Integer Programming Examples

Project Selection

An artist has been approached about a number of different projects for Christmas presents. There have been n projects requested on her website. For each project i, she would get paid c_i but the project would take a_i hours to complete. There are only b working hours left before Christmas so the artist may not be able to accept all possible projects.

Let x_i be a variable that can have value 0 or 1 representing whether or not the artist will work on project i. Then the artist can find the best subset of projects to accept by solving the following integer program.

$$\begin{aligned} \text{maximize } & \sum_{i=1}^{n} c_i x_i \\ \text{subject to } & \sum_{i=1}^{n} a_i x_i \leq b \\ & x_i \in \{0,1\} \quad \forall i = 1,2,\ldots,n \end{aligned}$$

Suppose we have any feasible solution x to this problem. Let S be the set of projects with $x_i = 1$. Note that the objective function will give the total payment she will get for selecting all projects in S and the linear constraint implies that the projects in S will be completed before Christmas.

Factory Production

A factory is trying to decide how to produce a certain product. If the factory produces the product using a more enviromentally friendly production method (method a), then they will get c dollars per unit produced. Using this method, each unit produced requires m_a minutes to produce and there are only 8 hours of production time each day. On the other hand, if the factory produces the product using the standard method (method b) they still get c dollars per unit produced but they have to pay a fine of f dollars per day of production. In addition, each unit only takes $m_b < m_a$ minutes to produce. The factory wants to choose the method that maximizes their profit.

Let $y \in \{0,1\}$ be a variable for whether or not they use method a and let x be a variable representing how many units are produced per day. If $y = 1$ and the factory uses method a, then we want to add the constraint that

$$m_a x \leq 480$$

and if $y = 0$ and the factory uses method b, then we want to add the constraint that

$$m_b x \leq 480.$$

We can do this by using the following constraint

$$x \leq (480/m_a)y + (480/m_b)(1-y).$$

This type of constraint is called an *if-then* constraint and shows some of the power of using integer variables! Thus, the factory can determine how many units to produce per day and which method to use by solving the following integer program.

$$
\begin{aligned}
&\text{maximize } cx - fy \\
&\text{subject to } x \le (480/m_a)y + (480/m_b)(1-y) \\
&\qquad\qquad\quad x \ge 0, \text{ integer} \\
&\qquad\qquad\quad y \in \{0,1\}
\end{aligned}
$$

Factory Scheduling

In the planning of the monthly production for the next four months a company must, in each month, operate either a normal shift, an extended shift, or not produce within that month. A normal shift costs 20000 per month and can produce up to 500 units per month. An extended shift costs 30000 per month and can produce up to 700 units per month. However, union rules state that an extended shift cannot operate two months in a row. If the company operates an extended shift within a month, it must produce at least 550 units.

The company must meet demand d_i for each month $i = 1, \ldots, 4$ and starts with 500 units in stock. The cost of holding stock is estimated to be 200 dollars per unit per month (based on the stock held at the end of each month). The company's goal is to schedule shifts to meet the demand constraints but minimize cost.

For each month we have a variable $x_i \in \{0,1\}$ representing whether or not the company operates an extended shift in month i. Furthermore, we let $p_i \ge 0$ be a variable representing how many units we produce in that month and $h_i \ge 0$ be a variable representing how many units we must hold between months.

Given the variables above, the cost of any solution is given by

$$\sum_{i=1}^{4} [20000 + 10000x_i] + \sum_{i=1}^{4} 200h_i.$$

To constrain that we cannot operate two extended shifts in one month, we can add in the following constraints

$$
\begin{aligned}
x_2 &\le 1 - x_1 \\
x_3 &\le 1 - x_2 \\
x_4 &\le 1 - x_3
\end{aligned}
$$

If $x_i = 1$, then we can produce between 550 and 700 units. If $x_i = 0$, then we can produce between 0 and 500 units. This requirement is captured in the following linear constraints.

$$550x_i \le p_i \le 500(1 - x_i) + 700x_i \quad (i = 1, \ldots, 4).$$

Lastly, we need constraints to capture how many units we need to hold each month. Note in the first month we start with 500 units and produce p_1 units. However, d_1 units are purchased. Therefore,

$$h_1 = 500 + p_1 - d_1.$$

Continuing this trend,

$$
\begin{aligned}
h_2 &= h_1 + p_2 - d_2 \\
h_3 &= h_2 + p_3 - d_3 \\
h_4 &= h_3 + p_4 - d_4
\end{aligned}
$$

Note that since we have the constraint that $h_i \ge 0$ for all months, we will always meet demand.

Let $h_0 = 500$. Combining all these together, we get the following integer program

$$
\begin{aligned}
\text{minimize } & \sum_{i=1}^{4} [20000 + 10000x_i] + \sum_{i=1}^{4} 200h_i \\
\text{subject to } & x_i \leq 1 - x_{i-1} \quad (i = 2, 3, 4) \\
& 500x_i \leq p_i \leq 500(1 - x_i) + 700x_i \quad (i = 1, 2, 3, 4) \\
& h_i = h_{i-1} + p_i - d_i \quad (i = 1, 2, 3, 4) \\
& h_i \geq 0 \text{ integer} \quad (i = 1, 2, 3, 4) \\
& x_i \in \{0, 1\} \quad (i = 1, 2, 3, 4) \\
& p_i \geq 0 \text{ integer} \quad (i = 1, 2, 3, 4)
\end{aligned}
$$

Campus maps

Students keep getting lost around campus so Cornell University wants to provide put up maps around campus. There are m spots on campus where a map can be placed. The university wants to ensure that of the n buildings on campus, no building is futher than distance d away from a map. For each spot i and buidling j, let $d_{i,j}$ be the distance from spot i to building j. However, each router costs f dollars and the university wants to minimize the total cost.

For each $i = 1, \ldots, m$, let $y_i \in \{0, 1\}$ be a variable representing whether or not the university should place a map at spot i. Further, for each $i = 1, \ldots, m$ and $j = 1, \ldots, n$ let $x_{i,j} \in \{0, 1\}$ be a variable representing whether or not spot i serves building j.

The cost of any solution will be given by $\sum_{i=1}^{m} fy_i$. The university wants to ensure that each building is served by a map. That is, for each $j = 1, \ldots, n$

$$\sum_{i=1}^{m} x_{i,j} = 1$$

Furthermore, spot i can only serve that building if there is indeed a map there so we need to add the constraint that

$$x_{i,j} \leq y_i.$$

Lastly, we want to make sure that a building is served by a map within distance d.

$$\sum_{i=1}^{m} d_{i,j}x_{i,j} \leq d.$$

Putting this together, the university can decide where to place maps around campus by solving the following integer program.

$$
\begin{aligned}
\text{minimize } & \sum_{i=1}^{m} fy_i \\
\text{subject to } & \sum_{i=1}^{m} x_{i,j} = 1 \quad (j = 1, \ldots, n) \\
& \sum_{i=1}^{m} d_{i,j}x_{i,j} \leq d \quad (j = 1, \ldots, n) \\
& x_{i,j} \leq y_i \quad (i = 1, \ldots, m, j = 1, \ldots, n) \\
& y_i \in \{0, 1\} \quad (i = 1, \ldots, m) \\
& x_{i,j} \in \{0, 1\} \quad (i = 1, \ldots m, j = 1, \ldots, n)
\end{aligned}
$$