Chapter 1

Lemmas and proof

This chapter presents the lemmas that are necessary to prove the main theorem.

1.1 Part 1

Lemma 1. If we have that $\sigma * \varphi$ is a polynomial for all $\varphi \in \mathcal{C}_0^\infty$. Then the degree of the polynomial $\sigma * \varphi$ is finite, i.e. there exists an m $\in \mathbb{N}$ such that $deg(\sigma * \varphi) \leq m$ for all $\varphi \in \mathcal{C}_0^\infty$.

Proof. We first prove the claim in the case of $\varphi \in \mathcal{C}_0^\infty[a,b]$, where $\mathcal{C}_0^\infty[a,b]$ is the set of functions $\mathcal{C}_0^\infty$ with support in $[a,b]$ for any $a < b$.

Let ρ be a metric on $\mathcal{C}_0^\infty[a,b]$ defined by

$$\rho(\varphi_1, \varphi_2) = \sum_{n=0}^{\infty} 2^{-n} \frac{\|\varphi_1 - \varphi_2\|_n}{1 + \|\varphi_1 - \varphi_2\|_n}$$

where $\|\varphi\|_n = \sum_{j=0}^{n} \sup_{x\in[a,b]} |\varphi^{(j)}(x)|$. We can show that $(\mathcal{C}_0^\infty[a,b], \rho)$ is a complete metric space. By assumption, we have that $\sigma * \varphi$ is a polynomial (for any $\varphi \in \mathcal{C}_0^\infty[a,b]$).

Consider the following set, which has the property that we want to show.

$$V_k = \{\varphi \in \mathcal{C}_0^\infty[a,b] \,|\, deg(\sigma * \varphi) \leq k\}$$

Clearly, if $\varphi \in V_k$, then $deg(\sigma * \varphi) \leq k$. We want to show that $\mathcal{C}_0^\infty[a,b] \subseteq V_k$. This set fulfills the following properties, $V_k \subset V_{k+1}$, V_k is a closed subspace and $\cup_{k=0}^{\infty} V_k = \mathcal{C}_0^\infty[a,b]$. As $\mathcal{C}_0^\infty[a,b]$ is a complete metric space, for Blaire's Category Theorem (appendix) then there exists an integer m such that $V_m = \mathcal{C}_0^\infty[a,b]$.

For the general case where $\varphi \in \mathcal{C}_0^\infty$, we note that the number m does not deppend on the interval $[a,b]$.

□

Lemma 2. If $\sigma * \varphi$ is a polynomial such that $deg(\sigma * \varphi) \leq m$ for all $\varphi \in \mathcal{C}_0^\infty$, then σ is a polynomial of degree at most m.

Proof. If $\sigma * \varphi$ is a polynomial of degree m. For all $\varphi \in \mathcal{C}_0^\infty$, we have that

$$(\sigma * \varphi)^{(m+1)}(x) = \int \sigma(x-y)\varphi^{(m+1)}(y)\, dy = 0$$

From standard results in Distribution Theory, σ is itself a polynomial of degree at most m (a.e.). NO SE PQ $\square$

Conclusion: If we have that $\sigma * \varphi$ is a polynomial then σ is a polynomial. This contradicts the hypothesis. Therefore, $\sigma * \varphi$ will not be a polynomial.

1.2 Σ_1 dense in $\mathcal{C}(\mathbb{R})$

Lemma 3. For each $\varphi \in \mathcal{C}_0^\infty$, $\sigma * \varphi \in \overline{\Sigma_1}$.

Proof. Consider

$$h_m = \sum_{i=1}^{m} \varphi(y_i)\Delta y_i \sigma(x - y_i)$$

The sequence (h_m) satisfies $h_j \in \Sigma_1$ for $j = 1, ..., m$. $(w_i = 1, \theta_i = -y_i, \beta_i = \varphi(y_i)\Delta y_i)$.

Where $y_i = -\alpha + \frac{2i\alpha}{m}$, $\Delta y_i = \frac{2\alpha}{m}$ for $i = 1, ..., m$. Partition of the interval $[-\alpha, \alpha]$

We want to show that $h_m \rightrightarrows \sigma * \varphi$ in $[-\alpha, \alpha]$.

Given $\epsilon > 0$, we choose $\delta > 0$ such that $10\delta\|\sigma\|_{L^\infty\{-2\alpha,2\alpha\}}\|\varphi\|_{L^\infty} \leq \frac{\epsilon}{3}$. Note that ...

We know that $\sigma \in M$. Hence, for this given $\delta > 0$ and $[-\alpha, \alpha]$ interval, there exists $r(\delta)$ finite number of intervals the measure of whose union $\mathfrak{U}$ is δ such that σ is uniformly continuous on $[-2\alpha, 2\alpha]$. We now choose m_i sufficiently large so that

1. $m_1\delta > \alpha r(\delta)$. We can do this by Archimedes' principle.
2. From the uniform continuity of φ.
3. From the previous, σ is uniformly continuous on $[-2\alpha, 2\alpha]$.

We choose m such that $m = max\{m_1, m_2, m_3\}$.

Now, fix $x \in [-\alpha, \alpha]$. Set $\Delta_i = [y_{i-1},]$ where .. dibuix.

First, recall that,

$$\int \sigma(x-y)\varphi(y)dy = \sum_{i=1}^{m} \int_{\Delta_i} \sigma(x-y)\varphi(y)dy$$

Consider the following difference

$$
\begin{aligned}
\left| \int \sigma(x-y)\varphi(y)dy - \sum_{i=1}^{m} \int_{\Delta_i} \sigma(x-y_i)\varphi(y)dy \right| &= \\
&= \left| \sum_{i=1}^{m} \int_{\Delta_i} \sigma(x-y)\varphi(y)dy - \sum_{i=1}^{m} \int_{\Delta_i} \sigma(x-y_i)\varphi(y)dy \right| \\
&= \left| \sum_{i=1}^{m} \int_{\Delta_i} \varphi(y)\Big(\sigma(x-y) - \sigma(x-y_i)\Big)dy \right| \\
&\leq \sum_{i=1}^{m} \int_{\Delta_i} |\varphi(y)|\, |\sigma(x-y) - \sigma(x-y_i)|\, dy
\end{aligned}
$$

If $x - \Delta_i \cap U = \varnothing$. Since $x - y \notin U$, $x - y_i \notin U$ and $x - y_i \in [-2\alpha, 2\alpha]$, bc (2) we have

$$
\begin{aligned}
\sum_{i=1}^{m} \int_{\Delta_i} |\varphi(y)|\, |\sigma(x-y) - \sigma(x-y_i)|\, dy \leq \frac{\epsilon}{\|\varphi\|_{L_1}} \sum_{i=1}^{m} \int_{\Delta_i} |\varphi(y)| &= \\
&= \frac{\epsilon}{3\|\varphi\|_{L_1}} \int |\varphi(y)|\, dy \\
&= \frac{\epsilon}{3\|\varphi\|_{L_1}} |\varphi(y)|_{L_1} = \frac{\epsilon}{3}
\end{aligned}
$$

If $x - \Delta_i \cap U \neq \varnothing$

$\sum_i |\widetilde{\Delta_i}| = \sum_i |(x - \Delta_i \cap U)| \leq |U| + 2|\Delta_i| r(\delta) \leq \delta + 2 \cdot \frac{2\alpha}{m} r(\delta) \leq \delta + 4\delta = 5\delta$

True by our choice of m, satisfies $m\delta > \alpha r(\delta) \iff \delta > \frac{\alpha \cdot r(\delta)}{m}$

$$
\begin{aligned}
\sum_{i=1}^{m} \int_{\widetilde{\Delta_i}} |\varphi(y)|\, |\sigma(x-y) - \sigma(x-y_i)|\, dy &\leq \\
&\leq \sum_{i=1}^{m} \int_{\widetilde{\Delta_i}} \|\varphi\|_{L^\infty}\, 2\|\sigma\|_{L^\infty[-2\alpha,2\alpha]} \\
&= \|\varphi\|_{L^\infty}\, 2\|\sigma\|_{L^\infty[-2\alpha,2\alpha]} \sum_i |\widetilde{\Delta_i}| \\
&\leq \|\varphi\|_{L^\infty}\, 2\|\sigma\|_{L^\infty[-2\alpha,2\alpha]}\, 5\delta \leq \epsilon/3
\end{aligned}
$$

□

Lemma 4. If $\sigma \in \mathcal{C}^\infty$, then Σ_1 is dense in $\mathcal{C}(\mathbb{R})$.

Proof. We recall that set $\Sigma_1 = span\{\sigma(w \cdot x + \theta) : w \in \mathbb{R}, \theta \in \mathbb{R}\}$. We can write any function $h \in \Sigma_1$ as $h = \sum_i \beta_i \sigma_i(w_i x + \theta_i) = \beta_1 \sigma_1(w_1 x + \theta_1) + \ldots$

$\frac{\sigma([w+h]x+\theta)-\sigma(wx+\theta)}{h} \in \Sigma_1$ because is a linear combination, where $\beta_1 = \frac{1}{h}, \beta_2 = \frac{-1}{h}$....
By hypothesis, we have $\sigma \in \mathcal{C}^\infty$. By definition of derivative we have

$$\frac{d}{dw}\sigma(wx+\theta) = \lim_{h\to 0}\frac{\sigma([w+h]x+\theta)-\sigma(wx+\theta)}{h} \in \overline{\Sigma_1}^{*}$$

Because the limit of a set belongs to the closure of the set.
By the same argument, $\frac{d^k}{dw^k}\sigma(wx+\theta) \in \overline{\Sigma_1}$ for all $k \in \mathbb{N}, w, \theta \in \mathbb{R}$.
We observe that $\frac{d}{dw}\sigma(wx+\theta) = \sigma'(wx+\theta)\cdot x$. If we differentiate this expression k times, we obtain

$$\frac{d^k}{dw^k}\sigma(wx+\theta) = \sigma^{(k)}(wx+\theta)\cdot x^k$$

Since σ is not a polynomial (theorem hypothesis) then there exists a $\theta_k \in \mathbb{R}$ such that $\sigma^{(k)}(\theta_k) \neq 0$
Lets see.**** If σ is not a polynomial and $\sigma \in \mathcal{C}^\infty$, lets assume that $\nexists \theta_k \in \mathbb{R}$ such that $\sigma^{(k)}(\theta_k) \neq 0$. This means that the k-th derivative at every point is 0, i.e, $\sigma^{(k)}(\theta) = 0 \ \forall \theta \in \mathbb{R}$. If we integrate k times, $\int \sigma^{(k)} = \int 0 \iff \sigma^{(k-1)} = C$, $\int \sigma^{(k-1)} = \int C \iff \sigma^{(k-2)} = Cw$, then we end up σ is a polynomial. Contradiction. Therefore, there always exists a point where the derivative does not vanish.
Thus, we evaluate at this point θ_k where the derivative does not vanish.

$$\sigma^{(k)}(\theta_k)\cdot x^k = \frac{d^k}{dw^k}\sigma(wx+\theta)\Big|_{w=0,\theta=\theta_k} \in \overline{\Sigma_1}$$

That implies that $\overline{\Sigma_1}$ contains all polynomials, because the expression $\sigma^{(k)}(\theta_k)x^k$ generates all polynomials. By the Weierstrass theorem, it follows that Σ_1 contains... falta mirar. □

Lemma 5. If for some $\varphi \in \mathcal{C}_0^\infty$ we have that $\sigma * \varphi$ is not a polynomial, then Σ_1 is dense in $\mathcal{C}(\mathbb{R})$

Proof. From Lemma 3, $\sigma * \varphi \in$ □

1.3 Σ_n dense in $\mathcal{C}(\mathbb{R}^n)$

Lemma 6. If Σ_1 is dense in $\mathcal{C}(\mathbb{R})$, then Σ_n is dense in $\mathcal{C}(\mathbb{R}^n)$.

Proof. Let $V := span\{f(ax) : a \in \mathbb{R}^n, f \in \mathcal{C}(\mathbb{R})\}$. V is dense in $\mathcal{C}(\mathbb{R}^n)$.
Let $g \in \mathcal{C}(\mathbb{R})$, for any compact subset $K \subset \mathbb{R}^n$, V dense in $\mathcal{C}(K)$. That is, given $\epsilon > 0$, there exist $f_i \in \mathcal{C}(\mathbb{R})$ and $a_i \in \mathbb{R}^n$ $_{i=1,\dots,k}$ such that □

1.4 Proof of the theorem

Proof.

*$\overline{\Sigma_1}$ denotes the clausure of the set Σ_1

$\Rightarrow$ To prove the implication, we will use proof by contrapositive. We will see the following. If σ is a polynomial then Σ_n is not dense in $\mathcal{C}(\mathbb{R}^n)$. Let σ be a polynomial of degree k, then $\sigma(wx+\theta)$ is a polynomial of degree k for every w, θ. We have $\Sigma_n = span\{\sigma(w \cdot x + \theta) : w \in \mathbb{R}^n, \theta \in \mathbb{R}\}$ that is the set of algebraic polynomials of degree at most k.

Σ_n is not dens in $\mathcal{C}(\mathbb{R}^n)$ if for a function $f(x) \in \mathcal{C}(\mathbb{R}^n)$ we can find $\epsilon > 0$ and K such that $\|p - f\| > \epsilon$ for all p polynomial of degree k. For example, let $f(x) = cos(x)$, and $p(x) = \sigma(wx+\theta)$ that has degree k. This implies has maximum k roots. We can find a interval where there are k+1 roots.

$\Leftarrow$ Recapitulem el que hem vist als lemes ..

$\square$

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