AUTOREGRESSIVE MODELS

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Autoregressive (AR) Models

- Often you can forecast a series based solely on the past values of Y_t.
- We are going to focus on the most basic case – only one lag value of Y_t – called an AR(1) model:

$$Y_t = \omega + \phi Y_{t-1} + e_t$$

Autoregressive (AR) Models

- This relationship between *t* and *t-1* exists for all one time period differences across the data set.

$$Y_t = \omega + \phi Y_{t-1} + e_t$$

$$Y_{t-1} = \omega + \phi Y_{t-2} + e_{t-1}$$

$$Y_{t-2} = \omega + \phi Y_{t-3} + e_{t-2}$$

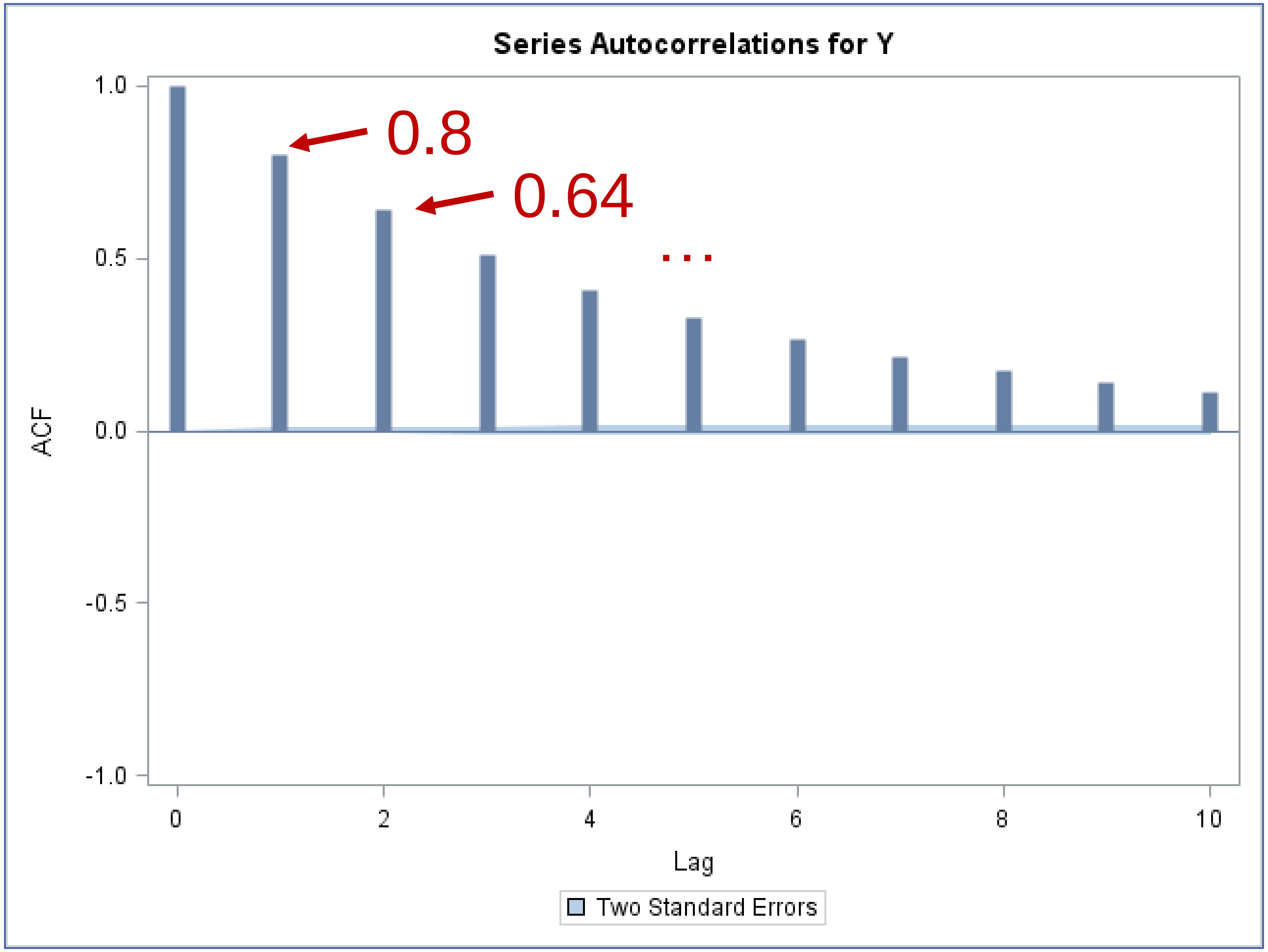
Correlation Functions for AR(1)

- The ACF decreases exponentially as the number of lags increases.
- The PACF has a significant spike at the first lag, followed by nothing after.
- The IACF has a significant spike at the first lag, followed by nothing after.

- Let's examine the following AR(1) model:

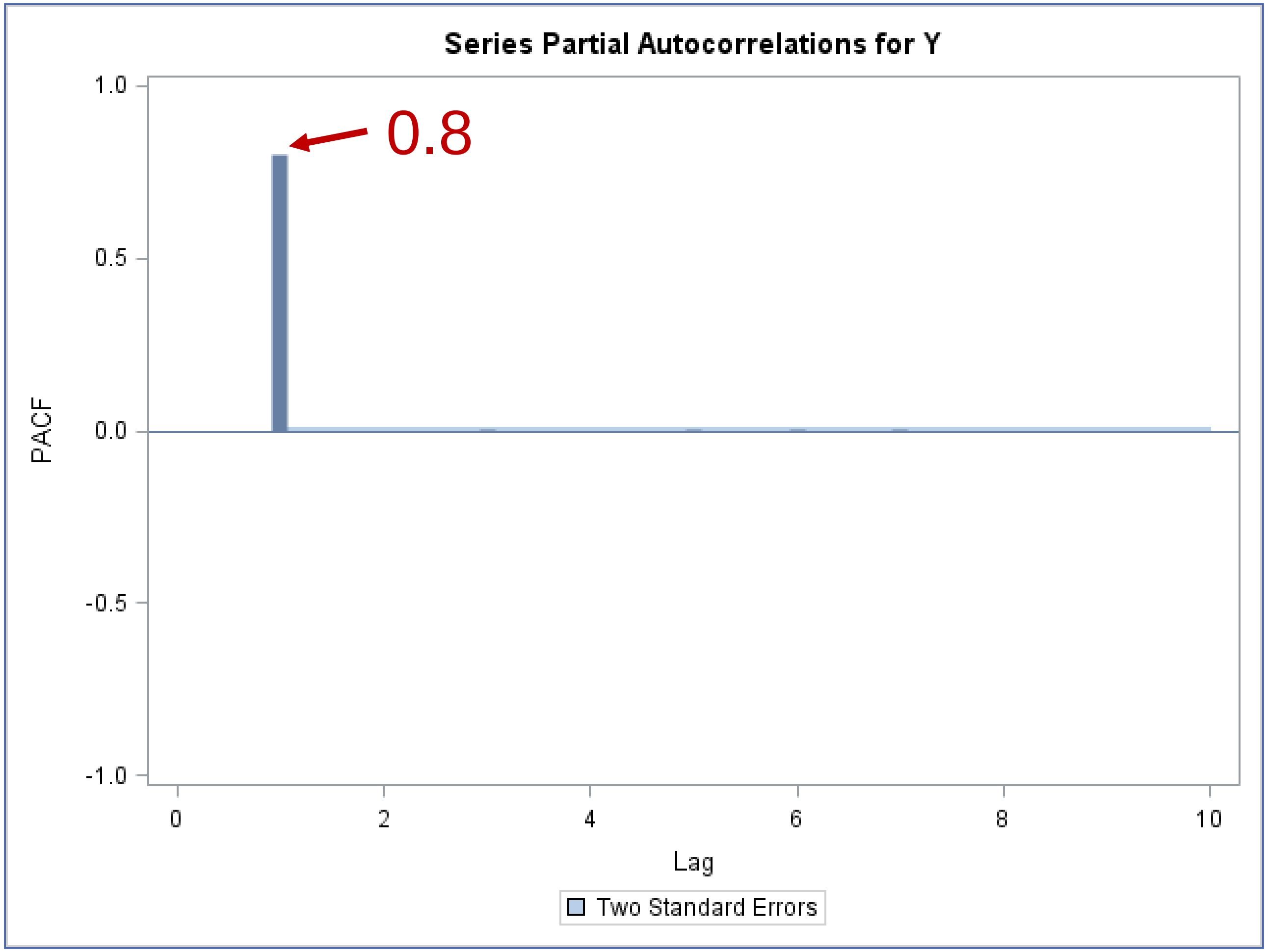
$$Y_t = 0 + 0.8Y_{t-1} + e_t$$

AR(1) – ACF

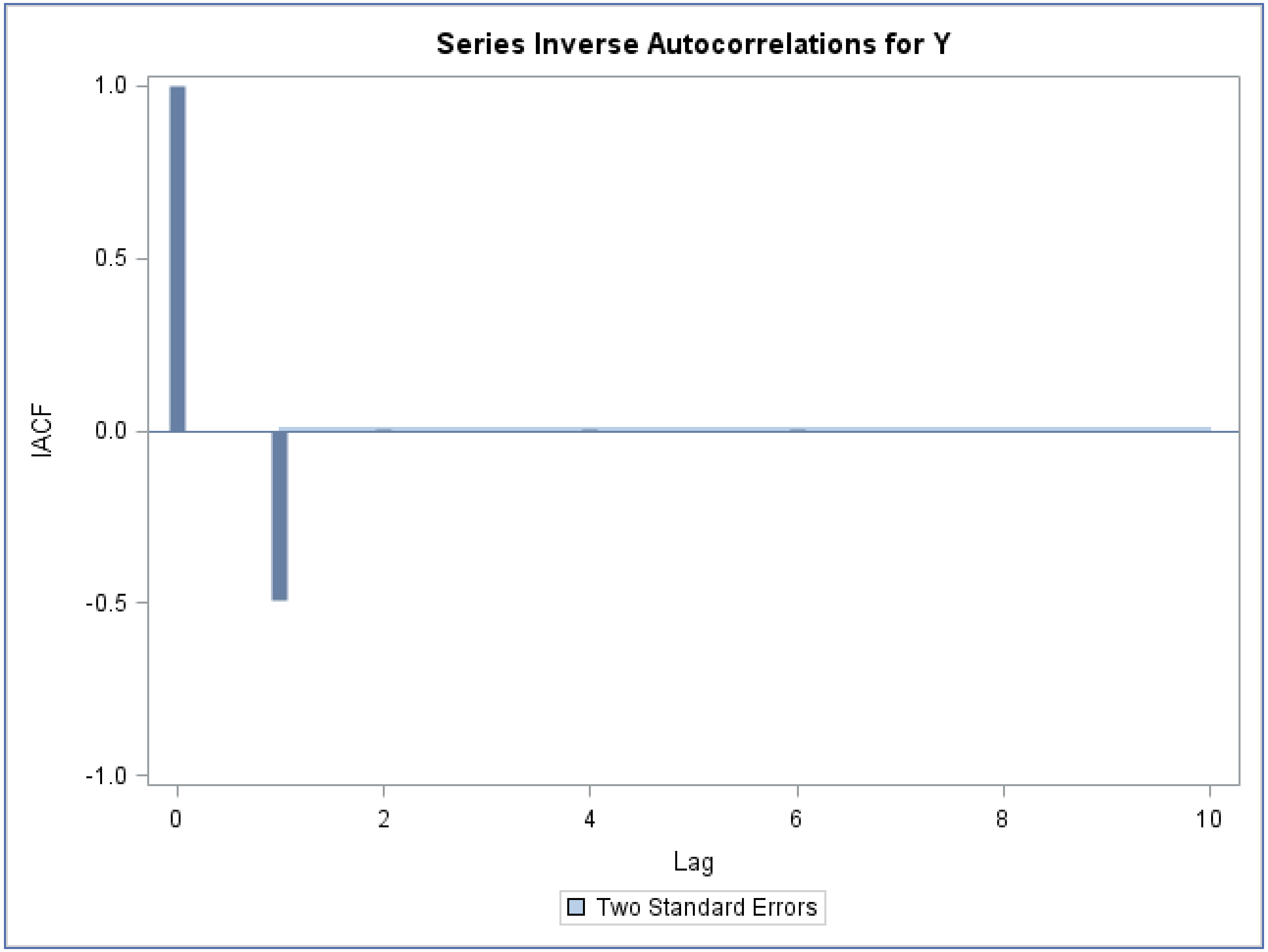


Series Autocorrelations for Y
1.0
0.5
0.0
-0.5
-1.0
ACF
0
2
4
6
8
10
Lag
Two Standard Errors
0.8
0.64
…

AR(1) – PACF


Series Partial Autocorrelations for Y
0.8
1.0
0.5
0.0
-0.5
-1.0
PACF
0
2
4
6
8
10
Lag
Two Standard Errors

AR(1) – IACF


Series Inverse Autocorrelations for Y
1.0
0.5
0.0
-0.5
-1.0
IACF
0
2
4
6
8
10
Lag
Two Standard Errors

Autoregressive (AR(1)) Models

- So the effect of shocks that happened long ago has little effect on the present *IF* the value for $|\phi| < 1$.
- This goes back to our idea of stationarity – the dependence of previous observations declines over time.
- There is a pattern for AR(1) models when it comes to stationarity.
- If ϕ=1, then Random Walk and NOT Autoregressive model
- If ϕ>1, then today depends on tomorrow (doesn't really make sense)

AR(2) Model

- A time series that is a linear function of *2* past values plus error is called an autoregressive process of order 2 – AR(2).

$$Y_t = \omega + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t$$

AR(2) Model

- There is a pattern for all AR models when it comes to stationarity.
- So the effect of shocks that happened long ago has little effect on the present *IF* the value for $|\phi_1 + \phi_2| < 1$.

AR(p) Model

- A time series that is a linear function of *p* past values plus error is called an autoregressive process of order p – AR(p).

$$Y_t = \omega + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t$$

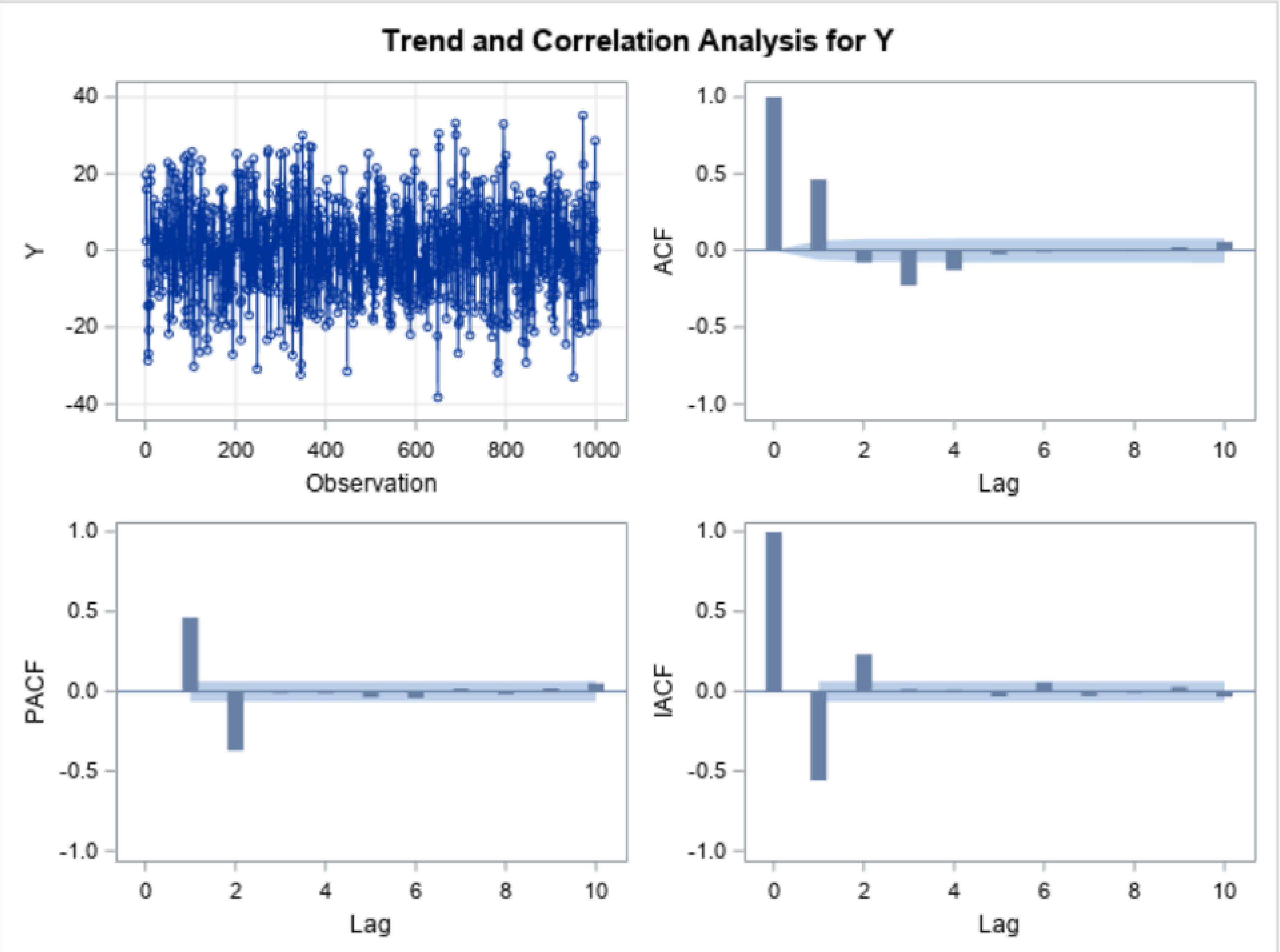
- More complicated restrictions on ϕ_i's (software will warn you when this becomes an issue)

Correlation Functions for AR(p)

- The ACF can have a variety of patterns.
- The PACF has a significant spike at the significant lags up to *p* lags, followed by nothing after.
- The IACF has a significant spike at the significant lags up to *p* lags, followed by nothing after.

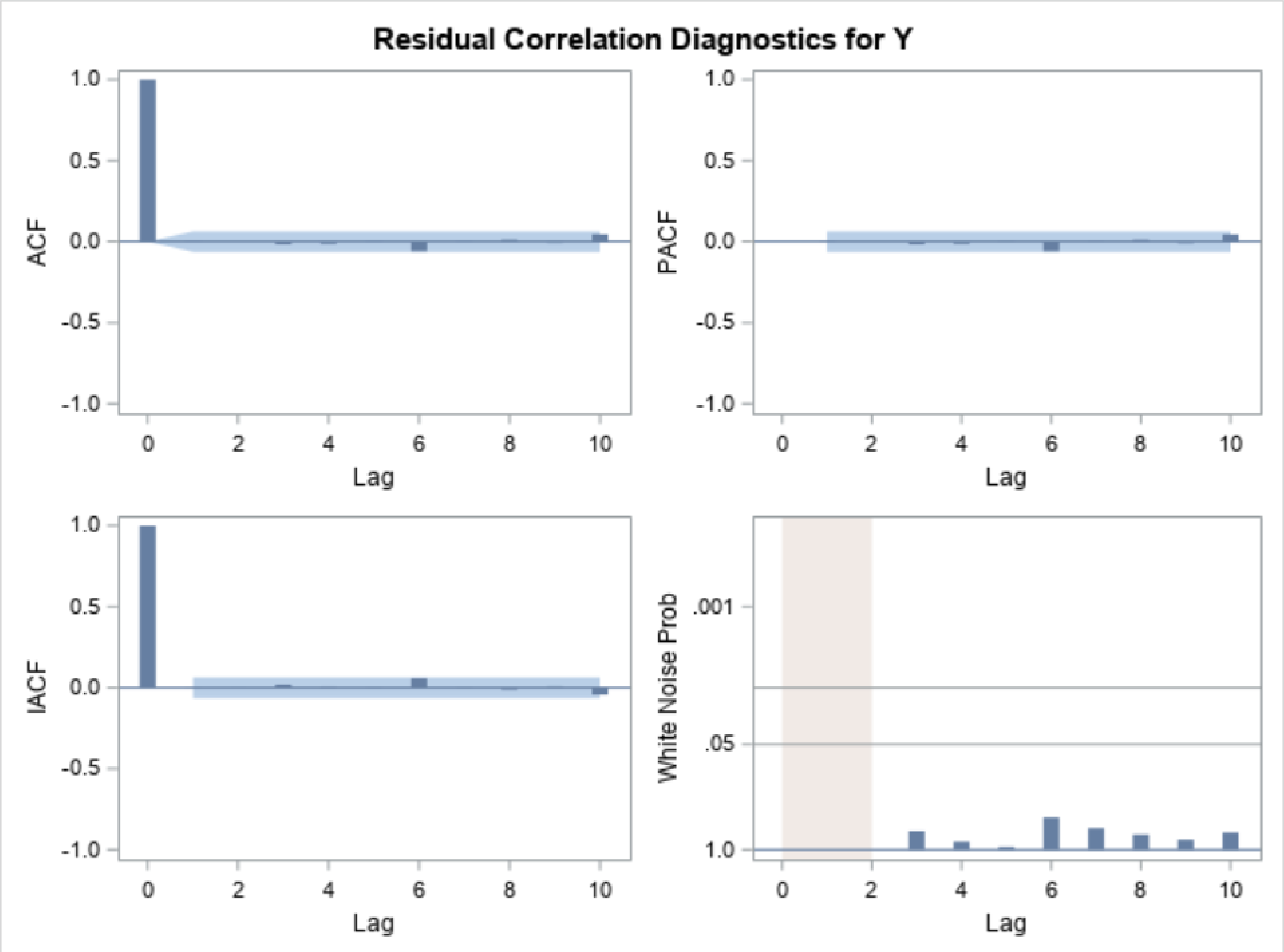
Autoregressive Models – SAS

```
proc arima data=Time.AR2 plot=all;
     identify var=y nlag=10;
     estimate p=2 method=ML;
run;
quit;
```


Trend and Correlation Analysis for Y
Y
Observation
ACF
Lag
PACF
Lag
IACF
Lag

Maximum Likelihood Estimation					
Parameter	Estimate	Standard Error	t Value	Approx Pr > |t|	Lag
MU	-0.13721	0.41922	-0.33	0.7434	0
AR1,1	0.64055	0.02941	21.78	<.0001	1
AR1,2	-0.37595	0.02939	-12.79	<.0001	2

Constant Estimate	-0.1009
Variance Estimate	95.06935
Std Error Estimate	9.750351
AIC	7396.028
SBC	7410.751
Number of Residuals	1000


Residual Correlation Diagnostics for Y
ACF
PACF
IACF
White Noise Prob
Lag
1.0
0.5
0.0
-0.5
-1.0
.001
.05
0
2
4
6
8
10

Model for variable Y	
Estimated Mean	-0.13721

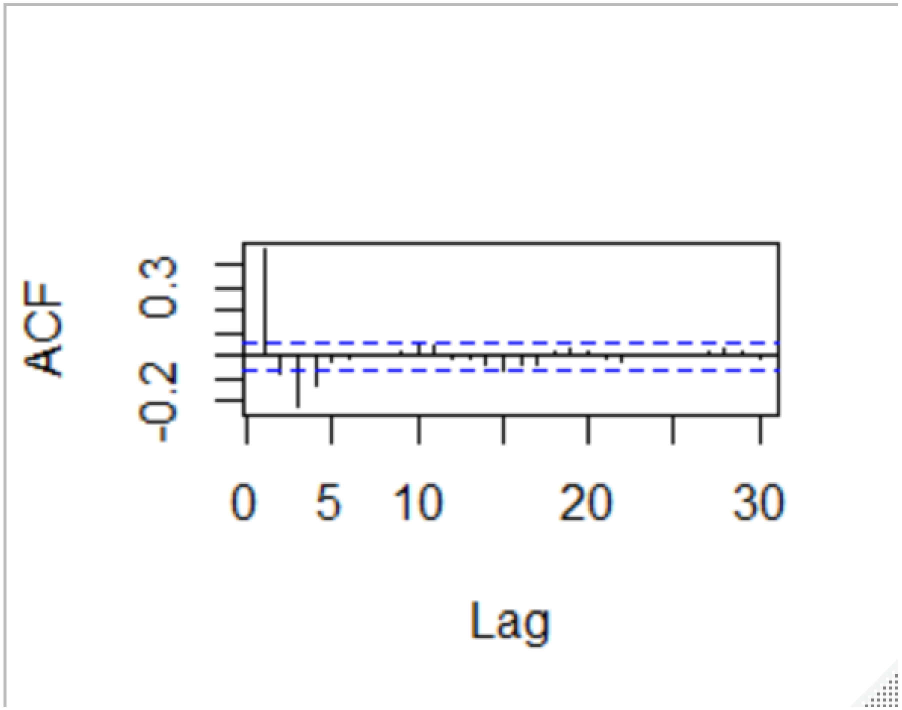
Autoregressive Factors	
Factor 1:	1 - 0.64055 B**(1) + 0.37595 B**(2)

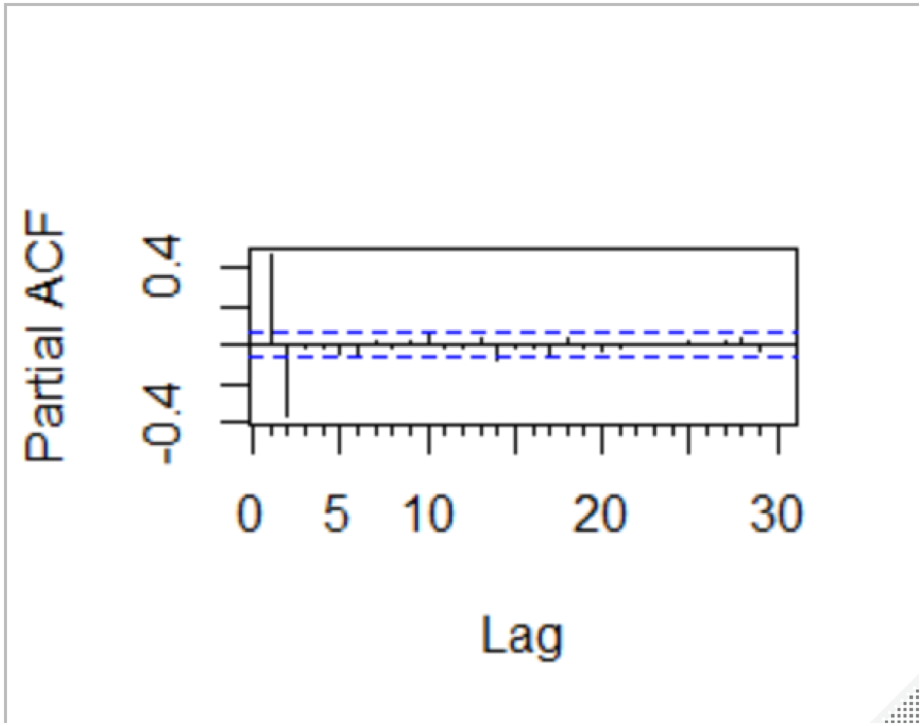
Autoregressive Models – R

```
Y <- ts(AR2$Y)
AR.Model <- Arima(Y, order=c(2,0,0))
```

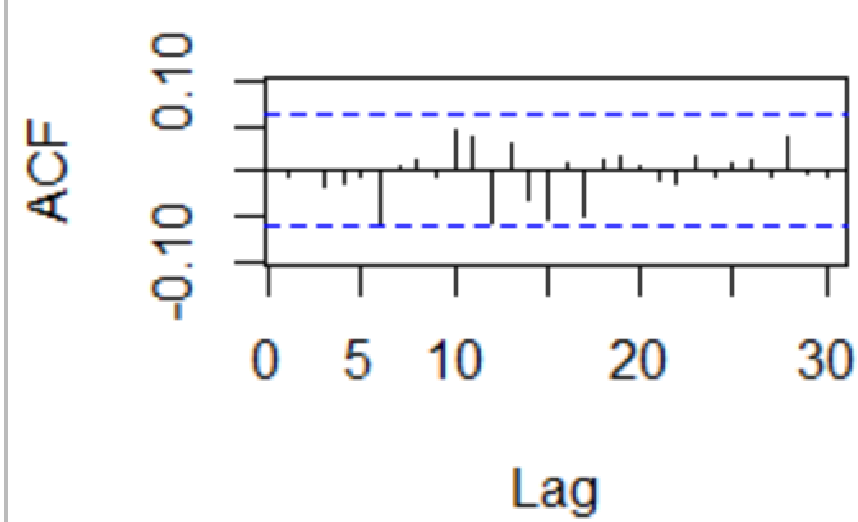
```
ARIMA(2,0,0) with non-zero mean

Coefficients:
         ar1      ar2     mean
      0.6406  -0.3760  -0.1371
s.e.  0.0294   0.0294   0.4187

sigma^2 estimated as 95.07:  log likelihood=-3695.01
AIC=7398.03   AICc=7398.07   BIC=7417.66

Training set error measures:
                       ME          RMSE       MAE       MPE         MAPE
Training -0.003995096 9.735715 7.710788 48.02355 313.2086
                     MASE          ACF1
Training set 0.7905136        -0.004541939
```

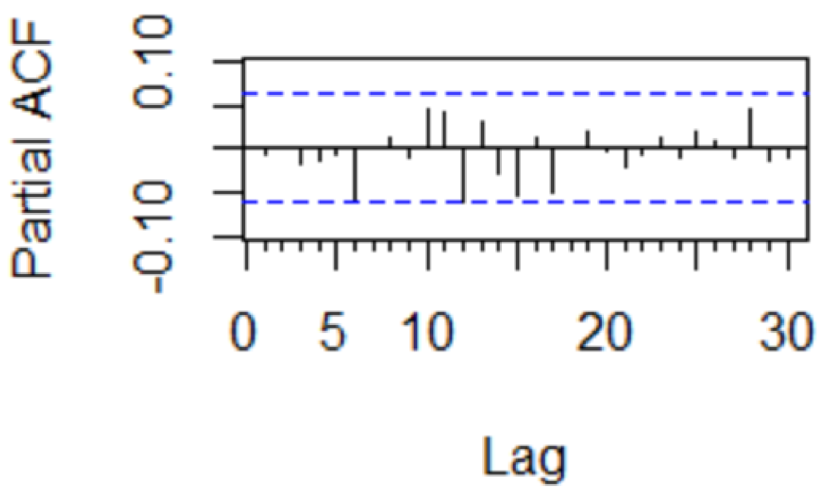
```
Acf(Y, main = "")$acf
Pacf(Y, main = "")$acf
```





Acf(AR.Model$residuals, main = "")$acf
Pacf(AR.Model$residuals, main = "")$acf





MOVING AVERAGE MODELS

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MOVING AVERAGE MODELS

Moving Average (MA) Models

- You can also forecast a series based solely on the past *error* values.
- This kind of model is better for describing events whose effect only lasts for short periods of time.
- We are going to focus on the most basic case – only one error lag value of e_t, called an MA(1) model:

$$Y_t = \omega + e_t + \theta e_{t-1}$$

MA(1) Model

- Therefore, for an MA(1) model, individual “shocks” only last for a short time.

$$Y_{t-1} = \omega + e_{t-1} + \theta e_{t-2}$$

MA(1) Model

- This is true for all observations (each observation is dependent on the error from the previous observation).
- In the MA model, we do not have the restrictions that we did on the AR models.

$$Y_t = \omega + e_t + \theta e_{t-1}$$

$$Y_{t-1} = \omega + e_{t-1} + \theta e_{t-2}$$

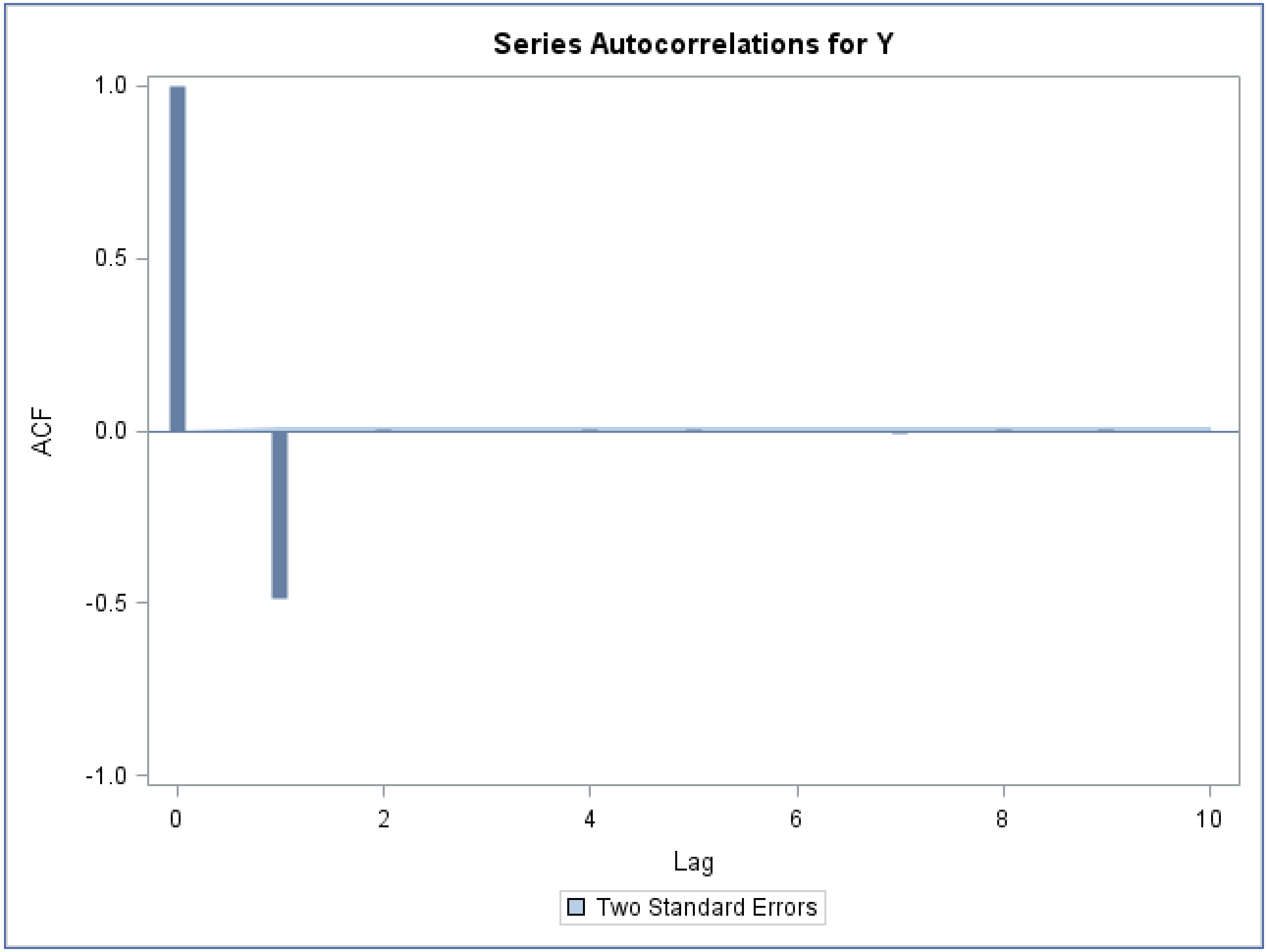
Correlation Functions for MA(1)

- The ACF has a significant spike at the first lag, followed by nothing after.
- The PACF decreases exponentially as the number of lags increases.
- The IACF decreases exponentially as the number of lags increases.

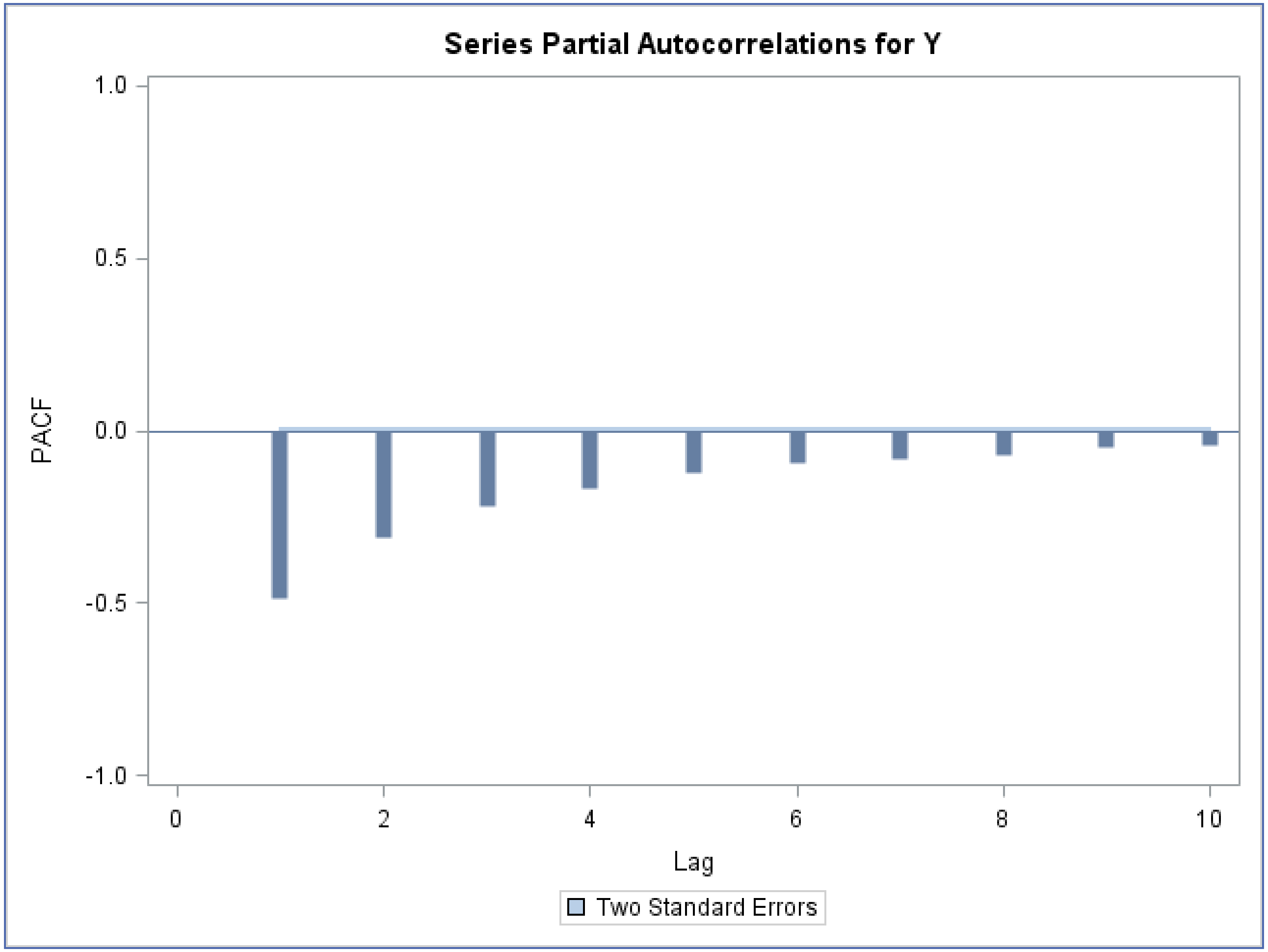
- Let's examine the following MA(1) model:

$$Y_t = 0 + e_t - 0.8e_{t-1}$$

MA(1) – ACF

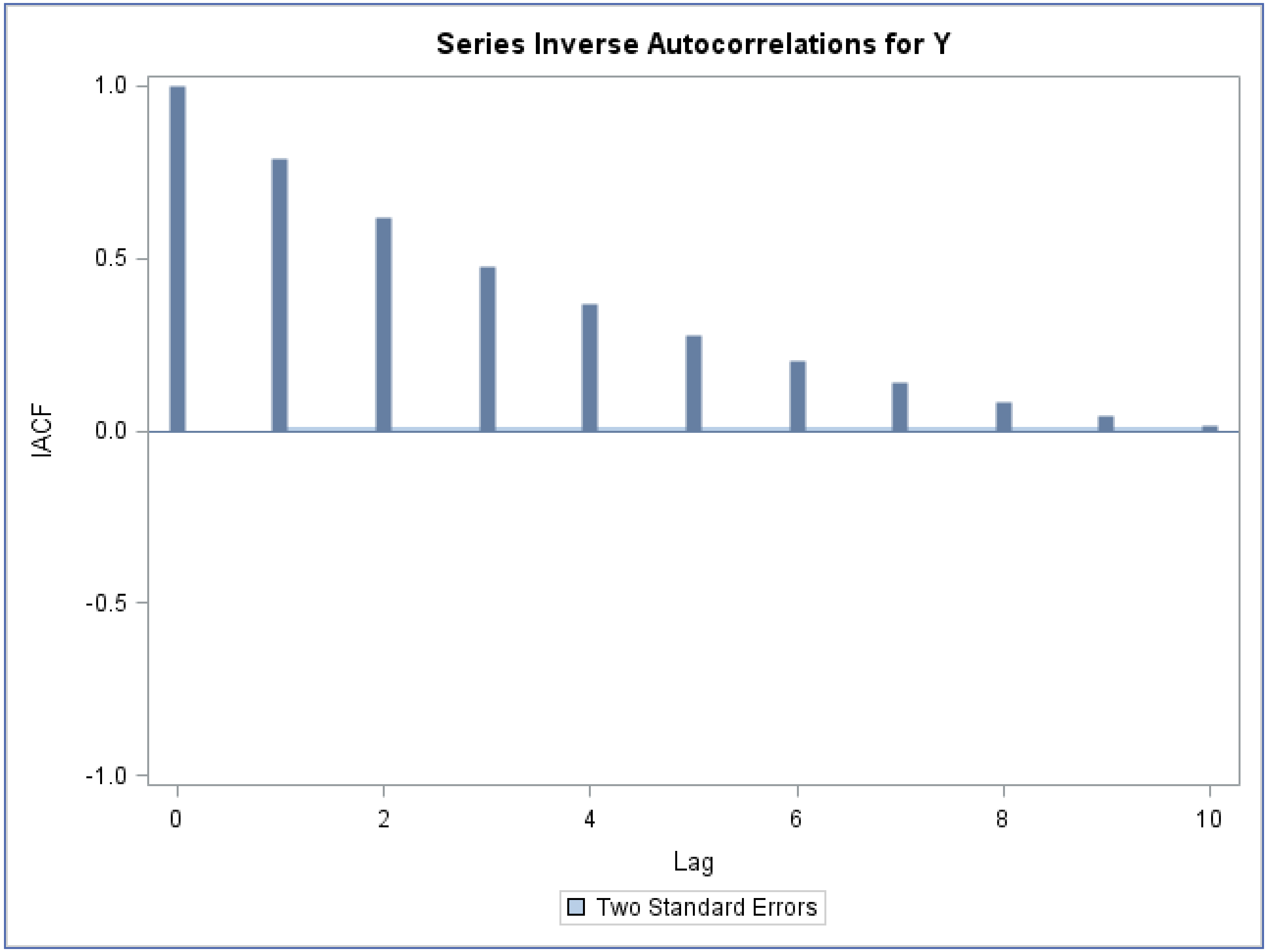

Series Autocorrelations for Y
1.0
0.5
0.0
-0.5
-1.0
ACF
0
2
4
6
8
10
Lag
Two Standard Errors

MA(1) – PACF



Series Partial Autocorrelations for Y
1.0
0.5
0.0
-0.5
-1.0
PACF
0
2
4
6
8
10
Lag
Two Standard Errors

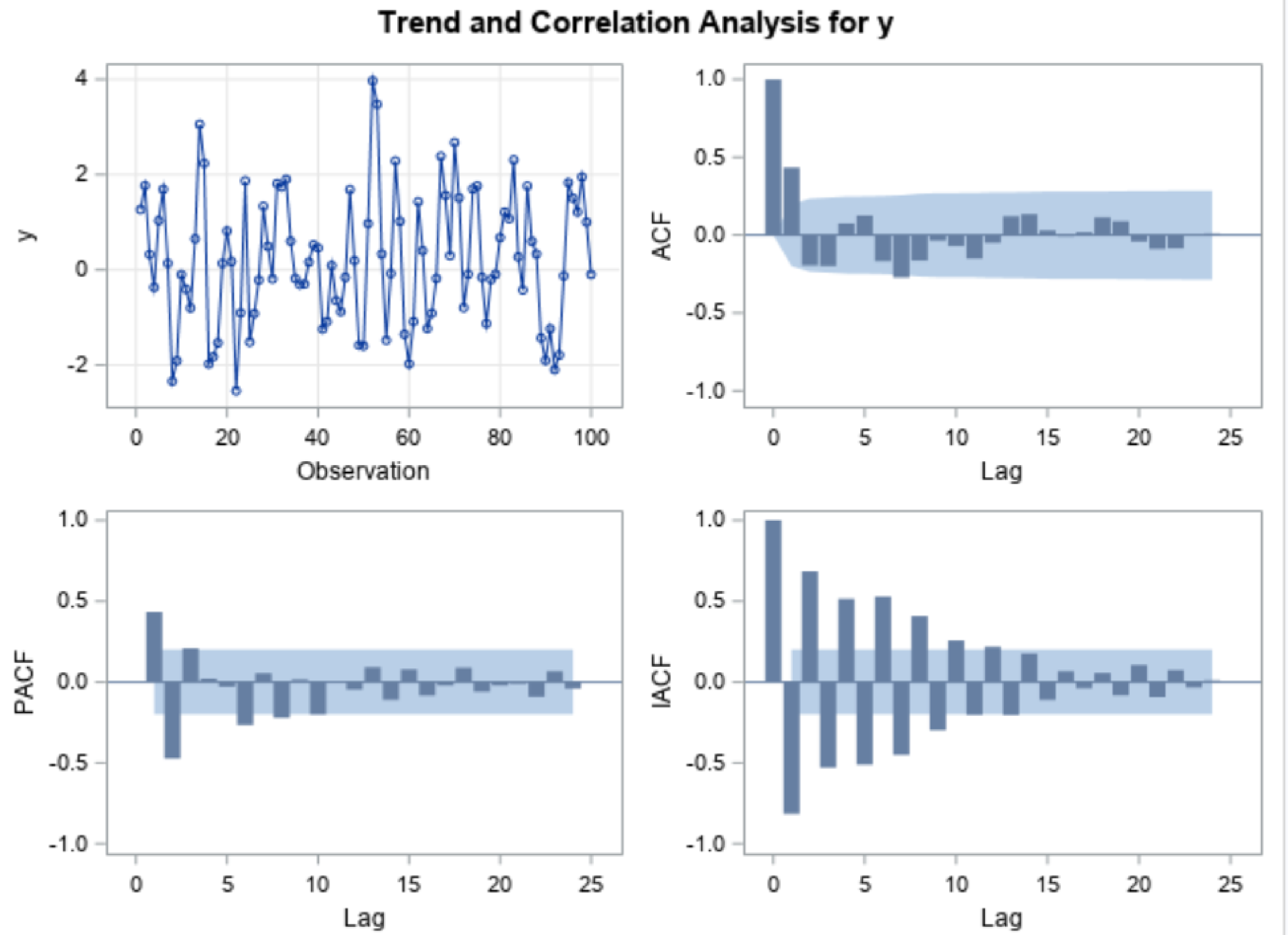
MA(1) – IACF



Series Inverse Autocorrelations for Y
1.0
0.5
0.0
-0.5
-1.0
IACF
0
2
4
6
8
10
Lag
Two Standard Errors

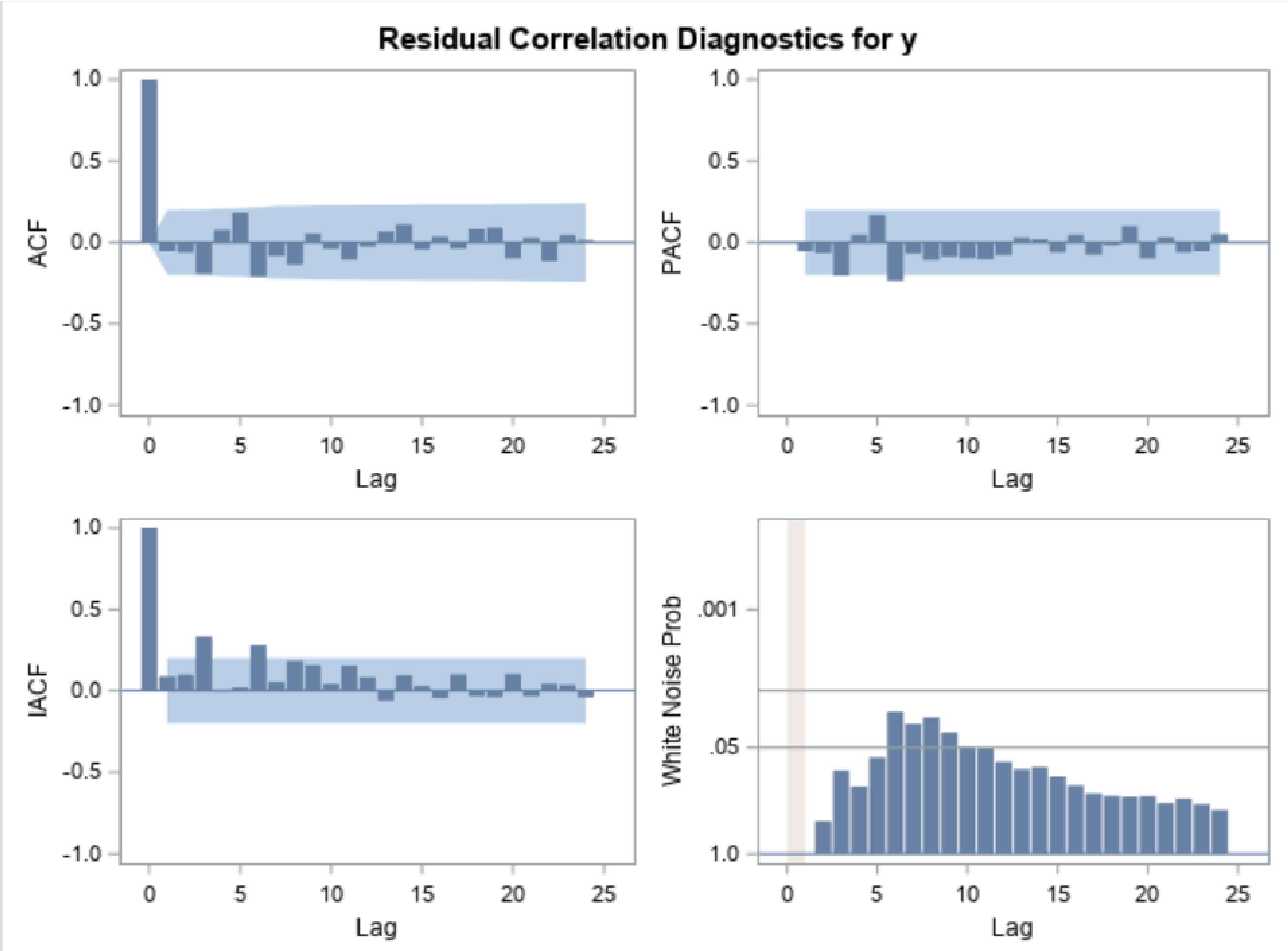
Moving Average Models – SAS

```
proc arima data=Time.sim_ma1;
identify var=y;
estimate q=1 method=ML;
run;
quit;
```


Trend and Correlation Analysis for y
y
Observation
ACF
Lag
PACF
Lag
IACF
Lag

Maximum Likelihood Estimation					
Parameter	Estimate	Standard Error	t Value	Approx Pr > |t|	Lag
MU	0.22282	0.20024	1.11	0.2658	0
MA1,1	-0.87171	0.05197	-16.77	<.0001	1

Constant Estimate	0.222819
Variance Estimate	1.172149
Std Error Estimate	1.082658
AIC	303.0779
SBC	308.2882
Number of Residuals	100

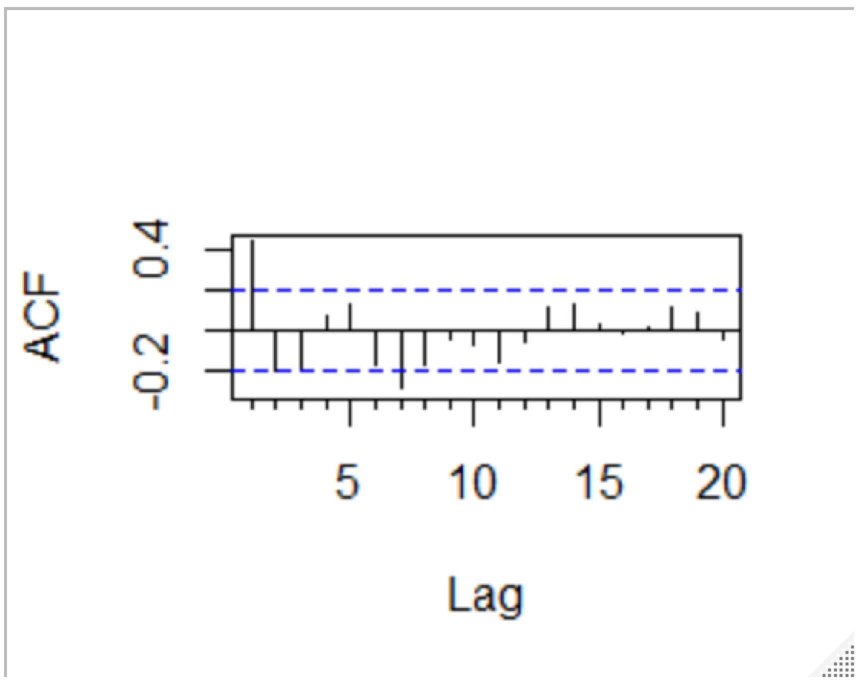

Residual Correlation Diagnostics for y
ACF
PACF
IACF
White Noise Prob
Lag
1.0
0.5
0.0
-0.5
-1.0
.001
.05
0
5
10
15
20
25

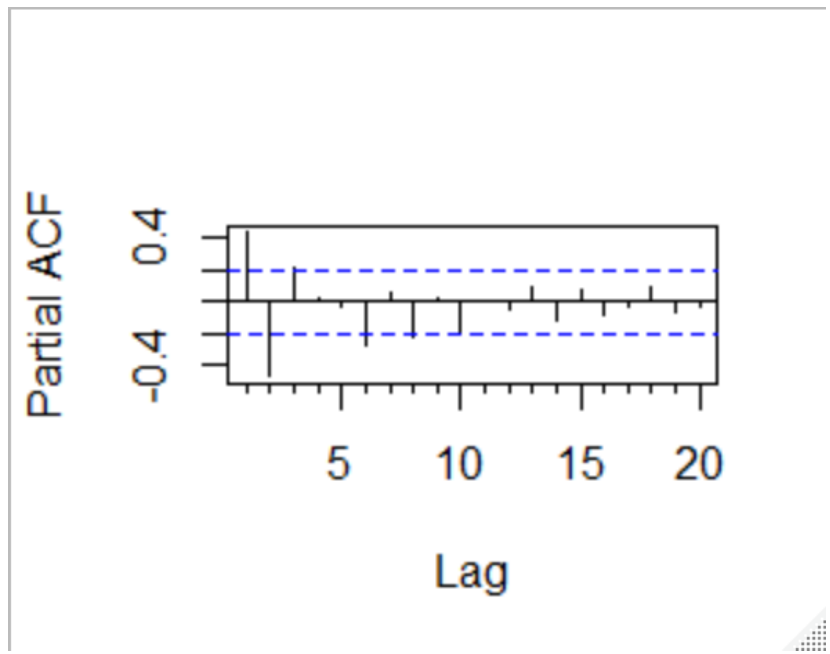
Moving Average Models – R

```
Acf(y, main = "")$acf
Pacf(y, main = "")$acf
MA.Model <- Arima(y, order = c(0, 0, 1))
Acf(MA.Model$residuals, main = "")$acf
Pacf(MA.Model$residuals, main = "")$acf
```

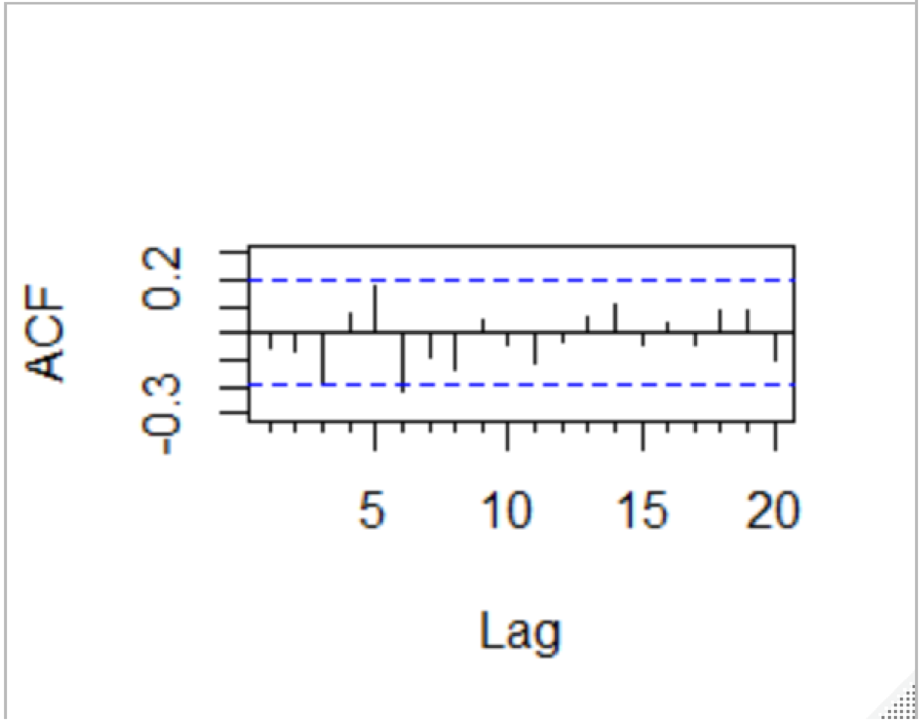
Coefficients:

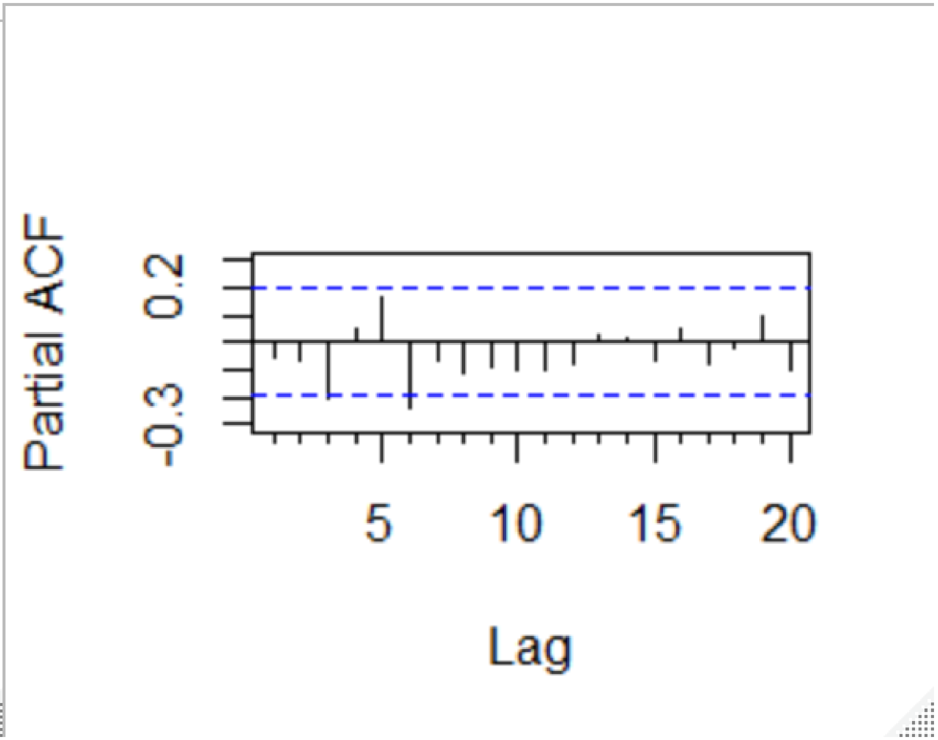
	ma1	mean
	0.8722	0.2228
s.e.	0.0653	0.1997





Residuals from MA model





MA(q) Model

- A time series that is a linear function of *q* past errors is called a moving average process of order q – called an MA(q).

$$Y_t = \omega + e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q}$$

Correlation Functions for MA(q)

- The ACF has a significant spike at the significant lags up to lag q, followed by nothing after.
- The PACF can have a variety of patterns.
- The IACF can have a variety of patterns.

Code for higher order models

```
AR.Model <- Arima(Y, order = c(4, 0, 0))
######If you want to skip some values:
AR.Model <- Arima(Y, order = c(2, 0, 0),fixed=c(0,NA,NA))
```

```
proc arima data=Time.AR2 plot=all;
        identify var=y nlag=10;
        estimate p=2 method=ML;
        estimate p=(2) method=ML;
        estimate p=(1,2,4) method=ML;
run;
quit;
```

Some notes about AR and MA models

- Any AR(p) model can be rewritten as an MA(∞).
- If the MA(q) model is invertible, then this MA(q) model can be rewritten as an AR(∞).
- Software should warn you if model is not invertible, if there is no convergence or any other issues….pay attention to the log and any warnings that you encounter when fitting these models.
- Depending on how software parameterizes equations, parameters can have different signs.