UNIONS OF INVERSE LIMITS OF SET-VALUED FUNCTIONS

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Abstract

I have found that the inverse limit of the union of an onto function $\mathbf{f}$ and any (non-empty) partial or total function $\mathbf{g}$ such that $\mathbf{f} \cap \mathbf{g} = \emptyset$ must be a strict superset of the union of inverse limits of the separate functions. It is also the case that for a function $\mathbf{f} : [0,1] \to [0,1]$ such that $\mathbf{f} : x \mapsto \{a\}$, if $\mathbf{g}$ is any partial or total function with $\mathbf{f} \cap \mathbf{g} = \emptyset$ and $\varprojlim(\mathbf{f} \cup \mathbf{g}) = \varprojlim \mathbf{f} \cup \varprojlim \mathbf{g}$ then $\mathbf{g}$ cannot intersect the lines $x = a$ or $y = x$ at any point other than (a, a).

Aims

I wish to investigate the inverse limit of the unions of functions. In particular, I want to consider conditions for which the inverse limit of the union of functions is equal to the union of the inverse limits.

Definitions

We say that $S = (x_0, x_1, x_2, ...)$ is a member of the inverse limit of $\mathbf{f}$ if $x_i \in \mathbf{f}(x_{i+1})$ for $i = 0, 1, 2, ...$; we denote this as $S \in \varprojlim \mathbf{f}$.

Given two set-valued functions $\mathbf{f} : X \to \mathcal{P}(X)$ and $\mathbf{g} : X \to \mathcal{P}(X)$, we define operations on sets in the usual way. For example, we define $\mathbf{f} \cup \mathbf{g}$ as follows: $\forall x \in X, (\mathbf{f} \cup \mathbf{g})(x) := \mathbf{f}(x) \cup \mathbf{g}(x)$

We define a partial function to be a (set-valued) function whose domain is a subset of the set on which it is formally defined. For example, $\mathbf{f} : X \to \mathcal{P}(X)$ where $\mathbf{f}(x)$ is defined only for $x \in X'$ where $X' \subseteq X$. If $X' = X$, we call this a total function.

Given a function $\mathbf{f} : X \to \mathcal{P}(X)$, we define the graph to be $\{(x, y) \,|\, y \in \mathbf{f}(x)\}$ and denote this $G(\mathbf{f})$.

We use the notation $\mathbb{N} := \{1, 2, 3, ...\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, ...\}$. We use $\supset$ to denote a strict superset; $A \supset B$ iff $A \neq B$ and $\forall x$, $x \in B \Rightarrow x \in A$. We use $\supseteq$ to denote a subset; $A \supseteq B$ iff $\forall x$, $x \in B \Rightarrow x \in A$.

Methods

It is intuitive that when we take the union of two functions, the inverse limit of the resulting function is a superset of the union of the inverse limits of the two functions.

This suggests that we consider under what circumstances the inverse limit of the union is *exactly* the union of the inverse limits. Formally:

$\varprojlim \mathbf{f} \cup \varprojlim \mathbf{g} = \varprojlim(\mathbf{f} \cup \mathbf{g})$

In *Figure 1*, we have $\varprojlim \mathbf{f} = \varprojlim \mathbf{g} = \varprojlim(\mathbf{f} \cup \mathbf{g}) = \varprojlim \mathbf{f} \cup \varprojlim \mathbf{g} = \{(0.5, 0.5, 0.5, ...)\}$; the inverse limit of the union of $\mathbf{f}$ and $\mathbf{g}$ is the same as the union of the inverse limits. This simply looks like one point in Hilbert space.

However, in *Figure 2*, we have $\mathbf{f} : x \mapsto \{a\}$ with $\varprojlim \mathbf{f} = \{(a, a, a, ...)\}$ and $\mathbf{g} : x \mapsto \{b\}$ (where $a \neq b$) with $\varprojlim \mathbf{g} = \{(b, b, b, ...)\}$. The inverse limit of $\mathbf{f} \cup \mathbf{g}$ contains the point $(a, b, b, ...)$ which occurs in neither of the inverse limits of $\mathbf{f}$ nor $\mathbf{g}$. So $\varprojlim(\mathbf{f} \cup \mathbf{g}) \neq \varprojlim \mathbf{f} \cup \varprojlim \mathbf{g}$. It is interesting to note that $\varprojlim(\mathbf{f} \cup \mathbf{g})$ is a Cantor set, whereas $\varprojlim \mathbf{f} \cup \varprojlim \mathbf{g}$ is simply the union of two points in Hilbert space.

I started thinking about properties of functions that would give an indication to what the inverse limit of the union is. I conjectured that the inverse limit of the union of two *onto* functions would necessarily be a strict superset of the union of the inverse limits of the separate functions.

For example, consider *Figure 3.1*. Here, we have $\mathbf{f} : [0, 1] \to [0, 1]$ with $\mathbf{f} : x \mapsto \{x\}$ and $\mathbf{g} : [0, 1] \to [0, 1]$ with $\mathbf{g} : x \mapsto \{1 - x\}$. $\varprojlim \mathbf{f} = \{(x, x, x, ...) \,|\, x \in [0, 1]\}$ and $\varprojlim \mathbf{g} = \{(x, 1 - x, x, 1 - x, ...) \,|\, x \in [0, 1]\}$. Consider $\mathrm{S} = (0, 1, 1, ...)$. Then $\mathrm{S} \notin \varprojlim \mathbf{f}$ and $\mathrm{S} \notin \varprojlim \mathbf{g}$ but $\mathrm{S} \in \varprojlim(\mathbf{f} \cup \mathbf{g})$. So the inverse limit of the union of $\mathbf{f}$ and $\mathbf{g}$ is a strict superset of the union of the inverse limits of $\mathbf{f}$ and $\mathbf{g}$. In this example, $\varprojlim(\mathbf{f} \cup \mathbf{g})$ is a Cantor fan, whereas $\varprojlim \mathbf{f} \cup \varprojlim \mathbf{g}$ is two arcs that intersect at the point $(0.5, 0.5, ...)$ (see *Figure 3.2*).

However, it is not always the case that $\varprojlim(\mathbf{f} \cup \mathbf{g}) \supset \varprojlim \mathbf{f} \cup \varprojlim \mathbf{g}$ for two onto functions $\mathbf{f}$ and $\mathbf{g}$. In *Figure 4.1*,

- $\varprojlim \mathbf{f} = \{(x, x, x, ...) \,|\, x \in [0, 1]\} \cup \{(\underbrace{b, b, ..., b}_{\text{n times}}, a, a, ...) \,|\, n \in \mathbb{N}\}$ This is a union of an arc and a convergent sequence.
- $\varprojlim \mathbf{g} = \{(x, x, x, ...) \,|\, x \in [0, 1]\} \cup \{(\underbrace{d, d, ..., d}_{\text{m times}}, c, c, ...) \,|\, m \in \mathbb{N}\}$ This is a union of an arc and a convergent sequence.
- $\varprojlim(\mathbf{f} \cup \mathbf{g}) = \{(x, x, x, ...) \,|\, x \in [0, 1]\} \cup \{(\underbrace{b, b, ..., b}_{\text{n times}}, a, a, ...) \,|\, n \in \mathbb{N}\} \cup \{(\underbrace{d, d, ..., d}_{\text{m times}}, c, c, ...) \,|\, m \in \mathbb{N}\} = \varprojlim \mathbf{f} \cup \varprojlim \mathbf{g}$. This is a union of an arc and two convergent sequences (see *Figure 4.2*).

I notice that there is a significant intersection between $\mathbf{f}$ and $\mathbf{g}$ in the example above. It is reasonable to hypothesize that this is the reason that $\varprojlim(\mathbf{f} \cup \mathbf{g}) = \varprojlim \mathbf{f} \cup \varprojlim \mathbf{g}$.

Lemma 1. *The inverse limit of the union of an onto function* $\boldsymbol{f}$ *and any (non-empty) partial or total function* $\boldsymbol{g}$ *such that* $\boldsymbol{f} \cap \boldsymbol{g} = \emptyset$ *must be a strict superset of the union of inverse limits of the separate functions.*

Proof. Pick some x_0 in the range of $\mathbf{g}$. Then by definition there must exist some x such that $x_0 \in \mathbf{g}(x)$. Call one such value x_1

Then, there must exist some x_2 such that $x_1 \in \mathbf{f}(x_2)$ since $\mathbf{f}$ is onto. Similarly, there must such some x_3 such that $x_2 \in \mathbf{f}(x_3)$ and so on.

So consider $(x_0, x_1, x_2, x_3, ...)$

- This is not in the inverse limit of $\mathbf{f}$. x_1 following directly on from x_0 cannot occur in any member of the inverse limit of $\mathbf{f}$ since the point (x_1, x_0) does not appear in the graph of $\mathbf{f}$. We know it can't appear because this point is in the graph of $\mathbf{g}$ and we know $\mathbf{f} \cap \mathbf{g} = \emptyset$.
- Similarly, this is not in the inverse limit of $\mathbf{g}$ since we have x_2 following directly on from x_1. As (x_2, x_1) is in the graph of $\mathbf{f}$, it is not in the graph of $\mathbf{g}$ and thus will never appear in a member of the inverse limit of $\mathbf{g}$.
- This does appear in the inverse limit of $\mathbf{f} \cup \mathbf{g}$ since (x_1, x_0), (x_2, x_1) and (x_{i+1}, x_i) for i = 2, 3, 4, ... are all in the graph of $\mathbf{f} \cup \mathbf{g}$.

$\square$

To try to characterise the other cases, I started by looking at unions of a constant function with simple partial functions. I define the function $\mathbf{F} : [0,1] \to [0,1]$ to be $\mathbf{F} : x \mapsto \{a\}$

Here I consider unions of $\mathbf{F}$ with a partial function $\mathbf{g}$ that consists of just one point. We know that $\underleftarrow{\lim}\, \mathbf{F} = \{(a, a, a, ...)\}$. There are three cases:

- Suppose the point is on the line $y = x$ at (x_0, x_0) where $x_0 \neq a$ (see *Figure 5*).
  - ▶ $\underleftarrow{\lim}(\mathbf{F} \cup \mathbf{g}) = \{(a, a, ...)\} \cup \{(\underbrace{a, a, ..., a}_{n \text{ times}}, x_0, x_0, ...) \mid n \in \mathbb{N}_0\}$. This is a converging sequence (see *Figure 5.2*).
  - ▶ $\underleftarrow{\lim}\, \mathbf{F} \cup \underleftarrow{\lim}\, \mathbf{g} = \{(a, a, a, ...)\} \cup \{(x_0, x_0, x_0, ...)\}$
  - ▶ So $\underleftarrow{\lim}(\mathbf{F} \cup \mathbf{g}) \supset \underleftarrow{\lim}\, \mathbf{F} \cup \underleftarrow{\lim}\, \mathbf{g}$. This is apparent in *Figure 5.2*.
- Suppose the point is on the line $x = a$ at (a, x_0) where $x_0 \neq a$ (see *Figure 6*).
  - ▶ $\underleftarrow{\lim}(\mathbf{F} \cup \mathbf{g}) = \{(a, a, a, ...)\} \cup$
    $\{(\underbrace{a, a, ..., a}_{n_0 \text{ times}}, x_0, a, a, ...) \mid n_0 \in \mathbb{N}_0\} \cup$
    $\{(\underbrace{a, a, ..., a}_{n_0 \text{ times}}, x_0, \underbrace{a, a, ..., a}_{n_1 \text{ times}}, x_0, a, a, ...) \mid n_0 \in \mathbb{N}_0, n_1 \in \mathbb{N}\} \cup$
    $\{(\underbrace{a, a, ..., a}_{n_0 \text{ times}}, x_0, \underbrace{a, a, ..., a}_{n_1 \text{ times}}, x_0, \underbrace{a, a, ..., a}_{n_2 \text{ times}}, x_0, a, a, ...) \mid n_0 \in \mathbb{N}_0;\, n_1, n_2 \in \mathbb{N}\} \cup ...$ This is a fractal (see *Figure 6.2*). Here, the maroon point is $(a, a, a...)$; the black points are a convergent sequence with n_0 increasing clockwise to the maroon point; the dark grey points are a convergent sequence with n_1 increasing clockwise to the black points and so on.
  - ▶ $\underleftarrow{\lim}\, \mathbf{F} \cup \underleftarrow{\lim}\, \mathbf{g} = \{(a, a, a, ...)\}$, this is the maroon point in *Figure 6.2*.
  - ▶ So clearly $\underleftarrow{\lim}(\mathbf{F} \cup \mathbf{g}) \supset \underleftarrow{\lim}\, \mathbf{F} \cup \underleftarrow{\lim}\, \mathbf{g}$
- Suppose the point is on neither the line $y = x$ nor $x = a$ at (x_0, x_1) where $x_0, x_1 \neq a$ (see *Figure 7*).
  - ▶ Then $\underleftarrow{\lim}(\mathbf{F} \cup \mathbf{g}) = \{(a, a, a, ...)\} = \underleftarrow{\lim}\, \mathbf{F} = \underleftarrow{\lim}\, \mathbf{F} \cup \underleftarrow{\lim}\, \mathbf{g}$. This is one point in Hilbert space.

So it would seem that, in order for $\underleftarrow{\lim}\, \mathbf{F} \cup \underleftarrow{\lim}\, \mathbf{g}$ to equal $\underleftarrow{\lim}(\mathbf{F} \cup \mathbf{g})$, $\mathbf{g}$ cannot intersect the lines $y = x$ or $x = a$, except at the point (a, a). This is not the case, however. Suppose $a = 1$. Then we have a counterexample in *Figure 8*:

- $\varprojlim \mathbf{f} = \{(1,1,1,...)\}$. This looks like one point in Hilbert space.
- $\varprojlim \mathbf{g} = \{(1,1,1,...)\} \cup$
$\{(\underbrace{1,1,...,1}_{n_0 \text{ times}},0,1,1,...) \,|\, n_0 \in \mathbb{N}_0\} \cup$
$\{(\underbrace{1,1,...,1}_{n_0 \text{ times}},0,\underbrace{1,1,...,1}_{n_1 \text{ times}},0,1,1,...) \,|\, n_0 \in \mathbb{N}_0, n_1 \in \mathbb{N}\} \cup$
$\{(\underbrace{1,1,...,1}_{n_0 \text{ times}},0,\underbrace{1,1,...,1}_{n_1 \text{ times}},0,\underbrace{1,1,...,1}_{n_2 \text{ times}},0,1,1,...) \,|\, n_0 \in \mathbb{N}_0;\, n_1, n_2 \in \mathbb{N}\} \cup ...$
$= \varprojlim(\mathbf{f} \cup \mathbf{g}) = \varprojlim \mathbf{f} \cup \varprojlim \mathbf{g}$. This looks like the fractal in *Figure 6.2.*

I conjecture that this, again, is due to the large intersection between $\mathbf{f}$ and $\mathbf{g}$.

Lemma 2. *Consider the function $\boldsymbol{F} : [0,1] \to [0,1]$ to be $\boldsymbol{F} :\, x \mapsto \{a\}$. Suppose $\boldsymbol{g}$ is a non-empty partial function such that $\boldsymbol{F} \cap \boldsymbol{g} = \emptyset$ and $\varprojlim \boldsymbol{F} \cup \varprojlim \boldsymbol{g} = \varprojlim(\boldsymbol{F} \cup \boldsymbol{g})$. Then $\boldsymbol{g}$ must not intersect the lines $x = a$ or $y = x$.*

Proof.

- Suppose that $\mathbf{g}$ intersects the line $x = a$ at the point $(a,\, x_0)$ for $x_0 \neq a$. Then $S = (a, x_0, a, x_0, ...) \in \varprojlim(\mathbf{f} \cup \mathbf{g})$. This is not in the inverse limit of $\mathbf{f}$ since $\varprojlim \mathbf{f} = \{(a, a, ...)\}$. It is also not in the inverse limit of $\mathbf{g}$. For us to get an a followed by x_0 in a member of the inverse limit of a function, the point $(x_0,\, a)$ must occur in the graph. We know $(x_0,\, a) \notin G(\mathbf{g})$ since $(x_0,\, a) \in G(\mathbf{f})$ and $\mathbf{f} \cap \mathbf{g} = \emptyset$. So S $\notin \varprojlim \mathbf{g}$ and thus $\varprojlim \mathbf{f} \cup \varprojlim \mathbf{g}$ is a strict subset of $\varprojlim(\mathbf{f} \cup \mathbf{g})$.
- Suppose that $\mathbf{g}$ intersects the line $y = x$ at the point $(x_0,\, x_0)$ for $x_0 \neq a$. Then $(a, x_0, x_0, ...) \in \varprojlim(\mathbf{f} \cup \mathbf{g})$. This is not in the inverse limit of $\mathbf{f}$ since $\varprojlim \mathbf{f} = \{(a, a, ...)\}$. It is also not in the inverse limit of $\mathbf{g}$. For us to get an a followed by x_0 in a member of the inverse limit of a function, the point $(x_0,\, a)$ must occur in the graph. We know $(x_0,\, a) \notin G(\mathbf{g})$ since $(x_0,\, a) \in G(\mathbf{f})$ and $\mathbf{f} \cap \mathbf{g} = \emptyset$. So $S \notin \varprojlim \mathbf{g}$ and thus $\varprojlim \mathbf{f} \cup \varprojlim \mathbf{g}$ is a strict subset of $\varprojlim(\mathbf{f} \cup \mathbf{g})$.

Thus if $\mathbf{g}$ intersects $y = x$ or $x = a$ at any point other than (a, a), $\varprojlim(\mathbf{f} \cup \mathbf{g}) \supset \varprojlim \mathbf{f} \cup \varprojlim \mathbf{g}$. So the contrapositive is necessarily true. □

Discussions

Characterising the inverse limits of unions of functions is complex. I have found preliminary results regarding constant and onto functions as well as several examples of functions (with various properties) for which the inverse limit of the union is exactly equal to the union of the inverse limits of the functions, and other examples for which it is a strict superset. However, there is still much to understand. It would be interesting to consider other properties of functions (for example upper or lower semicontinuity) that give information about the inverse limit of the union of a function with that property and another function.

Figures

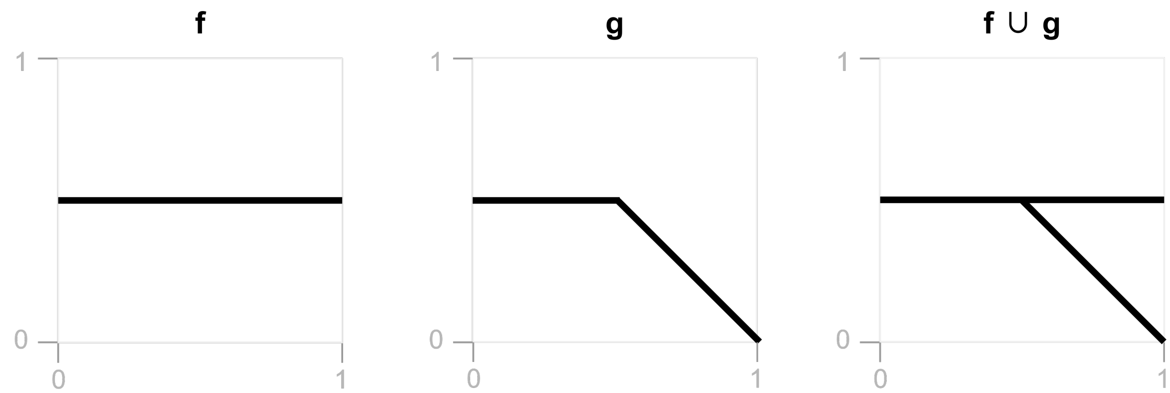

Figure 1

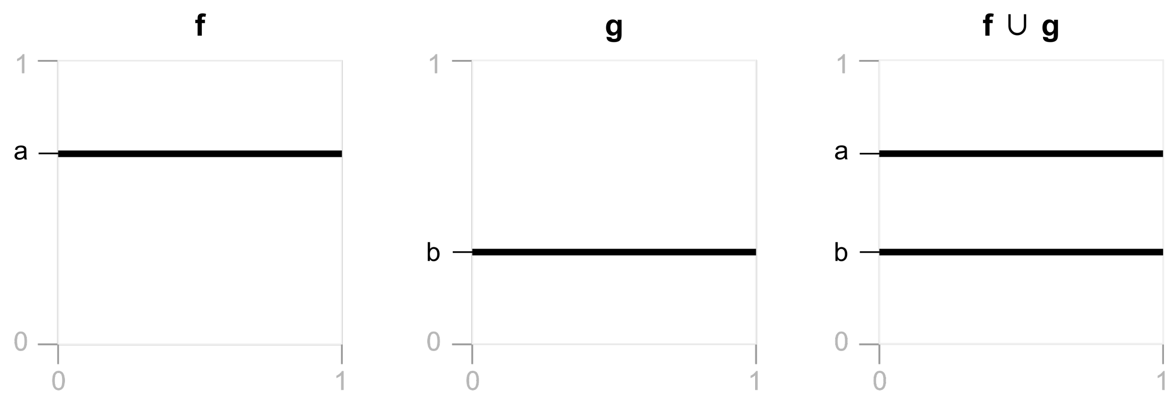

Figure 2

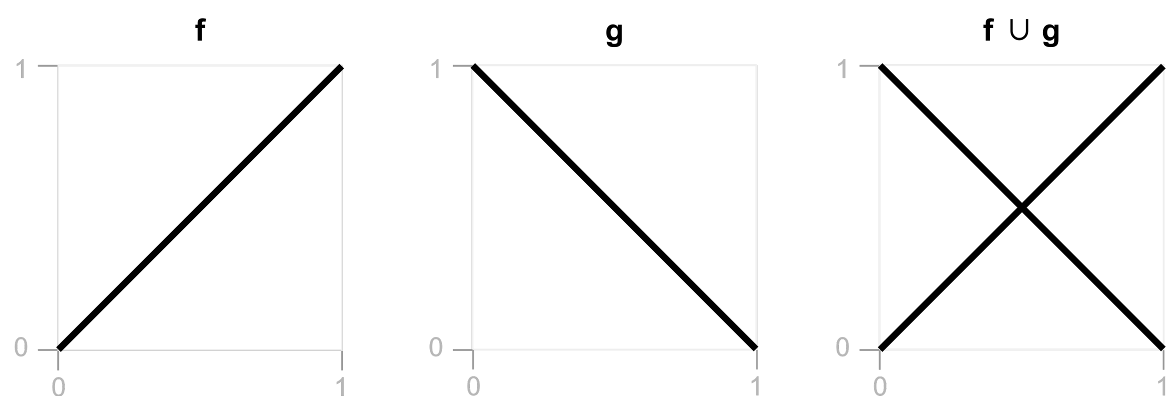

Figure 3.1

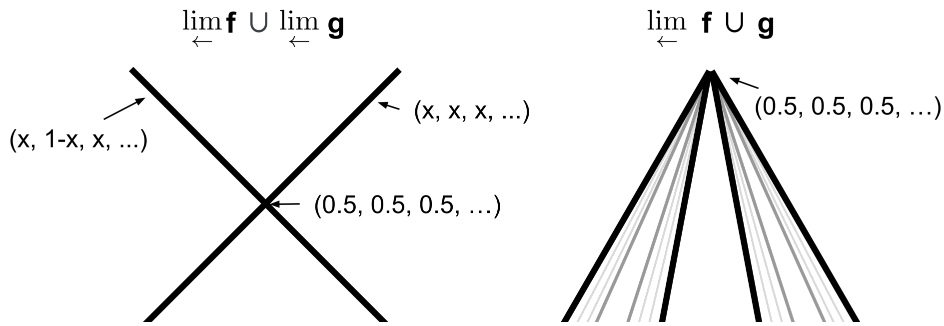

Figure 3.2

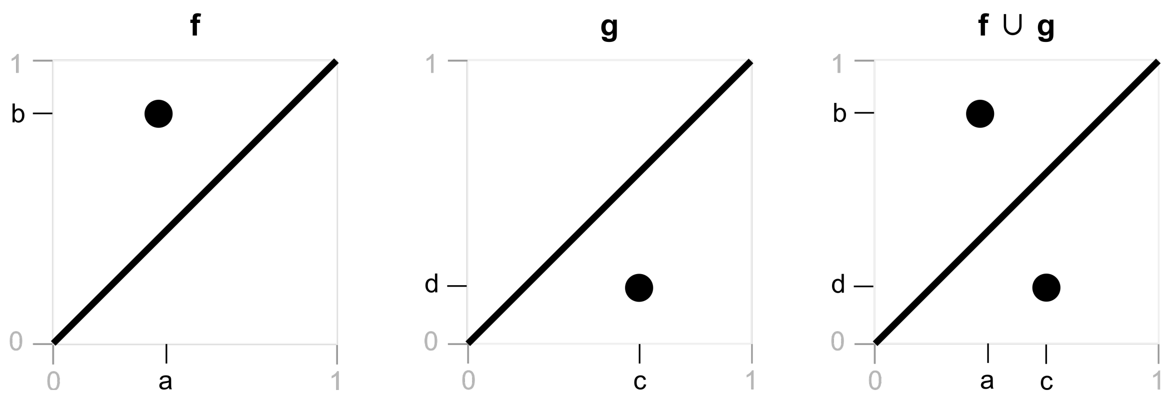

Figure 4.1

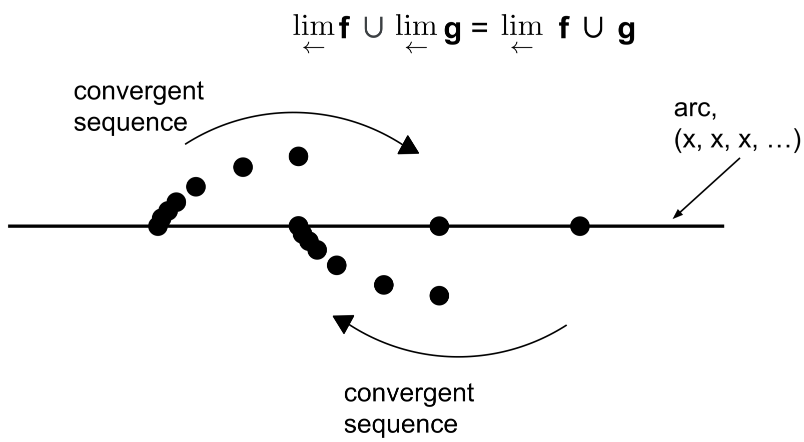

Figure 4.2

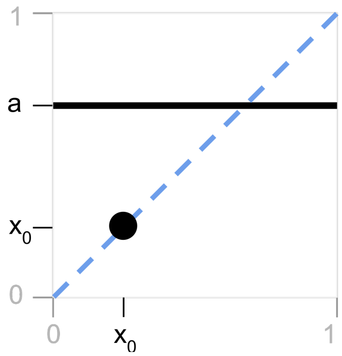

Figure 5

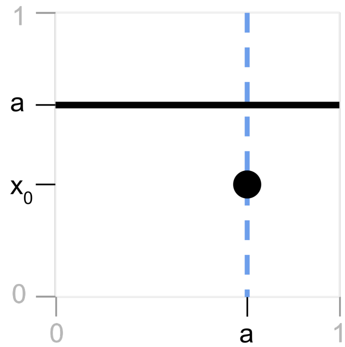

Figure 6

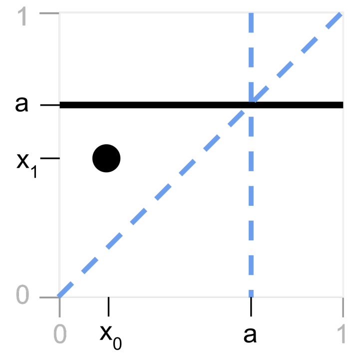

Figure 7

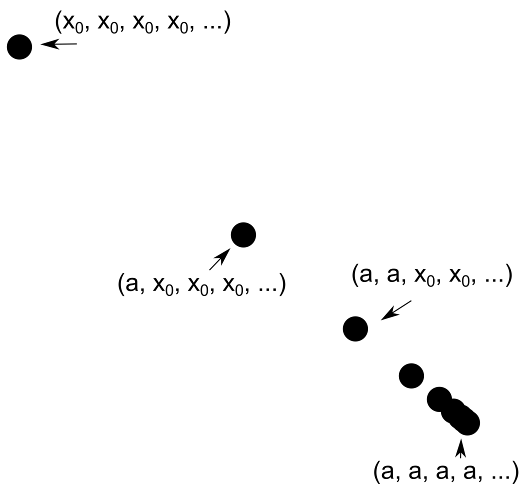

Figure 5.2

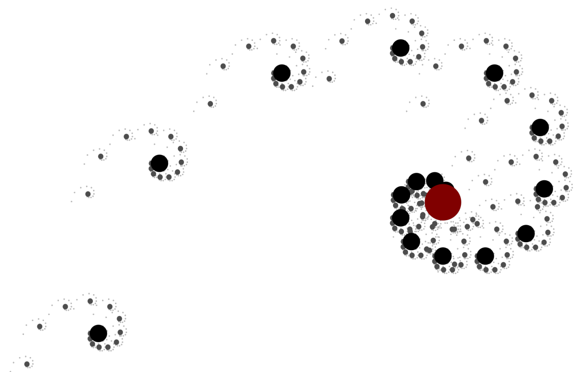

Figure 6.2

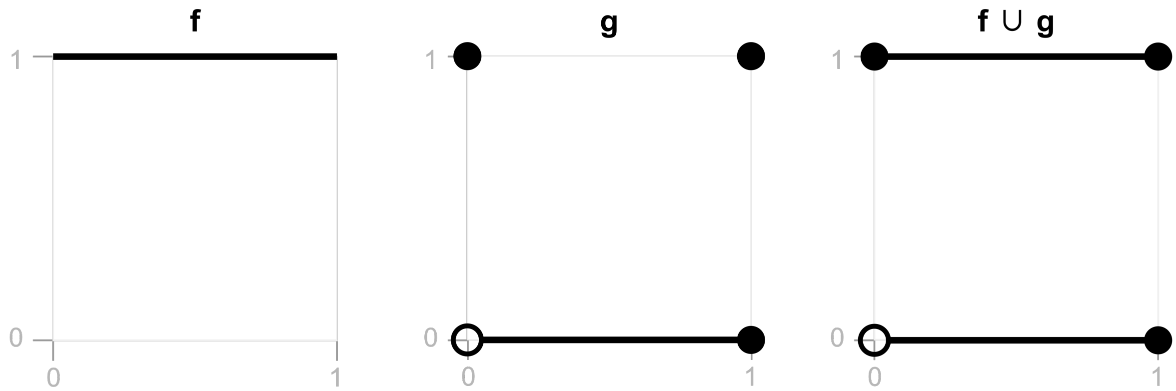

Figure 8

Bibliography

[1] W. T. Ingram, *An Introduction to Inverse Limits with Set-valued Functions*, Springer, New York, 2012.

[2] W.T.Ingram and W. S. Mahavier, *Inverse Limits: From Continua to Chaos*, Springer, New York, 2012.