Perfect Matching Problem

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Chapter 1

Introduction to the Problem of Perfect Matching

Graph Theory is a branch of Mathematics and Computer Science whose goal is the study of graphs. These ones are objects which allows to schematize situations, processes in order to analize them in quantitative and algorithmic terms. They are also object of study mainly in Computer Science thanks to the development of specific algorithms. In graph theory there are a lot of problems: one of the most famous is about the Perfect Matching. Before we introduce it, we are going to recall some useful definition, notions and results.

1.1 Some Generalities about Graphs

Definizione 1.1. We define *graph* a ordered pair $G = (V, E)$ comprising a set V of vertices (or nodes) together with a set E of edges (or arcs) which are 2-element subsets of V.

Given a graph G, with |V(G)| we denote the order of G, i.e the number of vertices of G while with |E(G)| we denote the size of G, i.e the number of edges of G

A vertex $v \in V$ is incident to an edge $e \in E$ se $v \in e$.
Two edges $e_1, e_2 \in E$ are incident (or adjacent) if they are a common vertex.
Two vertices $v_1, v_2 \in V$ are adjacent if there exist an edge e $\in E$ such that $e = v_1v_2$.

Definizione 1.2. Let $G = (V, E)$ e $G' = (V', E')$ be two graphs. If $V' \subseteq V$ and $E' \subseteq E$, then G' is a subgraph of G. Moreover, if $G \neq G$, then we say that G' is a proper subgraph of G.

Definizione 1.3. Let $G = (V, E)$ be a graph and let $v \in V$ be a vertex. The degree of a vertex v is defined as

$$d_G(V) := |N_G(v)|$$

where $N_G(v) := \{w \in V : vw \in E\}$ is the neighbour of v. In other words, it is the number of edges incident to the vertex $v \in V$. Next, we define

$$\delta(G) := \min_{v \in V} d_G(v) \in \mathbb{Z}$$

the minimum degree of G, i.e. the degree of the vertex of less incident edges and

$$\Delta(G) := \max_{v \in V} d_G(v) \in \mathbb{Z}$$

the maximum degree of G, i.e. the degree of the vertex with more incident edges. Furthermore, we denote the average degree of G as

$$d(G) := \frac{1}{|V|} \sum_{v \in V} d_G(v)$$

Obviously, we have that $\delta(G) \leq d(G) \leq \Delta(G)$.

Definizione 1.4. A graph $G = (V, E)$ is said to be K-regular if

$$d_G = k \qquad \forall v \in V$$

In other words, if all vertices have the same degree. As a consequence, we have $\delta = d = \Delta$. In particular, 3-regular graphs are called also cubic graphs.

Definizione 1.5. A path is a graph $P = (V, E)$ on the form $V = \{x_0x_1, x_1x_2, \ldots, x_{k-1}x_k\}$ where
x_0 e xk are called ends of P;
$x_2, \ldots, x_{k-1}$ are the inner vertices of P;
The number of edges is the length of P;
With P_k we mean a path of lenght k.

Definizione 1.6. A graph is said to be complete if every vertex is linked to all remaining vertices . It is denoted with K_n (with $n \in N$ the number of vertices)

Definizione 1.7. A graph $G = (V, E)$ is bipartite if its vertex set V can be partitioned into two disjoint subsets $V = V_1 \cup V_2$ such that every edge $e \in E$ has the form v_1v_2, with $v_1 \in V_1$ and $v_2 \in V_2$.

Definizione 1.8. We define a planar graph G a graph such that it can be rapresented in a plane in order that they do not admite edges who intersect each other.
The complete graphs K_5 e $K_{3,3}$ are non-planar graphs.

We report some example:

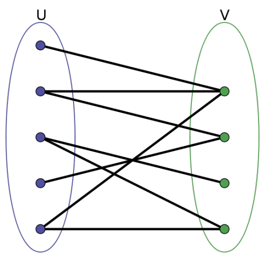

Figure 1.1: Bipartite graph

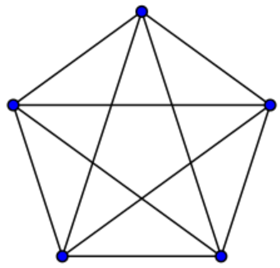

Figure 1.2: Complete graph (K_5)

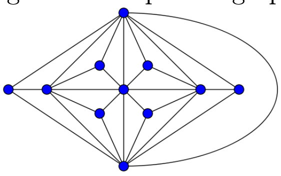

Figure 1.3: Planar graph

1.2 Matching and Perfect Matching

Given a graph G, we want to find as many incident edges as possible.

Definizione 1.9. Given a graph $G = (V, E)$, a matching $M(G)$ in G is a set of pairwise non-adjacent edges; that is, no two edges share a common vertex. Or, in an equivalent form, if every vertex of G is incident to at most an edge in M, i.e. $deg(v) <= 1 \forall v \in G$.
This means that, in a matching M(G), all vertices can have either degree 0 or 1. In particular

- se deg(v) = 1, then the vertex v is matched (or saturated) if it is an endpoint of one of the edges in the matching.
- se deg(v) = 0, then the vertex v is unmatched.

In a matching, two edges cannot be incident: if they were, then the vertex incident to these two edges would have degree 2, but this is not possible by definition.

Definizione 1.10. Given $G = (V, E)$, a maximal matching is a matching M of a graph G with the property that if any edge not in M is added to M, it is no longer a matching, that is, M is maximal if it is not a subset of any other matching in graph G

Definizione 1.11. A maximum matching M(G) is a matching that contains the largest possible number of edges. There may be many maximum matchings and such number is called matching number. Note that every maximum matching is maximal, but not every maximal matching is a maximum matching.

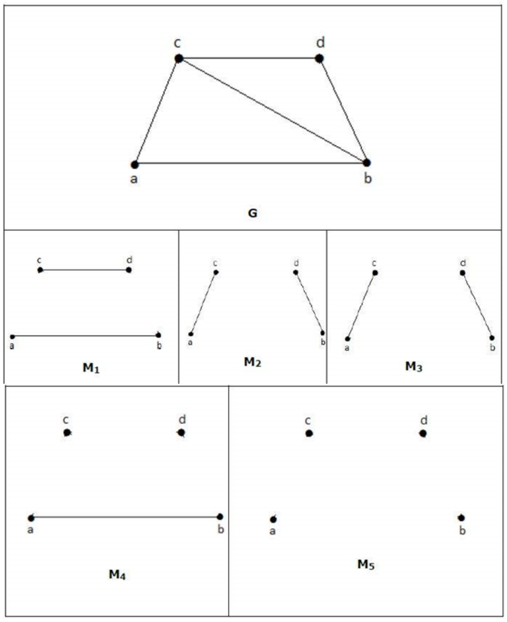

Figure 1.4: Examples of Matching of a graph G

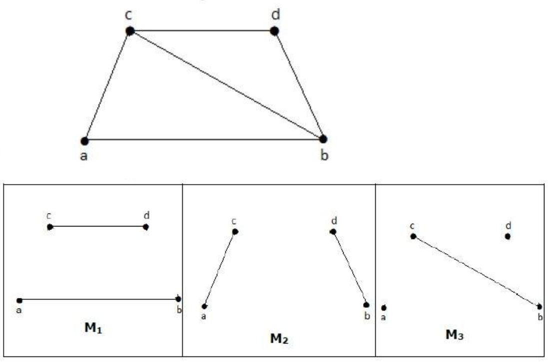

Figure 1.5: M_1, M_2, M_3 are maximal matching for G

The following figure shows examples of maximum matchings in the same three graphs.

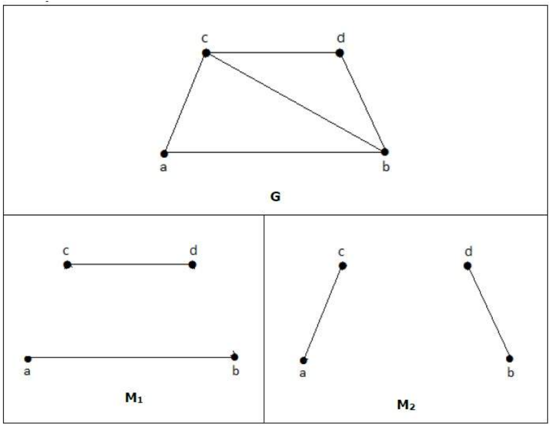

Figure 1.6: M_1 e M_2 are maximum matching for G and the matching number is 2. Hence, using the graph G, we can form only subgraphs with at most 2 edges. For this reason we have 2 as matching number

Definizione 1.12. A matching M(G) of a graph G is said to be perfect if every vertex $v \in G$ is incident to exactly one edge $e \in M$, i.e. if $deg(v) = 1 \forall v \in V$

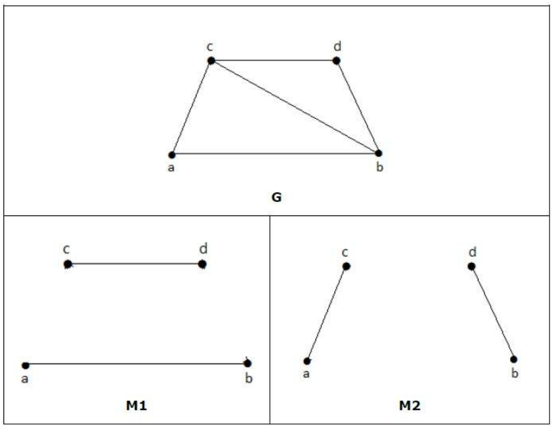

Figure 1.7: M_1 e M_2 are examples of perfect matching for G

Osservazione 1. From the definition above, we deduce:

1. Every perfect matching of graph is also a maximum matching of graph, because there is no chance of adding one more edge in a perfect matching graph.. As a consequence, it is also maximal.

2. A maximum matching of graph need not be perfect.

3. If a graph G has a perfect matching, then the number of vertices |V(G)| is even. If it were odd, then the last vertex pairs with the other vertex, and finally there remains a single vertex which cannot be paired with any other vertex for which the degree is zero. It clearly violates the perfect matching principle.

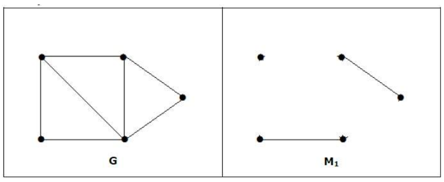

Figure 1.8: The converse of the above statement need not be true. If G has even number of vertices, then M_1 need not be perfect.

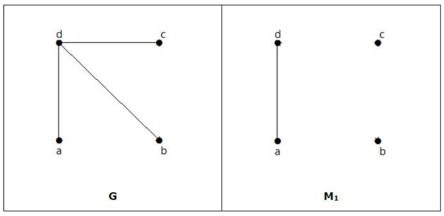

Figure 1.9: It is matching, but it is not a perfect match, even though it has even number of vertices

1.3 Some examples of Perfect Matching

To model some problems about Perfect Matching, you often use bipartite graphs. Let $G = (V, E)$ be a bipartite graph with bipartition $\{A, B\}$, i.e $V = A \cup B$ and $A \cap B = \emptyset$ and all edges link vertices from A to B. Our aim is to find a matching M in G with as many edges as possible.

Definizione 1.13. A path in G which starts in A at an unmatched vertex and then contains, alternately, edges from E $\setminus M$ and from M is called alternating path with respect to M.
An alternating path P that ends in an unmatched vertex B is called an augmenting path.

To solve some problems about combinations, you use concepts about graph theory. In this section, we are going to deal with some simple examples of perfect matching and how to solve them. For this reason, we state a very useful result

Teorema 1.3.1 (Hall, 1935)**.** *A bipartite graph G admits a matching A if and only if*

$$|N(S)| \geq |S| \qquad \forall S \subseteq A$$

Proof. See [1]. □

Esempio 1.1. Suppose I have 6 gifts (labeled 1,2,3,4,5,6) to give to 5 friends (Alice, Bob, Charles, Dot, Edward). Can i distribute one gift to each person so that everyone gets something they wish? Certainly, this depends on the preferences of my friends. If none of them like any of my gifts, then I am out of luck. But even if they all like some gifts, I may still not be able to give them out satisfactorily. For instance, if none of them like gifts 5 or 6, then I will have only 4 gifts to give to my 5 friends and so the problem admits no solutions. We exclude these two particular cases.

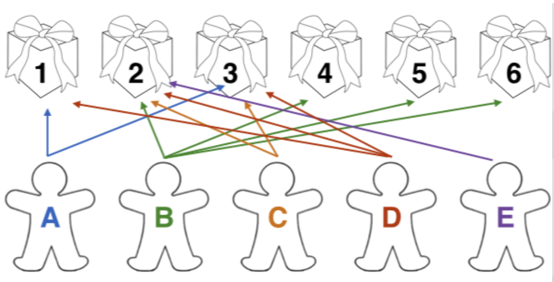

Figure 1.10: Can every child receive the gift he/she prefers?

We can model this situation as a graph partioned into two sets: the set of the gifts and the set of the friends. By definition, it is clear that the graph is bipartite. An edge is associated to each child which denotes the preferences about the gifts (for instance, Alice wishes gift 1 or 3 and so on). Let's check if it satisfies or not Hall's theorem: we consider a subset $X = \{A, C, D, E\}$, hence $|X| = 4$ and as a consequence $N(X) = \{1, 2, 3\}$, so $|X| = 3$. From this, we deduce that $|X| \geq |N(X)|$, thus Hall's condition has been violated and so there no exists a matching. In other words, it is not possible to distribute to everyone the desired gift.

Another example is the vertex cover problem.

Definizione 1.14. Let $G = (V, E)$ be a graph. A vertex cover of G is a subset $U \subseteq V$ such that every edge $e \in E$ is incident to a vertex $v \in U$.

This means that every vertex in the graph is touching at least one edge. Vertex cover is a topic in graph theory that has applications in matching problems and optimization problems. A vertex cover might be a good approach to a problem where all of the edges in a graph need to be included in the solution. In particular, you ask to find minimum vertex cover.

Osservazione 2. We just report some properties of Vertex Cover:

- The set of all vertices is a vertex cover;
- Endpoints of a maximal matching form a vertex cover;

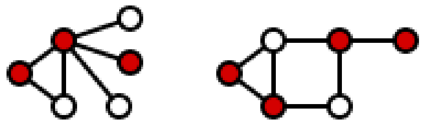

Figure 1.11: Examples of vertex cover

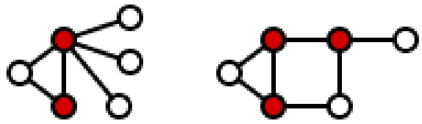

Figure 1.12: Examples of minimum vertex cover

- The complete bipartite graph $K_{m,n}$ has a minimum vertex cover given by $min\,\{m, n\}$.

The following result establishes a link between vertex cover and perfect matching. In particular

Teorema 1.3.2 (König, 1931)**.** *In a bipartite graph G, the number of edges in a maximum matching is equal to the number of vertices in a minimum vertex cover.*

Proof. See [1]. □

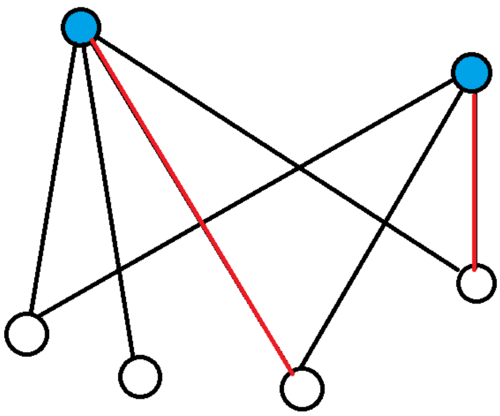

Figure 1.13: Example of application of König Theorem: red vertices are the minimum vertex cover and blue edges are the maximum matching

Bibliography

[1] Reinhard Diestel, "Graph Theory", Electionic Edition 2005

[2] John M. Harris, Jeffry L. Hirst, Michael J.Mossinghoff, "Combinatorics and Graph Theory", Springer Edition 2008