Dantzig-Wolfe Decomposition

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1 Some recalls from Linear Programming (LP)

A linear programming problem is a problem to find the *minimum* or the *maximum* of a linear function under some linear equality or inequaltiy constraints. For example, a LP problem is the following:

Find two numbers $x_1 \geq 0$ and $x_2 \geq 0$ that maximize $2x_1 + 3x_2$ under the following constraints:

$$\begin{aligned} -2x_1 + 3x_2 &\leq 3 \\ 8x_1 - 3x_2 &\geq 8 \\ 3x_1 + 4x_2 &= 7. \end{aligned}$$

The x_i are the decision variables, they must satisfy the constraints.

As you can see, there can be three kinds of linear constraints. To use an algorithm to solve such kind of problems, it is convenient to have the same structure for all constraints. For this aim, you can transform all three of them to constraints of the same form. We want to transform them in less-or-equal constraints:

- The first of them is alredy in the desired form.
- Multiply the second of them by -1. You get $-8x + 3y \leq -8$, and also this constraint now has the desired form.
- The third form of constraints can be replaced by two inequality constraints: $3x + 4y \leq 7$ and $3x + 4y \geq 7$. The first of this new constraints is already in the desired form, and we already know by the previous point how to transform the second of this new constraints to the desired form.

This new but equivalent form is called the standard form. Thus, you can transform every LP into the standard form. So, the standard maximum problem is formulated as follows

$$\begin{aligned} \max \ & c^T x \\ \text{s.t. } & Ax \leq b \\ & x_i \geq 0 \quad \forall i = 1, 2, \ldots, n \end{aligned} \tag{1}$$

where $x, c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

If you transform your LP problem into the standard form, you can use the simplex method to solve it. This is one of the main reason why you want to transform a linear programming problem.

Lets now recall some Definitions and important Theorems of LP:

Definition 1. *The function to maximize is called the **objective function**.*

Definition 2. *The second kind of constraints are called the **nonnegativity constraints**.*

Definition 3. *A vector x of variables is said to be **feasible** if it satisfies the constraints.*

Definition 4. *The set of feasible vectors is called the **constraint set**. The constraint set is a convex set since it is the intersection of halfplanes defined by A.*

Definition 5. *A linear programming problem is said to be **feasible** if the constraint set is not empty, otherwise it is said to be unfeasible.*

Definition 6. *The constraint set can be **bounded** or **unbounded**.*

Definition 7. *A feasible vector x is called an **optimal vector** if the objective function achieves at x the maximum. The optimal vector is not necessary unique, and it is always a vertex or extreme ray of the constraint set.*

Duality

To every linear program (1) there is a dual linear program with which it is connected. The dual problem, with A, b and c from (1), is:

$$\begin{aligned} \min \ & y^T b \\ \text{s.t.} \ & y^T A \geq c^T \\ & y_j \geq 0 \quad \forall j = 1, 2, \ldots, m \end{aligned} \tag{2}$$

So, in the dual problem (2) every variable y_j is associated to a constraint from the primal problem (1). In this forumulation (2) is also called the standard minimum problem. Let's now recall some useful properties of the dual.

Properties

Theorem 1. *If x is feasible for the standard maximum problem (1) and if y is feasible for its dual (2), then*

$$c^T x \leq y^T b \tag{3}$$

Proof.

$$c^T x \overset{(1)}{\leq} y^T A x \overset{(2)}{\leq} y^T b$$

where the first inequality follows from to the linear constraints of (1) and the second inequality follows from the linear constraints from (2). □

This Theorem is useful since you can deduce upper or lower bounds for your problem. The following Theorem gives you information about boundedness of the constraint set.

Theorem 2. *If a standard problem and its dual are both feasible, then both are bounded feasible.*

Proof. If y is feasible for the minimum problem, then (3) shows that $y^T b$ is an upper bound for the values of $c^T x$ for x feasible for the maximum problem. On the other hand, if x is feasible for the maximum problem, then (3) shows that $c^T x$ is a lower bound for the values of $y^T b$ for y feasible for the minimum problem. □

Theorem 3. *If there exists feasible $x^\star$ and $y^\star$ for a standard maximum problem* (1) *and its dual* (2) *such that*

$$c^T x^\star = y^{\star T} b, \tag{4}$$

then both are optimal for their respective problems.

Proof. If x is any feasible vector for (1), then $c^T x \leq y^{\star T} b = c^T x^\star$ which shows that $x^\star$ is optimal.

If y is any feasible vector for (2), then $y^{\star T} b = c^T x^\star \leq y^T b$ which shows that $y^\star$ is optimal. □

Theorem 4. ***(Duality Theorem)*** *If a standard linear programming problem is bounded feasible, then so is its dual, their values are equal, and there exists optimal vectors for both problems.*

The proof of this very important Theorem is longer than the proofs of the previous results and is done via the Simplex Algorithm, which is a method to solve linear programming problems.

Theorem 5. ***(Equilibrium Theorem)*** *Let $x^\star$ and $y^\star$ be feasible vectors for a standard maximum problem* (1) *and its dual* (2) *respectively. Then $x^\star$ and $y^\star$ are optimal, if and only if,*

$$y_j^\star = 0 \quad \forall j \textit{ for which } \quad \sum_{i=1}^{n} a_{ij} x_i^\star < b_j \tag{5}$$

$$x_i^\star = 0 \quad \forall i \textit{ for which } \quad \sum_{j=1}^{m} y_j^\star a_{ij} > c_i \tag{6}$$

Proof. Since $x^\star$ is a feasible vector, Equation (5) implies that if $y_j^\star \neq 0$ we have that $\sum_{i=1}^n a_{ij}x_i^\star = b_j$ thus,

$$\sum_{j=1}^m y_j^\star b_j = \sum_{j=1}^m y_j^\star \sum_{i=1}^n a_{ij}x_i^\star = \sum_{j=1}^m \sum_{i=1}^n y_j^\star a_{ij}x_i^\star.$$

Similarly, equation (6) gives

$$\sum_{j=1}^m \sum_{i=1}^n y_j^\star a_{ij}x_i^\star = \sum_{i=1}^n c_i x_i^\star.$$

Putting together this two equations, it follows by (4) that $x^\star$ and $y^\star$ are optimal solutions.

On the other hand, by (3) we have that

$$\sum_{i=1}^n c_i x_i \leq \sum_{j=1}^m \sum_{i=1}^n y_j a_{ij} x_i \leq \sum_{j=1}^m y_j b_j \tag{7}$$

and since $y^\star$ and $x^\star$ are optimal, we have equalities in the previous equation. Therefore,

$$\sum_{i=1}^n \left(c_i - \sum_{j=1}^m y_j^\star a_{ij} \right) x_i^\star = 0$$

Since $y^\star$ and $x^\star$ is a feasible solution, the term in the sum is non negative, thus the sum can be zero only if each term is zero. Therefore it follows (6). (5) can be obtained in the same way using the other inequality in (7). □

Since in the dual problem every variable is associated to a constraint, this Theorem tells us that if the optimal solution is not on the ith constraint, in the dual problem the variable y_i is zero.

2 Dantzig-Wolfe Decomposition

2.1 Introduction

Suppose you have a standard minimization LP problem of the following particular form:

$$
\begin{array}{lccccccccl}
\min & c_1^T x_1 & + & c_2^T x_2 & + & \dots & + & c_m^T x_m & & \\
\text{s.t.} & A_{01}x_1 & + & A_{02}x_2 & + & \dots & + & A_{0m}x_m & = & b_0 \\
& A_{11}x_1 & & & & & & & \leq & b_1 \\
& & & A_{22}x_2 & & & & & \leq & b_2 \\
& & & & & \dots & & & & \\
& & & & & & & A_{mm}x_m & \leq & b_m \\
& & & & & & & x_j & \geq & 0 \qquad \forall j = 1, 2, \dots m
\end{array} \tag{8}
$$

where $x_i, c_i \in \mathbb{R}^{n_i}, A_{ij} \in \mathbb{R}^{k_i \times n_j}, b_i \in \mathbb{R}^{k_i}$. Thus x_i and c_i are vectors, A_{ij} matrices of dimension $k_i \times n_j$. The first k_0 constraints, which depends on the matrices A_{0j}, are called the *coupling constraints*. The other remaining m systems of k_i constraints are indipendent of each other since they depend on different sets of variables. So we have $k_0 + k_1 + \dots + k_m = K$ constraints and $n_1 + n_2 + \dots + n_m$ variables.

Now we want take advantage of the particular form of the problem to solve it more easily and faster.

2.2 Idea

The main observation is that for every c a small problem like the following

$$
\begin{aligned}
\min \ & c^T x_i \\
\text{s.t.} \ & A_{ii}x_i \leq b_i^T \\
& x_i \geq 0
\end{aligned}
$$

can be easily solved by the Simplex Algorithm. We define now the constraint set of this problem as

$$Q_i = \{x_i \in \mathbb{R}^{n_i} : A_{ii}x_i \leq b_i, x_i \geq 0\}.$$

Since the constraint set is always convex (it is an intersection of half-planes) it can be described by its vertices v_{ij} if it is bounded or by its vertices v_{ij} and extreme rays w_{il} if it is unbounded:

Theorem 6. ***(Minkowski representation Theorem)*** *Every polyhedron $P \in \mathbb{R}^n$ can be represented in the form*

$$P = \left\{ x \in \mathbb{R}^n : x = \sum_{j=1}^{N} \lambda_j v_j + \sum_{l=1}^{L} \mu_l w_l, \quad \sum_{j=1}^{N} \lambda_j = 1, \quad \lambda_j, \mu_l \in \mathbb{R}^+ \right\}$$

where

$$\{v_j, j = 1, \ldots, N\} \text{ are the extreme points of } P$$
$$\{w_l, l = 1, \ldots, L\} \text{ are the extreme rays of } P$$

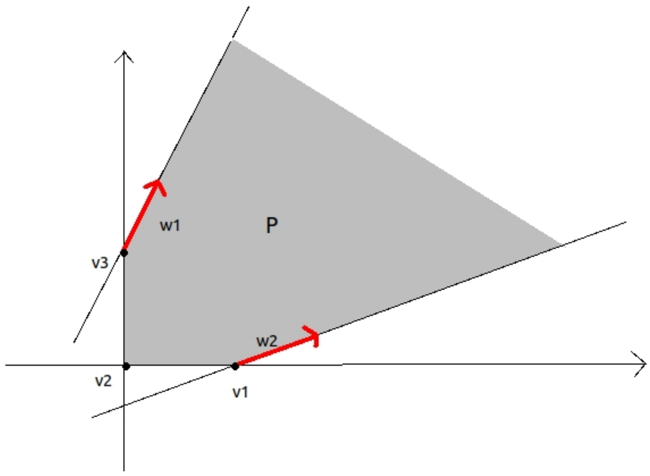


Figure 1: Graphical example of Minkowskis representation Theorem

So, every $x_i \in Q_i$ can be written as

$$x_i = \sum_{j=1}^{N_i} \lambda_{ij} v_{ij} + \sum_{l=1}^{L_i} \mu_{il} w_{il} \tag{9}$$

Recall that if Q_i is bounded, the μ_{il} are 0 and you have a simpler notation. N_i is the number of extreme points (or equivaltently, the number of vertices) of Q_i and L_i is the number of extreme rays of Q_i. Remember also that n_i is the dimension of Q_i.

If we insert now (9) into (8), we obtain a different but equivalent LP problem, called the *master problem*,

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{m} \left(\sum_{j=1}^{N_i} \lambda_{ij}(c_i^T v_{ij}) + \sum_{l=1}^{L_i} \mu_{il}(c_i^T w_{il}) \right) \\
\text{s.t.} \quad & \sum_{i=1}^{m} \left(\sum_{j=1}^{N_i} \lambda_{ij}(A_{0i} v_{ij}) + \sum_{l=1}^{L_i} \mu_{il}(A_{0i} w_{il}) \right) = b_0 \\
& \sum_{j=1}^{N_i} \lambda_{ij} = 1 \quad \forall i = 1, \ldots, m \\
& \lambda_{ij}, \mu_{ij} \geq 0 \quad \forall i, j
\end{aligned}
\tag{10}
$$

where the second kind of constraints follow from the fact that if x_i satisfies $A_{ii}x_i \leq b_i$, which means that $x_i \in Q_i$, then by the Minkowski representation Theorem, x_i is convex combination of the λ_{ij} (therefore this constraints) plus a combination of the μ_{il} (no particular constraints for this variables).

We can observe the following things:

- This problem (10) is equivalent to the original problem (8).
- We have now other decision variables: The new ones are the weights of the extreme points (λ_{ij}), and the weights of the extreme rays (μ_{ij}).
- The number of decision variables is now huge, much more than in the original formulation. To deal with this new big amount of variables, to solve the problem we will use the *Revised Simplex Method*, which ignores a great part of them. The next sections explain this method and the final algorithm to solve problem (10).
- But in contrast, the number of constraints is smaller: The original problem has $\sum_{i=0}^{m} k_i$ constraints, while the new formulation has only $k_0 + m$ constraints, since the block constraints $A_{ii}x_i \leq b_i$ of k_i constraints are now replaced by the single constraint $\sum_{j=1}^{N_i} \lambda_{ij} = 1$.

2.3 Revised Simplex Algorithm

A disadvantage of the Simplex Algorithm is that if you have a lot of variables and constraints, at every step you have to update the entire tableau. But to improve your solution with the simplex method you need only one negative reduced cost. The revised simplex method eliminates this disadvantage.

Consider the standard minimum problem (2), and introduce slack variables to obtain the following form:

$$\begin{aligned} \min\ & c^T x \\ \text{s.t. } & Ax = b \\ & x_j \geq 0 \quad \forall j \end{aligned} \tag{11}$$

Observe that (10) is in this form.

We obtain a matrix $A \in \mathbb{R}^{n\times m}$ with $m \gg n$. So we can write $A = (B, R)$[1] where B is a basis of A, i.e. a square and invertible matrix. We write also $x = (x_B, x_R)$. Now, since $Ax = b$, we have

$$\begin{aligned} Bx_B + Rx_R &= b \\ Bx_B &= b - Rx_R \\ x_B &= B^{-1}b - B^{-1}Rx_R \end{aligned}$$

Thus we have expressed x_B in function of x_R. To deal with less variables, we let all $x_i \in x_R$ be 0. So our initial solution, a basic solution, is

$$x_B = B^{-1}b,$$

which of course has to be greater than zero to be a feasible solution.

Now we split also the cost vector c in (c_B, c_R), and we want to minimize $c^T x$. Using the two previous results, it follows that

$$\begin{aligned} c^T x &= c_B x_B + c_R x_R \\ &= c_B(B^{-1}b - B^{-1}Rx_R) + c_R x_R \\ &= c_B B^{-1}b + (c_R - c_B B^{-1}R)x_R \end{aligned}$$

So we get an optimality condition: since we are minimizing and all decision variables x_i have to be greater than 0, we have an optimal solution if the vector $c_R - c_B B^{-1}R$ is positive since we set all non basis variables $x_R = 0$. If there is a negative entry we have to update the basis[2]. The coefficient of x_j in $(c_R - c_B B^{-1}R)x_R$ is $d_j := c_j - c_B B^{-1}a_j$ where a_j is the j^{th} column of matrix A. d_j is called the reduced cost of variable x_j.

Therefore, problem (11) is solved by applying the simplex method iteratively to the much smaller problem

[1] B doesn't have to be the first columns of A, but any set of columns can be taken, but then c, b and x need to be reordered.

[2] For the scope of this paper it is not important how to update the basis in general.

$$
\begin{aligned}
\min \ & c_B^T x_B \\
\text{s.t. } & Bx_B = b_B \\
& x_B \geq 0
\end{aligned}
\tag{12}
$$

where you have to update the basis at every step to find the optimal basis. To check that $(x_B, 0)$ is optimal for (11), you have to check that $d_j > 0 \ \forall j$. To simplify the calculations of d_j, let $y^T := c_B^T B^{-1}$. You can see that y are the dual variables of (12), thus if we solve (12) with the simplex method, it gives us as output x_B, but also y. So

$$
d_j = c_j - y^T a_j \tag{13}
$$

2.4 Algorithm

Now we have all ingredients to solve (10). The revised simplex method turns out to be useful for this aim since we have much more variables than constraints.

To start we need a basis. There is no particular rule which basis (or equivalently, which vertices or extreme rays) to take. The only constraint is that the inital basic solution you choose has to be feasible, take care about this! This beginning problem with a reduced number of variables is called the *restricted master problem* which is solved with the simplex method, where we denote the optimal vector by $\widehat{x}$.

To check if the output from the restricted master problem is also optimal for the entire problem (if you are lucky and choose directly the optimal basis), you have to calculate the reduced cost $\overline{\lambda_{ij}}$ of variable λ_{ij} and the reduced cost $\overline{\mu_{il}}$ of variable μ_{il} via the formula given by (13) and you have to check if they are non negative. By solving the restricted master problem we get from the simplex method not only $\widehat{x}$, but also the vector y, the dual variables form the coupling constraints and the vector z, the dual variables from the simple constraints. In the notation of (10), (13) becomes

$$
\begin{aligned}
\overline{\lambda_{ij}} &= c_i^T v_{ij} - ((A_{0i} v_{ij})^T y + e_i^T z) \\
&= c_i^T v_{ij} - (A_{0i} v_{ij})^T y - z_i \\
&= (c_i - A_{0i}^T y)^T v_{ij} - z_i
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
\overline{\mu_{il}} &= c_i^T w_{il} - ((A_{0i} w_{il})^T) \\
&= c_i^T w_{il} - (A_{0i} w_{il})^T y \\
&= (c_i - A_{0i}^T y)^T w_{il}
\end{aligned}
\tag{15}
$$

But calculating every reduced cost would be an enormous work since we have a lot of variables. So instead of calculating for every variable the reduced cost, we solve the following m subproblems:

$$\begin{aligned} \min \ & (c_i - A_{0i}^T y)^T x \\ \text{s.t.} \ & A_{ii} = b_i \\ & x \geq 0 \end{aligned} \tag{16}$$

It can be done in this way since the reduced costs are equal or similar to the form $(c_i - A_{0i}^T y)^T x$, where x is or v_{ij} or w_{il}. Since the simplex method gives as output of this i^{th} subproblem a vertex or extreme ray of Q_i, we can find for every subproblem the smallest reduced cost.

So, the optimal solution of this i^{th} subproblem is or a vertex v_{ik}, if the objective function value is bounded, or an extreme ray w_{ik}, if the objective function value is $-\infty$, of the set Q_i.

Therefore, we have three possibilities:

1. The objective function value is $-\infty$:
   - Simplex output: an extreme ray w_{ik} with $(c_i - A_{0i}^T y)^T w_{ik} < 0$
   - Conclusion: the reduced cost $\overline{\mu_{ik}}$ of μ_{ik} is negative.
   - Action: introduce the variable μ_{ik} in the restricted master problem.
2. The objective function value is finite, but less then z_i:
   - Simplex output: extreme point v_{ik} with $(c_i - A_{0i}^T y)^T v_{ij} < z_i$
   - Conclusion: the reduced cost $\overline{\lambda_{ik}}$ of λ_{ik} is negative.
   - Action: introduce the variable λ_{ik} in the restricted master problem.
3. The objective function value is finite and bigger or equal then z_i:
   - Conclusion: $(c_i - A_{0i}^T y)^T v_{ij} \geq z_i$ for all extreme points v_{ik} and $(c_i - A_{0i}^T y)^T w_{ik} \geq 0$ for all extreme rays w_{ik}
   - Action: Terminate, you have found an optimal basis since all reduced costs are non negative.

Summary

0. Find an initial basic feasible solution (take care about this, see chapter 2.3) and create the initial restricted master problem.

1. Solve the restricted master problem (with the simplex method), and store the dual variables y and z.

2. Solve the m subproblems. If the optimal cost of subproblem i is bigger or equal than z_i (case 3), terminate with optimal solution (9) where the λ_{ij} and μ_{ij} are the solutions of the restricted master problem.

3. If subproblem i is unbounded, add variable μ_{il} to the restricted master problem.

4. If subproblem i has bounded optimal cost less than z_i, add variable λ_{ij} to the restricted master problem.

5. Generate a column associated with the entering variable and continue with step one.

2.5 Bounds

Let us think about upper and lower bounds for this method. Let $s^\star$ be the best value for the objective function, i.e. $s^\star = c^T x^\star$ where $x^\star$ is the optimal vector of problem (8).

Upper bound It is easy to see, that the solution given by the restricted master problem is an upper bound of the master problem, since it is a feasible solution of the problem.

Lower bound To deduce a lower bound, we use the dual maximization problem, i.e. (3). Now, the dual problem of the master problem (10) would be

$$\begin{aligned} \max\ & y^T b_0 + z_1 + z_2 + \cdots + z_m \\ \text{s.t.}\ & y^T A_{0i} v_{ij} + z_i \le c_i^T v_{ij} \quad \forall i, \forall j \in N_i \\ & y^T A_{0i} w_{il} \le c_i^T w_{il} \qquad \forall i, \forall l \in L_i. \end{aligned} \tag{17}$$

Let s_i the optimal objective function value for every subproblem i and $\overline{y}, \overline{z}$ be the optimal dual vectors for the restricted master problem. We have that, if s_i is bounded, by (16), that

$$\begin{aligned} s_i &\le c_i^T v_{ij} - \overline{y}^T A_{0i} v_{ij} \quad \forall j \in N_i \\ 0 &\le c_i^T w_{il} - \overline{y}^T A_{0i} w_{il} \quad \forall l \in L_i \end{aligned}$$

It follows that $\overline{y}$ and the s_i are feasible solutions of (17), but not necessary optimal solutions. Therefore we have

$$s^\star \ge \overline{y}^T b_0 + s_1 + s_2 + \cdots + s_m$$

Let now s be the optimal objective function value for the restricted master problem. We use (4) again to deduce that $s = \overline{y}^T b_0 + \overline{z_1} + \cdots + \overline{z_m}$. Plug this result in the above equation and you obtain a lower bound:

$$s^\star \ge s + (s_1 - \overline{z_1}) + (s_2 - \overline{z_2}) + \cdots + (s_m - \overline{z_m}) \tag{18}$$

So, we have that

$$s + (s_1 - \overline{z_1}) + (s_2 - \overline{z_2}) + \cdots + (s_m - \overline{z_m}) \le s^\star \le s$$

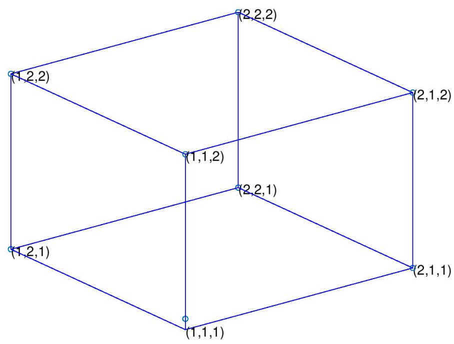
2.6 Examples

Now let us do some small numerical examples to understand the procedure better.

Example 1

$$\min \quad -4x_1 - x_2 - 6x_3 \tag{19}$$
$$\text{s.t. } 3x_1 + 2x_2 + 4x_3 = 17 \tag{20}$$
$$1 \leq x_1 \leq 2 \tag{21}$$
$$1 \leq x_2 \leq 2 \tag{22}$$
$$1 \leq x_3 \leq 2 \tag{23}$$

The last constraints (21) - (23) tell us that the constraint set is bounded, so all weights μ_{ij} of the extreme rays w_{ij} are zero. Now we have to choose which constraint is the coupling constraint and which are the other constraints, i.e. the matrices A_{0i} and A_{ii}. We decide to let $m = 1$, $A_{01} = [3 \;\; 2 \;\; 4]$, $b_0 = 17$, $c^T = [-4 \;\; -1 \;\; -6]$ and the set Q_1 be defined by constraint (21), (22), (23). So Q_i is a cube with 8 vertices. As explained before, we don't want to introduce all 8 vertices as new variables, so we start by choosing an admissible basis. We pick the vertices $v_1^T = (2, 2, 2)$ and $v_2^T = (1, 1, 2)$, which are feasible vertices (see chapter 2.3 to determine how many vertices you need and to verify if these two vertices are feasible).



Now we can transform our original problem into the restricted master problem:

$$\begin{aligned}
\min\ & (c_1^T v_1)\lambda_1 + (c_1^T v_2)\lambda_2 \\
\text{s.t.}\ & (A_{01}v_1)\lambda_1 + (A_{01}v_2)\lambda_2 = 17 \\
& \lambda_1 + \lambda_2 = 1 \\
& \lambda_1, \lambda_2 \geq 0
\end{aligned}$$

which becomes

$$\begin{aligned}
\min\ & -22\lambda_1 - 17\lambda_2 \\
\text{s.t.}\ & 18\lambda_1 + 13\lambda_2 = 17 \\
& \lambda_1 + \lambda_2 = 1 \\
& \lambda_1, \lambda_2 \geq 0
\end{aligned}$$

We solve this problem now with the simplex method, and we get as optimal vector $\lambda = (\lambda_1, \lambda_2) = (0.8, 0.2)$, and optimal dual vector $t = (y, z) = (-1, -4)$. (You can check this result, since by (4), $-21 = c^T\lambda = tb^T = -21$.)

Now we have to solve the subproblems. In this case, since $m = 1$, we have only one subproblem with coefficients given by $c^T - y^T A_{01}$:

$$[-4\ \ -1\ \ -6] - (-1)[3\ \ 2\ \ 4] = [-1\ \ 1\ \ -2].$$

So the subproblem is

$$\begin{aligned}
\min\ & -x_1 + x_2 - 2x_3 \\
& 1 \leq x_1 \leq 2 \\
& 1 \leq x_2 \leq 2 \\
& 1 \leq x_3 \leq 2
\end{aligned}$$

The optimal vector of this subproblem is $v_3^T = (2, 1, 2)$ with objective function value $s_1 = -5$ which is smaller than $z = -4$. Thus we introduce v_3 in the basis. Now we update the restricted master problem, inserting variable λ_3 which correspondents to vertex v_3. The restricted master problem becomes

$$\begin{aligned}
\min\ & -22\lambda_1 - 17\lambda_2 - 21\lambda_3 \\
\text{s.t.}\ & 18\lambda_1 + 13\lambda_2 + 16\lambda_3 = 17 \\
& \lambda_1 + \lambda_2 + \lambda_3 = 1 \\
& \lambda_1, \lambda_2, \lambda_3 \geq 0
\end{aligned}$$

The optimal vector is $\lambda_1 = 0.5$, $\lambda_3 = 0.5$ with dual $y = -0.5$ and $z = -13$. Now again we have to solve the new subproblem:

$$[-4 \ \ -1 \ \ -6] - (-0.5)[3 \ \ 2 \ \ 4] = [-2.5 \ \ 0 \ \ -4].$$

$$\begin{aligned} \min \ \ & -2.5x_1 - 4x_3 \\ & 1 \le x_1 \le 2 \\ & 1 \le x_2 \le 2 \\ & 1 \le x_3 \le 2 \end{aligned}$$

The optimal vector of it is $v_4 = (2, 2, 2)$ with objective function value $s_1 = -13$, so we stop here since $z = -13$, and the optimal vector of example 1 is

$$x = 0.5v_1 + 0.5v_3 = [2 \ \ 1.5 \ \ 2]^T$$

with optimal objective function value $s^\star = -21.5$. Note, that at the first iteration, if we look at the bounds,

$$-22 = -21 + (-5 - (-4)) = s + (s_1 - \overline{z_1}) \le s^\star \le s = -21$$

Let's do now another example, with an unbounded constraint set.

Example 2

$$\begin{aligned} \min \ \ & -5x_1 + x_2 \\ \text{s.t.} \ \ & x_1 \le 8 \\ & x_1 - x_2 \le 4 \\ & 2x_1 - x_2 \le 10 \\ & x_1, x_2 \le 0 \end{aligned}$$

This problem is not in the form to solve it, i.e. we have no equality constraint which we can use as the coupling constraint. Therefore we introduce a slack variable x_3 to overcome this obstacle:

$$\begin{aligned} \min \ \ & -5x_1 + x_2 && (24) \\ \text{s.t.} \ \ & x_1 + x_3 = 8 && (25) \\ & x_1 - x_2 \le 4 && (26) \\ & 2x_1 - x_2 \le 10 && (27) \\ & x_1, x_2, x_3 \le 0 && (28) \end{aligned}$$

As you can see, we have now three variables. Moreover, you can divide the constraints in two pieces with independent variables: The first part would be constraint (26), (27) and the nonnegativity constraints $x_1, x_2 \geq 0$. The second part would be only the constraint $x_3 \geq 0$. So we have two sets of independent variables, Q_1 and Q_2. By analyising these two sets, you can figure out that Q_1 has three extreme points, $v_{11} = (6, 2), v_{12} = (4, 0)$ and $v_{13} = (0, 0)$, and two extreme rays, $w_{11} = (1, 2)$ and $w_{12} = (0, 1)$. Q_2 has simply one extreme ray $w_{21} = 1$. So in total, if we would transform the problem with these new variables, we would have six variables and two constraints.

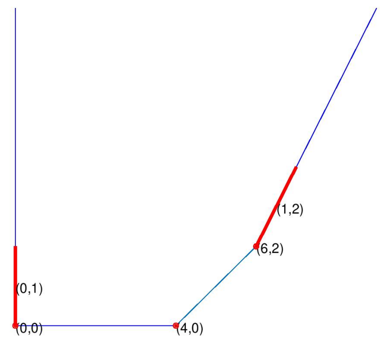


Figure 2: Set Q_1, as you can see it is unbounded.

Now we have to take an initial basis. We decide to take the extreme point v_{11} and the extreme ray w_{21}. We transform the problem into the restricted master problem where you calculate the coeficients in the same way as in the last example:

$$
\begin{aligned}
\min \quad & -28\lambda_{11} \\
\text{s.t.} \quad & 6\lambda_{11} + \mu_{21} = 8 \\
& \lambda_{11} = 1 \\
& \lambda_{11}, \mu_{21} \geq 0
\end{aligned}
$$

If we solve this problem with the simplex method, we obtain the optimal vector with $\lambda_{11} = 1, \mu_{21} = 2$ and with the optimal duals $y = 0$ and $z = -28$.

Again, now we have to calculate the coefficients for the subproblems:

For subproblem 1, we have $c_1^T - y^T A_{01} = [-5 \;\; 1] - (0)[1 \;\; 0] = [-5 \;\; 1]$, and for subproblem 2 we have $c_2^T - y^T A_{02} = 0 - (0)(1) = 0$. Therefore, we have only to solve subproblem 1 because subproblem 2 is trivial and gives a non negative reduced cost. Therefore, for subproblem 2 we know that we don't have to introduce a variable (This can also be obtained in a different way: Since we used all variables we have for representing Q_2, we can't improve the solution because we can not substract something since we don't have other variables.)

Thus, subproblem 1 is:

$$\begin{aligned} \min \ & -5x_1 + x_2 \\ & x_1 - x_2 \leq 4 \\ & 2x_1 - x_2 \leq 10 \\ & x_1, x_2 \leq 0 \end{aligned}$$

If we solve it with the simplex algorithm, we obtain as optimal vector the extreme ray $w_{11} = (1, 2)$ with objective function value $-\infty$. So we add variable μ_{11} to the restricted master problem, and we obtain:

$$\begin{aligned} \min \ & -28\lambda_{11} - 3\mu_{11} \\ \text{s.t. } & 6\lambda_{11} + \mu_{21} + \mu_{11} = 8 \\ & \lambda_{11} = 1 \\ & \lambda_{11}, \mu_{21}, \mu_{11} \geq 0 \end{aligned}$$

The simplex algorithm gives us the optimal vector with $\lambda_{11} = 1, \mu_{11} = 2$ and $\mu_{21} = 0$ with optimal duals $y = -3, z = -10$.

Again, to check optimality we have to check the reduced costs and therefore we have to solve subproblem 1, i.e.

$$\begin{aligned} \min \ & -2x_1 + x_2 \\ & x_1 - x_2 \leq 4 \\ & 2x_1 - x_2 \leq 10 \\ & x_1, x_2 \leq 0. \end{aligned}$$

The value objective function of the optimal vector $(8, 6)$ is $s_1 = -10$, which is (greater or) equal than z, thus the optimal solution of example 2 is

$$x = v_{11} + 2w_{11} = [8 \ \ 6]^T.$$

2.7 Dantzig-Wolfe in ILP

Consider the integer linear programming problem

$$\begin{aligned} s^\star =& \min\{c^T x : x \in X\} \\ X =& Y \cap Z \\ Y =& \{x : Dx \geq d\} \\ Z =& \{x : Bx \geq b\} \cap \mathbb{Z}^n_+ \end{aligned} \tag{29}$$

The idea to solve this with the Dantzig-Wolfe method is, to use the Minkowski representation Theorem for $conv(Z)$, i.e for the convex hull of Z.

If we reformulate it with the new variables, we get

$$\begin{aligned} s^\star = \min_{\lambda \geq 0} & \sum_{j=1}^{J} (c^T v_j)\lambda_j \\ \text{s.t. } & \sum_{j=1}^{J} (Dv_j)\lambda_j \geq d \\ & \sum_{j=1}^{J} \lambda_j = 1 \\ & \sum_{j=1}^{J} v_j \lambda_j \in \mathbb{Z}^n \end{aligned}$$

where the v_j are the vertices of $conv(Z)$ and J is the number of vertices of $conv(Z)$. This is called the *master problem.*

As before, to solve it we define the restricted master problem to deal with less variables at once. Thus, the restricted master problem is

$$\begin{aligned} s^{DWrestricted} = \min_{\lambda \geq 0} & \sum_{j=1}^{\overline{J}} (c^T v_j)\lambda_j \\ \text{s.t. } & \sum_{j=1}^{\overline{J}} (Dv_j)\lambda_j \geq d \\ & \sum_{j=1}^{\overline{J}} \lambda_j = 1 \end{aligned}$$

where $\overline{J} \subset J$. Let y be the dual variable of the first constraint and z be the dual variable of the second constraint of the restricted master problem.

As in the Dantzig-Wolfe algorithm in LP, to check optimality we need the reduced cost for every variable λ_j, and it is again $c^T v_j - y^T D v_j - t$. So, if we use the same trick to avoid to calculate all reduced costs,

$$\min_{Bx \geq b, x \in Z_+^n} (c^T - y^T D)x \tag{30}$$

has to be an easy ILP problem which we can solve quickly. Moreover, as in the LP case, we can deduce the same bounds, i.e. the solution of the restricted master problem is an upper bound for the master problem and $y^T d + s$ a lower bound.

Thus, to solve (29) we do the following steps:

1. Inizialize the upper bound $UB = +\infty$ and the lower bound $LB = -\infty$.
2. Find an initial basis v_1 and v_2 and Let $\overline{J} = \{v_1, v_2\}$.
3. Compute the restricted master problem with $\overline{J}$

    a. Solve the restricted master problem. Store its solution, the optimal dual vector (y, z) and update UB.

    b. Solve the subproblem $s = \min\{(c^T - y^T D)x : x \in Z\}$ and let v_k be the optimal vector. If $z = s$, stop the algorithm with optimal solution given by the restricted master problem, else add v_k to $\overline{J}$.

    c. Update the lower bound: $LB = \max\{LB, y^T d + s\}$. If $LB = UB$, stop the algorithm with optimal solution given by the restricted master problem.

    d. Repeat step 3.

But, anyway, if (30) is not easy to solve, this method is useless. In fact, the Dantzig-Wolfe decomposition is not that much used for ILP problems, because you don't know a priori if (30) is easy to solve or not.

3 AMPL code

3.1 The mod file

```
# DANTZIG-WOLFE DECOMPOSITION
# Bounded Problem

param N; #total number of variables DATA
param K; #total number of constraints DATA
param m; #number of systems         DATA

param ni{1..m}; #subdivision of variables, nj[j] = dimension;
         DATA
param ki{0..m}; #subdivision of constraints in sets of ki[i]
   constraints; DATA

#only two checks if parameters are set up in the right way
check: sum{j in 1..m} ni[j] = N;
check: sum{i in 0..m} ki[i] = K;



param b{i in 0..m,1..ki[i]}; #all constraint values,
   subdivised   DATA
param c{i in 1..m,1..ni[i]}; #all cost values, subdivised
          DATA

set index = {0..m,1..m}; #index (i,j) of matrices A[i,j]
set iindex within index; #specify what matrices DATA

param A{(i,j) in iindex, 1..ki[i], 1..ni[j]};#Matrices A[i,j]
   DATA

#cost of subproblem m
param cost_subproblem{1..m,1..ni[m]}; #    RUN
#to calculate it, you need the following:
#dual variable y (of coupling constraints)
param dual_cost_y{1..ki[0]}; #            RUN
#dual variabls z (of simple constraints)
param dual_cost_z{1..m}; #                RUN
```

```
var x{p in 1..m, 1..ni[p]} >= 0;


# THE m SUBPROBLEMS:

minimize Reduced_Cost{p in 1..m}:
   sum{i in 1..ni[p]} cost_subproblem[p,i]*x[p,i];


subject to Set_constraint{p in 1..m,j in 1..ki[p]}:
   sum{i in 1..ni[p]} A[p,p,j,i]*x[p,i] <= b[p,j];



# RESTRICTED MASTER PROBLEM




#number of vertices of set Q(i) (subproblem i),
param numberOfvertices{1..m} integer >= 0; #                RUN
#changes during execution,


#vertices values to corresponding lambdas,
#introduced with the RUN file
param vertices{p in 1..m, 1..numberOfvertices[p], 1..ni[p]}; #
    RUN

param numberOfInitialVertices{p in 1..m}; #DATA
param initial_vertices{p in 1..m, 1..numberOfvertices[p],
    1..ni[p]}; #DATA





#c_i*v_ik
param restricted_master_cost{p in 1..m,
    1..numberOfvertices[p]};# RUN
```

```
param restricted_couple_cost{p in 1..m,
   1..numberOfvertices[p], 1..ki[0]};#RUN


var lambda{p in 1..m, 1..numberOfvertices[p]} >= 0;


minimize Total_Restricted_Cost:
   sum{p in 1..m, k in 1..numberOfvertices[p]}
      restricted_master_cost[p,k]*lambda[p,k];

subject to Restricted_Coupled {j in 1..ki[0]}:
   sum{p in 1..m, k in 1..numberOfvertices[p]}
      restricted_couple_cost[p,k,j]*lambda[p,k] = b[0,j];


subject to Convex{p in 1..m}:
   sum{k in 1..numberOfvertices[p]} lambda[p,k] = 1;


#to check if the initial values are admissible




#
#var y{1..m,1..ki[0]};

#nothing to maximize, only to check if B^-1*b>=0
#
#maximize nothing: 0;

#
#subject to couple{i in 1..ki[0]}:
#  sum{p in 1..m, j in 1..numberOfvertices[p]}
#     restricted_couple_cost[p,j,i]*y[p,j] = b[0,i];

#subject to convex2{p in 1..m}:
#  sum{i in 1..numberOfvertices[p]} y[p,i] = 1;
```

3.2 The run file

```
# DANTZIG-WOLFE DECOMPOSITION
# on bounded sets


model DWgeneral.mod;
data DWgeneral.dat;

param nIter default 0;
param success default 0;
param mincost;
param initialChekparameter default 0;
#option omit_zero_rows 1;
#option display_1col 0;
option display_eps .000001;

# ----------------------------------------------------------

#Restricted Master
problem RMaster: Total_Restricted_Cost, lambda,
   Restricted_Coupled, Convex;

#Subproblems
problem Subproblem{p in 1..m}:
  Reduced_Cost[p],
  {j in 1..ni[p]} x[p,j],
  {i in 1..ki[p]} Set_constraint[p,i];


#to check if initial data is feasible
#problem initialCheck: nothing, y, couple, convex2;



let {i in 1..ki[0]} dual_cost_y[i] := 1;
let {p in 1..m} dual_cost_z[p] := 1;
let {p in 1..m} numberOfvertices[p] :=
   numberOfInitialVertices[p];
```

```
let {p in 1..m, i in 1..ni[p]} cost_subproblem[p,i] :=
      c[p,i] - (sum{j in 1..ki[0]} dual_cost_y[j]*A[0,p,j,i]);

let{p in 1..m,i in 1..numberOfvertices[p],j in 1..ni[p]}
    vertices[p,i,j] := initial_vertices[p,i,j];



#At the beginning step we need the costs for the RM:
let {p in 1..m,i in 1..numberOfvertices[p]}
   restricted_master_cost[p,i] := sum{j in 1..ni[p]}
       c[p,j]*vertices[p,i,j];



let {p in 1..m,j in 1..ki[0], k in 1..numberOfvertices[p]}
   restricted_couple_cost[p,k,j] :=
      sum{i in 1..ni[p]} A[0,p,j,i]*vertices[p,k,i];

#a check if you have the right amount of beginning variables
    (not necessary?)
if sum{p in 1..m} numberOfvertices[p] < ki[0] + m then {
   printf "\nNOT EXACT NUMBER OF INITIAL VECTORS.\n";
   };


####ONLY CHEK, NOT NECESSARY
#solve initialCheck;

#for {p in 1..m} {
#  for {j in 1..numberOfvertices[p]} {
#     if y[p,j] >= 0 then {
#        let initialChekparameter := initialChekparameter + 1;
#     }
#  }
#}
```

```
#if initialChekparameter < ki[0]+m then {
#  printf "\nNO CORRECT INITIAL DATA! POSSIBLY WRONG
    OUTPUT.\n";
#}

####CHECK UP TO HERE

repeat {

   let nIter := nIter + 1;
   printf "\nITERATION %d\n", nIter;

   #let {p in 1..m,j in 1..ki[0], k in 1..numberOfvertices[p]}
   #restricted_couple_cost[p,k,j] :=
   #  sum{i in 1..ni[p]} A[0,p,j,i]*vertices[p,k,i];

   display restricted_master_cost;
   display restricted_couple_cost;
   solve RMaster;

   printf "\nRestrictedMaster solution nr. %d\n", nIter;
   display lambda;

   let {i in 1..ki[0]} dual_cost_y[i] :=
      Restricted_Coupled[i].dual;#lambda dual
      Restricted_Couple;
   let {p in 1..m} dual_cost_z[p] := Convex[p].dual;#lambda
      dual convex

   printf "\nDual variables nr. %d\n", nIter;
   display dual_cost_z;
   display dual_cost_y;

   let {p in 1..m,i in 1..ni[p]} cost_subproblem[p,i] :=
     c[p,i] - (sum{j in 1..ki[0]} dual_cost_y[j]*A[0,p,j,i]);

   display cost_subproblem;


   for {p in 1..m} {
```

```
      solve Subproblem[p];

      if Reduced_Cost[p] < dual_cost_z[p] -0.00001 then {
         printf "\nNew vertex of supbroblem %d\n", p;
         display {i in 1..ni[p]} x[p,i];
         let numberOfvertices[p] := numberOfvertices[p] + 1;
         let{i in 1..ni[p]} vertices[p, numberOfvertices[p], i]
             := x[p,i];
         let restricted_master_cost[p,numberOfvertices[p]] :=
            sum{j in 1..ni[p]}
               c[p,j]*vertices[p,numberOfvertices[p],j];

         display restricted_master_cost;

      }

      else {
         let success := success+1;
      };
   };
   if success = m then {
      printf "\nOPTIMAL SOLUTION:\n";

      let {p in 1..m,i in 1..ni[p]}
         x[p,i] := sum{j in 1..numberOfvertices[p]}
            lambda[p,j]*vertices[p,j,i];
         display x;
         printf "\nOBJECTIVE FUNCTION VALUE:\n";
         let mincost := sum{p in 1..m,i in 1..ni[p]}
             x[p,i]*c[p,i];
         display mincost;
      break;
   };
   let success := 0;

   let {p in 1..m,j in 1..ki[0], k in 1..numberOfvertices[p]}
   restricted_couple_cost[p,k,j] :=
      sum{i in 1..ni[p]} A[0,p,j,i]*vertices[p,k,i];
};
```

3.3 Examples for data files

3.3.1 Example 1

```
# DANTZIG-WOLFE DECOMPOSITION
# Bounded Problem

param N := 3; #total number of variables
param K := 7; #total number of constraints
param m := 1; #number of systems

#subdivision of variables, nj[j] = dimension;
param ni :=
1 3;

#subdivision of constraints in sets of ki[i] constraints;
param ki :=
0 1
1 6;

#all constraint values, subdivised
#b{i in 0..m,1..ki[i]}
param b :=
   [0,*] :=
     1 17
   [1,*] :=
     1 2
     2 2
     3 2
     4 -1
     5 -1
     6 -1;

#all cost values, subdivised
#{i in 1..m,1..ni[i]}
param c :=
   [1,*] :=
     1 -4
     2 -1
     3 -6;
```

```
#within index; #specify what matrices, can done by RUN
set iindex :=
(0,1)
(1,1);

#{(i,j) in iindex, 1..ki[i], 1..ni[j]}
param A :=
   [0,1,*,*]: 1 2 3 :=
     1 3 2 4
   [1,1,*,*]: 1 2 3:=
     1 1 0 0
     2 0 1 0
     3 0 0 1
     4 -1 0 0
     5 0 -1 0
     6 0 0 -1;


param numberOfInitialVertices := #{p in 1..m};
1 2;

#{p in 1..m, 1..numberOfvertices[p], 1..ni[p]};
param initial_vertices :=
      [1,*,*]: 1 2 3:=
        1 2 2 2
        2 1 1 2;
```

3.3.2 Example 2

```
# DANTZIG-WOLFE DECOMPOSITION
# Bounded Problem

param N := 5; #total number of variables
param K := 8; #total number of constraints
param m := 2; #number of systems

#subdivision of variables, nj[j] = dimension;
param ni :=
1 2
2 3;

#subdivision of constraints in sets of ki[i] constraints;
param ki :=
0 2
1 2
2 4;

#all constraint values, subdivised
#b{i in 0..m,1..ki[i]}
param b :=
   [0,*] :=
      1 4
      2 1
   [1,*] :=
      1 7
      2 8
   [2,*] :=
      1 3
      2 7
      3 5
      4 3;

#all cost values, subdivised
#{i in 1..m,1..ni[i]}
param c :=
   [1,*] :=
      1 -4
```

```
      2 -2
   [2,*] :=
      1 -2
      2 -4
      3 -1;

#within index; #specify what matrices, can done by RUN
set iindex :=
(0,1)
(0,2)
(1,1)
(2,2);

#{(i,j) in iindex, 1..ki[i], 1..ni[j]}
param A :=
   [0,1,*,*]: 1 2 :=
      1 1 2
      2 1 2
   [0,2,*,*]: 1 2 3 :=
      1 3 2 -4
      2 -3 2 -1
   [1,1,*,*]: 1 2 :=
      1 4 2
      2 1 2
   [2,2,*,*]: 1 2 3 :=
      1 1 0 1
      2 2 3 1
      3 3 -1 1
      4 2 -1 1;



param numberOfInitialVertices := #{p in 1..m};
1 2
2 2;

#{p in 1..m, 1..numberOfvertices[p], 1..ni[p]};
param initial_vertices :=
      [1,*,*]: 1 2 :=
         1 1 1.5
         2 1.75 0
```

```
[2,*,*]: 1 2 3:=
   1 2 1 0
   2 0 1.33333333 3;
```

3.4 Output

3.4.1 Example 1

```
ITERATION 1
restricted_master_cost :=
1 1   -22
1 2   -17
;

restricted_couple_cost :=
1 1 1   18
1 2 1   13
;

MINOS 5.51: optimal solution found.
1 iterations, objective -21

RestrictedMaster solution nr. 1
lambda :=
1 1   0.8
1 2   0.2
;


Dual variables nr. 1
dual_cost_z [*] :=
1  -4
;

dual_cost_y [*] :=
1  -1
;

cost_subproblem :=
1 1   -1
1 2    1
1 3   -2
;

MINOS 5.51: optimal solution found.
```

```
2 iterations, objective -5

New vertex of supbroblem 1
x[p,i] [*] :=
1  2
2  1
3  2
;

restricted_master_cost :=
1 1   -22
1 2   -17
1 3   -21
;


ITERATION 2
restricted_master_cost :=
1 1   -22
1 2   -17
1 3   -21
;

restricted_couple_cost :=
1 1 1   18
1 2 1   13
1 3 1   16
;

MINOS 5.51: optimal solution found.
1 iterations, objective -21.5

RestrictedMaster solution nr. 2
lambda :=
1 1   0.5
1 2   0
1 3   0.5
;


Dual variables nr. 2
```

```
dual_cost_z [*] :=
1  -13
;

dual_cost_y [*] :=
1  -0.5
;

cost_subproblem :=
1 1   -2.5
1 2    0
1 3   -4
;

MINOS 5.51: optimal solution found.
0 iterations, objective -13

OPTIMAL SOLUTION:
x :=
1 1   2
1 2   1.5
1 3   2
;


OBJECTIVE FUNCTION VALUE:
mincost = -21.5
```

3.4.2 Example 2

```
ITERATION 1
restricted_master_cost :=
1 1   -7
1 2   -7
2 1   -8
2 2   -8.33333
;

restricted_couple_cost :=
1 1 1    4
1 1 2    4
1 2 1    1.75
1 2 2    1.75
2 1 1    8
2 1 2   -4
2 2 1   -9.33333
2 2 2   -0.333333
;

MINOS 5.51: optimal solution found.
2 iterations, objective -15.14285714

RestrictedMaster solution nr. 1
lambda :=
1 1   0.746032
1 2   0.253968
2 1   0.571429
2 2   0.428571
;


Dual variables nr. 1
dual_cost_z [*] :=
1  -7
2  -8.19048
;

dual_cost_y [*] :=
```

```
1   0.015873
2  -0.015873
;

cost_subproblem :=
1 1   -4
1 2   -2
2 1   -2.09524
2 2   -4
2 3   -0.952381
;

MINOS 5.51: optimal solution found.
1 iterations, objective -7
MINOS 5.51: optimal solution found.
1 iterations, objective -9.333333333

New vertex of supbroblem 2
x[p,i] [*] :=
1  0
2  2.33333
3  0
;

restricted_master_cost :=
1 1   -7
1 2   -7
2 1   -8
2 2   -8.33333
2 3   -9.33333
;


ITERATION 2
restricted_master_cost :=
1 1   -7
1 2   -7
2 1   -8
2 2   -8.33333
2 3   -9.33333
;
```

```
restricted_couple_cost :=
1 1 1   4
1 1 2   4
1 2 1   1.75
1 2 2   1.75
2 1 1   8
2 1 2  -4
2 2 1  -9.33333
2 2 2  -0.333333
2 3 1   4.66667
2 3 2   4.66667
;

MINOS 5.51: optimal solution found.
1 iterations, objective -15.43478261

RestrictedMaster solution nr. 2
lambda :=
1 1   0
1 2   1
2 1   0.461957
2 2   0.282609
2 3   0.255435
;


Dual variables nr. 2
dual_cost_z [*] :=
1  -6.69565
2  -8.52174
;

dual_cost_y [*] :=
1  -0.0144928
2  -0.15942
;

cost_subproblem :=
1 1  -3.82609
1 2  -1.65217
```

```
2 1   -2.43478
2 2   -3.65217
2 3   -1.21739
;

MINOS 5.51: optimal solution found.
0 iterations, objective -6.695652172
MINOS 5.51: optimal solution found.
0 iterations, objective -8.521739125

OPTIMAL SOLUTION:
x :=
1 1   1.75
1 2   0
2 1   0.923913
2 2   1.43478
2 3   0.847826
;


OBJECTIVE FUNCTION VALUE:
mincost = -15.4348
```

---

4 References

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