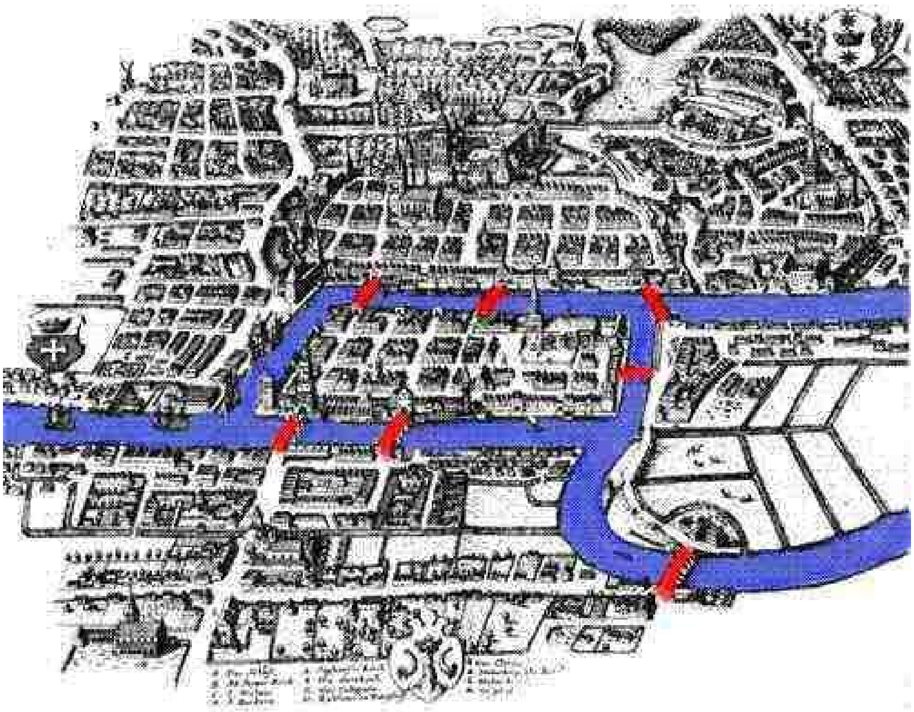
Seven bridges: the origin of graph theory

Difficulty level: intermediate

Keywords:
- Graph theory
- Eulerian graphs
- Euler's theorems

0) Introduction

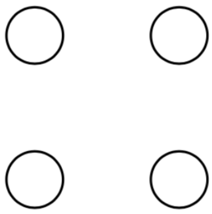


Given this map of a city crossed by a river, is it possible to cross all bridges once and only once? Motivate the answer.

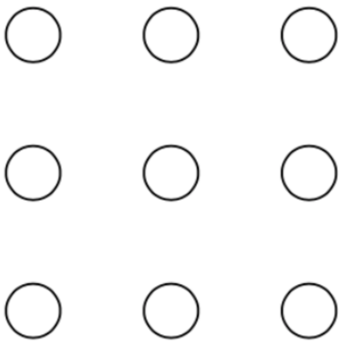
1) A first leap into the past: the “9 points” game

In the early years of school, did you ever challenge your classmates to “connect the dots” without ever removing the pen from the paper?

Let's start with four dots: can you connect them all without ever removing the pen from the paper? Indicate the starting point and the final point, and also mark the direction of travel of the connections you will draw. Not that you can only use straight lines.

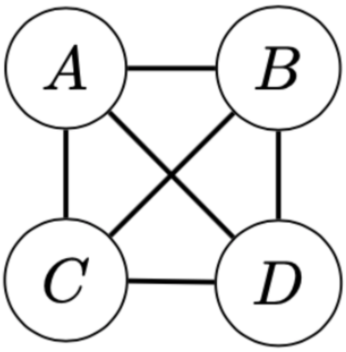
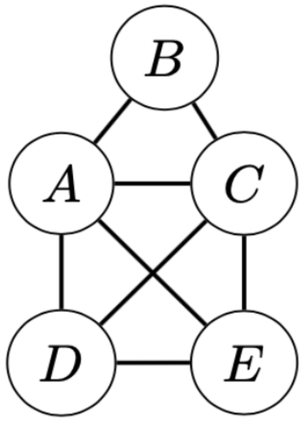
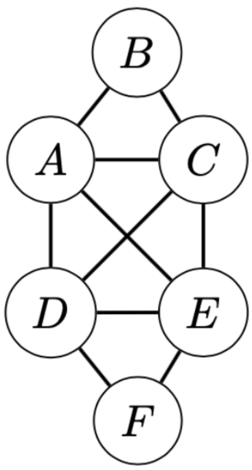
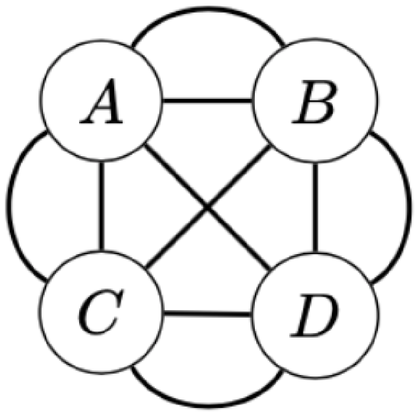


What if the dots were the following nine instead? Again, you can only use straight lines (actually, only four).



2) Not only dots (or vertices), but also edges...

Let us now consider some graphs. Can you connect all the dots without ever removing the pen from the paper, **going through each edge once and only once**? As before, indicate the starting and final vertex, and also mark the direction of travel of each edge.

a) 	b) 
c) 	d) 

In which of these graphs have you been able to pass on each edge only once without ever removing the pen from the paper?

a)	b)	c)	d)

How many ways have you managed to do this? That is, is the solution unique or are there several? Give the answer for each graph.

a)	b)	c)	d)

If there are multiple solutions, what do they have different (or in common) with each other?

In graph theory, what is the official name of what you have drawn for each graph?

3) A few formal definitions

Given a graph *G = (V, E)*:

- a path is said to be **simple** if the vertices are all different from each other;
- a path is actually a **cycle** when the first and last vertex coincide and the path is composed of at least three edges;
- a path is called **Eulerian** when it runs through each edge of the graph only once;
- when an Eulerian path begins and ends in the same vertex, it is actually called **Eulerian cycle**;
- *G* is **semi-Eulerian** when it contains an Eulerian path;
- *G* is **Eulerian** when it contains an Eulerian cycle.

In Activities 1 and 2, you were asked to determine the existence of a path…

____

If, given a graph, you say that it admits this type of path, how can you prove and support your claim?

____

If you do not find an Eulerian path in a graph, can you say with certainty that such a path does not exist? Why?

____

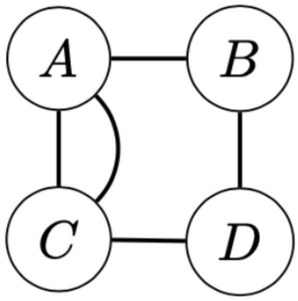
Can you establish a minimum number of attempts with which you can say that there is no Eulerian path in a graph? If so, how? If not, why?

____

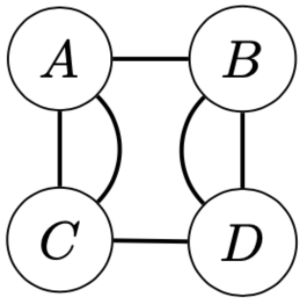
4) Let’s better analyse the graphs

We add to the collection of graphs previously examined – from a) to d) – the following three graphs:

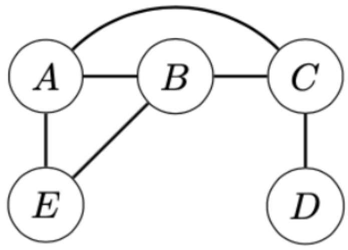
e)



f)



g)



For each of graphs a) to g), complete the following table.

Graph	Does it admit an Eulerian path?	Number of vertices	Number of edges	Degree of each vertex
a)				
b)				
c)				
d)				
e)				
f)				
g)				

Looking at the data collected in the table, what do you notice?

5) Towards a general method

Look carefully at the properties of graphs that admit an Eulerian path and those of graphs that do not. What regularities can you find? Try to distinguish between the starting and ending vertices, and the intermediate vertices (also called *intermediate vertices*).

You should have discovered that, if a certain property holds for a graph, then it admits (or does not admit) the existence of an Eulerian path. Try to write your theorem in a formal and rigorous way here below, and explain it by considering any graph.

If you have managed to answer the previous question, then most likely you have just stated one of Euler's two theorems, the one concerning Eulerian paths. Thus, you will always be able to determine when a graph admits an Eulerian path or not. Does the rule you have found apply to all the example graphs proposed above?

In which of the example graphs the Eulerian path you have found is actually an **Eulerian cycle**?

a)	b)	c)	d)	e)	f)	g)

Does the theorem you have stated allow us to even determine if a graph contains an Eulerian cycle? If so, why? If not, how could one start from that theorem to arrive at formulating a second theorem to express a property to identify graphs in which there is at least one Eulerian cycle? Also, check the statement of your first theorem.

Are you really sure that your theorems hold for any graph? Are there any further conditions that need to be imposed? Why?

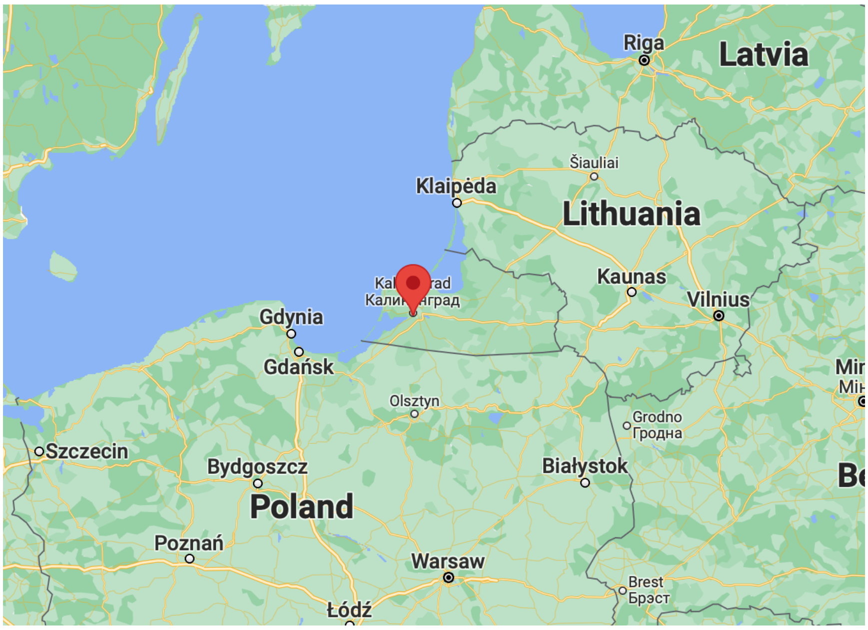
6) Examples invented by you

Here below, draw three graphs, different from all the previous ones, each with at least four vertices and five edges, such that:

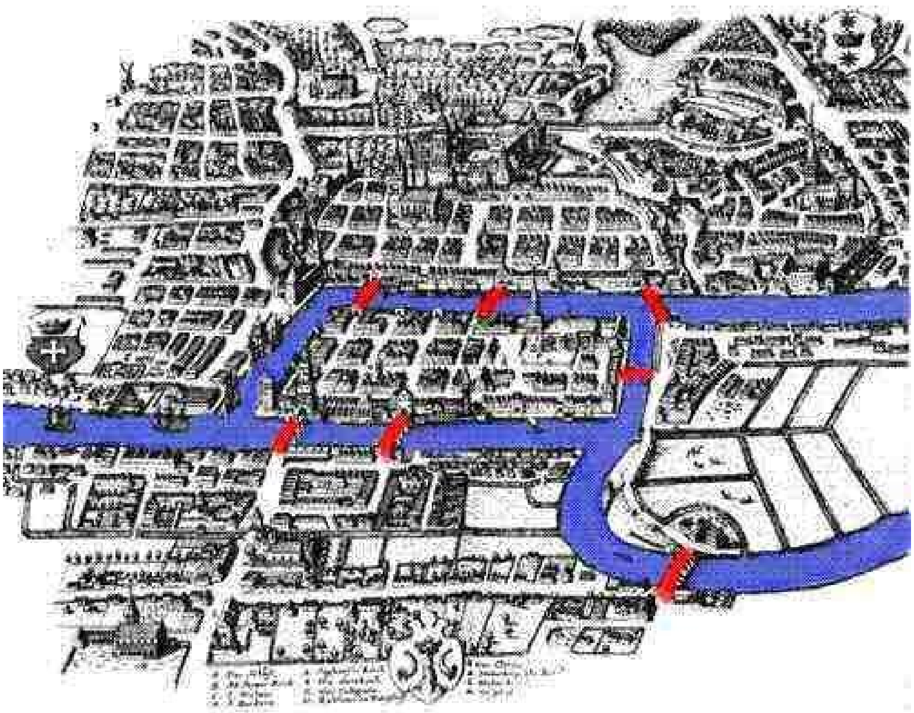
- the first admits an Eulerian path;
- the second admits an Eulerian cycle;
- the third does not admit an Eulerian path.

7) Another leap into the past: 1736

The Russian city *Kaliningrad*, located on the Baltic Sea between Poland and Lithuania, actually belonged to Prussia in the eighteenth century, and was known as **Königsberg**.



Königsberg is crossed by the Pregel River and some of its tributaries, which form two large islands. In the eighteenth century, these islands were connected to each other and to other areas of the city by seven bridges.



Adapted from "*The Liar Paradox and the Towers of Hanoi: The Ten Greatest Math Puzzles of All Time"* by Marcel Danesi:

"*The inhabitants of the city often wondered whether it was possible to take a walk starting from any point in the city, then cross each bridge once and only once, and return to the starting point. No one had ever succeeded in their intent, but on the other hand, no one was able to give an explanation of why this seemed impossible. Euler was fascinated by the question and turned it into one of the greatest puzzles of all time:*

> *In the city of Königsberg, is it possible to cross each of the seven bridges over the Pregel River, which connect two islands to each other and to the mainland, without crossing the same bridge twice?*

[...] He started by reducing the map of the area to a schematic form, known as a graph, and reformulating the riddle…"

With respect to the terminology you have learned so far, how do you think Euler reformulated the riddle in more formal mathematical terms?

____

____

____

____

____

Draw the graph corresponding to the map of the seven bridges of Königsberg:

Is it possible to take the walk of the seven bridges of Königsberg, as the inhabitants of the Prussian city intended? Motivate your answer by referring to the theorems discovered previously.

____

____

____

____

____

When, in 1736, Euler presented his solution to the problem of the Königsberg bridges to the Russian Academy and wrote the scientific article entitled "*Solutio problematis ad geometriam situs pertinentis*", the discipline of graph theory was officially born. Indeed, it was the first time that the term "*graph*" was ever used.

Also, Euler's result is also considered as one of the very first results of another branch of mathematics, more particularly of geometry: *topology*.
Topology studies the properties of figures and mathematical objects that do not change when a deformation is performed. Indeed, some geometric problems depend, more than on shape, only on the existing *connections* between objects. If you keep the set of edges unchanged (if you don't add or remove any), you can think of a graph as a topological object. The two Euler theorems you discovered do not depend on any kind of measurement. There are therefore some intersections between graph theory and topology.

8) Returning to the present…

Describe a real situation in which it could be very useful to know whether the corresponding graph admits (or does not) an Eulerian path or cycle.

____

____

____

____

____

____

____

____

____

____

____

____