Behavior-Aware Queueing: The Finite-Buffer Setting with Many Strategic Servers: Technical Online Appendix

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In this technical online appendix, we provide proofs for the results related to the Erlang B and C formulae in the Electronic Companion (EC) of the manuscript titled: "Behavior-Aware Queueing: The Finite-Buffer Setting with Many Strategic Servers". Specifically, we prove Lemmas EC.1, EC.2, EC.3, and EC.13. For the reader's convenience, we reiterate the statements in this file. Throughout, we use the notation $\rho = \frac{\lambda}{\mu}$.

Lemma EC.1 **(Monotonicity of Erlang B and C)**. *The following hold:*

(a) $ErlB(N,\rho)$ *is strictly decreasing in* N *and strictly increasing in* ρ*;*

(b) $ErlC(N,\rho)$ *is strictly decreasing in* N *and strictly increasing in* ρ*.*

Lemma EC.2 **(More Properties of Erlang B and C)**. *The following hold:*

(a) $ErlC(N,\rho) = N\left(\frac{N-\rho}{ErlB(N,\rho)} + \rho\right)^{-1}$.

(b) $ErlC(N,\rho)\begin{cases} <1, & \rho<N \\ =1, & \rho=N \\ >1, & \rho>N \end{cases}$. *Moreover,* $\lim_{\rho\downarrow 0} ErlC(N,\rho)=0$ *and* $\lim_{\rho\to\infty} ErlC(N,\rho)/\rho = 1$.

(c) $\frac{1-ErlC(N,\rho)}{N-\rho} \in (0,1)$, $\forall \rho>0$.

Lemma EC.3 **(Derivative of Erlang C)**. $\frac{\partial ErlC(N,\rho)}{\partial \rho} = ErlC(N,\rho)\left(\frac{1-ErlC(N,\rho)}{N-\rho} + \frac{N-\rho}{\rho}\right)$.

EC.1. Proof of Lemma EC.1

(a) follows from Problems 4 and 6 in Whitt (2002).

(b) follows from Problem 2 in Whitt (2002).

EC.2. Proof of Lemma EC.2

(a) From (1.7) in Whitt (2002),

$$ErlB(N,\rho) = \frac{\rho ErlB(N-1,\rho)}{N+\rho B(N-1,\rho)},$$

which implies

$$\frac{1}{ErlB(N-1,\rho)} = \left(\frac{1}{ErlB(N,\rho)} - 1\right)\frac{\rho}{N}. \tag{EC.1}$$

From (2.6) in Whitt (2002),

$$ErlC(N,\rho) = \frac{\frac{\frac{\rho}{N}ErlB(N-1,\rho)}{1-\frac{\rho}{N}}}{1+\frac{\frac{\rho}{N}ErlB(N-1,\rho)}{1-\frac{\rho}{N}}} = \frac{\frac{\rho}{N}ErlB(N-1,\rho)}{1-\frac{\rho}{N}+\frac{\rho}{N}ErlB(N-1,\rho)} = \frac{\frac{\rho}{N}}{\frac{1-\frac{\rho}{N}}{ErlB(N-1,\rho)}+\frac{\rho}{N}},$$

which, by (EC.1), evaluates to

$$\frac{\frac{\rho}{N}}{\left(1-\frac{\rho}{N}\right)\left(\frac{1}{ErlB(N,\rho)}-1\right)\frac{\rho}{N}+\frac{\rho}{N}}=\frac{1}{\left(1-\frac{\rho}{N}\right)\left(\frac{1}{ErlB(N,\rho)}-1\right)+1}=\frac{N}{\frac{N-\rho}{ErlB(N,\rho)}+\rho}.$$

Hence,

$$ErlC(N,\rho)=N\left(\frac{N-\rho}{ErlB(N,\rho)}+\rho\right)^{-1}.$$

(b) Letting $\rho=N$ in the Erlang-C formula gives

$$ErlC(N,N)=\frac{\frac{N^N}{N!}\frac{N}{N-N}}{\sum_{i=0}^{N-1}\frac{N^i}{i!}+\frac{N^N}{N!}\frac{N}{N-N}}=1.$$

From Lemma EC.1(b), $ErlC(N,\rho)$ is strictly increasing in ρ. Thus, $ErlC(N,\rho)<1$ when $\rho<N$, and $ErlC(N,\rho)>1$ when $\rho>N$.

Moreover, note that $\lim_{\rho\downarrow 0} ErlB(N,\rho)=0$, then it follows from part (a) that

$$\lim_{\rho\downarrow 0} ErlC(N,\rho)=\lim_{\rho\downarrow 0}\frac{N}{\frac{N-\rho}{ErlB(N,\rho)}+\rho}=0.$$

Similarly, note that $\lim_{\rho\to\infty} ErlB(N,\rho)=1$, then it follows from part (a) that

$$\lim_{\rho\to\infty} ErlC(N,\rho)/\rho=\lim_{\rho\to\infty}\frac{N\cdot ErlB(N,\rho)}{\frac{N}{\rho}-1+ErlB(N,\rho)}=1.$$

(c) Expanding $ErlC(N,\rho)$ as finite summations yields

$$\frac{1-ErlC\,(N,\rho)}{N-\rho}=\frac{\frac{1}{N}\sum_{i=0}^{N-1}\frac{\rho^i}{i!}}{\left(1-\frac{\rho}{N}\right)\sum_{i=0}^{N-1}\frac{\rho^i}{i!}+\frac{\rho^N}{N!}}=\frac{\frac{1}{N}\sum_{i=0}^{N-1}\frac{\rho^i}{i!}}{\sum_{i=0}^{N}\frac{\rho^i}{i!}-\frac{\rho}{N}\sum_{i=0}^{N-1}\frac{\rho^i}{i!}}=\frac{\sum_{i=0}^{N-1}\frac{1}{N}\frac{\rho^i}{i!}}{\sum_{i=0}^{N-1}\left(1-\frac{i}{N}\right)\frac{\rho^i}{i!}}.$$

Since $0<\frac{1}{N}<1-\frac{i}{N}$, $\forall i<N-1$ and $\frac{1}{N}=1-\frac{i}{N}$ for $i=N-1$, it follows that $\frac{1-ErlC(N,\rho)}{N-\rho}\in(0,1)$, $\forall\rho>0$ and $\forall N\geq 2$.

EC.3. Proof of Lemma EC.3

Differentiating $ErlC(N,\rho)$ with respect to ρ yields

$$\begin{aligned}\frac{\partial ErlC(N,\rho)}{\partial\rho}&=\left(\sum_{i=0}^{N-1}\frac{\rho^i}{i!}+\frac{\rho^N}{N!}\frac{N}{N-\rho}\right)^{-2}\Bigg[\left(\frac{N\rho^{N-1}}{N!}\frac{N}{N-\rho}+\frac{\rho^N}{N!}\frac{N}{(N-\rho)^2}\right)\left(\sum_{i=0}^{N-1}\frac{\rho^i}{i!}+\frac{\rho^N}{N!}\frac{N}{N-\rho}\right)-\\&\quad\left(\frac{\rho^N}{N!}\frac{N}{N-\rho}\right)\left(\sum_{i=0}^{N-1}\frac{i\rho^{i-1}}{i!}+\frac{N\rho^{N-1}}{N!}\frac{N}{N-\rho}+\frac{\rho^N}{N!}\frac{N}{(N-\rho)^2}\right)\Bigg]\\&=\left(\sum_{i=0}^{N-1}\frac{\rho^i}{i!}+\frac{\rho^N}{N!}\frac{N}{N-\rho}\right)^{-2}\left[\left(\frac{N\rho^{N-1}}{N!}\frac{N}{N-\rho}+\frac{\rho^N}{N!}\frac{N}{(N-\rho)^2}\right)\left(\sum_{i=0}^{N-1}\frac{\rho^i}{i!}\right)-\left(\frac{\rho^N}{N!}\frac{N}{N-\rho}\right)\sum_{i=0}^{N-1}\frac{i\rho^{i-1}}{i!}\right]\\&=\frac{\frac{\rho^N}{N!}\frac{N}{N-\rho}}{\sum_{i=0}^{N-1}\frac{\rho^i}{i!}+\frac{\rho^N}{N!}\frac{N}{N-\rho}}\,\frac{\left(\frac{N}{\rho}+\frac{1}{N-\rho}\right)\left(\sum_{i=0}^{N-1}\frac{\rho^i}{i!}\right)-\sum_{i=0}^{N-1}\frac{i\rho^{i-1}}{i!}}{\sum_{i=0}^{N-1}\frac{\rho^i}{i!}+\frac{\rho^N}{N!}\frac{N}{N-\rho}}\end{aligned}$$

$$
\begin{aligned}
&= ErlC(N,\rho)\frac{\left(\frac{1}{N-\rho}\sum_{i=0}^{N-1}\frac{\rho^i}{i!}\right)+\left(\left(\frac{N}{\rho}-1\right)\sum_{i=0}^{N-1}\frac{\rho^i}{i!}+\sum_{i=0}^{N-1}\frac{\rho^i}{i!}-\sum_{i=0}^{N-1}\frac{i\rho^{i-1}}{i!}\right)}{\sum_{i=0}^{N-1}\frac{\rho^i}{i!}+\frac{\rho^N}{N!}\frac{N}{N-\rho}} \\
&= ErlC(N,\rho)\frac{\left(\frac{1}{N-\rho}\sum_{i=0}^{N-1}\frac{\rho^i}{i!}\right)+\left(\left(\frac{N}{\rho}-1\right)\sum_{i=0}^{N-1}\frac{\rho^i}{i!}+\sum_{i=0}^{N-1}\frac{\rho^i}{i!}-\sum_{i=0}^{N-1}\frac{i\rho^{i-1}}{i!}\right)}{\sum_{i=0}^{N-1}\frac{\rho^i}{i!}+\frac{\rho^N}{N!}\frac{N}{N-\rho}} \\
&= ErlC(N,\rho)\left[\frac{1}{N-\rho}\frac{\sum_{i=0}^{N-1}\frac{\rho^i}{i!}}{\sum_{i=0}^{N-1}\frac{\rho^i}{i!}+\frac{\rho^N}{N!}\frac{N}{N-\rho}}+\frac{\left(\frac{N}{\rho}-1\right)\sum_{i=0}^{N-1}\frac{\rho^i}{i!}+\frac{\rho^{N-1}}{(N-1)!}}{\sum_{i=0}^{N-1}\frac{\rho^i}{i!}+\frac{\rho^N}{N!}\frac{N}{N-\rho}}\right] \\
&= ErlC(N,\rho)\left(\frac{1-ErlC(N,\rho)}{N-\rho}+\frac{N-\rho}{\rho}\right).
\end{aligned}
$$

Thus,

$$
\frac{\partial ErlC(N,\rho)}{\partial\rho} = ErlC(N,\rho)\left(\frac{1-ErlC(N,\rho)}{N-\rho}+\frac{N-\rho}{\rho}\right).
$$

Lemma EC.13 **(Asymptotic Properties of Functions of Erlang Formulae)**. *The following hold under linear staffing (14).*

(a) If $\mu < a$, then $\lim_{\lambda\to\infty} \frac{ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right)}{\lambda} = \frac{(a-\mu)^2}{a^2\mu}$. That is, $ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right)$ converges to ∞ linearly fast as $\lambda\to\infty$.

(b) If $\mu > a$, then $\lim_{\lambda\to\infty} P(\lambda) ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right) = 0$, where $P(\lambda)$ represents any polynomial in λ. That is, $ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right)$ converges to zero super-polynomially fast as $\lambda\to\infty$.

(c) If $\left|N^\lambda - \frac{\lambda}{a}\right| \in \mathcal{O}(\sqrt{\lambda})$, then $\lim_{\lambda\to\infty} \sqrt{\lambda}\left(\frac{1-ErlC\left(N^\lambda, \frac{\lambda}{a}\right)}{N^\lambda - \frac{\lambda}{a}}\right) \in (0,\infty)$ and $\lim_{\lambda\to\infty} \sqrt{\lambda}\ ErlB\left(N^\lambda, \frac{\lambda}{a}\right) \in (0,\infty)$.

(d) $\lim_{\lambda\to\infty} \lambda\left(\frac{1-ErlC\left(N^\lambda, \frac{\lambda}{a}\right)}{N^\lambda - \frac{\lambda}{a}}\right) = \infty$.

EC.4. Proof of Lemma EC.13

(a) Using the lower and upper bounds for Erlang C given in Propositions 3 and 4 in Harel (1988) when $\frac{\lambda}{\mu} \geq N^\lambda$ are given by

$$ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right) \geq \frac{(N^\lambda)^2}{2\frac{\lambda}{\mu}}\left[\left(\frac{\lambda}{N^\lambda\mu} - 1\right)^2 + \frac{2}{N^\lambda}\frac{\lambda}{N^\lambda\mu} + \left(\frac{\lambda}{N^\lambda\mu} - 1\right)\sqrt{\left(\frac{\lambda}{N^\lambda\mu} - 1\right)^2 + \frac{4}{N^\lambda}\frac{\lambda}{N^\lambda\mu}}\right],$$

and

$$ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right) \leq \frac{(N^\lambda)^2}{2\frac{\lambda}{\mu}}\left[\left(\frac{\lambda}{N^\lambda\mu} - 1\right)^2 + \frac{2}{N^\lambda}\frac{\lambda}{N^\lambda\mu} + \left(\frac{\lambda}{N^\lambda\mu} - 1\right)\sqrt{\left(\frac{\lambda}{N^\lambda\mu} - 1\right)^2 + \frac{4}{N^\lambda}\left(\frac{\lambda}{N^\lambda\mu} + \frac{1}{N^\lambda} + 1\right)}\right].$$

When $\mu < a$, $\frac{\lambda}{\mu} > N^\lambda$ for all large enough λ. Thus, it follows that

$$\lim_{\lambda\to\infty} \frac{ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right)}{\lambda} \geq \frac{\mu}{2a^2}\left[\left(\frac{a}{\mu} - 1\right) + 0 + \left(\frac{a}{\mu} - 1\right)\sqrt{\left(\frac{a}{\mu} - 1\right)^2 + 0}\right] = \frac{\mu}{a^2}\left(\frac{a}{\mu} - 1\right)^2,$$

and

$$\lim_{\lambda\to\infty} \frac{ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right)}{\lambda} \leq \frac{\mu}{2a^2}\left[\left(\frac{a}{\mu} - 1\right)^2 + 0 + \left(\frac{a}{\mu} - 1\right)\sqrt{\left(\frac{a}{\mu} - 1\right)^2 + 0}\right] = \frac{\mu}{a^2}\left(\frac{a}{\mu} - 1\right)^2,$$

noting that $\frac{\lambda}{N^\lambda\mu} \to \frac{a}{\mu}$ as $\lambda\to\infty$.

Together the above two inequalities imply

$$\lim_{\lambda\to\infty} \frac{ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right)}{\lambda} = \frac{\mu}{a^2}\left(\frac{a}{\mu} - 1\right)^2 = \frac{(a-\mu)^2}{a^2\mu},$$

meaning that $ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right)$ converges to ∞ linearly fast as $\lambda\to\infty$.

(b) Using the upper bound for Erlang C given in (14) in Proposition 2 of Harel (2010), when $N^\lambda \geq 2$ and $\frac{\lambda}{\mu} < N^\lambda$,

$$ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right) < \left(\frac{\lambda}{N^\lambda \mu}\right)^{\sqrt{N^\lambda}}. \tag{EC.2}$$

Under linear staffing (14), $\mu > a$ implies $\frac{\lambda}{N^\lambda \mu} < 1$ for all large enough λ, which satisfies the condition for bound (EC.2). Therefore, for any polynomial $P(\lambda)$, under (14),

$$P(\lambda) ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right) < P(\lambda)\left(\frac{\lambda}{N^\lambda \mu}\right)^{\sqrt{N^\lambda}} \to 0, \quad \text{as } \lambda \to \infty,$$

because $\frac{\rho}{N^\lambda} < 1$ for large enough λ, and exponential decay dominates polynomial growth. Therefore, by non-negativity, $\lim_{\lambda\to\infty} P(\lambda) ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right) = 0$. That is, under (12), $ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right)$ asymptotically decays to zero in a super-polynomial fashion, when $\mu > a$.

If $\left|N^\lambda - \frac{\lambda}{a}\right| \in \mathcal{O}(\sqrt{\lambda})$, then $\lim_{\lambda\to\infty} \sqrt{\lambda}\left(\frac{1 - ErlC\left(N^\lambda, \frac{\lambda}{a}\right)}{N^\lambda - \frac{\lambda}{a}}\right) \in (0,\infty)$ and $\lim_{\lambda\to\infty} \sqrt{\lambda}\, ErlB\left(N^\lambda, \frac{\lambda}{a}\right) \in (0,\infty)$.

(c): We first evaluate $\lim_{\lambda\to\infty} \sqrt{\lambda}\, ErlB\left(N^\lambda, \frac{\lambda}{a}\right)$ when $|N^\lambda - \frac{\lambda}{a}| \in \mathcal{O}(\sqrt{\lambda})$. From Proposition 1 in Harel (2010), using the upper and lower bounds for Erlang B when $N^\lambda \geq 2$,

$$\begin{aligned} ErlB\left(N^\lambda, \frac{\lambda}{a}\right) &\leq \frac{-2\frac{\lambda}{N^\lambda a} - \left(N^\lambda - \frac{\lambda}{a}\right) + \sqrt{\left(N^\lambda - \frac{\lambda}{a}\right)^2 + 4\frac{\lambda}{a}}}{2\left(1 - \frac{1}{N^\lambda}\right)\frac{\lambda}{a}} \\ &= \frac{-\frac{2}{N^\lambda}\frac{\lambda}{N^\lambda a} - \left(1 - \frac{\lambda}{N^\lambda a}\right) + \sqrt{\left(1 - \frac{\lambda}{N^\lambda a}\right)^2 + \frac{4}{N^\lambda}\frac{\lambda}{N^\lambda a}}}{2\left(1 - \frac{1}{N^\lambda}\right)\frac{\lambda}{N^\lambda a}}, \end{aligned}$$

and

$$\begin{aligned} ErlB\left(N^\lambda, \frac{\lambda}{a}\right) &\geq \frac{-2 - 3\left(N^\lambda - \frac{\lambda}{a}\right) + \sqrt{\left(N^\lambda - \frac{\lambda}{a}\right)^2 + 4N^\lambda + 4\frac{\lambda}{a} + 4}}{4\frac{\lambda}{a}} \\ &= \frac{-\frac{2}{N^\lambda} - 3\left(1 - \frac{\lambda}{N^\lambda a}\right) + \sqrt{\left(1 - \frac{\lambda}{N^\lambda a}\right)^2 + \frac{4}{N^\lambda} + \frac{4}{N^\lambda}\frac{\lambda}{N^\lambda a} + \frac{4}{(N^\lambda)^2}}}{4\frac{\lambda}{N^\lambda a}}. \end{aligned}$$

Note that the denominators of the upper and lower bounds converge to 2 and 4, respectively, as $\lambda \to \infty$. It suffices to investigate the numerators of the upper and lower bounds multiplied by $\sqrt{\lambda}$.

Multiplying the numerator of the upper bound for $ErlB$ by $\sqrt{\lambda}$ yields

$$\begin{aligned} &\sqrt{\lambda}\left(-\frac{2}{N^\lambda}\frac{\lambda}{N^\lambda a} + \left(\frac{\lambda}{N^\lambda a} - 1\right) + \sqrt{\left(\frac{\lambda}{N^\lambda a} - 1\right)^2 + \frac{4}{N^\lambda}\frac{\lambda}{N^\lambda a}}\right) \\ &= -\frac{2\frac{\lambda}{a}\sqrt{\lambda}}{\left(\frac{\lambda}{a} + \mathcal{O}(\sqrt{\lambda})\right)^2} - \frac{\mathcal{O}(\sqrt{\lambda})\sqrt{\lambda}}{\frac{\lambda}{a} + \mathcal{O}(\sqrt{\lambda})} + \sqrt{\frac{(\mathcal{O}(\sqrt{\lambda}))^2 + 4\frac{\lambda}{a}}{\left(\frac{\lambda}{a} + \mathcal{O}(\sqrt{\lambda})\right)^2}}\sqrt{\lambda} \\ &= -\frac{\frac{1}{a}\frac{2}{\sqrt{\lambda}}}{\left(\frac{1}{a} + \frac{\mathcal{O}(\sqrt{\lambda})}{\lambda}\right)^2} - \frac{\frac{\mathcal{O}(\sqrt{\lambda})}{\sqrt{\lambda}}}{\frac{1}{a} + \frac{\mathcal{O}(\sqrt{\lambda})}{\lambda}} + \sqrt{\frac{\left(\frac{\mathcal{O}(\sqrt{\lambda})}{\sqrt{\lambda}}\right)^2 + \frac{4}{a}}{\left(\frac{1}{a} + \frac{\mathcal{O}(\sqrt{\lambda})}{\lambda}\right)^2}} \in (0,\infty), \end{aligned}$$

where the limit of the first term is 0, the limit of the second and third terms both lie in $[0,\infty)$, and one can easily see that they cannot take value 0 at the same time. Thus, the upper bound for $\lim_{\lambda\to\infty}\sqrt{\lambda}\ ErlB\left(N^\lambda,\frac{\lambda}{a}\right)$ lies in $(0,\infty)$.

Similarly, we multiply the numerator of the lower bound for $ErlB$ by $\sqrt{\lambda}$ and get

$$\begin{aligned}
&\sqrt{\lambda}\cdot\left(-\frac{2}{N^\lambda}-3\left(1-\frac{\lambda}{N^\lambda a}\right)+\sqrt{\left(1-\frac{\lambda}{N^\lambda a}\right)^2+\frac{4}{N^\lambda}+\frac{4}{N^\lambda}\frac{\lambda}{N^\lambda a}+\frac{4}{(N^\lambda)^2}}\right)\\
&=-\frac{2\sqrt{\lambda}}{\frac{\lambda}{a}+\mathcal{O}(\sqrt{\lambda})}-\frac{3\sqrt{\lambda}\ \mathcal{O}(\sqrt{\lambda})}{\frac{\lambda}{a}+\mathcal{O}(\sqrt{\lambda})}+\sqrt{\frac{(\mathcal{O}(\sqrt{\lambda}))^2+4\left(\frac{\lambda}{a}+\mathcal{O}(\sqrt{\lambda})\right)+4\frac{\lambda}{a}+4}{\left(\frac{\lambda}{a}+\mathcal{O}(\sqrt{\lambda})\right)^2}}\sqrt{\lambda}\\
&=-\frac{2}{\frac{\sqrt{\lambda}}{a}+\frac{\mathcal{O}(\sqrt{\lambda})}{\sqrt{\lambda}}}-\frac{3\ \frac{\mathcal{O}(\sqrt{\lambda})}{\sqrt{\lambda}}}{\frac{1}{a}+\frac{\mathcal{O}(\sqrt{\lambda})}{\lambda}}+\sqrt{\frac{\left(\frac{\mathcal{O}(\sqrt{\lambda})}{\sqrt{\lambda}}\right)^2+4\left(\frac{1}{a}+\frac{\mathcal{O}(\sqrt{\lambda})}{\lambda}\right)+\frac{4}{a}+\frac{4}{\lambda}}{\left(\frac{1}{a}+\frac{\mathcal{O}(\sqrt{\lambda})}{\lambda}\right)^2}}\in(0,\infty),
\end{aligned}$$

where the limit of the first term is 0, the limit of the second and third terms both lie in $[0,\infty)$, and one can easily see that they cannot take value 0 at the same time. Thus, the lower bound for $\lim_{\lambda\to\infty}\sqrt{\lambda}\ ErlB\left(N^\lambda,\frac{\lambda}{a}\right)$ lies in $(0,\infty)$.

Combining the upper and lower bounds for $\lim_{\lambda\to\infty}\sqrt{\lambda}\ ErlB\left(N^\lambda,\frac{\lambda}{a}\right)$ yields

$$\lim_{\lambda\to\infty}\sqrt{\lambda}\ ErlB\left(N^\lambda,\frac{\lambda}{a}\right)\in(0,\infty).$$

Now, using the relationship between the Erlang-B and C formulae (from Lemma EC.2 (a)):

$$\sqrt{\lambda}\left(\frac{1-ErlC(N^\lambda,\frac{\lambda}{a})}{N^\lambda-\frac{\lambda}{a}}\right)=\sqrt{\lambda}\frac{1-\frac{N^\lambda}{\frac{N^\lambda-\frac{\lambda}{a}}{ErlB(N^\lambda,\frac{\lambda}{a})}+\frac{\lambda}{a}}}{N^\lambda-\frac{\lambda}{a}}=\frac{\sqrt{\lambda}(1-ErlB(N^\lambda,\frac{\lambda}{a}))}{N^\lambda-\frac{\lambda}{a}+\frac{\lambda}{a}ErlB(N^\lambda,\frac{\lambda}{a})}=\frac{1-ErlB(N^\lambda,\frac{\lambda}{a})}{\frac{N^\lambda-\frac{\lambda}{a}}{\sqrt{\lambda}}+\frac{\sqrt{\lambda}}{a}ErlB(N^\lambda,\frac{\lambda}{a})}.$$

Note that $\lim_{\lambda\to\infty}ErlB(N^\lambda,\frac{\lambda}{a})=0$ (from Lemma EC.12 (a)), and $\lim_{\lambda\to\infty}\sqrt{\lambda}ErlB(N^\lambda,\frac{\lambda}{a})\in(0,\infty)$ when $|N^\lambda-\frac{\lambda}{a}|=\mathcal{O}(\sqrt{\lambda})$ (as shown in the first part of the proof). Hence, $\lim_{\lambda\to\infty}\sqrt{\lambda}\left(\frac{1-ErlC(N^\lambda,\frac{\lambda}{a})}{N^\lambda-\frac{\lambda}{a}}\right)\in(0,\infty)$.

(d) Note that

$$\begin{aligned}
\lim_{\lambda\to\infty}\lambda\left(\frac{1-ErlC(N^\lambda.\frac{\lambda}{a})}{N^\lambda-\frac{\lambda}{a}}\right)&\overset{(i)}{=}\lim_{\lambda\to\infty}\lambda\frac{1-\frac{N^\lambda}{\frac{N^\lambda-\frac{\lambda}{a}}{ErlB(N^\lambda,\frac{\lambda}{a})}+\frac{\lambda}{a}}}{N^\lambda-\frac{\lambda}{a}}=\lim_{\lambda\to\infty}\frac{\lambda(1-ErlB(N^\lambda,\frac{\lambda}{a}))}{N^\lambda-\frac{\lambda}{a}+\frac{\lambda}{a}ErlB(N^\lambda,\frac{\lambda}{a})}\\
&=\lim_{\lambda\to\infty}\frac{1-ErlB(N^\lambda,\frac{\lambda}{a})}{\frac{N^\lambda-\frac{\lambda}{a}}{\lambda}+\frac{1}{a}ErlB(N^\lambda,\frac{\lambda}{a})}\overset{(ii)}{=}\infty,
\end{aligned}$$

where (i) follows from the relationship between the Erlang-B and C formulae (from Lemma EC.2 (a)), (ii) follows by the fact that $\lim_{\lambda\to\infty}ErlB(N^\lambda,\frac{\lambda}{a})=0$ (from Lemma EC.12 (a)).

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