Unions of Inverse Limits on Set-Valued Functions

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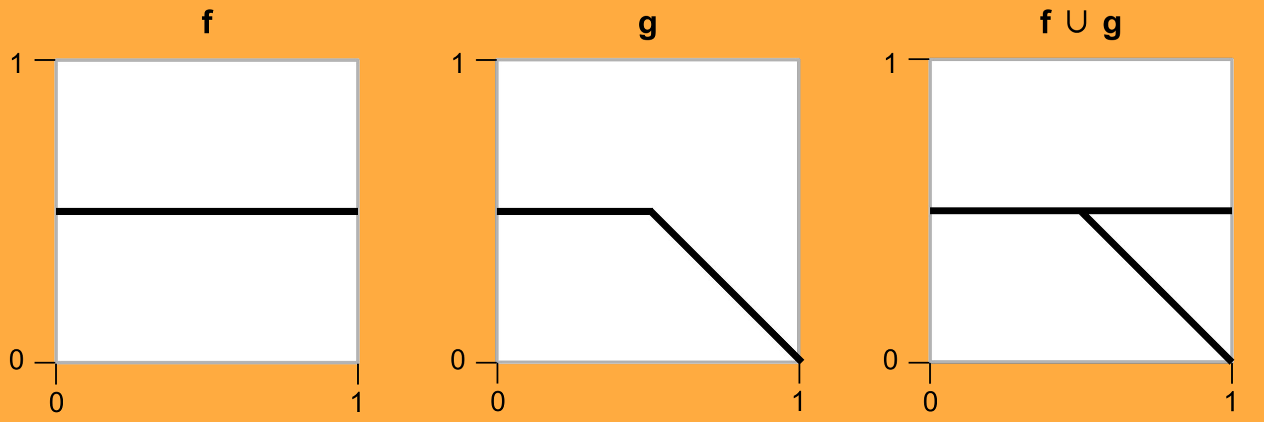
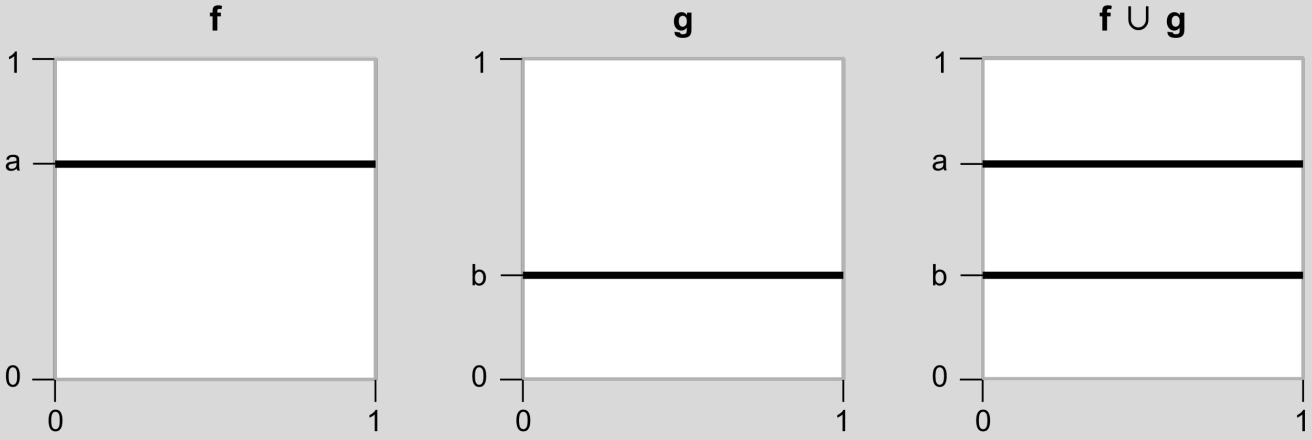
- We know that $\varprojlim \mathbf{f} \cup \varprojlim \mathbf{g} \subseteq \varprojlim(\mathbf{f} \cup \mathbf{g})$
- So when does $\varprojlim \mathbf{f} \cup \varprojlim \mathbf{g} = \varprojlim(\mathbf{f} \cup \mathbf{g})$?



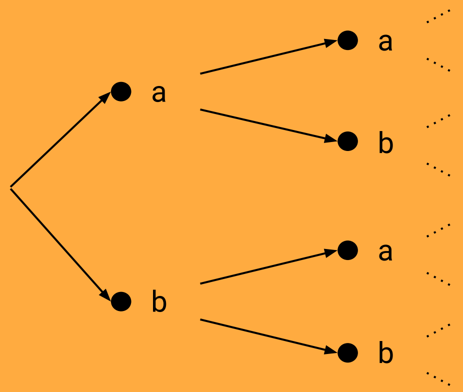
$$
\begin{aligned}
\varprojlim \mathbf{f} &= \varprojlim \mathbf{g} \\
&= \varprojlim \mathbf{f} \cup \varprojlim \mathbf{g} \\
&= \varprojlim(\mathbf{f} \cup \mathbf{g}) \\
&= \{(0.5, 0.5, 0.5, \dots)\}
\end{aligned}
$$



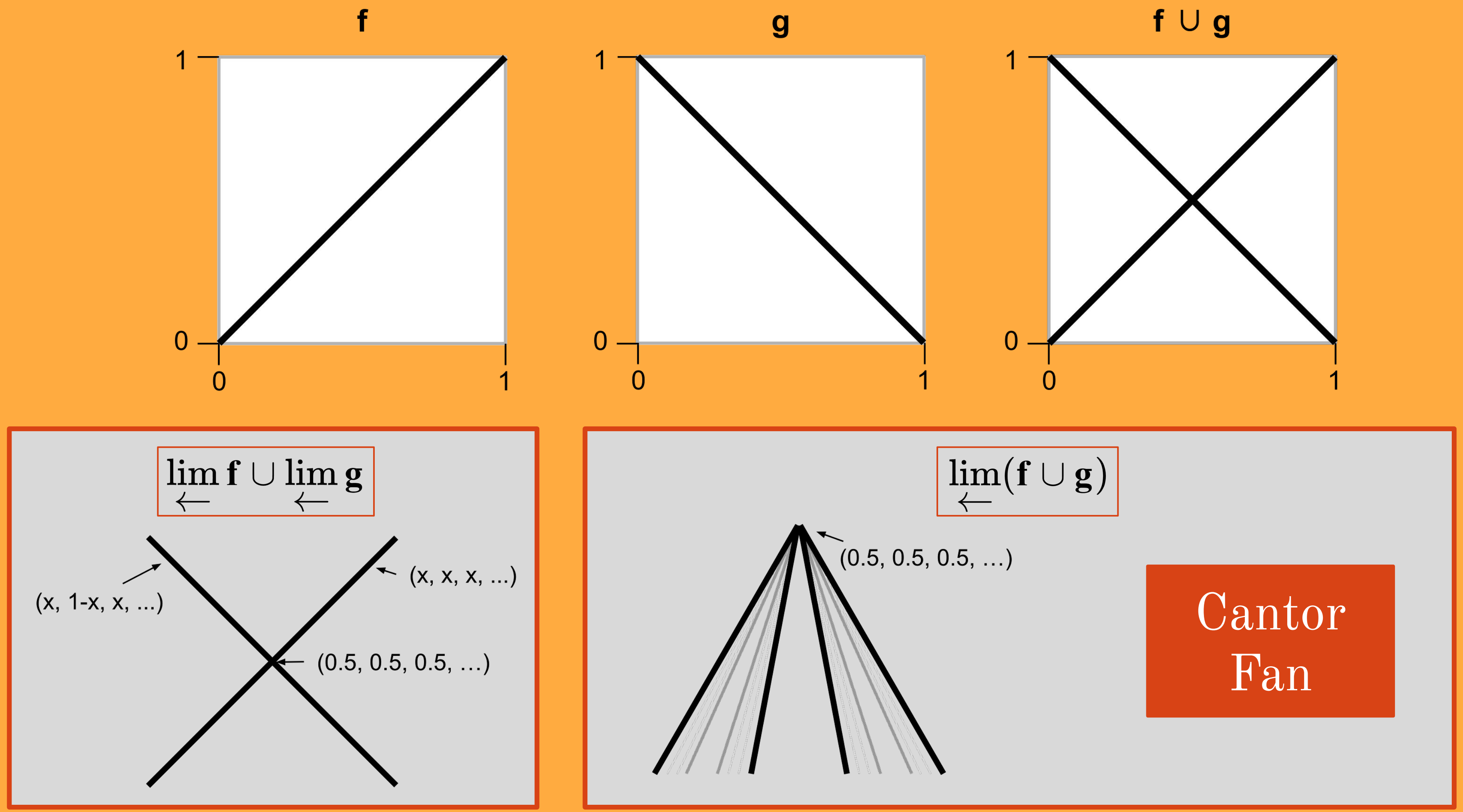
$$\varprojlim \mathbf{f} = \{(a, a, a, \dots)\}$$

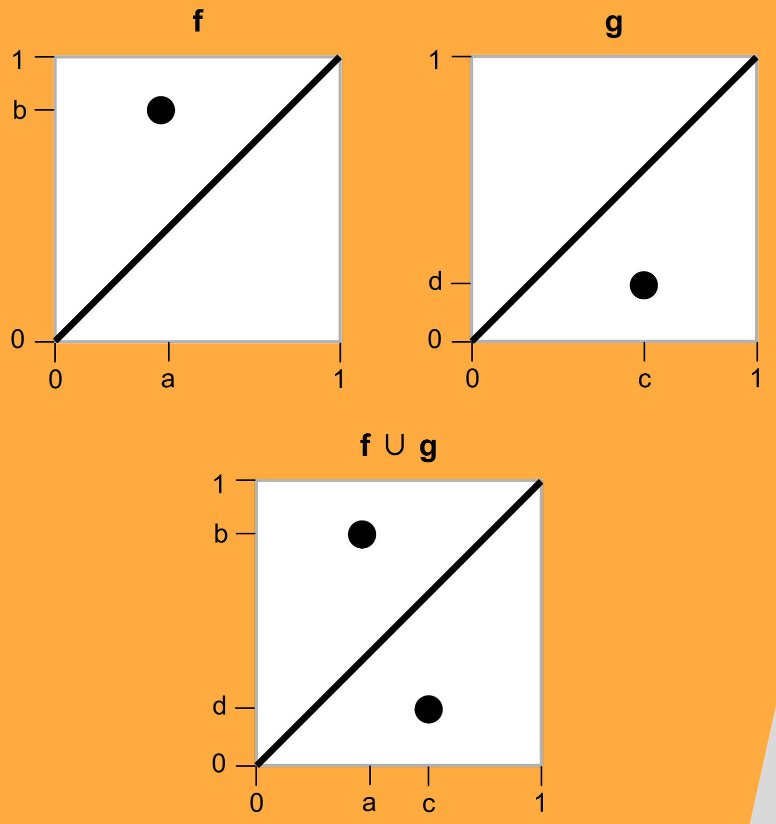
$$\varprojlim \mathbf{g} = \{(b, b, b, \dots)\}$$

$$\varprojlim \mathbf{f} \cup \mathbf{g} =$$

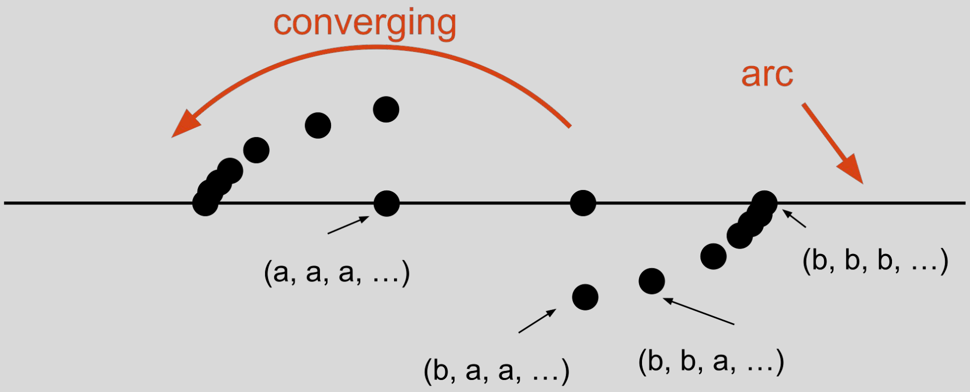


Cantor set


f
g
f ∪ g
1
0
0
1
$\varprojlim \mathbf{f} \cup \varprojlim \mathbf{g}$
(x, 1-x, x, ...)
(x, x, x, ...)
(0.5, 0.5, 0.5, …)
$\varprojlim(\mathbf{f} \cup \mathbf{g})$
(0.5, 0.5, 0.5, …)
Cantor Fan



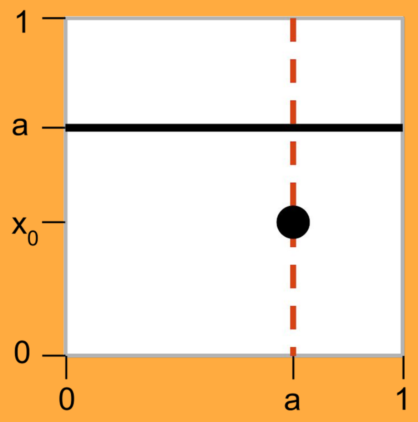
$$\varprojlim \mathbf{f} \cup \varprojlim \mathbf{g} = \varprojlim(\mathbf{f} \cup \mathbf{g})$$

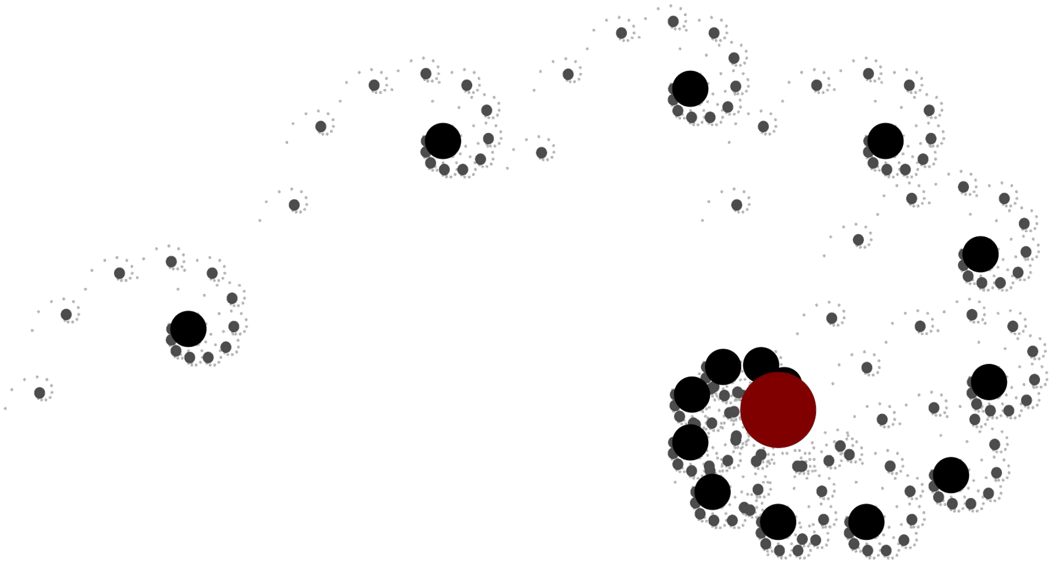


The inverse limit of the union of an onto function and any (non-empty) partial or total function $\mathbf{g}$ such that $\mathbf{f} \cap \mathbf{g} = \varnothing$ must be a strict superset of the union of inverse limits of the separate functions.

Consider the line y = a

- What happens if we add one point?





$$\{(a, a, \dots)\} \cup$$

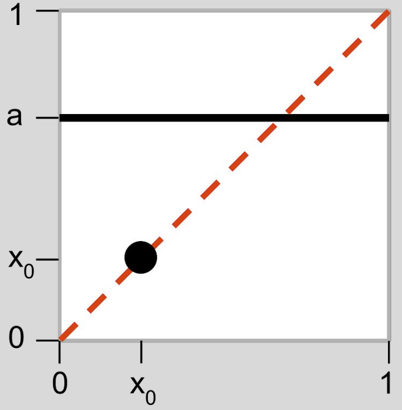
$$\{(\underbrace{a, a, \dots, a}_{n_0 \text{ times}}, x_0, a, a, \dots) \mid n_0 \in \mathbb{N}_0\} \cup$$

$$\{(\underbrace{a, a, \dots, a}_{n_0 \text{ times}}, x_0, \underbrace{a, a, \dots, a}_{n_1 \text{ times}}, x_0, a, a, \dots) \mid n_0 \in \mathbb{N}_0, n_1 \in \mathbb{N}\} \cup$$

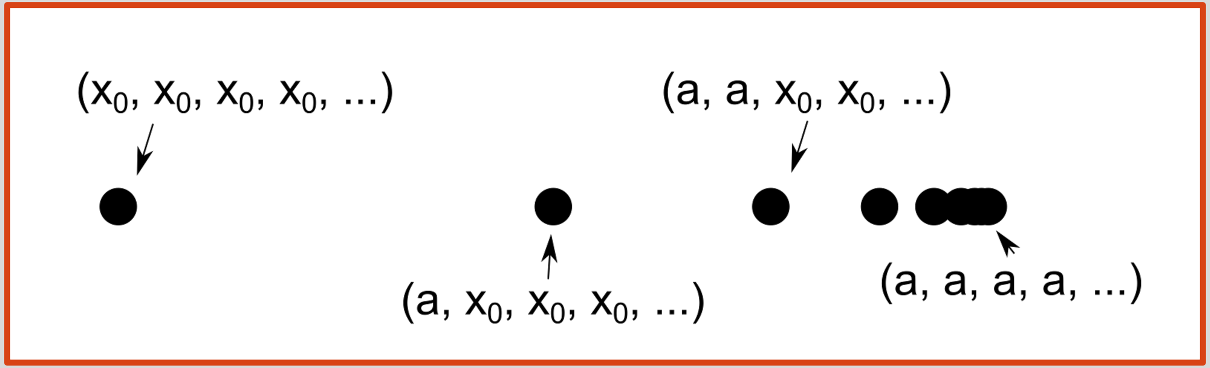
$$\{(\underbrace{a, a, \dots, a}_{n_0 \text{ times}}, x_0, \underbrace{a, a, \dots, a}_{n_1 \text{ times}}, x_0, \underbrace{a, a, \dots, a}_{n_2 \text{ times}}, x_0, a, a, \dots) \mid n_0 \in \mathbb{N}_0; n_1, n_2 \in \mathbb{N}\} \cup \dots$$

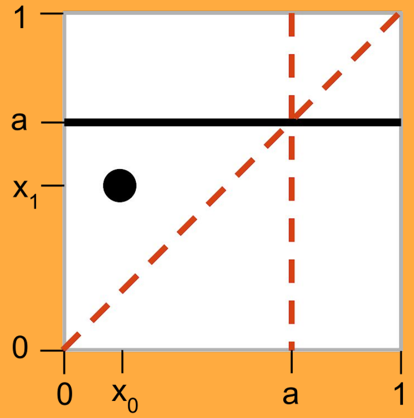
Consider the line y = a

- What happens if we add one point?



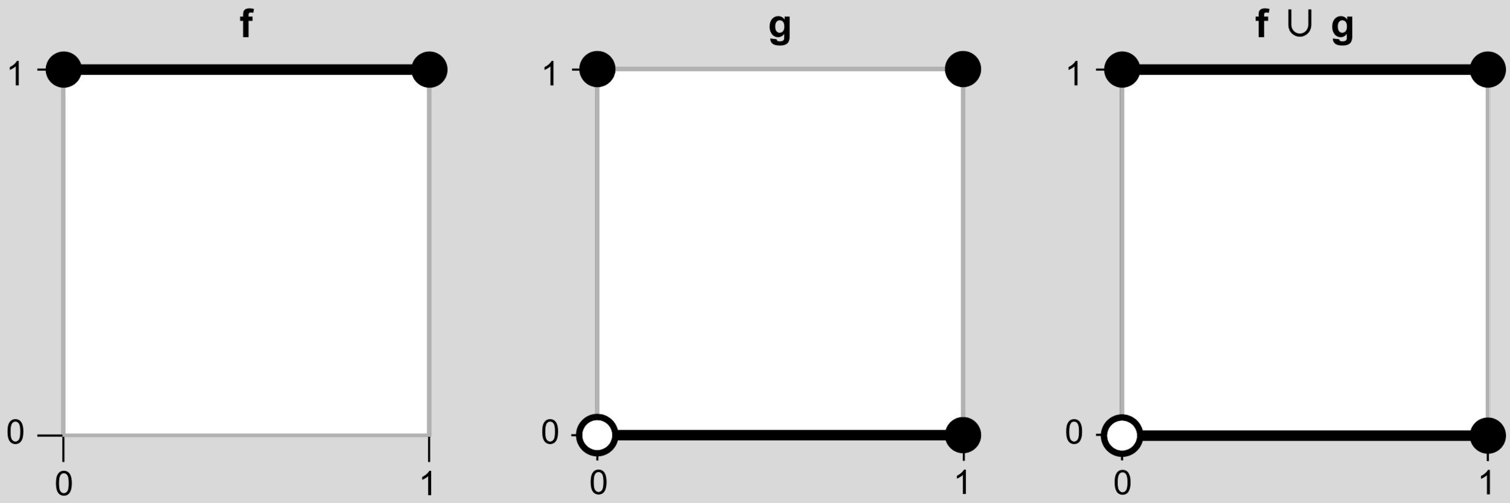
$$\{(a, a, a, \ldots)\} \cup \{(\underbrace{a, a, \ldots, a}_{n \text{ times}}, x_0, x_0, \ldots) \mid n \in \mathbb{N}_0\}$$





$$\{(a, a, a \ldots)\}$$

- So it would seem that, in order for $\underset{\leftarrow}{\lim}(\mathbf{f} \cup \mathbf{g})$ to equal $\underset{\leftarrow}{\lim}\,\mathbf{f} \cup \underset{\leftarrow}{\lim}\,\mathbf{g}$, $\mathbf{g}$ must not intersect the lines $y = x$ or $x = a$, except at the point (a, a)
- Suppose $a = 1$



Consider the function $\mathbf{f} : x \mapsto \{a\}$.
Suppose $\mathbf{g}$ is a (partial) function such that $f \cap g = \varnothing$. If $\mathbf{g}$ intersects the lines $x = a$ or $y = x$, then $\underset{\leftarrow}{\lim}\,\mathbf{f} \cup \underset{\leftarrow}{\lim}\,\mathbf{g} \neq \underset{\leftarrow}{\lim}(\mathbf{f} \cup \mathbf{g})$

Thank you!