Chapter 1

Multilayer Feedforward Networks

1.1 Function Approximation

In this paper we take $C(\mathbb{R}^n)$ to be the family of "real world" functions that one may wish to approximate. If we can show that a given set of functions F is dense in $C(\mathbb{R}^n)$, we can conclude that for every continuous function $g \in C(\mathbb{R}^n)$ and each compact set $K \subset \mathbb{R}^n$, there is a function $f \in F$ such that f is a good approximation to g on K.

Definition 1. A *metric* (or *distance*) on a set X is a function $d : X \times X \to \mathbb{R}$ such that forall $s, t \in X$ the following properties are satisfied:

1. $d(s,t) \geq 0$ and $d(s,t) = 0$ if and only if $s = t$.
2. $d(s,t) = d(t,s)$.
3. $d(s,t) \leq d(s,u) + d(u,t)$ (*triangular inequality*).

A *metric space* is a pair (X, d), where X is a set and d is a distance in X.

If we take X to be a set of functions, the metric $d(f,g)$ will enable us to measure the distance between functions $f, g \in X$.

Definition 2. We denote the support of a function u by $supp(u) = \{x | u(x) \neq 0\}$

Definition 3. Let f, g be real-valued functions with compact support. We define the *convolution* of f with g as

$$(f * g)(x) = \int f(x-t)g(t)\,dt$$

1.2 Lebesgue measure

An essential part of measure theory is the calculation of lengths, areas, volumes, etc. We are already familiar with the Riemann integral, and now we aim to introduce a more general integral, the Lebesgue integral, that comes from the Lebesgue measure.

Definition 4. A box in $\mathbb{R}^d$ is a set of the form

$$Q = [a_1, b_1] \times ... \times [a_d, b_d] = \prod_{i=1}^{d} [a_i, b_i]$$

The volume of the box is

$$vol(Q) = (b_1, a_1)...(b_d - a_d) = \prod_{i=1}^{d} (b_i - a_i)$$

The *exterior measure* (or outer measure) of a set $E \subseteq \mathbb{R}^d$ is

$$|E|^* = \inf\{\sum_k vol(Q_k)\}$$

where the infimum is taken over all finite or countable collection of boxes Q_k such that $E \subseteq \cup_k Q_k$

Definition 5. A set $E \subseteq \mathbb{R}^n$ is *Lebesgue mesurable* (or mesurable) if $\forall \epsilon > 0$, there exist U open set such that $E \subseteq U$ and $|U \setminus E|^* < \epsilon$

Definition 6. A function u defined almost everywhere on a measurable set $\Omega \in \mathbb{R}^n$ is said to be *essentially bounded* on Ω if $|u(x)|$ is bounded almost everywhere on Ω. We denote $u \in L^\infty(\Omega)$ with the norm

$$\|u\|_{L^\infty(\Omega)} = inf(\lambda | \{x : |u(x)| \geq \lambda\} = 0) = ess \sup_{x \in \Omega} |u(x)|$$

We have that $L^\infty(\mathbb{R})$ is the space of essentially bounded functions.

Examples and counterexamples of functions essentially bounded.

- $f : \Omega \rightarrow$

Definition 7. A function u defined almost everywhere on a domain Ω (a domain is an open set in $\mathbb{R}^n$) is said to be *locally essentially bounded* on Ω if for every compact set $K \subset \Omega$, $u \in L^\infty(K)$. We denote $u \in L^\infty_{loc}(K)$.

Definition 8. We say that a set of functions $F \subset L^\infty_{loc}(\mathbb{R})$ is *dense* in $C(\mathbb{R}^n)$ if for every function $g \in C(\mathbb{R}^n)$ and for every compact $K \subset \mathbb{R}^n$, there exist a sequence of functions $f_j \in F$ such that

$$\lim_{j \to \infty} \|g - f_j\|_{L^\infty(K)} = 0.$$

1.3 Results

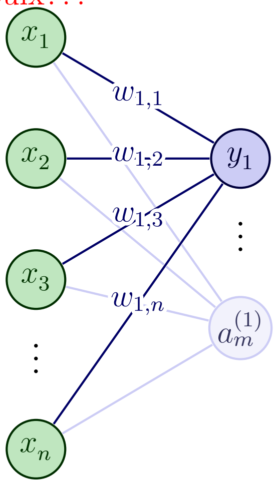
Definition 9. Let $\mathcal{M}$ denote the set of functions which are in $L^\infty_{loc}(\mathbb{R})$ and have the following property. The closure of the set of points of discontinuity of any function in $\mathcal{M}$ is of zero Lebesgue measure.

Proposition 10. (This implies that) for any $\sigma \in \mathcal{M}$, interval $[a, b]$. and $\delta > 0$, there exists a finite number of open intervals, the union of which we denote by U, of measure δ, such that σ is uniformly continuous on $[a, b]/U$.

Definition 11. $\mathcal{C}^\infty_0$ functions $\mathcal{C}^\infty$ with compact support.

1.4 Multilayer Feedforward Network

Multilayer feedforward networks are a type of artificial neural network that consist of several layers of interconnected nodes, with each node taking input from the previous layer and producing output for the next layer. The general architecture of a multilayer feedforward network, MFN, consist of: input layer: n-input units, one/more hidden layers : intermediate processing units, output layer: m output-units. dibuix???



$$\begin{pmatrix} a_1^{(1)} \\ a_2^{(1)} \\ \vdots \\ a_m^{(1)} \end{pmatrix} = \beta_j \cdot \sigma \left[\begin{pmatrix} w_{1,0} & w_{1,1} & \dots & w_{1,n} \\ w_{2,0} & w_{2,1} & \dots & w_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m,0} & w_{m,1} & \dots & w_{m,n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_m \end{pmatrix} \right]$$

$$a^{(1)} = \beta_j \cdot \sigma \left(\mathbf{W}^{(0)} a^{(0)} - \theta_{\mathbf{j}} \right)$$

Definition 12. (Multilayer feedforward networks) The function that a MFN compute is:

$$f(x) = \sum_{j=1}^{k} \beta_j \cdot \sigma(w_j \cdot x - \theta_j)$$

where $x \in \mathbb{R}^n$ is the input vector, $k \in \mathbb{N}$ is the number of processing units in the hidden layer, $w_j \in \mathbb{R}^n$ is the weight vector that connects the input to processing unit j in the hidden layer, $\sigma : \mathbb{R} \to \mathbb{R}$ is an activation function applied element-wise to the vector $w_j^T x - \theta_j$, where $\theta_j \in \mathbb{R}$ is the threshold (or bias) associated with processing unit j in the hidden layer, and $\beta_j \in \mathbb{R}$ is the weight that connects processing unit j in the hidden layer to the output of the network.

Let N_w be the family of all functions implied by the network's architecture. If we can show that N_w is dense in $C(\mathbb{R}^n)$, we can conclude that for every continuous function $g \in C(\mathbb{R}^n)$ and each compact set $K \subset \mathbb{R}^n$, there is a function $f \in N_w$ such that f is a good approximation to g on K.

Under which necessary and sufficient conditions on σ will the family of networks N_w be capable of approximating to any desired accuracy any given continuous function?

1.5 Theorem

Theorem 13. *Let $\sigma \in M$. Set*

$$\Sigma_n = span\{\sigma(w \cdot x + \theta) : w \in \mathbb{R}^n, \theta \in \mathbb{R}\}$$

Then Σ_n is dense in $\mathcal{C}(\mathbb{R}^n)$ if and only if σ is not an algebraic polynomial.

1.5.1 Why does not contradict the Weierstrass approximation theorem?

Theorem 14. *(Weierstrass approximation theorem). Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then, there exists polynomials $p_n \in \mathcal{R}[x]$ such that the sequence (p_n) converge uniformly to f on $[a, b]$.*

Corollary 15. The set of polynomial functions $\mathcal{R}^n[x]$ is dense in the space of continuous functions on a compact set $K \subset \mathbb{R}^n$, $\mathcal{C}(K)$. So any continuous function on a compact set can be approximated arbitrarly well by a polynomial.

The theorem states that: if Σ_n is dense in $\mathcal{C}(\mathbb{R}^n)$ then σ is not an algebraic polynomial. But why this statment does not contradict the Weierstrass approximation theorem ? This does not work because σ has degree fixed k, then any element in the set Σ_n has degree at most k. Hence, the set Σ_n is a finite vector space and can not be dense in $\mathcal{C}(\mathbb{R}^n)$. Not all contiunous functions can be apparoximated with a polynimial of degree fixed, for example: (comment per afegir : per exemple una funcio que sigui continua que no es pugui approximar per un polinomi de com a molt grau k , una k tingui grau mes gran que k polinomi de k+1 ??)

1.5.2 Previous results

The activation functions that were reported thus far in the literature.

Theorem 16. *(Hornik Theorem 1). Standard multilayer feedforward networks with a bounded and nonconstant activation function can approximate any function in $L^p(\mu)$ arbitrary well, given a sufficiently large number of hidden units.*

Theorem 17. *(Hornik Theorem 2) Standard multilayer feedforward networks with a continuous, bounded and nonconstant activation function can approximate any continuous function on X arbitrarily well (with respect to the uniform distance) given a sufficiently large number of hidden units.*

The theorem generalize in particular Hornik's Theorem 2 by establishing necessary and sufficientconditions for universal approximation. Diferencies ? ??

1.6 Results

Definition 18. The set $L^p(\mu)$ contains all mesurable functions f such that:

$$\|f\|_{L^p(\mu)} = \left(\int_{R^n} |f(x)|^p d\mu(x) \right)^{1/p} < \infty$$

Proposition 19. Let μ be a non-negative finite measure on $\mathbb{R}$ with compact support, absolutely continous with respect to Lebesgue measure. Then Σ_n is dense in $L_p(\mu)$, $1 \leq p < \infty$, if and only of, σ is not a polynomial.