Equation differentiel of nth order has at most a singularity of the first kind at z_0 iff z_0 is a regular singular point for that equation

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Theory of Ordinary Differential Equations

Abstract

If z_0 is a singular point of the first kind of a differential equation of second order, then z_0 is a regular singular point of same equation. We will show that converse of this theorem is also true with giving examples and proof of the converse theorem.

Keywords: Differential equation, singular point, regular singular point, singular point of the first kind, analytic point

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1 Linear Differential Equations

In this chapter we'll give elementary definitions related with the topic.

Definition 1.1. *A linear differential equation of the order n is of the form*

$$a_0(z)\frac{d^n w}{dz^n} + a_1(z)\frac{d^{n-1}w}{dz^{n-1}} + \cdots + a_{n-1}(z)\frac{d^w}{dz} + a_n(z)w = b(z) \tag{1.1}$$

where the coefficients $a_k(z)$ and $b_k(z)$ are single-valued analytic functions.

In next chapters, we'll continue with differential equations of second order. (n=2)

Definition 1.2. *A point z_0 is called an ordinary point or analytic point for the differential equation*

$$a_0(z)w'' + a_1(z)w' + a_0(z)w = 0$$

if and only if $a_0(z_0) \neq 0$.

But we are interested in non-analytic (singular) points instead of analytic point. z_0 is a singular point of the equation (1.0.1) if and only if $a_0(z_0) = 0$.

Definition 1.3. *$f(z)$ is said to have an isolated singulary at z_0 if and only if z_0 is analytic in a deleted neightborhood D of z_0 but is not analytic at z_0.*

We will continue with giving definition of a pole, essential singularity and removable singularity.

Definition 1.4. *Suppose f has an isolated singularity at z_0, then*

- *If there exists a function g which is analytic at z_0 and $f(z) = g(z)$ for all z in some deleted neighborhood of z_0 then we say f has a removable singularity at z_0.*
- *If, for $z \neq z_0$, f can be written in the form $\frac{A(z)}{B(z)}$ where A and B are analytic at z_0 and $A(z_0) \neq 0$, $B(z_0) = 0$, we say that f has a pole at z_0.*
- *If f has neither a removable singularity nor a pole at z_0, we say that f has an essential singularity at z_0.*

2 Linear Systems with Isolated Singularities: Singularities of the First Kind

We will analyze the linear system

$$w' = A(z)w$$

in this chapter where A is a nxn(for order n, and $2x2$ for order 2)(complex valued) matrix with at most an isolated singularity at some point z_0.

2.1 Classification of Singularities

If A has a singularity at z_0 then z_0 is called a singular point for the system

$$w' = A(z)w. \tag{2.1}$$

If A has at most a pole at z_0 (that is either A is analytic at z_0 or has a pole at z_0), but is analytic for $0 < |z - z_0| < a$ for $a > 0$, then A may be written in the following form

$$A(z) = (z - z_0)^{-\mu-1}B(z) \tag{2.2}$$

where μ is an integer, B analytic for $|z - z_0| < a$ and $B(z_0) \neq 0$. Now lets look at different cases of μ.
When $\mu \leq -1$, since $-\mu - 1 \geq 0$, it is clear that then A is analytic at z_0 and hence every fundamental matrix of (2.1) is analytic in $|z - z_0| < a$. That's why, if $\mu \leq -1$, the point z_0 is called an analytic point for (2.1).
If $\mu \geq 0$ the integer μ is called the rank of the singularity. It turns out that there is a significant difference between the cases $\mu = 0$ and $\mu \geq 1$. Therefore, while μ=0 the point z_0 will be called a "singular point of the first kind" or while $\mu \geq 1$, the point will be called a "singular point of the second kind" for (2.1). It follows that any fundamental matrix Φ of (2.1), where A has an isolated singularity at z_0 is of the form

$$\Phi(z) = S(z)(z - z_0)^P$$

where S is single valued, analytic in $0 < |z - z_0| < a$, and P is a constant matrix. If S has at most a pole at z_0, then z_0 will be called as "regular singular point" for (2.1). And if z_0 is a regular singular point for (2.1) then S can be written in the form

$$S(z) = (z - z_0)^{-k}\tilde{S}(z)$$

where k is an integer, $\tilde{S}$ analytic at z_0 and $\tilde{S} \neq 0$. Consequently, Φ can be written as

$$\Phi(z) = \tilde{S}(z)(z - z_0)^{P-kE}.$$

Theorem 2.1. *If z_0 is a singular point of the first kind for* (2.1)*, then it is a regular singular point for* (2.1)*.*

Proof. The proof will be given for the case $z_0 = 0$. By hypothesis, we can rewrite the system (2.1) under form

$$w' = z^{-1}\tilde{A}(z)w \tag{2.3}$$

where $\tilde{A}$ is analytic for $0 \leq |z| < a$, $a > 0$, and $\tilde{A}(0) \neq 0$. If Φ is any fundamental matrix of (2.3), it must be shown that in the representation $\Phi = Sz^P$. (see Thm (1.1) at additifs) S is either analytic or has a pole at $z = 0$. This will be done by showing that there exists a positive integer m s.t. $z^m S$ is bounded in a neighborhood of $z = 0$, and, by a theorem of Riemann, this implies the result. We'll show existance of m in additives part(4.1 - Existance of such m)

□

Converse of theorem (2.1) is not true for $n > 1$ (but true for $n \leq 1$ which is our theorem (3.2) , for example, you can see for $n = 2$ with considering the system

$$w' = (z^{-2}C_1 + C_2)w$$

where

$$C_1 = \begin{pmatrix} 0 & 0 \\ \frac{-3}{16} & 0 \end{pmatrix}, C_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

This system has at $z = 0$ a singularity of the second kind with rank $\mu = 1$. A fundamental matrix Φ for this system is readily seen to be given by

$$\Phi(z) = \begin{pmatrix} z^{\frac{1}{4}} & z^{\frac{3}{4}} \\ \frac{1}{4}z^{\frac{-3}{4}} & \frac{3}{4}z^{\frac{-1}{4}} \end{pmatrix}.$$

If S and R are defined by

$$S = \begin{pmatrix} z & z \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}.$$

and

$$R = \begin{pmatrix} \frac{-3}{4} & 0 \\ 0 & \frac{-1}{4} \end{pmatrix}.$$

it is seen that $\Phi = Sz^R$, and from this representation of Φ it follows that $z = 0$ is a regular singular point.

3 Equation of 2nd Order

Our but in this chapter is showing that differential equation of 2nd order has at most a singularity of the first kind at z_0 if and only if z_0 is a regular singular point for that equation. This is true for differential equations of nth order aswell but we'll study over 2nd order differential equations in this paper. Now consider equation of second order

$$\sum_{m=0}^{2} a_{2-m}(z) w^{(m)} = 0 \tag{3.1}$$

where a_k are single valued and analytic in a punctured vicinity of a point z_0. If any of the a_k have a singularity at z_0 then z_0 is called a singular point of (3.1), otherwise z_0 is called an analytic point for differential equation (3.1). Analogous to the definition of a singular point of the first kind for a system of the first order, the one were saying z_0 is a singular point of the first kind for (3.1) if z_0 is a singular point for (3.1) and the coefficients (a_k) of (3.1) have the form

$$a_k(z) = (z - z_0)^{-k} b_k(z) \tag{3.2}$$

for $k = 1, 2$ where the b_k are analytic at z_0. The equation (3.1) is said to have at most a singularity of the first kind at z_0 if z_0 is either an analytic point or a singular point of the first kind for (3.1).
If z_0 is a singular point of the first kind for (3.1), then z_0 may not be a singular point of the first kind for the first-order system associated with (3.1). Ofcourse this is true only in the case where coefficients a_k have at most simple poles at z_0. However, there does exist a first-order system equation in relation with (3.1) which satisfies property that if z_0 is a singular point of the first kind then z_0 is a singular point of the first kind for the system.

Suppose (3.1) has at most, a singularity of the first kind at z_0, and let φ be any solution of (3.1). Define the vector $\hat{\varphi} = (\varphi_1, \varphi_2)$ with components φ_1, φ_2 such that

$$\varphi_k = (z - z_0)^{k-1} \varphi^{(k-1)} \tag{3.3}$$

for $k = 1, 2$, then since

$$\varphi_k' = (k-1)(z - z_0)^{k-2} \varphi^{(k-1)} + (z - z_0)^{k-1} \varphi^{(k)}$$

we have

$$\begin{aligned}
(z-z_0)\varphi_1' &= \varphi_2 \\
(z-z_0)\varphi_2' &= \varphi_2 - \sum_{m=1}^{2} b_{2-m+1}(z)\varphi_m \\
&= \varphi_2 - (b_2(z)\varphi_1 + b_1(z)\varphi_2) \\
&= (1-b_1(z))\varphi_2 - b_2(z)\varphi_1
\end{aligned} \tag{3.4}$$

So, by these equations we'll show that $\hat{\varphi} = (\varphi_1, \varphi_2)$ is a solution of the linear equation

$$w' = A(z)w \tag{3.5}$$

where $A(z)$ is a matrix of length $2x2$ with structure

$$A(z) = (z-z_0)^{-1}\begin{pmatrix} 0 & 1 \\ -b_2 & -b_1+1 \end{pmatrix} \tag{3.6}$$

Proof.

$$\begin{aligned}
A(z)w &= (z-z_0)^{-1}.\begin{pmatrix} 0 & 1 \\ -b_2 & -b_1+1 \end{pmatrix} \cdot \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \\
&= (z-z_0)^{-1}.\begin{pmatrix} \varphi_1 \\ -b_2\varphi_1 + (1-b_1)\varphi_2 \end{pmatrix} \\
&= \begin{pmatrix} \varphi_1' \\ \varphi_2' \end{pmatrix} \\
&= w'
\end{aligned} \tag{3.7}$$

□

Therefore, $w = (\varphi_1, \varphi_2)$ is a solution of (3.1). Obviously

$$(z-z_0)A(z)$$

is analytic at z_0 and does not vanish there, so as a result, the system (3.5) has singularity of the first kind at z_0. Consequently, by theorem (2.1), the point z_0 is a regular singular point for (3.5). If every solution of (3.1) can be expressed in a punctured vicinity of z_0 as a finite linear combination of the terms in form

$$(z-z_0)^r(log(z-z_0))p(z)$$

where r is a constant(in general, complex) and p is analytic at z_0 such that $p(z_0) \neq 0$ then z_0 is said to be regular singular point for (3.1). Thus the above argument proves the following analogue of theorem (2.1)

Theorem 3.1. *If* (3.1) *has at most a singularity of the first kind at* z_0 *then* z_0 *is a regular singular point for* (3.1).

Our but was showing converse of this theorem is also true which is

Theorem 3.2. *If* z_0 *is a regular singular point of equation* (3.1) *then* (3.1) *has at most a singularity of the first kind at* z_0.

Proof. Suppose the b_k are related to the a_k in (3.1) via (3.2). Here it is not assumed the b_k are analytic at z_0, but it is true that the b_k are analytic and single-valued in a punctured vicinity of z_0. It is clear then that the system (3.5), (3.6) meets the requirements of Theorem(4.1-Additives). Since the element in the first row of any solution vector of (3.5) is a solution of (3.1), it follows that there exists a solution φ_1 of (5.1) near z_0 of the form

$$\varphi_1(z) = (z - z_0)^r p(z)$$

where p is a single-valued and analytic in a punctured vicinity of z_0. But since z_0 is a regular singular point, this solution must be in form

$$\varphi_1(z) = (z - z_0)^s q(z) \tag{3.8}$$

where s is a constant and q is analytic at z_0, $q(z_0) \neq 0$. If φ is any solution of (3.1) near z_0, and

$$\varphi = \varphi_1 \psi$$

(we're making variation of parameters), then ψ must be a solution of the following equation

$$\sum_{m=0}^{2} c_{2-m}(z) \tilde{w}^{(m)} = 0 \tag{3.9}$$

where the coefficients c_k are defined with

$$c_{k-m} = a_{k-m}\varphi_1 + (m+1)a_{k-m-1}\varphi_1' + \cdots + \binom{k-1}{k-m-1} a_1 \varphi_1^{(k-m-1)} + \binom{k}{m} \varphi^{(k-m)} \tag{3.10}$$

for $m = 0, 1, 2$. However, from (3.10), for $m = 0$,

$$c_k = a_k\varphi_1 + a_{k-1}\varphi_1' + \cdots + a_1\varphi_1^{k-1} + \varphi_1^{(k)}$$

for $k \in 0, 1, 2$ is zero for φ_1 satisfies (3.1). Hence (3.9) actually is a linear equation of order $2-1 = 1$ for $\tilde{w}$. Letting u $= \tilde{w}'$, and dividing (3.9) through by φ_1, there results an equation

$$\sum_{m=0}^{1} d_{2-m-1}(z) u^{(m)} = d_1(z)u + d_0(z)u' = 0 \tag{3.11}$$

where

$$d_0 = 1$$

and

$$d_1 = \frac{c_1}{\varphi_1} = a_1 + \frac{a_0 \varphi_1'}{\varphi_1}. \tag{3.12}$$

Consider differential equation

$$w' + a_1(z)w = 0 \tag{3.13}$$

where a_1 is analytic and single-valued in a punctured vicinity(deleted neighborhood) of z_0. If the solution φ_1 of the form (3.8) is substituted back into (5.13), we'll obtain

$$(z - z_0)a_1(z) = -s - (z - z_0)\frac{q'(z)}{q(z)}.$$

Therefore $(z - z_0)a_1(z)$ is analytic at z_0, which proves the theorem for order 1. Since z_0 is a regular singular point for (3.1), it is also one for (3.11). (3.11) has as solution as function $(\varphi_2/\varphi_1)'$ where φ_1, φ_2 are linearly independant solutions of (3.1), with φ_1 being the function in (3.8). If φ_1 and φ_2 are dependants then there is no constant c_2 such that $c_2(\varphi_2/\varphi_1)' = 0$. There follows the linear dependance of φ_1 and φ_2 is impossible. Thus $(\varphi_2/\varphi_1)'$ is fundamental solution for (3.11). The derivative $(\varphi_2/\varphi_1)'$ is, by hypothesis, sum of expression of the type

$$(z - z_0)^a (log(z - z_0))^b \frac{\tilde{p(z)}}{p(z)}$$

where a is a constant, b an integer, $p(z_0) \neq 0$, $\tilde{p}$ analytic at z_0. Therefore the coefficients d_k have at z_0 at most a pole of order k. So in (3.12) a_1 has at most a pole of order 1 which proves the theorem. □

So we have shown that if z_0 is a regular singular point of differential equation of order 2(or n), then z_0 is a singular point of the first kind of that differential equation and converse of this also true.

4 Additives

Theorem 4.1. *If A in* (3.5) *is a single-valued and analytic in a punctured vicinity of z_0,* $0 < |z - z_0| < a$, *then every fundamental matrix Φ of (3.5) has the form*

$$\Phi(z) = S(z)(z - z_0)^P$$

where S is single-valued, analytic on $0 < |z - z_0| < a$, and P is a constant matrix.

4.1 Existance of such m for theorem 2.1

Proof. We continue proof from where we left. Let φ be any nonzero vector solution of (2.3) and let $\tilde{\varphi}(\rho, \theta) := \varphi(\rho e^{i\theta})$, $r = ||\tilde{\varphi}||$. Then

$$\frac{d\tilde{\varphi}}{d\rho}(p, \theta) = \frac{d\varphi}{dz}(\rho e^{i\theta})e^{i\theta}$$

and thus

$$\left|\frac{d\tilde{\varphi}}{d\rho}(p, \theta)\right| \leq \left|\frac{d\varphi}{dz}(\rho e^{i\theta})e^{i\theta}\right| \leq |\tilde{A}(\rho e^{i\theta})|\frac{r(\rho, \theta)}{\rho}$$

also

$$\left|\frac{dr}{d\rho}\right| \leq \left|\frac{d\tilde{\varphi}}{d\rho}\right|.$$

Therefore, if $|\tilde{A}| \leq c$ for $|z| \leq \rho_1 < a$,

$$\left|\frac{dr}{d\rho}\right| \leq \frac{cr}{\rho},$$

for $(0 < \rho \leq \rho_1)$. From this follows

$$\frac{dr}{d\rho} + \frac{cr}{\rho} \geq 0$$

and hence for same ρ_1 and ρ,

$$\rho_1^c r(\rho_1, \theta) - \rho^c r(\rho, \theta) \geq 0.$$

If M denotes the maximum of $r(\rho_1, \theta)$ for $0 \leq \theta \leq 2\pi$, then

$$\left|\varphi(\rho e^{i\theta})\right| = r(\rho, \theta) \leq \frac{\rho_1^c r(\rho_1, \theta)}{\rho^c} \leq \frac{M\rho_1^c}{\rho^c}.$$

Thus if Φ is a fundamental matrix for (2.3), there exists a constant $d > 0$ such that if $z = \rho e^{i\theta}$

$$|\Phi(z)| \leq \frac{d}{\rho^c} \tag{4.1}$$

for $(0 \leq \theta \leq 2\pi, 0 < |z| \leq \rho_1)$. It remains to appraise the term z^{-P} in the representation $S = \Phi z^{-P}$. The one has $z^{-P} = e^{(-log z)P} = e^{(-log\rho)P}e^{-i\theta P}$, and so we have,

$$\left|z^{-P}\right| \leq \left|e^{(-log\rho)P}\right|\left|e^{-i\theta P}\right|.$$

Now

$$\left| e^{-(log\rho)P} \right| \leq (n-1) + e^{|log\rho|P}$$

and if $0 < \rho < 1$,

$$\left| e^{-(log\rho)P} \right| \leq (n-1) + e^{-(log\rho)|P|} \leq n\rho^{-P}.$$

Also, if $0 \leq \theta \leq 2\pi$,

$$\left| e^{-i\theta P} \right| \leq (n-1) + e^{2\pi|P|}.$$

So, there results

$$\left| z^{-P} \right| \leq n\rho^{-|P|}((n-1) + e^{2\pi|P|})$$

provided $0 < \rho < 1$, $0 \leq \theta \leq 2\pi$. Combining this with (2.4), one obtains finally

$$\rho^{c+|P|} \left| S(z) \right| \leq \tilde{d},$$

for $0 < \rho < min(1, \rho_1)$ and $0 \leq \theta \leq 2\pi$, where $\tilde{d}$ is a constant independant of z in the range $0 < |z| < min(1, \rho_1)$. Therefore a positive integer m can be chosen so large that $z^m S$ bounded in neighborhood of $z = 0$, thus completing the proof of the theorem. $\square$

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