

# CTA200 2021 Assignment 2

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## 1 Question 1

### 1.1 Methods

For this problem, the derivative of  $f(x) = \sin(x)$  was approximated using two methods. The first method was taking the derivative using first principles about a point  $x_0$  with infinitesimal step  $h$  as  $h \rightarrow 0$ ,

$$d_x f|_{x_0} \approx \frac{f_{x_0+h} - f_{x_0}}{h}. \quad (1)$$

A better approximation can be found when  $h$  is instead finite using

$$d_x f|_{x_0} \approx \frac{f_{x_0+h} - f_{x_0-h}}{2h}. \quad (2)$$

The derivative of  $f(x) = \sin(x)$  for  $x_0 = 0.1$  was taken using both methods for increasingly small values of  $h$ . The error of the numerical derivative was compared to the analytical derivative was plotted as a function of  $h$  on a double log plot for each method, as seen in Figure 1.2.

### 1.2 Analysis

The values for the approximation of  $\frac{d}{dx} \sin(x)|_{x_0}$  for increasingly small values of  $h$  are shown in Table 1.

Table 1: Approximation of  $\frac{d}{dx} \sin(x)|_{x_0}$  for  $x_0 = 0.1$

$h$	Method 1	Method 2
1e-1	0.9883591414823306	0.9933466539753061
1e-3	0.994954082739849	0.995003999444008
1e-9	0.9950041623962845	0.9950041623962845
1e-15	0.9992007221626408	0.9922618282587335

As can be seen from the table, both methods give similar results for larger values of  $h$ , but as  $h$  decreases these values diverge from one another. From the plot of the error in Figure 1.2, we can see that Method 2 gives a better approximation of the true derivative of  $f(x) = \sin(x)$  since its error goes to zero much quicker than that for Method 1. The steeper slope of the plot for Method 2 indicates that the approximation of the derivative about  $x_0$  converges more quickly than the approximation using Method 1.

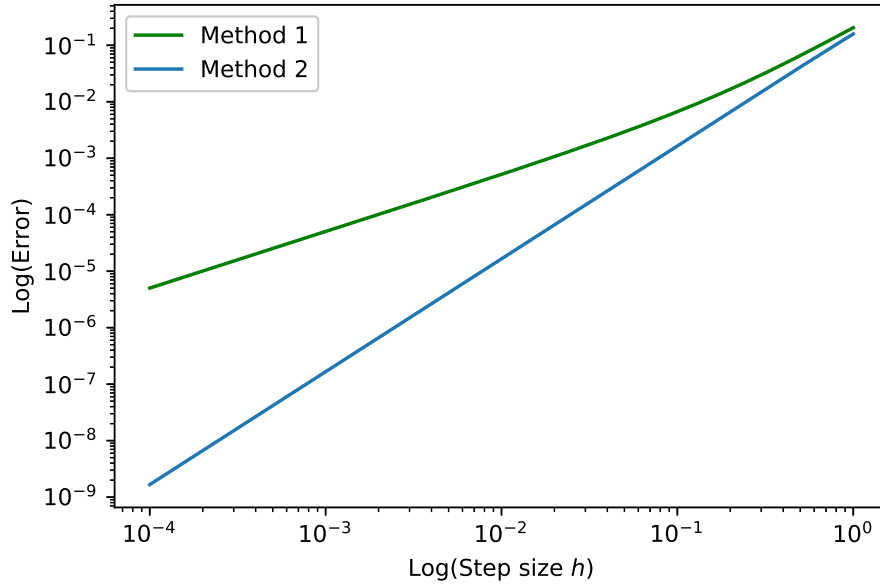


Figure 1: Loglog plot of error in numerical approximation of the derivative of  $f(x) = \sin(x)$  using Method 1 (green) and Method 2 (blue).

## 2 Question 2

### 2.1 Methods

The equation

$$z_{i+1} = z_i^2 + c \quad (3)$$

where  $c = x + iy$  for  $-2 < x < 2$  and  $-2 < y < 2$ , and  $z_0 = 0$ , will converge for some points in the complex plane  $c$  and diverge for others. 2000 values for both  $x$  and  $y$  were generated and then iterated through Equation 3 255 times. If  $|z^2|$  remained bounded (i.e.  $|z^2| < \text{a large number}$ ) for all iterations,  $z$  was said to diverge at that point. Otherwise,  $z$  was said to converge. The data for each iteration at each point was stored in arrays and plotted both as a Boolean function of divergence (see Figure 2.2) and as a gradient indicating the iteration at which  $z$  diverges at a given point  $c$  (see Figure 2.2).

### 2.2 Analysis

The plots of the results of the Boolean function of divergence and the gradient indicating the iteration at which  $z$  diverges at a given point  $c$  can be seen in Figures 2.2 and 2.2, respectively. The resolution of the images is sharper for a larger set of values of  $x$  and  $y$ , however adding these iterations dramatically increases the computation time. The figures show the Mandelbrot set.

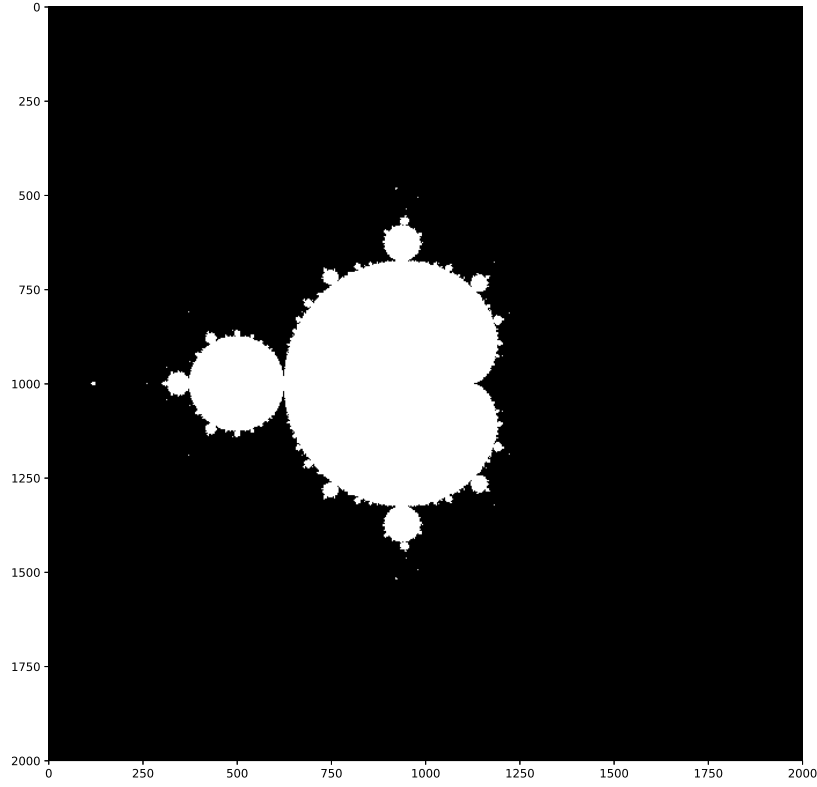


Figure 2: Plot of  $z_{i+1} = z_i^2 + c$ , colouring is Boolean-based on whether a given point  $c = x + iy$  converges (white) or diverges (black) after 255 iterations.

### 3 Question 3

#### 3.1 Methods

In this problem, the SIR model for the spread of infectious diseases was analyzed using a system of 3 first order ODE's:

$$\frac{dS}{dt} = -\frac{\beta SI}{N}, \quad (4)$$

$$\frac{dI}{dt} = \frac{\beta SI}{N} - \gamma I, \quad (5)$$

$$\frac{dR}{dt} = \gamma I \quad (6)$$

representing the individuals in a fixed population size  $N = 1000$  who are susceptible to the disease but not yet infected, infected, and recovered from the disease with immunity (sometimes called “removed”), respectively. The parameter  $\beta$  describes the rate of infection of exposed individuals, and the parameter  $\gamma$

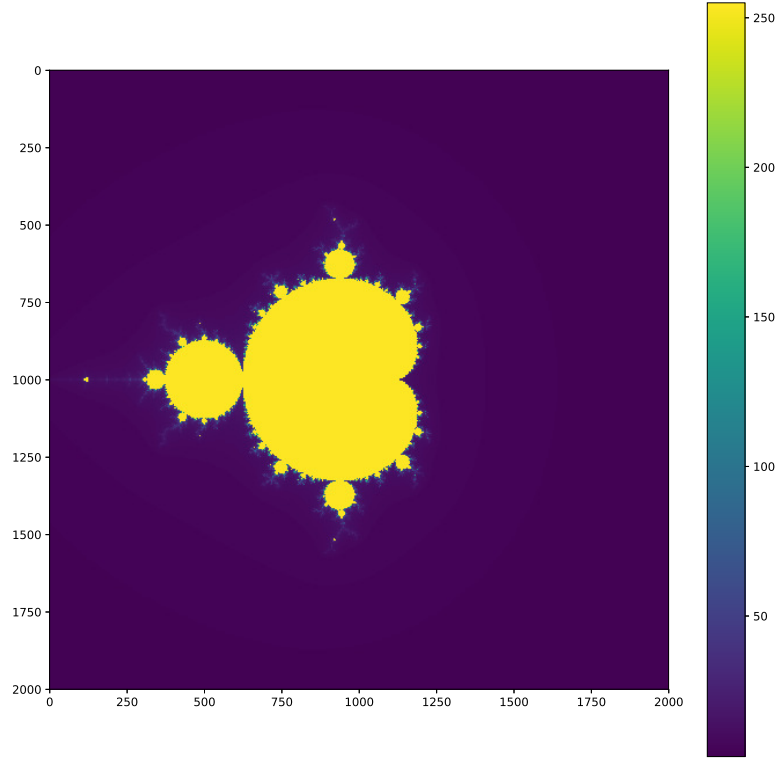


Figure 3: Plot of  $z_{i+1} = z_i^2 + c$ , colourmap is based on the iteration at which the given point  $c = x + iy$  diverges - colouring is yellow if  $c$  converges after 255 iterations.

describes the inverse of the average length of infection.

The system of equations was solved using the ODE integrator `dopri5` imported from the `Scipy` module and the initial conditions  $S(0) = 999$ ,  $I(0) = 1$ , and  $R(0) = 0$ . Various combinations of values for the parameters of  $\beta$  and  $\gamma$  were used to solve the equations. The solutions can be seen in the four plots of Figure 3.2.

### 3.2 Analysis

Four combinations of values for *beta* and *gamma* were chosen to demonstrate drastic differences in the results. A high probability of infection ( $\beta = 0.9$ ) and a long recovery period ( $\gamma = 10$ ) days are simulated in the top left plot. This combination shows a virus that infects over half of the population but would be virtually wiped out after 50 days from the initial infection. The top right plot shows a simulation of an infection with a moderate rate of infection ( $\beta = 0.7$ ) and a short period of infection ( $\gamma \approx 1.5$  days), resulting in a disease that also takes 50 days to be wiped out from the community but is only contracted by about  $1/10^{\text{th}}$  of the population. A model with a moderate infection rate ( $\beta = 0.5$ ) and a very short recovery period ( $\gamma \approx 1$  day) is shown in the bottom left plot. This simulation shows a disease that infects nearly none of the overall population and does not seem to have a significant impact on the community. The plot in the bottom right shows a simulation with a low infection rate ( $\beta = 0.2$ ) and a long recovery period

( $\gamma = 100$  days). This model shows a virus that infects almost every individual in the population, and takes a very long time to be wiped from the community.

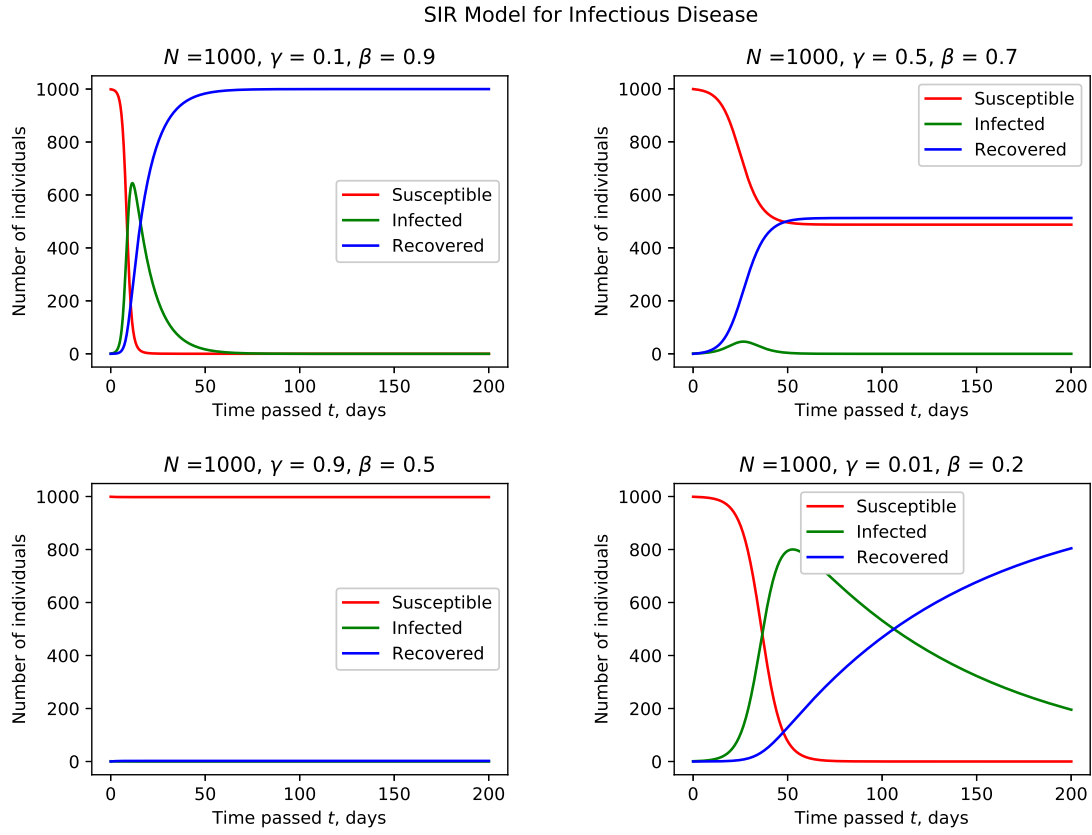


Figure 4: Plots of SIR model for infectious diseases for various parameters recovery rate,  $\gamma$ , and infection rate,  $\beta$ .