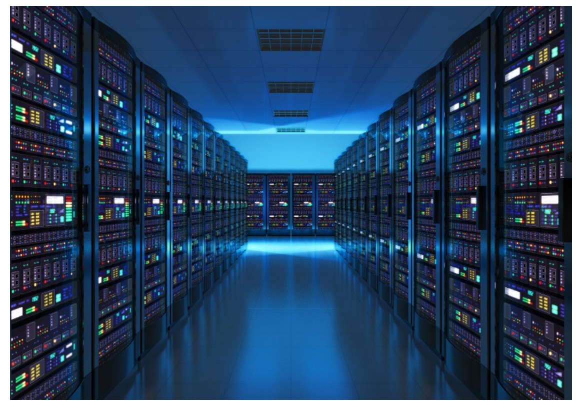
# Motivation






| | **Traditional Queueing (Manufacture/Computer Science)** | **Behavior-Aware Queueing (Service Operations)** |
|---|---|---|
| | Arrival and service processes are assumed to be exogenous | Arrival and service processes are endogenously determined by customer and worker's utility functions |
| **Demand side** | Inanimate jobs | Human customers<br>⟹ may not wait in queue forever [Naor (1969), Knudsen (1972)] |
| **Supply side** | Inanimate machines | Human workers<br>⟹ service rates are not exogeneous<br>speedup (social pressure)<br>slowdown (fatigue or social loafing) |

# Strategic Arrivals

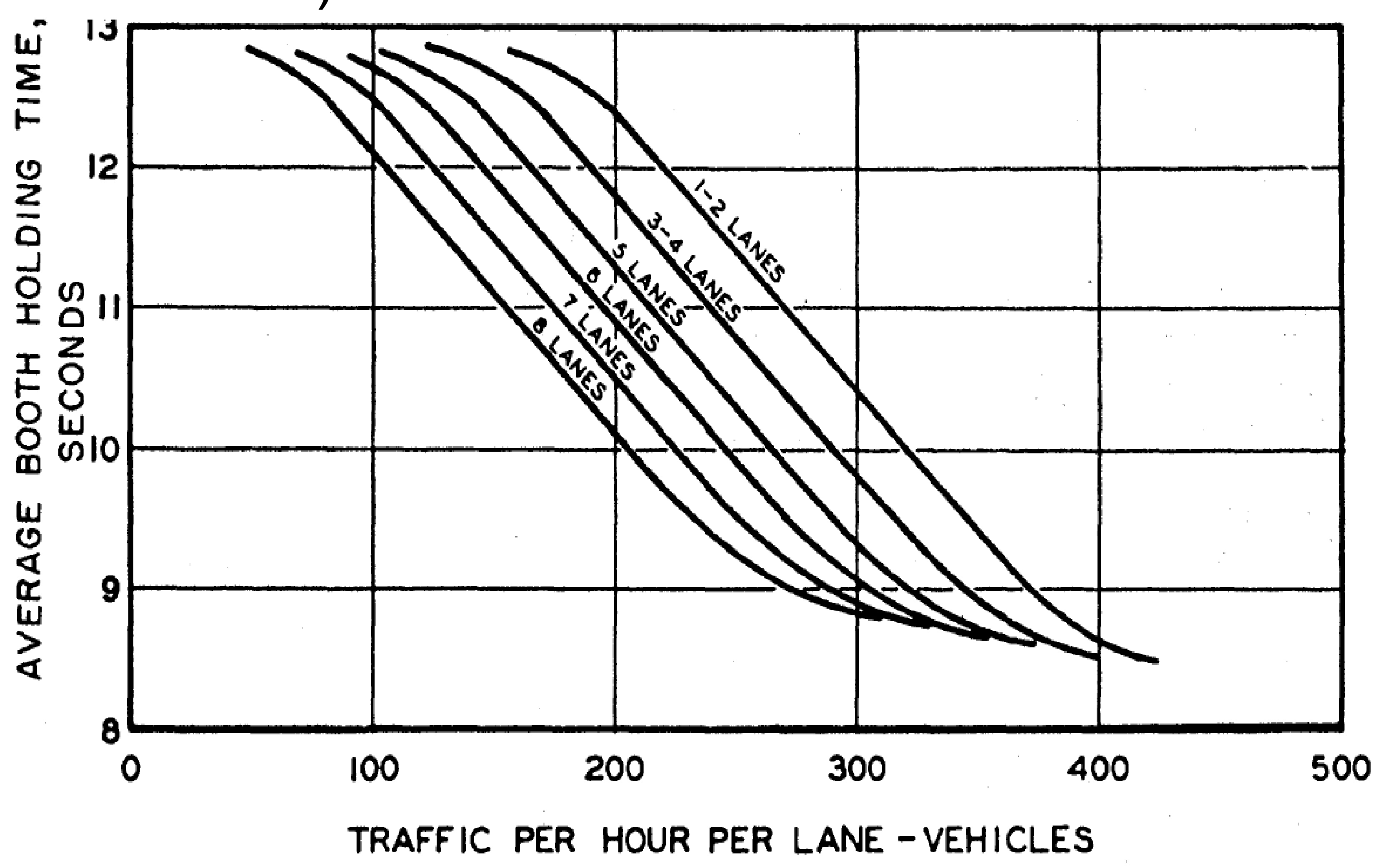
Congestion impacts joining decision.

Foundational Econometrica papers
Naor (1969), Knudsen (1972)

# Strategic Servers

Congestion impacts service time.


(Service time)




Edie (1954), Fig 7: Average booth holding time per vehicle at George Washington Bridge

Larger goal is to understand the behavior of strategic customers, strategic servers, and their interactions.

# Objective

To develop an analytical queueing model to investigate how server work speed is affected by system design decisions concerning

- How many servers to staff and how much to pay them;
- Whether and when to turn away customers.

# Literature Review

**Empirical literature**

[Delasay et al., 2019] (survey)

[Kc & Terwiesch, 2009]

[Staats & Gino, 2012]

[Mas & Morretti, 2009]

**Large system asymptotic analysis**

[Ibrahim, 2018]

[Dong and Ibrahim, 2020]

[Zhan and Ward, 2019]

[Gopalakrishnan et al., 2016]

**Queueing game**

[Hassin and Haviv, 2003] (survey book)

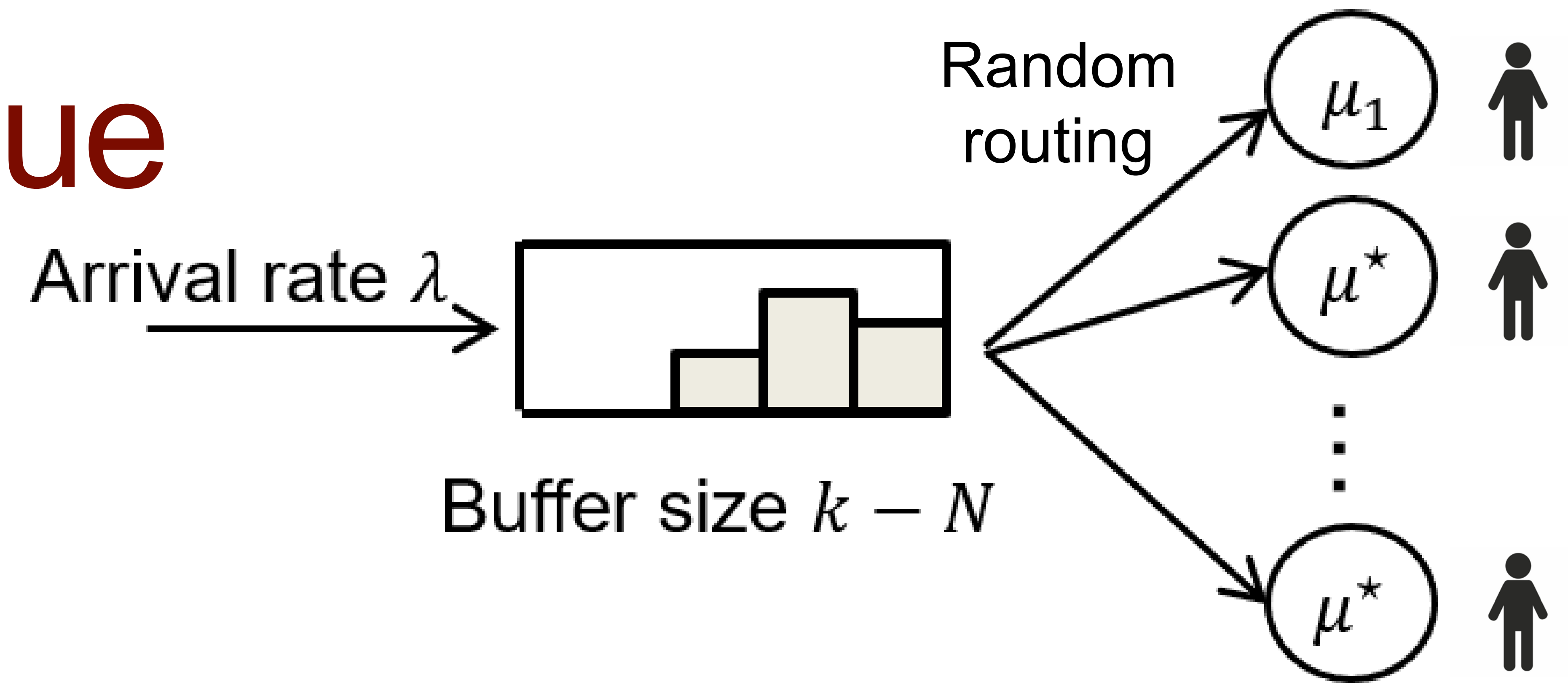
[Hassin, 2016] (survey book)

[Allon and Kremer, 2018] (survey chapter)

# Outline

- Model
- Asymptotic Analysis
- Looking Ahead

# Model: Strategic Server $M/M/N/k$ Queue




**Definition: Nash Equilibrium Service Rate**

The servers want to choose rates $\vec{\mu} = (\mu_1, \mu_2, \ldots, \mu_N)$ that satisfy

$$U_i(\vec{\mu}) = \max_{\mu_i \geq 0} \underbrace{p\mu_i B_i(\vec{\mu})}_{\text{Payment}} + \underbrace{vI_i(\vec{\mu})}_{\text{Idleness}} - \underbrace{c(\mu_i)}_{\text{Effort cost}},$$

**Definition: Symmetric Equilibrium**

$\mu^\star \in \text{argmax}_{\mu_1 \geq 0}\, U_1(\mu_1, \mu^\star)$ where

$$U_1(\mu_1, \mu) = p\mu_1 B_1(\mu_1, \mu) + vI_1(\mu_1, \mu) - c(\mu_1).$$

Notation:
$N$ = number of servers
$\lambda$ = Poisson arrival rate
$\mu_i$ = service rate of server $i$
$k$ = system size
$k - N$ = buffer, or waiting room size

**Individual Rationality:** $U(\mu^\star, \mu^\star) \geq 0$

Notation and Assumption:
$B_i(\vec{\mu})$ = Long-run average fraction of time servers are busy;
$I_i(\vec{\mu})$ = Long-run average fraction of time servers are idle, as opposed to working to serve customers;
$c(\mu)$ = Effort cost function that is continuous, differentiable, strictly increasing, strictly convex, and has $c(0) = 0$.

# Equilibrium Analysis

Equilibrium Analysis Steps: (1) Satisfy first-order condition (FOC); (2) Global maximum.

$$\left.\frac{\partial U(\mu_1,\mu)}{\partial \mu_1}\right|_{\mu_1=\mu} = 0 \quad \Leftrightarrow \quad p\big(1 - I(\mu,\mu)\big) + (v - p\mu)\left.\frac{\partial I}{\partial \mu_1}\right|_{\mu_1=\mu} = c'(\mu).$$

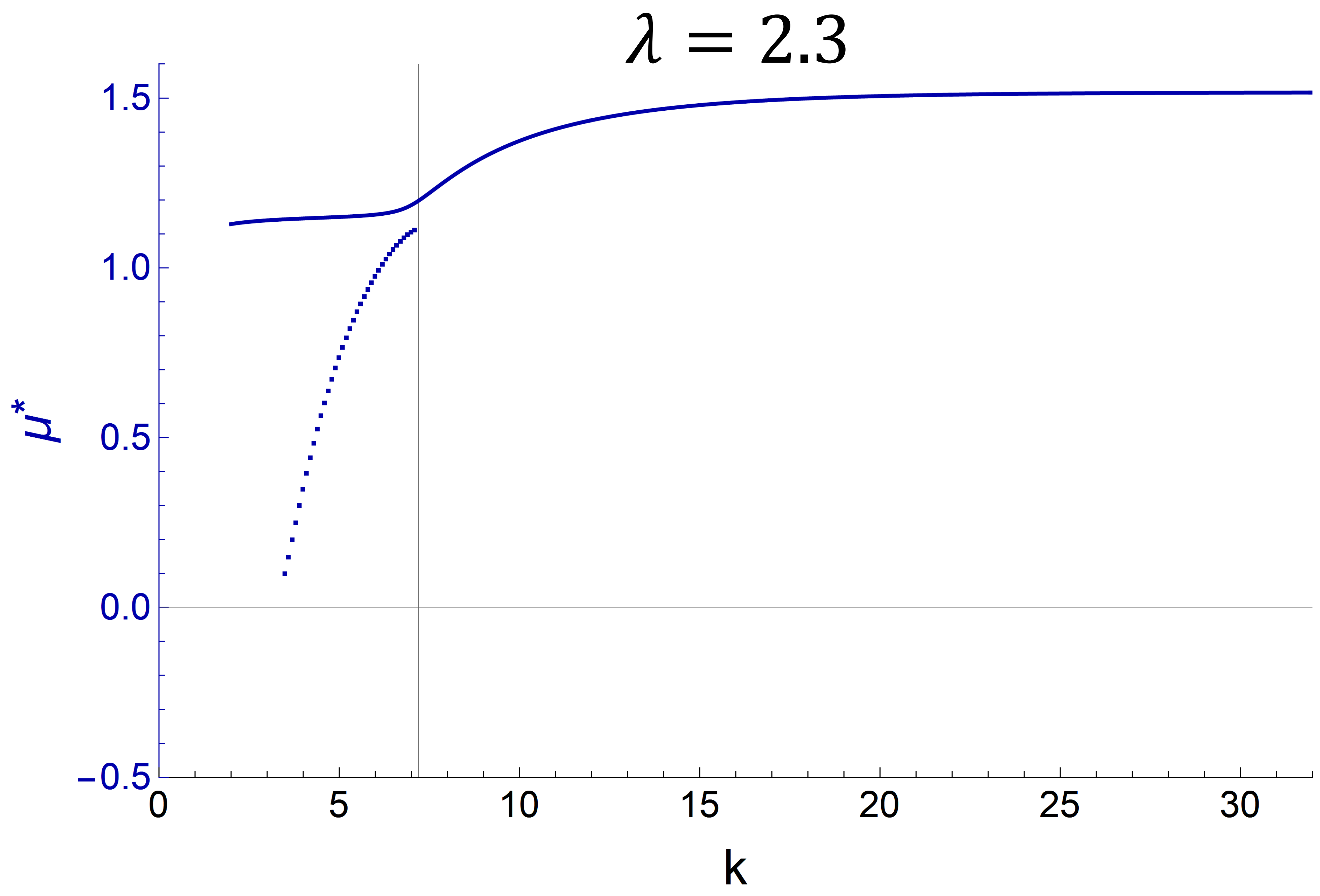
**Lemma**: In an $M/M/N/k$ system with $N-1$ servers operating at rate $\mu > 0$, and a tagged server with rate $\mu_1 > 0$.

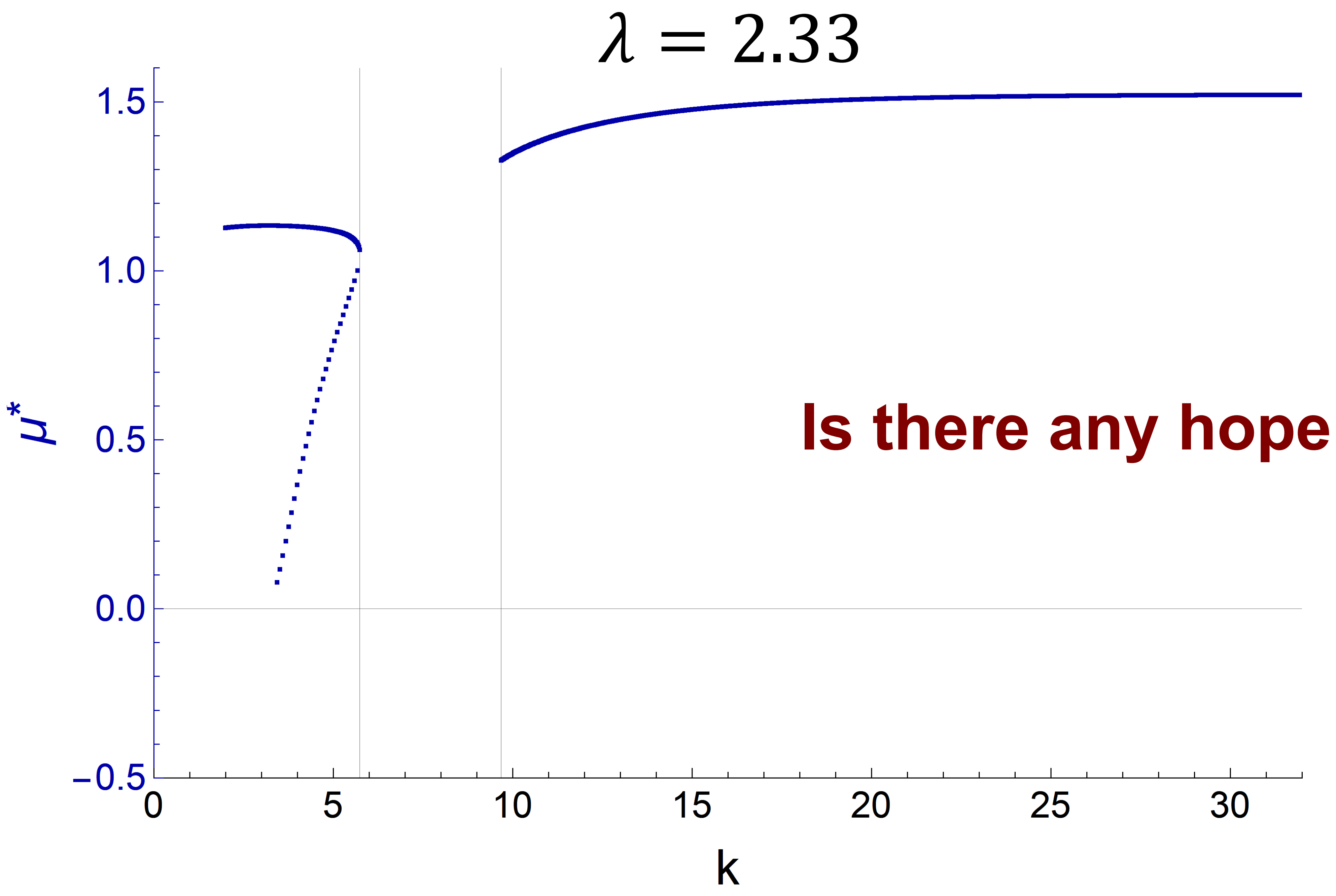
$$I(\mu_1,\mu) = \left(1 + \rho\frac{\mu}{\mu_1}\left(\frac{1-C}{N-\rho} + \left(1 - \left(\frac{\rho}{N - \left(1 - \frac{\mu_1}{\mu}\right)}\right)^{k-N}\right)\frac{C}{(N-\rho) - \left(1 - \frac{\mu_1}{\mu}\right)}\right)\right)^{-1},$$

where $\rho = \frac{\lambda}{\mu}$ and $C \coloneqq ErlC(N,\rho) = \frac{\frac{\rho^N}{N!}\cdot\frac{N}{N-\rho}}{\sum_{j=0}^{N-1}\frac{\rho^j}{j!} + \frac{\rho^N}{N!}\cdot\frac{N}{N-\rho}}$.

**How do solutions to the FOC that are equilibria behave?**

# $M/M/2/k$ System: $N = 2, p = 0, v = 1, c(\mu) = \frac{3}{32}\mu^2$




**Is there any hope for analysis?**

- Not unique
- Not continuous
- Not monotonic

# Outline

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# Asymptotic Analysis

Consider a sequence of $M/M/N^{\lambda}/k^{\lambda}$ systems, and let $\lambda$ become large:

- $N^{\lambda}$: the staffing level
- $k^{\lambda} \geq N^{\lambda}$: the system size
- $\mu^{\star,\lambda}$: prelimit equilibrium

Linear staffing with parameter $a$

**Lemma:** Fix $\mu > 0$. If $\boxed{N^{\lambda} = \frac{1}{a}\lambda + o(\lambda)}$, for $a > 0$,

$$\lim_{\lambda\to\infty} I^{\lambda}(\mu,\mu) = \left[1 - \frac{a}{\mu}\right]^{+} \text{ and } \lim_{\lambda\to\infty} \left.\frac{\partial I^{\lambda}(\mu_1,\mu)}{\partial \mu_1}\right|_{\mu_1=\mu} = \frac{a[\mu-a]^{+}}{\mu^3}.$$

Otherwise, if $N^{\lambda} = f(\lambda) + o\big(f(\lambda)\big)$ for $f(\lambda) = o(\lambda)$ or $f(\lambda) = \omega(\lambda)$, then $\lim_{\lambda\to\infty} \left.\frac{\partial I^{\lambda}}{\partial \mu_1}\right|_{\mu_1=\mu} = 0$ or $1$ (degenerate).

**Proposition:** For large enough $\lambda$, $\frac{\partial^2 U^{\lambda}(\mu_1,\mu)}{\partial \mu_1^2} < 0$ for all $\mu_1 > 0$ and $\mu > 0$.

# Asymptotic Analysis: Existence

$$N^\lambda = \frac{1}{a}\lambda + o(\lambda)$$

**FOC:** $p\left(1 - I^\lambda(\mu,\mu)\right) + (v - p\mu)\left.\frac{\partial I^\lambda(\mu_1,\mu)}{\partial \mu_1}\right|_{\mu_1=\mu} = c'(\mu).$

**Limiting FOC:** $p\left(1 - \left[1 - \frac{a}{\mu}\right]^+\right) + (v - p\mu)\frac{a[\mu - a]^+}{\mu^3} = c'(\mu)$

**Theorem:** There exists $\bar{a}(p,v)$ such that:

Underloaded Equilibria $\left(\frac{a}{\mu} < 1\right)$ → $n_u$; Overloaded Equilibria $\left(\frac{a}{\mu} > 1\right)$ → $n_o$

$p \le c'(0)$

| | $n_u$ | $n_o$ | |
|---|---|---|---|
| $a < \bar{a}(p,v)$ | 2 | 0 | ← Blue |
| $a > \bar{a}(p,v)$ | 0 | 0 | ← Purple |

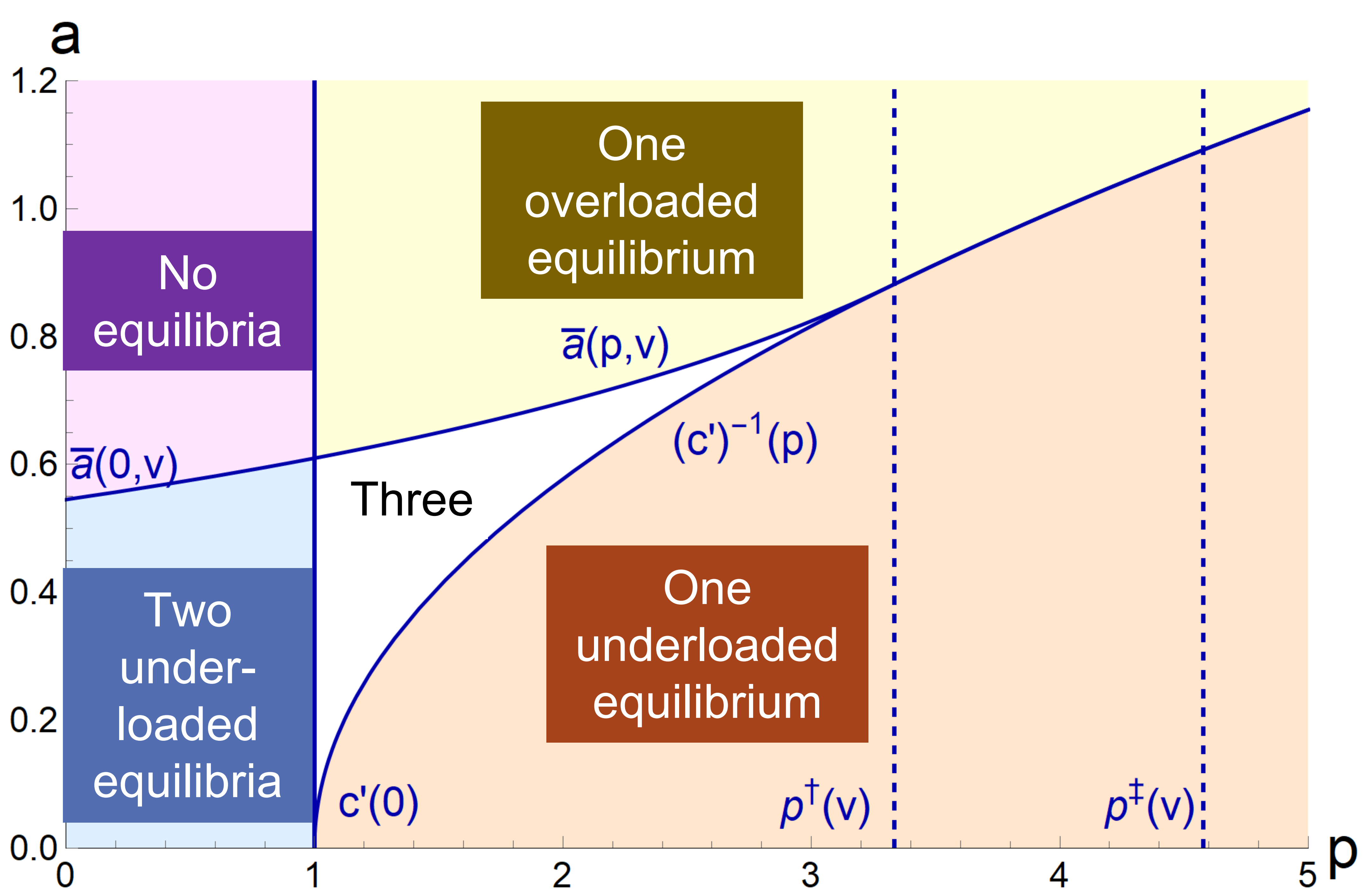
(ignoring boundary behavior; $a = \bar{a}(p,v)$ iff $n_c = 1$)

Critically loaded equilibria $\left(\frac{a}{\mu} = 1\right)$

$p > c'(0)$

| | $n_u$ | $n_o$ | |
|---|---|---|---|
| $a < (c')^{-1}(p)$ | 1 | 0 | ← Orange |
| $(c')^{-1}(p) < a < \bar{a}(p,v)$ | 2 | 1 | ← White |
| $\bar{a}(p,v) < a$ | 0 | 1 | ← Yellow |

(ignoring boundary behavior; $a = (c')^{-1}(p)$ iff $n_c = 1$)




$c(\mu) = \mu^3 + \mu$ and $v = 10$

**Takeaway**: The system manager must either staff enough servers or pay them enough to ensure equilibrium existence.

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# Asymptotic Analysis: Multiplicity

**Lemma (Equilibrium Selection):**
If $\mu_1^\star$, $\mu_2^\star$ are two distinct limiting equilibria with $\mu_1^\star > \mu_2^\star$, then $U(\mu_1^\star, \mu_1^\star) > U(\mu_2^\star, \mu_2^\star)$.

**Takeaway:** Servers prefer the faster equilibrium. The self-interested behavior of servers is aligned with the system manager's interest in better system performance as measured by less waiting time or larger throughput.

# Asymptotic Analysis: Monotonicity

**Proposition (Monotonicity of $a$):**

**Overloaded equilibria** (solves $p = c'(\mu)$):

$\mu_o^\star(a, p; v)$ does not depend on $a$.

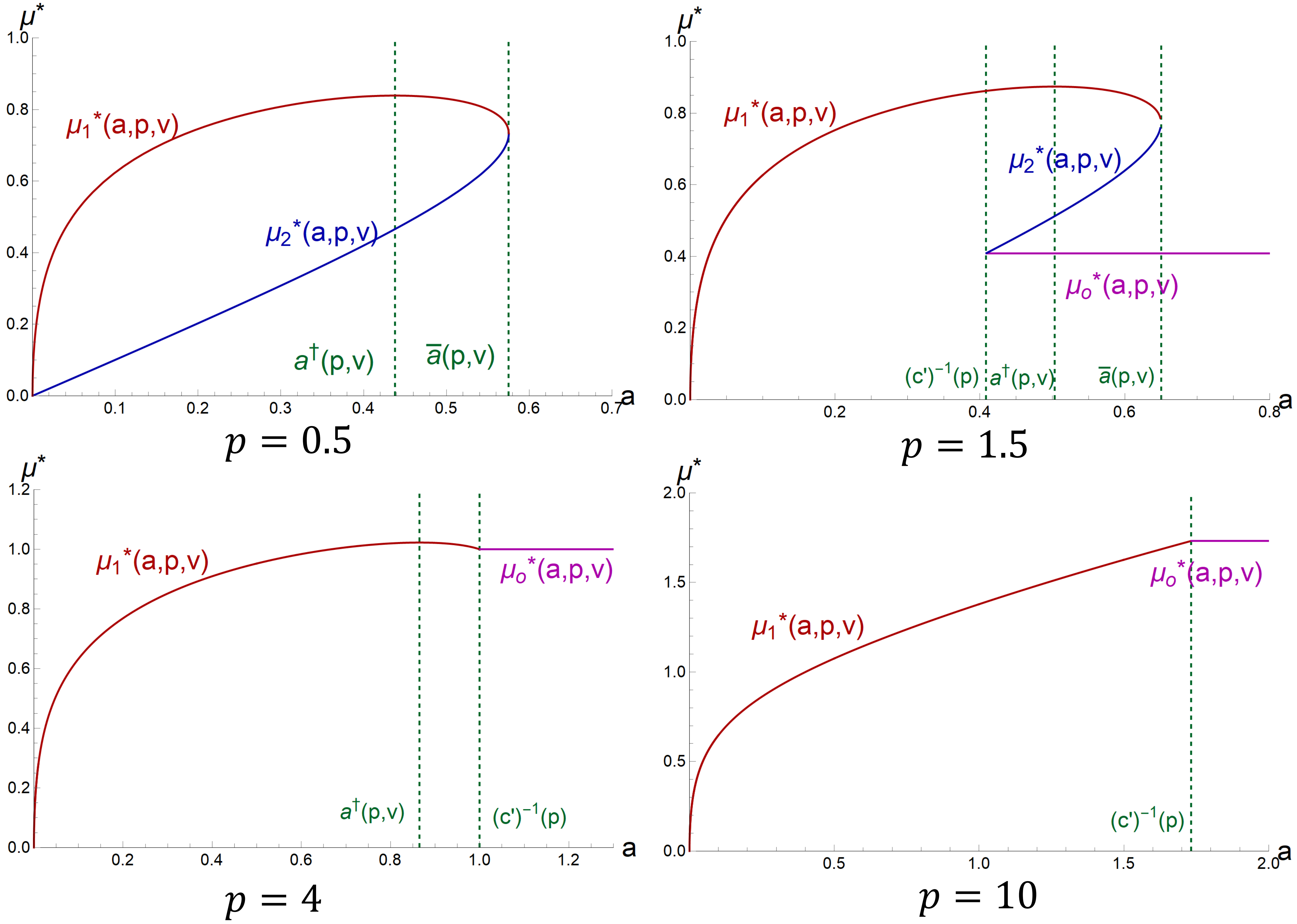
**Underloaded equilibria:**

(i) $\mu_1^\star(a, p; v)$ is strictly increasing in $a$, and then strictly decreasing for smaller $p$; otherwise, $\mu_1^\star(a, p; v)$ is strictly increasing.

(ii) $\mu_2^\star(a, p; v)$ is strictly increasing in $a$.

**Takeaway**: When servers are not paid enough, increasing workload beyond a tipping point may result in a sharp drop in system performance due to server "rebellion".

**Takeaway**: Large enough payment can (1) incentivize servers to work rather than rebel, when the workload is very high; (2) guarantee monotonic increasing behavior.




$c(\mu) = \mu^3 + \mu$

# Asymptotic Analysis: Monotonicity

**Proposition (Monotonicity of $p$):**

**Overloaded equilibria** (solves $p = c'(\mu)$)**:**

$\mu_o^\star(a, p, v)$ is strictly increasing in $p$.

**Underloaded equilibria:**

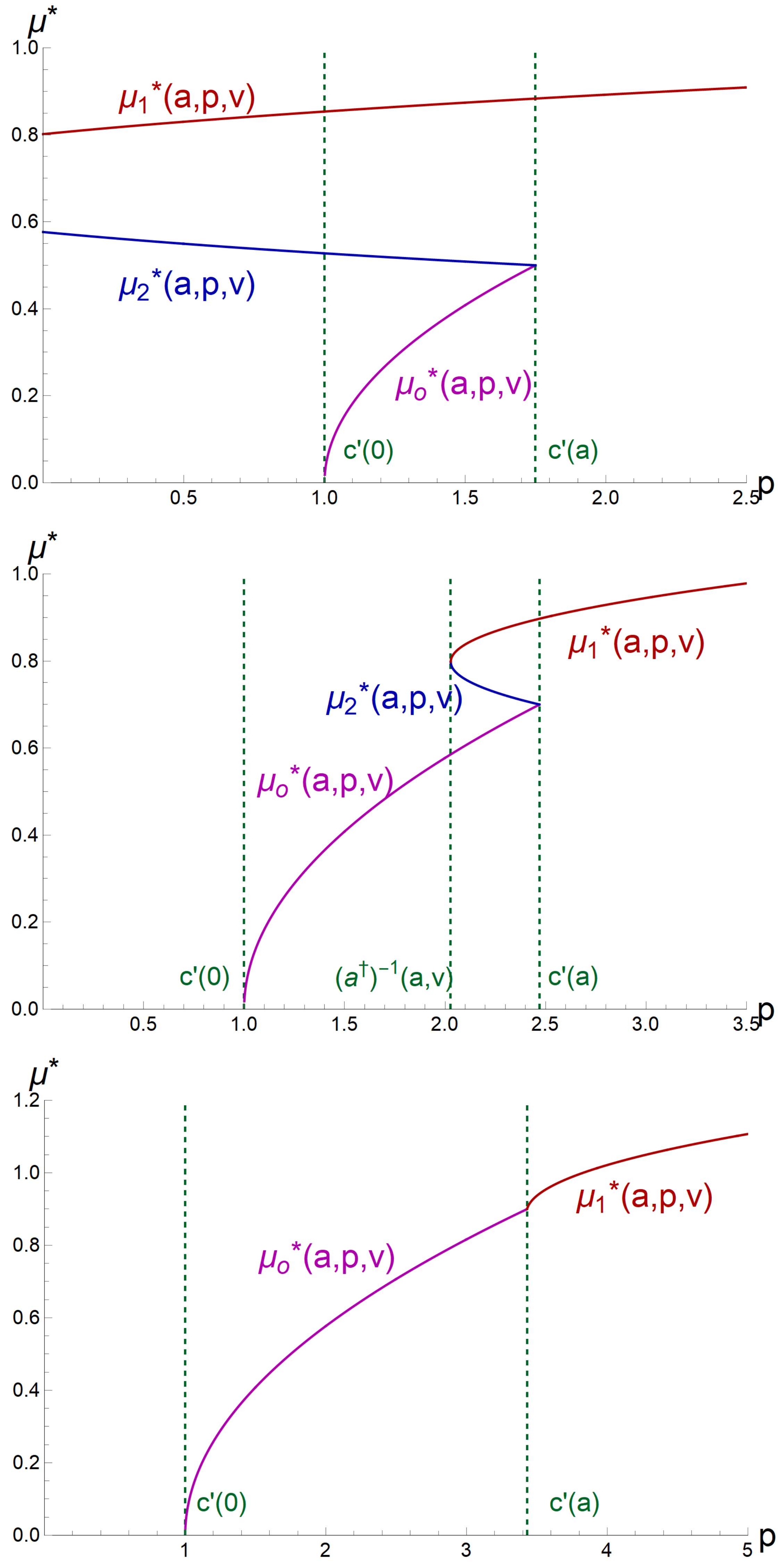
$\mu_1^\star(a, p, v)$ is strictly increasing in $p$,
and $\mu_2^\star(a, p, v)$ is strictly decreasing in $p$.

**Takeaway:** Higher wage could cause servers to speed up, and could cause servers to slow down.

$a = 0.5$

$a = 0.7$

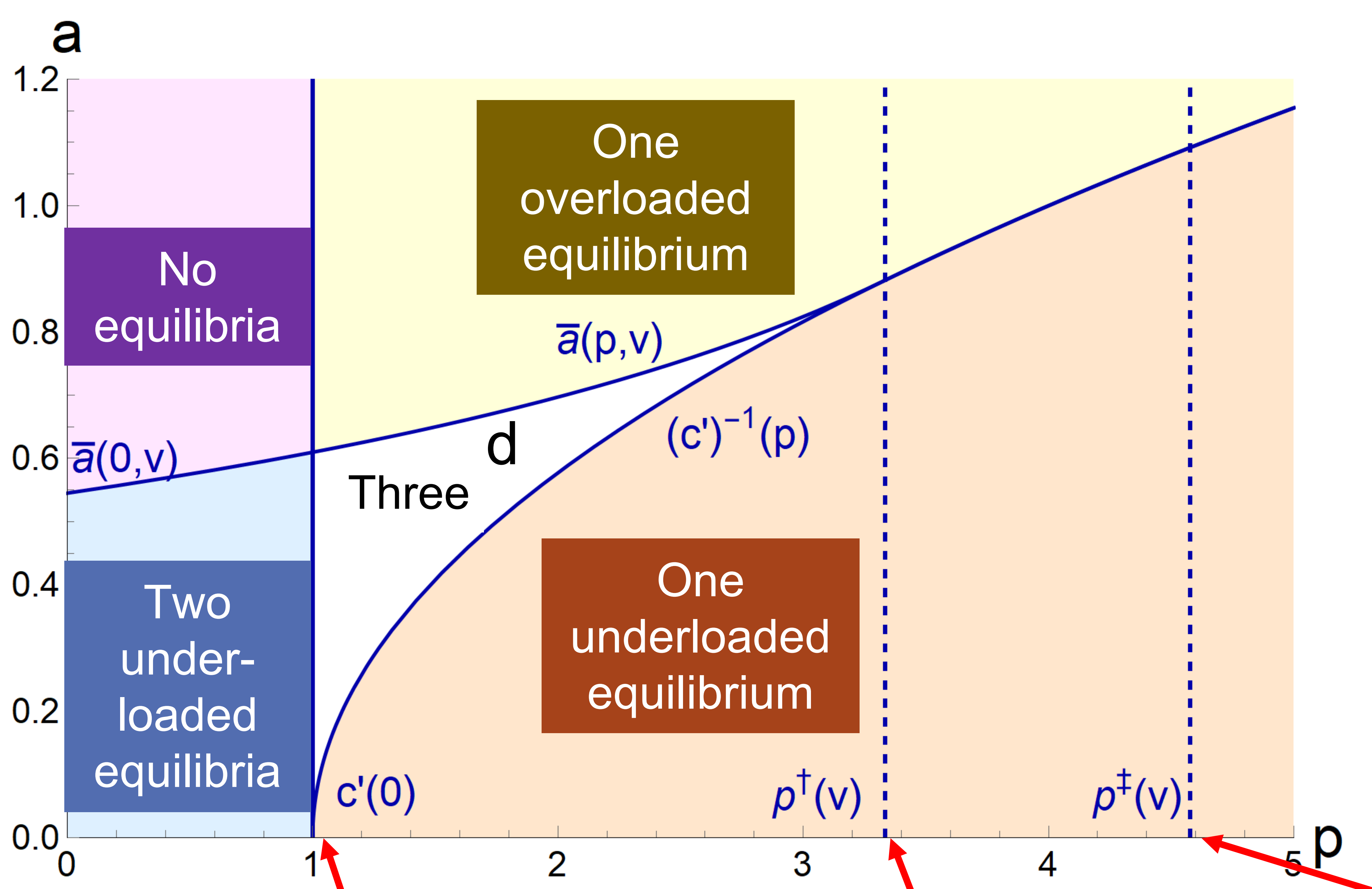
$a = 0.9$




$c(\mu) = \mu^3 + \mu$

# Asymptotic Analysis: Summary

$c(\mu) = \mu^3 + \mu$ and $v = 10$




$p > c'(0)$
guarantee existence

$p > p^{\dagger}(v)$
guarantee uniqueness

$p > p^{\ddagger}(v)$
guarantee monotonicity

**Takeaway:** More desirable equilibrium properties for all staffing levels come at increasing costs to the manager.

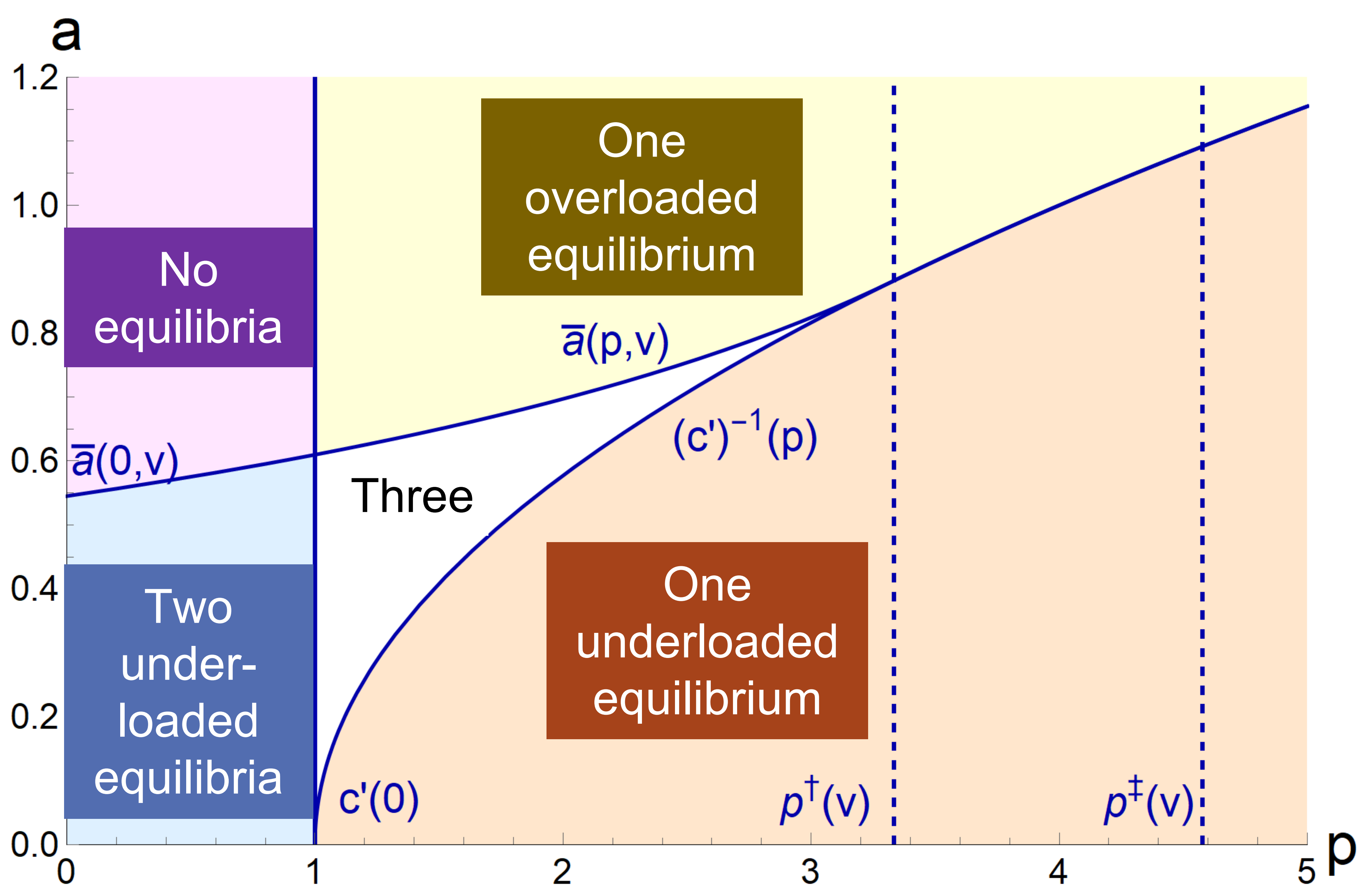
# Asymptotic Analysis: Convergence Theorem

**Theorem:** The following holds for all large enough $\lambda$:

- If $n_o = 0$, then $n_o^\lambda = 0$.
- If $n_u = 0$, then $n_u^\lambda = 0$.

- If $n_o = 1$, then $n_o^\lambda \geq 1$.
- If $n_u = 2$, then $n_u^\lambda \geq 2$.
- If $n_u = 1$, then $n_u^\lambda \geq 1$.

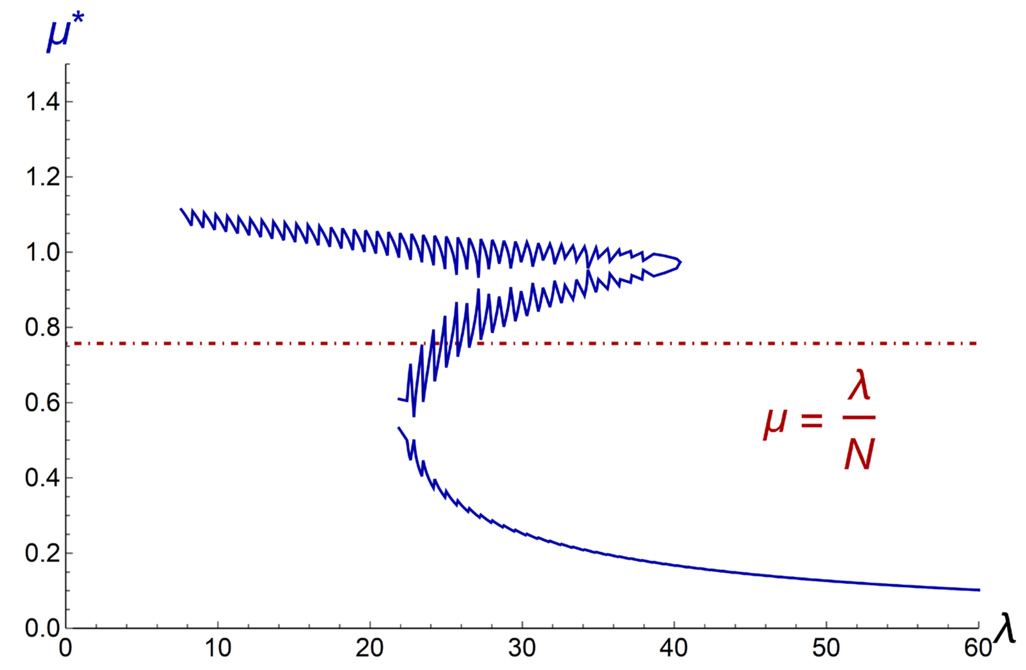
Furthermore, for any limit equilibria, there exists a prelimit equilibria that is close.



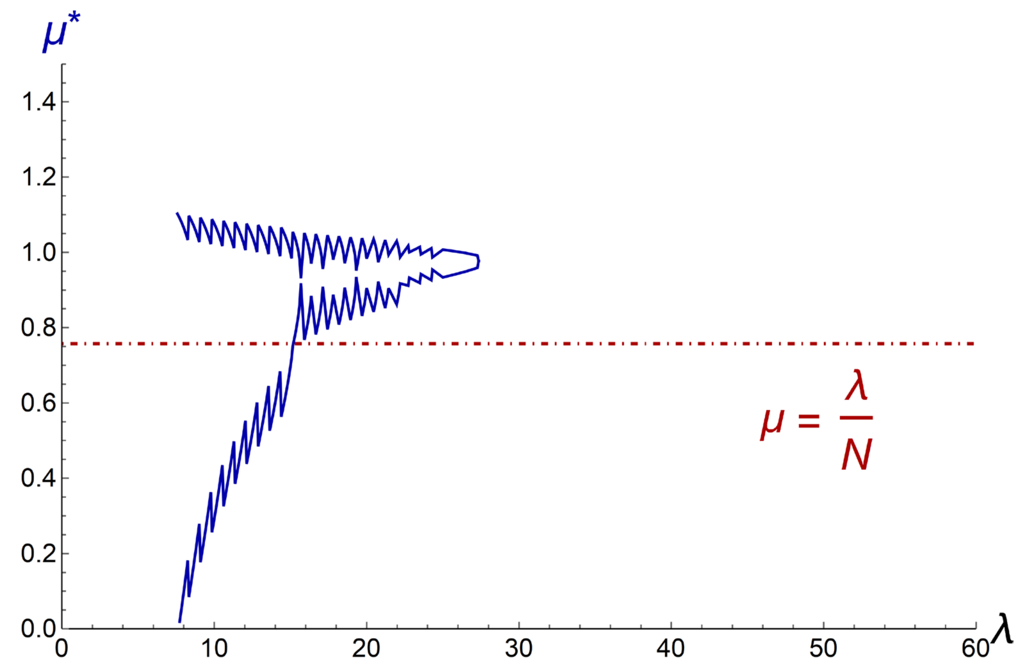
# Asymptotic Analysis: Convergence Theorem

Case: No limiting underloaded equilibrium.

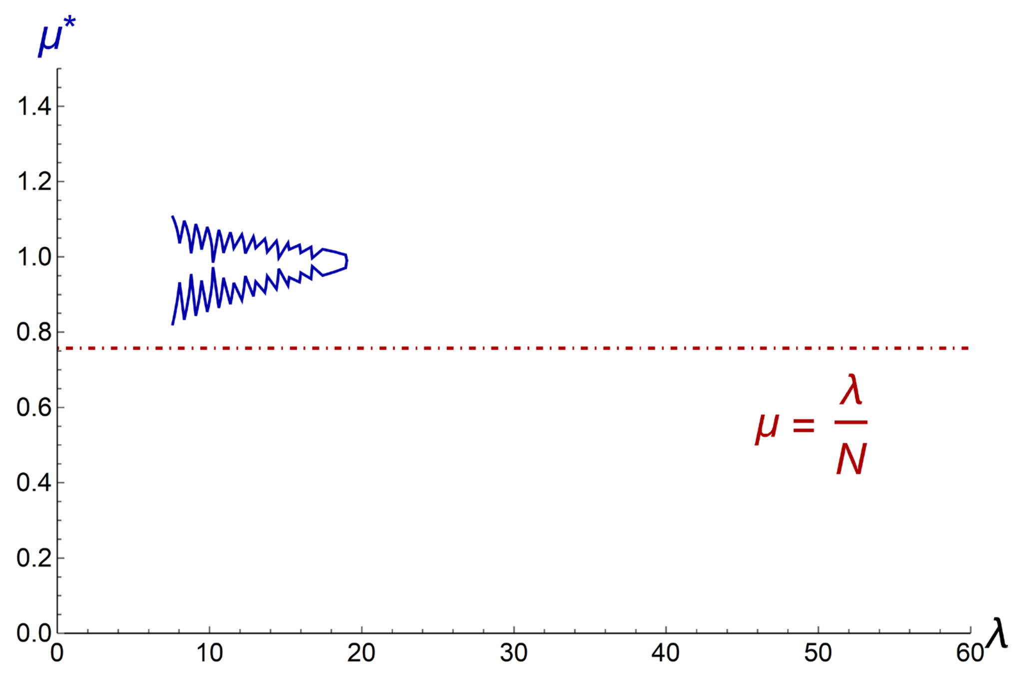
$$\boldsymbol{p = 0.5, v = 10, c(\mu) = \mu^3 + \mu}$$




$N^\lambda = \lfloor 1.4\lambda \rfloor, k^\lambda = \lfloor 1.01 N^\lambda \rfloor$

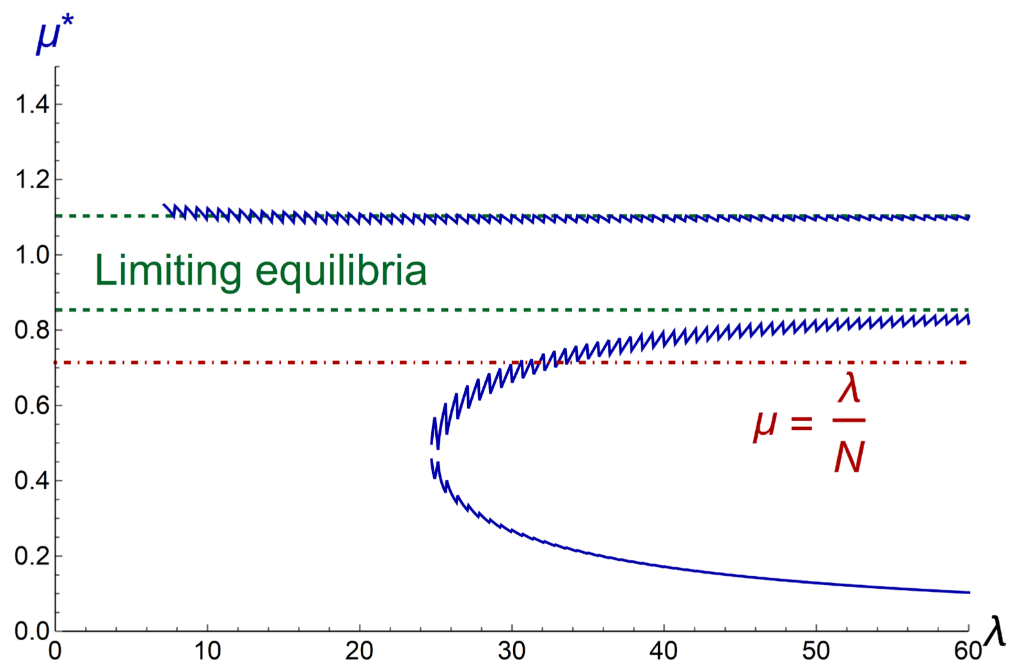



$N^\lambda = \lfloor 1.4\lambda \rfloor, k^\lambda = \lfloor 1.1 N^\lambda \rfloor$

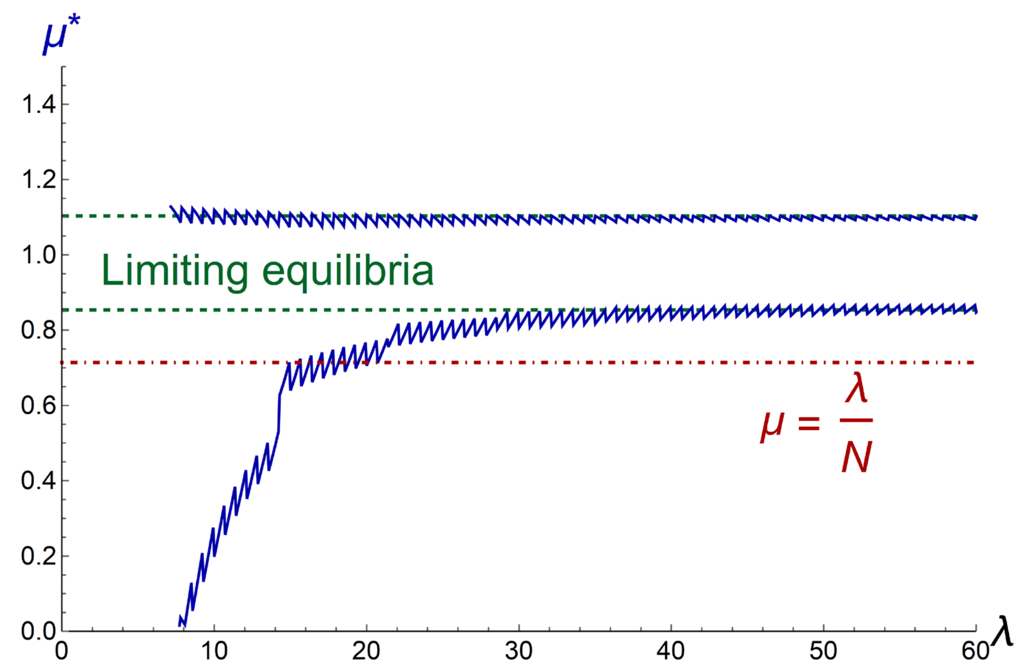



$N^\lambda = \lfloor 1.4\lambda \rfloor, k^\lambda = \lfloor 3 N^\lambda \rfloor$

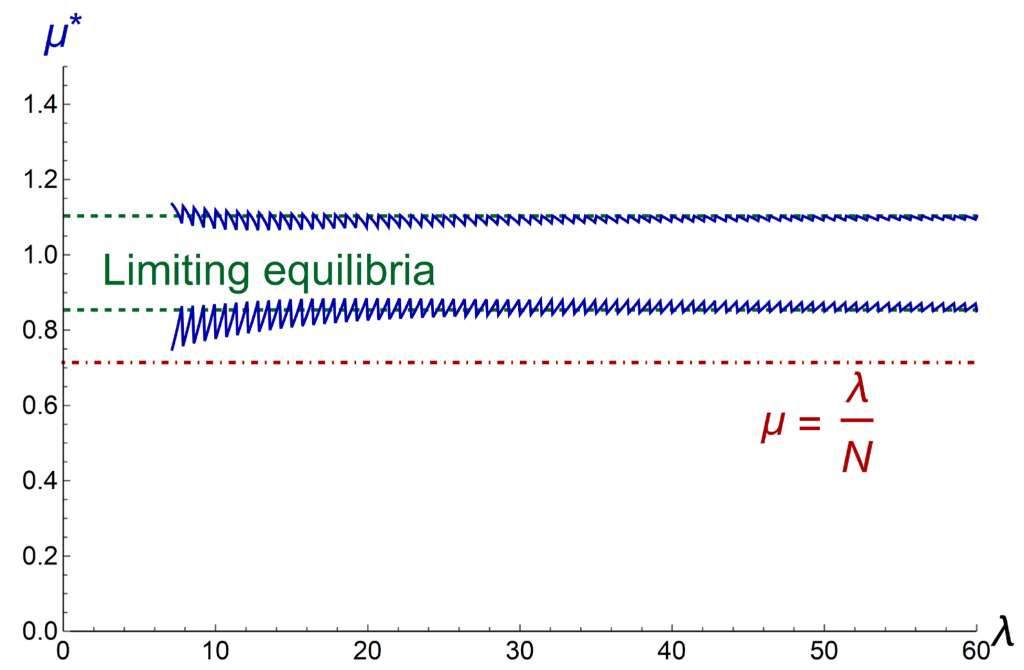
Case: Two underloaded limiting equilibria.




$N^\lambda = \lfloor 1.32\lambda \rfloor, k^\lambda = \lfloor 1.01 N^\lambda \rfloor$




$N^\lambda = \lfloor 1.32\lambda \rfloor, k^\lambda = \lfloor 1.1 N^\lambda \rfloor$




$N^\lambda = \lfloor 1.32\lambda \rfloor, k^\lambda = \lfloor 3 N^\lambda \rfloor$

**Limiting FOC:** $p\left(1 - \left[1 - \frac{a}{\mu}\right]^+\right) + (v - p\mu)\frac{a[\mu - a]^+}{\mu^3} = c'(\mu)$ $\boxed{\boldsymbol{N^\lambda = \frac{1}{a}\lambda + o(\lambda)}}$

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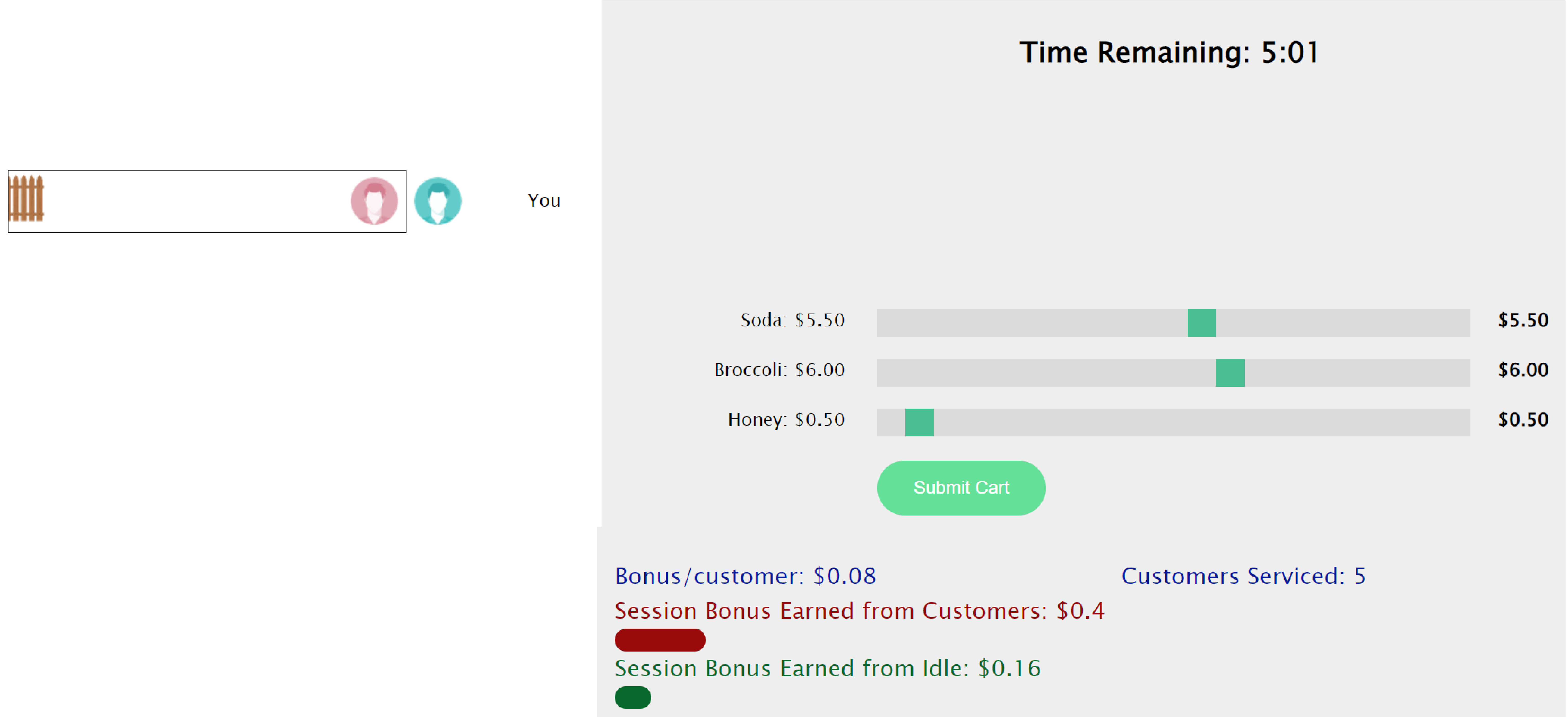
# Ongoing Work: Experiment to test hypotheses

**H1**: The servers work speed is increasing in payment.

**H2**: The servers work speed is first increasing and then decreasing, as the workload grows.

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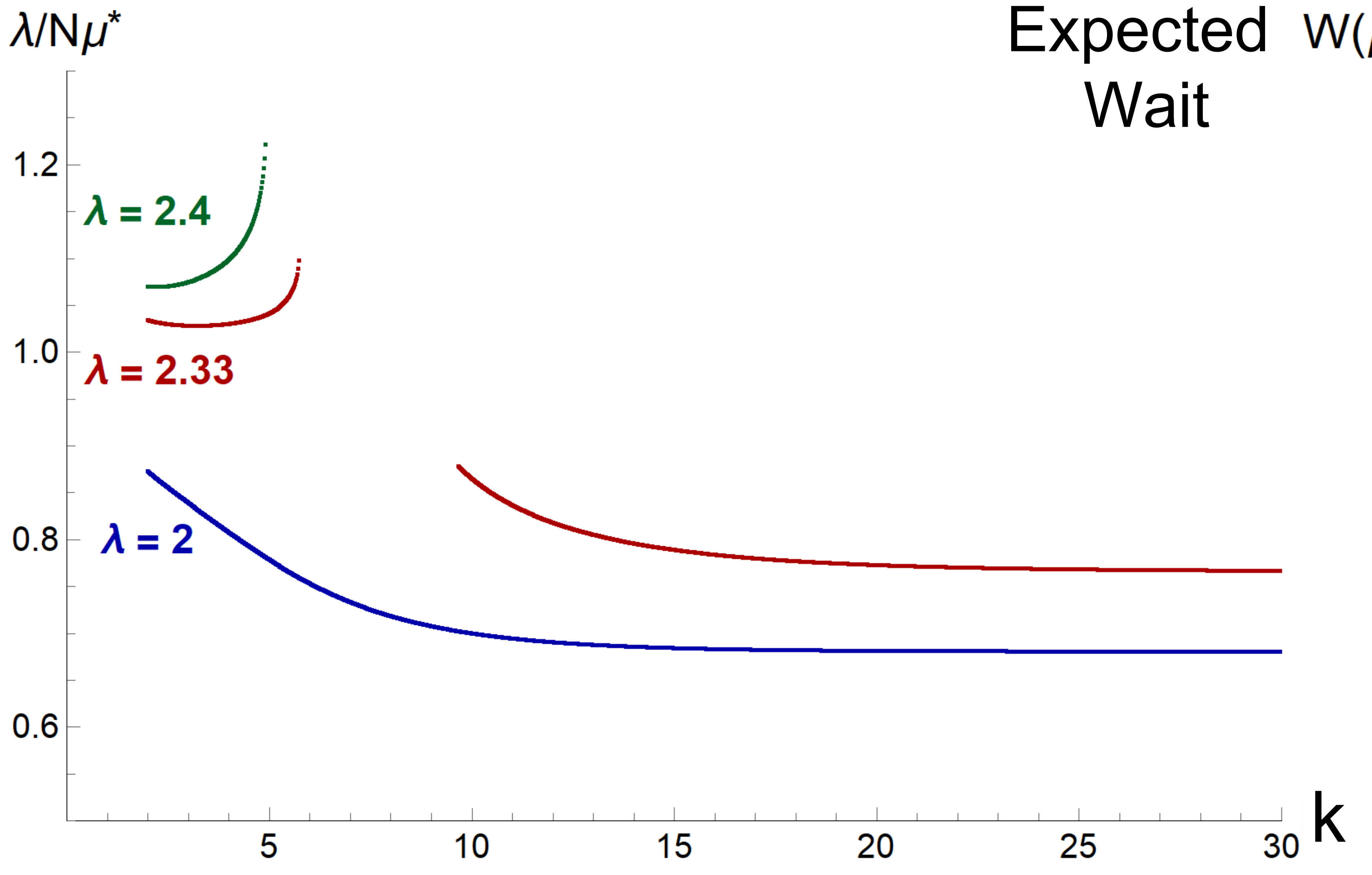
# Ongoing Work: Consequences for system design

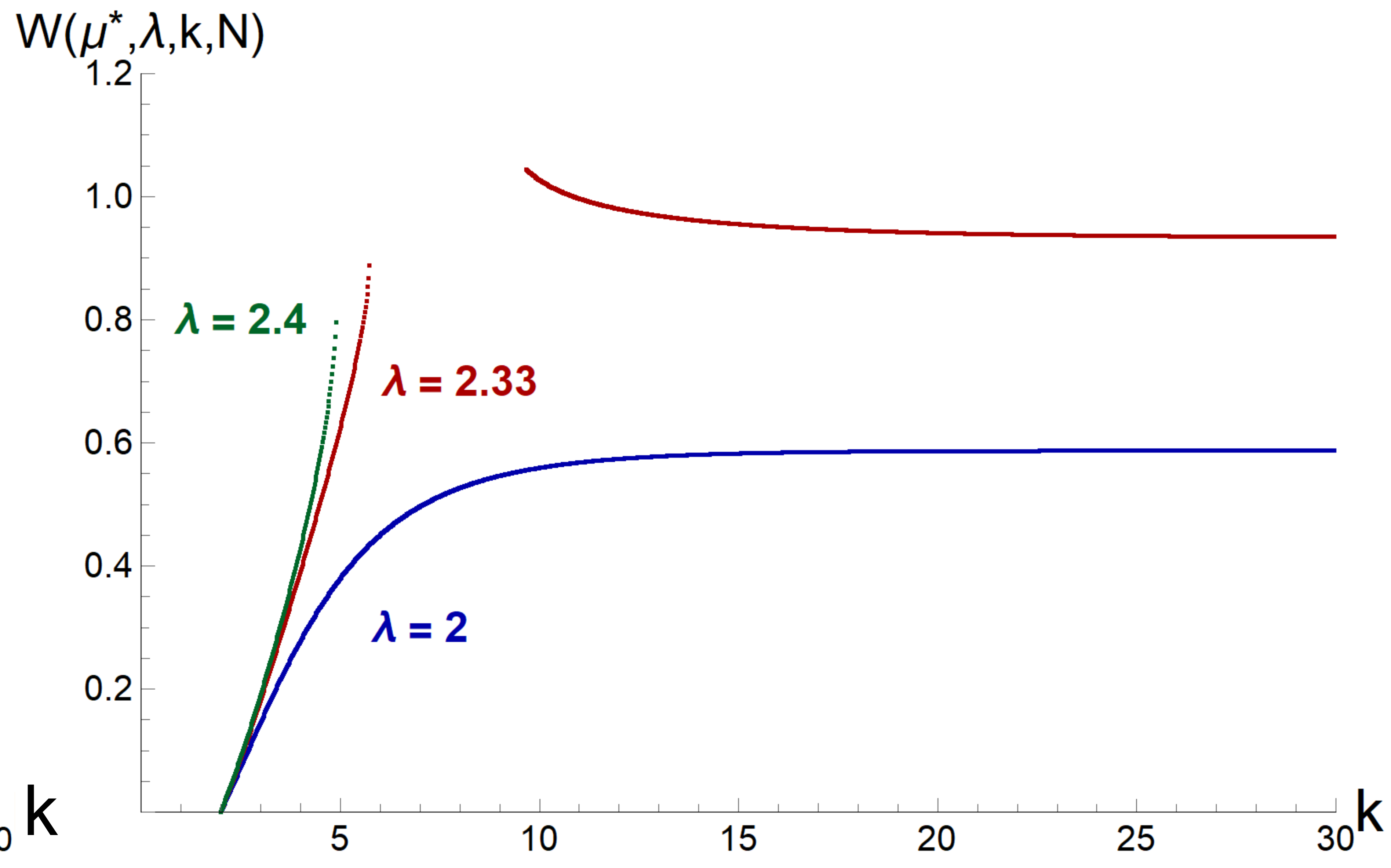
- Admission control, social welfare optimization, staffing optimization
- Queueing models that endogenize server behavior require revisiting system design “rules of thumb”.

  Ex: Does decreased load lead to decreased waiting?

System Load




Expected Wait




$$N = 2, p = 0$$
$$v = 1, c(\mu) = \frac{3}{32}\mu^2$$

# Ongoing Work: Strategic Arrival and Server Interactions

**The Customer Side:**

- A customer who arrives to find $i$ customers in the system joins if and only if $\overset{\text{Value for service}}{R} - \overset{\text{Cost of waiting}}{C} \underbrace{\left( \frac{i - N + 1}{N\mu} + \frac{1}{\mu} \right)}_{\text{Expected time in system}} \geq 0.$

- Customer equilibrium strategy is to join if and only if there are no more than $k^* - 1$ customers in the system, where $k^* = \left\lfloor \frac{RN\mu}{C} \right\rfloor$.

**Open Question: Joint Equilibrium** † **:**

$(k^*, \mu^*)$ is a Nash equilibrium if and only if $\bullet\ k^* = \left\lfloor \frac{RN\mu^*}{C} \right\rfloor \geq N$, and, $\bullet\ \mu^* \in \arg\max_{\mu_i \geq 0} U_i(\mu_i, \mu^*; \lambda, N, k^*)$.

† Only such paper is Chung, Ahn, and Righter (2020), which is restricted to $N = 1$ setting.

This paper provides foundation to study such interaction where customers' decisions endogenously induce a finite buffer.

# Summary

- Analytically studied some nontraditional server behavior documented in empirical works.
- We studied a many-strategic-server finite-buffer ($M/M/N/k$) queueing system.
- Asymptotic analysis allows us to characterize when equilibria exist, and to analyze their behavior, for large arrival rate.
- Equilibria may or may not exist, and may not be monotonic, which has consequences for system design.

# Thanks!
# Q&A

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