Analysis of Brownian Motion

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1 Introduction

Brownian motion is a physical phenomenon that was first discovered in 1827 by Robert Brown. It describes the apparent random motion of small particles that are submerged in fluid or gas. Albert Einstein in 1905 hypothesized that this phenomenon was due to the collision of the molecules inside the medium with the particle, which was the first evidence of the existence of atoms and molecules[1].

The first mathematical construction of the Brownian motion was developed by Norbert Wiener. The Wiener process is a continuous-time stochastic process, that is often called the standard Brownian motion due to its historical connection with Brownian motion. However, its application extends to applied mathematics, noise analysis in electrical engineering, filter theory, and finance. An example of a 2 dimensional Wiener process is shown in Fig. 1.

In the following paper, we attempt to reconstruct the Wiener process with Fourier representation, and Haar representation.

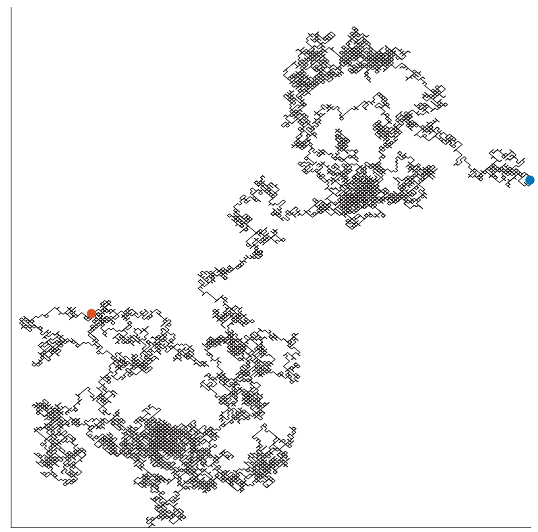

Figure 1: A single realization of a 2 Dimensional Wiener process. The red dot represent the starting location and the blue dot is the ending location.

2 Preliminaries

Before we can begin our analysis on Brownian motion, we must first recall some basic concepts of statistics and stochastic process.

2.1 Probability space

Definition 2.1. A **probability space**[2] $(\Omega, \mathcal{F}, \mathbb{P})$ consists of three components :

(i) A sample space Ω; an arbitrary non-empty set.

(ii) A set of events $\mathcal{F}$ that forms a $\sigma-$algebra; that is a set of subsets of Ω such that

- $\Omega \in \mathcal{F}$
- $\mathcal{F}$ is closed under complements: if A $\in \mathcal{F}$then $A^c = \Omega \setminus A \in \mathcal{F}$

- $\mathcal{F}$ is closed under countable unions: if $A_i \in \mathcal{F}, i = 1, 2, ...$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

(iii) A mapping $\mathbb{P}$ from $\mathcal{F}$ to $[0, 1]$, referred to as a probability measure, that satisfies

- $\mathbb{P}(\Omega) = 1$
- $A_i \in \mathcal{F}, i = 1, 2, ...$ are pairwise disjoint, i.e. $A_i \cap A_j = \emptyset$ for $i \neq j$ then

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

Definition 2.2. A random variable X[2] is normally distributed with mean μ and variance σ^2 if

$$\mathbb{P}\{X \leq x\} = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{x} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt$$

for all $x \in \mathbb{R}$.

Definition 2.3. A collection of random variables is called a **Gaussian process**[4], if the joint distribution of any finite number of its members is Gaussian. In other words, a Gaussian process is a $\mathbb{R}^d-$ valued stochastic process with continuous time (or with index) t such that $(X(t_0), X(t_q), ..., X(t_n))^\top$ is a $(n+1)$-dimensional Gaussian random vector for any $0 \leq t_0 < t_1 < ... < t_n$.

2.2 Wiener Process

Definition 2.4. A real-valued stochastic process $\{W(t) : t \geq 0\}$ is called a **Wiener Process** [3] if the following holds:

(i) $W(0) = 0$ a.s. (almost surely)

(ii) If $0 \leq t_0 \leq t_1 \leq ... \leq t_m$ for $1 \leq j \leq m$ the increments $W_{t_j} - W_{t_{j-1}}$, are independent random variables.

(iii) If $0 \leq s < t$, then $W_t - W_s \in \mathcal{N}(0, t-s)$

(iv) The paths of W are almost surely continuous (i.e $\exists$ a measurable set such that $P(A) = 1$ and $\forall \omega \in A$, the map $t \mapsto W_t(\omega)$ is continuous).

More generally, if we replace condition (i) with the requirement that $W_0 = x$ a.s. for some $x \in \mathbb{R}$, we say that W is a wiener process starting from x.

2.3 Properties of Wiener Process

The following are some basic properties of a one-dimensional Wiener Process [3]:

(i) The probability density function of W_t is the following: $f_{W_t} = \frac{1}{\sqrt{2\pi t}} e^{-x^2/(2t)}$ with mean 0, variance t.

(ii) $\mathbb{E}[W_t] = 0$

(iii) $\mathrm{Var}[W_t] = \mathbb{E}[W_t^2] - \mathbb{E}^2[W_t] = \mathbb{E}[W_t^2] = t$

(iv) $\mathrm{Cov}(W_s W_t) = \min\{s, t\}$

The following proposition [4] is an equivalent characterization for a Wiener process. Representing the definition of a Wiener Process in this manner will help us with the proofs to come.

Proposition 1. *Suppose $Y = \{Y_t\}_{t \geq 0}$ is a Gaussian process with almost surely continuous paths (as in condition of definition 1.3). Then Y is a Wiener process if and only if, $\forall s, t \in [0, \infty), \mathbb{E}[Y_s] = 0$ and $\mathbb{E}[Y_s Y_t] = \min(s, t)$.*

2.4 Brownian Bridge

A Brownian bridge[6], $B_t := (W_t|W_1 = 0), t \in [0,\ 1]$, is a continuous-time stochastic process, similar to Wiener process, subject to the condition that it is bounded at the origin and the end of the interval, as shown in Fig. 2. It has the following properties:

(i) $B_0 = B_1 = 0$

(ii) $\mathbb{E}(B_t) = 0$ for $t \in [0, 1]$

(iii) $\mathrm{cov}(B_s, B_t) = \min\{s, t\} - st$, for $s, t \in [0, 1]$

(iv) B_t is continuous on $[0, 1]$

Proposition 2. *The Brownian Bridge is a centered Gaussian Process and its covariance is* $K_B(s,t) = min\{s,t\} - (s \cdot t)$

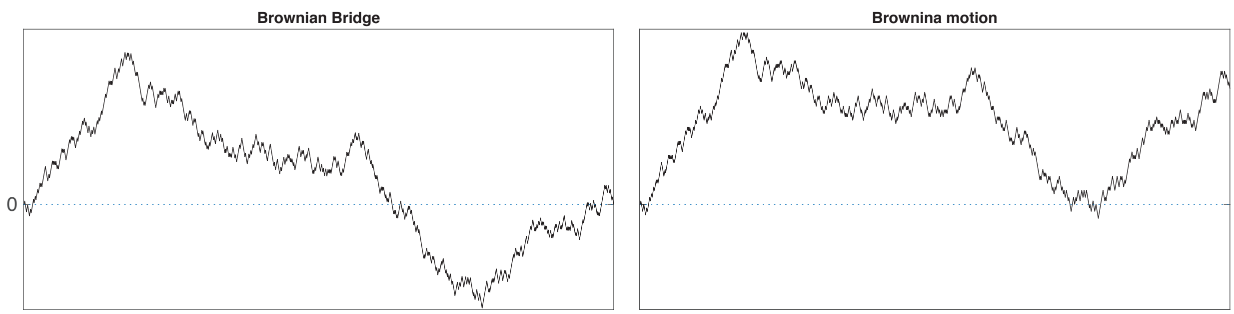

Figure 2: The Brownian bridge B_t on the left and the Brownian motion W_t on the right.

Brownian bridge has a very simple relation to Brownian motion. If $W(t)$ is a standard Wiener process, representing a Brownian motion then

$$B(t) = W(t) - \frac{t}{T}W(T)$$

is a Brownian bridge for $t \in [0,\ T]$. Similarly if $B(t)$ is a Brownian bridge and Z is a standard normal random variable independent of B, then

$$W(t) = B(t/T) + \frac{t}{\sqrt{T}}Z.$$

3 Fourier Series Representation

We will show the Fourier Series representation of the Brownian Bridge. Note that this proof is heavily influenced by Giordano Giambartolomei [7].
We first introduce the following theorem, which will allow us to represent the Brownian Bridge as a Fourier Series.

Theorem 3.1 (Karhunen-Loève Expansion). *Let $X = \{X_t\}$, $t \in [a,b] \subset \mathbb{R}$, where $a,b < \infty$, be zero-mean square-integrable stochastic process with a continuous covariance function $K_X(t,\tau)$. Then letting e_k be an orthonormal set of basis in L^2, then $\forall t \in [a,b]$ we may decompose*

$$X_t = \sum_{k=1}^{\infty} Z_k e_k(t)$$

where the convergence is in L^2 sense, uniform in t and

$$Z_k = \int_a^b X_t e_k(t) dt$$

The $\{e_k\}_{j=1}^{\infty}$ can be found by solving for the eigenfunctions of the Hilbert-Schmidt integral operator A on $L^2[a,b]$, where $(Af)(t) = \int_a^b K_X(t,\tau)f(\tau)d\tau$. Moreover the $\{Z_k\}$ are pairwise orthogonal random variables with zero mean and variance λ_k, where λ_k is the eigenvalue corresponding to the eigenfunction e_k. Now, we wish to show that we can represent the Brownian Bridge as the following using the Karhunen-Loève Expansion on the unit interval [0,1].

Theorem 3.2 (Fourier representation of Brownian Bridge). *Let B_t denote our Brownian Bridge. Then we can write*

$$B_t = \frac{\sqrt{2}}{\pi} \sum_{k=1}^{\infty} \frac{Z_k^*}{k} \sin(k\pi t)$$

where

$$Z_k = \int_0^1 B_t \sqrt{2} \sin(k\pi t) dt,$$
$$\lambda_k = \frac{1}{k^2\pi^2}, k \in \mathbb{N}, Z_k^* = k\pi Z_k$$

and the convergence in mean square is almost surely.

Proof. Note: Brownian Bridge satisfies all the properties needed for Theorem 3.1 and by proposition 2 we have that its covariance is $K_B(s,t) = min(s,t) - (s \cdot t)$. Thus we must verify the eigenvectors/values and verify that B_t has Fourier coefficients Z_k, as stated above. First we find the eigenvalues using $Af = \lambda f$, where $(Af)(t) = \int_0^1 K_B(s,t)f(s)ds$ and, $K_B(s,t) = \min(s,t) - st$

$$\int_0^1 [\min(s,t) - st] e(s) ds = \lambda e(t)$$
$$\int_0^t (s - st) e(s) ds + \int_t^1 (t - st) e(s) ds = \lambda e(t)$$
$$(1-t) \int_0^t s \cdot e(s) ds + t \int_t^1 (1-s) e(s) ds = \lambda e(t)$$

Take the derivatives of both sides with respect to t to obtain the following:

$$-\int_0^t s\cdot e(s)ds + (1-t)e(t) + \int_t^1 (1-s)e(s)ds - (1-t)t\cdot e(t) = \lambda e'(t)$$

$$\int_t^1 e(s)ds - \int_0^1 s\cdot e(s)ds = \lambda e'(t) \tag{1}$$

We then take the derivative of both sides with respect to t again to obtain the following:

$$-e(t) = \lambda e''(t)$$

its characteristic polynomial: $x^2\lambda + 1 = 0$
we obtain the following roots : $x_1 = \frac{i}{\sqrt{\lambda}}, x_2 = \frac{-i}{\sqrt{\lambda}}$
$\Rightarrow e(t) = a\sin(\frac{t}{\sqrt{\lambda}}) + b\cos(\frac{t}{\sqrt{\lambda}})$
if $\lambda = 0 \Rightarrow -e(t) = \lambda e''(t) \equiv 0$, which means 0 is not an eigenvalue, and we can proceed to determine a,b and λ.
For t=0, we have e(0)=b, but

$$e(0) = \frac{1}{\lambda}\int_0^1 [\min(0,s) - s\cdot 0]e(s)ds = 0$$

Thus b=0 and $e(t) = a\sin(\frac{t}{\sqrt{\lambda}})$, which we now substitute in equation (1) to obtain the following:

$$a\int_t^1 \sin(\frac{s}{\sqrt{\lambda}})ds - a\int_0^1 s\cdot\sin(\frac{s}{\sqrt{\lambda}})ds = \lambda\frac{a}{\sqrt{\lambda}}\cos(\frac{t}{\sqrt{\lambda}})$$

let $z = \frac{t}{\sqrt{\lambda}} \Rightarrow dz = \frac{1}{\sqrt{\lambda}}ds$

$$\Rightarrow \sqrt{\lambda}\int_{\frac{t}{\sqrt{\lambda}}}^{\frac{1}{\sqrt{\lambda}}} \sin(z)dz - \lambda\int_0^{\frac{1}{\sqrt{\lambda}}} z\cdot\sin(z)dz = \frac{\lambda}{\sqrt{\lambda}}\cos(\frac{t}{\sqrt{\lambda}})$$

$$\Big[-\lambda\cos(z)\Big]_{\frac{t}{\sqrt{\lambda}}}^{\frac{t}{\sqrt{\lambda}}} - \lambda\sqrt{\lambda}\Big\{-z\cdot\cos(z)\Big|_0^{\frac{1}{\sqrt{\lambda}}} + \int_0^{\frac{1}{\sqrt{\lambda}}}\cos(z)dz\Big\} = \lambda\cos(\frac{t}{\lambda})$$

$$\cos(\frac{t}{\sqrt{\lambda}}) - \cos(\frac{-1}{\sqrt{\lambda}}) + \cos(\frac{1}{\lambda}) - \sqrt{\lambda}\sin(\frac{1}{\lambda}) = \cos(\frac{t}{\sqrt{\lambda}})$$

$$\cos(\frac{t}{\sqrt{\lambda}}) - \sqrt{\lambda}\sin(\frac{1}{\sqrt{\lambda}}) = \cos(\frac{t}{\sqrt{\lambda}})$$

For t = 1 we have the following:

$$\cos(\frac{1}{\sqrt{\lambda}}) - \sqrt{\lambda}\sin(\frac{1}{\sqrt{\lambda}}) = \cos(\frac{1}{\sqrt{\lambda}})$$

$$\Rightarrow \sin(\frac{1}{\sqrt{\lambda}}) = 0$$

$$\Rightarrow \frac{1}{\sqrt{\lambda}} = k\pi, k\in\mathbb{N}$$

$\therefore$ The positive eigenvalues are: $\lambda_k = \frac{1}{k^2\pi^2}, k\in\mathbb{N}$ and the eigenfunctions are: $e_k(t) = a\cdot\sin(tk\pi)$ where a must be a normalized constant by the orthonormality condition on the eigenfunctions. Hence:

$$\frac{1}{a^2} = ||\sin(tk\pi)||_2^2$$

$$= \int_0^1 \sin^2(tk\pi)dt$$

let $z = tk\pi \Rightarrow dz = k\pi dt$

$$= \frac{1}{k\pi}\int_0^{k\pi} \sin^2(z)dz$$

$$= \frac{1}{k\pi}\int_0^{k\pi} \frac{1-\cos(2z)}{2}dz$$

let $u = 2z \Rightarrow dz = 2du$

$$= \frac{1}{4k\pi}\int_0^{2k\pi} 1-\cos(u)du$$

$$= \frac{1}{4k\pi}[2k\pi - \sin(2k\pi)] = \frac{1}{2}$$

$\therefore$ we have that $a = \sqrt{2}$. Thus $e_k(s) = a \cdot \sin(\frac{s}{\sqrt{\lambda_k}}) = \sqrt{2}\sin(sk\pi)$.
Now applying the Hilbert-Schmidt integral operator A to $e_k(t)$ we obtain:

$$\int_0^1 K_B(s,t)e_k(t)$$

$$\int_0^1 [\min(s,t) - st]e_k(s)ds$$

$$\sqrt{2}\int_0^t (s-st)\sin(sk\pi)ds + \sqrt{2}\int_t^1 (t-st)\sin(sk\pi)ds$$

$$\sqrt{2}\Big[\int_0^t s\cdot\sin(sk\pi)ds - t\int_0^1 s\cdot\sin(sk\pi)ds + t\int_t^1 \sin(sk\pi)ds\Big]$$

let $z = sk\pi \Rightarrow dz = k\pi ds$, thus we have:

$$\frac{\sqrt{2}}{k\pi}\Big[\frac{1}{k\pi}\int_0^{tk\pi} z\cdot\sin(z)dz - \frac{t}{k\pi}\int_0^{k\pi} z\cdot\sin(z)dz + t\int_{tk\pi}^{t\pi}\cos(z)dz\Big]$$

$$\frac{\sqrt{2}}{k\pi}\Big\{\Big[-\frac{1}{k\pi}z\cdot\cos(z)\Big]_0^{tk\pi} + \frac{1}{k\pi}\int_0^{tk\pi}\cos(z)dz + \Big[\frac{t}{k\pi}z\cdot\cos(z)\Big]_0^{k\pi} - \frac{t}{k\pi}\int_0^{k\pi}\cos(z)dz\Big\}$$

$$\frac{\sqrt{2}}{k\pi}\Big\{-t\cos(tk\pi) + \frac{1}{k\pi}\sin(tk\pi) + t\cdot\cos(k\pi) - t\cdot\cos(k\pi) + t\cdot\cos(tk\pi)\Big\}$$

$$\frac{\sqrt{2}}{k^2\pi^2}\sin(tk\pi) = \lambda_k e_k(t).$$

Thus we have: $Z_k = \frac{\langle e_k, f\rangle}{\langle e_k, e_k\rangle} = \int_0^1 B_t\sqrt{2}\sin(k\pi t)dt$ as required.
Now placing the following values back into the KLE theorem we obtain:

$$B_t = \sum_{k=1}^{\infty}\frac{\sqrt{2}}{\pi k}\ Z_k^*\sin(k\pi t)$$

where $Z_k = \int_0^1 B_t\sqrt{2}\sin(k\pi t)dt, \lambda_k = \frac{1}{k^2\pi^2}, k \in \mathbb{N}, Z_k^* = k\pi Z_k$ and the convergence in mean square is almost surely, as required. $\square$

4 Haar Wavelet Representation

Fourier Transform becomes a cumbersome tool that requires many coefficients to represent a localized event. Thus, although we represent Brownian motion as a Fourier series, it is sometimes best to represent it as a wavelet. Wavelets are well localized and need few coefficients to represent local transient structures.

We now show that we can represent Brownian Motion as the following using Haar wavelets on the unit interval [0,1]. First, we shall define some things to make our representation clearer. Note that this section is heavily influenced by Peter Rudzis [4], Maria Pereyra and Lesley Ward [5].

Definition 4.1. Haar wavelet $h(x)$ on unit interval [0,1] is defined as follows:

$$h(x) = -\chi_{[0,\frac{1}{2})}(x) + \chi_{[\frac{1}{2},1]}(x).$$

where $\chi_{[0,1]}(x)$ is the characteristic function defined as:

$$\chi_{[0,1]}(x) = \begin{cases} 1 & \text{if } x \in [0,1] \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 3. *The family* $\{h_{j,k}(x) := 2^{j/2}h(2^jx-k)\}_{j,k\in\mathbb{Z}}$ *forms an orthonormal basis of* $L^2(\mathbb{R})$

Proof. We begin by noting:

$$h_{j,k}(x) := -2^{j/2}\chi_{[0,\frac{1}{2})}(x) + 2^{j/2}\chi_{[\frac{1}{2},1]}(x).$$
$$h_{j,k}(x) := -2^{j/2}\chi_{[\frac{k}{2^j},\frac{k+1/2}{2^j})}(x) + 2^{j/2}\chi_{[\frac{k+1/2}{2^j},\frac{k+1}{2^j})}(x).$$

where the intervals of the characteristic functions are dyadic intervals of length $\frac{1}{2^{j+1}}$ and where a dyadic interval is defined as :

$$I_{j,k} = [k2^{-j}, (k+1)2^{-j}].$$

We wish to show that the integral of our Haar family is 0. First note that :

$$\int_{\mathbb{R}} h_{j,k}(x)dx = \int {}_{\mathbb{R}} 2^{j/2}h(2^jx-k)dx$$

let $y = 2^jx - k \Rightarrow dy = 2^jdx$.

$$\int_{\mathbb{R}} h_{j,k}(x)dx = 2^{j/2}\cdot 2^{-j}\int_{\mathbb{R}} h(t)dy$$
$$\int_{\mathbb{R}} h_{j,k}(x)dx = 2^{j/2}\cdot 2^{-j}\int_{\mathbb{R}} -\chi_{[0,\frac{1}{2})}(y) + \chi_{[\frac{1}{2},1]}(y).$$
$$\int_{\mathbb{R}} h_{j,k}(x)dx = 2^{j/2}\cdot 2^{-j}[-\frac{1}{2} + 0 + 1 - \frac{1}{2}] = 0.$$

Now, wish to show that: $< h_{j,k}, h_{j',k'} >= \delta_{j,j',k,k'} = \begin{cases} 1 & \text{if } j = j' \text{ and } k = k' \\ 0 & \text{otherwise} \end{cases}$

<u>**Case 1:**</u> let $j = j', k = k'$, then:

$$\begin{aligned}\int_{\mathbb{R}} h_{j,k}(x)\overline{h_{j',k'}}(x)dx &= \int_{\mathbb{R}} h_{j,k}^2(x)dx \\ &= (2^{j/2})^2\int_{\mathbb{R}} h(2^jx-k)^2dx \\ &= (2^{j/2})^2\left[\int_{\mathbb{R}} \chi_{[0,\frac{1}{2})}(2^jx-k)^2 - \chi_{[0,\frac{1}{2})}(2^jx-k)\chi_{[\frac{1}{2},1]}(2^jx-k) + \chi_{[\frac{1}{2},1]}(2^jx-k)^2dx\right]\end{aligned}$$

let $\beta = 2^j x - k \Rightarrow d\beta = 2^j dx$

$$\begin{aligned}
&= \int_{\mathbb{R}} \chi_{[0,\frac{1}{2})}(\beta)^2 - \chi_{[0,\frac{1}{2})}(\beta)\chi_{[\frac{1}{2},1]}(\beta) + \chi_{[\frac{1}{2},1]}(\beta)^2 d\beta \\
&= \frac{\alpha}{\alpha}\left[\int_0^{1/2} 1 d\beta - \int_{1/2}^{1/2} 1 d\beta + \int_{1/2}^{1} 1 d\beta\right] \\
&= [\frac{1}{2} - 0 + 1 - \frac{1}{2}] \\
&= 1
\end{aligned}$$

as required.

<u>Case 2:</u> let $j \neq j'$, WLOG assume $j > j'$, so $h_{j,k}$ is finer than $h_{j',k'}$ then:

$h_{j,k}$ is supported on on $\left[\frac{k}{2^j}, \frac{k+1}{2^j}\right)$ and $h_{j,k}$ is supported on on $\left[\frac{k'}{2^{j'}}, \frac{k'+1}{2^{j'}}\right)$. We use the properties of dyadic intervals. The $h_{j,k}$, $h_{j',k'}$ are constant on all smaller dyadic intervals. Since $j > j'$, $h_{j,k}$ is supported on a dyadic interval where $h_{j'k'}$ is constant since $\int h_{j,k}(x)dx = 0$ This means that

$$\langle h_{j,k}, h_{j',k'} \rangle = 0$$

<u>Case 3:</u> let $j = j'$ but $k \neq k'$
In this case the support of hj, k and $h_{j',k'}$ are disjoint, so

$$\langle h_{j,k}, h_{j',k'} \rangle = 0$$

Therefore $\{h_{j,k}\}_{j,k\in\mathbb{Z}}$ is an orthonormal set

□

In order to represent Brownian Motion as Haar wavelets as clearly as possible, we must first introduce another way of representing the Haar functions on the unit interval [0,1].
We begin with an indexed collection of subintervals of the unit interval.
Let $\mathbb{I} = \{I_{ij} : 0 \leq i < \infty \text{ and } 0 \leq j \leq 2^i - 1\}$
Let each $I_{i0} = \frac{1}{2}I_{i-1,0} = \{x/2 : x \in I_{i-1,0}\}$ then $I_{ij} = I_{i0} + j2^{-i} = \{x + j2^{-i} : x \in I_{ij}\}$.
Notice that: $\forall i, j, I_{ij} = I_{i+1,2j} \cup I_{i+1,2j+1}$. Using induction, we obtain: $\forall k \geq 0$,

$$I_{ij} = I_{i+k,2^k j} \cup I_{i+k,2^k j+1} \cup ... \cup I_{i+k,2^k j+2^k-1}$$

where the intervals are disjoint.
Now we can represent the Haar functions in the following way:
For $0 \leq i < \infty$ and $0 \leq j \leq 2^{i-1} - 1$, we define functions $\varphi_{ij} : [0,1] \to \mathbb{R}$ as follows. Let $\varphi_{00} \equiv 1$ and for $i \geq 1$ define

$$\varphi_{ij}(x) = \begin{cases} 2^{i-1}/2 & \text{if } x \in I_{i,2j} \\ -2^{i-1}/2 & \text{if } x \in I_{i,2j+1} \\ 0 & \text{otherwise} \end{cases}$$

where $\varphi_{ij}(x)$ are the Haar functions and the support of each function $\varphi_{ij}(x)$ is $I_{i,2j} \cup I_{i,2j+1} = I_{i-1,j}$.

Now that we have represented our Haar functions in this manner, we can now begin our Haar wavelet representation of Brownian Motion.
We start with a collection of Haar functions $\{\varphi_{ij}(x) : 0 \leq i < \infty \text{ and } 0 \leq j \leq 2^{i-1}\}$ and $\forall \varphi_{ij}$, define the function Ψ_{ij} on [0,1] by:

$$\Psi_{ij}(t) = \int_0^t \varphi_{ij}(s)ds$$

The following lemma gives us a useful bound for the functions $\Psi_{ij}(t)$. This will come in handy when we present our proof.

Lemma 4.1. *$\forall i, j$, we have that $0 \leq \Psi_{ij} \leq 2^{-(i+1)/2}$ and supp$(\varphi_{ij}) = I_{i-1,j}$.*

Proposition 4. *(Borel-Cantelli Lemma). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and suppose $\{A_n\}_{n=1}^{\infty}$ is a sequence of sets in $\mathcal{F}$ s.t. $\sum \mathbb{P}(A_n) < \infty$. Then $\mathbb{P}(\cap_{n=1}^{\infty} \cup_{j=n}^{\infty} A_n) = 0$.*

Having established these results, we can now state the following theorem:

Theorem 4.2 (Haar Representation of Brownian Motion)**.** *$\forall \varphi_{ij}$, define $\Psi_{ij}(t) = \int_0^t \varphi_{ij}(s)ds$ for $t \in [0,1]$ and let $\{Y_{ij}\}$ be a collection of independent $\mathcal{N}(0,1)$ random variables. Define $V_0(t) = Y_{00}\Psi_{00}$, and for each $i \geq 1$ define:*

$$V_i(t) = \sum_{j=1}^{2^{i-1}} Y_{ij}\Psi_{ij}(t)$$

Let X_t denote our Brownian Motion, where

$$X_t = \sum_{i=1}^{\infty} V_i(t)$$

Proof. We wish to show that $X = \{X_t\}_{t\in[0,1]}$ is a Wiener process.
To do this, we must show that X_t satisfies the properties stated in proposition 1, $\forall t \in [0,1]$.
Thus we must show that the following properties hold:

(i) $X = \{X_t\}_{t\in[0,1]}$ is a Gaussian process with almost surely continuous paths.(i.e $\exists$ a measurable set such that $P(A) = 1$ and $\forall \omega \in A$, the map $t \mapsto X_t(\omega)$ is continuous).

(ii) $\forall s, t \in [0,1], \mathbb{E}[X_s] = 0$ and $\mathbb{E}[X_sX_t] = \min(s,t)$.

<u>Property (i):</u>
$\forall t$ fixed, X_t is a linear combination of Gaussian random variables and hence by proposition, $X = \{X_t\}_{t\in[0,1]}$ is Gaussian.
Now we must show that $X = \{X_t\}_{t\in[0,1]}$ has almost surely continuous paths.
Consider the sequence of sets $\{A_i\}_{i=1}^{\infty}$ defined by $A_i = (|V_i(t)| > i^{-2}$ for some $t \in [0,1])$.
We wish to show that the sets A_i are measurable.
Note: for some $t \in [0,1]$, we have $|V_i(t)| > i^{-2}$ if and only if $|V_i(t)| > i^{-2}$ for some $t \in [0,1] \cap \mathbb{Q}$, by continuity of $V_i(t)$ $\forall \omega$ fixed.
Therefore, $\forall i \geq 1, A_i = \cup_{t\in[0,1]\cap\mathbb{Q}}\{\omega : |V_i(t)| > i^{-2}\}$, which is a countable union of measurable sets and hence measurable.
Now, $\forall i \geq 1$, if $|V_i(t)| > i^{-2}$, then $|Y_{ij}|\Psi_{ij}(t) > i^{-2}$ for some $1 \leq j \leq 2^{i-1}$.
This is because $|V_i(t)| \leq \sum_j |Y_{ij}|\Psi_{ij}(t)$ and the $\Psi_{ij}(t)$'s have disjoint support by Lemma 4.1.
Then $|Y_{ij}|2^{-(i+1)/2} > i^{-2}$ for some $1 \leq j \leq 2^{i-1}$ by the bound obtained in Lemma 4.1.
Thus, it follows that $\forall i \geq 1$,

$$\mathbb{P}(A_i) \leq \mathbb{P}\left(\left|Y_{ij}2^{-(i+1)/2} > i^2\right| \text{ for some } 1 \leq j \leq 2^{i-1}\right)$$

$$\leq 2^{i-1}\mathbb{P}(|Y_{ij}| \geq 2^{-(i+1)/2}i^{-2})$$
$$\leq 2^{i-1}\exp(-2^i i^{-4})$$

Note that $\sum 2^{i-1}\exp -2^i i^{-4}$ converges by the root test. Thus $\sum_{i=1}^{\infty}\mathbb{P}(A_i) < \infty$ by Weierestrass M-test and by the Borel-Cantelli Lemma, $\mathbb{P}(\cap_{n=1}^{\infty}\cup_{j=n}^{\infty} A_n) = 0$. We have shown that with probability 1, there are only finitely many $i \geq 1$ such that for some $t \in [0,1], |V_i(t)| \geq i^{-2}$. Hence, $\sum_{i=1}^{\infty} V_i(t)$ converges uniformly on [0,1] with probability 1. Uniform convergence implies that $\sum_{i=1}^{\infty} V_i(t)$ is continuous in t, because each of the $V_i(t)$'s is continuous in t, thus we have shown that $X = \{X_t\}_{t\in[0,1]}$ is a Gaussian process with almost surely continuous paths.

Property (ii):

Let $s, t \in [0,1]$, we have that:

We have that $\mathbb{E}[X_s] = 0$ by definition.

Now, observe that $\forall N > 0$

$$\mathbb{E}\big[\big(\sum_{i=0}^{N} V_i(s)\big)\big(\sum_{k=0}^{N} V_k(t)\big)\big] = \sum_{i,k=0}^{N}\mathbb{E}[V_i(s)V_k(t)]$$

Since Y_{ij}'s are independent standard normal random variables, we have $\mathbb{E}[Y_{ij}Y_{kl}] = \delta_{ij,kl}$:

$$= \sum_{i,k=0}^{N}\sum_{\substack{0\leq j\leq 2^{i-1}-1 \\ 0\leq l\leq 2^{i-1}-1}}\mathbb{E}[Y_{ij}Y_{kl}\Psi_{ij}(s)\Psi_{kl}(t)] = \sum_{i=0}^{N}\sum_{j=0}^{2^{i-1}-1}\Psi_{ij}(s)\Psi_{ij}(t)$$

Note:

(i) $\langle f, g\rangle = \int_0^1 f(t)\overline{g(t)}dt$, where $f, g \in L^2[0,1]$ is the real Hilbert space inner product of two functions.

(ii)If $\{e_j\}_{j=1}^{\infty}$ is a complete orthonormal system in $L^2[0,1]$, then $\forall f \in L^2[0,1]$ we have the following:

$$\sum_{j=1}^{N}\langle f, e_j\rangle e_j \to f$$

in L^2 as $N \to \infty$.

$\forall i \geq 0$ and $0 \leq j \leq 2^{i-1} - 1$, observe that by definition $\Psi_{ij}(t) = \langle \chi_{[0,t]}, \varphi_{ij}\rangle$

Therefore we have that:

$$\sum_{i=0}^{N}\sum_{j=0}^{2^{i-1}-1}\varphi_{ij}(s)\varphi_{ij}(t)$$

Notice, $\lim_{n\to\infty}(\sum_{i=0}^{N} V_i(s))(\sum_{k=0}^{\infty} V_k(t)) = X_s X_t$

Thus

$$\mathbb{E}[(\sum_{i=0}^{N} V_i(s))(\sum_{k=0}^{\infty} V_k(t))] = \mathbb{E}[X_s X_t] = \sum_{i,j}\langle \chi_{[0,s]}, \varphi_{ij}\rangle\langle \chi_{[0,t]}, \varphi_{ij}\rangle$$

Using bilinear property of inner products we have that

$$\mathbb{E}[X_s X_t] = \left\langle \chi_{[0,s]}, \sum_{i,j}\langle \chi_{[0,t]}\varphi_{ij}\rangle\varphi_{ij}\right\rangle$$
$$= \langle \chi_{[0,s]}, \chi_{[0,t]}\rangle = \min\{s, t\}$$

as required.

□

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