Mean-Variance Portfolio Optimization

Abstract

This report accompanies the code in the corresponding GitHub repository. We present a mean-variance analysis of a portfolio of equities, generating many random weightings and producing a scatter plot of the expected returns and variances of the corresponding portfolios. We see that this produces the expected Markowitz bullet. We also compute the feasible set; the boundary of the Markowitz bullet and the set of portfolios for which the variance is minimized for a given expected return. We then plot the efficient frontier; a subset of the feasible portfolios that maximises return for a given level of risk. Finally, we compare the performance of a randomly selected portfolio (away from the feasible set) against a portfolio of comparable risk on the efficient frontier.

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1 Introduction

Portfolio optimisation is a mathematical approach to the construction of a portfolio that achieves certain specified objectives. For instance, by using some portfolio optimisation technique, an investor can work out how much of their capital they need to invest in each asset (termed the *portfolio weights*) in order to achieve a desired level of expected return. Alternatively, the investor may want to find portfolio weights that maximise their expected return for a given level of risk that they're willing to take on.

One approach to portfolio optimisation, and the one we hope to elucidate in this report, is so-called *Mean-Variance* or *Markowitz Optimisation* [1]. Markowitz optimisation is a set of techniques that allow an investor to construct a portfolio of maximised expected return for some given level of risk, or alternatively a portfolio of minimised risk for some target level of expected return. A portfolio that gives the maximum possible expected return for a given level of risk, or achieves the minimum possible risk for a given target expected return, is said to be an *Efficient Portfolio*. The naïve investor's intuition may indicate that the relationship between risk and return is linear; namely that in order to achieve higher expected returns an investor must take on more risk. In fact, using Markowitz optimisation one can show that the relationship between risk (which is taken to be the variance in portfolio return) and return is parabolic, not linear. In this way, it is possible for an investor to increase expected returns while taking on the same amount of risk with respect to some randomly selected portfolio[1].

In this report, we perform a mean-variance analysis of a small portfolio of assets. In chapter 2 we explain the optimisation procedure, for both a 2-asset portfolio and a portfolio of an arbitrary number of assets under the constraint that the investor wishes to achieve a particular target return. In chapter 3 we give an overview of the code in the corresponding GitHub repository, and present the resulting plots showing the Markowitz bullet, the feasible set of portfolios and the efficient frontier. We also compare the performance of a randomly selected portfolio in the Markowitz bullet against a portfolio of the same risk on the efficient frontier. We conclude with a summary of the report and a few directions for future work.

[1] If the randomly selected portfolio is particularly "bad", the investor can even *decrease* their risk while simultaneously *increasing* their expected returns by choosing the portfolio weights more carefully. This will be elucidated later in the report.

2 Mean-Variance Optimisation

Markowitz optimisation begins with a number of basic assumptions, the most important of which being:

1. The only source of risk is the variance in the portfolio's returns.
2. The investor is rational (they will never invest in a particular portfolio if there exists another with higher returns and the same risk) and risk-averse (they would prefer to decrease risk if possible).
3. Analysis is based on a single period of investment, during which the parameters of the probability distribution from which asset returns are drawn are constant[2].
4. An investor wants to either maximise return for a desired level of risk, or minimise risk for a desired level of return.

As a simple illustrative example of the optimisation procedure, we give a pen-and-paper analysis of a 2-asset portfolio, aiming to minimise the risk without any constraint on the expected return. After this, we will again minimise the risk, but this time for an m-asset portfolio under the constraint that we wish to achieve a certain target return, α_0.

2.1 2-Asset Portfolio - Minimising Risk

Consider a portfolio consisting of just two assets, with returns R_1 and R_2 (both random variables). We represent the return of the portfolio by a random vector

$$\vec{R} := (R_1, R_2), \tag{2.1}$$

with the expected return being given by

$$\mathbb{E}[\vec{R}] \equiv \vec{\alpha} = (\alpha_1, \alpha_2)^T \equiv (\mathbb{E}[R_1], \mathbb{E}[R_2])^T. \tag{2.2}$$

We also compute the covariance matrix of the returns,

$$\mathbb{C}\mathrm{ov}(\vec{R}) \equiv \Sigma = \begin{pmatrix} \mathbb{V}\mathrm{ar}(R_1) & \mathbb{C}\mathrm{ov}(R_1, R_2) \\ \mathbb{C}\mathrm{ov}(R_1, R_2) & \mathbb{V}\mathrm{ar}(R_2) \end{pmatrix} \equiv \begin{pmatrix} \sigma_1^2 & \mathbb{C}\mathrm{ov}(R_1, R_2) \\ \mathbb{C}\mathrm{ov}(R_1, R_2) & \sigma_2^2 \end{pmatrix}. \tag{2.3}$$

We define a *Portfolio Weighting* to be a vector of weights characterising the fraction of the portfolio held in each asset, namely

$$\vec{w} = (w_1, w_2), \tag{2.4}$$

with the constraint that

$$\sum_{i=1}^{2} w_i \overset{!}{=} 1. \tag{2.5}$$

[2]Namely, the expected return and variance of the portfolio are taken to be constant over the investment period.

For the simple case of two assets, we therefore have

$$w_1 \equiv w, \;\; w_2 = 1 - w. \tag{2.6}$$

We can compute the return of the portfolio as a whole by taking the dot product of the weight vector with the return vector;

$$R_w := \vec{w}^T \vec{R} = \sum_{i=1}^{2} w_i R_i. \tag{2.7}$$

R_w is thus a random vector, with the expected return of the portfolio given by

$$R_w = (1-w)R_1 + wR_2 \implies \mathbb{E}[R_w] = (1-w)\alpha_1 + w\alpha_2. \tag{2.8}$$

The variance of the portfolio, which we take as a proxy for risk, is then given by

$$\mathbb{V}\mathrm{ar}(R_w) \equiv \sigma_w^2 = (1-w)^2\sigma_1^2 + w^2\sigma_2^2 + 2w(1-w)\mathbb{C}\mathrm{ov}(R_1, R_2). \tag{2.9}$$

Let the two assets in the portfolio have Pearson correlation ρ. We then have

$$\sigma_w^2 = (1-w)^2\sigma_1^2 + w^2\sigma_2^2 + 2w(1-w)\rho\sigma_1\sigma_2. \tag{2.10}$$

Now, suppose the investor wants to minimise the risk they're taking on by investing in the portfolio, with no constraint on the return they'd like to achieve. We can minimise the portfolio variance by computing

$$\frac{\partial \sigma_w^2}{\partial w} \overset{!}{=} 0 \implies w = \frac{\sigma_1^2 - \sigma_1\sigma_2\rho}{\mathbb{V}\mathrm{ar}(R_1 - R_2)} \tag{2.11}$$

Consider the case that the assets are perfectly uncorrelated; $\rho = 0$. In this case, we find that the expected return of the portfolio is

$$\mathbb{E}[R_{w_{\rho=0}}] = \frac{1}{\sigma_1^2 + \sigma_2^2}\left(\sigma_2^2\alpha_1 + \sigma_1^2\alpha_2\right). \tag{2.12}$$

Let the parameters have the following values:

$$\mathbb{E}[R_1] \equiv \alpha_1 = 0.15, \;\; \sigma_1 = 0.2 \tag{2.13}$$

$$\mathbb{E}[R_2] \equiv \alpha_2 = 0.25, \;\; \sigma_2 = 0.3. \tag{2.14}$$

Then, the expected return and variance of the portfolio are given by

$$\mathbb{E}[R_{w_{\rho=0}}] \approx 0.18, \;\; \sigma^2_{w_{\rho=0}} \approx 0.028. \tag{2.15}$$

We thus notice something interesting - if we'd chosen to just invest in asset 1, we'd achieve an expected return of 0.15 and a variance of $0.2^2 = 0.04$. However, by investing in the two-asset portfolio and choosing the weights to minimise the portfolio variance, we can achieve higher returns despite taking on lower risk.

2.2 m-Asset Portfolio with Constraint on Return

Let us now consider the case of an m asset portfolio, where again we would like to minimise the risk of the portfolio. However, here we would like to achieve a certain target expected return α_0, and minimise the risk subject to this target return. In other words, we are looking for a solution to the problem of *"what's the smallest amount of risk I need to take on to achieve my target return?"*.

We can phrase this as a constrained optimisation problem. Namely, we aim to

$$\begin{aligned} &\text{Minimise: } \mathbb{V}\text{ar}(R_w) \equiv \sigma_w^2 = \vec{w}^T \Sigma \vec{w}, \\ &\text{Subject to Constraints: } \vec{w}^T \vec{\alpha} = \alpha_0, \;\; \vec{w}^T \mathbb{I} = 1. \end{aligned} \tag{2.16}$$

To solve this problem, we can use the method of Lagrange multipliers, first writing down the Lagrangian

$$\mathcal{L}(\vec{w}, \lambda_1, \lambda_2) = \frac{1}{2} \vec{w}^T \Sigma \vec{w} + \lambda_1 (\vec{w}^T \vec{\alpha} - \alpha_0) + \lambda_2 (\vec{w}^T \mathbb{I} - 1). \tag{2.17}$$

Now, we take the first derivatives with respect to the parameters[3]

$$\frac{\partial \mathcal{L}}{\partial \vec{w}} = \Sigma \vec{w} + \lambda_1 \vec{\alpha} + \lambda \mathbb{I} \stackrel{!}{=} 0, \tag{2.18}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_1} = \vec{w}^T \vec{\alpha} - \alpha_0 \stackrel{!}{=} 0, \tag{2.19}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_2} = \vec{w}^T \mathbb{I} - 1 \stackrel{!}{=} 0. \tag{2.20}$$

Solving the first equation results in

$$\vec{w}_0 = \lambda_1 \Sigma^{-1} \vec{\alpha} + \lambda_2 \Sigma^{-1} \mathbb{I}, \tag{2.21}$$

with the corresponding minimum variance being given by

$$\sigma_0^2 = \vec{w}^T \Sigma \vec{w} \tag{2.22}$$

$$= \ldots$$

$$= \lambda_1^2 \vec{\alpha}^T \Sigma^{-1} \vec{\alpha} + \lambda_2^2 \mathbb{I}\, \Sigma^{-1} \mathbb{I} + 2\lambda_1 \lambda_2 \vec{\alpha}^T \Sigma^{-1} \mathbb{I} \tag{2.23}$$

$$= (\lambda_1, \lambda_2) \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}, \tag{2.24}$$

with

$$a := \vec{\alpha}^T \Sigma^{-1} \vec{\alpha}, \quad b := \vec{\alpha}^T \Sigma^{-1} \mathbb{I}, \quad c := \mathbb{I}\, \Sigma^{-1} \mathbb{I}. \tag{2.25}$$

Solving the second and third equations gives

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}^{-1} \begin{pmatrix} \alpha_0 \\ 1 \end{pmatrix}, \tag{2.26}$$

and thus we find that

$$\sigma_0^2 = \frac{1}{ac - b^2} (c\alpha_0^2 - 2b\alpha_0 + a). \tag{2.27}$$

[3] Note that we are indeed *minimising* rather than *maximising* here; the second derivative of the Lagrangian with respect to the weights gives $\frac{\partial^2 \mathcal{L}}{\partial \vec{w} \partial \vec{w}^T} = \Sigma$, which by definition of a covariance matrix is non-negative.

Namely, we see that the relationship between the minimum variance σ_0^2 and the target return α_0 is not linear, but rather traces out a parabola in the (σ_0^2, α_0) plane.

In the following section, we present the results of a simulation of a large number of portfolios, and demonstrate that indeed the relationship between σ_0^2 and α_0 is parabolic.

3 Code & Results

In this section we give an overview of the code, and present the results of the Mean-Variance analysis.

The portfolio we are aiming to optimise consists of seven assets; SPY, QQQ, AMZN, AAPL, HES, KO, and MRK. The code begins by defining a number of functions that will prove useful in the analysis later on, for such purposes as downloading historical price data, computing random portfolio weights, and computing portfolio performance.

We compute the expected log returns for the assets in the portfolio, as well as their covariance matrix. We then compute a matrix of $70,000$ different sets of portfolio weights, before calculating the expected return, variance and Sharpe ratio for each corresponding portfolio.

We then plot the variance and expected return of each portfolio on a scatter plot, which results in the so-called "Markowitz Bullet". The plot is shown in figure 1.

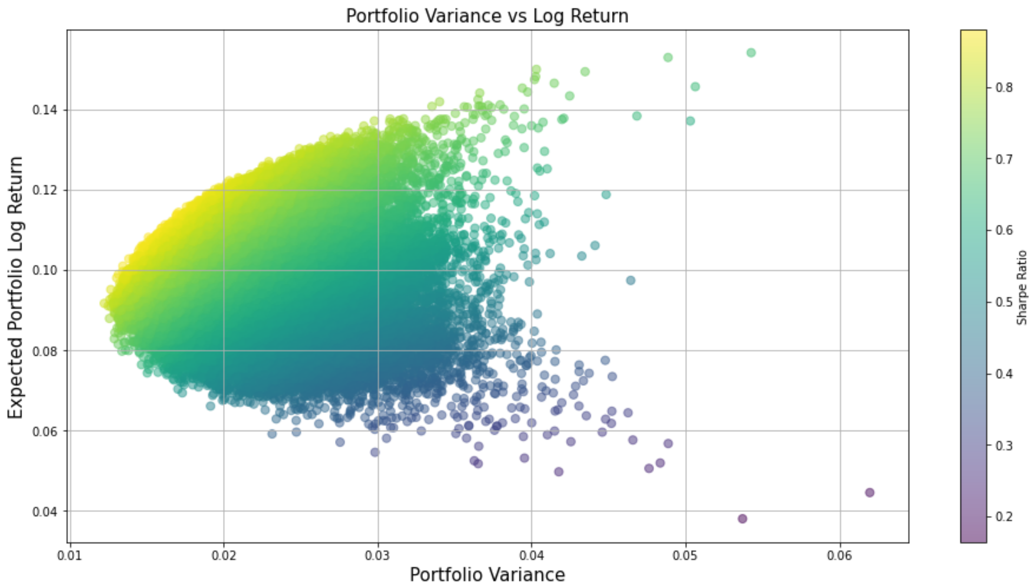

Figure 1: A plot showing the variance and expected return of the 70,000 random portfolios, resulting in the classic "Markowitz Bullet" shape. The plot is colour-coded according to Sharpe ratio.

The next part of the code computes the *feasible set*, the set of portfolios that minimise the risk for a given target return with no regard to rationality of the investor. We use the *Optimize* package from *Scipy* to solve the constrained minimisation problem defined in the previous section, and plot the feasible set on top of the scatter plot in figure 1. The resulting plot is shown in figure 2.

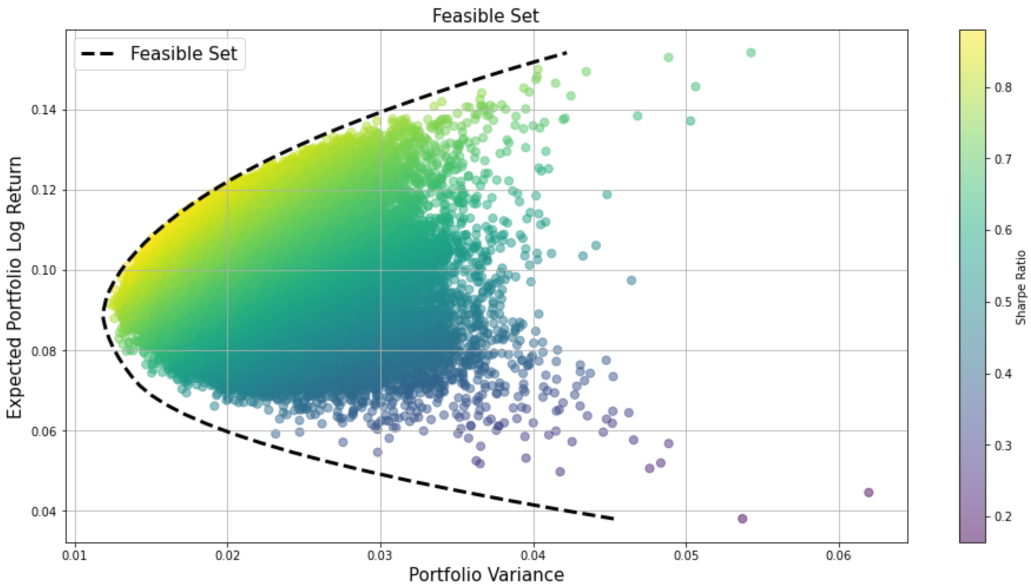

Figure 2: A plot showing the feasible set of portfolios. We see that as expected, the feasible set forms a parabola in the (σ_0^2, α_0) plane.

Inspection of the feasible set reveals that any portfolio on the bottom half of the feasible set can, by more careful weighting, be pushed towards lower risk and higher returns. As such, a rational investor would never invest in such portfolios. This leads us to the idea of the *Efficient Frontier*, which enables us to compute the maximum possible return for a desired risk tolerance, or the lowest possible risk for a desired target return. Formally, the efficient frontier is the set of portfolios satisfying the condition that there exists no other portfolio with that particular variance, but a higher expected return. The efficient frontier is simply the upper-half of the feasible set, and it shown in figure 3.

Finally we compare the performance of a randomly selected portfolio in the Markowitz bullet, with the performance of a portfolio with the same risk but on the efficient frontier. We see that, as expected, the efficient portfolio performs better than the random portfolio. The relevant plot is shown in figure 4.

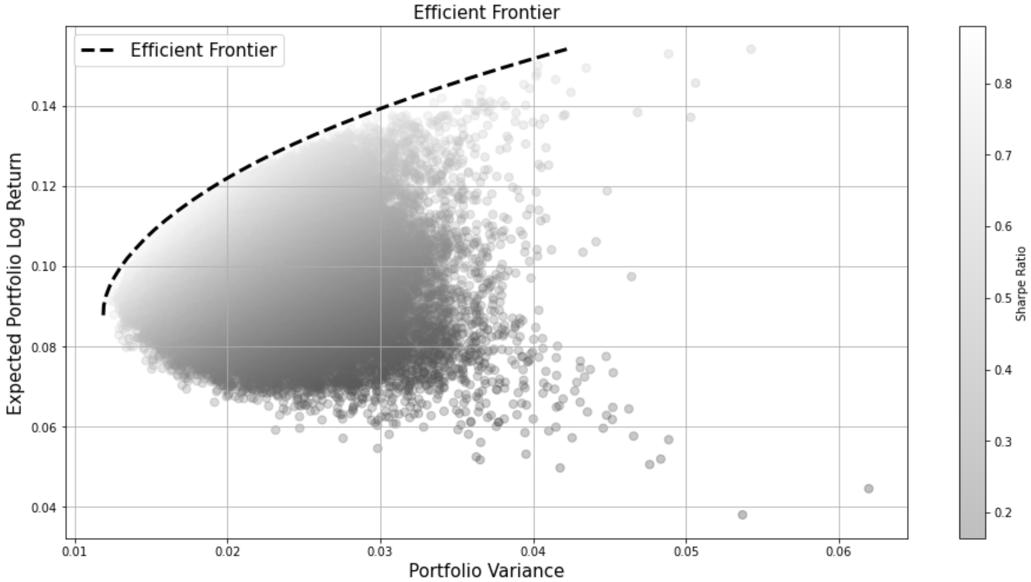

Figure 3: A plot showing the efficient frontier; the set of portfolios for which there exists no other portfolio with the same risk, but higher expected return.

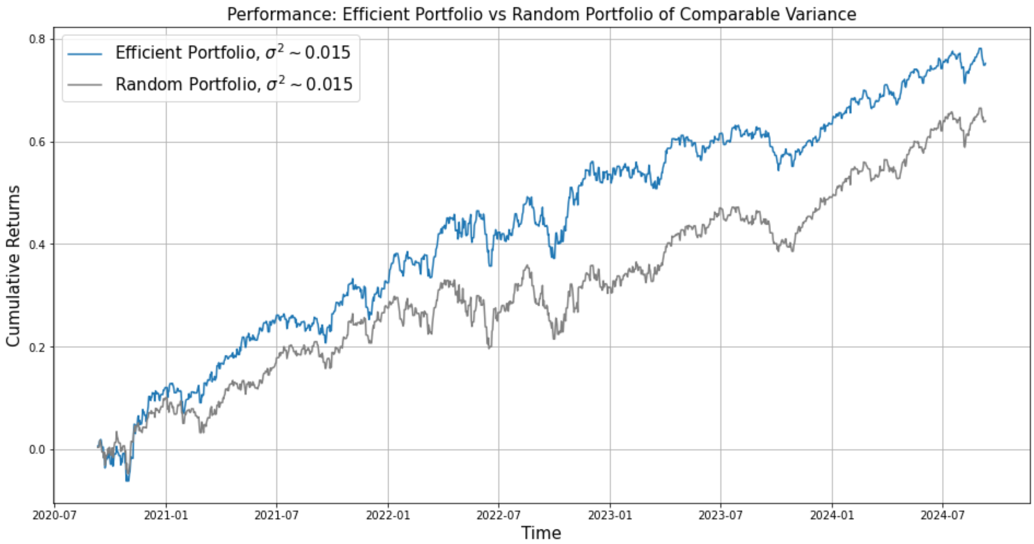

Figure 4: A plot showing the performance of a portfolio constructed with randomly selected weights, compared to that of an efficient portfolio of comparable risk. We see that the efficient portfolio, despite being of the same risk, performs better.

4 Conclusion

4.1 Short Summary

In this report, we performed a mean-variance analysis of a small portfolio of assets. We generated a large number of portfolio weights, computed the mean-variance characteristics of each resulting portfolio, and showed that the portfolios produce the characteristic Markowitz bullet. We found a set of weights that minimise the risk for a given target return, and plotted the resulting feasible set of portfolios. From this, we plotted the efficient frontier; the subset of the feasible set that a rational investor would consider investing in. Finally, we compared the performance of a randomly selected portfolio with that of an efficient portfolio of comparable risk, and found that the efficient portfolio performed better.

4.2 Directions for Further Work

We assumed in this report that all of our available capital is invested in the portfolio of risky assets. As such, we took the risk-free rate in our analysis to be $r = 0$. In the presence of a riskless asset the efficient frontier is not a parabola, but rather the upper part of a tangent line that touches the parabola at the highest possible Sharpe ratio. It would be interesting to repeat the above analysis for the case of an investor in a portfolio consisting of both risky and risk-free assets.

In addition, the above analysis uses portfolio variance (ultimately, random fluctuations in prices) as a proxy for risk, and makes no attempt to incorporate the *source* of such fluctuations into the model. It would be interesting to investigate whether or not other portfolio optimisation techniques allow for the incorporation of more sophisticated measures of risk.

References

[1] Markowitz, H. (1952), *Portfolio Selection*. The Journal of Finance, 7: 77-91. https://doi.org/10.1111/j.1540-6261.1952.tb01525.x