Discussion of the article J.-P. Nadal, D. Phan, M. B. Gordon & J.Vannimenus, *Multiple equilibria in a monopoly market with heterogeneous agents and externalities*, Quantitative Finance 5, 557-568 (2005)

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We explore the behaviour of a simple market model where a large number of agents face a binary choice under social influence. We study the specific case of agents choosing to buy or not to buy a single product sold by a unique seller, i.e. a monopolist, being subjected to a global, homogeneous social influence. We argue this condition are analog to applying a mean-field approach in the Random Field Ising model from statistical physics. We find the monopolist's optimization program for maximizing his profit. In this effort, We build a phase diagram for the solutions of the maximization program and ilustrate a first-order phase transition for the order parameter. Finally, we exhibit the existence of hysteresis in the phase transition.

INTRODUCTION

We make decisions every day. In the case of binary decisions, such as buying or not a product, interceding or not in an ethically compromised situation, what are the conditions under which we select one option or the other? How to take into account the cost of each option, our personal preferences and the social influence others exert on us? Is there a way to make a quantitative model and explain or predict empirical results? This and other questions have been addressed by the physicists in the last decades with arguable success [1–9].

In this work we discuss one of the articles that have paved the way toward this goal: J.-P. Nadal, D. Phan, M. B. Gordon & J.Vannimenus *Multiple equilibria in a monopoly market with heterogeneous agents and externalities*, Quantitative Finance 5, 557-568 (2005) [1]. The authors study a simple market model composed by a large number of agents that face the decision of buying or not a certain product sold by a single seller (monopolist). While choosing, the agents are considered under global, homogeneous social influence.

We begin with an exposition of the model proposed by the authors to study binary choices under global, homogeneous social influence. Then we use it to deduce the optimization program the monopolist has to follow to maximize his profit. In this effort, we build a phase diagram displaying the solutions of the maximization process for the two control parameters of the model, exhibiting a first-order phase transition for the order parameter. At the end, we illustrate the existence of hysteresis in the phase transition.

SIMPLE MODEL FOR BINARY CHOICES UNDER SOCIAL INFLUENCE

The authors consider a set Ω_N of N agents. Each agent $i \in \Omega_N$ has either the option to buy ($\omega_i = 1$) or not to buy ($\omega_i = 0$) a specific product sold by a single seller (monopolist). Each agent chooses ω_i in order to maximize her/his surplus function

$$\max_{\omega_i \in 0,1} V_i = \max_{\omega_i \in 0,1} \omega_i (H_i + \sum_{k \in \vartheta_i} J_{ik}\omega_k - P) \qquad (1)$$

where P is the price of the product and H_i represents the idiosyncratic willingness-to-pay (IWP): without social influence, it is the maximum price the agent is willing to pay for the good. The sum $\sum_{k\in\vartheta_i} J_{ik}\omega_k$ is the social influence exerted on agent i by the subset $\vartheta_i \in \Omega_N$ of neighbor agents, being J_{ik} the weight of social influence exerted on agent i by agent $k \in \vartheta_i$. We can see that Equation (1) follows the form of a modified Ising model. Table I illustrates how to properly translate the economic and psychological variables relevant for a binary choice problem into their corresponding counterparts in statistical physics.

It is assumed that every agent i is subjected to a homogeneous social influence $J_{ik} = J/n$ from his neighbors $k \in \vartheta_i$ with $|\vartheta_i| = n$. In this case, each agent buys if

$$H_i > P - \frac{J}{n} \sum_{k\in\vartheta_i} \omega_k. \qquad (2)$$

Notice that we are considering here that the customers are myopic, in the sense that they will always buy if condition (2) holds. Only the monopolist is considered as a cognitive agent, who sets the price in order to maximize his profit.

Annealed vs. quenched disorder

According to the characteristics of the parameter H_i, two different approaches can be distinguished. One follows a psychological approach, proposed by Thurstone (1927) [10], called Thurstone Random Utility Model (TRUM). The other case comes from Economics, inspired by the works of Manski (1977) [11] and McFadden (1984) [12], and is called Quenched Random Utility Model (QRUM).

TABLE I. Correspondence between economic and psychological variables with statistical physics variables.

Economics & Psychology Discrete choice theory	Statistical Physics Ising and Potts models
Random Utility Models (RUM)	Random Fields Ising Model (RFIM)
N agents (customers)	N Ising spins
Binary choice buy or not buy: $\omega_i = \{0,1\}$	Binary state $S_i = \pm 1$
Posted price P	Global external field
Idiosyncratic willingness-to-pay H_i	Local external field
Social influence Positive externality $J_{ij} > 0$	Interactions Ferromagnetic coupling
Rational agents: i buys if $V_i = H_i - P + \sum_{k\in\vartheta_i} J_{ik}\omega_k > 0$	Ground state at $T=0$ $\sum_{i=1}^{N} \langle S_i \rangle \neq 0$

In the TRUM case, the H_i are considered i.i.d. time-dependent random variables. At each instant of time t, the $H_i(t)$'s are drawn from the same probability distribution described by its mean value H and the cumulative probability function of the deviations from the mean,

$$F(z) \equiv \mathcal{P}(H_i - H \leqslant z). \tag{3}$$

We assume that the mean value H is a "macroscopic" variable of the market in the sense that it can be can be observed by the monopolist. It has been shown [13] that in binary choice problems the random component of personal preferences that can not be observed by an external agent (the deviations from the mean herein) follows a logistic distribution. So the H_i's are logistically distributed with mean H, variance $\sigma^2 = \pi^2/(3\beta^2)$ and cumulative probability distribution

$$F(z) = \frac{1}{1+e^{-\beta z}}. \tag{4}$$

In the QRUM case, the H_i's are randomly distributed over the population Ω_N but their values remain fixed in time. Assuming a logistic distribution with mean H and variance $\sigma^2 = \pi^2/(3\beta^2)$, then at each instant of time t, the empirical distribution of the preferences H_i will be the same for the TRUM and the QRUM case. Taking the limit of a very large population, that distribution will become equal to the logistic distribution.

Following the analogy introduced in Table I, we can see that these two descriptions resemble two kinds of disorder found in Statistical Mechanics.

The TRUM corresponds to a case of annealed disorder. Having time-varying H_i's is analog to having stochastic (thermal) noise in the Ising spins. For the case of a logistic distribution, it corresponds to an Ising model with a uniform external field $H-P$ at a temperature $T = 1/\beta$. This description allows the existence of a well defined stationary value for the aggregate fraction of consumers —the order parameter of the problem.

On the other hand, the QRUM is a case of quenched disorder. In particular, it corresponds to a Random Field Ising Model (RFIM) with local time-independent random fields H_i and $T=0$ —no noise, then deterministic dynamics.

MEAN-FIELD APPROACH TO THE BINARY CHOICE PROBLEM WITH SOCIAL INFLUENCE

Hereafter the results will be presented for the QRUM case with homogeneous interactions and full connectivity. This corresponds to using a mean-field approach in the RFIM. Within this framework, the goal of this section is to find the optimization program the monopolist has to follow in order to maximize his profit.

Aggregate demand

We work within the QRUM picture, hence assuming that idiosyncratic preferences H_i remain fixed during the considered time. The IWP's can be then expressed as

$$H_i = H + \theta_i, \tag{5}$$

where H is the mean value over the population and θ_i is the idiosyncratic deviation from H.

We also assume homogeneous interactions and full connectivity among agents. The consequence of homogeneous interactions was already presented in Equation (2). Full connectivity means that the buying decision of agent i is influenced by the buying decision of all other agents. From (1) and (2), we can write

$$J_{ik} = \frac{J}{n} = \frac{J}{N-1} \underset{\lim\limits_{N\to\infty}}{=} \frac{J}{N}. \tag{6}$$

We can rewrite the condition of buying (2) as

$$\theta_i > P - H - J\eta \tag{7}$$

where $\eta = \eta(P)$ is the fraction of buyers that choose to buy at a given price P, namely:

$$\eta \equiv \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \omega_k. \tag{8}$$

Let's reformulate the condition of buying (7) as

$$\omega_i = 1 \iff \theta_i > z, \quad z \equiv P - H - J\eta. \tag{9}$$

We can find η as a fixed point. From (3), (5) and (8):

$$\begin{aligned}
\mathcal{P}(\theta_i \leqslant z) &\equiv F(z) \\
\mathcal{P}(\theta_i \geqslant z) &= 1 - F(z) \\
\lim_{N\to\infty} \frac{\sum_{k=1}^{N} \omega_k}{N} &= 1 - F(z) \\
\eta &= 1 - F(z) \qquad (10)\\
\eta &= 1 - F(P - H - J\eta) \qquad (11)
\end{aligned}$$

This result is general for any cumulative distribution of θ_i. Using the logistic distribution we obtain

$$\eta = \frac{1}{1 + e^{\beta(P-H-J\eta)}}. \quad (12)$$

Using Equation (11) we can write η as an implicit function of the price P:

$$\Phi(\eta, P) = \eta(P) + F(P - H - J\eta(P)) - 1 = 0. \quad (13)$$

Equation (12) relates the market penetration rate η with the product price P, hence an *inverse demand function* can be found [14]:

$$P^d(\eta) = H + J\eta + \frac{1}{\beta}\ln\left(\frac{1-\eta}{\eta}\right). \quad (14)$$

At given values of β, J and H, for most values of P Equation (12) has a unique solution $\eta(P)$. However, as will be discussed later, when $\beta J > 4 = \beta J_B$, there is a range of prices for which this equation has two stable and one unstable solutions.

Monopolist's supply maximizing profit

Now we are interested in finding the monopolist's optimal supply that maximizes his profit. We define the profit p in the usual way as

$$p \equiv P - C \quad (15)$$

where C is the cost for the production of each unit sold. Since $P - H = (P - C) - (H - C)$, we define

$$h \equiv H - C \quad (16)$$

and we rewrite z from Equation (9) as

$$z = p - h - J\eta \quad (17)$$

and taking

$$p^d(\eta) \equiv P^d(\eta) - C. \quad (18)$$

Each agent can buy only one unit of good, so the total expected profit for the monopolist is $pN\eta = N\Pi(p)$ with

$$\Pi(p) = p\,\eta(p) \quad (19)$$

where $\eta(p)$ is the solution of the implicit function (13). In order to optimize his profit the monopolist has to solve the following maximization problem:

$$p_{\mathrm{M}} = \arg\max_p \Pi(p). \quad (20)$$

If there is no discontinuity in the demand curve $\eta(p)$ (hence $\beta J < 4 = \beta J_B$ as mentioned before), we have

$$\left.\frac{\mathrm{d}\Pi(p)}{\mathrm{d}p}\right|_{p_{\mathrm{M}}} = 0 \quad (21)$$

$$\left[\eta(p) + p\frac{\mathrm{d}\eta(p)}{\mathrm{d}p}\right]_{p_{\mathrm{M}}} = 0$$

$$\left.\frac{\mathrm{d}\eta(p)}{\mathrm{d}p}\right|_{p_{\mathrm{M}}} = -\left.\frac{\eta(p)}{p}\right|_{p_{\mathrm{M}}} \quad (22)$$

From (13), we find the differential of the implicit function $\mathrm{d}\Phi(\eta, p) = 0$ and obtain

$$\frac{\mathrm{d}\eta(p)}{\mathrm{d}p} = -\frac{\partial\Phi/\partial p}{\partial\Phi/\partial\eta} = -\frac{\frac{\partial F}{\partial z}\cdot 1}{1 - J\frac{\partial F}{\partial z}} = \frac{-f(z)}{1 - Jf(z)} \quad (23)$$

where $f(z)$ is the logistic probability density function defined as

$$f(z) \equiv \frac{\mathrm{d}F(z)}{\mathrm{d}z} = \frac{\beta e^{-\beta z}}{\left(1 + e^{-\beta z}\right)^2}. \quad (24)$$

From (22) and (23) we obtain at $p = p_{\mathrm{M}}$:

$$\left.\frac{f(z)}{1 - Jf(z)}\right|_{p_{\mathrm{M}}} = \left.\frac{\eta(p)}{p}\right|_{p_{\mathrm{M}}} \quad (25)$$

Since the monopolist sees the demand level η, we may use (10) to rewrite Equation (24):

$$\begin{aligned} f(z) &= \beta\frac{1}{1+e^{-\beta z}}\frac{e^{-\beta z}}{1+e^{-\beta z}} \\ f(z) &= \beta\frac{1}{1+e^{-\beta z}}\frac{1}{1+e^{+\beta z}} \\ f(z) &= \beta F(z)\left(1 - F(z)\right) \\ f(z) &= \beta\left(1 - \eta\right)\eta. \end{aligned} \quad (26)$$

We may define $p = p^s(\eta)$ and write Equation (25) as

$$\begin{aligned} p^s(\eta) &= \eta\,\frac{1 - J\beta(1-\eta)\eta}{\beta(1-\eta)\eta} \\ p^s(\eta) &\equiv \frac{1}{\beta(1-\eta)} - J\eta. \end{aligned} \quad (27)$$

The function $p^s(\eta)$ was computed assuming the monopolist knows the penetration rate η, and it relates this variable with the profit (with price implicitly included), so it plays the role of an *implied* inverse supply function.

We may find the maximum profit, p_{M}, and the fraction of buyers that yields this profit, η_{M}, as the intersection between the inverse demand function (14) and the implied inverse supply function (27):

$$p_{\mathrm{M}} = p^d(\eta_{\mathrm{M}}) = p^s(\eta_{\mathrm{M}}). \quad (28)$$

The profit maxima are the solutions of (28) satisfying

$$\frac{\mathrm{d}^2\Pi(p)}{\mathrm{d}p^2} < 0. \quad (29)$$

Computing the second derivative $\partial^2\eta(p)/\partial p^2$ in the same spirit than Equation (23), using the explicit form of $f(z)$ (24) and considering Equation (22), we obtain

$$\frac{\mathrm{d}^2\Pi(p)}{\mathrm{d}p^2} = -2\frac{\eta}{p}\left[1 + \frac{2\eta - 1}{2\beta p(1-\eta)^2}\right] \quad (30)$$

and condition (29) holds if

$$\frac{1 - 2\eta}{2\beta p(1-\eta)^2} < 1. \quad (31)$$

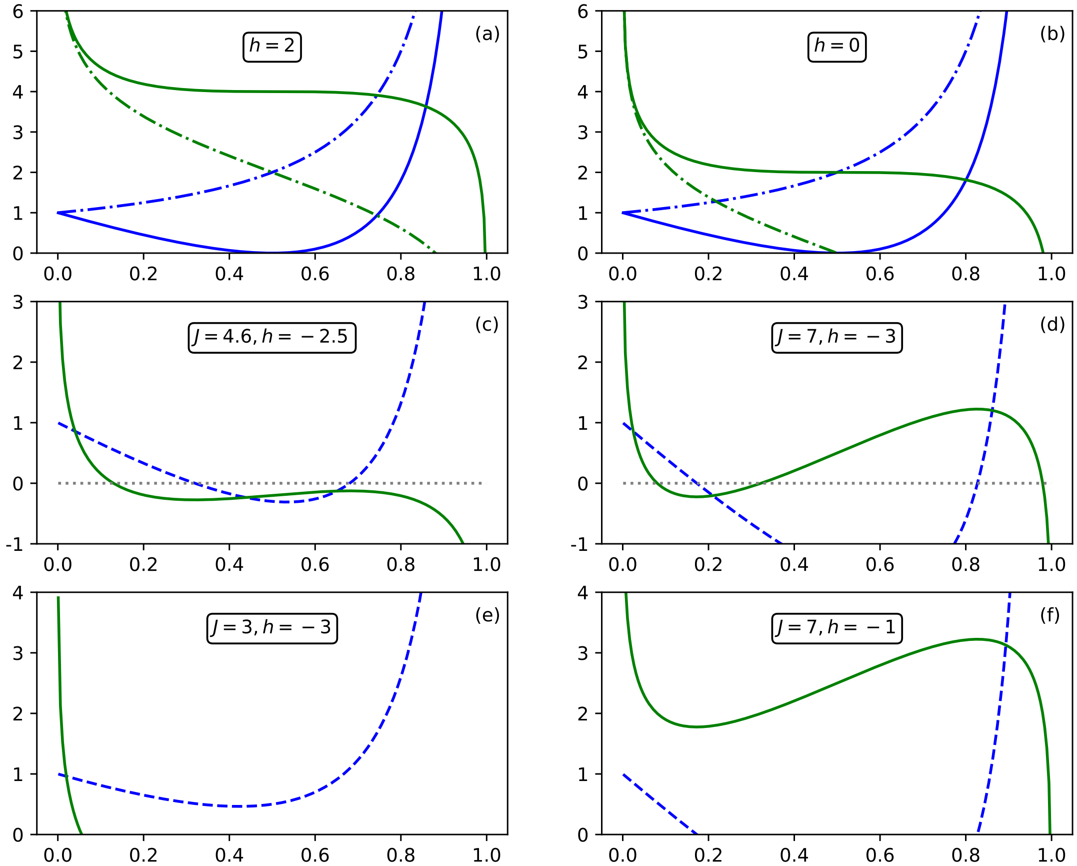

FIG. 1. Inverse demand $p^d(\eta)$ (blue) and implied supply $p^s(\eta)$ (green) functions for different values of h and J ($\beta=1$, $C=0$ hence $h=H$). The solutions for the monopolist's maximization problem η_M are found at the intersection of the two curves. (a) and (b) illustrate the difference between absence of externality ($J=0$, dashsotted lines) and strong externality ($J=4$, solid lines). (a) shows the case of a high mean value of the IWP's over the population ($h=2$) and (b) the case where the population is neutral ($h=0$). (c)-(f) correspond to the points shown in the phase diagram in Figure 2. They all correspond to negative values of mean willingness-to-buy ($h<0$) so that only a small fraction of the population is willing to buy. (c) and (d) are placed in the *coexistence region* where two solutions exist for the profit maximization problem. (c) corresponds to solutions with negative profit, which is undesired by the monopolist. (d) has only one relevant (positive) solution with large η. (e) corresponds to the situation where only one solution exists for a small value of η. (f) represent a situation with high social effect; one single solution exists for a large fraction of buyers and high price.

We can check that for $\eta > 1/2$ this condition always holds, so the corresponding solutions will always be local maxima. For a general value of $\eta \in [0,1]$, plugging Equation (27) into condition (31) we obtain

$$\left[1+\beta J\eta(\eta-1)\right]\left[-1+2\beta J\eta(\eta-1)^2\right] < 0. \tag{32}$$

The term $\beta J\eta(\eta-1)$ lies in the interval $[-0.25, 0]\,\forall\eta$, hence the left term in (32) is always negative. Then the condition (31) boils down to

$$2\beta J\eta(1-\eta)^2 < 1. \tag{33}$$

For $\beta J > 27/8 = \beta J_A$, the monopolist has to find $p_{\rm M}$ that realizes the program:

$$p_{\rm M} : \max\left\{\Pi_-(p_-^M), \Pi_+(p_+^M)\right\} \tag{34}$$

$$p_+^M = \arg\max_p \Pi_+(p), \quad \Pi_+(p) \equiv p\,\eta_+(p) \tag{35}$$

$$p_-^M = \arg\max_p \Pi_-(p), \quad \Pi_-(p) \equiv p\,\eta_-(p) \tag{36}$$

where the subscript $+$ and $-$ refer to the solutions of (12) for a fraction of buyers larger, and respectively smaller, that 1/2.

Figure 1 shows the solutions of Equation (28) for different values of J and h.

PHASE DIAGRAM

We can see that the maximum profit (20) depends only on the products βJ and βh. Indeed, from (12) and (19), we can see that (20) reads

$$\Pi(p) = p\eta(p) = \frac{p}{1 + e^{-\beta h - \beta J \eta(p) + \beta p}}. \quad (37)$$

Then βJ and βh are the parameters of our problem and we can build a phase diagram for the solutions of the maximization problem (28). The parameter β is related to the inverse of the variance of the logistic distribution of the personal preferences θ_i and it sets the scale of the problem. For simplicity, we will discuss the phase diagram for values of J and h measured in units $1/\beta$, which is equivalent to setting $\beta = 1$ during the analysis.

Figure 2 shows the phase diagram. The regions 1 and 2 contain two solutions for the maximum profit. For the case of $h > 0$, that is most of the customers have the preference to buy the product, there is a single solution η_M that varies continuously for all the parameters values. It corresponds to the majority of people buying the product and the monopolist maximizing its profit by selling at low price.

The case of $h < 0$ is more interesting. Suppose we start with really low values of h. Because of the distribution of the IWP's, only a small fraction of the customers are willing to buy the product. Here the monopolist's largest profit is achieved by selling at high price to a small number of customers. Now we start increasing h. If $J < 27/8 = J_A$ the solution η_M varies continuously from low to high values as h increases.

If $J > 27/8 = J_A$, when h increases we hit the lower boundary of the 1-2 region, $h = h_-(J)$, with solution $\eta = \eta_- < 1/3$. In this point, a new maximum of the profit appears for $\eta = \eta_+ > 1/3$. We call $h = h_+(J)$ the upper boundary of region 1. The two curves $h_-(J)$ and $h_+(J)$ merge at the singular point A, at which $\eta = \eta_- = \eta_+ = 1/3$, $J_A = 27/8$, and $h_A = -3/4 - \ln(2)$.

Expanding the analytical expressions of J and h (skipped here for the sake of simplicity) around $\eta = 1/3$, we can find that both curves $h_-(J)$ and $h_+(J)$ are cotangent in A with slope $-2/3$. A straight segment of slope $-2/3$ starting in A can be seen as the transition line of a *first-order phase transition*: at a critical value $J_c(h)$ of the control parameter, the fraction of buyers jumps from a low to a high value. Before the transition, for a value of J larger than $J_-(h) < J_c(h)$ —that is the bottom part of the 1-2 region—Equation (28) already presents several solutions. Two of them are local maxima and one a local minimum. The global maximum corresponds to the solution of high price with few buyers for $J < J_c$, and that of low price and many buyers for $J > J_c$.

If $J > 4 = J_B$, when h increases we hit the lower boundary of region 2. In this case the local maximum for η_+ appears with a negative profit, being of null profit at point B for $\eta_+ = 1/2$. This situation is not desired by the monopolist. The profit becomes positive on the boundary between the regions 1 and 2.

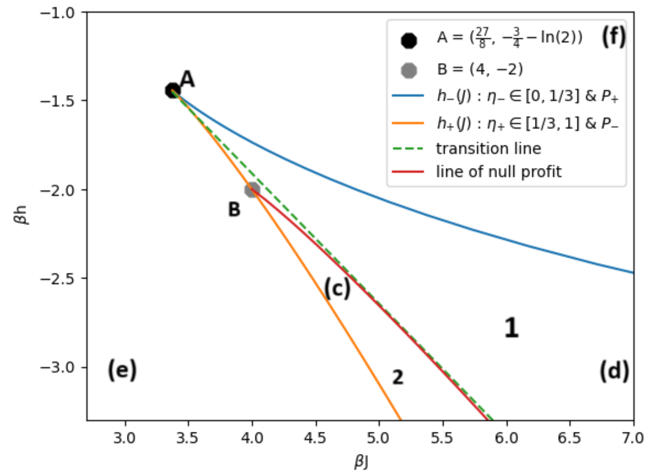

FIG. 2. Phase diagram for the solutions of the profit maximization program (28) as a function of the two control parameters βJ and βh. Regions 1 and 2 correspond to the regions where multiple solutions exist; outside these regions, a single continuous solution exists. Regions 1 and 2 are bounded by the functions $h_+(J)$ and $h_-(J)$. These two functions merge cotangently at the critical point A with slope -2/3. The green dashed line that departs from A with slope $-2/3$ can be regarded as the transition line of a first-order phase transition. At point B the profit is null. Region 2 contains the solutions with negative profit, thus being non-desirable for the monopolist.

From this results it can be seen that if the social influence is strong enough ($J > J_A = 27/8$), as the mean social willingness-to-buy increases, the cost decreases or the social influence increases more, the optimal strategy of the monopolist changes abruptly from a regime of high price with few buyers to a regime of low price with many buyers.

HYSTERESIS

Up to now we have made a static analysis of our problem. We can study its dynamics by considering that the fraction of buyers $\eta(t)$ at time t depends upon $\eta(t-1)$. In what we call synchronous dynamics, at each instant of time t, every agent updates his decision based on the average collective decision in the previous instant of time. From Equation (11) we can write

$$\eta(t) = 1 - F(P - H - J\eta(t-1)). \quad (38)$$

To discuss the dynamics, the authors cited the results of a previous article by their own research group, Phan et al. [2]. In that paper they used an agent-based simulation software called *Moduleco*, developed at the end of the last century and already deprecated, to study the dynamics of a finite number of agents situated in a lattice and with a finite number of neighbors.

So far in the article we have discussed the case of a mean-field approach (global, homogeneous externality),

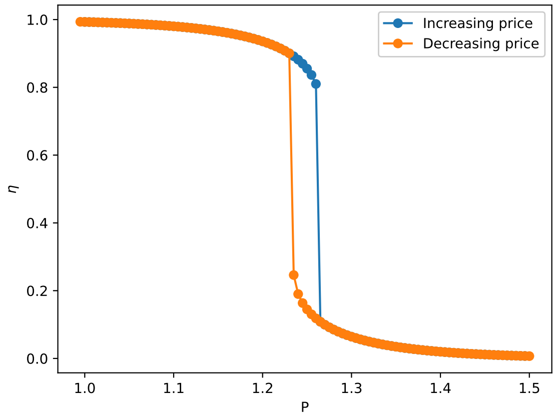

FIG. 3. Equilibrium value of the number of buyers $\eta(P)$ for increasing and decreasing values of the price P. The rest of parameters are $H = 1$, $J = 0.5$ and $\beta = 10$. Abrupt changes in the value of η (a case of *avalanche*) take place at two different values of the price depending on the direction of variation. The blue dots correspond to increasing values of the price, and the orange dots to decreasing prices. This is an example of hysteresis.

where the details of the lattice play no role. For this reason we have preferred taking a different route for reproducing Figure 4a from the original article [1].

We know Equation (12) represents the fixed point η for a given price, that is the equilibrium value of η after a sufficiently long time has elapsed. We find the root of Equation (12) for each value of $P \in [1.0, 1.5]$. Standard root finding methods require an initial value of the variable to look for the nearest root. When price is increasing, we set the initial value to 1, since we know that at low price there will exist a big fraction of buyers. On the opposite side, when price is decreasing we set the initial value equal to 0.

Figure 3 shows the fixed points $\eta(P)$ for increasing (blue dots) and decreasing (orange dots) price P with steps 10^{-2}. The values of the rest of the parameters correspond to $H = 1$, $J = 0.5$ and $\beta = 10$. We can see that the price at which the first-order phase transition occurs depends on the direction of variation of the price, so it depends on the history of the system. This phenomenon is called hysteresis, characteristic in first-order phase transitions. Along the downstream trajectory (with increasing prices, blue curve) the externality effect induces a strong resistance of the demand against a decrease in the number of customers. The inverse is valid in the opposite direction.

In both cases, the abrupt change in the value of η represents a unique, huge avalanche. For a system where agents interact only with a finite neighborhood, we see a series of successive, smaller avalanches until the system reaches equilibrium for each value of P (see Figure 4b in the original article [1]).

The monopolist achieves the biggest profit by selling at the highest possible price to the largest number of customers. This means that he follows a risky optimization program: the price that maximizes his profit is just below the price that makes the demand fall abruptly.

The existence of the avalanches is controlled by the value of the parameter β. We can see in Figure 4a that for a small value of β (large σ) there is no avalanche. As we increase β, we find the first avalanche for $\beta = 9$ (Figure 4b). For larger values of β, the size of the gap between the two prices at which the avalanche occurs increases, see Figure 4c.

DISCUSSION AND CONCLUSIONS

In this work we discussed the article J.-P. Nadal *et al.* (2005) [1], where we studied a simple model for binary choices under social influence. Other models has been proposed in the literature for multiple (non-binary) choices, for example in [7].

We discussed two different ways to quantitatively model human preferences: the Thurstone Random Utility Model (TRUM) and the Quenched Random Utility Model (QRUM). We argued the analogy of these two models with the annealed and quenched disorders found in statistical physics. In particular, the QRUM case corresponds to a Random Field Isign Model (RFIM) with local time-independent random fields and zero temperature.

By considering that idiosyncratic preferences are random over the population but fixed in the considered time, corresponding to the QRUM picture, we studied the problem of binary choices made by a large number of agents under social influence. In particular, the case where all agents, submitted to homogeneous external influence and full connectivity, have to choose to buy or not buy a single product sold by a unique seller, i.e. a monopolist.

It was found the monopolist's optimization program for maximizing his profit under this scenario. In this effort, a first-order phase transition was exhibited: when the social influence is strong enough, there is a regime where, upon increasing the mean willingness-to-pay or decreasing the production costs, the optimal monopolist's strategy abruptly jumps from selling at high price to small number of buyers, to selling at low price to a large number of buyers. In the mean field approach, this transition takes place through a unique, huge avalanche for the value of the order parameter. This seems to support the idea of using the RFIM at zero temperature for describing the system, since this model has proven to be adequate for describing material cracking that takes place through a series of avalanches [15].

We showed that at fixed control parameters and varying the price, the value of the price at which the phase transition occurs depends on whether the price was being

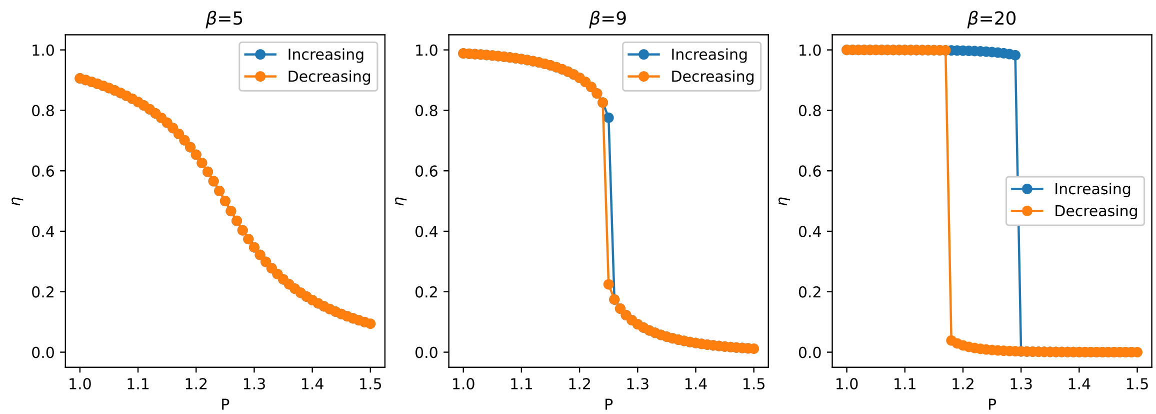

FIG. 4. Dependence of η with the price for different values of β. (a) There is no avalanche for a small values of β. (b) The first avalanche occurs for $\beta = 9$. (c) For larger values of β, the size of the gap between the two prices at which the avalanche takes place increases.

increased or decreased, i.e. depends on the history of the system. This phenomenon is called hysteresis, and it is typical in systems exhibiting first-order phase transitions.

In later works of the authors' research group [4, 5] they have deepen the study of this topic. For example, they have noticeably expanded the analysis of the phase diagram, discussing multiple strategies of the monopolist together with their risks and outcomes.

The predicted value of -2/3 for the slope of the transition line (see Figure 2) has been in good agreement with empirical data of birth rate in Europe in the period 1950-2000 and cell phone adoption in five European countries in the period 1994-2003 [9]. This is considered a good support of the model presented herein since it is hard to find another argument that explains this behaviour [16].

We would like to make a final, ethical remark. The original article was made assuming people always buy if a certain condition holds and was focused on discussing the different strategies a monopolist, i.e. a large company that controls the whole market, can follow to maximize its profit. In our case, we would like instead to use our scientific knowledge and efforts toward finding ways to improve the quality of life of the majority of people, specially those that are in the disadvantageous side of inequality.

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