ENGRI 1101	Engineering Applications of OR	Fall 2016	Handout 8

Into to Linear Programming

Suppose that I am trying to decide how much orange juice and cereal to consume at breakfast. I will let $x_1 \geq 0$ be a variable repesenting how many servings of orange juice I drink and $x_2 \geq 0$ be a variable representing how many servings of cereal I eat. Each serving of orange juice gives me 70% of my daily value of vitamin C but has 7 grams of sugar. Each serving of cereal gives me 10% of my daily value of vitamin C and has 1.2 grams of sugar.

To meet my daily value of vitamin C I would need

$$0.7x_1 + 0.1x_2 \geq 1.$$

However, I want to limit the sugar to at most 9 grams at breakfast so I also would like

$$7x_1 + 1.2x_2 \leq 9.$$

If x_1 and x_2 satisfy these two constraints I would consider them a feasible breakfast to eat. There are many ways I could meet these two requirements. However, each serving of orange juice costs me 50 cents whereas a serving of Cheerios costs me 25 cents. Therefore, I want to find a feasible breakfast that minimizes my cost of $50x_1 + 25x_2$.

Therefore, I want to minimize $50x_1 + 25x_2$ subject to

$$\begin{aligned} 0.7x_1 + 0.1x_2 &\geq 1 \\ 7x_1 + 1.2x_2 &\leq 9 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

Notice that the objective function and constraints are *linear* in my decision variables x_1 and x_2. That is they are linear functions of these variables. This is a linear programming problem.

Linear Programs

A linear programming (LP) problem is defined as a problem of maxmizing or minimizing a linear objective function subject to linear constraints. In any linear program, we will have **decision variables** $x_1, x_2, \ldots, x_n$ whose values we want to determine.

The goal will be to set the values of $x_1, \ldots, x_n$ to maximize or minimize some **linear objective function**

$$c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

where the c_i's are numbers we know ahead of time (like 50 and 25 in the above example).

However, the x_i variables are subject to some **linear constraints** which can be of the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n \leq b,$$

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n \geq b,$$

or

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b.$$

Again, the value of each a_i and b are given numbers. For example, we often require that $x_i \geq 0$ for $i = 1, 2, \ldots n$. These are called **non-negativity constraints**. Any solution $(x_1, x_2, \ldots, x_n)$ is a **feasible solution** if it satisfies the given linear constraints.

An example linear program is

$$\begin{aligned} \text{maximize } & x_1 + x_2 \\ \text{subject to } & x_1 + 2x_2 \leq 4 \\ & 4x_1 + 2x_2 \leq 12 \\ & -x_1 + x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

One feasible solution is $(8/3, 2/3)$ and has objective function value $8/3 + 2/3 = 10/3$ whereas the solution $(-1, 0)$ is not feasible. In fact, $(8/3, 2/3)$ has the highest objective function value of any feasible solution. We call this solution the **optimal solution** and 10/3 the **optimal value** (or optimal objective).

Feasible Region and Graphical Method

Consider the following linear program

$$\begin{aligned} \text{maximize } & 2x_1 + x_2 \\ \text{subject to } & x_1 + x_2 \leq 5 \\ & x_1, x_2 \geq 0 \end{aligned}$$

We can easily see that the optimal solution to this LP is $(5, 0)$ with optimal value 10. However, sometimes the LP might not be so easy to solve. Graphing the set of feasible points can sometimes help.

Suppose a company wants to decide how many tables and chairs to produce per day. Every table earns the company 12 dollars whereas a chair earns them 10 dollars. Each day, they can produce at most four tables per day and at most three chairs. Furthermore, they only have six employees and can produce at most six total pieces of furniture a day.

To represent this as a linear program we let variable x_1 represent how many tables per day to produce and x_2 represent how many chairs to produce. This yields the following linear program:

$$\begin{aligned} \text{maximize } & 12x_1 + 10x_2 \\ \text{subject to } & x_1 \leq 4 \\ & x_2 \leq 3 \\ & x_1 + x_2 \leq 6 \\ & x_1, x_2 \geq 0 \end{aligned}$$

By graphing each of these constraints, we know all feasible solutions will be in the region indicated in Figure 1. The region containing all feasible solutions is called the **feasible region**. Now we could try to look at all points in this region to determine the best possible one but there are infinitely many! Many of these points will actually have the same objective function value. For example if we want to find all points with objective function value 20 we need to find points such that

$$12x_1 + 10x_2 = 20.$$

This is just a line through the feasible region given in Figure 2. We call this line an **isoprofit line** since all points will have the same "profit" or objective function.

Let $Z = 12x_1 + 10x_2$. We can graph all points with objective function value $Z = 30$, $Z = 40$, etc. As shown in Figure 3, these lines will be parallel. In addition, as we increase Z, we can now easily tell that the point $(3, 3)$ at the intersection of the lines $x_1 = 3$ and $x_1 + x_2 = 6$ will be the optimal point with an objective function value of 66.

This method of finding the optimal objective function value for linear programs is called the **graphical method.** Unfortunately it becomes much more complicated with more than two variables. To solve larger linear programs we will use an algorithm called the *simplex method*. Notice that our optimal solution was at a corner of the feasible region. The simplex method takes advantage of this property and goes from one corner to another to find the best solution.

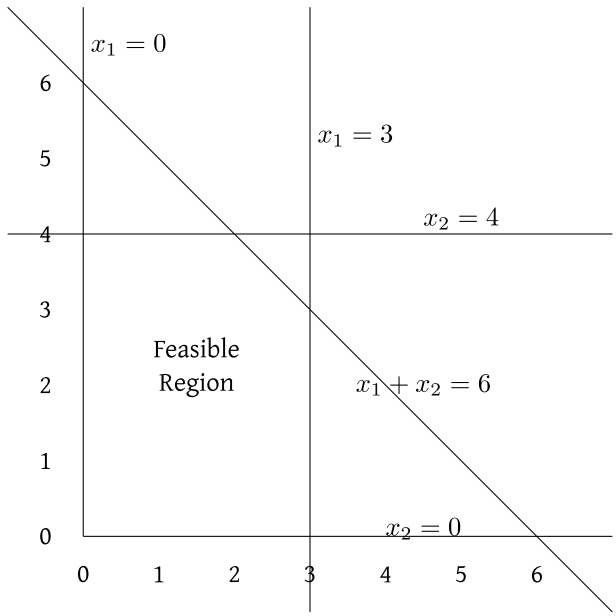

Figure 1: The feasible region of a linear program found by graphing the constraints.

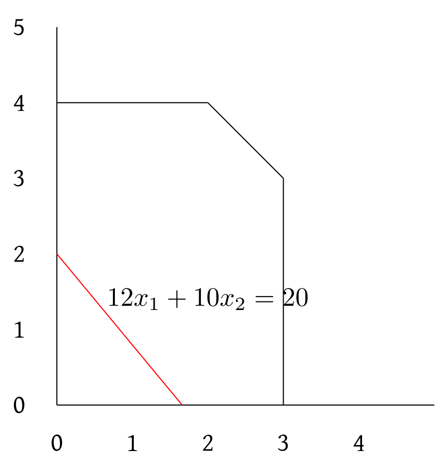

Figure 2: Plotting all feasible points with objective function value 20.

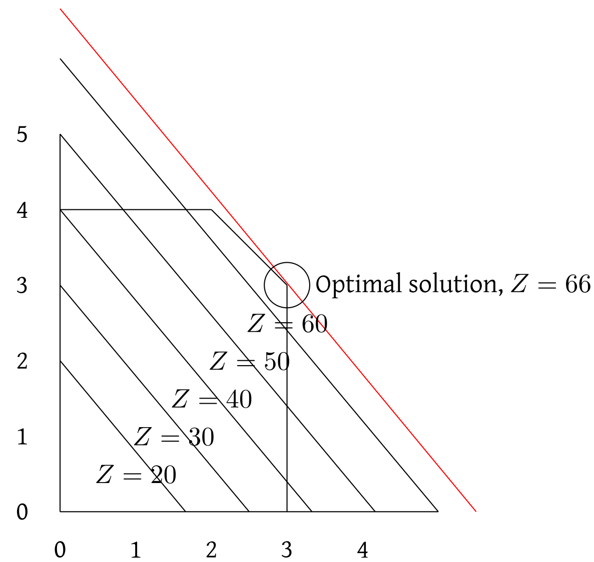

Figure 3: Plotting several isoprofit lines for different objective function values.

LP Example 1: Assignment Problem

A company has n workers available and m projects available that day. The value of person i working on project j for the full day is $v_{i,j}$. A person cannot work more than the given day and any project can only be worked on by one person at a time. The company wants to assign workers to jobs to maximize the value.

For each worker $i = 1, \ldots, n$ and job $j = 1, \ldots, m$ let $x_{i,j}$ be a variable representing what proportion of the day worker i will work on project j. The value of any such assignment is given by

$$\sum_{i=1}^{n} \sum_{j=1}^{m} v_{i,j} x_{i,j}.$$

We need to make sure the proportions for each worker add up to at most one. For each worker i, this can be represented as the linear constraint

$$\sum_{j=1}^{m} x_{i,j} \leq 1.$$

Furthermore, we need to make sure people don't overlap on a project by adding the constraint

$$\sum_{i=1}^{n} x_{i,j} \leq 1$$

for each job $j = 1, \ldots, m$.

Putting these together, we can get the optimal assignment of workers to project by solving the following linear program.

$$
\begin{aligned}
\text{maximize } & \sum_{i=1}^{n}\sum_{j=1}^{m} v_{i,j}x_{i,j} \\
\text{subject to } & \sum_{j=1}^{m} x_{i,j} \leq 1 \quad (i = 1, \ldots, n) \\
& \sum_{i=1}^{n} x_{i,j} \leq 1 \quad (j = 1, \ldots, m) \\
& x_{i,j} \geq 0 \quad (i = 1, \ldots, n; j = 1, \ldots m)
\end{aligned}
$$

LP Example 2: Cookie Recipe

Betsy really enjoys these cookies she buys at her local grocery shop. However, they are quite expensive and Betsy is a great baker. Given the nutrition label, Betsy wants to determine the recipe. Suppose there are n ingredients such that ingredient 1 is listed first on the label then ingredient 2, etc. In addition, there are m nutrient values listed.

For each nutrient j, let b_j be the amount (in grams) of that nutrient listed on the cookie's nutrition label. Furthermore, for each ingredient i and nutrient j, let $c_{i,j}$ be the grams of nutrient j per gram of ingredient i.

Betsy lets variable $x_i \geq 0$ be the amount of ingredient i for $i = 1, 2 \ldots, n$ to put into her cookie recipe. Given the ordering of the ingredients on the label, Betsy adds the linear constraints

$$x_1 \geq x_2 \geq \ldots \geq x_n.$$

Furthermore, for each nutrient $j = 1, 2, \ldots, m$, Betsy adds the constraint

$$\sum_{i=1}^{n} c_{i,j}x_i = b_j$$

to match the nutrition label.

Since Betsy is just interested in finding a feasible solution, all possible solutions have the same objective function value to her. Therefore, to find a potential recipe for the cookies Betsy will solve the following linear program.

$$
\begin{aligned}
\text{maximize } & 0 \\
\text{subject to } & \sum_{i=1}^{n} c_{i,j}x_i = b_j \quad (j = 1, 2, \ldots m) \\
& x_i \geq x_{i+1} \quad (i = 1, \ldots, n-1) \\
& x_i \geq 0 \quad (i = 1, \ldots, n)
\end{aligned}
$$

Even though this problem has a very odd format it still counts as a linear program!