The Heston Model

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Abstract

This report accompanies the code in the corresponding GitHub repository. We present an implementation of the famous *Heston Model*, a stochastic volatility model for European option pricing. In the model, the volatility follows an Ornstein-Uhlenbeck process, while the underlying follows a process resembling a Geometric Brownian Motion. We simulate many paths for the volatility and the underlying processes, using Monte Carlo simulation to compute a range of call and put option prices for various strikes and maturities. We compute the implied volatility as a function of strike price for calls and puts, showing that the Heston model captures volatility smiles. We also plot volatility surfaces from the model.

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1 Introduction

The Heston model is a stochastic volatility model consisting of two processes; the volatility follows a mean-reverting Ornstein-Uhlenbeck process, while the stock price follows a process resembling a Geometric Brownian Motion, albeit with non-constant volatility. The canonical Black-Scholes options pricing model assumes the highly unrealistic assumption of a constant volatility, and so the Heston model represents a significant improvement over the Black-Scholes model.

The model captures a number of features that Black-Scholes does not. One of the most striking improvements from the Black-Scholes model is the prediction of a non-flat implied volatility curve. Implied volatility (IV) is the volatility implied by real-world option prices in the market, by "working backwards" through the Black-Scholes model. Real-world IV is a function of strike price K and time to maturity T; if we plot the IV against K we see a phenomenon known as a *Volatility Smile/Smirk/Skew* (depending on the shape of the curve). A volatility smile, for instance, indicates that deep in-the-money and out-of-the-money options tend to have a higher implied volatility than those at-the-money.

If we use the Black-Scholes model to compute option prices, imagine that those prices are market prices, and work back through the Black-Scholes model to get an IV, we will trivially get that the IV is constant since constant volatility is an assumption of the Black-Scholes model. In particular, we find that IV is not a function of the strike price or time to maturity. Thus, we say that the Black-Scholes model predicts a flat implied volatility curve. This is in contrast to the Heston model; if we compute option prices with the Heston model and then use these to work backwards through Black-Scholes to get an IV, we find that the IV depends non-trivially on both the strike price and the time to maturity. In this project we use Monte Carlo simulation to compute the prices of various European call and put options, before computing the corresponding implied volatility smiles and surfaces.

2 The Heston Model

The starting point for the Heston model is two Stochastic Differential Equations (SDEs),

$$dS_t = rS_t dt + \sqrt{\nu_t} S_t dW_t^{S,\rho}, \tag{2.1}$$

$$d\nu_t = \kappa(\theta - \nu_t)dt + \xi\sqrt{\nu_t} dW_t^{\nu,\rho}, \tag{2.2}$$

representing the underlying stock price and the volatilty of the stock, respectively. The two Brownian Motions are taken to be correlated, with pearson correlation coefficient ρ. The other parameters in the model are

- r - the risk-free rate. This is the rate at which a riskless instrument such as a bond or a bank account appreciates over time.
- κ - the mean-reversion rate. This captures the speed at which the volatility process will move back towards its long-term mean after perturbations away from said mean.
- θ - the long-term mean of the volatility.

- ξ - the volatility of the volatility process.

Note that here we are using the risk-neutral measure $\mathbb{Q}$, under which the stock price is a $\mathbb{Q}$-martingale and the price of an option can be computed as the expectation value of the option's payoff function at maturity, discounted to today[1]

The leverage effect is a phenomenon whereby periods of high volatility tend to result in negative asset returns; namely that prices and volatility are inversely correlated. In the Heston model we capture this effect by choosing the correlation coefficient ρ, characterising the correlation between the two Brownian motions, to be negative.

3 Simulation

3.1 Generating pairs of correlated Brownian Motion increments

The first step in the simulation is to generate M pairs of correlated Brownian Motions to use in our SDEs, where M is the number of paths we intend on simulating.

First, consider a real-valued bivariate normally distributed random vector $\vec{X}$,

$$\vec{X} = \begin{pmatrix} X \\ Y \end{pmatrix} \sim \mathcal{N}(\vec{\mu}, \Sigma), \tag{3.1}$$

where

$$\vec{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \tag{3.2}$$

is a vector of the means of X and Y, and

$$\Sigma = \begin{pmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{YX} & \sigma_Y^2, \end{pmatrix} \tag{3.3}$$

is the covariance matrix. We use the notation

$$\sigma_{XY} \equiv \mathbb{C}\text{ov}(X, Y). \tag{3.4}$$

For $\vec{X}$ a bivariate normally distributed random vector, a fact is that the components of $\vec{X}$ are also normally distributed. Namely in our case,

$$\vec{X} \sim \mathcal{N}(\vec{\mu}, \Sigma) \implies X \sim \mathcal{N}(\mu_1, \sigma_X^2), \;\; Y \sim \mathcal{N}(\mu_2, \sigma_Y^2). \tag{3.5}$$

In the case pertinent to the Heston model, we are interested in $X \sim \mathcal{N}(0, \Delta t)$ and $Y \sim \mathcal{N}(0, \Delta t)$, since Brownian motion increments are distributed as

$$W_{t+\Delta t} - W_t \sim \mathcal{N}(0, \Delta t). \tag{3.6}$$

[1]We are perhaps not starting at step 0 here, which would be to begin by writing down the SDEs under the real-world measure and then performing a change of measure to $\mathbb{Q}$. We skip this step for simplicity.

Given $\vec{X} \sim \mathcal{N}(\vec{\mu}, \Sigma)$, how do we define a random vector $\vec{\hat{X}}$ such that its components are distributed as $\hat{X} \sim \mathcal{N}(0, \Delta t)$ and $\hat{Y} \sim \mathcal{N}(0, \Delta t)$?

Firstly, we require that $\vec{\mu} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. We also require

$$\Sigma = \begin{pmatrix} \Delta t & \sigma_{XY} \\ \sigma_{YX} & \Delta t, \end{pmatrix} \tag{3.7}$$

$$= \Delta t \begin{pmatrix} 1 & \frac{\sigma_{XY}}{\Delta t} \\ \frac{\sigma_{YX}}{\Delta t} & 1, \end{pmatrix} \tag{3.8}$$

$$= \Delta t \begin{pmatrix} 1 & \rho \\ \rho & 1, \end{pmatrix} \tag{3.9}$$

$$\equiv \Delta t \, \tilde{\Sigma}, \tag{3.10}$$

where we used that the Pearson correlation coefficient is defined to be

$$\rho := \frac{\sigma_{XY}}{\sigma_X \sigma_Y}. \tag{3.11}$$

Trivially, we have that

$$\vec{X} \sim \mathcal{N}(\vec{\mu}, \Sigma) \iff \vec{X} \sim \mathcal{N}(\vec{\mu}, \Delta t \, \tilde{\Sigma}). \tag{3.12}$$

We now define a new random vector

$$\vec{\hat{X}} := \frac{1}{\sqrt{\Delta t}} \vec{X}, \tag{3.13}$$

and ask how $\vec{\hat{X}}$ is distributed. Using

$$\mathbb{V}\mathrm{ar}(\alpha M) = \alpha^2 \mathbb{V}\mathrm{ar}(M), \tag{3.14}$$

$$\mathbb{C}\mathrm{ov}(\alpha M, \beta N) = \alpha\beta \mathbb{C}\mathrm{ov}(M, N), \tag{3.15}$$

$$\tag{3.16}$$

it can be checked that the covariance matrix corresponding to $\vec{\hat{X}}$ is exactly $\tilde{\Sigma}$, and that $\vec{\hat{X}}$ has mean $\vec{\mu} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Therefore, we have found that if we have a random vector $\vec{\hat{X}} \sim \mathcal{N}(\vec{0}, \tilde{\Sigma})$ and we define $\vec{X} := \sqrt{\Delta t} \vec{\hat{X}}$, the components of $\vec{X}$ will be distributed as

$$X \sim \mathcal{N}(0, \Delta t), \; Y \sim \mathcal{N}(0, \Delta t). \tag{3.17}$$

The bottom line is that if we generate pairs of correlated stochastic processes by sampling from a bivariate normal distribution

$$\mathcal{N}\left(\vec{0}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right), \tag{3.18}$$

and we multiply each individual process by $\sqrt{\Delta t}$, the result will have distribution $\mathcal{N}(0, \Delta t)$. This is exactly what we do in the code to generate pairs of correlated Brownian motion increments.

3.2 Code walk-through and plots

The code opens by defining some parameters for generating correlated Brownian motions, as well as for simulating the volatility and stock price processes. The time until expiry of the contract T is taken to be 1 year, and the number of time steps we simulate over is taken to correspond to 252 trading days. The function *heston_model* takes the parameters of the model as inputs, and returns the stock price process and the volatility process as outputs. It begins by defining the time step

$$\Delta t := \frac{T}{N}, \tag{3.19}$$

where N is the total number of steps, before defining the mean and covariance matrices $\vec{\mu} = \vec{0}$ and $\tilde{\Sigma}$ defined in the previous subsection.

We then generate $N \times M$ pairs of correlated Brownian motion increments by taking $N \times M$ draws from the bivariate normal distribution (where M is the number of pairs of paths we want to simulate). The result is a numpy array of shape $(N, M, 2)$, and so slicing appropriately we can split the array into two seperate arrays; one consisting of Brownian motion paths for the volatility process and one for the underlying process, each correlated with Pearson correlation coefficient ρ. An example plot for a single pair of correlated processes is shown in figure 1. We then begin the

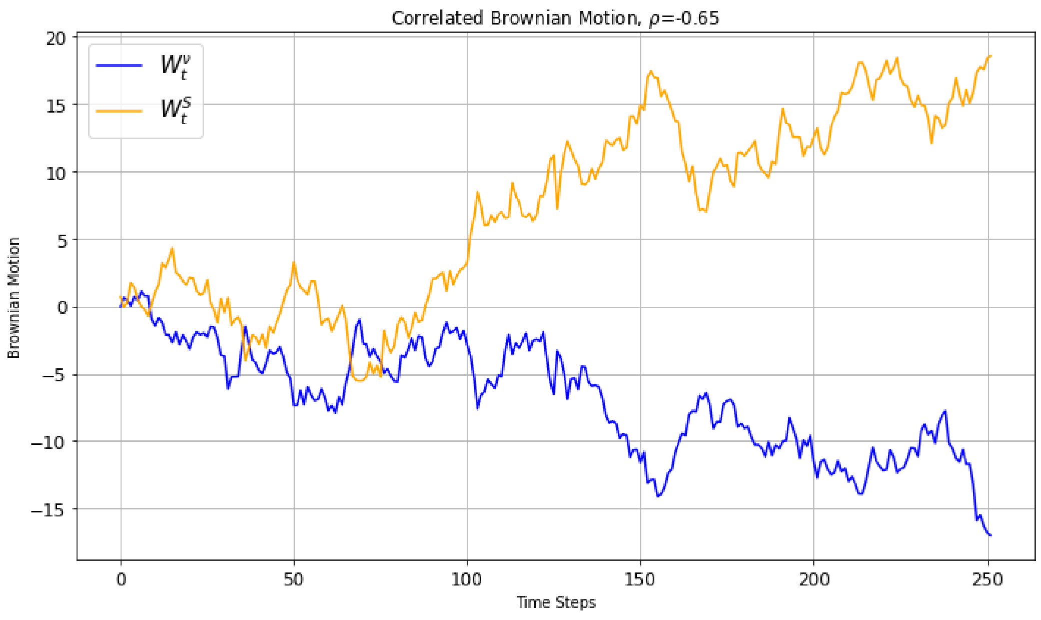


Figure 1: A pair of correlated Brownian motion increments, with correlation coefficient $\rho = -0.65$. We choose a negative ρ in order to capture the leverage effect.

Monte Carlo method for solving the SDEs. We use the Euler discretisation scheme, whereby 2.1 and

2.2 become

$$S_{t+\Delta t} = S_t + rS_t\Delta t + \sqrt{\nu_t}S_t\underbrace{(W^S_{t+\Delta t} - W^S_t)}_{\sim\mathcal{N}(0,\Delta t)}, \tag{3.20}$$

$$\nu_{t+\Delta t} = \nu_t + \kappa(\theta - \nu_t)\Delta t + \xi\sqrt{\nu_t}\underbrace{(W^\nu_{t+\Delta t} - W^\nu_t)}_{\sim\mathcal{N}(0,\Delta t)}. \tag{3.21}$$

The code looks slightly different to this, with the stochastic term in each multiplied by $\sqrt{\Delta t}$, for the reason described in the previous subsection. The function closes by returning the M price and volatility paths. We then plot these, resulting in figure 2.

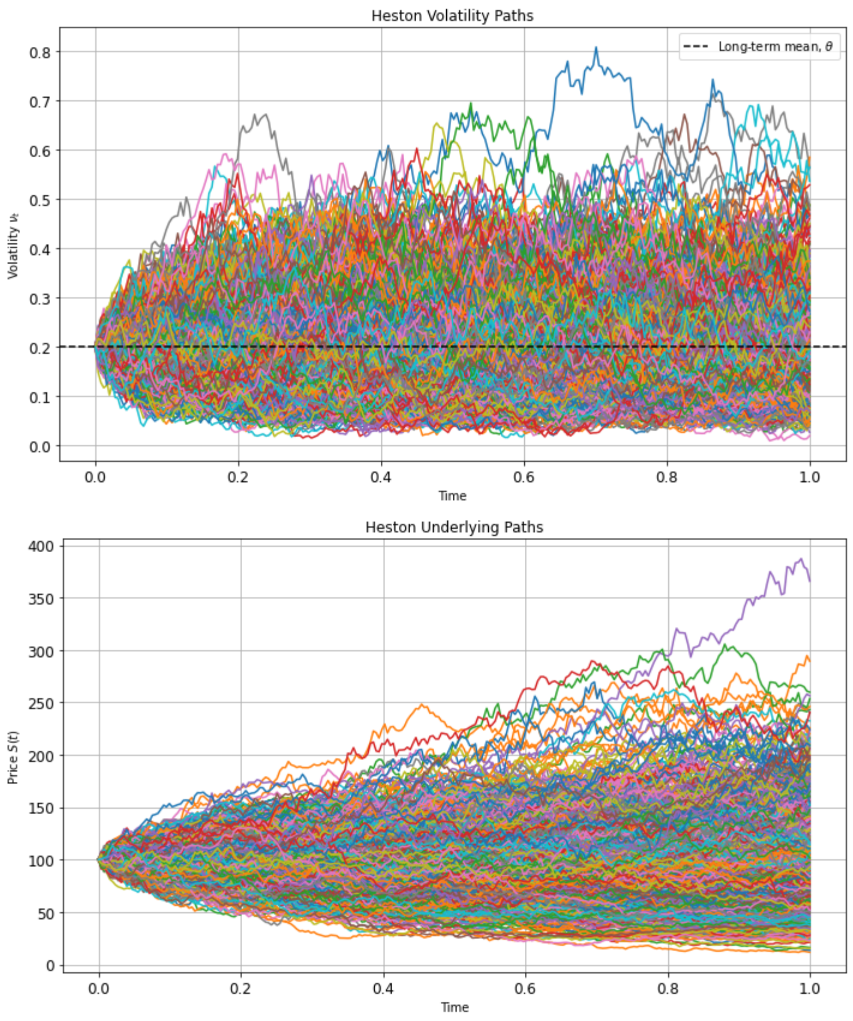


Figure 2: The volatility and underlying stock paths computed under the Heston model.

From here we compute prices of calls and puts for a variety of strike prices. We use that under the risk-neutral measure the price of an option is the expected value of the payoff function, discounted to today. Namely, we compute

$$V(T) = e^{-rT}\mathbb{E}_{\mathbb{Q}}[\max(0, f(S_T, K))], \tag{3.22}$$

where

$$f(S_T, K) := \begin{cases} S_T - K \text{ for a call} \\ K - S_T \text{ for a put,} \end{cases} \tag{3.23}$$

simply by taking the mean of the final prices of each underlying path. We then use these prices to compute the implied volatility for calls and puts using the *py_vollib_vectorized* package, and plot the results in figure 3.

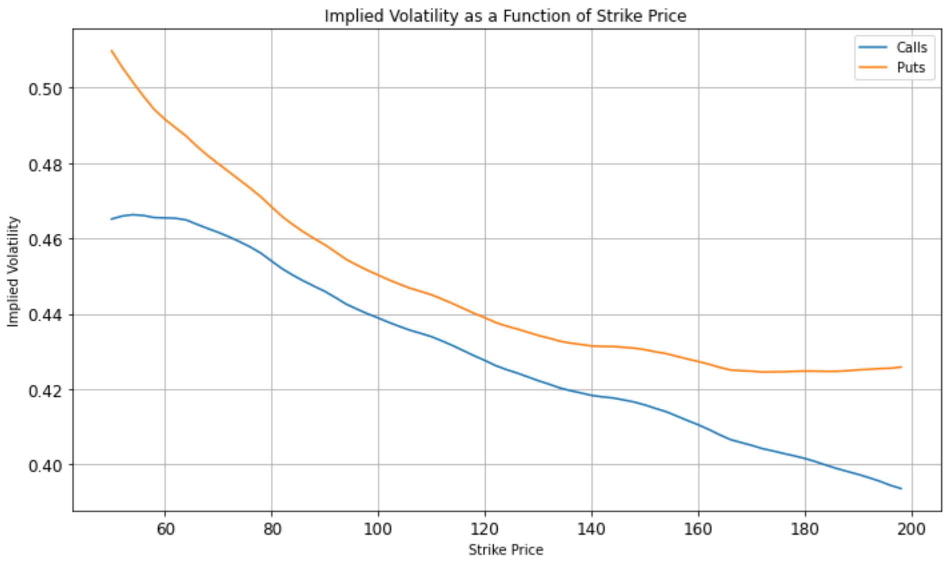
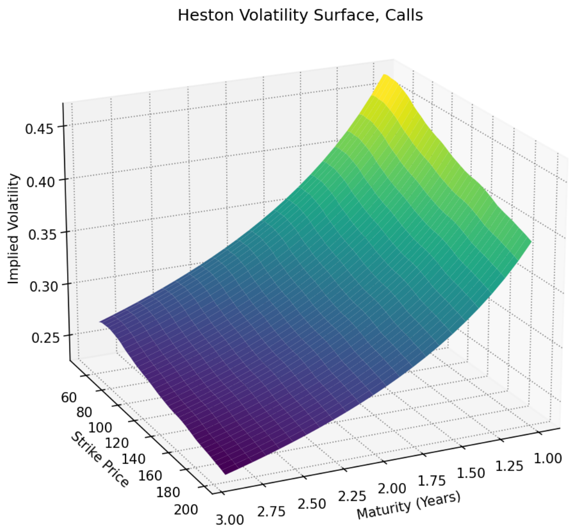


Figure 3: Implied volatility for call and put options as a function of strike price under the Heston model. These plots were generated with a time to maturity of 1 year. We see that the model captures IV skews.

Finally, we compute implied volatilities as a function of both the strike price and the time to maturity, and plot the resulting volatility surfaces for calls and puts. These are shown in figure 4.

4 Conclusion

In this report, we detailed an implementation of the Heston model - a stochastic volatility model for options pricing. We used it to simulate the dynamics of an underlying stock process and its volatility via Monte Carlo simulation, before computing prices of European call and put options. We then computed the implied volatility for a variety of strike prices and maturities, and showed that the Heston model captures IV skews.



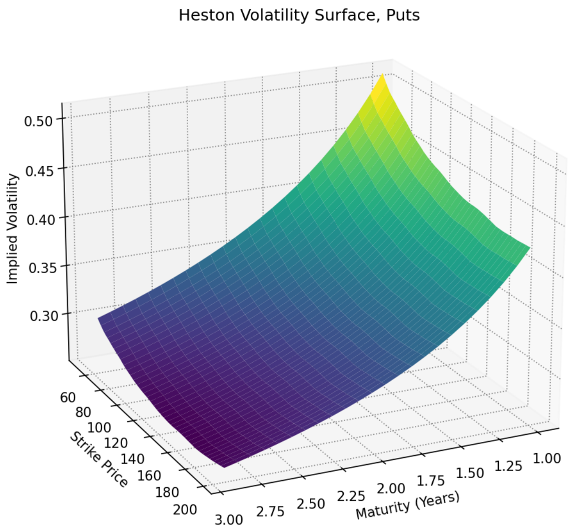


Figure 4: Implied volatility surfaces for call and put options under the Heston model.