Master's thesis

Point processes: theory and modeling

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April-June 2025

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1 Introduction

In this document, I am trying to summarize the concepts and articles that I studied during my internship with Jean-Christophe Breton and which I may need during my PhD thesis. The first part introduces the basics of point processes, giving the definitions of some useful tools to study them. The second part gives three classical examples of point processes, respectively attractive, repulsive and neither to get an idea of what these notions can be for a point process. The third part is a recap of an article that introduces a model of random grains and gives convergence results when the number of grains goes to infinity and their volume goes to zero, if the centers of the grains are distributed as a Poisson process. One of the goals of my PhD thesis is to generalize this type of results by considering more general distributions for the centers of the grains, like the ones presented in the second part, perhaps using the tools from the first part.

2 Point processes: definitions and tools

We start by introducing the notion of point processes, and by presenting some tools that will be useful for the rest, or that I studied because they may be interesting for my future PhD thesis.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let us denote $L_0(\Omega)$ the set of random variables of this space.

2.1 Definitions

Let us start with the definitions and basic properties of random measures and in particular of point processes. This subsection is inspired by [1], [2], [6] and [7].

Let E be a locally compact, complete and separable metric space, which is endowed with the corresponding Borelian σ-field denoted as $\mathcal{E}$. Usually, we consider $E = \mathbb{R}^d$.

Definition 2.1.
- *A positive measure μ on E is said to be locally finite if for any bounded Borel set B of E, $\mu(B) < +\infty$. Let us denote $\mathcal{M}(E)$ the set of positive locally finite measures on E.*
- *Let us denote $\mathcal{M}_p(E)$ the set of point measures of $\mathcal{M}(E)$, that is to say the set of integer-valued measures of $\mathcal{M}(E)$.*
- *A measure μ of $\mathcal{M}_p(E)$ is said to be simple if for all $x \in E$, $\mu(\{x\}) \in \{0, 1\}$. Let us denote $\mathcal{M}_p^*(E)$ the set of simple point measures on E.*

In particular, given the assumptions on E, a locally finite measure on E is also a σ-finite measure, since the space can be written as an union of countably many compact sets (i.e. it is σ-compact).

Proposition 2.2. *A measure $\mu \in \mathcal{M}(E)$ belongs to $\mathcal{M}_p(E)$ if and only if μ is of the form*

$$\mu = \sum_{i \in I} k_i \delta_{x_i}, \tag{1}$$

where I is countable, $X = \{x_i,\, i \in I\}$ is a locally finite subset of E (meaning that for any bounded Borel set B of E, the cardinality of $B \cap X$ is finite), and $(k_i)_{i \in I}$ is a sequence of positive integers.
Moreover, if $\mu \in \mathcal{M}_p(E)$, then μ is simple if and only if every k_i appearing in (1) is equal to 1.

Remark 2.3. *This proposition says that if a measure $\mu \in \mathcal{M}_p(E)$ of the form (1) is simple, then it can be identified with the subset $X = \{x_i,\, i \in I\}$. Indeed, μ can be viewed as the counting measure associated to X. In the rest, we will use both of these points of view to study simple point measures.*

The set $\mathcal{M}(E)$ is endowed with the σ-algebra $\mathbb{M} = \sigma(\pi_B,\, B \in \mathcal{E})$ where for $B \in \mathcal{E}$ and $\mu \in \mathcal{M}(E)$, $\pi_B(\mu) = \mu(B)$. The set $\mathcal{M}_p(E)$ is endowed with the restriction of $\mathbb{M}$, let us denote it $\mathbb{M}_p$.

Definition 2.4. *A random measure, respectively a point process, on $(E, \mathcal{E})$ is a random variable with values in $(\mathcal{M}(E), \mathbb{M})$, respectively in $(\mathcal{M}_p(E), \mathbb{M}_p)$. A point process X is simple when $\mathbb{P}(X \in \mathcal{M}_p^*(E)) = 1$.*

Remark 2.5. *According to the previous remark, a simple point process can be seen as a random locally finite subset of E.*

Definition 2.6. *Let μ be a random measure on $(E, \mathcal{E})$.*

- *We define the intensity measure (or mean measure) of μ as the measure α defined on $(E, \mathcal{E})$ by*

$$\alpha(B) = \mathbb{E}[\mu(B)], \quad B \in \mathcal{E}.$$

- *We define the Laplace transform of μ as the functional $\mathcal{L}_\mu$ defined on $\mathcal{F}_+(E) := \{f : E \longrightarrow \mathbb{R}_+ \text{ measurable}\}$ by*

$$\mathcal{L}_\mu(f) = \mathbb{E}\left[\exp\left(-\int_E f(x)\mu(\mathrm{d}x)\right)\right], \quad f \in \mathcal{F}_+(E).$$

Generally, the distribution of a random measure is not characterized by its intensity measure. However, we will see later that, for instance, it is the case if the considered random measure is a Poisson point process. Conversely, the Laplace transform of a random measure determines its distribution as we will see after.

Since $\mathbb{M}$ is generated by cylinder sets of the form $\{\mu \in \mathcal{M}(E) : \mu(B_k) \in A_k,\ 1 \leq k \leq n\}$ for $B_1, \ldots, B_n \in \mathcal{E}$ and $A_1, \ldots, A_n \in \mathcal{B}(\mathbb{R})$, an application of Dynkin theorem gives the following result.

Proposition 2.7. *The distribution of a random measure μ on $(E, \mathcal{E})$ is determined by its finite-dimensional distributions, that is to say by the distribution of the random vectors $(\mu(B_1), \ldots, \mu(B_n))$ for $n \in \mathbb{N}^*$ and $B_1, \ldots, B_n$ Borel sets. Actually, it is sufficient to consider the distribution of the random vectors $(\mu(B_1), \ldots, \mu(B_n))$ for $n \in \mathbb{N}^*$ and $B_1, \ldots, B_n$ **bounded** Borel sets.*

Corollary 2.8. *The distribution of a random measure μ on $(E, \mathcal{E})$ is determined by its Laplace transform $\mathcal{L}_\mu$.*

Proof. Let $n \in \mathbb{N}^*$ and $B_1, \ldots, B_n$ be bounded Borel sets. The distribution of the random vector $(\mu(B_1), \ldots, \mu(B_n))$ is determined by its Laplace transform $\mathcal{L}_{(\mu(B_1),\ldots,\mu(B_n))}$ defined by

$$\mathcal{L}_{(\mu(B_1),\ldots,\mu(B_n))}(t_1, \ldots, t_n) = \mathbb{E}[\exp(-(t_1\mu(B_1) + \cdots + t_n\mu(B_n))], \quad t_1, \ldots, t_n \in \mathbb{R}_+,$$

see for instance [9] (Theorem 5.3). Yet, for $t_1, \ldots t_n \in \mathbb{R}_+$, we can rewrite $\mathcal{L}_{(\mu(B_1),\ldots,\mu(B_n))}(t_1, \ldots, t_n)$ in the following way:

$$\mathcal{L}_{(\mu(B_1),\ldots,\mu(B_n))}(t_1, \ldots, t_n) = \mathcal{L}_\mu\left(\sum_{i=1}^n t_i \mathbb{1}_{B_i}\right).$$

Thus, the distribution of $(\mu(B_1), \ldots, \mu(B_n))$ is determined by $\mathcal{L}_\mu$, and by the previous proposition, the distribution of such vectors determines the distribution of μ, which concludes the proof. □

Now that we have these definitions, let us study some useful tools to investigate point processes. In the sequel, all point processes are supposed to be simple, and will be considered as random point measures as well as random locally finite subsets.

2.2 Joint intensities

We have already seen that the distribution of a random measure is characterized by its finite-dimensional distributions. Here, we want to find another way to describe the distribution of a point process. This subsection is based on the results of [4], [6] and [7].

Let $\mu \in \mathcal{M}(E)$ be a measure on E.

Definition 2.9. *A point process X on E is said to admit joint intensities with respect to μ if for any $k \geq 1$, there exists a function $\rho^{(k)} : E^k \to [0, +\infty)$ such that for all mutually disjoint Borel sets $B_1, \dots, B_k$,*

$$\mathbb{E}\left[\prod_{i=1}^{k} X(B_i)\right] = \int_{\prod_{i=1}^{k} B_k} \rho^{(k)}(x_1, \dots, x_k)\mu(\mathrm{d}x_1)\dots\mu(\mathrm{d}x_k). \tag{2}$$

In addition, we require $\rho^{(k)}(x_1, \dots, x_k) = 0$ if there exist $i \neq j$ such that $x_i = x_j$, i.e. $\rho^{(k)}$ vanishes on the diagonal.

Let us make the link between these functions and some measures that will be useful later to define the Palm distributions of a point process. The proof of the following result can be found in [13] (see Lemma 27) for the case $E = \mathbb{C}$, and can be done with the same method for the general case.

Proposition 2.10. *Let X be a point process on E. For $k \geq 1$, we define the k^{th} order factorial moment measure $\alpha^{(k)}$ on E^k by*

$$\alpha^{(k)}(B) = \mathbb{E}[\#(B \cap X^{(k)})],$$

where $B \subset E^k$ is a Borel set, and where $X^{(k)}$ designates the set of k-tuples of distinct points of X. If $B = \prod_{i=1}^{k} B_i$, where $B_1, \dots, B_k \in \mathcal{E}$, then this can be rewritten as

$$\alpha^{(k)}(B) = \alpha^{(k)}\left(\prod_{i=1}^{k} B_i\right) = \mathbb{E}\left[\sum_{x_1,\dots,x_k \in X}^{\neq} \mathbb{1}(x_1 \in B_1, \dots, x_k \in B_k)\right],$$

where the sign $\neq$ means that $x_1, \dots, x_k$ are pairwise distinct. If X admits joint intensities with respect to μ, then for any $k \geq 1$ and any Borel set $B \subset E^k$, we have

$$\alpha^{(k)}(B) = \int_B \rho^{(k)}(x_1, \dots, x_k)\mu(\mathrm{d}x_1)\dots\mu(\mathrm{d}x_k),$$

i.e. the k^{th} order factorial moment measure is absolutely continuous with respect to μ^k on E^k, with density $\rho^{(k)}$.

The measure $\alpha^{(1)}$ is the intensity measure α defined in the first section. If X admits joint intensities, α is absolutely continuous with respect to μ, and the corresponding density $\rho^{(1)}$ (or ρ for simplicity) is called the intensity function.

Remark 2.11. *Why is the measure $\alpha^{(k)}$ called like this ? If B is of the form $B = \prod_{i=1}^{r} B_i^{k_i}$ with $B_1, \dots, B_r$ disjoint and $\sum_{i=1}^{r} k_i = k$, then*

$$\alpha^{(k)}(B) = \mathbb{E}[\#(B \cap X^{(k)})] = \mathbb{E}\left[\prod_{i=1}^{r} (X(B_i))_{k_i}\right], \tag{3}$$

where

$$(X(B_i))_{k_i} = \binom{X(B_i)}{k_i} k_i! = X(B_i)(X(B_i) - 1)\dots(X(B_i) - k_i + 1).$$

In particular, if $B = B_1^k$,

$$\alpha^{(k)}(B) = \mathbb{E}[(X(B_1))_k],$$

i.e. $\alpha^{(k)}(B)$ is equal to the k^{th} factorial moment of $X(B_1)$.

For $x_1, \dots, x_k \in E$, $\rho^{(k)}(x_1, \dots, x_k)$ can be seen as the infinitesimal probability that the points $x_1, \dots, x_k$ belong to the process X. We will see later another expression for joint intensities in the case where the processes are finite that will make this even clearer.

The initial goal of this section was to find another way than the probabilities of cylinder sets to describe the distribution of a point process, but do these functions characterize the distribution of a point process ?

Proposition 2.12. *Let X be a point process on E such that X admits joint intensities (with respect to μ) and such that for every compact set $K \subset E$, there exists $c > 0$ such that for any $k \geq 1$,*

$$\mathbb{P}(X(K) \geq k) \leq e^{-ck}. \tag{4}$$

Then the joint intensities characterize the distribution of X.

Proof. We have already seen that X is characterized by its finite-dimensional distributions for (bounded) Borel subsets (cf Proposition 2.7), but actually we can do better than this. There exists a countable class $\mathcal{E}_0$ of relatively compact Borel subsets of E such that the distribution of any random measure μ on E is characterized by the distribution of the random vectors $(\mu(B_1), \dots, \mu(B_n))$ for $n \in \mathbb{N}^*$ and $B_1, \dots, B_n \in \mathcal{E}_0$, see Lemma 1.3.1, Corollary 1.3.4 and Lemma 1.3.6 of [1]. Then, it is sufficient to prove that the joint intensities characterize the distribution of this type of random vectors.
Let $B_1, \dots, B_n \in \mathcal{E}_0$. Let $i \in [\![1, n]\!]$. Since B_i is relatively compact, there exists a compact D_i such that $B_i \subset D_i$. Then if we denote c_i the constant given by the hypothesis for $K = D_i$, we have for any $k \geq 1$,

$$\mathbb{P}(X(B_i) \geq k) \leq \mathbb{P}(X(D_i) \geq k) \leq e^{-c_i k}.$$

Then, let us show that this implies that the function $t \mapsto \mathbb{E}[e^{tX(B_i)}]$ (which we will also call Laplace transform for the proof despite the difference in definition compared to the rest of the document) is finite on $(-c_i, c_i)$.
Let $t \in (-c_i, c_i)$. Then

$$\mathbb{E}\left[e^{tX(B_i)}\right] = \sum_{k=1}^{+\infty} (e^{tk} - e^{t(k-1)}) \mathbb{P}(X(B_i) \geq k) + 1.$$

Indeed,

$$\begin{aligned}\sum_{k=1}^{+\infty} (e^{tk} - e^{t(k-1)}) \mathbb{P}(X(B_i) \geq k) &= \sum_{k=1}^{+\infty} e^{tk} \mathbb{P}(X(B_i) \geq k) \; - \sum_{k=0}^{+\infty} e^{tk} \mathbb{P}(X(B_i) \geq k+1) \\ &= \sum_{k=0}^{+\infty} e^{tk} \underbrace{(\mathbb{P}(X(B_i) \geq k) - \mathbb{P}(X(B_i) \geq k+1))}_{=\mathbb{P}(X(B_i)=k)} - 1.\end{aligned}$$

So, by hypothesis,

$$\mathbb{E}\left[e^{tX(B_i)}\right] \leq \sum_{k=1}^{+\infty} (e^{tk} - e^{t(k-1)}) e^{-c_i k} + 1 = (1 - e^{-t}) \sum_{k=1}^{+\infty} (e^{t-c_i})^k + 1 < +\infty.$$

Let $c = \min_{1 \leq i \leq n} c_i$ and $s_1, \dots, s_n$ such that $s := \sum_{i=1}^{n} |s_i| \in (-c, c) \backslash \{0\}$. Then by convexity,

$$\mathbb{E}\left[\exp(s_1 X(B_1) + \cdots + s_n X(B_n))\right] \leq \sum_{i=1}^{n} \frac{|s_i|}{s} \mathbb{E}[\exp(\mathrm{sgn}(s_i) s X(B_i))] < +\infty.$$

Thus, the Laplace transform of $(X(B_1), \dots, X(B_n))$ is finite in a neighborhood of zero, so it characterizes the distribution of the random vector. Moreover, thanks to Fubini's theorem,

$$\mathbb{E}\left[\exp(s_1 X(B_1) + \cdots + s_n X(B_n))\right] = \sum_{k=0}^{+\infty} \mathbb{E}\left[\frac{(s_1 X(B_1) + \cdots + s_n X(B_n))^k}{k!}\right],$$

so the Laplace transform of $(X(B_1), \dots, X(B_n))$ is determined by moments of the form (3) (by decomposing the $(B_i)_{1 \leq i \leq n}$ as disjoint sets and then using for each term of the form $X(B)^k$ the following formula: $X(B)^k = \sum_{i=0}^{k} s(n,k)(X(B))_k$, where the coefficients are Stirling numbers of the second kind), so determined by the joint intensities. □

From the definition of joint intensities, and by classical arguments of measure theory (linearity and monotone convergence), we obtain the Campbell formula.

Proposition 2.13. *Let X be a point process on E such that X admits joint intensities (with respect to μ). Let $h : E^n \longrightarrow \mathbb{R}_+$ be a measurable function. Then*

$$\mathbb{E}\left[\sum_{x_1,\dots,x_n\in X}^{\neq} h(x_1,\dots,x_n)\right] = \int_{E^n} h(x_1,\dots,x_n)\alpha^{(n)}(\mathrm{d}x_1\dots\mathrm{d}x_n) \tag{5}$$

$$= \int_{E^n} h(x_1,\dots,x_n)\rho^{(n)}(x_1,\dots,x_n)\mu(\mathrm{d}x_1\dots\mathrm{d}x_n). \tag{6}$$

The formula still hold true for functions that are integrable with respect to $\alpha^{(n)}$.

We now have the definition and basic properties of joint intensities. Before moving on, let us see a function that is related to joint intensities and that allows us to give a fair idea of what "attractiveness" or "repulsiveness" of a point process can be.

Definition 2.14. *Let X be a point process on E such that X admits joint intensities (with respect to μ). We define the correlation function by*

$$g(u,v) = \frac{\rho^{(2)}(u,v)}{\rho(u)\rho(v)},$$

for $u, v \in E$ such that $\rho(u)\rho(v) > 0$, and $g(u,v) = 0$ otherwise.

Thanks to the interpretation of $\rho^{(k)}(x_1,\dots,x_k)$ as the infinitesimal probability that the points $x_1,\dots,x_k$ belong to the process X, $g(u,v)\rho(u)$ can be seen as the conditional intensity of X at u given that v belongs to X, and $g(u,v)$ can be interpreted as follows: if $g(u,v) > 1$, then the fact that X has a point at v implies that it is more likely to have a point at u (and vice versa by symmetry) and in this case there is an idea of attractiveness; if $g(u,v) < 1$, then the opposite holds and the fact that X has a point at u implies that it is less likely to have a point at v (and vice versa) and in this case there is an idea of repulsiveness.

This interpretation will be even clearer with the object that will be introduced in the following section, that allows us to define a notion of conditional distribution of a point process X given that a fixed point belongs to it.

2.3 Palm distributions

As announced before, let us define the notion of Palm distributions of a point process, which models the distributions of the process, given that some fixed points belong to it. This notion will allow us to better understand the idea of attractiveness or repulsiveness for a point process, with a reinterpretation of the correlation function, or with applications on the examples of the next section.
This subsection will mainly follow the chapter 3 of [1], and will sometimes use results of [4].

Proposition 2.15. *Let X be a point process on E such that its mean measure α is σ-finite. We define the Campbell measure of X as the measure defined on $E \times \mathcal{M}_p(E)$ by*

$$C_X(B \times L) = \mathbb{E}[X(B)\mathbb{1}(X \in L)], \quad B \in \mathcal{E},\ L \in \mathbb{M}_p.$$

This measure is σ-finite, and it admits a disintegration with respect to α:

$$C_X(B \times L) = \int_B P_X^x(L)\alpha(\mathrm{d}x), \quad B \in \mathcal{E},\ L \in \mathbb{M}_p,$$

where $P_X(.)$ is a probability kernel from E to $\mathcal{M}_p(E)$. The kernel $\{P_X^x(.)\}_{x\in E}$ is unique up to a α null set.
Let $x \in E$. Then P_X^x is a probability measure on $\mathcal{M}_p(E)$ called Palm distribution of X at x, and a point process X_x distributed according to P_X^x is called Palm version of X at x.

Remark 2.16. *If $\mathbb{P}(X \in L) = 1$, then $C_X(B \times L) = \mathbb{E}[X(B)] = \alpha(B)$, so $P_X^x(L) = \mathbb{P}(X_x \in L) = 1$ for α-almost all x. Thus, if X is simple, then for α-almost all x, X_x is also simple.*

Remark 2.17. *One can show that the distribution of a random measure X on E is determined by any family of Palm distribution $(P_X^x)_{x\in E}$.*

As before with the Campbell's formula, thanks to classical arguments of measure theory, we deduce the Campbell-Little-Mecke formula.

Theorem 2.18 (Campbell-Little-Mecke)**.** *Let X be a point process on E, such that its mean measure α is σ-finite. With the previous notations, for any measurable function $h : E \times \mathcal{M}_p(E) \longrightarrow \mathbb{R}_+$, we have*

$$\mathbb{E}\left[\sum_{x\in X} h(x, X)\right] = \int_{E\times\mathcal{M}_p(E)} h(x, Y)P_X^x(\mathrm{d}Y)\alpha(\mathrm{d}x). \tag{7}$$

The formula still hold true for functions such that

$$\mathbb{E}\left[\sum_{x\in X} |h(x, X)|\right] < +\infty \quad \textit{or equivalently} \quad \int_{E\times\mathcal{M}_p(E)} |h(x, Y)|P_X^x(\mathrm{d}Y)\alpha(\mathrm{d}x) < +\infty.$$

Now, let us see a proposition that legitimates the interpretation of P_X^x as the conditional distribution of the point process X given that X has a point at x.

Proposition 2.19. *Let X be a point process on E, such that its mean measure α is σ-finite. Then, with the previous notations, for α-almost every $x \in E$,*

$$\mathbb{P}(X_x(\{x\}) \geq 1) = 1.$$

Thus, we can define the reduced Palm distribution of X at x, which represents the distribution of $X\backslash\{x\}$, given that x belongs to X.

Definition 2.20. *Let X be a point process on E, such that its mean measure α is σ-finite. Then, the point process*

$$X_x^! := X_x\backslash\{x\}$$

is called reduced Palm version of X at x, and its distribution is called reduced Palm distribution of X at x, and denoted $P_X^{!x}$.

We also have a Campbell-Little-Mecke formula for this distribution, that comes from the previous one.

Proposition 2.21 (Campbell-Little-Mecke)**.** *Let X be a point process on E, such that its mean measure α is σ-finite. With the previous notations, for any measurable function $h : E \times \mathcal{M}_p(E) \longrightarrow \mathbb{R}_+$, we have*

$$\mathbb{E}\left[\sum_{x\in X} h(x, X\backslash\{x\})\right] = \int_{E\times\mathcal{M}_p(E)} h(x, Y)P_X^{!x}(\mathrm{d}Y)\alpha(\mathrm{d}x). \tag{8}$$

The formula still hold true for functions such that

$$\mathbb{E}\left[\sum_{x\in X} |h(x, X\backslash\{x\})|\right] < +\infty \quad \textit{or equivalently} \quad \int_{E\times\mathcal{M}_p(E)} |h(x, Y)|P_X^{!x}(\mathrm{d}Y)\alpha(\mathrm{d}x) < +\infty.$$

Remark 2.22. *In the case of stationary processes, the reduced Palm distributions can be expressed differently. We consider $E = \mathbb{R}^d$, and X be a stationary point process on E, i.e. its distribution is invariant under translation. This implies that its intensity measure α is a multiple of the Lebesgue measure, i.e. there exists $\rho \in \mathbb{R}_+^*$ such that*

for any Borel set B, $\alpha(B) = \rho|B|$, where $|B|$ is the Lebesgue measure of B.
Then, we have an explicit expression for $P_X^{!x}$: for $F \in \mathbb{M}_p$, we can take

$$P_X^{!0}(F) = \frac{1}{\rho|B|}\mathbb{E}\left[\sum_{x\in X} \mathbb{1}(x \in B, X\backslash\{x\} - x \in F)\right], \quad \textit{and} \quad P_X^{!x}(F) = P_X^{!0}(F - x), \tag{9}$$

where B is a Borel set with $|B| > 0$, $0 = (0, \dots, 0)$ designates the origin of $\mathbb{R}^d$, and if A is a set, then $A - x := \{y - x,\, y \in A\}$. By stationarity, this quantity does not depend on B. Indeed, by classical arguments of measure theory, it is easy to check that this expression implies that for any measurable function $h : E \times \mathcal{M}_p(E) \longrightarrow \mathbb{R}_+$,

$$\mathbb{E}\left[\sum_{x\in X} h(x, X\backslash\{x\})\right] = \int_{E\times\mathcal{M}_p(E)} h(x, Y + x)\rho P_X^{!0}(\mathrm{d}Y)\mathrm{d}x = \int_{E\times\mathcal{M}_p(E)} h(x, Y + x)P_X^{!0}(\mathrm{d}Y)\alpha(\mathrm{d}x),$$

and thus the distribution verifies (8).
Once again, this legitimates the previous interpretation of Palm distributions: let us assume that we want to estimate $P_X^{!0}(F)$ (for a fixed F) from an observation of X. Let B be a bounded (but large) Borel set. Since X is stationary, for $x \in X$, $X - x$ is an observation of the conditional distribution of X given that the origin belongs to X. Then, a natural estimator of $P_X^{!0}(F)$ is the following empirical mean:

$$\frac{1}{X(B)}\sum_{x\in X\cap B} \mathbb{1}(X\backslash\{x\} - x \in F) \approx \frac{1}{\rho|B|}\sum_{x\in X\cap B} \mathbb{1}(X\backslash\{x\} - x \in F),$$

if B is large. Then, we find the natural estimator of (9).
Finally, (9) *implies that $X_x^! - x \overset{d}{=} X_0^!$, thus $P_X^{!0}$ can be seen as the conditional distribution of X given a "typical" point of X.*

Before moving on, let us see a proposition that gives a strategy to determine the Palm distribution of a random measure.

Proposition 2.23. *Let X be a point process on E, such that its mean measure α is σ-finite. Then for any $f, g \in \mathcal{F}_+(E)$ such that g is α-integrable, we have*

$$\frac{\partial}{\partial t}\mathcal{L}_X(f + tg)\Big|_{t=0} = \mathbb{E}\left[-\left(\int_E g(x)X(\mathrm{d}x)\right)\exp\left(-\int_E f(x)X(\mathrm{d}x)\right)\right] = -\int_E g(x)\mathcal{L}_{X_x}(f)\alpha(\mathrm{d}x).$$

Moreover, this equality characterizes $\mathcal{L}_{X_x}$; if there exists a family $\mathcal{L}_x : \mathcal{F}_+(E) \longrightarrow \mathbb{R}_+$ such that

$$\frac{\partial}{\partial t}\mathcal{L}_X(f + tg)\big|_{t=0} = -\int_E g(x)\mathcal{L}_x(f)\alpha(\mathrm{d}x),$$

then for α-almost every $x \in E$, we have $\mathcal{L}_x = \mathcal{L}_{X_x}$.

With the same method, we can define a distribution that represents the distribution of a point process X (or of $X\backslash\{x_1, \dots, x_n\}$), given that $x_1, \dots, x_n$ belong to X.

In the sequel, $\mathcal{E}^{\otimes n}$ stands for the product σ-algebra of E^n, and if X is a point process, X^n stands for its n^{th} power in the sense of products of measures; from the set point of view, this represents the set of n-tuples of points of X.

Proposition 2.24. *Let $n \in \mathbb{N}^*$. Let X be a random process on E such that the intensity measure α_n of X^n is σ-finite. We define the n^{th} Campbell measure of X as the measure defined on $E^n \times \mathcal{M}_p(E)$ by*

$$C_X^n(B \times L) = \mathbb{E}[X^n(B)\mathbb{1}(X \in L)], \quad B \in \mathcal{E}^{\otimes n},\ L \in \mathbb{M}_p.$$

This measure is σ-finite, and it admits a disintegration with respect to α_n:

$$C_X^n(B \times L) = \int_B P_X^{(x_1,\dots,x_n)}(L)\alpha_n(\mathrm{d}x_1 \dots \mathrm{d}x_n), \quad B \in \mathcal{E}^{\otimes n},\ L \in \mathbb{M}_p,$$

where $P_X(.)$ is a probability kernel from E^n to $\mathcal{M}_p(E)$.
Let $(x_1,\dots,x_n) \in E^n$. Then $P_X^{(x_1,\dots,x_n)}$ is a probability measure on $\mathcal{M}_p(E)$ called n^{th} Palm distribution of X at $(x_1,\dots,x_n)$, and a point process $X_{(x_1,\dots,x_n)}$ distributed according to $P_X^{(x_1,\dots,x_n)}$ is called n^{th} Palm version of X at $(x_1,\dots,x_n)$.

Once again, the following proposition legitimates the interpretation of $P_X^{(x_1,\dots,x_n)}$ as the conditional distribution of X given that $x_1,\dots,x_n$ belong to the process.

Proposition 2.25. *Let X be a random process on E such that the intensity measure α_n of X^n is σ-finite. Then, with the previous notations, for α_n-almost every $(x_1,\dots,x_n) \in E^n$ pairwise distinct,*

$$\mathbb{P}(\forall i \in [\![1,n]\!], X_{(x_1\dots,x_n)}(\{x_i\}) \geq 1) = 1.$$

From then on, there is a small difference to define the reduced Palm distributions, because the hypothesis on the process required to define them is slightly different and is about the n^{th} order factorial moment measure, that is different from α_n if $n > 1$. However, we will see after that the interpretation of the reduced Palm distribution is the same as before.

Proposition 2.26. *Let $n \in \mathbb{N}^*$. Let X be a random process on E such that the intensity measure $\alpha^{(n)}$ of $X^{(n)}$ is σ-finite. We define the n^{th} reduced Campbell measure of X as the measure defined on $E^n \times \mathcal{M}_p(E)$ by*

$$C_X^{(n)}(B \times L) = \mathbb{E}\left[\sum_{x_1,\dots,x_n \in X}^{\neq} \mathbb{1}((x_1,\dots,x_n) \in B, X\backslash\{x_1,\dots,x_n\} \in L\right], \quad B \in \mathcal{E}^{\otimes n},\ L \in \mathbb{M}_p.$$

This measure is σ-finite, and it admits a disintegration with respect to $\alpha^{(n)}$:

$$C_X^{(n)}(B \times L) = \int_B P_X^{!(x_1,\dots,x_n)}(L)\alpha^{(n)}(\mathrm{d}x_1 \dots \mathrm{d}x_n), \quad B \in \mathcal{E}^{\otimes n},\ L \in \mathbb{M}_p,$$

where $P_X^!(.)$ is a probability kernel from E^n to $\mathcal{M}_p(E)$.
Let $(x_1,\dots,x_n) \in E^n$. Then $P_X^{!(x_1,\dots,x_n)}$ is a probability measure on $\mathcal{M}_p(E)$ called n^{th} reduced Palm distribution of X at $(x_1,\dots,x_n)$, and a point process $X^!_{(x_1,\dots,x_n)}$ distributed according to $P_X^{(x_1,\dots,x_n)}$ is called n^{th} reduced Palm version of X at $(x_1\dots,x_n)$.

Proposition 2.27. *Let X be a random process on E such that the intensity measure $\alpha^{(n)}$ of $X^{(n)}$ is σ-finite. Then, with the previous notations, for $\alpha^{(n)}$-almost every $(x_1,\dots,x_n) \in E^n$ pairwise distinct,*

$$X^!_{(x_1\dots,x_n)} \overset{d}{=} X_{(x_1\dots,x_n)}\backslash\{x_1,\dots,x_n\},$$

meaning they have the same distribution.

Thus, the n^{th} reduced Palm version of X at $(x_1,\dots,x_n)$ can still be interpreted as the conditional distribution of $X_{(x_1,\dots,x_n)}\backslash\{x_1,\dots,x_n\}$ given that $x_1,\dots,x_n$ belong to X.

We still have Campbell-Little-Mecke formulas for n^{th} (reduced) Palm distributions.

Theorem 2.28 (Campbell-Little-Mecke)**.** *Let X be a point process on E, such that α_n (respectively $\alpha^{(n)}$) is σ-finite. With the previous notations, for any measurable function $h : E^n \times \mathcal{M}_p(E) \longrightarrow \mathbb{R}_+$, we have*

$$\mathbb{E}\left[\sum_{x_1,\dots,x_n \in X} h(x_1,\dots,x_n,X)\right] = \int_{E^n \times \mathcal{M}_p(E)} h(x_1,\dots,x_n,Y)P_X^{(x_1,\dots,x_n)}(\mathrm{d}Y)\alpha_n(\mathrm{d}x_1 \dots \mathrm{d}x_n), \tag{10}$$

respectively

$$\mathbb{E}\left[\sum_{x_1,\dots,x_n\in X}^{\neq} h(x_1,\dots,x_n,X\backslash\{x_1,\dots,x_n\})\right] = \int_{E^n\times\mathcal{M}_p(E)} h(x_1,\dots,x_n,Y)P_X^{!(x_1,\dots,x_n)}(\mathrm{d}Y)\alpha^{(n)}(\mathrm{d}x_1\dots\mathrm{d}x_n). \tag{11}$$

As before, the formulas still hold true for functions such that the same integrals with $|h|$ are finite.

The following proposition legitimates a reasonable intuition: conditioning on $x_1,\dots,x_n \in X$ and then conditioning on $y_1,\dots,y_m \in X$ is the same as conditioning directly on $x_1,\dots,x_n,y_1,\dots,y_m \in X$.

Proposition 2.29. *Let X be a random process on E such that $\alpha^{(n+m)}$ is σ-finite. Then for any $A\in\mathcal{E}^{\otimes n}, B\in\mathcal{E}^{\otimes m}$,*

$$\alpha^{(n+m)}(A\times B) = \int_A \alpha^{!(m)}_{(x_1,\dots,x_n)}(B)\alpha^{(n)}(\mathrm{d}x_1\dots\mathrm{d}x_n),$$

where $\alpha^{!(m)}_{(x_1,\dots,x_n)}$ is the m^{th} order factorial moment measure of $X^!_{(x_1,\dots,x_n)}$, that is to say the intensity measure of $(X^!_{(x_1,\dots,x_n)})^{(m)}$. Moreover, for $\alpha^{(n+m)}$-almost every $(x_1,\dots,x_n,y_1,\dots,y_m)\in E^{n+m}$,

$$(X^!_{(x_1,\dots,x_n)})^!_{(y_1,\dots,y_m)} \overset{d}{=} X^!_{(x_1,\dots,x_n,y_1,\dots,y_m)}.$$

This proposition allows us to make a link with joint intensities, and, as announced before, to give a new expression of the correlation function that makes its interpretation clearer. Thanks to the interpretation of joint intensities and Palm distributions, the following proposition can be seen as a formula of the type $\mathbb{P}(A\cap B) = \mathbb{P}(B|A)\mathbb{P}(A)$, with $A=\{x_1,\dots,x_n\in X\}$ and $B=\{y_1,\dots,y_m\in X\}$. This result follows directly from the previous one.

Proposition 2.30. *Let $\mu\in\mathcal{M}(E)$ a measure on E. Let X be a point process on E such that X admits joint intensities with respect to μ. Then for $\alpha^{(n)}$-almost every $(x_1,\dots,x_n)\in E^n$ pairwise distinct, $X^!_{(x_1,\dots,x_n)}$ admits joint intensities $(\rho^{(m)}_{(x_1,\dots,x_n)})_{m\in\mathbb{N}^*}$ and for $\alpha^{(m)}$-almost every $(y_1,\dots,y_m)\in(E\backslash\{x_1,\dots,x_n\})^m$ pairwise distinct, we have*

$$\rho^{(m+n)}(x_1,\dots,x_n,y_1,\dots,y_m) = \rho^{(m)}_{(x_1,\dots,x_n)}(y_1,\dots,y_m)\rho^{(n)}(x_1,\dots,x_n). \tag{12}$$

Moreover, for $\alpha^{(2)}$-almost every $(u,v)\in E^2$ different such that $\rho(u)\rho(v)>0$, we have

$$g(u,v) = \frac{\rho_u(v)}{\rho(v)} = \frac{\rho_v(u)}{\rho(u)}. \tag{13}$$

This rewriting of $g(u,v)$ makes its interpretation even clearer: with the previous interpretations of joint intensities, $g(u,v)$ can be seen as the ratio between the probability that u belongs to X given that v belongs to X and the probability that u belongs to X (and vice versa): if $g(u,v)>1$, it means that X is more likely to have a point at u if it has a point of v; conversely, if $g(u,v)<1$, then X is less likely to have a point at u if it has a point of v. These two situations can be interpreted as a certain attractiveness or repulsiveness of the process.

In the remaining of this document, we will use these tools on examples of point processes to guess if they are more attractive or repulsive (or neither).

3 Examples of point processes

3.1 Poisson point processes

Among all point processes, the Poisson point process is the one that has been studied the most, in particular because calculations are simpler compared to other processes, since the Poisson process is characterized by a property of independence.

Every result of this subsection and its proof can be found in [9] or in [10].

Definition 3.1. *Let α be a σ-finite measure on $(E, \mathcal{E})$. A (simple) point process N on E is said to be a Poisson point process with intensity α if*

1. *for any $A \in \mathcal{E}$, the random variable $N(A)$ has the Poisson distribution $\mathcal{P}(\alpha(A))$,*
2. *for any disjoint subsets $A_1, \ldots, A_n \in \mathcal{E}$, the random variables $N(A_1), \ldots, N(A_n)$ are independent.*

In this case, α is the intensity measure of N.

If $E = \mathbb{R}^d$, and if α is absolutely continuous with respect to the Lebesgue measure with density ρ, ρ is also called intensity of N. (This is the same intensity function $\rho = \rho^{(1)}$ (with respect to $\mu = \text{Leb}$) as in the first section.) If ρ is a constant, then we say that the process N is a uniform or homogeneous Poisson process, this designation will be justified in the rest.

Let us give some results of existence and uniqueness of this type of point processes.

Proposition 3.2.

- *The distribution of a Poisson point process is completely determined by its intensity measure.*
- *Let α be a σ-finite, non atomic measure on $(E, \mathcal{E})$. Then there exists a Poisson point process with intensity α.*

Proposition 3.3. *Let N be a Poisson process on E with intensity α. Then the Laplace transform of N is given by*

$$\mathcal{L}_N(f) = \exp\left(-\int_E (1 - e^{-f(x)})\alpha(\mathrm{d}x)\right). \tag{14}$$

For the rest, we will need notions of integration with respect to a Poisson process. If N is a Poisson process with intensity α, and $f : E \longrightarrow \mathbb{R}$ is a (deterministic) measurable function, we denote, when it exists,

$$N(f) := \int_E f(x)N(\mathrm{d}x) = \sum_{x \in X} f(x),$$

and

$$(N - \alpha)(f) := \int_E f(x)(N - \alpha)(\mathrm{d}x) = \sum_{x \in X} f(x) - \int_E f(x)\alpha(\mathrm{d}x) = \sum_{x \in X} f(x) - \mathbb{E}\left[\sum_{x \in X} f(x)\right].$$

Proposition 3.4. *Let N be a Poisson process on E with intensity α and let $f : E \longrightarrow \mathbb{R}$ and $g : E \longrightarrow \mathbb{R}$ be (deterministic) measurable functions. Then*

- *$N(f)$ exists (i.e. the sum is absolutely convergent with probability 1) if and only if*

$$\int_E (|f(x)| \wedge 1)\alpha(\mathrm{d}x) < +\infty, \tag{15}$$

and in this case,

$$\mathbb{E}\left[\exp(iN(f))\right] = \exp\left(\int_E (e^{if(x)} - 1)\alpha(\mathrm{d}x)\right); \tag{16}$$

- *$(N - \alpha)(f)$ exists (i.e. the integral is absolutely convergent with probability 1) if and only if*

$$\int_E (|f(x)| \wedge f(x)^2)\alpha(\mathrm{d}x) < +\infty, \tag{17}$$

and in this case

$$\mathbb{E}\left[\exp(i(N - \alpha)(f))\right] = \exp\left(\int_E (e^{if(x)} - 1 - if(x))\alpha(\mathrm{d}x)\right); \tag{18}$$

- *if $f, g \in L^2(\alpha)$, then*

$$\mathrm{Cov}(N(f), N(g)) = \mathbb{E}[(N-\alpha)(f), (N-\alpha)(g)] = \int_E f(x)g(x)\alpha(\mathrm{d}x).$$

The following property justifies the naming "uniform Poisson process" introduced previously: in the case where $E = \mathbb{R}^d$ and $\alpha = \rho\,\mathrm{Leb}$, the points are distributed uniformly in the space.

Proposition 3.5. *Let N be a Poisson process on E with intensity α. Let B be a Borel set of E such that $0 < \alpha(B) < +\infty$. Then conditionally on the event $\{N(B) = n\}$, the points of N in B have the same distribution as n independent random variables with distribution $\dfrac{\mathbb{1}(x \in B)\alpha(\mathrm{d}x)}{\alpha(B)}$.*
In particular, if $E = \mathbb{R}^d$ and $\alpha = \rho\,\mathrm{Leb}$, then conditionally on the event $\{N(B) = n\}$, the points of N in B have the same distribution as n independent random variables with distribution $\dfrac{\mathbb{1}(x \in B)\mathrm{d}x}{|B|}$, i.e. they are uniformly distributed.

Corollary 3.6. *Let N be a Poisson process on E with intensity α. Let B be a Borel set of E such that $0 < \alpha(B) < +\infty$. Then for any measurable function $h : \mathcal{M}_p(E) \longrightarrow \mathbb{R}_+$, we have*

$$\mathbb{E}[h(X \cap B)] = \sum_{k=0}^{+\infty} \mathbb{P}(\#(X \cap B) = k)\mathbb{E}[h(X \cap B)|\#(X \cap B) = k] \tag{19}$$

$$= \sum_{k=0}^{+\infty} \frac{\exp(-\alpha(B))}{k!} \int_{B^k} h(\{x_1, \dots, x_k\})\alpha(\mathrm{d}x_1)\dots\alpha(\mathrm{d}x_k), \tag{20}$$

where the term for $k = 0$ is $\exp(-\alpha(B))h(\emptyset)$. In particular, if $E = \mathbb{R}^d$ and if α is absolutely continuous with respect to the Lebesgue measure with intensity function ρ, then

$$\mathbb{E}[h(X \cap B)] = \sum_{k=0}^{+\infty} \frac{\exp(-\alpha(B))}{k!} \int_{B^k} h(\{x_1, \dots, x_k\})\rho(x_1)\dots\rho(x_k)\mathrm{d}x_1\dots\mathrm{d}x_k. \tag{21}$$

The following proposition shows that, unlike some of the following examples, the Poisson process is neither attractive nor repulsive: conditioning on the process having a point at x does not modify the distribution of the process. This is linked to the independence property of the Poisson process. In fact, this property characterizes the Poisson process among the point processes.

Theorem 3.7 (Slivnyak-Mecke)**.** *Let X be a point process on E such that its mean measure α is σ-finite. Then X is a Poisson point process if and only if for α-almost every $x \in E$,*

$$X_x^! \overset{d}{=} X.$$

Actually, this property characterizes the Poisson process among all the random measures, and it can be proven in the same way.

Proof. "⇒": Let us suppose that X is a Poisson process, with intensity α. By Proposition 3.3, the Laplace transform of X is given by

$$\mathcal{L}_X(f) = \exp\left(-\int_E (1 - e^{-f(x)})\alpha(\mathrm{d}x)\right), \quad f \in \mathcal{F}_+(E).$$

Let $f \in \mathcal{F}_+(E)$ and B be a bounded Borel set of E. Then,

$$\mathcal{L}_X(f + t\mathbb{1}_B) = \exp\left(-\int_E (1 - e^{-f(x)-t\mathbb{1}_B(x)})\alpha(\mathrm{d}x)\right),$$

so

$$\frac{\partial}{\partial t}\mathcal{L}_X(f+t\mathbb{1}_B) = \mathcal{L}_X(f+t\mathbb{1}_B)\frac{\partial}{\partial t}\left(\int_E \underbrace{-(1-e^{-f(x)-t\mathbb{1}_B(x)})}_{:=g(x,t)}\alpha(\mathrm{d}x)\right).$$

Yet,

$$\left|\frac{\partial}{\partial t}g(x,t)\right| = \mathbb{1}_B(x)e^{-f(x)-t\mathbb{1}_B(x)} \leq \mathbb{1}_B(x),$$

and $\mathbb{1}_B$ is integrable with respect to α since B is bounded. Hence, we can invert the derivation and the integral, and

$$\frac{\partial}{\partial t}\mathcal{L}_X(f+t\mathbb{1}_B) = -\mathcal{L}_X(f+t\mathbb{1}_B)\int_E \mathbb{1}_B(x)e^{-f(x)-t\mathbb{1}_B(x)}\alpha(\mathrm{d}x).$$

Thus,

$$\frac{\partial}{\partial t}\mathcal{L}_X(f+t\mathbb{1}_B)\Big|_{t=0} = -\int_B \mathcal{L}_X(f)e^{-f(x)}\alpha(\mathrm{d}x).$$

Now, we know from Proposition 2.23 that

$$\frac{\partial}{\partial t}\mathcal{L}_X(f+t\mathbb{1}_B)\Big|_{t=0} = -\int_B \mathcal{L}_{X_x}(f)\alpha(\mathrm{d}x).$$

Hence,

$$\int_B \mathcal{L}_X(f)e^{-f(x)}\alpha(\mathrm{d}x) = \int_B \mathcal{L}_{X_x}(f)\alpha(\mathrm{d}x).$$

This is true for any bounded Borel subset, so it is also true for any Borel subset (by writing the subset as an increasing limit of bounded Borel subsets and using the monotone convergence theorem), and it is in particular true for the Borel subsets $\{x \in E,\ \mathcal{L}_X(f)e^{-f(x)} \leq \mathcal{L}_{X_x}(f)\}$ and $\{x \in E,\ \mathcal{L}_X(f)e^{-f(x)} > \mathcal{L}_{X_x}(f)\}$, for which this equality allows us to conclude that for $\alpha-$almost every $x \in E$,

$$\mathcal{L}_{X_x}(f) = \mathcal{L}_X(f)e^{-f(x)} = \mathcal{L}_{X+\delta_x}(f).$$

Since this is true for any $f \in \mathcal{F}_+(E)$, then $\mathcal{L}_{X_x} = \mathcal{L}_{X+\delta_x}$, and this implies by Corollary 2.8 that $X_x \overset{d}{=} X + \delta_x$, i.e. $X_x^! \overset{d}{=} X$.

"$\Leftarrow$": Let us suppose that X is such that for $\alpha-$almost every $x \in E$, $X_x^! \overset{d}{=} X$, ie $X_x \overset{d}{=} X + \delta_x$. Thanks to Corollary 2.8, to prove that X is a Poisson process, we are going to prove that its Laplace transform is given by

$$\mathcal{L}_X(f) = \exp\left(-\int_E (1-e^{-f(x)})\alpha(\mathrm{d}x)\right), \quad f \in \mathcal{F}_+(E). \tag{22}$$

Actually, we are going to show the equality only for $\mathcal{F}_+(E)$ bounded and whose support is bounded, which is sufficient because for a general $f \in \mathcal{F}_+(E)$, we can write f as a limit of such functions and use the monotone convergence theorem to conclude.

Then, let $f \in \mathcal{F}_+(E)$ be a bounded function such that its support is bounded. Let us show the equality (22) by showing that $F : t \in \mathbb{R}_+ \mapsto \mathcal{L}_X(tf)$ satisfies a differential equation. For $r > 0$, we denote $\tau_r : t \mapsto t + r$. Let $t > 0$, then

$$F'(t) = F' \circ \tau_s(t)\Big|_{s=0}.$$

Yet, for $s > 0$,

$$F' \circ \tau_s(t) = F'(t+s) = F' \circ \tau_t(s) = (F \circ \tau_t)'(s),$$

so

$$F'(t) = (F \circ \tau_t)'(s)|_{s=0} = \frac{\partial}{\partial s}\mathcal{L}_X(tf+sf)|_{s=0}.$$

Thus, using Proposition 2.23,

$$F'(t) = -\int_E f(x)\mathcal{L}_{X_x}(tf)\alpha(\mathrm{d}x),$$

so by hypothesis,

$$F'(t) = -\int_E f(x)\mathcal{L}_{X+\delta_x}(tf)\alpha(\mathrm{d}x) = -\mathcal{L}_X(tf)\int_E f(x)e^{-tf(x)}\alpha(\mathrm{d}x) = -a(t)F(t),$$

where

$$a(t) := \int_E f(x)e^{-tf(x)}\alpha(\mathrm{d}x).$$

Yet,

$$|f(x)e^{-tf(x)}| \leq |f(x)|, \tag{23}$$

and f is integrable with respect to α since it is supposed to be bounded and with bounded support. So, since $t \mapsto f(x)e^{-tf(x)}$ is continuous, a is continuous on $\mathbb{R}_+$, and the differential equation $F' = aF$ with the initial condition $F(0) = 1$ admits an unique solution. Hence F is given by

$$\begin{aligned} F(t) &= \exp\left(-\int_0^t a(s)\mathrm{d}s\right) = \exp\left(-\int_0^t\int_E f(x)e^{-sf(x)}\alpha(\mathrm{d}x)\mathrm{d}s\right) \\ &= \exp\left(-\int_E\left(\int_0^t f(x)e^{-sf(x)}\mathrm{d}s\right)\alpha(\mathrm{d}x)\right) = \exp\left(-\int_E(1-e^{-tf(x)})\alpha(\mathrm{d}x)\right). \end{aligned}$$

where for the third equality, we use Fubini's theorem, justified by (23). Then, in particular

$$F(1) = \mathcal{L}_X(f) = \exp\left(-\int_E(1-e^{-f(x)})\alpha(\mathrm{d}x)\right),$$

which concludes. □

In particular, if N is a Poisson process with intensity α absolutely continuous with respect to a certain measure μ on E with density ρ, then thanks to (12), we deduce that N admits joint intensities $(\rho^{(n)})_{n\in\mathbb{N}^*}$ with respect to μ, and for α^n-almost every $(x_1,\dots,x_n) \in E^n$ pairwise distinct, we have

$$\rho^{(n)}(x_1,\dots,x_n) = \rho(x_1)\dots\rho(x_n). \tag{24}$$

Especially, we have for $\alpha^{(2)}$-almost every $(u,v) \in E^2$ distinct,

$$g(u,v) = 1.$$

With the previous interpretation of g, this confirms the vision of the Poisson process as a process neither attractive or repulsive.

3.2 Reinterpretation of joint intensities and Palm distributions in the finite case

The goal of this subsection is to give, in a nice context, an easy expression of the quantities defined in the previous section, that makes clearer the previous interpretations. This subsection is inspired by [4].

For this part, we assume that E is a bounded Borel set of $\mathbb{R}^d$. Let N be a Poisson process on E with intensity function $\rho \equiv 1$ with respect to the Lebesgue measure. Let X be a point process on E such that the distribution of X is absolutely continuous with respect to the distribution of N with density f. Then for any measurable function $h : \mathcal{M}_p(E) \longrightarrow \mathbb{R}_+$,

$$\mathbb{E}[h(X)] = \mathbb{E}[h(N)f(N)], \tag{25}$$

so, by (21),

$$\mathbb{E}[h(X)] = \sum_{k=0}^{+\infty} \frac{\exp(-|E|)}{k!} \int_{E^k} h(\{x_1, \ldots, x_k\}) f(\{x_1, \ldots, x_k\}) \mathrm{d}x_1 \ldots \mathrm{d}x_k. \tag{26}$$

Thus, for $x_1, \ldots, x_k \in E$, the quantity $\exp(-|E|) f(\{x_1, \ldots, x_n\})$ can be interpreted as the infinitesimal probability that the process X is equal to the set $\{x_1, \ldots, x_n\}$. The term $\exp(-|E|)$ will disappear in the sequel because, since we want to consider conditional distributions, we will consider ratios of this type of quantities.

Definition 3.8. *With the previous notations, assume that f is hereditary, i.e. that for any pairwise distinct $x_0, x_1 \ldots, x_n \in E$, $f(\{x_0, x_1 \ldots, x_n\}) > 0$ implies $f(\{x_1 \ldots, x_n\}) > 0$. We define the n^{th} order Papangelou conditional intensity by*

$$\lambda^{(n)}(x_1, \ldots, x_n, Y) = \frac{f(Y \cup \{x_1, \ldots, x_n\})}{f(Y)},$$

for pairwise distinct $x_1, \ldots, x_n \in E$, $Y \in \mathcal{M}_p(E \backslash \{x_1, \ldots, x_n\})$, with the writing convention $\frac{0}{0} = 0$.

With the previous interpretation of f, $\lambda^{(n)}(x_1, \ldots, x_n, Y)$ can be interpreted as the (infinitesimal) conditional probability that $x_1, \ldots, x_n$ belong to X given that $X \cap (E \backslash \{x_1, \ldots, x_n\})$ is equal to Y.

Proposition 3.9. *With the previous setting and notations, X admits joint intensities $(\rho^{(n)})_{n \in \mathbb{N}^*}$ with respect to the Lebesgue measure, and for pairwise distinct $x_1, \ldots, x_n \in E$, we have*

$$\rho^{(n)}(x_1, \ldots, x_n) = \mathbb{E}[f(Z \cup \{x_1, \ldots, x_n\})],$$

and if f is hereditary, this can be rewritten as follows

$$\rho^{(n)}(x_1, \ldots, x_n) = \mathbb{E}[\lambda^{(n)}(x_1, \ldots, x_n, X)].$$

With the previous interpretation of $\lambda^{(n)}$, $\rho^{(n)}(x_1, \ldots, x_n)$ can be interpreted as the probability that $x_1, \ldots, x_n$ belong to X, and we return to the previous interpretation of $\rho^{(n)}$.

Proof. Let $h : E^n \longrightarrow \mathbb{R}_+$ be a measurable function. We define, for $Y \in \mathcal{M}_p(E)$,

$$H(Y) = \sum_{x_1, \ldots, x_n \in Y}^{\neq} h(\{x_1, \ldots, x_n\}).$$

Then, by using (26) and (21),

$$\begin{aligned}
&\mathbb{E}\left[\sum_{x_1, \ldots, x_n \in X}^{\neq} h(\{x_1, \ldots, x_n\})\right] = \mathbb{E}[H(X)] \\
&= \sum_{k=0}^{+\infty} \frac{\exp(-|E|)}{k!} \int_{E^k} \underbrace{H(\{x_1, \ldots, x_k\})}_{=0 \text{ if } k<n} f(\{x_1, \ldots, x_k\}) \mathrm{d}x_1 \ldots \mathrm{d}x_k \\
&= \sum_{k=n}^{+\infty} \frac{\exp(-|E|)}{k!} \int_{E^k} \left(\sum_{x_{i_1}, \ldots, x_{i_n} \in \{x_1, \ldots, x_k\}}^{\neq} h(x_{i_1}, \ldots, x_{i_n}) \right) f(\{x_1, \ldots, x_k\}) \mathrm{d}x_1 \ldots \mathrm{d}x_k \\
&= \sum_{k=n}^{+\infty} \frac{\exp(-|E|)}{k!} \int_{E^k} \frac{k!}{(k-n)!} h(x_1, \ldots, x_n) f(\{x_1, \ldots, x_k\}) \mathrm{d}x_1 \ldots \mathrm{d}x_k \\
&= \int_{E^n} h(x_1, \ldots, x_n) \left(\sum_{k=n}^{+\infty} \frac{\exp(-|E|)}{(k-n)!} \int_{E^{k-n}} f(\{x_1, \ldots, x_n\} \cup \{x_{n+1}, \ldots, x_k\}) \mathrm{d}x_{n+1} \ldots \mathrm{d}x_k \right) \mathrm{d}x_1 \ldots \mathrm{d}x_n
\end{aligned}$$

$$= \int_{E^n} h(x_1, \dots, x_n) \left(\sum_{k=0}^{+\infty} \frac{\exp(-|E|)}{k!} \int_{E^k} f(\{x_1, \dots, x_n\} \cup \{y_1, \dots, y_k\}) \mathrm{d}y_1 \dots \mathrm{d}y_k \right) \mathrm{d}x_1 \dots \mathrm{d}x_n$$
$$= \int_{E^n} h(x_1, \dots, x_n) \mathbb{E}[f(N \cup \{x_1, \dots, x_n\})] \mathrm{d}x_1 \dots \mathrm{d}x_n.$$

Thus, $(x_1, \dots, x_n) \mapsto \mathbb{E}[f(Z \cup \{x_1, \dots, x_n\})]$ satisfies (6), so it is equal to $\rho^{(n)}$. The rewriting in the case where f is hereditary is clear thanks to (25). □

Proposition 3.10. *With the previous setting and notations, for pairwise $x_1, \dots, x_n \in E$, the n^{th} reduced Palm distribution of X at $(x_1, \dots, x_n)$, previously denoted as $P_X^{!(x_1,\dots,x_n)}$, is absolutely continuous with respect to the distribution of N with density $f_{(x_1,\dots,x_n)}$, given by*

$$f_{(x_1,\dots,x_n)}(Y) = \frac{f(Y \cup \{x_1, \dots, x_n\})}{\rho^{(n)}(x_1, \dots, x_n)},$$

for $Y \in \mathcal{M}_p(E \backslash \{x_1, \dots, x_n\})$.

Proof. The idea is the same as for the previous proof. Let $h : E^n \times \mathcal{M}_p(E) \longrightarrow \mathbb{R}_+$ be a measurable function. We define, for $Y \in \mathcal{M}_p(E)$,

$$\tilde{H}(Y) = \sum_{x_1,\dots,x_n \in Y}^{\neq} h(\{x_1, \dots, x_n, Y \backslash \{x_1, , \dots, x_n\}\}).$$

Then, by using again (26) and (21),

$$\mathbb{E}\left[\sum_{x_1,\dots,x_n \in X}^{\neq} h(\{x_1, \dots, x_n, X \backslash \{x_1, \dots, x_n\}) \right] = \mathbb{E}[\tilde{H}(X)]$$
$$= \sum_{k=0}^{+\infty} \frac{\exp(-|E|)}{k!} \int_{E^k} \underbrace{\tilde{H}(\{x_1, \dots, x_k\})}_{=0 \text{ if } k<n} f(\{x_1, \dots, x_k\}) \mathrm{d}x_1 \dots \mathrm{d}x_k$$
$$= \sum_{k=n}^{+\infty} \frac{\exp(-|E|)}{(k-n)!} \int_{E^k} h(x_1, \dots, x_n, \{x_{n+1}, \dots, x_k\}) f(\{x_1, \dots, x_k\}) \mathrm{d}x_1 \dots \mathrm{d}x_k$$
$$= \int_{S^n} \left(\sum_{k=0}^{+\infty} \frac{\exp(-|E|)}{k!} \int_{S^k} h(x_1, \dots, x_n, \{y_1, \dots, y_k\}) f(\{x_1, \dots, x_n\} \cup \{y_1, \dots, y_k\}) \mathrm{d}y_1 \dots \mathrm{d}y_k \right) \mathrm{d}x_1 \dots \mathrm{d}x_n$$
$$= \int_{S^n} \mathbb{E}[h(x_1, \dots, x_n, N) f(N \cup \{x_1, \dots, x_n\})] \mathrm{d}x_1 \dots \mathrm{d}x_n$$
$$= \int_{S^n} \mathbb{E}[h(x_1, \dots, x_n, N) f_{(x_1,\dots,x_n)}(N)] \underbrace{\rho^{(n)}(x_1, \dots, x_n) \mathrm{d}x_1 \dots \mathrm{d}x_n}_{=\alpha^{(n)}(\mathrm{d}x_1 \dots \mathrm{d}x_n)}$$
$$= \int_{S^n \times \mathcal{M}_p(E)} h(x_1, \dots, x_n, Y) f_{(x_1,\dots,x_n)}(Y) P_N(\mathrm{d}Y) \alpha^{(n)}(\mathrm{d}x_1 \dots \mathrm{d}x_n),$$

where P_N designates the distribution of N. Thus, comparing this to formula (11), we conclude that the n^{th} reduced Palm distribution of X at $(x_1, \dots, x_n)$ is absolutely continuous with respect to P_N with density $f_{(x_1,\dots,x_n)}$. □

With the previous interpretations of f and $\rho^{(n)}$, $f_{(x_1,\dots,x_n)}$ can be interpreted as the ratio between the probability that X is equal to $Y \cup \{x_1, \dots, x_n\}$ and the probability that $x_1, \dots, x_n$ belong to X, thus it can be seen as a conditional probability that $X \backslash \{x_1, \dots, x_n\}$ is equal to Y, given that $x_1, \dots, x_n$ belong to X. Hence, this reinforces the idea that $P_X^{!(x_1,\dots,x_n)}$ represents the conditional distribution of $X \backslash \{x_1, \dots, x_n\}$ given that $x_1, \dots, x_n \in X$.

3.3 Cox processes

From the Poisson point process, we can built point processes that are attractive or repulsive, according to the definition we gave before: it is the case for Cox processes. We assume again for this part that $E \subset \mathbb{R}^d$. This subsection is based on the results of [3], [12] and [4].

Definition 3.11. *Let $(\Lambda(x))_{x\in E}$ be a non-negative random field on E such that Λ is locally integrable a.s., i.e. for any bounded Borel set B, the integral $\int_B \Lambda(x)\mathrm{d}x$ is finite a.s. Then a point process X is said to be a Cox process driven by Λ if the distribution of X given Λ is the distribution of a Poisson process with intensity Λ.*

For a more formal definition of this object, see [1].

By conditioning on Λ, we deduce from (24) that if Λ has moments of any order, then X admits joint intensities with respect to the Lebesgue measure.

Proposition 3.12. *Let $(\Lambda(x))_{x\in E}$ be a non-negative random field on E such that Λ is locally integrable a.s. and X a Cox process driven by Λ. If Λ has moments of any order, then X admits joint intensities given by*

$$\rho^{(n)}(x_1,\dots,x_n) = \mathbb{E}\left[\prod_{i=1}^{n}\Lambda(x_i)\right], \tag{27}$$

for $n \in \mathbb{N}^*$ and pairwise distinct $x_1,\dots,x_n \in E$.
In particular, g is given by

$$g(u,v) = \frac{\mathbb{E}[\Lambda(u)\Lambda(v)]}{\mathbb{E}[\Lambda(u)]\mathbb{E}[\Lambda(v)]}, \tag{28}$$

for distinct $u, v \in E$.

In particular, we deduce from (28) that, according to the definitions we gave previously, a Cox process can be attractive or repulsive depending on the covariance of Λ. However, Cox processes are mostly used to describe aggregated point patterns, for instance in biology, so the most practical models of Cox processes actually verify that $g \geq 1$. Let us look at a classical example of a particular Cox process.

Definition 3.13. *Let $(Y(x))_{x\in E}$ be a Gaussian process on E with mean function μ and covariance function c such that the random field Λ defined by $\Lambda(x) := \exp(Y(x))$ is locally integrable. Then a point process X is said to be a log Gaussian Cox process (LGCP) if it is a Cox process driven by Λ.*

Proposition 3.14. *With the previous notations and hypothesis, the first intensity of X is given by*

$$\rho(x) = \rho^{(1)}(x) = \exp\left(\mu(x) + \frac{c(x,x)}{2}\right),$$

for $x \in E$ and the correlation function of X is given by

$$g(u,v) = e^{c(u,v)},$$

for distinct $u, v \in E$. The higher order joint intensities are given by

$$\rho^{(n)}(x_1,\dots,x_n) = \left(\prod_{i=1}^{n}\rho(x_i)\right)\left(\prod_{1\leq i<j\leq n} g(x_i,x_j)\right),$$

for $n \geq 2$ and pairwise distinct $x_1,\dots,x_n \in E$.

Proof. Let $x \in E$. Then $Y(x)$ is normally distributed with mean $\mu(x)$ and variance $c(x,x)$, so thanks to (27),

$$\rho(x) = \mathbb{E}[\Lambda(x)] = \mathbb{E}[\exp(Y(x))] = \exp\left(\mu(x) + \frac{1}{2}c(x,x)\right).$$

More generally, let pairwise distinct $x_1, \dots, x_n \in E$, then $\sum_{i=1}^{n} Y(x_i)$ is normally distributed with mean $\sum_{i=1}^{n} \mu(x_i)$ and variance $\sum_{i=1}^{n} c(x_i,x_i) + 2\sum_{1\le i<j\le n} c(x_i,x_j)$, so thanks to (27) again,

$$\begin{aligned}
\rho^{(n)}(x_1,\dots,x_n) &= \mathbb{E}\left[\prod_{i=1}^{n}\exp(Y(x_i))\right] = \mathbb{E}\left[\exp\left(\sum_{i=1}^{n} Y(x_i)\right)\right] \\
&= \exp\left(\sum_{i=1}^{n}\mu(x_i) + \frac{1}{2}\left(\sum_{i=1}^{n} c(x_i,x_i) + 2\sum_{1\le i<j\le n} c(x_i,x_j)\right)\right) \\
&= \prod_{i=1}^{n} \underbrace{\exp\left(\mu(x_i) + \frac{1}{2}c(x_i,x_i)\right)}_{\rho(x_i)} \times \prod_{1\le i<j\le n}\exp(c(x_i,x_j)).
\end{aligned}$$

In particular, for $i \neq j$,

$$g(x_i,x_j) = \frac{\rho^{(2)}(x_i,x_j)}{\rho(x_i)\rho(x_j)},$$

which concludes. □

Thus, for $c \ge 0$, which is an usual setting, we have $g \ge 1$, which means, as seen before, that the process is attractive. This attractiveness is also clear while studying the Palm distributions of the process. Indeed, the following result shows that more generally, if $c \ge 0$, for any distinct $x, x_1, \dots, x_n$, we have

$$\rho_{(x_1,\dots,x_n)}(x) \ge \rho(x).$$

Proposition 3.15. *Let pairwise $x_1, \dots, x_n \in E$. With the previous notations and hypothesis, $X^{!}_{(x_1,\dots,x_n)}$ has the same joint intensities as a LGCP for a gaussian process with mean function $\mu_{(x_1,\dots,x_n)}$ defined by*

$$\mu_{(x_1,\dots,x_n)}(x) = \mu(x) + \sum_{i=1}^{n} c(x,x_i), \quad x \in E,$$

and covariance function c.

Proof. We use (12) with the previous result: for pairwise distinct $y_1, \dots, y_m \in E\backslash\{x_1,\dots,x_n\}$,

$$\begin{aligned}
\rho^{(m)}_{(x_1,\dots,x_n)}(y_1,\dots,y_m) &= \frac{\rho^{(m+n)}(x_1,\dots,x_n,y_1,\dots,y_m)}{\rho^{(n)}(x_1,\dots,x_n)} \\
&= \left(\prod_{j=1}^{m}\rho(y_j)\right)\left(\prod_{1\le i<j\le m} g(y_i,y_j)\right)\left(\prod_{\substack{1\le i\le n\\ 1\le j\le m}} g(x_i,y_j)\right) \\
&= \left(\prod_{j=1}^{m}\left(\rho(y_j)\prod_{1\le i\le n} g(x_i,y_j)\right)\right)\left(\prod_{1\le i<j\le m} g(y_i,y_j)\right)
\end{aligned}$$

$$= \left(\prod_{j=1}^{m} \exp\left(\underbrace{\mu(y_j) + \sum_{i=1}^{n} c(x_i, y_j)}_{=\mu_{(x_1,\dots,x_n)}(y_j)} + \frac{c(y_j, y_j)}{2} \right) \right) \left(\prod_{1 \le i < j \le m} g(y_i, y_j) \right),$$

which concludes. □

From this result, we could have concluded that there was equality in distribution between $X^!_{(x_1,\dots,x_n)}$ and the corresponding LGCP using Proposition 2.12, but the hypothesis 4 is not clearly verified. However, it is still possible to conclude that there is equality in law by dodging this problem, see [3].

Proposition 3.16. *Let pairwise $x_1, \dots, x_n \in E$. With the previous notations and hypothesis, $X^!_{(x_1,\dots,x_n)}$ is a LGCP for Gaussian process $Y_{(x_1,\dots,x_n)}$ with mean function $\mu_{(x_1,\dots,x_n)}$ and covariance c.*

Actually, we can take

$$Y_{(x_1,\dots,x_n)}(x) = Y(x) + \sum_{i=1}^{n} c(x, x_i),$$

and one can show (for instance by showing the equality of Laplace transforms, using expressions of Laplace transform for Cox processes and for thinning of point processes, see Corollary 2.3.2. and Proposition 2.2.6. of [1]) that if $c \ge 0$, X has the same distribution as an independent thinning of $X^!_{(x_1,\dots,x_n)}$ with inclusion probabilities

$$p(x) = \exp\left(- \sum_{i=1}^{n} c(x, x_i) \right).$$

This again justifies the attractiveness of the process: conditioning on X having points at $x_1, \dots, x_n$ attracts so many points that some should be removed randomly from the obtained process to get back on our feet.

Now that we have encountered an example of attractive process, let us see an example of a repulsive process.

3.4 Determinantal processes

Let us introduce repulsive point processes, that come from quantum mechanics : determinantal processes. This subsection follows [1] and [7].

Let $\mu \in \mathcal{M}(E)$.

Definition 3.17. *Let $K : E^2 \longrightarrow \mathbb{C}$ a measurable function. A (simple) point process X on E is said to be a determinantal process with background measure μ and kernel K if it admits joint intensities with respect to μ given by*

$$\rho^{(n)}(x_1, \dots, x_n) = \det(K(x_i, x_j))_{1 \le i,j \le n}, \tag{29}$$

for $n \ge 1$ and μ^n-almost every $x_1, \dots, x_n \in E$.

So that the joint intensities (29) are well defined, K must verify some hypothesis: the joint intensities must be non negative, defined almost everywhere and locally integrable. Let us give a setting for which there is hope that a process with such intensities will be defined. To do this, let us take a quick look at what we will need about integral operators later on.

Reminder 3.18. *Let $K : E^2 \longrightarrow \mathbb{C}$ be a measurable function. Assume $K \in L^2(E^2, \mu^2)$. Then we can associate to K an operator $\mathcal{K}$ defined on $L^2(E, \mu)$ by*

$$\mathcal{K}(f)(x) = \int_E K(x, y) f(y) \mu(\mathrm{d}y), \quad x \in E,\ f \in L^2(E, \mu).$$

Then this operator is compact, so the set of its eigenvalues is at most countable, 0 is the only possible accumulation point and every non-zero eigenvalue has finite multiplicity.
The operator $\mathcal{K}$ is hermitian if and only if for μ^2-almost every $(x,y) \in E^2$,

$$K(x,y) = \overline{K(y,x)}.$$

In this case, there exists an orthonormal basis of $L^2(E,\mu)$ composed of eigenvectors $(\varphi_n)_{n\in\mathbb{N}^*}$ of $\mathcal{K}$ with associated real eigenvalues $(\lambda_n)_{n\in\mathbb{N}^*}$ such that for μ^2-almost every $(x,y) \in E^2$,

$$K(x,y) = \sum_{n\in\mathbb{N}^*} \lambda_n \varphi_n(x)\overline{\varphi_n(y)}.$$

The hermitian operator $\mathcal{K}$ is said to be trace class if

$$\sum_{n\in\mathbb{N}^*} |\lambda_n| < +\infty.$$

The hermitian operator $\mathcal{K}$ is non-negative definite if and only if for all $n \geq 1$, $\lambda_n \geq 0$.

With the previous notations, let us make the following hypothesis on K:

- $K \in L^2(E^2, \mu^2)$,
- for μ^2-almost every $(x,y) \in E^2$, $K(x,y) = \overline{K(y,x)}$,
- for μ^n-almost every $x_1, \dots, x_n \in E^n$, $\det(K(x_i,x_j))_{1\leq i,j\leq n} \geq 0$,
- for all $n \geq 1$, $\lambda_n \geq 0$ and $\sum_{n\in\mathbb{N}^*} |\lambda_n| < +\infty$.

In the following, these hypotheses will be denoted as $(\mathcal{H})$. From the previous reminder, this implies that $\mathcal{K}$ is hermitian, non-negative definite and trace class. One can show that it is sufficient to guarantee that for any $n \geq 1$, the joint intensity defined by (29) is well defined and non-negative μ^n-almost everywhere and locally integrable. However, it may not guarantee the uniqueness or the existence of a process with such joint intensities. Let us deal with uniqueness first.

Proposition 3.19. *Let K satisfy hypotheses $(\mathcal{H})$. Assume that there exists a point process X with joint intensities given by (29). Then for any compact $B \subset E$, there exists $c_1, c_2 > 0$ such that for any $k \geq 1$,*

$$\mathbb{P}(X(B) \geq k) \leq c_1 e^{-c_2 k}.$$

Thus, there exists (in law) at most one determinantal process with background measure μ and kernel K.

Proof. Let us use Hadamard's inequality: let $A = (a_{i,j})_{1\leq i,j\leq n}$ be a hermitian non-negative definite matrix, then

$$\det(A) \leq \prod_{i=1}^{n} a_{i,i}. \tag{30}$$

Let $B \subset E$ be a compact subset. Since for μ^n-almost every $x_1, \dots, x_n \in E$, the matrix $(K(x_i,x_j))_{1\leq i,j\leq n}$ is hermitian non-negative definite, we have

$$\begin{aligned}\mathbb{E}\left[\binom{X(B)}{n} n!\right] &= \int_{B^n} \det(K(x_i,x_j))_{1\leq i,j\leq n}\, \mu(\mathrm{d}x_1)\dots\mu(\mathrm{d}x_n) \\ &\leq \int_{B^n} \left(\prod_{i=1}^{n} K(x_i,x_i)\right) \mu(\mathrm{d}x_1)\dots\mu(\mathrm{d}x_n) = \left(\underbrace{\int_B K(x,x)\mu(\mathrm{d}x)}_{:=c}\right)^n.\end{aligned}$$

Hence, for $s > 0$, we have by Fubini's theorem

$$\mathbb{E}[(1+s)^{X(B)}] = \mathbb{E}\left[\sum_{n=0}^{+\infty} \binom{X(B)}{n} s^n\right] \quad (\text{since } \binom{X(B)}{n} = 0 \text{ for } n > X(B))$$
$$= \sum_{n=0}^{+\infty} \mathbb{E}\left[\binom{X(B)}{n}\right] s^n \leq \sum_{n=0}^{+\infty} \frac{c^n}{n!} s^n = e^{sc}.$$

And then, by Markov's inequality, for any $k \geq 1$, we have

$$\begin{aligned}\mathbb{P}(X(B) \geq k) &= \mathbb{P}((1+s)^{X(B)} \geq (1+s)^k) \\ &\leq (1+s)^{-k}\, \mathbb{E}[(1+s)^{X(B)}] \\ &\leq \underbrace{(1+s)}_{:=e^{c_2}}{}^{-k}\, \underbrace{e^{sc}}_{:=c_1},\end{aligned}$$

which concludes for the wanted inequality. The uniqueness (in law) follows from this inequality and from the Proposition 2.12. □

For the existence of a determinantal process given a kernel K, the proofs are more technical and we will not go into details here, see [7] or [1] for more details. Note that hypotheses $(\mathcal{H})$ are not necessary for K to be the kernel of a determinantal process, for instance K does not necessarily need to be hermitian.

Theorem 3.20. *Let K satisfy hypotheses $(\mathcal{H})$. If the spectrum of $\mathcal{K}$ is contained in $[0,1]$, then there exists a determinantal process with kernel K. In particular, it is the case if K is a projection kernel, i.e. if all the non-zero eigenvalues are equal to 1.*

Now, let us see a classical example of determinantal process given by spectrum of some random matrices. Examples of these processes can also be found in many other contexts, with random walks, randomly chosen spanning trees of graphs, zeros of random functions, etc.

Theorem 3.21 (Wigner)**.** *Let A be a $n \times n$ matrix whose coefficients are independent and identically distributed standard complex gaussian random variables (i.e. distributed as the sum $X + iY$ of two independent centered random variables normally distributed with variance $\frac{1}{2}$). Let $M = \frac{A+A^*}{\sqrt{2}}$. Then the set of eigenvalues of M is a determinantal point process on $\mathbb{R}$ with background measure*

$$\mu(\mathrm{d}x) = \frac{1}{\sqrt{(2\pi)}} e^{-\frac{x^2}{2}}\, \mathrm{d}x,$$

and kernel

$$K_n(x,y) = \sum_{k=0}^{n} H_k(x) H_k(y),$$

where $(H_k)_{k\in\mathbb{N}}$ are Hermite polynomials.

Remark 3.22. *As mentioned in the beginning of this part and as we will see after, determinantal processes are repulsive. This makes sense given this example. Indeed, even in the deterministic case, we can observe a phenomenon of repulsiveness for the set of eigenvalues of a matrix. Let $A = \begin{pmatrix} a_1 & a_3 \\ a_3 & a_4 \end{pmatrix}$ be a (deterministic) symmetric matrix with $a_3 \neq 0$. Then if λ_1 and λ_2 designate the two eigenvalues of A, we have $|\lambda_1 - \lambda_2| \geq 2|a_3| > 0$.*

The following proposition highlights the repulsiveness of determinantal processes.

Proposition 3.23. *Let X be a determinantal process with kernel K satisfying hypotheses $(\mathcal{H})$. Then for μ^n-almost every $(x_1, \ldots, x_n) \in E^n$,*

$$\rho^{(n)}(x_1, \ldots, x_n) \leq \rho(x_1) \ldots \rho(x_n).$$

In particular, for μ^2-almost every $(u, v) \in E^2$ distinct,

$$g(u,v) \leq 1.$$

Proof. This follows once again from Hadamard's inequality (30). □

Finally, one can show (see [1]) that the reduced Palm version of a determinantal process at a fixed point is still a determinantal process.

Proposition 3.24. *Let X be a determinantal process with kernel K satisfying hypotheses $(\mathcal{H})$. Then for μ-almost every $x \in E$, the reduced Palm version X_x of X at x is a determinantal process with background measure μ and kernel K_x given by*

$$K_x(u,v) = \frac{1}{K(x,x)} \det \begin{pmatrix} K(u,v) & K(u,x) \\ K(x,v) & K(x,x) \end{pmatrix} = K(u,v) - \frac{K(u,x)K(x,v)}{K(x,x)},$$

for μ^2-almost every $(u,v) \in E^2$. More generally, for μ^n-almost every $(x_1,\dots,x_n) \in E^n$ pairwise distinct, the n^{th} reduced Palm version $X_{(x_1,\dots,x_n)}$ of X at $(x_1,\dots,x_n)$ is a determinantal process with background measure μ and kernel $K_{(x_1,\dots,x_n)}$ that can be expressed from K.

4 Random grains model and convergence results

In this section, we will change topics a bit by discussing a random grains model, that is close to the models that I should study during my future PhD thesis. All this section follows [8] and [2].

4.1 Presentation of the model

The goal is to look at a random grains model: first we consider a certain point process, which will represent the centers of the grains, and then we associate a mark with each point of the process, i.e. a random variable representing the volume of the grain associated with that point. Once the model has been defined, we will then look at what happens at the limit if we add more and more centers (and therefore more grains) while making the volume of the grains go to zero.

Let $C \subset \mathbb{R}^d$ be a bounded measurable set such that $|C| = 1$ and $|\partial C| = 0$, where $\partial C = \bar{C} \backslash \mathring{C}$. For instance, one can take $C = B(0, r_0)$, with the correct choice of r_0 to satisfy $|C| = 1$.

Let X be a Poisson process with intensity measure $\lambda \mathrm{d}x$, it represents the centers of the grains. With each $x \in X$, we associate a random variable V_x that will control the volume of the grain associated to x. We assume that the (V_x) are independent copies of a positive random variable V such that $\mathbb{E}[V] = 1$. Actually, the volume of the grains will be given by ρV_x, where $\rho > 0$ will be the parameter on which we will act to make the volume of the grains go to 0. Let us denote F the probability distribution of V. Then the probability distribution of ρV is given by $F_\rho(v) = F(\frac{v}{\rho})$ for any $v > 0$ and ρ is the mean volume of the grains. Thus, the process $N_{\lambda,\rho} := ((x, \rho V_x))_{x \in X}$ is a Poisson process on $\mathbb{R}^d \times \mathbb{R}_+$ with intensity measure $\lambda \mathrm{d}x F_\rho(\mathrm{d}v)$, see Marking theorem in [10].

Then, we want to study for instance the number of grains covering a certain point $y \in \mathbb{R}^d$ given by

$$J_{\lambda,\rho}(y) := \#\{x \in X,\ y \in x + (\rho V_x)^{1/d} C\},$$

or the measure of the place taken by the grains within a certain measurable set $A \subset \mathbb{R}^d$, given by

$$J_{\lambda,\rho}(A) := \sum_{x \in X} |A \cap (x + (\rho V_x)^{1/d} C)|.$$

More generally, we want to study

$$J_{\lambda,\rho}(\phi) := \sum_{x \in X} \phi(x + (\rho V_x)^{1/d} C),$$

where ϕ is a measure on $\mathbb{R}^d$. The previous examples correspond to $J_{\lambda,\rho}(\phi)$ for $\phi = \delta_x$ and $\phi = |A \cap .|$ respectively. Then $J_{\lambda,\rho}(\phi)$ can be written as the following stochastic integral with respect to $N_{\lambda,\rho}$:

$$J_{\lambda,\rho}(\phi) = \int_{\mathbb{R}^d} \int_{\mathbb{R}_+} \phi(x + v^{1/d} C) N_{\lambda,\rho}(\mathrm{d}x, \mathrm{d}v). \tag{31}$$

To ensure the good definition of this object, we will work with the following linear space:

$$M^1 := \{\phi \text{ signed measure on } \mathbb{R}^d,\ \|\phi\|_1 < +\infty\},$$

where $\|\phi\|_1$ is the total variation of ϕ. Indeed, for $\phi \in M^1$, we have by Fubini's theorem,

$$\begin{aligned}
\int_{\mathbb{R}^d} \int_{\mathbb{R}_+} |\phi(x + v^{1/d} C)| \lambda \mathrm{d}x F_\rho(\mathrm{d}v) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} \underbrace{\mathbb{1}(y \in (x + v^{1/d} C))}_{\mathbb{1}\left(\frac{y-x}{v^{1/d}} \in C\right)} |\phi|(\mathrm{d}y) \lambda \mathrm{d}x F_\rho(\mathrm{d}v) \\
&= \lambda \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbb{1}\left(\frac{y-x}{v^{1/d}} \in C\right) \mathrm{d}x |\phi|(\mathrm{d}y) F_\rho(\mathrm{d}v) \\
&= \lambda \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} \underbrace{\int_{\mathbb{R}^d} v \mathbb{1}\left(u \in C\right) \mathrm{d}u}_{=v} |\phi|(\mathrm{d}y) F_\rho(\mathrm{d}v) \\
&\leq \lambda \|\phi\|_1 \int_{\mathbb{R}_+} v F_\rho(\mathrm{d}v) = \lambda \|\phi\|_1 \rho < +\infty,
\end{aligned}$$

where $|\phi|$ designates the total variation measure of ϕ. So by Proposition 3.4, the integral in (31) is well defined for $\phi \in M^1$. In the sequel, if $\phi \in L^1(R^d)$, we will identify ϕ with its associated measure in M^1 defined by $\phi(x)\mathrm{d}x$. In the same way, for any measurable set A, we identify A with $\mathbb{1}_A \in L^1(R^d) \subset M^1$.
In the following, the convergence results will be stated for measures in more restricted subsets of M^1, which we define here.

Definition 4.1. *Let $\alpha \in (0,1)$. We define*

$$M^\alpha := \left\{ \phi \in M^1,\ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\phi|(\mathrm{d}x)|\phi|(\mathrm{d}y)}{|x-y|^{(1-\alpha)d}} < +\infty \right\}.$$

Proposition 4.2. *Let $0 < \alpha_1 < \alpha_2 < 1$. Then*

$$L^1 \cap L^2 \subset M^{\alpha_1} \subset M^{\alpha_2} \subset M^1.$$

Now, let us see a condition for the model to have long-range dependence.

Lemma 4.3. *For $r > 0$, let us denote B_r the open ball centered at the origin with radius r. We say that $J_{\lambda,\rho}$ has long-range dependence if*

$$\lim_{r \to +\infty} |\mathrm{Cov}\left(J_{\lambda,\rho}(B_1), J_{\lambda,\rho}(B_r \backslash B_1)\right)| = +\infty. \tag{32}$$

Then $J_{\lambda,\rho}$ has long-range dependence if and only if $\mathbb{E}[V^2] = +\infty$.

Proof. For $r > 0$, by Proposition 3.4 we have

$$\mathrm{Cov}\left(J_{\lambda,\rho}(B_1), J_{\lambda,\rho}(B_r \backslash B_1)\right) = \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} |B_1 \cap (x + (\rho V_x)^{1/d} C)||(B_r \backslash B_1) \cap (x + (\rho V_x)^{1/d} C)| \lambda \mathrm{d}x F_\rho(\mathrm{d}v),$$

so thanks to the dominated convergence theorem,

$$L := \lim_{r \to +\infty} |\mathrm{Cov}\left(J_{\lambda,\rho}(B_1), J_{\lambda,\rho}(B_r \backslash B_1)\right)| = \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} |B_1 \cap (x + (\rho V_x)^{1/d} C)||B_1^c \cap (x + (\rho V_x)^{\frac{1}{d}} C)| \lambda \mathrm{d}x F_\rho(\mathrm{d}v).$$

First, we have

$$|B_1 \cap (x + (\rho V_x)^{1/d} C)| \leq v,$$

and by Fubini's theorem,

$$\begin{aligned}\int_{\mathbb{R}^d} |B_1^c \cap (x + (\rho V_x)^{1/d} C)| \mathrm{d}x &= \int_{\mathbb{R}^d} \int_{B_1} \mathbb{1}(y \in x + v^{1/d} C) \mathrm{d}y \mathrm{d}x \\ &= \int_{B_1} \underbrace{\int_{\mathbb{R}^d} \mathbb{1}\left(\frac{y-x}{v^{1/d}} \in C\right) \mathrm{d}x}_{=v} \mathrm{d}y = v|B_1|.\end{aligned}$$

Hence, we have

$$L \leq |B_1| \lambda \int_{\mathbb{R}_+} v^2 F_\rho(\mathrm{d}v) = |B_1| \lambda \rho^2 \mathbb{E}[V^2].$$

Furthermore,

$$|B_1 \cap (x + (\rho V_x)^{1/d} C)| \geq v - |B_1|,$$

so

$$\begin{aligned}L &\geq \lambda \int_{\mathbb{R}_+} (v - |B_1|) \int_{\mathbb{R}^d} |B_1 \cap (x + (\rho V_x)^{1/d} C)| \mathrm{d}x F_\rho(\mathrm{d}v) \\ &= \lambda |B_1| \int_{\mathbb{R}_+} v(v - |B_1|) F_\rho(\mathrm{d}v) = \lambda |B_1| (\rho^2 \mathbb{E}[V^2] - \rho).\end{aligned}$$

Thus, we have

$$\lambda |B_1| (\rho^2 \mathbb{E}[V^2] - \rho) \leq L \leq |B_1| \lambda \rho^2 \mathbb{E}[V^2],$$

which gives the equivalence. □

In the following, we will consider the case without long-range dependence and one case with long-range dependence, and we will see that unlike the first case, in the second we will have several limit theorems, depending on the number of "large" grains of the model.

Now, let us see what happens when λ goes to $+\infty$ and ρ goes to 0, that is to say when the number of grains goes to infinity, while the volume of the grains goes to 0. In the following limits, λ will be considered as a function of ρ such that $\lambda(\rho) \xrightarrow[\rho \to 0]{} +\infty$..

4.2 Convergence result without long-range dependence

Let us start with the case where the model has no long-range dependence. This case is easier because we do not have to separate the cases according to the number of 'large' grains, as we will have to do later, since the correlation between the grains is at short distance.

Before stating a theorem in this case, let us recall what a stable random measure is, a notion which will appear in what follows. For more details, see [14].

Reminder 4.4. *A real random variable X (considered non-degenerate in the sequel) is said to be stable if for all $(A, B) \in \mathbb{R}_+^2$, there exists $C \in \mathbb{R}_+$ and $D \in \mathbb{R}$ such that*

$$AX_1 + BX_2 \overset{d}{=} CX + D, \tag{33}$$

where X_1 and X_2 are independent copies of X. We say that X is symmetric stable if X is stable and symmetric ($X \overset{d}{=} -X$). If X is stable then there exists $\alpha \in (0, 2]$ such that the number C in (33) satisfies

$$C^\alpha = A^\alpha + B^\alpha,$$

and then X is said to be α-stable and α is called the index of stability of X.
Actually, stable random variables are the only ones that can be obtained as limit in distribution of variables of the form

$$\frac{Y_1 + \cdots + Y_n}{v_n} + a_n,$$

where $(Y_n)_{n\in\mathbb{N}^*}$ is a sequence of independent and identically distributed random variables, $(v_n)_{n\in\mathbb{N}^*}$ is a sequence of positive real numbers, and $(a_n)_{n\in\mathbb{N}^*}$ is a sequence of real numbers.
If X is stable, there exist unique $\alpha \in (0,2]$, $\beta \in [-1,1]$, $\sigma \in \mathbb{R}_+$ and $\mu \in \mathbb{R}$ such that the characteristic function of X is given by

$$\mathbb{E}\left[e^{i\theta X}\right] = \begin{cases} \exp\left(-\sigma^\alpha |\theta|^\alpha \left(1 - i\beta \operatorname{sgn}(\theta) \tan\left(\frac{\pi\alpha}{2}\right)\right) + i\mu\theta\right) & \text{if } \alpha \neq 1, \\ \exp\left(-\sigma|\theta| \left(1 + i\frac{2\beta}{\pi} \operatorname{sin}(\theta)\ln(\theta)\right) + i\mu\theta\right) & \text{if } \alpha = 1. \end{cases}$$

Then α is the index of stability, and we denote $X \sim S_\alpha(\sigma, \beta, \mu)$. For $\alpha = 2$, we obtain the characteristic function of a gaussian random variable. In this case, by convention, we take $\beta = 0$.
Let $(E, \mathcal{E}, m)$ be a measure space, $\beta : E \longrightarrow [-1,1]$ be a measurable function. Let us denote $\mathcal{E}_0 := \{B \in \mathcal{E}, m(B) < +\infty\}$. If $M : \mathcal{E}_0 \longrightarrow L_0(\Omega)$ is such that

1. *if $(B_n)_{n\in\mathbb{N}^*}$ are disjoint subsets such that $\bigcup_{n=1}^{+\infty} B_n \in \mathcal{E}_0$, then*

$$M\left(\bigcup_{n=1}^{+\infty} B_n\right) = \sum_{n=1}^{+\infty} M(B_n),$$

2. *if $B_1, \ldots, B_n$ are disjoint subsets, then the random variables $M(B_1), \ldots, M(B_n)$ are independent,*
3. *if $B \in \mathcal{E}_0$, then*

$$M(B) \sim S_\alpha\left(m(B)^{1/\alpha}, \frac{\int_B \beta(x) m(\mathrm{d}x)}{m(B)}, 0\right),$$

then M is called an α-stable random measure on $(E, \mathcal{E})$ with control measure m and skewness intensity β. One can show that such a measure exists. In the following, we consider only $\alpha \in (1,2]$.
Then, we can define integration with respect to M: for each $f \in L^\alpha(E,m)$,

$$I(f) := \int_E f(x) M(\mathrm{d}x)$$

exists as a limit in probability, and

$$I(f) \sim S_\alpha(\sigma_f, \beta_f, 0),$$

where

$$\sigma_f = \left(\int_E |f(x)|^\alpha m(\mathrm{d}x)\right)^{1/\alpha},$$

and

$$\beta_f = \frac{\int_E f(x)^{<\alpha>} \beta(x) m(\mathrm{d}x)}{\int_E |f(x)|^\alpha m(\mathrm{d}x)},$$

where

$$x^{<\alpha>} = \begin{cases} x^\alpha & \text{if } x \geq 0, \\ -|x|^\alpha & \text{if } x < 0. \end{cases}$$

In particular,

$$\mathbb{E}[\exp(i\theta I(f))] = \exp\left(-\int_E |\theta f(x)|^\alpha \left(1 - i\beta(x) \operatorname{sgn}(\theta f(x)) \tan\left(\frac{\pi\alpha}{2}\right)\right) m(\mathrm{d}x)\right).$$

In the case $\alpha = 2$, i.e. M is a gaussian measure with control measure m, we have

$$\mathbb{E}[\exp(i\theta I(f))] = \exp\left(-\theta^2 \int_E f(x)^2 m(\mathrm{d}x)\right).$$

In this and the following theorems, we will consider limit in the sense of finite-dimensional distributions, i.e. for $M \subset M^1$, and for F_ρ, F_0 random functionals on M^1, we will denote

$$F_\rho(\phi) \xrightarrow[\rho\to 0]{M} F_0(\phi)$$

if

$$\forall n \geq 1, \quad \forall \phi_1, \ldots, \phi_n \in M, \quad (F_\rho(\phi_1), \ldots, F_\rho(\phi_n)) \xrightarrow[\rho\to 0]{d} (F_0(\phi_1), \ldots, F_0(\phi_n)),$$

where $\xrightarrow{d}$ designates the convergence in distribution.

Theorem 4.5. *With the previous notations, assume that $\mathbb{E}[V^2] < +\infty$. Then*

$$\frac{J_{\lambda,\rho}(\phi) - \mathbb{E}[J_{\lambda,\rho}(\phi)]}{\rho(\lambda \mathbb{E}[V^2])^{1/2}} \xrightarrow[\rho\to 0]{L^1 \cap L^2} W(\phi),$$

where W is the centered Gaussian random linear functional on L^2 with covariance given by

$$\mathbb{E}[W(\phi)W(\psi)] = \int_{\mathbb{R}^d} \phi(x)\psi(x)\mathrm{d}x,$$

i.e. with the previous vocabulary, W is the gaussian random measure with Lebesgue control measure.

Let us prove this result, but before entering into the proof, let us see two lemmas which will be useful to us.

Lemma 4.6. *Let us denote $f : \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{C} \\ u & \longmapsto & e^{iu} - 1 - iu \end{array}$. Then for any $v \in \mathbb{R}$, we have*

$$|\psi(u)| \leq \frac{u^2}{2} \quad \text{and} \quad \left|\psi(u) + \frac{u^2}{2}\right| \leq \frac{|u|^3}{6}.$$

Lemma 4.7. *If $\phi : \mathbb{R}^d \longrightarrow \mathbb{R}$ is a locally integrable function, we define the average function m_ϕ by*

$$m_\phi(x, v) = \frac{1}{v} \int_{x + v^{1/d} C} \phi(y)\mathrm{d}y, \quad v > 0, \ x \in \mathbb{R}^d$$

and the maximal function ϕ_* of ϕ by

$$\phi_*(x) = \sup_{v>0} m_\phi(x, v), \quad x \in \mathbb{R}^d.$$

Then,

1. *if $\phi \in L^1$, then for almost every $x \in \mathbb{R}^d$, $\lim_{v\to 0} m_\phi(x, v) = \phi(x)$ and $\phi_*(x) < +\infty$;*
2. *if for $p > 1$, $\phi \in L^p$, then $\phi_* \in L^p$.*

Let us move on to the proof of the theorem.

Proof. Since $J_{\lambda,\rho}$ and W are linear, thanks to Cramér–Wold device, it is sufficient to show the convergence of the one-dimensional distributions to prove the convergence of every finite-dimensional distributions.
Let us denote $b := \rho(\lambda \mathbb{E}[V^2])^{1/2}$. Let $\phi \in L^1 \cap L^2$. Thanks to Proposition 3.4,

$$\mathbb{E}\left[i\frac{J_{\lambda,\rho}(\phi) - \mathbb{E}[J_{\lambda,\rho}(\phi)]}{b}\right] = \exp\left(\int_{\mathbb{R}^d} \int_{\mathbb{R}_+} \psi\left(\frac{\phi(x + v^{1/d}C)}{b}\right) \lambda F_\rho(\mathrm{d}v)\mathrm{d}x\right),$$

where ψ is the function defined in Lemma 4.6.
Here,

$$\phi(x+v^{1/d}C)=\int_{x+v^{1/d}C}\phi(y)\mathrm{d}y=vm_\phi(x,v),$$

with the notations of Lemma 4.7. Thus,

$$\begin{aligned}\mathbb{E}\left[i\frac{J_{\lambda,\rho}(\phi)-\mathbb{E}[J_{\lambda,\rho}(\phi)]}{b}\right]&=\exp\left(\int_{\mathbb{R}^d}\int_{\mathbb{R}_+}\lambda\psi\left(\frac{vm_\phi(x,v)}{b}\right)F_\rho(\mathrm{d}v)\mathrm{d}x\right)\\&=\exp\left(\int_{\mathbb{R}^d}\int_{\mathbb{R}_+}\lambda\psi\left(\frac{vm_\phi(x,\rho v)}{(\lambda\mathbb{E}[V^2])^{1/2}}\right)F(\mathrm{d}v)\mathrm{d}x\right).\end{aligned}$$

Let us find the limit of this quantity using the dominated convergence theorem. On the one hand, according to Lemma 4.7, for $v>0$ and almost every $x\in\mathbb{R}^d$, $\lim\limits_{\rho\to0}m_\phi(x,\rho v)=\phi(x)$. Moreover, by Lemma 4.7, we have

$$\psi(u)=-\frac{1}{2}u^2+\varepsilon(u),\quad\text{where}\quad\varepsilon(u)\leq\frac{|u|^3}{6}.$$

Here $u=\frac{vm_\phi(x,\rho v)}{(\lambda\mathbb{E}[V^2])^{1/2}}$, so

$$\varepsilon(u)\leq\frac{v^3m_\phi(x,\rho v)^3}{\mathbb{E}[V^2]^{3/2}\lambda^{1/2}}\xrightarrow[\rho\to0]{}0,$$

and thus

$$\lim_{\rho\to0}\lambda\psi\left(\frac{vm_\phi(x,\rho v)}{(\lambda\mathbb{E}[V^2])^{1/2}}\right)=-\frac{v^2\phi(x)^2}{2\mathbb{E}[V^2]}.$$

On the other hand, again with the notations of Lemma 4.7, we have by Lemma 4.6,

$$\lambda\psi\left(\frac{vm_\phi(x,\rho v)}{(\lambda\mathbb{E}[V^2])^{1/2}}\right)\leq\frac{v^2\phi_*(x)^2}{2\mathbb{E}[V^2]}.$$

Since $\phi\in L^2$, thanks to Lemma 4.7, we know that $\phi_*\in L^2$, so $(x,v)\mapsto v^2\phi_*(x)^2$ is integrable with respect to $\mathrm{d}xF(\mathrm{d}v)$. Thus, thanks to the dominated convergence theorem, we have

$$\lim_{\rho\to0}\mathbb{E}\left[i\frac{J_{\lambda,\rho}(\phi)-\mathbb{E}[J_{\lambda,\rho}(\phi)]}{b}\right]=\exp\left(-\int_{\mathbb{R}^d}\int_{\mathbb{R}_+}-\frac{v^2\phi(x)^2}{2\mathbb{E}[V^2]}F(\mathrm{d}v)\mathrm{d}x\right)=\exp\left(-\int_E\phi(x)^2\mathrm{d}x\right)=\mathbb{E}[\exp(iW(\phi))],$$

which concludes thanks to Lévy's convergence theorem. □

Now let us see what can happen if the model has some long-range dependence.

4.3 Convergence result with long-range dependence

To study a case with long-range dependence, let us assume that the distribution F of V has a regularly varying tail of index $\gamma\in(1,2)$, i.e. that for all $a>0$,

$$\lim_{u\to+\infty}\frac{\bar{F}(au)}{\bar{F}(u)}=a^{-\gamma},$$

where $\bar{F}(u):1-F(u)$. In particular, this implies that $\mathbb{E}[V^2]=+\infty$ (for details on this type of random variables, see [11]), so according to Lemma 4.3, $J_{\lambda,\rho}$ has long-range dependence.
As mentioned before, the convergence results will depend on the number of "large" grains contained in the process: if there are few large grains, then at the limit they do not play a role, and it is as if we only see points; whereas if we have a lot of large grains, they will be still "visible" at the limit, and since the information spreads at long

range thanks to those large grains, we will still have at the limit a process with long-range dependence.
Thus, we have to study the number of "large" grains of the model, and more precisely, let us consider the expected number of grains with volume larger than one that cover the origin:

$$\begin{aligned}\mathbb{E}[\#\{x \in X :\ \rho V_x > 1 \text{ and } 0 \in x + (\rho V_x)^{1/d} C\}] &= \mathbb{E}\left[\sum_{x \in X} \mathbb{1}(\rho V_x > 1,\ 0 \in x + (\rho V_x)^{1/d} C)\right] \\ &= \int\int_{\{(x,v):\, v>1,\, 0 \in x + v^{1/d} C)\}} \lambda \mathrm{d}x F_\rho(\mathrm{d}v) \\ &= \lambda \int_1^{+\infty} \underbrace{\int_{\mathbb{R}^d} \mathbb{1}\left(\frac{-x}{v^{1/d}} \in C\right) \mathrm{d}x}_{=v} F_\rho(\mathrm{d}v) = \lambda \int_1^{+\infty} v F_\rho(\mathrm{d}v).\end{aligned}$$

One can show, thanks to Karamata's theorem (see [11]) that

$$\mathbb{E}[\#\{x \in X :\ \rho V_x > 1 \text{ and } 0 \in x + (\rho V_x)^{1/d} C\}] \sim_{\rho \to 0} \frac{\lambda \bar{F}_\rho(1)}{1 - \gamma^{-1}}.$$

Thus, as we will see, the limit of the model when $\rho \to 0$ and $\lambda \to +\infty$ depends on the behavior of $\lambda \bar{F}_\rho(1)$ at the limit:

- if $\lambda \bar{F}_\rho(1) \to +\infty$, then there are many large grains in the model, that control the limit behavior: we call this regime "large-grain scaling",
- if $\lambda \bar{F}_\rho(1) \to 0$, then the large grains disappear at the limit, and the limit behavior is controlled by the small grains: we call this regime "small-grain scaling",
- if $\lambda \bar{F}_\rho(1) \to \sigma_0$, then it is an intermediate regime, and the limit behavior takes in account large and small grains: we call this regime "intermediate scaling".

We will not prove the following theorem in this document, but the idea is the same as for the theorem in the previous part: to prove the convergence of the finite-dimensional distribution, it is sufficient to prove the one-dimensional distribution thanks to Cramér–Wold device, and to do this, one can show the convergence of the characteristic functions.

Theorem 4.8. *With the previous notations, let $\alpha \in (0, 2 - \gamma)$.*

1. *(Large-grain scaling) If $\lambda \bar{F}_\rho(1) \to +\infty$, then*

$$\frac{J_{\lambda,\rho}(\phi) - \mathbb{E}[J_{\lambda,\rho}(\phi)]}{(\gamma \lambda \bar{F}_\rho(1))^{1/2}} \xrightarrow[\rho \to 0]{M^\alpha} W_{\gamma,C}(\phi),$$

where $W_{\gamma,C}$ is the centered gaussian random linear functional on $M^{2-\gamma}$ with covariance given by

$$\mathbb{E}[W_{\gamma,C}(\phi) W_{\gamma,C}(\psi)] = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K_{\gamma,C}(x - y) \phi(\mathrm{d}x) \psi(\mathrm{d}y),$$

where

$$K_{\gamma,C}(x) = \int_{\mathbb{R}_+} |(v^{-1/d} x + C) \cap C| v^{-\gamma} \mathrm{d}v.$$

2. *(Intermediate scaling) If $\lambda \bar{F}_\rho(1) \to \sigma_0$, then*

$$J_{\lambda,\rho}(\phi) - \mathbb{E}[J_{\lambda,\rho}(\phi)] \xrightarrow[\rho \to 0]{M^\alpha} J^*_{\gamma,C}(\phi_\sigma),$$

where ϕ_σ is defined by

$$\phi_\sigma(A) = \phi(\sigma A) \quad \textit{with} \quad \sigma = (\gamma\sigma_0)^{1/((\gamma-1)d)},$$

and where for $\phi \in M^\alpha$,

$$J^*_{\gamma,C}(\phi) = \int_{\mathbb{R}^d}\int_{\mathbb{R}_+} \phi(x + v^{1/d}C)(N_\gamma(\mathrm{d}x, \mathrm{d}v) - \mathrm{d}x v^{-\gamma-1}\mathrm{d}v),$$

with N_γ a Poisson process on $\mathbb{R}^d \times \mathbb{R}_+$ with intensity measure $\mathrm{d}x v^{-\gamma-1}\mathrm{d}v$.

3. *(Small-grain scaling) If $\lambda\bar{F}_\rho(1) \to 0$, then*

$$\frac{J_{\lambda,\rho}(\phi) - \mathbb{E}[J_{\lambda,\rho}(\phi)]}{c_\gamma(1/\bar{F}_\rho)^{-1}(\lambda\gamma)} \xrightarrow[\rho\to 0]{L^1\cap L^2} \Lambda_\gamma(\phi),$$

where Λ_γ is the $\gamma-$stable random measure with Lebesgue control measure and skewness intensity $\beta \equiv 1$,

$$(1/\bar{F}_\rho)^{-1}(u) = \inf\{v : (1/\bar{F}_\rho)(v) \geq u\}$$

and

$$c_\gamma = \left(-\frac{\Gamma(2-\gamma)}{\gamma(\gamma-1)}\cos\left(\frac{\pi\gamma}{2}\right)\right)^{1/\gamma}.$$

Remarks 4.9.

- *In the third case, we are forced to consider measures that are absolutely continuous with respect to the Lebesgue measure, because at the limit, the process is too chaotic for an atomic measure to capture any information: the behavior is controlled by small grains, that "looks" like points at the limits, and we need to consider a regular measure to get some information on the limit process.*
- *If C is symmetric around this origin, for instance if $C = B(0,1)$, then $W_{\gamma,C}$ can be expressed more easily from W, the gaussian random measure with Lebesgue control measure appearing in Theorem 4.5: for $\phi \in M^{2-\gamma}$,*

$$W_{\gamma,C}(\phi) \overset{d}{=} c\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\phi(\mathrm{d}y)}{|x-y|^{\gamma d/2}} W(\mathrm{d}x),$$

where $c > 0$ is a constant.

- *One can show that the limit in the large-grain scaling has long-range dependence, in the same sense as (32): since there are still large grains at the limit, they continue to spread the correlation at long range.*

This type of model has already been studied also in the case where the centers of the grains are distributed for example as a determinantal process, see [2]. One of the goals of my future PhD thesis would be to generalize this kind of results by considering more general distributions, and studying the impact of a potential attractiveness or repulsiveness of the processes considered.

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