Universitat Autònoma
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Treball Final de
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Machine learning: mathematical foundations

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Any
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Convocatòria
Juny

To my colleagues,

...

Abstract

Exemple d'abstract no definitiu, x ficar algo.

Abstract – In various studies, researchers have described the activation functions that allow multilayer feedforward networks to act as universal approximators. However, we demonstrate that most of the characterizations presented in the literature are specific instances of the following general finding: A typical multilayer feedforward network that employs a locally bounded piecewise continuous activation function can accurately approximate any continuous function to an arbitrary degree if and only if the activation function is non-polynomial.

Resum

blslalblallb en català

Preface

Inpirat per **?**.

blablalbla amb glosary Universitat Autònoma de Barcelona (UAB) i ara curt UAB.

Contents

Chapter 1

Introduction

ficar una quote que quedi bé

— John S. Bell, *Against Measurement*

Back in the day, computers seemed to be limited to exact computation tasks. However, over time, researchers started pushing the boundaries of what computers can do, eventually leading to make them do the sorts of things that minds can do. Today, computers can perform complex tasks such as natural language processing, speech recognition, and image synthesis, and can learn and adapt through machine learning algorithms. These advancements have allowed computers to enhance human decision-making in fields such as healthcare, finance, and transportation.

Some of these (e.g. reasoning) are normally described as "intelligent." Others (e.g. vision) aren't. But all involve psychological skills—such as perception, association, prediction, planning, motor control—that enable humans and animals to attain their goals.

3. on les matematiques prenen lloc ? pq son importants . Can we suggest conjectures, relationships , theorems between fields ??? using ml as a tool to see unexpected relationships.

ML might become a bycicle for the mind !!

MATHS USING MCH LEARNING <–> ML USING MATHS

Chapter 2

Artificial Intelligence

2.1 What is Artificial Intelligence

2.2 What is Machine Learning

Machine Learning is the science of programming computers so they can learn from data.

(with the aim to solve a problem without being explicitly programmed.)

For example,
Classification and regression problem.

2.2.1 Artificial neural networks ?

Artificial neural networks are a class of machine learning models that are inspired by the structure and function of the human brain. They are composed of interconnected processing units called artificial neurons, which are organized in layers and are capable of learning and generalizing from data.

2.3 Types of Learning

We can think about learning as the way we understand it as a human. We can classify a learning problem based on the degree of feedback. The three main types are:

1. Supervised learning, where we have immidiate feedback.
2. Reinforcement learning, where we have indirect feedback. For example when we are playing the game of chess.
3. Unsupervised learning, where we have non feedback signal. For example, deducing which dog belongs to each owner.

Example 1. Example of a supervised learning task. Recognition of a letter. What we are trying to learn is a probability distribution fuction

$$f : \{0, ..., 255\}^{28\times 28} \longrightarrow \text{probability distribution on } \{0, 1, ..., 9\}$$

2.4 Function Learning

Important principle I: Many supervised learning tasks are about function learning.

Example 2. Example of a classification problem. We want to classify if an image is a dog or not a dog. We would like to produce a value which is correlated with the probability of this image being a dog or not a dog. We can approach the problem in the following way. We want to find a function that takes very high values when dog-image and very low val ues when non dog images and takes the value 0 when its uncertain.

$$d : \mathbb{R}^{\#\text{pixels in image}} \to \mathbb{R}$$

such that $\mathbb{P}(d(\text{image}))$ = probability that the image is a dog.

That is what we mean by many problems can be recast as function learning. Note that there is not a god-given reason why this function should exist. We know that certain points in space, and they have certain values associated to them, but we dont know that there is some big function.

Important principle II: Sometimes function learning can be recast as a classification problem.

Binary classification problem. Rather learning $\mu : \mathbb{R}^{\#\text{bits}} \to \mathbb{R}$ where big values correspond to likely and small values to unlikely. It is better to learn $\mu : \mathbb{R}^{\#\text{bits}} \to$ probability distribution on $\{-1, 0, 1\}$. In number theory the function $\mu(n)$ it is called Möebius function

Chapter 3

Multilayer Feedforward Networks

3.1 Function Approximation

In this paper we take $C(\mathbb{R}^n)$ to be the family of "real world" functions that one may wish to approximate. If we can show that a given set of functions F is dense in $C(\mathbb{R}^n)$, we can conclude that for every continuous function $g \in C(\mathbb{R}^n)$ and each compact set $K \subset \mathbb{R}^n$, there is a function $f \in F$ such that f is a good approximation to g on K.

Definition 1. A *metric* (or *distance*) on a set X is a function $d : X \times X \to \mathbb{R}$ such that forall $s, t \in X$ the following properties are satisfied:

1. $d(s,t) \geq 0$ and $d(s,t) = 0$ if and only if $s = t$.
2. $d(s,t) = d(t,s)$.
3. $d(s,t) \leq d(s,u) + d(u,t)$ (*triangular inequality*).

A *metric space* is a pair (X, d), where X is a set and d is a distance in X.

If we take X to be a set of functions, the metric $d(f,g)$ will enable us to measure the distance between functions $f, g \in X$.

Definition 2. We denote the support of a function u by $supp(u) = \{x | u(x) \neq 0\}$

Definition 3. Let f, g be real-valued functions with compact support. We define the *convolution* of f with g as

$$(f * g)(x) = \int f(x-t)g(t)\,dt$$

3.2 Lebesgue measure

An essential part of measure theory is the calculation of lengths, areas, volumes, etc. We are already familiar with the Riemann integral, and now we aim to introduce a more general integral, the Lebesgue integral, that comes from the Lebesgue measure.

Definition 4. A box in $\mathbb{R}^d$ is a set of the form

$$Q = [a_1, b_1] \times ... \times [a_d, b_d] = \prod_{i=1}^{d} [a_i, b_i]$$

The volume of the box is

$$vol(Q) = (b_1, a_1)...(b_d - a_d) = \prod_{i=1}^{d} (b_i - a_i)$$

The *exterior measure* (or outer measure) of a set $E \subseteq \mathbb{R}^d$ is

$$|E|^* = \inf\{\sum_k vol(Q_k)\}$$

where the infimum is taken over all finite or countable collection of boxes Q_k such that $E \subseteq \cup_k Q_k$

Definition 5. A set $E \subseteq \mathbb{R}^n$ is *Lebesgue mesurable* (or mesurable) if $\forall \epsilon > 0$, there exist U open set such that $E \subseteq U$ and $|U \setminus E|^* < \epsilon$

Definition 6. A function u defined almost everywhere on a measurable set $\Omega \in \mathbb{R}^n$ is said to be *essentially bounded* on Ω if $|u(x)|$ is bounded almost everywhere on Ω. We denote $u \in L^\infty(\Omega)$ with the norm

$$\|u\|_{L^\infty(\Omega)} = inf(\lambda | \{x : |u(x)| \geq \lambda\} = 0) = ess \sup_{x \in \Omega} |u(x)|$$

We have that $L^\infty(\mathbb{R})$ is the space of essentially bounded functions.

Examples and counterexamples of functions essentially bounded.

- $f : \Omega \to$

Definition 7. A function u defined almost everywhere on a domain Ω (a domain is an open set in $\mathbb{R}^n$) is said to be *locally essentially bounded* on Ω if for every compact set $K \subset \Omega$, $u \in L^\infty(K)$. We denote $u \in L^\infty_{loc}(K)$.

Definition 8. We say that a set of functions $F \subset L^\infty_{loc}(\mathbb{R})$ is *dense* in $C(\mathbb{R}^n)$ if for every function $g \in C(\mathbb{R}^n)$ and for every compact $K \subset \mathbb{R}^n$, there exist a sequence of functions $f_j \in F$ such that

$$\lim_{j \to \infty} \|g - f_j\|_{L^\infty(K)} = 0.$$

3.3 Results

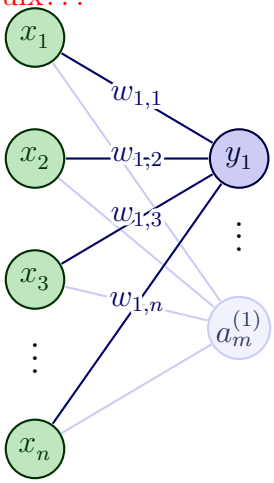
Definition 9. Let $\mathcal{M}$ denote the set of functions which are in $L^\infty_{loc}(\mathbb{R})$ and have the following property. The closure of the set of points of discontinuity of any function in $\mathcal{M}$ is of zero Lebesgue measure.

Proposition 10. (This implies that) for any $\sigma \in \mathcal{M}$, interval $[a, b]$. and $\delta > 0$, there exists a finite number of open intervals, the union of which we denote by U, of measure δ, such that σ is uniformly continuous on $[a, b]/U$.

Definition 11. $\mathcal{C}_0^\infty$ functions $\mathcal{C}^\infty$ with compact support.

3.4 Multilayer Feedforward Network

Multilayer feedforward networks are a type of artificial neural network that consist of several layers of interconnected nodes, with each node taking input from the previous layer and producing output for the next layer. The general architecture of a multilayer feedforward network, MFN, consist of: input layer: n-input units, one/more hidden layers : intermediate processing units, output layer: m output-units. dibuix???



$$\begin{pmatrix} a_1^{(1)} \\ a_2^{(1)} \\ \vdots \\ a_m^{(1)} \end{pmatrix} = \beta_j \cdot \sigma \left[\begin{pmatrix} w_{1,0} & w_{1,1} & \dots & w_{1,n} \\ w_{2,0} & w_{2,1} & \dots & w_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m,0} & w_{m,1} & \dots & w_{m,n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_m \end{pmatrix} \right]$$

$$a^{(1)} = \beta_j \cdot \sigma \left(\mathbf{W}^{(0)} a^{(0)} - \theta_{\mathbf{j}} \right)$$

Definition 12. (Multilayer feedforward networks) The function that a MFN compute is:

$$f(x) = \sum_{j=1}^{k} \beta_j \cdot \sigma(w_j \cdot x - \theta_j)$$

where $x \in \mathbb{R}^n$ is the input vector, $k \in \mathbb{N}$ is the number of processing units in the hidden layer, $w_j \in \mathbb{R}^n$ is the weight vector that connects the input to processing unit j in the hidden layer, $\sigma : \mathbb{R} \to \mathbb{R}$ is an activation function applied element-wise to the vector $w_j^T x - \theta_j$, where $\theta_j \in \mathbb{R}$ is the threshold (or bias) associated with processing unit j in the hidden layer, and $\beta_j \in \mathbb{R}$ is the weight that connects processing unit j in the hidden layer to the output of the network.

Let N_w be the family of all functions implied by the network's architecture. If we can show that N_w is dense in $C(\mathbb{R}^n)$, we can conclude that for every continuous function $g \in C(\mathbb{R}^n)$ and each compact set $K \subset \mathbb{R}^n$, there is a function $f \in N_w$ such that f is a good approximation to g on K.

Under which necessary and sufficient conditions on σ will the family of networks N_w be capable of approximating to any desired accuracy any given continuous function?

3.5 Theorem

Theorem 13. *Let $\sigma \in M$. Set*

$$\Sigma_n = span\{\sigma(w \cdot x + \theta) : w \in \mathbb{R}^n, \theta \in \mathbb{R}\}$$

Then Σ_n is dense in $\mathcal{C}(\mathbb{R}^n)$ if and only if σ is not an algebraic polynomial.

3.5.1 Why does not contradict the Weierstrass approximation theorem?

Theorem 14. *(Weierstrass approximation theorem). Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then, there exists polynomials $p_n \in \mathcal{R}[x]$ such that the sequence (p_n) converge uniformly to f on $[a, b]$.*

Corollary 15. The set of polynomial functions $\mathcal{R}^n[x]$ is dense in the space of continuous functions on a compact set $K \subset \mathbb{R}^n$, $\mathcal{C}(K)$. So any continuous function on a compact set can be approximated arbitrarly well by a polynomial.

The theorem states that: if Σ_n is dense in $\mathcal{C}(\mathbb{R}^n)$ then σ is not an algebraic polynomial. But why this statment does not contradict the Weierstrass approximation theorem ? This does not work because σ has degree fixed k, then any element in the set Σ_n has degree at most k. Hence, the set Σ_n is a finite vector space and can not be dense in $\mathcal{C}(\mathbb{R}^n)$. Not all contiunous functions can be apparoximated with a polynimial of degree fixed, for example: (comment per afegir : per exemple una funcio que sigui continua que no es pugui approximar per un polinomi de com a molt grau k , una k tingui grau mes gran que k polinomi de k+1 ??)

3.5.2 Previous results

The activation functions that were reported thus far in the literature.

Theorem 16. *(Hornik Theorem 1). Standard multilayer feedforward networks with a bounded and nonconstant activation function can approximate any function in $L^p(\mu)$ arbitrary well, given a sufficiently large number of hidden units.*

Theorem 17. *(Hornik Theorem 2) Standard multilayer feedforward networks with a continuous, bounded and nonconstant activation function can approximate any continuous function on X arbitrarily well (with respect to the uniform distance) given a sufficiently large number of hidden units.*

The theorem generalize in particular Hornik's Theorem 2 by establishing necessary and sufficientconditions for universal approximation. Diferencies ? ??

3.6 Results

Definition 18. The set $L^p(\mu)$ contains all mesurable functions f such that:

$$\|f\|_{L^p}(\mu) = \Big(\int_{R^n} |f(x)|^p d\mu(x) \Big)^{1/p} < \infty$$

Proposition 19. Let μ be a non-negative finite measure on $\mathbb{R}$ with compact support, absolutely continous with respect to Lebesgue measure. Then Σ_n is dense in $L_p(\mu)$, $1 \leq p < \infty$, if and only of, σ is not a polynomial.

Chapter 4

Lemmas and proof

This chapter presents the lemmas that are necessary to prove the main theorem. The problem of approximating a function g on some compact K of $\mathbb{R}^n$ from Σ_n, can be divided into two parts. One part is the approximation of the form $\sum_i f_i(a^i \cdot x)$ where f_i are functions in $C(\mathbb{R})$. The other is the approximation of f_i on the appropiate set from Σ_1. !!canviar

4.1 Part 1

Lemma 1. If we have that $\sigma * \varphi$ is a polynomial for all $\varphi \in \mathcal{C}_0^\infty$. Then the degree of the polynomial $\sigma * \varphi$ is finite, i.e. there exists an m $\in \mathbb{N}$ such that $deg(\sigma * \varphi) \leq m$ for all $\varphi \in \mathcal{C}_0^\infty$.

Proof. We first prove the claim in the case of $\varphi \in \mathcal{C}_0^\infty[a,b]$, where $\mathcal{C}_0^\infty[a,b]$ is the set of functions $\mathcal{C}_0^\infty$ with support in $[a,b]$ for any $a < b$.

Let ρ be a metric on $\mathcal{C}_0^\infty[a,b]$ defined by

$$\rho(\varphi_1, \varphi_2) = \sum_{n=0}^{\infty} 2^{-n} \frac{\|\varphi_1 - \varphi_2\|_n}{1 + \|\varphi_1 - \varphi_2\|_n}$$

where $\|\varphi\|_n = \sum_{j=0}^{n} \sup_{x\in[a,b]} |\varphi^{(j)}(x)|$. We can show that $(\mathcal{C}_0^\infty[a,b], \rho)$ is a complete metric space. By assumption, we have that $\sigma * \varphi$ is a polynomial (for any $\varphi \in \mathcal{C}_0^\infty[a,b]$).

Consider the following set, which has the property that we want to show.

$$V_k = \{\varphi \in \mathcal{C}_0^\infty[a,b] \,|\, deg(\sigma * \varphi) \leq k\}$$

Clearly, if $\varphi \in V_k$, then $deg(\sigma * \varphi) \leq k$. We want to show that $\mathcal{C}_0^\infty[a,b] \subseteq V_k$. This set fulfills the following properties, $V_k \subset V_{k+1}$, V_k is a closed subspace and $\cup_{k=0}^{\infty} V_k = \mathcal{C}_0^\infty[a,b]$. As $\mathcal{C}_0^\infty[a,b]$ is a complete metric space, for Blaire's Category Theorem (appendix !!Revisar) then there exists an integer m such that $V_m = \mathcal{C}_0^\infty[a,b]$.

For the general case where $\varphi \in \mathcal{C}_0^\infty$, we note that the number m does not deppend on the interval $[a,b]$. !! acabar

□

Lemma 2. If $\sigma * \varphi$ is a polynomial such that $deg(\sigma * \varphi) \leq m$ for all $\varphi \in \mathcal{C}_0^\infty$, then σ is a polynomial of degree at most m.

Proof. If $\sigma * \varphi$ is a polynomial of degree m. For all $\varphi \in \mathcal{C}_0^\infty$, we have that

$$(\sigma * \varphi)^{(m+1)}(x) = \int \sigma(x-y)\varphi^{(m+1)}(y)\,dy = 0$$

From standard results in Distribution Theory, σ is itself a polynomial of degree at most m (a.e.). !!ho he buscat i no he trobat res perquè implica que sigma polinomi que la integral sigui 0 □

Conclusion: If we have that $\sigma * \varphi$ is a polynomial then σ is a polynomial. This contradicts the hypothesis. Therefore, $\sigma * \varphi$ will not be a polynomial.

4.2 Σ_1 dense in $\mathcal{C}(\mathbb{R})$

Lemma 3. For each $\varphi \in \mathcal{C}_0^\infty$, $\sigma * \varphi \in \overline{\Sigma_1}$.

Proof. Consider

$$h_m = \sum_{i=1}^{m} \varphi(y_i)\Delta y_i \sigma(x - y_i)$$

The sequence (h_m) satisfies $h_j \in \Sigma_1$ for $j = 1, ..., m$. $(w_i = 1, \theta_i = -y_i, \beta_i = \varphi(y_i)\Delta y_i)$.

Where $y_i = -\alpha + \frac{2i\alpha}{m}$, $\Delta y_i = \frac{2\alpha}{m}$ for $i = 1, ..., m$. Partition of the interval $[-\alpha, \alpha]$

We want to show that $h_m \rightrightarrows \sigma * \varphi$ in $[-\alpha, \alpha]$.

Given $\epsilon > 0$, we choose $\delta > 0$ such that $10\delta\|\sigma\|_{L^\infty\{-2\alpha,2\alpha\}}\|\varphi\|_{L^\infty} \leq \frac{\epsilon}{3}$. Note that ...

We know that $\sigma \in M$. Hence, for this given $\delta > 0$ and $[-\alpha, \alpha]$ interval, there exists $r(\delta)$ finite number of intervals the measure of whose union $\mathcal{U}$ is δ such that σ is uniformly continuous on $[-2\alpha, 2\alpha]$. We now choose m_i sufficiently large so that

1. $m_1\delta > \alpha r(\delta)$. We can do this by Archimedes' principle.
2. From the uniform continuity of φ.
3. From the previous, σ is uniformly continuous on $[-2\alpha, 2\alpha]$.

We choose m such that $m = max\{m_1, m_2, m_3\}$.

Now, fix $x \in [-\alpha, \alpha]$. Set $\Delta_i = [y_{i-1},]$ where .. dibuix.

First, recall that,

$$\int \sigma(x-y)\varphi(y)dy = \sum_{i=1}^{m} \int_{\Delta_i} \sigma(x-y)\varphi(y)dy$$

Consider the following difference

$$\left| \int \sigma(x-y)\varphi(y)dy - \sum_{i=1}^{m} \int_{\Delta_i} \sigma(x-y_i)\varphi(y)dy \right| =$$

$$= \left| \sum_{i=1}^{m} \int_{\Delta_i} \sigma(x-y)\varphi(y)dy - \sum_{i=1}^{m} \int_{\Delta_i} \sigma(x-y_i)\varphi(y)dy \right|$$

$$= \left| \sum_{i=1}^{m} \int_{\Delta_i} \varphi(y)\Big(\sigma(x-y) - \sigma(x-y_i)\Big)dy \right|$$

$$\leq \sum_{i=1}^{m} \int_{\Delta_i} |\varphi(y)|\,|\sigma(x-y) - \sigma(x-y_i)|\,dy$$

If $x - \Delta_i \cap U = \varnothing$. Since $x - y \notin U$, $x - y_i \notin U$ and $x - y_i \in [-2\alpha, 2\alpha]$, bc (2) we have

$$\sum_{i=1}^{m} \int_{\Delta_i} |\varphi(y)|\,|\sigma(x-y) - \sigma(x-y_i)|\,dy \leq \frac{\epsilon}{\|\varphi\|_{L_1}} \sum_{i=1}^{m} \int_{\Delta_i} |\varphi(y)| =$$

$$= \frac{\epsilon}{3\|\varphi\|_{L_1}} \int |\varphi(y)|\,dy$$

$$= \frac{\epsilon}{3\|\varphi\|_{L_1}} |\varphi(y)|_{L_1} = \frac{\epsilon}{3}$$

If $x - \Delta_i \cap U \neq \varnothing$

$\sum_i |\widetilde{\Delta_i}| = \sum_i |(x - \Delta_i \cap U)| \leq |U| + 2|\Delta_i| r(\delta) \leq \delta + 2 \cdot \frac{2\alpha}{m} r(\delta) \leq \delta + 4\delta = 5\delta$

True by our choice of m, satisfies $m\delta > \alpha r(\delta) \iff \delta > \frac{\alpha \cdot r(\delta)}{m}$

$$\sum_{i=1}^{m} \int_{\widetilde{\Delta_i}} |\varphi(y)|\,|\sigma(x-y) - \sigma(x-y_i)|\,dy \leq$$

$$\leq \sum_{i=1}^{m} \int_{\widetilde{\Delta_i}} \|\varphi\|_{L^\infty} 2\|\sigma\|_{L^\infty[-2\alpha,2\alpha]}$$

$$= \|\varphi\|_{L^\infty} 2\|\sigma\|_{L^\infty[-2\alpha,2\alpha]} \sum_i |\widetilde{\Delta_i}|$$

$$\leq \|\varphi\|_{L^\infty} 2\|\sigma\|_{L^\infty[-2\alpha,2\alpha]}\, 5\delta \leq \epsilon/3$$

!! Falta acabar □

Lemma 4. If $\sigma \in \mathcal{C}^\infty$, then Σ_1 is dense in $\mathcal{C}(\mathbb{R})$.

Proof. We recall that set $\Sigma_1 = span\{\sigma(w \cdot x + \theta) : w \in \mathbb{R}, \theta \in \mathbb{R}\}$. We can write any function $h \in \Sigma_1$ as $h = \sum_i \beta_i \sigma_i(w_i x + \theta_i) = \beta_1 \sigma_1(w_1 x + \theta_1) + ...$

We can see that $\frac{\sigma([w+h]x+\theta) - \sigma(wx+\theta)}{h} \in \Sigma_1$ because is a linear combination, where

$\beta_1 = \frac{1}{h}, \beta_2 = \frac{-1}{h}$.
By hypothesis, we have $\sigma \in \mathcal{C}^\infty$. By definition of derivative we have

$$\frac{d}{dw}\sigma(wx+\theta) = \lim_{h\to 0}\frac{\sigma([w+h]x+\theta)-\sigma(wx+\theta)}{h} \in \overline{\Sigma_1}^*$$

Because the limit of a set belongs to the closure of the set.
By the same argument, $\frac{d^k}{dw^k}\sigma(wx+\theta) \in \overline{\Sigma_1}$ for all $k \in \mathbb{N}, w, \theta \in \mathbb{R}$.
We observe that $\frac{d}{dw}\sigma(wx+\theta) = \sigma'(wx+\theta)\cdot x$. If we differentiate this expression k times, we obtain

$$\frac{d^k}{dw^k}\sigma(wx+\theta) = \sigma^{(k)}(wx+\theta)\cdot x^k$$

We will see by reduction to absurdity that if σ is not a polynomial (by hypothesis) then there exists a $\theta_k \in \mathbb{R}$ such that $\sigma^{(k)}(\theta_k) \neq 0$.
If σ is not a polynomial and $\sigma \in \mathcal{C}^\infty$, lets assume that $\nexists \theta_k \in \mathbb{R}$ such that $\sigma^{(k)}(\theta_k) \neq 0$.
This means that the k-th derivative at every point is 0,

$$\sigma^{(k)}(\theta) = 0 \quad \forall \theta \in \mathbb{R}$$

If we integrate k times this expression,

$$\int \sigma^{(k)} = \int 0 \Rightarrow \sigma^{(k-1)} = C$$

,

$$\int \sigma^{(k-1)} = \int C \Rightarrow \sigma^{(k-2)} = Cw$$

, then we end up σ is a polynomial. Contradiction. Therefore, there always exists a point where the derivative does not vanish.
Thus, we evaluate at the point where the derivative does not vanish, we call it θ_k.

$$\sigma^{(k)}(\theta_k)\cdot x^k = \frac{d^k}{dw^k}\sigma(wx+\theta)\Big|_{w=0,\theta=\theta_k} \in \overline{\Sigma_1}$$

That implies that $\overline{\Sigma_1}$ contains all polynomials, because the expression $\sigma^{(k)}(\theta_k)x^k$ generates all polynomials. By the Weierstrass theorem, it follows that Σ_1 contains... falta acabar. □

Lemma 5. If for some $\varphi \in \mathcal{C}_0^\infty$ we have that $\sigma * \varphi$ is not a polynomial, then Σ_1 is dense in $\mathcal{C}(\mathbb{R})$

Proof. From Lemma 3, $\sigma * \varphi \in \overline{\Sigma_1}$. Clearly, $\sigma * \varphi(wx+\theta) \in \overline{\Sigma_1}$, for each $\theta \in \mathbb{R}$. For σ and $\varphi \in \mathcal{C}_0^\infty$ we have that $\sigma * \varphi \in \mathcal{C}^\infty$ □

*$\overline{\Sigma_1}$ denotes the clausure of the set Σ_1

4.3 Σ_1 is dense in $\mathcal{C}(\mathbb{R})$, then Σ_n is dense in $\mathcal{C}(\mathbb{R}^n)$

We will proof that approximating a $\mathcal{C}(\mathbb{R})$ function with one from the set Σ_1 implies approximating a function $\mathcal{C}(\mathbb{R}^n)$ from the set Σ_n. Therefore, it is only necessary to approximate a continuous function. We can see this from the density characterization:

Lemma 6. If Σ_1 is dense in $\mathcal{C}(\mathbb{R})$, then Σ_n is dense in $\mathcal{C}(\mathbb{R}^n)$.

Proof. Let

$$V := span\{f(ax) : a \in \mathbb{R}^n, f \in \mathcal{C}(\mathbb{R})\}$$

We shall see that V is dense in $\mathcal{C}(\mathbb{R}^n)$. If we show that V contains the polynomials (are dense in $C(R^n)$ for weierstrass theorem) ja estaria.

Let $L(a)$ denote the span of the n rows of a for each $a \in \mathbb{R}^n$. Set $L(R^n) = \cup L(a)$. Let

$$H_k^n = \{\sum c_m s^m\}$$

denote the set of homogeneous polynomials of n variables of total degree k, and

$$H^n = \cup_{k=0}^{\infty} H_k^n$$

the set of all homogeneous polynomials of n variables.

Assume that for a given $k \in \mathbb{N}$ no non-trivial $p \in H_k^n \subseteq V$ for all $k \in \mathbb{Z}$, then V contains all polynomials. For that we have V dense in $\mathcal{C}(\mathbb{R}^n)$. Now, we only need to show that $H_k^n \subseteq V$. SOS

Let $g \in \mathcal{C}(\mathbb{R})$, for any compact subset $K \subset \mathbb{R}^n$, V dense in $\mathcal{C}(K)$. That is, given $\epsilon > 0$, there exist $f_i \in \mathcal{C}(\mathbb{R})$ and $a_i \in \mathbb{R}^n$ $_{i=1,...,k}$ such that

$$\left|g(x) - \sum_{i=1}^{k} f_i(a^i \cdot x)\right| < \frac{\epsilon}{2}$$

for all $x \in K$. We now consider the set of all the points in the compact K multiplied by the vector a^i. That is $\{a^i \cdot x | x \in K\} \subseteq [\alpha_i, \beta_i]$ for some finite interval $[\alpha_i, \beta_i]$, $i = 1, ..., k$. By hipothesis Σ_1 dense in $\mathcal{C}(\mathbb{R})$, specifically Σ_1 is dense in $[\alpha_i, \beta_i]$ $i = 1, ..., k$. Hence there exist constants c_{ij}, w_{ij} and θ_{ij}, $j = 1, ..., m_i$, $i = 1, ..., k$ such that

$$\left| f_i(y) - \sum_{j=1}^{m} c_{ij}\sigma(w_{ij}y + \theta_{ij})\right| < \frac{\epsilon}{2k}$$

for all $x \in K$.

Therefore,

$$\left|g(x) - \sum_{i=1}^{k}\sum_{j=1}^{m} c_{ij}\sigma(w_{ij}(a^i \cdot x) + \theta_{ij})\right| < \epsilon$$

□

We showed that to approximate a $\mathcal{C}(\mathbb{R}^n)$ function we only need to approximate a $\mathcal{C}(\mathbb{R})$ function with the set Σ_1.

4.4 Proof of the theorem

Proof.

$\Rightarrow$ To prove the implication, we will use proof by contrapositive. We will see the following. If σ is a polynomial then Σ_n is not dense in $\mathcal{C}(\mathbb{R}^n)$. Let σ be a polynomial of degree k, then $\sigma(wx+\theta)$ is a polynomial of degree k for every w, θ. We have $\Sigma_n = span\{\sigma(w \cdot x + \theta) : w \in \mathbb{R}^n, \theta \in \mathbb{R}\}$ that is the set of algebraic polynomials of degree at most k. Σ_n is not dens in $\mathcal{C}(\mathbb{R}^n)$ if for a function $f(x) \in \mathcal{C}(\mathbb{R}^n)$ we can find $\epsilon > 0$ and K such that $\|p - f\| > \epsilon$ for all p polynomial of degree k. For example, let $f(x) = cos(x)$, and let $p(x) = \sigma(wx+\theta)$ that has degree at most k. This implies has maximum k roots. We can find a interval where $cos(x)$ has k+1 roots. Therefore, Σ_n is not dense in $\mathcal{C}(\mathbb{R}^n)$.

$\Leftarrow$ Recapitulem el que hem vist als lemes ..

□

Leshno et al. [1993]

Chapter 5

Results

$$t = x + y \tag{5.1}$$

Chapter 6

Conclusions

> It is a mistake to confound strangeness with mystery.
>
> — Sherlock Holmes, *A Study in Scarlet*

6.1 Summary

6.2 Outlook and Future Work

Hem trobat:

- Aaaaa
- Bbbbbb

Chapter 7

References

M. Leshno, V. Y. Lin, A. Pinkus, and S. Schocken. Multilayer feedforward networks with a nonpolynomial activation function can approximate any function. *Neural Networks*, 6(6):861–867, 1993.

Appendix A

Theory used

Definition 20. Riemann integral reminder. The Riemann integral is a method for calculating the volume under a curve of a continuous function on a closed, bounded domain in R^n. The method involves dividing the domain into smaller subregions and approximating the volume of each subregion with a rectangular solid whose height is the function value at a specific point in the subregion. The Riemann sum is the sum of the volumes of all the rectangular solids, and as the size of the subregions approaches zero, the Riemann sum converges to the Riemann integral.

Definition 21. Let Σ be a σ-algebra over a set Ω. A *measure* over Ω is any function

$$\mu : \Sigma \longrightarrow [0, \infty]$$

satisfying the following properties:

1. $\mu(\varnothing) = 0$.
2. σ-*additivity*: If $(A_n) \in \Sigma$ are pairwise disjoint, then:

$$\mu\left(\bigsqcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

Definition 22. A metric space (X, d) is said to be *complete* if every Cauchy sequence in X converges to a point in X.

Definition 23. We say that a property holds almost everywhere (a.e.) if the set of points that doesn't hold it is null.

Definition 24. $\varphi : I \to \mathbb{R}$ is uniformly continuous on I if $\forall \epsilon > 0 \exists\ \delta > 0$ such that $|\varphi(x) - \varphi(y)| < \epsilon$ whenever $|x - y| < \delta$

A.1 Blaire's category theorem

Definition 25. Let A be a subset of the metric space (X, d). A is said to be *nowhere dense* if for every (nonempty) open subset $U \subseteq X$, the intersection $U \cap \overline{A}$ is not dense in U, meaning that U contains a point that is not in the closure of A.

Definition 26. A set it is said to be category I if it can be written as a countable union of nowhere-dense sets. Otherwise it is said to be of *category II*

Theorem 27. *(Blaire's Category Theorem) Any complete metric space is of category II.*

Therefore, if we have $\mathcal{C}_0^\infty[a,b]$ complete metric space, we know that is of category II, i.e. $\mathcal{C}_0^\infty[a,b]$ cannot be written as a countable union of nowhere-dense sets. We have $\cup_{k=0}^\infty V_k = \mathcal{C}_0^\infty[a,b]$. Therefore, some V_m contains a nonempty open set. V_m is a vector space thus $V_m = \mathcal{C}_0^\infty[a,b]$. no entenc el final