Chapter 1

Lemmas and proof

This chapter presents the lemmas that are necessary to prove the main theorem. The problem of approximating a function g on some compact K of $\mathbb{R}^n$ from Σ_n, can be divided into two parts. One part is the approximation of the form $\sum_i f_i(a^i \cdot x)$ where f_i are functions in $C(\mathbb{R})$. The other is the approximation of f_i on the appropiate set from Σ_1. !!canviar

1.1 $\sigma * \varphi$ is not a polynomial.

Lemma 1. If we have that $\sigma * \varphi$ is a polynomial for all $\varphi \in \mathcal{C}_0^\infty$. Then the degree of the polynomial $\sigma * \varphi$ is finite, i.e. there exists an m $\in \mathbb{N}$ such that $deg(\sigma * \varphi) \leq m$ for all $\varphi \in \mathcal{C}_0^\infty$.

Proof. We first prove the claim in the case of $\varphi \in \mathcal{C}_0^\infty[a,b]$, where $\mathcal{C}_0^\infty[a,b]$ is the set of functions $\mathcal{C}_0^\infty$ with support in $[a,b]$ for any $a < b$.

Let ρ be a metric on $\mathcal{C}_0^\infty[a,b]$ defined by

$$\rho(\varphi_1, \varphi_2) = \sum_{n=0}^{\infty} 2^{-n} \frac{\|\varphi_1 - \varphi_2\|_n}{1 + \|\varphi_1 - \varphi_2\|_n}$$

where $\|\varphi\|_n = \sum_{j=0}^{n} \sup_{x\in[a,b]} |\varphi^{(j)}(x)|$. We can show that $(\mathcal{C}_0^\infty[a,b], \rho)$ is a complete metric space (Fréchet space). By assumption, we have that $\sigma * \varphi$ is a polynomial (for any $\varphi \in \mathcal{C}_0^\infty[a,b]$).

Consider the following set, which has the property that we want to show.

$$V_k = \{\varphi \in \mathcal{C}_0^\infty[a,b] \,|\, deg(\sigma * \varphi) \leq k\}$$

Clearly, if $\varphi \in V_k$, then $deg(\sigma * \varphi) \leq k$. We want to show that $\mathcal{C}_0^\infty[a,b] \subseteq V_k$. This set fulfills the following properties, $V_k \subset V_{k+1}$, V_k is a closed subspace and $\cup_{k=0}^{\infty} V_k = \mathcal{C}_0^\infty[a,b]$. As $\mathcal{C}_0^\infty[a,b]$ is a complete metric space, for Blaire's Category Theorem then there exists an integer m such that $V_m = \mathcal{C}_0^\infty[a,b]$.

For the general case where $\varphi \in \mathcal{C}_0^\infty$, we note that the number m does not deppend on the interval $[a,b]$. !! acabar

$\square$

Lemma 2. If $\sigma * \varphi$ is a polynomial such that $deg(\sigma * \varphi) \leq m$ for all $\varphi \in \mathcal{C}_0^\infty$, then σ is a polynomial of degree at most m.

Proof. If $\sigma * \varphi$ is a polynomial of degree m. For all $\varphi \in \mathcal{C}_0^\infty$, we have that

$$(\sigma * \varphi)^{(m+1)}(x) = \int \sigma(x-y)\varphi^{(m+1)}(y)\, dy = 0$$

From standard results in Distribution Theory, σ is itself a polynomial of degree at most m (a.e.). !!ho he buscat i no he trobat res perquè implica que sigma polinomi que la integral sigui 0 $\square$

Conclusion: If we have that $\sigma * \varphi$ is a polynomial then σ is a polynomial. This contradicts the hypothesis. Therefore, $\sigma * \varphi$ will not be a polynomial.

1.2 $\sigma * \varphi \in \overline{\Sigma_1}$

Lemma 3. For each $\varphi \in \mathcal{C}_0^\infty$, $\sigma * \varphi \in \overline{\Sigma_1}$.

Proof. Consider

$$h_m = \sum_{i=1}^{m} \varphi(y_i)\Delta y_i \sigma(x - y_i)$$

The sequence (h_m) satisfies $h_j \in \Sigma_1$ for $j = 1, ..., m$. $(w_i = 1, \theta_i = -y_i, \beta_i = \varphi(y_i)\Delta y_i)$.

Where $y_i = -\alpha + \frac{2i\alpha}{m}$, $\Delta y_i = \frac{2\alpha}{m}$ for $i = 1, ..., m$. Partition of the interval $[-\alpha, \alpha]$

We want to show that $h_m \rightrightarrows \sigma * \varphi$ in $[-\alpha, \alpha]$.

Given $\epsilon > 0$, we choose $\delta > 0$ such that $10\delta\|\sigma\|_{L^\infty\{-2\alpha,2\alpha\}}\|\varphi\|_{L^\infty} \leq \frac{\epsilon}{3}$. Note that ...

We know that $\sigma \in M$. Hence, for this given $\delta > 0$ and $[-\alpha, \alpha]$ interval, there exists $r(\delta)$ finite number of intervals the measure of whose union $\mathfrak{U}$ is δ such that σ is uniformly continuous on $[-2\alpha, 2\alpha]$. We now choose m_i sufficiently large so that

1. $m_1\delta > \alpha r(\delta)$. We can do this by Archimedes' principle.
2. From the uniform continuity of φ.
3. From the previous, σ is uniformly continuous on $[-2\alpha, 2\alpha]$.

We choose m such that $m = max\{m_1, m_2, m_3\}$.

Now, fix $x \in [-\alpha, \alpha]$. Set $\Delta_i = [y_{i-1}, y_i]$ where $y_0 = \alpha$ .. dibuix.

First, recall that,

$$\int \sigma(x-y)\varphi(y)dy = \sum_{i=1}^{m} \int_{\Delta_i} \sigma(x-y)\varphi(y)dy$$

Consider the following difference

$$\begin{aligned}
&\left| \int \sigma(x-y)\varphi(y)dy - \sum_{i=1}^{m} \int_{\Delta_i} \sigma(x-y_i)\varphi(y)dy \right| = \\
&\qquad = \left| \sum_{i=1}^{m} \int_{\Delta_i} \sigma(x-y)\varphi(y)dy - \sum_{i=1}^{m} \int_{\Delta_i} \sigma(x-y_i)\varphi(y)dy \right| \\
&\qquad = \left| \sum_{i=1}^{m} \int_{\Delta_i} \varphi(y)\Big(\sigma(x-y) - \sigma(x-y_i)\Big)dy \right| \\
&\qquad \leq \sum_{i=1}^{m} \int_{\Delta_i} |\varphi(y)|\,|\sigma(x-y) - \sigma(x-y_i)|\,dy
\end{aligned}$$

If $x - \Delta_i \cap U = \varnothing$. Since $x - y \notin U$, $x - y_i \notin U$ and $x - y_i \in [-2\alpha, 2\alpha]$, bc (2) we have

$$\begin{aligned}
\sum_{i=1}^{m} \int_{\Delta_i} |\varphi(y)|\,|\sigma(x-y) - \sigma(x-y_i)|\,dy &\leq \frac{\epsilon}{\|\varphi\|_{L_1}} \sum_{i=1}^{m} \int_{\Delta_i} |\varphi(y)| = \\
&= \frac{\epsilon}{3\|\varphi\|_{L_1}} \int |\varphi(y)|\,dy \\
&= \frac{\epsilon}{3\|\varphi\|_{L_1}} |\varphi(y)|_{L_1} = \frac{\epsilon}{3}
\end{aligned}$$

If $x - \Delta_i \cap U \neq \varnothing$

$\sum_i |\widetilde{\Delta_i}| = \sum_i |(x - \Delta_i \cap U)| \leq |U| + 2|\Delta_i| r(\delta) \leq \delta + 2 \cdot \frac{2\alpha}{m} r(\delta) \leq \delta + 4\delta = 5\delta$

True by our choice of m, satisfies $m\delta > \alpha r(\delta) \iff \delta > \frac{\alpha \cdot r(\delta)}{m}$

$$\begin{aligned}
&\sum_{i=1}^{m} \int_{\widetilde{\Delta}_i} |\varphi(y)|\,|\sigma(x-y) - \sigma(x-y_i)|\,dy \leq \\
&\qquad \leq \sum_{i=1}^{m} \int_{\widetilde{\Delta}_i} \|\varphi\|_{L^\infty}\, 2\|\sigma\|_{L^\infty[-2\alpha,2\alpha]} \\
&\qquad = \|\varphi\|_{L^\infty}\, 2\|\sigma\|_{L^\infty[-2\alpha,2\alpha]} \sum_i |\widetilde{\Delta}_i| \\
&\qquad \leq \|\varphi\|_{L^\infty}\, 2\|\sigma\|_{L^\infty[-2\alpha,2\alpha]}\, 5\delta \leq \epsilon/3
\end{aligned}$$

$$\left|\sum_{i=1}^{m}\int_{\Delta_i}\sigma(x-y_i)\varphi(y)dy - \sum_{i=1}^{m}\sigma(x-y_i)\varphi(y_i)\Delta y_i\right| =$$

$$= \left|\sum_{i=1}^{m}\int_{\Delta_i}\sigma(x-y_i)[\varphi(y)-\varphi(y_i)]dy\right|$$

$$\leq \sum_{i=1}^{m}\int_{\Delta_i}|\sigma(x-y_i)|\,|\varphi(y)-\varphi(y_i)|\,dy$$

$$\leq \sum_{i=1}^{m}\int_{\Delta_i}|\sigma(x-y_i)|\,dy\left[\frac{\epsilon/3}{2\alpha\|\sigma\|_{L^\infty[-2\alpha,2\alpha]}}\right] \leq \frac{\epsilon}{3}$$

Finally, we have the result $h_m \rightrightarrows \sigma * \varphi$ because

$$\left|\int\sigma(x-y)\varphi(y)dy - \sum_{i=1}^{m}\sigma(x-y_i)\varphi(y_i)\Delta y_i\right| \leq \epsilon$$

!! Falta acabar □

1.3 Σ_1 dense in $\mathcal{C}(\mathbb{R})$

Lemma 4. If $\sigma \in \mathcal{C}^\infty$, then Σ_1 is dense in $\mathcal{C}(\mathbb{R})$.

Proof. We recall that set $\Sigma_1 = span\{\sigma(w\cdot x+\theta) : w\in\mathbb{R}, \theta\in\mathbb{R}\}$. We can write any function $h\in\Sigma_1$ as $h = \sum_i \beta_i\sigma_i(w_i x+\theta_i) = \beta_1\sigma_1(w_1x+\theta_1)+...$

We can see that $\frac{\sigma([w+h]x+\theta)-\sigma(wx+\theta)}{h} \in \Sigma_1$ because is a linear combination, where $\beta_1 = \frac{1}{h}, \beta_2 = \frac{-1}{h}$.
By hypothesis, we have $\sigma\in\mathcal{C}^\infty$. By definition of derivative we have

$$\frac{d}{dw}\sigma(wx+\theta) = \lim_{h\to 0}\frac{\sigma([w+h]x+\theta)-\sigma(wx+\theta)}{h} \in \overline{\Sigma_1}^{*}$$

Because the limit of a set belongs to the closure of the set.
By the same argument, $\frac{d^k}{dw^k}\sigma(wx+\theta)\in\overline{\Sigma_1}$ for all $k\in\mathbb{N}, w,\theta\in\mathbb{R}$.
If we differentiate this expression k times, we obtain

$$\frac{d^k}{dw^k}\sigma(wx+\theta) = \sigma^{(k)}(wx+\theta)\cdot x^k$$

We will see by reduction to absurdity that if σ is not a polynomial (by hypothesis) then there exists a $\theta_k\in\ \mathbb{R}$ such that $\sigma^{(k)}(\theta_k)\neq 0$.
If σ is not a polynomial and $\sigma\in\mathcal{C}^\infty$, lets assume that $\nexists\theta_k\in\mathbb{R}$ such that $\sigma^{(k)}(\theta_k)\neq 0$.
This means that the k-th derivative at every point is 0,

$$\sigma^{(k)}(\theta) = 0 \quad \forall\theta\in\mathbb{R}$$

*$\overline{\Sigma_1}$ denotes the clausure of the set Σ_1

If we integrate k times this expression,

$$\int \sigma^{(k)} = \int 0 \Rightarrow \sigma^{(k-1)} = C$$

,

$$\int \sigma^{(k-1)} = \int C \Rightarrow \sigma^{(k-2)} = Cw$$

, then we end up σ is a polynomial. Contradiction. Therefore, there always exists a point where the derivative does not vanish.
Thus, we evaluate at the point where the derivative does not vanish, we call it θ_k.

$$\sigma^{(k)}(\theta_k) \cdot x^k = \frac{d^k}{dw^k}\sigma(wx+\theta)\Big|_{w=0,\theta=\theta_k} \in \overline{\Sigma_1}$$

This implies that $\overline{\Sigma_1}$ contains all polynomials, because the expression $\sigma^{(k)}(\theta_k)x^k$ generates all polynomials. By the Weierstrass theorem, we know that the polynomials are dense in $\mathcal{C}(\mathbb{R})$. This concludes that the set $\overline{\Sigma_1}$ contains a set which is dense in $\mathcal{C}(\mathbb{R})$, therefore Σ_1 is dense in $\mathcal{C}(\mathbb{R})$.

□

Lemma 5. If for some $\varphi \in \mathcal{C}_0^\infty$ we have that $\sigma * \varphi$ is not a polynomial, then Σ_1 is dense in $\mathcal{C}(\mathbb{R})$.

Proof. From Lemma 3, $\sigma * \varphi \in \overline{\Sigma_1}$. Clearly, $\sigma * \varphi(wx+\theta) \in \overline{\Sigma_1}$, for each $\theta \in \mathbb{R}$. For σ and $\varphi \in \mathcal{C}_0^\infty$ we have that $\sigma * \varphi \in \mathcal{C}^\infty$. (ho hem de veure!!!). From Lemma 4, if $\sigma * \varphi \in \mathcal{C}^\infty$, then Σ_1 dense in $\mathcal{C}(\mathbb{R}^n)$. ???? (nose si apliquem el lemma 4 amb sigma = sigma conv varphi o si sigma conv varphi de c infinit implica sigma de c infinlt aleshores apliquem el lemma 4 ??) □

1.4 Σ_1 is dense in $\mathcal{C}(\mathbb{R})$, then Σ_n is dense in $\mathcal{C}(\mathbb{R}^n)$

We will proof that approximating a $\mathcal{C}(\mathbb{R})$ function with one from the set Σ_1 implies approximating a function $\mathcal{C}(\mathbb{R}^n)$ from the set Σ_n. Therefore, it is only necessary to approximate a continuous function. We can see this from the density characterization:

Lemma 6. If Σ_1 is dense in $\mathcal{C}(\mathbb{R})$, then Σ_n is dense in $\mathcal{C}(\mathbb{R}^n)$.

Proof. Let

$$V := span\{f(ax) : a \in \mathbb{R}^n, f \in \mathcal{C}(\mathbb{R})\}$$

We shall see that V is dense in $\mathcal{C}(\mathbb{R}^n)$. If we show that V contains the polynomials (which are dense in $C(R^n)$ for Weierstrass Theorem) that would be enough.

!!mirar Let $L(a)$ denote the span of the n rows of a for each $a \in \mathbb{R}^n$. Set $L(R^n) = \cup L(a)$. Let

$$H_k^n = \{\sum c_m s^m\}$$

denote the set of homogeneous polynomials of n variables of total degree k, and

$$H^n = \cup_{k=0}^{\infty} H_k^n$$

the set of all homogeneous polynomials of n variables.

Assume that for a given $k \in \mathbb{N}$ no non-trivial $p \in H_k^n \subseteq V$ for all $k \in \mathbb{Z}$, then V contains all polynomials. For that we have V dense in $\mathcal{C}(\mathbb{R}^n)$. Now, we only need to show that $H_k^n \subseteq V$. SOS

Let $g \in \mathcal{C}(\mathbb{R})$, for any compact subset $K \subset \mathbb{R}^n$, V dense in $\mathcal{C}(K)$. That is, given $\epsilon > 0$, there exist $f_i \in \mathcal{C}(\mathbb{R})$ and $a_i \in \mathbb{R}^n$ $_{i=1,...,k}$ such that

$$\left|g(x) - \sum_{i=1}^{k} f_i(a^i \cdot x)\right| < \frac{\epsilon}{2}$$

for all $x \in K$. We now consider the set of all the points in the compact K multiplied by the vector a^i. That is $\{a^i \cdot x | x \in K\} \subseteq [\alpha_i, \beta_i]$ for some finite interval $[\alpha_i, \beta_i]$, $i = 1, ..., k$. By hipothesis Σ_1 dense in $\mathcal{C}(\mathbb{R})$, specifically Σ_1 is dense in $[\alpha_i, \beta_i]$ $i = 1, ..., k$. Hence there exist constants c_{ij}, w_{ij} and θ_{ij}, $j = 1, ..., m_i$, $i = 1, ..., k$ such that

$$\left|\ f_i(y) - \sum_{j=1}^{m} c_{ij}\sigma(w_{ij}y + \theta_{ij})\right| < \frac{\epsilon}{2k}$$

for all $x \in K$.

Therefore,

$$\left|g(x) - \sum_{i=1}^{k}\sum_{j=1}^{m} c_{ij}\sigma(w_{ij}(a^i \cdot x) + \theta_{ij})\right| < \epsilon$$

□

We showed that to approximate a $\mathcal{C}(\mathbb{R}^n)$ function we only need to approximate a $\mathcal{C}(\mathbb{R})$ function with the set Σ_1.

1.5 Proof of the theorem

Proof.

- $\Rightarrow$ To prove the implication, we will use proof by contrapositive. We will see the following. If σ is a polynomial then Σ_n is not dense in $\mathcal{C}(\mathbb{R}^n)$. Let σ be a polynomial of degree k, then $\sigma(wx + \theta)$ is a polynomial of degree k for every w, θ. We have $\Sigma_n = span\{\sigma(w \cdot x + \theta) : w \in \mathbb{R}^n, \theta \in \mathbb{R}\}$ that is the set of algebraic polynomials of degree at most k. Σ_n is not dens in $\mathcal{C}(\mathbb{R}^n)$ if for a function $f(x) \in \mathcal{C}(\mathbb{R}^n)$ we can find $\epsilon > 0$ and K such that $\|p - f\| > \epsilon$ for all p polynomial of degree k. For example, let $f(x) = cos(x)$, and let $p(x) = \sigma(wx + \theta)$ that has degree at most k. This implies has maximum k roots. We can find a interval where $cos(x)$ has k+1 roots. Therefore, Σ_n is not dense in $\mathcal{C}(\mathbb{R}^n)$.
- $\Leftarrow$ Recapitulem el que hem vist als lemes ..

□

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