UAB
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Treball Final de
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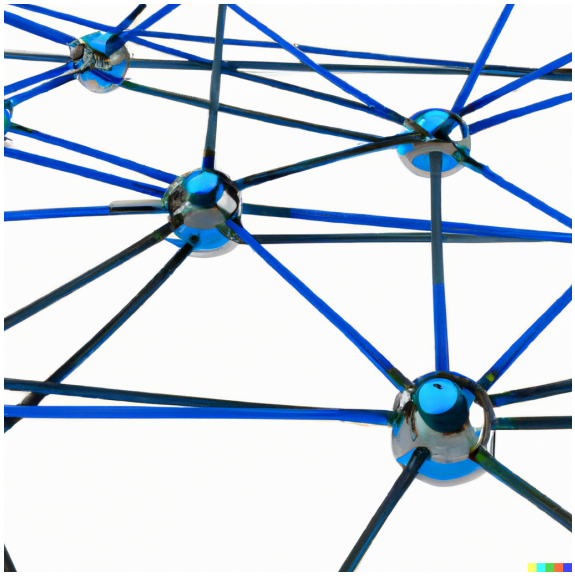
Machine learning: mathematical foundations

Alícia Chimeno Sarabia



Supervisor
Roberto Rubio

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Abstract

Nowadays, machine learning models are being applied more and more, but it is the task of mathematicians to understand their complex underlying principles. How can we ensure the existence of a predictive function from a given dataset? In this work, we will take an analytical approach to machine learning, emphasizing function approximation as a central component. This research seeks to address these concerns by exploring the mathematical foundations of function approximation in machine learning, with a specific focus on neural networks.

In particular, we delve into a significant finding, the theorem proved by Leshno-Lin-Pinkus-Schonken in 1993 [LLPS93], which states that a multilayer feedforward network equipped with a non-polynomial activation function can effectively approximate any continuous function. Our work revolves around understanding and reinterpreting the proof, while expanding and providing further details. Through this study, we aim to bridge the gap between the practical application of machine learning and the mathematical principles that underpin its success.

Resum

En l'actualitat, cada cop més s'apliquen models de machine learning, però és tasca dels matemàtics entendre el seu complex rerefons. Com podem garantir l'existència d'una funció predictiva a partir d'un conjunt de dades donat? En aquest treball prendrem una visió analitica al machine learning posant èmfasi en l'aproximació de funcions com a component central. Aquesta recerca pretén abordar aquestes qüestions, explorant els fonaments matemàtics de l'aproximació de funcions en l'aprenentatge automàtic, amb un focus específic en les xarxes neuronals.

En particular, aprofundim en una troballa important, el teorema demostrat per Leshno-Lin-Pinkus-Schonken el 1993, que afirma que una xarxa d'alimentació cap endavant multicapa equipada amb una funció d'activació no polinomial pot aproximar efectivament qualsevol funció contínua. El nostre treball gira entorn de comprendre i reinterpretar la demostració, alhora que s'amplia i es proporcionen més detalls. A través d'aquest estudi, pretenem establir un nexe entre l'aplicació pràctica de l'aprenentatge automàtic i els principis matemàtics que sustenten el seu èxit.

Preface

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Contents

Chapter 1

Introduction

> Computers are like a bicycle for our minds.
>
> — Steve Jobs, *Michael Lawrence Films*

Our brain is constantly classifying and recognizing. For instance, when we spot a dog on the street, one easy classification we can make is {dog, not dog}, which is probably too easy for our brain—it's almost instantaneous. However, things get a bit more complex when we read the teacher's whiteboard. What happens when we encounter a symbol that confuses us because it resembles another? We can interpret the mathematics behind this reasoning as the brain seeking/creating a function that provides us with the certainty of recognizing that particular letter. Eventually, we reach a point where we feel confident enough to write it down in our notes.

Artificial intelligence aims to replicate the remarkable capabilities of our brains. It seeks to develop computational models and algorithms that can perform tasks such as classification, recognition, and decision-making with a level of accuracy and efficiency comparable to human intelligence. When AI first emerged, one of the initial challenges was actually hand-written digit recognition, exemplified by the MNIST digits dataset. This dataset comprises 60,000 examples of handwritten digits from 0 to 9. To enable a machine learning model to recognize these digits, it must effectively map each image to its corresponding number. This problem naturally aligns with a mathematician's perspective of function learning, where the goal is to approximate a function based on a given dataset consisting of points in space.

Neural networks are widely employed in artificial intelligence to address various problems. The theory of function approximation using neural networks has a long history, dating back to the work by McCulloch and Pitts in the 1940s [MP43]. These pioneers laid the groundwork for understanding how neural networks can mimic the behavior of biological neurons to compute complex functions. Since then, significant advancements have been made in the design, training, and optimization of neural networks, enabling them to tackle increasingly challenging tasks.

This Bachelor's thesis aims to explore the [LLPS93] Theorem, a fundamental result in deep learning that establishes the necessary and sufficient conditions for an activation function to enable neural networks to act as universal approximators.

To motivate this theorem, in the first chapter, we start by introducing some basic concepts of machine learning, accompanied by examples of models such as linear regression and logistic regression. We then proceed with a detailed description of a neural network model, accompanied by its purely mathematical definition. In the following chapter, we dive deeper into function approximation, providing necessary definitions and results. The fourth and final chapter presents the main theorem along with its extensive and detailed proof. We pay special attention to..

Chapter 2

Machine Learning

2.1 Machine Learning Basics

Machine Learning focuses on the development of algorithms and models that enable computers to learn from data with the aim of making predictions without being explicitly programmed.

The machine learning model is built using one or more input variables which are also called **predictors** or independent variables. The output of this model is the **response** or dependent variable which we want to predict. Machine learning is about learning an approximate function that can be used to predict the value of response variable.

We can think about learning as the way we understand it as a human. We can classify a learning problem based on the degree of feedback. Machine learning models fall into three primary categories:

- Supervised learning, where we have immediate feedback.
- Reinforcement learning, where we have indirect feedback. For example when we are playing the game of chess.
- Unsupervised learning, where we have non-feedback signal. For example, deducing which dog belongs to each owner.

Machine learning models simplify reality for the purposes of understanding or prediction. This prediction can be either a numerical prediction or a classification prediction. Several machine learning algorithms are commonly used, for example to name a few: linear regression, logistic regression, decision trees, random forests...

2.1.1 Motivation

In order to motivate our study of machine learning, we are going to present some examples.

Example 1. Let us consider the following hypothetical scenario. Imagine that we are the Data Scientist of a big football club. The club needs a new main striker for the next season and we are tasked with evaluating each candidate and decide whether to sign them or not. We have loads of data from each player.

Player Attributes Last Season:	
Age	21
Matches Played	38
Goals	14
Assists	11
Expected goals	10.56
Shots on target	32
...	...

- Input: $x_c = (x_{c_1}, ..., x_{c_d})$ "attributes of the player".
- Output:
$$y = \begin{cases} \text{sign} \\ \text{not sign} \end{cases}$$
- Target function: f "ideal player signing formula".
- The dataset $\{(x_1, y_1), (x_2, y_2), ..., (x_n, y_n)\}$ consists of historical records of strikers, where x_i represents the player's attributes and y_i indicates the classification of whether they were signed or not.

We are looking for the function f such that $f(x_c) = y$.

An important aspect of machine learning is that *many supervised learning tasks are about function learning.* In general, a fundamental problem in machine learning can be defined as follows: given a dataset of the form $\{(x_i, y_i)\}_{i=1}^m \subset \mathbb{R}^n \times \mathbb{R}$, the goal is to find a model f that accurately predicts the output y_i for a given input x_i.

Example 2. An example of a supervised learning task is digit recognition. The objective is to identify handwritten digits (0-9) based on input images. In this task, we aim to learn a probability distribution function denoted as f, which maps a set of pixel values ranging from 0 (black) to 255 (white), representing a 28x28 image, to a probability distribution over the digits 0 to 9. In practice, we often learn the function
$$f : \{0, ..., 255\}^{28 \times 28} \longrightarrow \mathbb{R}^{10}$$
where big values represent that is very likely and small very unlikely.

Example 3. Example of a classification problem. We want to classify if an image is a dog or not a dog. We would like to produce a value which is correlated with the probability of this image being a dog or not a dog. We can approach the problem in the following way. We want to find a function that takes very high values when dog-image and very low values when non dog images and takes the value 0 when its uncertain. That function is

$$d : \mathbb{R}^{\#\text{pixels in image}} \to \mathbb{R}.$$

This is what we mean by many problems can be recast as function learning. Note that there is not a God-given reason why this function should exist. We know that certain points in space, and they have certain values associated to them, but we dont know that there is some function.

2.1.2 Linear Regression

A linear regression algorithm is used to predict numerical values, based on a linear relationship between different values. A simple linear model is defined by the following equation:

$$y_i = w_0 + w_1 x_i + \epsilon_i$$

where $i = 1, ..., n$.

Note that y is the dependent variable (response), x is the independent variable (predictor), w_0 is the intercept, w_1 is the slope coefficient and ϵ is the error term or the residual.

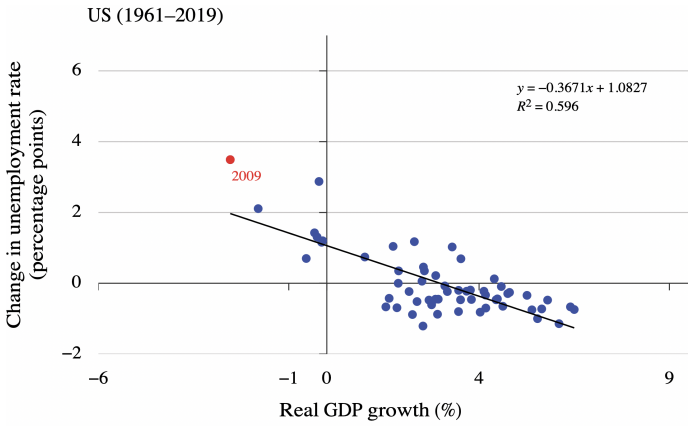

Figure 2.1: y response variable: unemployment rate , x predictor: GDP growth. [BS17]

We can add additional p predictors to a simple linear model, transforming it into a multivariate linear model, which we define as follows:

$$y_i = w_0 + w_1 x_{1i} + \ldots + w_p x_{pi} + \epsilon_i.$$

More commonly, the multivariate linear regression equation is expressed in matrix form as: $y = w^T x + \theta$.

2.1.3 Logistic Regression

Logistic regression is a model for predicting the probability that a binary response is 1. It is suitable for classification tasks, as well as for prediction of probabilities. From a statistical perspective, it is defined by assuming that the distribution of the binary response variable, y, given the features, x, follows a Bernoulli distribution with success probability p.

$$P(y = 1|X = x) = p \qquad \text{and} \qquad P(y = 0|X = x) = 1 - p.$$

We need to define the concept of sigmoid function that will be important along the work. A *sigmoid function* is a mathematical function that maps input values to a range between 0 and 1. We consider the following sigmoid function, the logit inverse function:

$$\text{logit} : (0, 1) \to \mathbb{R} \quad \text{and is expressed as:} \quad \text{logit}(x) = \log\left(\frac{x}{x-1}\right)$$

$$\text{logit}^{-1} : \mathbb{R} \to (0, 1) \quad \text{and is expressed as:} \quad \text{logit}^{-1}(x) = \frac{e^x}{1+e^x}$$

The linear predictor, $w^T x + \theta$, fluctuates between $(-\infty, \infty)$ where x represents all predictors in the model. To address this difference in scale, the outcome variable is transformed using the logit function. The logistic regression model assumes a linear (affine) relationship between the feature vector x_i and the log odds of p. Namely,

$$\text{logit}(p) = w^T x + \theta.$$

The logistic model can be alternatively expressed using the inverse logit function:

$$P(y = 1|X = x) = \text{logit}^{-1}(w^T x + \theta).$$

2.2 Multilayer Feedforward Networks

Artificial Neural Networks (ANN) are the quintessential deep learning models, especially *multilayer feedforward networks*. They are widely used for nonlinear function approximation. The goal of an artificial neural network is to approximate some function f^*. For example, for a classifier, $y = f^*(x)$ maps an input x to a category y.

The term *neural* refers to the fact that this model was originally inspired by how biological neurons process information.

The term *feedforward* indicates the direction of information flow within the network, moving only forward in contraposition to backwards. Each layer processes the input data and passes its output to the next layer, creating a sequence of transformations until the final output is produced.

The term *network* refers to the interconnected structure of artificial neurons. A multilayer network consists of multiple layers, including an input layer, one or more

hidden layers, and an output layer.

The architecture of the network entails determining its *depth*, *width*, and *activation functions* used. Depth is the number of hidden layers. Width is the number of units (nodes) on each hidden layer. The *activation function* defines how the weighted sum of the input is transformed into an output from a node in a layer of the network. Because the activation function plays a crucial role in our work, further details regarding its importance will be provided in the next section.

2.3 Architecture of a Multilayer Feedforward Network

2.3.1 Artificial neuron

The equation

$$y = \sigma(w^T x + \theta) \tag{2}$$

represents what we may call a *single layer of a deep learning model*, also called an *artificial neuron*. Observe that the artificial neuron is composed of an affine transformation $z = w^T x + \theta$ followed by a (generally) non-linear transformation $\sigma(z)$.

In more detail, $x \in \mathbb{R}^n$ is the input vector and represents a set of n features or predictors, $w \in \mathbb{R}^n$ is the weights vector where each element of the weights vector w_i corresponds to the importance assigned to the corresponding input feature x_i. θ is the bias and σ is the activation function. The result variable is a scalar output $y \in \mathbb{R}$.

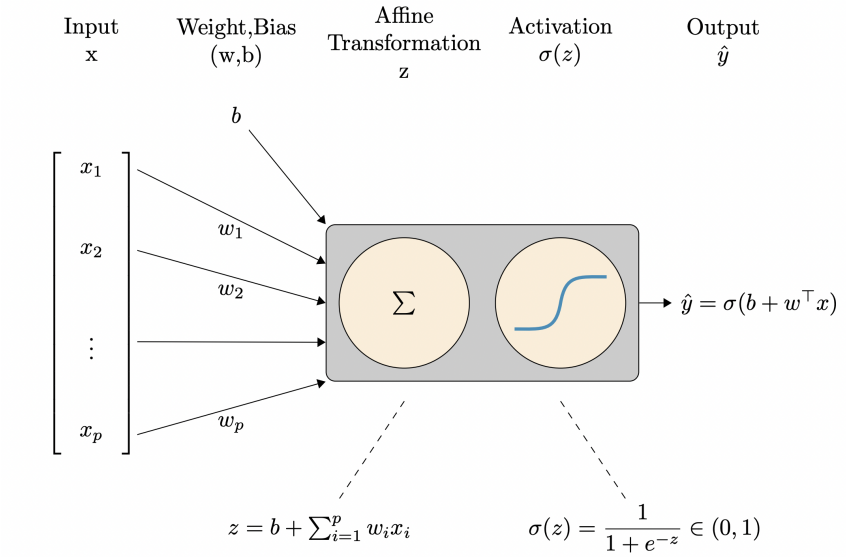

Figure 2.2: Components of an artificial neuron. [LMN23]

2.3.2 Activation Function

The introduction of the activation function in ANN was inspired by biological neural networks whose purpose is to decide whether a particular neuron fires or not. The simple addition of such function can tremendously help the network to exploit more, thereby learning faster. There are various activation functions proposed in the literature, and it is difficult to find the optimal activation function that can tackle any problem.

Note that a logistic regression is an artificial neuron where the activation function σ is logit^{-1}. Two widely popular activation functions are the Hyperbolic Tangent and Rectified Linear Unit (ReLU):

$$\text{Hyperbolic Tangent (tanh): } \mathbb{R} \to (-1, 1)$$

and

$$\text{ReLU} : \mathbb{R} \to (0, \infty) \quad \text{and is expressed as:} \quad \text{ReLU}(x) = \max(0, x)$$

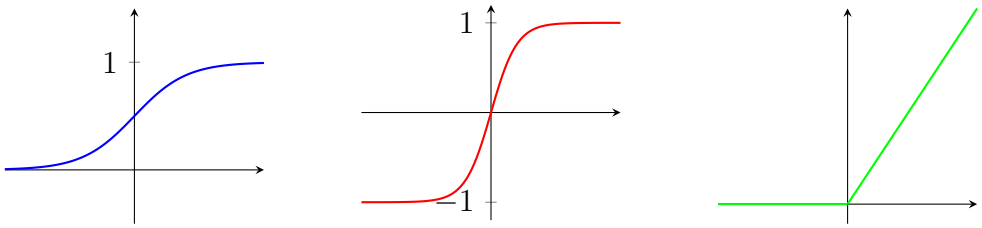

Figure 2.3: Graphs of the sigmoid, hyperbolic tangent, and ReLU functions.

2.3.3 Definition

The general architecture of a multilayer feedforward network consists of an input layer with n input-units, an output layer with m output-units, and one or more hidden layers consisting of intermediate processing units.

Because a mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ can be computed by m mappings : $f_j : \mathbb{R}^n \to \mathbb{R}$ it is (theoretically) sufficient to focus on networks with one output-unit only. In addition, since our findings require only a single hidden layer, we will assume hereafter that the network consists of three layers only: input, hidden, and output.

The Figure 2.4 show a more general network with several hidden layers and several outputs, meanwhile the Figure 2.4 depricates a simplifyed one with one output and one hidden layer.

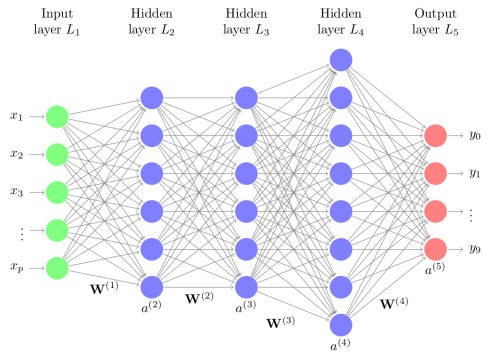

Figure 2.4: Neural Network with 3 hidden layers and 4 outputs.

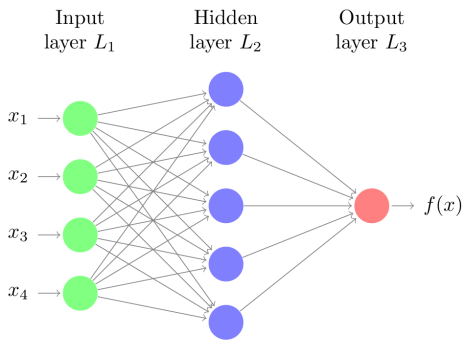

Figure 2.5: Neural Network with 1 hidden layer and 1 output.

Definition 1. A *multilayer feedforward network* is the function

$$f(x) = \sum_{j=1}^{k} \beta_j \cdot \sigma(w_j \cdot x - \theta_j)$$

where $x \in \mathbb{R}^n$ is the input vector, $k \in \mathbb{N}$ is the number of processing units in the hidden layer, $w_j \in \mathbb{R}^n$ is the weight vector that connects the input to processing unit j in the hidden layer, $\sigma : \mathbb{R} \to \mathbb{R}$ is an activation function, $\theta_j \in \mathbb{R}$ is the threshold (or bias) associated with processing unit j in the hidden layer, and $\beta_j \in \mathbb{R}$ is the weight that connects the processing unit j in the hidden layer to the output of the network.

Let N_w be the family of all functions that can be describe with a given network architecture. If we can show that N_w is dense in $C(\mathbb{R}^n)$, we can conclude that for every continuous function $g \in C(\mathbb{R}^n)$ and each compact set $K \subset \mathbb{R}^n$, there is a function $f \in N_w$ such that f is a good approximation to g on K.

The guiding question of the present work is: under which necessary and sufficient conditions on σ will the family of networks N_w be capable of approximating to any desired accuracy any given continuous function?

Chapter 3

Function Approximation

Creating a machine learning model to predict or classify from given data is a similar process than when we calculate a function from given points in the space. This is called function approximation and among the most famous techniques of function approximation, we find interpolation: such as Taylor polynomial and Chebyshev polynomial or also the splines approximation.

3.1 Definitions and some results

In this chapter we present some mathematical definitions and known results of function approximation. If we want to approximate functions, we need to recall the following notions: metric spaces, distance between functions and density. Also, we will introduce the Lebesgue space and the Baire's theorem which will be crucial for our understanding of the proof.

Definition 2. We dentoe by $\mathcal{C}(\mathbb{R}^n)$ the set of continuous functions defined on $\mathbb{R}^n$.

Definition 3. The support of a function u is denoted by

$$supp(u) = \overline{\{x|u(x) \neq 0\}}$$

Definition 4. We denote by $\mathcal{C}_0^\infty$ the set of infinitely differentiable functions $\mathcal{C}^\infty$, also called smooth functions, with compact support.

3.1.1 Metric spaces

Definition 5. A *metric* (or *distance*) on a set X is a function $d : X \times X \to \mathbb{R}$ such that for all $s, t, u \in X$ the following properties are satisfied:

1. $d(s,t) \geq 0$ and $d(s,t) = 0$ if and only if $s = t$.
2. $d(s,t) = d(t,s)$.
3. $d(s,t) \leq d(s,u) + d(u,t)$ (*triangle inequality*).

Definition 6. A *metric space* is a pair (X, d), where X is a set and d is a distance in X.

If we take X to be a set of functions, the metric $d(f, g)$ will enable us to measure the distance between functions $f, g \in X$.

Proposition 7. Let (X, d) be a metric space and $\mathcal{B}$ be a basis of open sets for X. Then, for all open set U in X, we have:

$$U = \bigcup \{B_x : x \in U, B_x \subseteq U, B_x \in \mathcal{B}\}.$$

This proposition states that any open set U in a metric space X can be expressed as the union of basis elements B_x.

Definition 8. Let (X, d) be a metric space. The closure of a set A is defined as follows:

$$closure(A) = \overline{A} = \{t \,|\, \forall \epsilon > 0, \exists a \in A,\, d(a, t) < \epsilon\}.$$

Proposition 9. Let (X, d) be a metric space and $A \subseteq X$. Then, $a \in \overline{A}$ if and only if there exists a sequence (a_n) in A such that for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(a_n, a) < \epsilon$ for all $n \geq N$.

Proposition 10 (Prop. 7)**.** [Tre67] Let ρ be a metric defined on the set $\mathcal{C}_0^\infty[a, b]$ as follows:

$$\rho(\varphi_1, \varphi_2) = \sum_{n=0}^{\infty} 2^{-n} \frac{\|\varphi_1 - \varphi_2\|_n}{1 + \|\varphi_1 - \varphi_2\|_n}$$

where

$$\|\varphi\|_n = \sum_{j=0}^{n} \sup_{x \in [a,b]} |\varphi^{(j)}(x)|.$$

Then the metric space $(\mathcal{C}_0^\infty[a, b], \rho)$ is complete, also known as a Fréchet space.

3.1.2 Baire's Theorem

The Baire's theorem is used in the proof, which is why we are going to give a brief overview.

Definition 11. Let (X, d) be a metric space and $A \subseteq X$ subset. A is said to be *nowhere dense* if for every (nonempty) open subset $U \subseteq X$, the intersection $U \cap \overline{A}$ is not dense in U, meaning that U contains a point that is not in the closure of A.

Definition 12. A set it is said to be category I if it can be written as a countable union of nowhere-dense sets. Otherwise it is said to be of *category II*.

Theorem 13. *(Baire's Category Theorem) [BN72]. Any complete metric space is of category II.*

3.1.3 Lebesgue space

Definition 14. Let Σ be a σ-algebra over a set Ω. A *measure* over Ω is any function

$$\mu : \Sigma \longrightarrow [0, \infty]$$

satisfying the following properties:

1. $\mu(\varnothing) = 0$.
2. *σ-additivity*: If $(A_n) \in \Sigma$ are pairwise disjoint, then:

$$\mu\left(\bigsqcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

Definition 15. A box in $\mathbb{R}^d$ is a set of the form

$$Q = [a_1, b_1] \times ... \times [a_d, b_d] = \prod_{i=1}^{d} [a_i, b_i]$$

and the volume of the box is

$$vol(Q) = (b_1, a_1)...(b_d - a_d) = \prod_{i=1}^{d} (b_i - a_i).$$

Definition 16. The *exterior measure* (or outer measure) of a set $E \subseteq \mathbb{R}^d$ is

$$|E|^* = \inf\{\sum_k vol(Q_k)\}$$

where the infimum is taken over all finite or countable collection of boxes $\{Q_k\}$ such that $E \subseteq \cup_k Q_k$

Definition 17. A set $E \subseteq \mathbb{R}^n$ is *Lebesgue mesurable* (or mesurable) if $\forall \epsilon > 0$, there exist U open set such that $E \subseteq U$ and $|U \setminus E|^* < \epsilon$

Definition 18. A set $N \subset \mathbb{R}^n$ is called a *null set* if $|N|^* = 0$

Definition 19. We say that a property holds almost everywhere (a.e.) if the set of points that doesn't hold it is null.

Definition 20. A function u that is measurable on $\Omega \in \mathbb{R}^n$ is said to be *essentially bounded* on Ω if there is a constant λ such that $|u(x)| \leq \lambda$ a.e on Ω. The greatest lower bound of such constants λ is called the essential supremum of $|u|$ on Ω and is denoted by $\operatorname{ess\,sup}_{x\in\Omega} |u(x)|$. We denote by $L^\infty(\Omega)$ the space of all functions u that are essentially bounded on Ω. We denote the norm on $L^\infty(\Omega)$ by $\|\|_{L^\infty(\Omega)}$ defined by

$$\|u\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x\in\Omega} |u(x)|.$$

Definition 21. A function u defined almost everywhere on a domain Ω (a domain is an open set in $\mathbb{R}^n$) is said to be *locally essentially bounded* on Ω if for every compact set $K \subset \Omega$, $u \in L^\infty(K)$. We denote $u \in L^\infty_{loc}(K)$.

Definition 22. Let $\mathcal{M}$ denote the set of functions which are in $L^\infty_{loc}(\mathbb{R})$ and have the following property. The closure of the set of points of discontinuity of any function in $\mathcal{M}$ is of zero Lebesgue measure.

This implies that for any $\sigma \in \mathcal{M}$, interval $[a, b]$. and $\delta > 0$, there exists a finite number of open intervals, the union of which we denote by U, of measure δ, such that σ is uniformly continuous on $[a, b]/U$.

Definition 23. We say that a set of functions $F \subset L^\infty_{loc}(\mathbb{R})$ is *dense* in $C(\mathbb{R}^n)$ if for every function $g \in C(\mathbb{R}^n)$ and for every compact $K \subset \mathbb{R}^n$, there exist a sequence of functions $f_j \in F$ such that

$$\lim_{j\to\infty} \|g - f_j\|_{L^\infty(K)} = 0.$$

Definition 24. $\varphi : I \to \mathbb{R}$ is uniformly continuous on I if $\forall \epsilon > 0 \exists\ \delta > 0$ such that $|\varphi(x) - \varphi(y)| < \epsilon$ whenever $|x - y| < \delta$.

3.1.4 Convolution

Definition 25. Let f, g be real-valued functions with compact support. We define the *convolution* of f with g as

$$(f * g)(x) = \int f(x - t)g(t)\, dt$$

Proposition 26. [AFF03] If f is a smooth function that is compactly supported and g is a distribution, then $f * g$ is a smooth function defined by

$$\int_{\mathbb{R}^d} f(y)g(x - y)\, dy = (f * g)(x) \in C^\infty(\mathbb{R}^d).$$

Proposition 27. Also we have

$$\frac{\partial}{\partial x_i}(f * g) = \frac{\partial f}{\partial x_i} * g = f * \frac{\partial g}{\partial x_i}.$$

3.1.5 Annihilator

Definition 28. Let V be a vector space over a field F, and $X \subseteq V$ subset. The *annihilator of* X is defined as the set of all linear functionals $V \to F$ that evaluate to zero on every element of X:

$$\operatorname{ann}(X) = \{\varphi \in V^* : \quad \text{for all}\, x \in X \quad \varphi(x) = 0\}$$

Definition 29. Let V be a finite-dimensional vector space over a field F, and let $F \subset V^*$. The annihilator of F is defined as the set of all vectors in V that are annihilated by every functional in F:

$$\operatorname{ann}(F) = \{x \in V : \quad \text{for all}\, \varphi \in F \quad \varphi(x) = 0\}$$

Chapter 4

Theorem and proof

The work revolves around the following theorem and its proof.

4.1 Theorem

Theorem 30. *Let $\sigma \in \mathcal{M}$. Set*

$$\Sigma_n = span\{\sigma(w \cdot x + \theta) : w \in \mathbb{R}^n, \theta \in \mathbb{R}\}.$$

Then Σ_n is dense in $\mathcal{C}(\mathbb{R}^n)$ if and only if σ is not a polynomial.

4.1.1 Previous results

There has been significant research on the approximation capabilities of feedforward networks prior to the proof of this theorem. Previous studies have demonstrated that if the activation functions of the network satisfy certain explicit assumptions, then the network can be proven to be as they call it, a universal approximator. For instance, [Hor91] have established two results, which are as follows:

Theorem 31. *(Hornik Theorem 1). Multilayer feedforward networks with a bounded and nonconstant activation function can approximate any function in $L^p(\mu)$ arbitrary well, given a sufficiently large number of hidden units.*

Theorem 32. *(Hornik Theorem 2) Multilayer feedforward networks with a continuous, bounded and nonconstant activation function can approximate any continuous function on X arbitrarily well (with respect to the uniform distance) given a sufficiently large number of hidden units.*

The Theorem 30 generalizes Hornik's Theorem 2 by establishing necessary and sufficient conditions for universal approximation. Note that the theorem merely requires "nonpolynomiality" in the activation function. Unlike Hornik's result, the activation functions do not need to be continuous or smooth. This has an important biological interpretation because the activation functions of real neurons may well be discontinuous or even non-elementary.

4.2 Proof

4.2.1 If σ is not a polynomial then Σ_n is dense in $\mathcal{C}(\mathbb{R}^n)$

Consider that σ is not an algebraic polynomial and we aim to show that Σ_n is dense in $\mathcal{C}(\mathbb{R}^n)$. In order to show that, we devided the proof in 3 steps, each one aims to proof one proposition that needs some previous Lemmas.

Step 1

Proposition 33. If σ is not a polinomial then $\sigma * \varphi$ is not a polynomial for some $\varphi \in \mathcal{C}_0^\infty$.

To proof this proposition we need the following two lemmas.

Lemma 34. If we have that $\sigma * \varphi$ is a polynomial for all $\varphi \in \mathcal{C}_0^\infty$, then the degree of the polynomial $\sigma * \varphi$ is finite, i.e. there exists an $m \in \mathbb{N}$ such that $deg(\sigma * \varphi) \leq m$ for all $\varphi \in \mathcal{C}_0^\infty$.

Proof. We first prove the claim in the case of $\varphi \in \mathcal{C}_0^\infty[a, b]$, for some $a < b$.

By Proposition **??** we have that $(\mathcal{C}_0^\infty[a, b], \rho)$ is a complete metric space.

Consider the following set,

$$V_k = \{\varphi \in \mathcal{C}_0^\infty[a, b] \,|\, deg(\sigma * \varphi) \leq k\}.$$

It is clear that this set $V_k \subseteq \mathcal{C}_0^\infty[a, b]$. We want to show that $\mathcal{C}_0^\infty[a, b] = V_k$.

The set V_k fulfills the following properties: $V_k \subset V_{k+1}$, V_k is a closed subspace, $\cup_{k=0}^\infty V_k = \mathcal{C}_0^\infty[a, b]$ and V_k is a vector space. We can easily see that $\mathcal{C}_0^\infty[a, b]$ is also a vector space.

As $\mathcal{C}_0^\infty[a, b]$ is a complete metric space, by Baire's Category Theorem 13, this set is of category II, i.e. $\mathcal{C}_0^\infty[a, b]$ cannot be written as a countable union of nowhere-dense sets. Recall that $\mathcal{C}_0^\infty[a, b]$ can be written as a countable union of V_k, therefore some V_m is not a nowhere-dense set, that is, there exists an open set $U \subseteq C_0^\infty[a, b]$ that is contained in the closure of V_m, but, as V_m is closed, for that we have that $U \subseteq V_m$. For topology results, any open set of a vector space contains a basis of the vector space, in our case U contains a baisis of $\mathcal{C}_0^\infty[a, b]$, and $U \subseteq V_m$, therefore V_m contains a baisis of $\mathcal{C}_0^\infty[a, b]$. Now we can conclude that $\mathcal{C}_0^\infty[a, b] \subseteq V_k$. And consequently $\mathcal{C}_0^\infty[a, b] = V_k$. This means that any $\varphi \in \mathcal{C}_0^\infty[a, b]$ also satisfies $\varphi \in V_k$, that means that the convolution $\sigma * \varphi$ has degree finite.

For the general case where $\varphi \in \mathcal{C}_0^\infty$, we note that the number m does not depend on the interval $[a, b]$. This can be seen as follows. By translation m depends at most of the length of the interval. Let $[A, B]$ be any interval. For $\varphi \in \mathcal{C}_0^\infty[A, B]$ we can find

$\varphi_i \in \mathcal{C}_0^\infty[a_i, b_i]$ for $i = 1, ..., k$ such that $[A, B] \subset \cup_{i=1}^k [a_i, b_i]$ where $b_i - a_i = b - a$ and $\varphi = \sum_{i=1}^k \varphi_i$ Thus

$$\sigma * \varphi = \sum_{i=1}^{k} \sigma * \varphi_i$$

and for every $i = 1, ..., k$ we have that $\sigma * \varphi$ is a polynomial of degree less than or equal to m. Therefore $deg(\sigma * \varphi) \leq m$. □

Lemma 35. If $\sigma * \varphi$ is a polynomial such that $deg(\sigma * \varphi) \leq m$ for all $\varphi \in \mathcal{C}_0^\infty$, then σ is a polynomial of degree at most m.

Proof. If $\sigma * \varphi$ is a polynomial of degree m. For all $\varphi \in \mathcal{C}_0^\infty$, using (27) we have that

$$(\sigma * \varphi)^{(m+1)}(x) = \int \sigma(x - y)\varphi^{(m+1)}(y)\, dy = 0$$

From standard results in Distribution Theory [pp 57 [Fri63]], σ is itself a polynomial of degree at most m (a.e.). □

Proof of Proposition 33. We will show the contrapositive. Suppose that the convolution $\sigma * \varphi$ is a polynomial for all $\varphi \in \mathcal{C}_0^\infty$, by Lemma 34 the degree of the convolution is finite. Now we have that $\sigma * \varphi$ is a polynomial of finite degree, by Lemma 35 we have that σ is a polynomial. □

Step 2

Proposition 36. If for some $\varphi \in \mathcal{C}_0^\infty$ we have that $\sigma * \varphi$ is not a polynomial, then Σ_1 is dense in $\mathcal{C}(\mathbb{R})$.

Lemma 37. For each $\varphi \in \mathcal{C}_0^\infty$, $\sigma * \varphi \in \overline{\Sigma_1}$.

Proof. We recall the definition of the set

$$\Sigma_1 = \text{span}\{\sigma(w \cdot x + \theta) : w \in \mathbb{R}, \theta \in \mathbb{R}\}. \tag{1}$$

Without loss of generality, assume that supp $\varphi \subseteq [-\alpha, \alpha]$. To show that $\sigma * \varphi \in \overline{\Sigma_1}$ we will use the characterization for elements in the closure. We will to prove that there exists a sequence in Σ_1 such that uniformly converges to $\sigma * \varphi$ on $[-\alpha, \alpha]$. Note that we usually denote for un$\rightrightarrows$)

We shall consider the following sequence:

$$h_m = \sum_{i=1}^{m} \varphi(y_i)\Delta y_i \sigma(x - y_i).$$

Which satisfies $h_j \in \Sigma_1$ for $j = 1, ..., m$. Note that we have $w_i = 1, \theta_i = -y_i$ and $\beta_i = \varphi(y_i)\Delta y_i$.

We will define a partition of the interval $[-\alpha, \alpha]$ to be the following, where

$$y_i = -\alpha + \frac{2i\alpha}{m} \quad i = 1, ..., m$$

and $\Delta y_i = \frac{2\alpha}{m}$.

Given $\epsilon > 0$, we choose $\delta > 0$ such that

$$10\delta \|\sigma\|_{L^\infty\{-2\alpha,2\alpha\}} \|\varphi\|_{L^\infty} \leq \frac{\epsilon}{3}.$$

We know that $\sigma \in M$. Hence, for the previous choosen $\delta > 0$ and $[-\alpha, \alpha]$ interval, there exists $r(\delta)$ finite number of intervals the measure of whose union U is δ such that σ is uniformly continuous on $[-2\alpha, 2\alpha]/U$. We now choose m_i sufficiently large so that

1. $m_1\delta > \alpha r(\delta)$. We can do this by Archimedes' principle.
2. From the uniform continuity of φ we know that

   if $|s - t| \leq \delta_2 = \frac{2\alpha}{m_2}$ then

   $$|\varphi(s) - \varphi(t)| \leq \epsilon_2 = \frac{\epsilon}{2\alpha \|\sigma\|_{L^\infty[-2\alpha,2\alpha]}}$$

3. From the previous, σ is uniformly continuous on $[-2\alpha, 2\alpha]/U$ thus we have,

   if $s, t \in [-2\alpha, 2\alpha]/U$ and $|s - t| \leq \delta_3 = \frac{2\alpha}{m_3}$ then

   $$|\sigma(s) - \sigma(t)| \leq \epsilon_3 = \frac{\epsilon}{\|\varphi\|_L}$$

We choose m such that $m = max\{m_1, m_2, m_3\}$.

Now, fix $x \in [-\alpha, \alpha]$. Set $\Delta_i = [y_{i-1}, y_i]$ where $y_0 = \alpha$.

First, recall that,

$$\int \sigma(x - y)\varphi(y)dy = \sum_{i=1}^{m} \int_{\Delta_i} \sigma(x - y)\varphi(y)dy.$$

Consider the following difference:

$$\left| \int \sigma(x-y)\varphi(y)dy - \sum_{i=1}^{m} \int_{\Delta_i} \sigma(x-y_i)\varphi(y)dy \right|$$
$$= \left| \sum_{i=1}^{m} \int_{\Delta_i} \sigma(x-y)\varphi(y)dy - \sum_{i=1}^{m} \int_{\Delta_i} \sigma(x-y_i)\varphi(y)dy \right|$$
$$= \left| \sum_{i=1}^{m} \int_{\Delta_i} \varphi(y)\Big(\sigma(x-y) - \sigma(x-y_i)\Big)dy \right|$$
$$\leq \sum_{i=1}^{m} \int_{\Delta_i} |\varphi(y)|\,|\sigma(x-y) - \sigma(x-y_i)|\,dy.$$

If $x - \Delta_i \cap U = \varnothing$. Since $x - y \notin U$, $x - y_i \notin U$ and $x - y_i \in [-2\alpha, 2\alpha]$. For choice of m in property 2, we have

$$\sum_{i=1}^{m} \int_{\Delta_i} |\varphi(y)|\,|\sigma(x-y) - \sigma(x-y_i)|\,dy \leq \frac{\epsilon}{\|\varphi\|_{L_1}} \sum_{i=1}^{m} \int_{\Delta_i} |\varphi(y)|$$
$$= \frac{\epsilon}{3\|\varphi\|_{L_1}} \int |\varphi(y)|\,dy = \frac{\epsilon}{3\|\varphi\|_{L_1}} |\varphi(y)|_{L_1} = \frac{\epsilon}{3}.$$

If $x - \Delta_i \cap U \neq \varnothing$ then we will denote such intervals by $\widetilde{\Delta_i}$.

$\sum_i |\widetilde{\Delta_i}| = \sum_i |(x - \Delta_i \cap U)| \leq |U| + 2|\Delta_i| r(\delta) \leq \delta + 2 \cdot \frac{2\alpha}{m} r(\delta) \leq \delta + 4\delta = 5\delta$

We used the property $m\delta > \alpha r(\delta)$, indeed $\delta > \frac{\alpha \cdot r(\delta)}{m}$.

$$\sum_{i=1}^{m} \int_{\widetilde{\Delta_i}} |\varphi(y)|\,|\sigma(x-y) - \sigma(x-y_i)|\,dy \leq \sum_{i=1}^{m} \int_{\widetilde{\Delta_i}} \|\varphi\|_{L^\infty} 2\|\sigma\|_{L^\infty[-2\alpha,2\alpha]}$$
$$= \|\varphi\|_{L^\infty} 2\|\sigma\|_{L^\infty[-2\alpha,2\alpha]} \sum_i |\widetilde{\Delta_i}|$$
$$\leq \|\varphi\|_{L^\infty} 2\|\sigma\|_{L^\infty[-2\alpha,2\alpha]}\, 5\delta \leq \frac{\epsilon}{3}$$

Moreover,

$$\left| \sum_{i=1}^{m} \int_{\Delta_i} \sigma(x-y_i)\varphi(y)dy - \sum_{i=1}^{m} \sigma(x-y_i)\varphi(y_i)\Delta y_i \right|$$
$$= \left| \sum_{i=1}^{m} \int_{\Delta_i} \sigma(x-y_i)[\varphi(y) - \varphi(y_i)]dy \right|$$
$$\leq \sum_{i=1}^{m} \int_{\Delta_i} |\sigma(x-y_i)|\,|\varphi(y) - \varphi(y_i)|\,dy$$
$$\leq \sum_{i=1}^{m} \int_{\Delta_i} |\sigma(x-y_i)|\,dy \left[\frac{\epsilon/3}{2\alpha\|\sigma\|_{L^\infty[-2\alpha,2\alpha]}} \right] \leq \frac{\epsilon}{3}$$

Finally, we have the result $h_m \rightrightarrows \sigma * \varphi$ because

$$\left| \int \sigma(x-y)\varphi(y)dy - \sum_{i=1}^{m} \sigma(x-y_i)\varphi(y_i)\Delta y_i \right| \leq \epsilon$$

for all $x \in [-\alpha, \alpha]$. □

Lemma 38. If $\sigma \in \mathcal{C}^\infty$, then Σ_1 is dense in $\mathcal{C}(\mathbb{R})$.

Proof. We suppose that $\sigma \in \mathcal{C}^\infty$ and recall by the theorem hypothesis σ is not a polynomial. We can write any function f of the set Σ_1 as

$$f = \sum_i \beta_i \sigma_i(w_i x + \theta_i) = \beta_1 \sigma_1(w_1 x + \theta_1) + ...$$

We can see that the function

$$\frac{\sigma([w+h]x+\theta) - \sigma(wx+\theta)}{h} \in \Sigma_1$$

because is a linear combination, where $\beta_1 = \frac{1}{h}, \beta_2 = \frac{-1}{h}$.

By hypothesis, $\sigma \in \mathcal{C}^\infty$. By definition of derivative we have

$$\frac{d}{dw}\sigma(wx+\theta) = \lim_{h\to 0} \frac{\sigma([w+h]x+\theta) - \sigma(wx+\theta)}{h} \in \overline{\Sigma_1}$$

because the limit of a set belongs to the closure of the set.

By the same argument,

$$\frac{d^k}{dw^k}\sigma(wx+\theta) \in \overline{\Sigma_1}$$

for all $k \in \mathbb{N}, w, \theta \in \mathbb{R}$.

If we differentiate this expression k times, we obtain

$$\frac{d^k}{dw^k}\sigma(wx+\theta) = \sigma^{(k)}(wx+\theta) \cdot x^k$$

We are going to see that if σ is not a polynomial (by hypothesis) then there exists a $\theta_k \in \mathbb{R}$ such that $\sigma^{(k)}(\theta_k) \neq 0$. To show that, let us assume that does not exist any $\theta_k \in \mathbb{R}$ such that $\sigma^{(k)}(\theta_k) \neq 0$. This means that the k-th derivative at every point is 0, i.e.

$$\sigma^{(k)}(\theta) = 0 \quad \forall \theta \in \mathbb{R}$$

If we integrate this expression, we will have $\int \sigma^{(k)} = \int 0$. This implies that

$$\sigma^{(k-1)}(x) = C_1$$

for some constant C_1, as integrating zero results in a constant. If we integrate again, we have:

$$\sigma^{(k-2)}(x) = C_1 x + C_2$$

for some constants C_1 and C_2.
Continuing this process, we arrive at

$$\sigma(x) = C_1 x^{k-1} + C_2 x^{k-2} + \ldots + C_{k-1} x + C_k$$

for constants $C_1, C_2, \ldots, C_k$. Hence, σ is a polynomial of degree $k-1$, which contradicts our assumption that σ is not a polynomial. Therefore, there always exists a point where the derivative does not vanish.

Thus, we evaluate at the point where the derivative does not vanish, we call it θ_k.

$$\sigma^{(k)}(\theta_k) \cdot x^k = \frac{d^k}{dw^k}\sigma(wx+\theta)\Big|_{w=0,\theta=\theta_k} \in \overline{\Sigma_1}$$

This implies that $\overline{\Sigma_1}$ contains all polynomials, because the expression $\sigma^{(k)}(\theta_k)x^k$ generates all polynomials. By the Weierstrass theorem, we know that the polynomials are dense in $\mathcal{C}(\mathbb{R})$. This concludes that the set $\overline{\Sigma_1}$ contains a set which is dense in $\mathcal{C}(\mathbb{R})$, therefore Σ_1 is dense in $\mathcal{C}(\mathbb{R})$. □

Proof of Proposition 36. From Lemma 37, $\sigma * \varphi \in \overline{\Sigma_1}$ for each $\varphi \in \mathcal{C}_0^\infty$. It immedialy follows that, $\sigma * \varphi(wx+\theta) \in \overline{\Sigma_1}$, for each $w, \theta \in \mathbb{R}$ and $\varphi \in \mathcal{C}_0^\infty$.
Now, we shall see the results in distributions 26 to proof the following result. For σ and $\varphi \in \mathcal{C}_0^\infty$, we have that $\sigma * \varphi \in \mathcal{C}^\infty$. From Lemma 38 applied in $\sigma = \sigma * \varphi$, if $\sigma * \varphi \in \mathcal{C}^\infty$, then Σ_1 dense in $\mathcal{C}(\mathbb{R})$. □

Step 3

We will prove that approximating a $\mathcal{C}(\mathbb{R})$ function with one from the set Σ_1 implies approximating a function $\mathcal{C}(\mathbb{R}^n)$ from the set Σ_n. We can see this from the density characterization.

Proposition 39. If Σ_1 is dense in $\mathcal{C}(\mathbb{R})$, then Σ_n is dense in $\mathcal{C}(\mathbb{R}^n)$.

Proof. The following proof is inspired by [CL92].

Consider the set

$$V := span\{f(ax) : a \in \mathbb{R}^n, f \in \mathcal{C}(\mathbb{R})\}.$$

First, we shall see that V is dense in $\mathcal{C}(\mathbb{R}^n)$. If we show that V contains the polynomials, which are dense in $C(\mathbb{R}^n)$ by Weierstrass Theorem, that will be enough.

In fact, we have the set

$$H = \langle (ax)^k \rangle = span\{p(ax) : a \in \mathbb{R}^n, p \in \mathbb{R}[x]\} \subseteq V.$$

We only need to show that $H := \mathbb{R}[x]$, in other words, that the polynomials of degree k can be generated by $(a \cdot x)^k$. For the isomorphism theorem, we know that

$$\mathbb{R}[x]^*/\text{ann(H)} \cong H^*$$

Since

$$D^{m_1} x^{m_2} = \delta_{m_1,m_2} k!,$$

we see that $\mathbb{R}[x]^*$ can be generated by $\langle D^{m_1} \rangle_{|m_1|=k}$. Consider any element of $\mathbb{R}[x]^*$, say $\sum \alpha_j D^{m_j}$ and suppose that annihilates H. That is

$$(\sum \alpha_j D^{m_j}) x^{m_j} = \alpha_j k! = 0.$$

This implies that for all j, $\alpha_j = 0$ and then the element $\sum \alpha_j D^{m_j} = 0$ For that reason, it means that $ann(H) = 0$ which implies that $\mathbb{R}[x]^* \cong H^*$, which is what we wanted to see. Thus, we have the set V dense in $\mathcal{C}(\mathbb{R})$.

Let $g \in \mathcal{C}(\mathbb{R}^n)$, for any compact subset $K \subset \mathbb{R}^n$, V dense in $\mathcal{C}(K)$. That is, given $\epsilon > 0$, there exist $f_i \in \mathcal{C}(\mathbb{R})$ and $a_i \in \mathbb{R}^n$ $_{i\,=\,1,\,...,\,k}$ such that

$$\left| g(x) - \sum_{i=1}^{k} f_i(a^i \cdot x) \right| < \frac{\epsilon}{2}$$

for all $x \in K$. We now consider the set of all the points in the compact K multiplied by the vector a^i. That is $\{a^i \cdot x | x \in K\} \subseteq [\alpha_i, \beta_i]$ for some finite interval $[\alpha_i, \beta_i]$, $i = 1, ..., k$. By hypothesis Σ_1 dense in $\mathcal{C}(\mathbb{R})$, specifically Σ_1 is dense in $[\alpha_i, \beta_i]$ $i = 1, ..., k$. Hence there exist constants c_{ij}, w_{ij} and θ_{ij}, $j = 1, ..., m_i$, $i = 1, ..., k$ such that

$$\left| f_i(y) - \sum_{j=1}^{m} c_{ij} \sigma(w_{ij} y + \theta_{ij}) \right| < \frac{\epsilon}{2k}$$

for all $y \in [\alpha_i, \beta_i]$.

Therefore,

$$\left| g(x) - \sum_{i=1}^{k} \sum_{j=1}^{m} c_{ij} \sigma(w_{ij}(a^i \cdot x) + \theta_{ij}) \right| < \epsilon$$

for all $x \in K$. We have shown that Σ_n is dense in $\mathcal{C}(\mathbb{R}^n)$ which is what we wanted. □

4.3 Proof of Theorem 30

Proof.

⇒ To prove this implication statement, we aim to show that if Σ_n is dense in $\mathcal{C}(\mathbb{R}^n)$, then σ is not a polynomial. We will proceed to prove the contrapositive

statement, assuming that σ is indeed a polynomial, and demonstrate that in this case, Σ_n cannot be dense in $\mathcal{C}(\mathbb{R}^n)$.

Let σ be a polynomial of degree k, then $\sigma(wx + \theta)$ is a polynomial of degree k for every w, θ. Recall that

$$\Sigma_n = span\{\sigma(w \cdot x + \theta) : w \in \mathbb{R}^n, \theta \in \mathbb{R}\}$$

that is the set of algebraic polynomials of degree at most k. To show that Σ_n is not dens in $\mathcal{C}(\mathbb{R}^n)$, for the definition of density, we need to find a function $f(x) \in \mathcal{C}(\mathbb{R}^n)$, $\epsilon > 0$ and K such that $\|p - f\| > \epsilon$ for all p polynomial of degree k. For example, let $f(x) = cos(x)$, and let $p(x) = \sigma(wx + \theta)$ that has degree at most k. This implies has maximum k roots. We can find a interval where $cos(x)$ has k+1 roots. Therefore, Σ_n is not dense in $\mathcal{C}(\mathbb{R}^n)$.

$\Leftarrow$ In order to prove this implication, we need to show that if σ is not an algebraic polynomial, then Σ_n is dense in $\mathcal{C}(\mathbb{R}^n)$.

By hypothesis, σ is not a polynomial, by Proposition 1 this implies that $\sigma * \varphi$ is not a polynomial for some $\varphi \in \mathcal{C}_0^\infty$. By Proposition 2 if $\sigma * \varphi$ is not a polynomial for some φ, then Σ_1 is dense in $\mathcal{C}(\mathbb{R})$. Finally in Proposition 3 we have showed that this implied that Σ_n is dense in $\mathcal{C}(\mathbb{R}^n)$.

□

4.4 About the theorem

4.4.1 Why does it not contradict the Weierstrass approximation theorem?

In mathematical analysis, we prove the Weierstrass approximation theorem, which we will now recall.

Theorem 40. *(Weierstrass approximation theorem) Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then, there exists polynomials $p_n \in \mathcal{R}[x]$ such that the sequence (p_n) converge uniformly to f on $[a, b]$.*

Corollary 41. The set of polynomial functions $\mathcal{R}^n[x]$ is dense in the space of continuous functions on a compact set $K \subset \mathbb{R}^n$, $\mathcal{C}(K)$. So any continuous function on a compact set can be approximated arbitrarly well by a polynomial.

The theorem 30 states that: if Σ_n is dense in $\mathcal{C}(\mathbb{R}^n)$ then σ is not an algebraic polynomial. But why this statment does not contradict the Weierstrass approximation theorem? This does not work because σ has degree fixed k, then any element in the set Σ_n has degree at most k. Hence, the set Σ_n is a finite vector space and can not be dense in $\mathcal{C}(\mathbb{R}^n)$. Not all contiunous functions can be approximated with a polynimial of degree fixed. We argued this in the proof where with the example of the *cosine* function. For example, if $k = 3$, we cannot approximate the cosine function with linear combinations of polynomials of maximum degree 3.

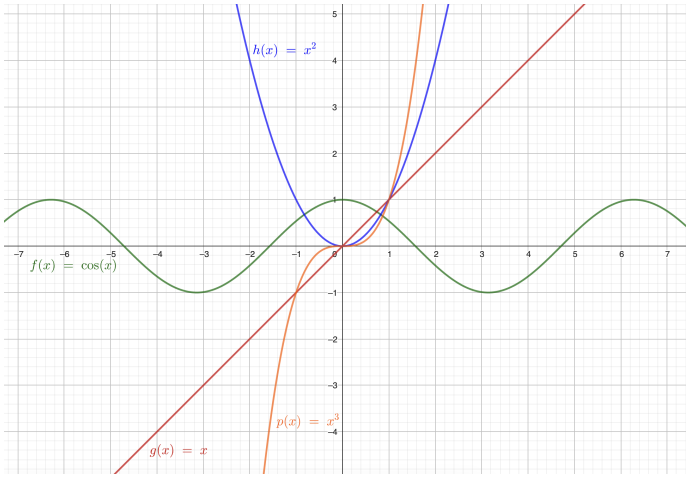

Figure 4.1: Cosine function cannot be approximated with linear combinations of polynomials of maximum degree 3.

4.4.2 Conclusion

The theorem only requires the activation function to be non-polynomial; we do not need continuity in sigma. Our finding that the activation function need not be continuous or smooth also has an important biological interpretation because the activation functions of real neurons may well be discontinuous or even non-elementary. Note that the ReLU function mentioned before is commonly regarded as one of the best activation functions in deep learning. We can see that it is non-smooth, and due to the proven theorem, we can still assure that it can be used in any case (because it is non-polynomial).

4.4.3 Corollaries

Definition 42. The set $L^p(\mu)$ contains all mesurable functions f such that:

$$\|f\|_{L^p}(\mu) = \Big(\int_{R^n} |f(x)|^p d\mu(x) \Big)^{1/p} < \infty.$$

Proposition 43. Let μ be a non-negative finite measure on $\mathbb{R}$ with compact support, absolutely continous with respect to Lebesgue measure. Then Σ_n is dense in $L_p(\mu)$, $1 \leq p < \infty$, if and only of, σ is not a polynomial.

Proposition 44. If $\sigma \in M$ is not a polynomial (a.e) then,

$$\Sigma_n(\mathcal{A}) = span\{\sigma(\lambda w \cdot x + \theta) : \lambda, \theta \in \mathbb{R}, w \in \mathcal{A}\ \}$$

is dense in $\mathcal{C}(\mathbb{R}^n)$ for some $\mathcal{A} \subset \mathbb{R}^n$ if and only if there does not exist a nontrivival polynomial vanishing on $\mathcal{A}$.

Chapter 5

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