Eigenvectors and Intro to PCA - Worksheet

Part One

1. Show that $\mathbf{v}$ is an eigenvector of $\mathbf{A}$ and find the corresponding eigenvalue:

   a. $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$

   b. $\mathbf{A} = \begin{pmatrix} -1 & 1 \\ 6 & 0 \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

   c. $\mathbf{A} = \begin{pmatrix} 4 & -2 \\ 5 & -7 \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

2. Can a rectangular matrix have eigenvalues/eigenvectors?

3. For the following matrix, determine the eigenvalue associated with the given eigenvector.

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

   From this eigenvalue, what can you conclude about the matrix $\mathbf{A}$?

4. The matrix $\mathbf{M}$ has eigenvectors $\mathbf{u}$ and $\mathbf{v}$. What is λ_1, the first eigenvalue for the matrix $\mathbf{M}$?

$$\mathbf{M} = \begin{pmatrix} -1 & 1 \\ 6 & 0 \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

5. For the previous problem, is the specific eigenvector ($\mathbf{u}$ or $\mathbf{v}$) the *only* eigenvector associated with λ_1?

Part Two

1. For the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 4 \\ -1 & 5 \end{pmatrix},$$

   a. Verify that the pairs

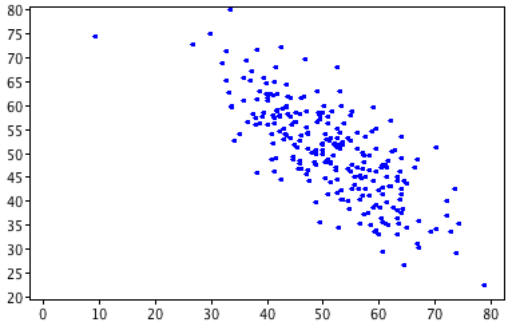
$$(\lambda_1 = 4, \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}) \quad \text{and} \quad (\lambda_2 = 1, \mathbf{v}_2 = \begin{pmatrix} 4 \\ 1 \end{pmatrix})$$

   are eigenpairs of $\mathbf{A}$.

   b. For each pair in part a, provide *another* eigenvector associated with that eigenvalue.

Part Three

1. For the following data plot, take your best guess and draw the direction vectors of the first and second principal components (the eigenvectors of the covariance matrix).



2. Suppose your data contained the variables *VO2_max*, *mile pace*, and *weight* in that order. The first principal component for this data is the eigenvector of the covariance matrix.

$$\begin{pmatrix} 0.69 \\ 0.61 \\ -0.38 \end{pmatrix}.$$

What would be the sign of the coordinate along this basis vector for an individual that had above average *VO2_max*, above average *mile pace* and below average *weight*? Explain.

List of Key Words/Phrases.

eigenvalue	directional variance
eigenvector	proportion of variance
eigenpair	correlation matrix
eigenspace	covariance matrix
$\lvert\lambda_1\rvert \geq \lvert\lambda_2\rvert \geq \lvert\lambda_3\rvert \geq \cdots \geq \lvert\lambda_n\rvert$	orthogonal projection
diagonalization	PCA loadings
eigenvalues of symmetric matrices	biplot
principal components	zero eigenvalues