Université de Rennes

M2 Seminar

Skohorod topology

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1 Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

The objective of our study of the Skorohod topology will be to get a version of the Donsker's theorem where the random functions considered are no more continuous but just cadlag. Before getting into our subject, let us remind of the Donsker's theorem in the continuous case, and of the necessary tools to prove it. Let us denote $C = C([0,1], \mathbb{R})$.

Theorem 1.1. *Let $(\xi_n)_{n\geq 1}$ be a sequence of independent and identically distributed random variables, such as $\mathbb{E}(\xi_1) = 0$ and $\mathrm{Var}(\xi_1) = 1$. Let S_n be the random variable defined as $S_n = \sum_{i=1}^{n} \xi_i$ if $n \geq 1$, and $S_0 = 0$. We can define the interpolated continuous process based on this sequence by the formula :*

$$X_t^n := \frac{S_{\lfloor nt \rfloor}}{\sqrt{n}} + \frac{\xi_{\lfloor nt \rfloor + 1}}{\sqrt{n}}(nt - \lfloor nt \rfloor),$$

for $t \in [0,1]$. Then the sequence of processes $(X^n)_{n\geq 1}$ converges in law with respect to the uniform topology on C to a standard Brownian motion B, i.e. $(X_t^n)_{t\in[0,1]} \Rightarrow (B_t)_{t\in[0,1]}$.

By the continuous mapping theorem, we know that in the continuous case, convergence in law implies convergence of the finite-dimensional distributions, but to prove the theorem, we need the converse of this fact. Indeed, the convergence of the finite-dimensional distributions is given by the central limit theorem. However, the converse is not true, and to obtain the desired result, we need the Prohorov's theorem. Before talking about this theorem, let us remind of the two concepts that it involves.

Definition 1.2. *Let $(P_n)_{n\geq 1}$ be a sequence of probability measures on a metric space. The sequence is said to be tight if for every $\varepsilon > 0$, there exists a compact set K_ε such that for all $n \geq 1$, $P_n(K_\varepsilon) \geq 1 - \varepsilon$.*

Definition 1.3. *Let $(P_n)_{n\geq 1}$ be a sequence of probability measures. The sequence is said to be relatively compact if every subsequence of $(P_n)_{n\geq 1}$ contains a weakly convergent subsequence.*

Theorem 1.4 (Prohorov)**.** *Let $(P_n)_{n\geq 1}$ be a sequence of probability measures on a separable complete metric space. Then $(P_n)_{n\geq 1}$ is tight if and only if it is relatively compact.*

This theorem allows us to establish the converse of the previous fact.

Corollary 1.5. *Let $(P_n)_{n\geq 1}$ and P be probability measures on C. If the finite-dimensional distributions of P_n converge weakly to those of P and if $(P_n)_{n\geq 1}$ is tight, then $P_n \Rightarrow P$.*

Thus, it is necessary to find characterizations of the compact sets of C. This is given by the Arzelà-Ascoli theorem. Finally, we can deduce, thanks to this theorem and the Prohorov's theorem, conditions on a sequence $(P_n)_{n\geq 1}$ of probability measures to be tight. All put together, these results allows us to prove the Donsker's theorem.

This small recap of the continuous case allows us to understand the necessary steps in order to adapt the Donsker's theorem in the cadlag case :

- ⋆ find a topology on the space of cadlag functions on $[0,1]$ which is separable and complete, to use the Prohorov's theorem,
- ⋆ get an equivalent of the Arzelà-Ascoli theorem to characterize the compact sets of the space,
- ⋆ use this characterization with the Prohorov's theorem to find conditions on a sequence $(P_n)_{n\geq 1}$ of probability measures to be tight, and therefore to be convergent,
- ⋆ apply this to get the Donsker's theorem.

Now that we have a plan to reach our goal, let's get to the heart of the matter.

2 The Skorohod topology

Definition 2.1. *A function $x : [0,1] \to \mathbb{R}$ is said to be cadlag ("continue à droite limite à gauche") if*

1. *for all $t \in [0,1)$, $x(t^+) = \lim_{s \searrow t} x(s)$ exists and $x(t^+) = x(t)$ (right-continuity),*
2. *for all $t \in (0,1]$, $x(t^-) = \lim_{s \nearrow t} x(s)$ exists (left-limits).*

Let $D = D[0,1]$ denote the set of cadlag functions on $[0,1]$.

For the rest, we will need a modulus of continuity to study this type of functions. First, let us define the classical one, used for continuous functions.

Definition 2.2. *For $x \in D$ and $T \subset [0,1]$, we define*

$$\omega_x(T) = \omega(x,T) = \sup_{s,t \in T} |x(s) - x(t)|,$$

and for $\delta \in (0,1)$, we define

$$\omega_x(\delta) = \omega(x,\delta) = \sup_{|s-t| \leq \delta} |x(s) - x(t)| = \sup_{0 \leq t \leq 1-\delta} \omega_x[t, t+\delta].$$

We know that any continuous function on $[0,1]$ is uniformly continuous. The following lemma shows that any cadlag function is almost uniformly continuous, apart from its jumps.

Lemma 2.3. *Let x be a function in D, and let $\varepsilon > 0$. There exist $t_0, t_1, \ldots t_n$ in $[0,1]$ such that $0 = t_0 < t_1 < \cdots < t_n = 1$ and*

$$\forall i \in [\![1, n]\!], \quad \omega_x[t_{i-1}, t_i) < \varepsilon.$$

This lemma implies that any function of D has at most countably many discontinuities, and is bounded.

This modulus of continuity is going to be useful in the rest, but we want another one that takes into account the jumps of cadlag functions.

Definition 2.4. *Let $\delta \in (0,1)$. A set of points $\{t_0, t_1, \ldots, t_n\}$ of $[0,1]$ is called a δ-sparse if $0 = t_0 < t_1 < \cdots < t_n = 1$ and $\min_{1 \leq i \leq n} (t_i - t_{i-1}) > \delta$. Let $x \in D$. We define*

$$\omega'_x(\delta) = \omega'(x,\delta) = \inf_{\{t_0,t_1,\ldots,t_n\}\ \delta-sparse} \max_{1 \leq i \leq n} \omega_x[t_{i-1}, t_i).$$

The previous lemma shows that for every $x \in D$, $\lim_{\delta \to 0} \omega'_x(\delta) = 0$. In fact, the converse is true : a function x on $[0,1]$ is in D if and only if $\lim_{\delta \to 0} \omega'_x(\delta) = 0$.

Now that we have those moduli that will allow us to work with cadlag functions, we want to define our desired topology, and to do this, we are going to define a metric on D. Unlike the uniform topology on C, we want the topology defined by this metric to allow a small deformation of time, in addition to a small deformation of space.

Definition 2.5. *Let Λ denote the set of strictly increasing, continuous functions $\lambda : [0,1] \to [0,1]$. Let I denote the identity map on $[0,1]$. For $x, y \in D$, let us define*

$$d(x,y) := \inf_{\lambda \in \Lambda} \{ \|\lambda - I\|_\infty \vee \|x - y \circ \lambda\|_\infty \}.$$

Then d is a metric, and the topology defined by this metric is called the Skorohod topology.

Remarks 2.6. ⋆ *The notion of convergence for this metric is going to be useful for the rest : a sequence $(x_n)_{n\geq 1}$ of D converges in this topology if and only if there exists a sequence $(\lambda_n)_{n\geq 1}$ of Λ such that $\|\lambda_n - I\|_\infty \xrightarrow[n\to+\infty]{} 0$ and $\|x_n \circ \lambda_n - x\|_\infty \xrightarrow[n\to+\infty]{} 0$.*

- ⋆ *The uniform convergence implies the convergence in the Skorohod topology (the previous assertion is then true by taking $\lambda_n = I$). But the other direction is wrong : for $a \in (0,1)$, the sequence $(x_n)_{n\geq 1}$ defined by $x_n = \mathbb{1}_{[0,a+\frac{1}{n})}$ converges in the Skorohod topology to $\mathbb{1}_{[0,a)}$ (take λ_n such that $\lambda_n(a) = a + \frac{1}{n}$ and λ_n is linear on $[0,a]$ and on $[a,1]$, then $x_n \circ \lambda_n = x$ and $\|\lambda_n - I\|_\infty = \frac{1}{n} \xrightarrow[n\to+\infty]{} 0$), but for all $n \geq 1$, $\|x_n - x\|_\infty = 1$.*
- ⋆ *For $x_n, x \in D$, $\lambda_n \in \Lambda$ and $t \in [0,1]$, we can write*

$$|x_n(t) - x(t)| \leq |x_n(t) - x(\lambda_n^{-1}(t))| + |x(\lambda_n^{-1}(t)) - x(t)|, \tag{1}$$

and then

$$\|x_n - x\|_\infty \leq \|x_n - x \circ \lambda_n^{-1}\|_\infty + \omega_x(\|\lambda_n^{-1} - I\|_\infty) = \|x_n \circ \lambda_n - x\|_\infty + \omega_x(\|\lambda_n^{-1} - I\|_\infty). \tag{2}$$

Thus, if a sequence $(x_n)_{n\geq 1}$ of C converges in the Skorohod topology to $x \in D$, then x is continuous (since $\|x_n \circ \lambda_n - x\|_\infty \xrightarrow[n\to+\infty]{} 0$, x is a uniform limit of continuous functions), and by (2), we deduce that the convergence is in the uniform topology. Finally, the uniform topology and the Skorohod topology coincide on C.
- ⋆ *The inequality (1) also implies that if a sequence $(x_n)_{n\geq 1}$ of D converges in the Skorohod topology to $x \in D$, then $x_n(t) \xrightarrow[n\to+\infty]{} x(t)$ if t is a continuity point of x.*

But this topology is not satisfactory with regard to the introduction.

Proposition 2.7. *The space D is not complete under the metric d.*

Proof. Let $(x_n)_{n\geq 1}$ be the sequence defined by $x_n = \mathbb{1}_{[0,\frac{1}{2^n})}$. Let $(\lambda_n)_{n\geq 1}$ such that $\lambda_n(\frac{1}{2^n}) = \frac{1}{2^{n+1}}$, and λ_n is linear on $[0, \frac{1}{2^n}]$ and on $[\frac{1}{2^n}, 1]$.
Then, $x_{n+1} \circ \lambda_n = x_n$, and $\|\lambda_n - I\|_\infty = |(\lambda_n - I)(\frac{1}{2^n})| = \frac{1}{2^{n+1}}$. Therefore, $d(x_n, x_{n+1}) \leq \frac{1}{2^{n+1}}$, and then for all $n \leq p$,

$$d(x_n, x_p) \leq \sum_{k=0}^{p-n-1} d(x_{n+k}, x_{n+k+1}) \leq \sum_{k=0}^{p-n-1} \frac{1}{2^{n+k+1}} = \sum_{k=n+1}^{p} \frac{1}{2^k} \leq \varepsilon,$$

for $n, p \geq N_\varepsilon$. Hence, $x_n = \mathbb{1}_{[0,\frac{1}{2^n})}$ is a Cauchy sequence in the metric d. However, for all $t > 0$, $x_n(t) \xrightarrow[n\to+\infty]{} 0$, but for all $n \geq 1$, $d(x_n, 0) = 1$, and $(x_n)_{n\geq 1}$ is not d-convergent. □

Since we want to apply the Prohorov's theorem, we need a complete topology. To do this, let us define another metric, that still allows the deformation of time, but in a more strict way.

Definition 2.8. *For $\lambda \in \Lambda$, let us define*

$$\|\lambda\|^\circ := \sup_{s<t} |\log\left(\frac{\lambda(t) - \lambda(s)}{t-s}\right)|.$$

For $x, y \in D$, let us define

$$d^\circ(x,y) := \inf_{\lambda\in\Lambda}\{\|\lambda\|^\circ \vee \|x - y \circ \lambda\|_\infty\}.$$

Then d° is still a metric, and this time, the deformation of time have to be really close to the identity map, in the sense that the slopes of its chords should be close to 1.

Example 2.1. *Let us come back to the sequence of the proof of proposition 2.7. If a function of* Λ *does not map* $\frac{1}{2^n}$ *to* $\frac{1}{2^{n+1}}$*, then* $\|x_{n+1} \circ \lambda_n - x_n\|_\infty = 1$*. Thus, the only way to make the distance* $d(x_{n+1}, x_n)$ *small is to take a function that maps* $\frac{1}{2^n}$ *to* $\frac{1}{2^{n+1}}$*, and* λ_n *defined as in the proof (linear everywhere else) is the function that has the smallest norm* $\|.\|^\circ$ *among the functions that maps* $\frac{1}{2^n}$ *to* $\frac{1}{2^{n+1}}$*. Hence,* $d^\circ(x_n, x_{n+1}) = \|\lambda_n\|^\circ = \log(2)$*, and* $(x_n)_{n\geq 1}$ *is not a Cauchy sequence in the metric* d° *: the slopes of the chords of* λ_n *are too big.*

Theorem 2.9. *The two metrics* d *and* d° *are topologically equivalent. The space* D *is separable under* d *and* d°*, and complete under* d°*.*

We have now reached the first objective announced in the introduction : we have a topology on D which is separable and complete. The second goal was to have an equivalent to the Arzelà-Ascoli theorem to characterize the compact sets of D. This is given by the following theorem, which is really close to the Arzelà-Ascoli theorem : the only differences are that ω_x is replaced by ω'_x, and that a control on just a single point is not enough for cadlag functions, because of the jumps that can be bigger and bigger (consider for instance $x_n = n\mathbb{1}_{[\frac{1}{2},1)}$).

Theorem 2.10. *A set* $A \subset D$ *is relatively compact in the Skorohod topology if and only if*

$$\sup_{x\in A} \|x\|_\infty < \infty \tag{3}$$

and

$$\lim_{\delta\to 0} \sup_{x\in A} \omega'_x(\delta) = 0. \tag{4}$$

Even if this theorem is what we wanted, we are going to look for another characterization of compacts sets, implying another modulus of continuity that is easier to work with, and that will be more adapted for the future tightness criteria.

Definition 2.11. *For* $x \in D$ *and* $\delta \in (0, 1)$*, we define*

$$\omega''_x(\delta) = \sup_{\substack{t_1\leq t\leq t_2\\ t_2-t_1\leq\delta}} \{|x(t) - x(t_1)| \wedge |x(t_2) - x(t)|\}.$$

Remark 2.12. *This modulus of continuity is bounded from above by the previous one : for* $x \in D$ *and* $\delta \in (0, 1)$*,*

$$\omega''_x(\delta) \leq \omega'_x(\delta).$$

However, there is no inequality in the opposite direction : for instance, let $(x_n)_{n\geq 1}$ *be the sequence defined as* $x_n = \mathbb{1}_{[0,\frac{1}{n})}$*. This sequence is such that for all* $\delta \in (0, 1)$ *and for all* $n \geq 1$*,* $\omega''_{x_n}(\delta) = 0$*, but also for all* $n \geq \frac{1}{\delta}$*,* $\omega'_{x_n}(\delta) = 1$*. This example also shows us that a condition only involving* ω''_x *is not sufficient : the set* $A = \{x_n, n \geq 1\}$ *is such that* $\lim_{\delta\to 0} \sup_{x\in A} \omega''_x(\delta) = 0$*, but it is not relatively compact according to the previous theorem. Thus, it is necessary to impose a condition on the behavior of the function near* 0 *and* 1*.*

Theorem 2.13. *A set* $A \subset D$ *is relatively compact in the Skorohod topology if and only if it satisfies the condition* (3) *and*

$$\begin{cases} \lim_{\delta\to 0} \sup_{x\in A} \omega''_x(\delta) = 0, \\ \lim_{\delta\to 0} \sup_{x\in A} |x(\delta) - x(0)| = 0, \\ \lim_{\delta\to 0} \sup_{x\in A} |x(1^-) - x(1 - \delta)| = 0. \end{cases} \tag{5}$$

We have now reached the second objective announced in the introduction : we have a good characterization of the compact sets of D. The next step is to make the link with the notions of weak convergence and tightness.

3 Weak convergence and tightness in D

Before talking about the notions of weak convergence and tightness, let us study projections that are going to be useful in the rest of this section.

Definition 3.1. *For $0 \le t_1 < \cdots < t_k \le 1$, let us define*

$$\pi_{t_1,\ldots,t_k} : \begin{array}{ccc} D & \longrightarrow & \mathbb{R}^k \\ x & \longmapsto & (x(t_1), \ldots, x(t_k)). \end{array}$$

Let $\mathcal{R}^k$ denote the Borel $\sigma-$algebra of $\mathbb{R}^k$, and $\mathcal{D}$ the Borel $\sigma-$algebra of D.

Proposition 3.2.

1. *The projections π_0 and π_1 are continuous. For $t \in (0,1)$ and $x \in D$, π_t is continuous at x if and only if x in continuous at t.*
2. *For all $0 \le t_1 < \cdots < t_k \le 1$, $\pi_{t_1,\ldots,t_k} : (D, \mathcal{D}) \longrightarrow (\mathbb{R}^k, \mathcal{R}^k)$ is measurable.*

Proof. Let us start by the continuity of π_0 and π_1. Let $x, y \in D$. Since every function of Λ maps 0 to 0, we have, for all $\lambda \in \Lambda$:

$$|\pi_0(x) - \pi_0(y)| = |x(0) - y(0)| = |x(0) - y(\lambda(0))| \le \|x - y \circ \lambda\|_\infty \vee \|\lambda - I\|_\infty.$$

Since this is true for any $\lambda \in \Lambda$, we can conclude

$$|\pi_0(x) - \pi_0(y)| \le d(x, y).$$

Thus, π_0 is continuous. Since every function λ of Λ maps 1 to 1, we can show in the same way that π_1 is continuous.

Let $t \in (0,1)$. Let $x \in D$ such that x is continuous at t. Let $(x_n)_{n\ge1}$ be a sequence of D such that $x_n \xrightarrow[n\to+\infty]{} x$ in the Skorohod topology. As we saw previously, since x is continuous at t, $x_n(t) \xrightarrow[n\to+\infty]{} x(t)$, that is to say $\pi_t(x_n) \xrightarrow[n\to+\infty]{} \pi_t(x)$, and π_t is continuous at x.
Now, let $x \in D$ such that x is not continuous at t. For $n \ge 1$, we define λ_n such that $\lambda_n(t) = t - \frac{1}{n}$, and λ_n is linear on $[0, t]$ and on $[t, 1]$. Let $(x_n)_{n\ge1}$ be the sequence defined by $x_n = x \circ \lambda_n$. Then $\|\lambda_n^{-1} - I\|_\infty = \frac{1}{n} \xrightarrow[n\to+\infty]{} 0$, and $\|x_n \circ \lambda_n^{-1} - x\|_\infty = 0$. Hence, $x_n \xrightarrow[n\to+\infty]{} x$ in the Skorohod topology, but $x_n(t) \xrightarrow[n\to+\infty]{} x(t^-) \ne x(t)$, and π_t is not continuous at x.

Let us move on the second point of the proposition. Since a function into $(\mathbb{R}^k, \mathcal{R}^k)$ is measurable if each component mapping is, we only need to show that for any $t \in [0,1]$, $\pi_t : (D, \mathcal{D}) \longrightarrow (\mathbb{R}, \mathcal{R})$ is measurable. For π_0 and π_1, this is clear since they are continuous. Let $t \in (0,1)$, and $x \in D$. For $0 < \varepsilon < 1-t$, let $h_\varepsilon(x) = \dfrac{1}{\varepsilon} \displaystyle\int_t^{t+\varepsilon} x(s)\mathrm{d}s$. Let $(x_n)_{n\ge1}$ be a sequence such that $x_n \xrightarrow[n\to+\infty]{} x$ in the Skorohod topology. As seen previously, $x_n(s) \xrightarrow[n\to+\infty]{} x(s)$ for all continuity points of x, that is to say for points outside a set of Lebesgue measure 0. Since the sequence is convergent, it is uniformly bounded. Thus, it follows from the dominated convergence theorem that $h_\varepsilon(x_n) \xrightarrow[n\to+\infty]{} h_\varepsilon(x)$, so h_ε is continuous, and therefore measurable. Moreover, by right-continuity, $h_{\frac{1}{m}}(x) \xrightarrow[m\to+\infty]{} x(t) = \pi_t(x)$. Finally, π_t is measurable as a limit of measurable functions. □

This shows that the modulus of continuity $\omega''(\cdot, \delta)$ is measurable. One can also shows that $\omega'(\cdot, \delta)$ is measurable.

The following theorem is technical, but will be useful to apply the Dynkin's theorem in the rest.

Theorem 3.3. *Let T be a subset of $[0,1]$. We define*

$$p[\pi_t,\, t \in T] := \{\pi_{t_1,\ldots,t_k}^{-1} H \mid k \in \mathbb{N}^*,\, t_1, \ldots, t_k \in T^k,\, H \in \mathcal{R}^k\},$$

and $\sigma[\pi_t,\, t \in T]$ the σ-algebra generated by $p[\pi_t,\, t \in T]$. Then $p[\pi_t,\, t \in T]$ is a π-system, and if T contains 1 and is dense in $[0,1]$, then $\sigma[\pi_t,\, t \in T] = \mathcal{D}$.

Now that we have those technical tools, we can study the notion of weak convergence in D. First, we see that, unlike the continuous case, convergence in law of processes does not necessary imply convergence of the finite-dimensional distribution, since the projections are not everywhere continuous on D. For the converse, it is the same as in the continuous case : it is also wrong, and we need an additional hypothesis such as tightness.
To state results linked to the convergence of finite-dimensional distributions, we need to define the following set.

Definition 3.4. *Let P be a probability measure on $(D, \mathcal{D})$. Let T_P be the set of $t \in [0,1]$ such that the projection π_t is continuous except at points forming a set of P-measure 0, that is to say :*

$$T_P = \{t \in [0,1],\ \pi_t \text{ is continuous a.s.}\}.$$

Proposition 3.5. *Let P be a probability measure on $(D, \mathcal{D})$. The set T_P contains 0 and 1 and its complement in $[0,1]$ is at most countable.*

Proof. Since π_0 and π_1 are continuous, it is clear that $0, 1 \in T_P$. For $t \in (0,1)$, we have seen that π_t is continuous at x if and only if x is continuous at t, therefore, t is in T_P if and only if $P(J_t) = 0$, where $J_t = \{x,\ x(t) \neq x(t^-)\}$. Let us show that there are at most countably many points in T_P^c. Let $\varepsilon > 0$ and $\delta > 0$. Let us define $J_t(\varepsilon) = \{x,\ |x(t) - x(t^-)| > \varepsilon\}$. Let us define also $A_{\varepsilon,\delta} = \{t \in (0,1),\ P(J_t(\varepsilon)) \geq \delta\}$ and $A_\varepsilon = \{t \in (0,1),\ P(J_t(\varepsilon)) > 0\}$. Assume that $A_{\varepsilon,\delta}$ is infinite : there exists a sequence $(t_n)_{n\in\mathbb{N}}$ of distinct elements of $A_{\varepsilon,\delta}$. Then,

$$P(\limsup_{n\to+\infty} J_{t_n}(\varepsilon)) = P\left(\bigcap_{n\geq 1}\bigcup_{k\geq n} J_{t_k}(\varepsilon)\right) = \lim_{n\to+\infty} P\left(\bigcup_{k\geq n} J_{t_k}(\varepsilon)\right) \geq \lim_{n\to+\infty} P(J_{t_n}(\varepsilon)) \geq \delta.$$

However, x is in $\limsup_{n\to+\infty} J_{t_n}(\varepsilon)$ means

$$\forall n \geq 1, \quad \exists k \geq n, \quad |x(t_k) - x(t_k^-)| \geq \varepsilon,$$

that implies that x has an infinite number of jumps exceeding ε, which is impossible according to Lemma 2.3. Thus, $A_{\varepsilon,\delta}$ is finite. Therefore, $A_\varepsilon = \displaystyle\bigcup_{\delta>0,\ \delta\in\mathbb{Q}} A_{\varepsilon,\delta}$ is at most countable. Finally, since $J_t(\varepsilon)$ increases when ε decreases, we have

$$P(J_t) = P\left(\bigcup_{\varepsilon>0,\ \varepsilon\in\mathbb{Q}} J_t(\varepsilon)\right) = \lim_{\varepsilon\to 0} P(J_t(\varepsilon)).$$

Thus, $T_P^c = \displaystyle\bigcup_{\varepsilon>0,\ \varepsilon\in\mathbb{Q}} A_\varepsilon$ is at most countable. ☐

The continuous mapping theorem directly gives us the following result.

Proposition 3.6. *Let $(P_n)_{n\geq 1}$ and P be probability measures on $(D, \mathcal{D})$. Let $t_1, \ldots, t_k \in T_P$. If $P_n \Rightarrow P$, then $P_n \pi_{t_1,\ldots,t_k}^{-1} \Rightarrow P\pi_{t_1,\ldots,t_k}^{-1}$.*

Example 3.1. *If t is not in T_P, then $P_n\pi_t^{-1} \Rightarrow P\pi_t^{-1}$ may not follows from $P_n \Rightarrow P$. For instance, take $P_n = \delta_{\mathbb{1}_{[0,t+\frac{1}{n})}}$ and $P = \delta_{\mathbb{1}_{[0,t)}}$. If f is continuous and bounded, then*

$$\int f \mathrm{d}P_n = f(\mathbb{1}_{[0,t+\frac{1}{n})}) \quad \text{and} \quad \int f \mathrm{d}P = f(\mathbb{1}_{[0,t)}).$$

Since $d(\mathbb{1}_{[0,t+\frac{1}{n})}, \mathbb{1}_{[0,t)}) \xrightarrow[n\to+\infty]{} 0$ (take λ_n such that $\lambda_n(t) = t + \frac{1}{n}$ and λ is linear on $[0,t]$ and on $[t,1]$), $\displaystyle\int f\mathrm{d}P_n \xrightarrow[n\to+\infty]{} \int f\mathrm{d}P$. Therefore $P_n \Rightarrow P$. However, for all $n \geq 1$, $P_n\pi_t^{-1} = \delta_1$ and $P\pi_t^{-1} = \delta_0$.

The converse is going to be the same as in the continuous case. Indeed, D is complete under the metric d°, which is topologically equivalent to d, therefore the convergence in law is the same for d and d°, and we can apply the Prohorov's theorem while working with the metric d.

Theorem 3.7. *Let $(P_n)_{n\geq 1}$ and P be probability measures on $(D,\mathcal{D})$. If $(P_n)_{n\geq 1}$ is tight, and for all $t_1,\ldots,t_k \in T_P$, $P_n\pi^{-1}_{t_1,\ldots,t_k} \Rightarrow P\pi^{-1}_{t_1,\ldots,t_k}$, then $P_n \Rightarrow P$.*

To prove the theorem, we are going to use this lemma.

Lemma 3.8. *Let $(P_n)_{n\geq 1}$ and P be probability measures. If each subsequence of $(P_n)_{n\geq 1}$ contains a further subsequence that converges weakly to P then $P_n \Rightarrow P$.*

Proof. Suppose that $P_n \not\Rightarrow P$, then there exists a continuous and bounded function f such that $\int f\mathrm{d}P_n \nrightarrow \int f\mathrm{d}P$. Hence, there exists $\varepsilon > 0$ and a subsequence $(P_{n_k})_{k\in\mathbb{N}}$ such that for all $k \in \mathbb{N}$, $|\displaystyle\int f\mathrm{d}P_{n_k} - \int f\mathrm{d}P| > \varepsilon$, and no further subsequence of this subsequence can converge weakly to P. $\square$

Let us move on to the proof of the theorem.

Proof. To prove that $P_n \Rightarrow P$, we just have to prove that if a subsequence converges weakly to a probability measure Q, then $P = Q$. Indeed since $(P_n)_{n\geq 1}$ is tight, according to the Prohorov's theorem, it is relatively compact, so each subsequence contains a further subsequence that converges weakly to some limit, and if the previous fact is true, then this limit must be P, and we can apply Lemma 3.8.

Let $(P_{n_i})_{i\in\mathbb{N}}$ be a subsequence of $(P_n)_{n\geq 1}$ that converges weakly to a probability measure Q. If $t_1,\ldots,t_k \in T_P$, then $P_{n_i}\pi^{-1}_{t_1,\ldots,t_k} \Rightarrow P\pi^{-1}_{t_1,\ldots,t_k}$ by hypothesis of the theorem. Moreover, since $P_{n_i} \Rightarrow Q$, we know that for all $t_1,\ldots,t_k \in T_Q$, $P_{n_i}\pi^{-1}_{t_1,\ldots,t_k} \Rightarrow Q\pi^{-1}_{t_1,\ldots,t_k}$. Therefore, if $t_1,\ldots,t_k$ are in $T_P \cap T_Q$, by uniqueness of the limit, $P\pi^{-1}_{t_1,\ldots,t_k} = Q\pi^{-1}_{t_1,\ldots,t_k}$. By taking up the notations of the theorem 3.3, P and Q coincide on $p[\pi_t,\, t \in T_P \cap T_Q]$. Moreover, $T_P \cap T_Q$ contains 1 and is dense in $[0,1]$, because its complement is at most countable according to Proposition 3.5. Hence, we can apply the theorem 3.3 : $p[\pi_t,\, t \in T_P \cap T_Q]$ is a π-system that generates $\mathcal{D}$, and it follows from the Dynkin's theorem that $P = Q$. $\square$

Now that we have this criterion of weak convergence, we want, as in the continuous case, criteria of tightness, and to do this, we will use the equivalents of Arzelà-Ascoli theorem that we obtained previously.

Theorem 3.9. *Let $(P_n)_{n\geq 1}$ be probability measures on $(D,\mathcal{D})$. The sequence $(P_n)_{n\geq 1}$ is tight if and only if*

$$\forall\eta > 0, \quad \exists a > 0, \quad \exists n_0 \in \mathbb{N}, \quad \forall n \geq n_0, \quad P_n(\{x,\, \|x\|_\infty > a\}) \leq \eta, \tag{6}$$

and

$$\forall\epsilon > 0, \quad \forall\eta > 0, \quad \exists\delta > 0, \quad \exists n_0 \in \mathbb{N}, \quad \forall n \geq n_0, \quad P_n(\{x,\, \omega'_x(\delta) > \varepsilon\}) \leq \eta. \tag{7}$$

Remark 3.10. *Those two conditions can be summarized as follows :*

$$\lim_{a\to+\infty}\limsup_{n\to+\infty} P_n(\{x,\, \|x\|_\infty > a\}) = 0$$

and for all $\varepsilon > 0$,

$$\lim_{\delta\to 0}\limsup_{n\to+\infty} P_n(\{x,\, \omega'_x(\delta) > \varepsilon\}) = 0.$$

Proof. First, suppose that $(P_n)_{n\geq 1}$ is tight. Let $\varepsilon > 0$ and $\eta > 0$. Let K be a compact set of $(D,\mathcal{D})$ such that for all $n \geq 1$, $P_n(K) > 1-\eta$. By Theorem 2.10, there exist $a > 0$ such that $K \subset \{x,\, \|x\|_\infty \leq a\}$ and $\delta > 0$ such that $K \subset \{x,\, \omega'_x(\delta) \leq \varepsilon\}$. This implies for all $n \geq 1$ $P_n(\{x,\, \|x\|_\infty > a\}) \leq \eta$ and $P_n(\{x,\, \omega'_x(\delta) \leq \varepsilon\}) \leq \eta$, which implies (6) and (7) for $n_0 = 1$.

Now, assume (6) and (7). First, let us show that we can take $n_0 = 1$. Let $\varepsilon > 0$ and $\eta > 0$, and let a, δ and n_0 given by (6) and (7). For $i < n_0$, the single measure P_i is relatively compact, so by the Prohorov's theorem, it is tight, and by the first part of the proof, there exist $a_i > 0$ and $\delta_i > 0$ such that $P_i(\{x,\, \|x\|_\infty > a_i\}) \leq \eta$ and $P_i(\{x,\, \omega'_x(\delta_i) > \varepsilon\}) \leq \eta$. Thus, even it means taking $a' = (a \vee \max\limits_{1\leq i<n_0} a_i)$ and $\delta' = (\delta \wedge \min\limits_{1\leq i<n_0} \delta_i)$, we can assume $n_0 = 1$.

Let $\eta > 0$. Let $a > 0$, given by (6) such that for all $n \geq 1$, $P_n(A) \geq 1 - \frac{\eta}{2}$, where $A := \{x,\ \|x\|_\infty \leq a\}$. For $k \geq 1$, let δ_k, given by (7) such that for all $n \geq 1$, $P_n(A_k) \geq 1 - \frac{\eta}{2^{k+1}}$ where $A_k := \{x,\ \omega'_x(\delta_i) < \frac{1}{k}\}$. We define $B := A \cap \bigcap_{k\geq 1} A_k$, and $K := \overline{B}$. Then for all $n \geq 1$,

$$P_n(K^c) \leq P_n(A^c) + \sum_{k\geq 1} P_n(A_k^c) \leq \frac{\eta}{2} + \frac{\eta}{2}\sum_{k\geq 1}\frac{1}{2^k} = \eta.$$

Moreover, K satisfies (3) and (4), so by Theorem 2.10, K is compact. Thus, $(P_n)_{n\geq 1}$ is tight. □

Using Theorem 2.13 instead of 2.10 to characterize compactness, the second condition can be replaced by another one. We can also replace the first condition by another one involving only boundedness on a dense set of $[0, 1]$.

Corollary 3.11. *In the previous theorem, the condition* (7) *can be replaced by*

$$\forall \epsilon > 0, \quad \forall \eta > 0, \quad \exists \delta > 0, \quad \exists n_0 \in \mathbb{N}, \quad \forall n \geq n_0, \quad \begin{cases} P_n(\{x,\ \omega''_x(\delta) > \varepsilon\}) \leq \eta, \\ P_n(\{x,\ |x(\delta) - x(0)| > \varepsilon\}) \leq \eta, \\ P_n(\{x,\ |x(1^-) - x(1-\delta)| > \varepsilon\}) \leq \eta. \end{cases} \tag{8}$$

Let T be a dense subset of $[0, 1]$, that contains 0 and 1. In the previous theorem, the condition (6) *can be replaced by :*

$$\forall t \in T, \quad \forall \eta > 0, \quad \exists a > 0, \quad \exists n_0 \in \mathbb{N}, \quad \forall n \geq n_0, \quad P_n(\{x,\ |x(t)| > a\}) \leq \eta. \tag{9}$$

In the condition (9), the role of the set T will be played by T_P for a limit distribution P. We will see in the following theorem that this condition is satisfied as soon as we have the convergence of finite-dimensional distributions. The rest of the following theorem tells us that the second and third conditions of (8) are verified in the cases that interest us. If X is a random function of D, let us write T_X for T_P where P is the distribution of X.

Theorem 3.12. *Let $(X^n)_{n\geq 1}$ and X be random functions of D. Suppose that for all $t_1, \ldots, t_k \in T_X$,*

$$(X^n_{t_1}, \ldots, X^n_{t_k}) \Rightarrow (X_{t_1}, \ldots, X_{t_k}).$$

Suppose further that

$$\forall \varepsilon > 0, \quad \lim_{\delta\to 0} \mathbb{P}(|X_1 - X_{1-\delta}| \geq \varepsilon) = 0 \tag{10}$$

and

$$\forall \epsilon > 0, \quad \forall \eta > 0, \quad \exists \delta > 0, \quad \exists n_0 \in \mathbb{N}, \quad \forall n \geq n_0, \quad \mathbb{P}(\omega''(X^n, \delta) > \varepsilon) \leq \eta. \tag{11}$$

Then $X^n \Rightarrow X$.

Proof. According to Theorem 3.7, we only need to prove that the sequence $(P_n)_{n\geq 1}$ is tight, where P_n is the distribution of X_n. To do this, we need to check (8) and (9), with $T = T_X$.
First, for each $t \in T_X$, since $P_n\pi_t^{-1} \Rightarrow P\pi_t^{-1}$, the sequence $(P_n\pi_t^{-1})_{n\geq 1}$ is relatively compact, and therefore, by the Prohorov's theorem, it is tight, which gives (9).
Since the first one is a hypothesis of the theorem, we only need to check the second and third conditions of (8). Since X is in D, X is continuous at 0 a.s., that is to say $X_\delta - X_0 \xrightarrow[\delta\to 0]{\text{p.s.}} 0$, so $X_\delta - X_0 \xrightarrow[\delta\to 0]{\mathbb{P}} 0$, which gives that for $\varepsilon > 0$ and $\eta > 0$, there exists $\delta > 0$ such that $\mathbb{P}(|X_\delta - X_0| \geq \varepsilon) \leq \frac{\eta}{2}$. Even if it means reducing δ, we can assume $\delta \in T_X$, and then $(X^n_0, X^n_\delta) \Rightarrow (X_0, X_\delta)$, so by the continuous mapping theorem, $|X^n_\delta - X^n_0| \Rightarrow |X_\delta - X_0|$. Thus, there exists $n_0 \geq 1$ such that for all $n \geq n_0$, $|\mathbb{P}(|X^n_\delta - X^n_0| \geq \varepsilon) - \mathbb{P}(|X_\delta - X_0| \geq \varepsilon)| \leq \frac{\eta}{2}$, and therefore $\mathbb{P}(|X^n_\delta - X^n_0| \geq \varepsilon) \leq \eta$.
Finally, it is going to be the same for the third condition, but we need to consider some technical details since the continuity at 1 is not necessarily guaranteed. The hypothesis (10) implies $X_1 - X_{1-\delta} \xrightarrow[\delta\to 0]{\mathbb{P}} 0$. Since X is in D, X

has a left limit at 1 : $X_{1^-} - X_{1-\delta} \xrightarrow[\delta\to 0]{\text{p.s.}} 0$, so $X_{1^-} - X_{1-\delta} \xrightarrow[\delta\to 0]{\mathbb{P}} 0$. By combining the two convergences, we obtain $X_1 = X_{1^-}$ a.s, that is to say X is continuous at 1 a.s. Thus, by the same reasoning as previously, for $\varepsilon > 0$ and $\eta > 0$, there exist $\delta > 0$ and $n_0 \geq 1$ such that for all $n \geq n_0$, $\mathbb{P}(|X_1^n - X_{1-\delta}^n| \geq \frac{\varepsilon}{3}) \leq \frac{\eta}{3}$. Then, for all $n \geq n_0$,

$$\mathbb{P}(|X_{1^-}^n - X_{1-\delta}^n| \geq \varepsilon) \leq \mathbb{P}(|X_1^n - X_{1-\delta}^n| \geq \frac{\varepsilon}{2}) + \mathbb{P}(|X_1^n - X_{1^-}^n| \geq \frac{\varepsilon}{2}) \leq \frac{\eta}{3} + \mathbb{P}(|X_1^n - X_{1^-}^n| \geq \frac{\varepsilon}{2}).$$

Moreover, since $|X_1^n - X_{1^-}^n| = \lim_{h\to 0} |X_1^n - X_{1-h}^n|$, if $|X_1^n - X_{1^-}^n| \geq \frac{\varepsilon}{2}$, there exists $h > 0$ such that for all $h_1 \leq h$, $|X_1^n - X_{1-h_1}^n| \geq \frac{\varepsilon}{3}$. Thus,

$$\begin{aligned}\mathbb{P}(|X_1^n - X_{1^-}^n| \geq \frac{\varepsilon}{2}) &\leq \mathbb{P}\left(\bigcup_{h>0} \bigcap_{h_1 \leq h} \{|X_1^n - X_{1-h_1}^n| \geq \frac{\varepsilon}{3}\}\right) = \lim_{h\to 0} \mathbb{P}\left(\bigcap_{h_1 \leq h} \{|X_1^n - X_{1-h_1}^n| \geq \frac{\varepsilon}{3}\}\right) \\ &\leq \lim_{h\to 0} \mathbb{P}(|X_1^n - X_{1-h}^n| \geq \frac{\varepsilon}{3}).\end{aligned}$$

Even if it means reducing δ, we have

$$\lim_{h\to 0} \mathbb{P}(|X_1^n - X_{1-h}^n| \geq \frac{\varepsilon}{3}) \leq \mathbb{P}(|X_1^n - X_{1-\delta}^n| \geq \frac{\varepsilon}{3}) + \frac{\eta}{3} \leq \frac{\eta}{3} + \frac{\eta}{3} = \frac{2\eta}{3}.$$

Finally, we get

$$\mathbb{P}(|X_{1^-}^n - X_{1-\delta}^n| \geq \varepsilon) \leq \frac{\eta}{3} + \frac{2\eta}{3} = \eta.$$

□

From this result, we can deduce the following theorem. Its proof is too technical and too long to appear here, but this criterion will be way easier to use than the previous ones, for instance to prove the Donsker's theorem in the rest.

Theorem 3.13. *Let $(X^n)_{n\geq 1}$ and X be random functions of D. Suppose that for all $t_1, \ldots, t_k \in T_X$,*

$$(X_{t_1}^n, \ldots, X_{t_k}^n) \Rightarrow (X_{t_1}, \ldots, X_{t_k}).$$

Suppose further that

$$X_1 - X_{1-\delta} \Rightarrow_{\delta\to 0} 0 \tag{12}$$

and that there exist $\beta \geq 0$, $\alpha > \frac{1}{2}$ and F a nondecreasing, continuous function on $[0, 1]$ such that

$$\forall r \leq s \leq t, \quad \forall n \geq 1, \quad \forall \lambda > 0, \quad \mathbb{P}(|X_s^n - X_r^n| \wedge |X_t^n - X_s^n| \geq \lambda) \leq \frac{1}{\lambda^{4\beta}}(F(t) - F(r))^{2\alpha}. \tag{13}$$

Then $X^n \Rightarrow X$.

Remark 3.14. *The condition (13) is satisfied if we have the following inequality for all $n \geq 1$ and for all $r \leq s \leq t$:*

$$\mathbb{E}(|X_s^n - X_r^n|^{2\beta} |X_t^n - X_s^n|^{2\beta}) \leq (F(t) - F(r))^{2\alpha}. \tag{14}$$

Indeed, it follows from the Markov's inequality

$$\mathbb{P}(|X_s^n - X_r^n| \wedge |X_t^n - X_s^n| \geq \lambda) \leq \mathbb{P}(|X_s^n - X_r^n||X_t^n - X_s^n| \geq \lambda^2) \leq \frac{\mathbb{E}(|X_s^n - X_r^n|^{2\beta}|X_t^n - X_s^n|^{2\beta})}{\lambda^{4\beta}}.$$

We have now reached the first objective announced in the introduction : we have a great criterion for a sequence of random functions of D to be weakly convergent. Therefore, we have all the tools we needed to adapt the Donsker's theorem in the cadlag case.

4 The Donsker's theorem

Theorem 4.1. *Let $(\xi_n)_{n\geq 1}$ be a sequence of independent and identically distributed random variables, such as $\mathbb{E}(\xi_1)=0$ and $\mathrm{Var}(\xi_1)=1$. Let S_n be the random variable defined as $S_n=\sum_{i=1}^{n}\xi_i$ if $n\geq 1$, and $S_0=0$. Let X^n be the random function of D defined by*

$$X^n_t := \frac{S_{\lfloor nt\rfloor}}{\sqrt{n}},$$

for $t\in[0,1]$. Then the sequence of processes $(X^n)_{n\geq 1}$ converges in law with respect to the Skorohod topology on D to a standard Brownian motion B, i.e. $(X^n_t)_{t\in[0,1]}\Rightarrow(B_t)_{t\in[0,1]}$.

Proof. We are going to use Theorem 3.13. First, let us prove the convergence of the finite-dimensional distributions. Since the Brownian motion is continuous, $T_B=[0,1]$. Let $0<t_1<\cdots<t_m\leq 1$. For the rest, let us note $t_0=0$. Let U_n and U be defined as $U_n=(X^n_{t_1},\dots,X^n_{t_m})$ and $U=(B_{t_1},\dots,B_{t_m})$. We want to prove that $U_n\Rightarrow U$. To prove this convergence, it is sufficient to prove that $V_n\Rightarrow V$, where $V_n=(X^n_{t_1},X^n_{t_2}-X^n_{t_1},\dots,X^n_{t_m}-X^n_{t_{m-1}})$ and $V=(B_{t_1},B_{t_2}-B_{t_1},\dots,B_{t_m}-B_{t_{m-1}})$. Indeed, since the function $f:\begin{array}{ccc}\mathbb{R}^m & \longrightarrow & \mathbb{R}^m\\ x & \longmapsto & y\end{array}$ where $y_k=\sum_{j=1}^{k}x_j$ is continuous, by the continuous mapping theorem, $V_n\Rightarrow V$ implies $U_n=f(V_n)\Rightarrow U=f(V)$.
Let $N\geq 1$ such that $\frac{1}{N}<\inf_{1\leq i\leq m}|t_i-t_{i-1}|$. Then, for all $n\geq N$, for all $1\leq i\leq m$,

$$X^n_{t_i}-X^n_{t_{i-1}}=\frac{1}{\sqrt{n}}\sum_{k=\lfloor nt_{i-1}\rfloor+1}^{\lfloor nt_i\rfloor}\xi_k.$$

Thus, the marginal variables of V_n are independent. Likewise, since the Brownian motion has independent increments, the marginal variables of V are independent. Therefore, it is sufficient to prove that each marginal variable converges weakly. Let $n\geq N$, and $1\leq i\leq m$. Then

$$X^n_{t_i}-X^n_{t_{i-1}}=\frac{1}{\sqrt{n}}\sum_{k=\lfloor nt_{i-1}\rfloor+1}^{\lfloor nt_i\rfloor}\xi_k=\sqrt{\frac{\lfloor nt_i\rfloor-\lfloor nt_{i-1}\rfloor}{n}}\times\frac{1}{\sqrt{\lfloor nt_i\rfloor-\lfloor nt_{i-1}\rfloor}}\sum_{k=\lfloor nt_{i-1}\rfloor+1}^{\lfloor nt_i\rfloor}\xi_k,$$

with $\sqrt{\frac{\lfloor nt_i\rfloor-\lfloor nt_{i-1}\rfloor}{n}}\xrightarrow[n\to+\infty]{}\sqrt{t_i-t_{i-1}}$ and by the central limit theorem, $\frac{1}{\sqrt{\lfloor nt_i\rfloor-\lfloor nt_{i-1}\rfloor}}\sum_{k=\lfloor nt_{i-1}\rfloor+1}^{\lfloor nt_i\rfloor}\xi_k\Rightarrow\mathcal{N}(0,1)$. Thus

$$X^n_{t_i}-X^n_{t_{i-1}}\Rightarrow\mathcal{N}(0,t_i-t_{i-1})=\mathcal{L}(B_{t_i}-B_{t_{i-1}}).$$

Finally, we have proved the convergence of the finite-dimensional distributions.

The condition (12) is satisfied since the Brownian motion is continuous. To prove that (13) is satisfied, we are going to prove that (14) is satisfied. Let $t_1\leq t\leq t_2$. Then

$$\mathbb{E}(|X^n_t-X^n_{t_1}|^2|X^n_{t_2}-X^n_t|^2)=\frac{1}{n^2}(\lfloor nt\rfloor-\lfloor nt_1\rfloor)(\lfloor nt_2\rfloor-\lfloor nt\rfloor).$$

Indeed, there are two possibilities : either ($\lfloor nt\rfloor=\lfloor nt_1\rfloor$ or $\lfloor nt_2\rfloor=\lfloor nt\rfloor$), then both sides of the previous equality are 0 and that is satisfied, or $\lfloor nt_1\rfloor<\lfloor nt\rfloor<\lfloor nt_2\rfloor$ and by independence of the $(\xi_i)_{i\geq 1}$, we then have :

$$\begin{aligned}\mathbb{E}(|X^n_t-X^n_{t_1}|^2|X^n_{t_2}-X^n_t|^2)&=\mathbb{E}\left(\left(\frac{1}{\sqrt{n}}\sum_{k=\lfloor nt_1\rfloor+1}^{\lfloor nt\rfloor}\xi_k\right)^2\right)\times\mathbb{E}\left(\left(\frac{1}{\sqrt{n}}\sum_{k=\lfloor nt\rfloor+1}^{\lfloor nt_2\rfloor}\xi_k\right)^2\right)\\&=\frac{1}{n^2}\left(\sum_{k=\lfloor nt_1\rfloor+1}^{\lfloor nt\rfloor}\mathbb{E}(\xi_k^2)\right)\times\left(\sum_{k=\lfloor nt\rfloor+1}^{\lfloor nt_2\rfloor}\mathbb{E}(\xi_k^2)\right)=\frac{1}{n^2}(\lfloor nt\rfloor-\lfloor nt_1\rfloor)(\lfloor nt_2\rfloor-\lfloor nt\rfloor).\end{aligned}$$

Then, we deduce that

$$\mathbb{E}(|X_t^n - X_{t_1}^n|^2 |X_{t_2}^n - X_t^n|^2) \leq 4(t_2 - t_1)^2.$$

Indeed, there are again two possibilities : either $t_2 - t_1 \geq \frac{1}{n}$, then

$$\mathbb{E}(|X_t^n - X_{t_1}^n|^2 |X_{t_2}^n - X_t^n|^2) \leq \left(\frac{\lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor}{n}\right)^2 \leq \left(t_2 - t_1 + \frac{1}{n}\right)^2 \leq 4(t_2 - t_1)^2,$$

or $t_2 - t_1 < \frac{1}{n}$, then t is either in $\left[\frac{\lfloor nt_1 \rfloor}{n}, \frac{\lfloor nt_1 \rfloor + 1}{n}\right)$ or in $\left[\frac{\lfloor nt_2 \rfloor}{n}, \frac{\lfloor nt_2 \rfloor + 1}{n}\right)$, and in both cases we obtain

$$\mathbb{E}(|X_t^n - X_{t_1}^n|^2 |X_{t_2}^n - X_t^n|^2) = 0 \leq 4(t_2 - t_1)^2.$$

Finally, we have obtained (14) with $\beta = 1$, $\alpha = 1$ and $F(t) = 2t$. Thus, we can apply Theorem 3.13, and $X^n \Rightarrow B$. $\square$

Finally, we have reached our goal, and to do this, we have revisited the notion of weak convergence and tightness for random functions of D. In this document, we have worked with a topology called the J_1-topology. This work can be also done with others topologies on D, this is developed in [3].

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