دانشکده‌ی علوم ریاضی

احتمال و کاربرد ۸ آبان ۱۳۹۸

تمرین : سری ۱

مدرّس: دکتر محسن شریفی تبار مهلت تحویل ۱۴ آبان

علیرضا درویشی.. ۹۶۱۰۹۶۷۴

مسأله ۱

الف)

$$P(B_i|A) = \frac{P(B_i)P(A|B_i)}{\sum_{j=1}^{j=k} P(B_j)P(A|B_i)}$$
$$P(1st\, black|2nd\, red) = \frac{P(1st\ black)P(2nd\ red|1st\ black)}{P(1st\ black)P(2nd\ red|1st\ black)+P(1st\ red)P(2nd\ red|1st\ red)}$$
$$\Rightarrow P(1st\, black|2nd\, red) = \frac{\frac{b}{r+b}\frac{r}{r+b+c}}{\frac{b}{r+b}\frac{r}{r+b+c}+\frac{r}{r+b}\frac{r+c}{r+b+c}} = \frac{b}{r+b+c}$$

ب)

$$for\ i\ : A_i = \{i \in A\ \& i \in B\} \parallel \{i \notin A\ \& i \in B\} \parallel \{i \notin A\ \& i \notin B\}$$
$$for\ i\ : p(A_i) = \frac{3}{4}$$
$$p(\bigcap_{i=1}^{n} A_i) = (\frac{3}{4})^n$$

ج)

$$p(E|E \cup F) = \frac{p(E\cap(E\cup F))}{p(E\cup F)}\frac{p(E)}{p(E\cup F)} = \frac{p(E\cap F)+p(E\cap F^c)}{p(F)+p(E\cap F^c)} \geq \frac{p(E\cap F)}{p(F)} = p(E|F)$$

مسأله ۲

$$\overbrace{?\ \ ?\ \ ?\ ....\ ?}^{n} : H\ \ T\ \ \overbrace{?\ ....\ ?}^{n-2}\ \ or\ \ T\ \ \overbrace{?\ \ ?\ ....\ ?}^{n-1}$$

$$\Rightarrow f_n = f_{n-1} + f_{n-2}$$
$$f_0 = 1$$
$$f_1 = 2$$

مسأله‌ی ۳

$n_A: \ number\ of\ heads\ in\ A$
$n_B: \ number\ of\ heads\ in\ B$
$p(n_A = n_B) = \sum_j p(n_A = j, n_B = j) = \sum_j p(n_A = j)p(n_B = j) =$
$\sum_j (\frac{1}{2})^k \binom{k}{j} (\frac{1}{2})^{n-k} \binom{n-k}{j} = (\frac{1}{2})^n \sum_j \binom{k}{j}\binom{n-k}{j} = (\frac{1}{2})^n \binom{n}{k} = p(n_A + n_B = k)$

مسأله‌ی ۴

$$p(X = x) = \frac{n}{n+m}\frac{n-1}{n+m-1}\cdots\frac{n-(x-1)}{n+m-(x-1)} \times \frac{m}{n+m-x} = \frac{m}{n+m-x}\frac{\frac{n!}{(n-x)!}}{\frac{(n+m)!}{(n+m-x)!}} = \frac{m}{n+m-x}\frac{\frac{(n+m-x)!}{(n-x)!}}{\frac{(n+m)!}{n!}} =$$
$$\frac{m}{n+m-x}\frac{\binom{n+m-x}{m}}{\binom{n+m}{m}} = \frac{\binom{n+m-1-x}{m-1}}{\binom{n+m}{m}}$$
$$E[X] = \sum_{x=1}^n xp(X = x) = \sum_{x=1}^n x\frac{\binom{n+m-1-x}{m-1}}{\binom{n+m}{m}} = \frac{1}{\binom{n+m}{m}}\sum_{x=1}^n x\binom{n+m-1-x}{m-1} =$$
$$\frac{1}{\binom{n+m}{m}}\sum_{j=1}^n \underbrace{\sum_{x=j}^n \binom{n+m-1-x}{m-1}}_{\sum_{k=m}^n \binom{k}{m} = \binom{n+1}{m+1}} = \frac{1}{\binom{n+m}{m}}\sum_{j=1}^n \binom{m-1+n-j+1}{m-1+1} =$$
$$\frac{1}{\binom{n+m}{m}}\underbrace{\sum_{j=1}^n \binom{m+n-j}{m}}_{\sum_{k=m}^n \binom{k}{m} = \binom{n+1}{m+1}} = \frac{1}{\binom{n+m}{m}}\binom{n+m}{m+1} = \frac{\frac{(n+m)!}{(m+1)!(n-1)!}}{\frac{(n+m)!}{n!m!}} = \boxed{\frac{n}{m+1}}$$

مسأله ۵

$f(x) = e^{-x^2/2}$
$f_1(x) = (1 + \frac{1}{x^2})e^{-x^2/2}$
$f_2(x) = \frac{x^4+2x^2-1}{x^4+2x^2+1}e^{-x^2/2}$
$\Rightarrow f_2(x) \le f(x) \le f_1(x) \Rightarrow \int_x^\infty f_2(y)dy \le \int_x^\infty f(y)dy \le \int_x^\infty f_1(y)dy$
$\Rightarrow \frac{1}{\sqrt{2\pi}}\frac{x}{x^2+1}e^{-\frac{x^2}{2}} \le p(Z \ge x) \le \frac{1}{\sqrt{2\pi}}\frac{1}{x}e^{-\frac{x^2}{2}}$

مسأله ۶

الف)

$$J = \begin{pmatrix} \partial x \\ \partial y \end{pmatrix}\begin{pmatrix} U & V \end{pmatrix} = \begin{pmatrix} U_x & V_x \\ U_y & V_y \end{pmatrix} = \begin{pmatrix} y & 1/y \\ x & -x/y^2 \end{pmatrix} = -2x/y = -2V$$
$$g(U, V) = f(x, y)\frac{1}{|J|} = \frac{1}{2U^2V}$$
$$g(U, V) = \begin{cases} \frac{1}{2U^2V} & 0 < V \ \&\ max(V, \frac{1}{V}) \le U \\ 0 & else \end{cases}$$

ب)

$$f_U(U) = \int_{V=\frac{1}{U}}^{U} g(U,V)dV = \frac{1}{U^2}ln(U)$$
$$f_V(V) = \int_U g(U,V)dV$$
$$f_V(V) = \begin{cases} \int_{U=V}^{\infty} g(U,V)dU = \frac{1}{2V^2} & V \geq 1 \\ \int_{U=\frac{1}{V}}^{\infty} g(U,V)dU = \frac{1}{2} & 0 < V \leq 1 \end{cases}$$

مسأله ۷

$$E[X_1 + X_2 + .. + X_n | X_1 + X_2 + .. + X_n = x] = x$$
$$E[X_1 + X_2 + .. + X_n | X_1 + X_2 + .. + X_n = x] = E[X_1 | X_1 + X_2 + .. + X_n = x] + E[X_2 | X_1 + X_2 + .. + X_n = x] + .. + E[X_n | X_1 + X_2 + .. + X_n = x] = nE[X_1 | X_1 + X_2 + .. + X_n = x]$$
$$\Rightarrow E[X_1 | X_1 + X_2 + .. + X_n = x] = \frac{x}{n}$$

مسأله ۸

الف)

$$x_i : \begin{cases} 1 & ith\ ball\ is\ red \\ 0 & else \end{cases},\ X = \sum_{i=1}^{12} x_i$$
$$y_i : \begin{cases} 1 & ith\ ball\ is\ blue \\ 0 & else \end{cases},\ Y = \sum_{i=1}^{12} y_i$$
$$Cov(X,Y) = E[XY] - E[X]E[Y] = E[\sum x_i \sum y_j] - E[\sum x_i]E[\sum y_i]$$
$$E[\sum x_i \sum y_i] = E[\sum x_i y_i] + E[\sum_{i \neq j} x_i y_j] = \sum \underbrace{E[x_i y_i]}_{=0} + \sum_{i \neq j} E[x_i y_j]$$
$$= 12 \times 11 \times E[x_i]E[y_i]$$
$$Cov(X,Y) = (12 \times 11 - 12 \times 12)E[x_i]E[y_i] = -12E[x_i]E[y_i] = -12 \frac{12 \times 10}{30} \frac{12 \times 8}{30} = \boxed{-153.6}$$

ب)

$$E[XY] = \iint_{x,y} xyf(x,y)dxdy = \frac{1}{2}\int_{y=0}^{\pi/2} ydy\{\int_{x=0}^{\pi/2} xsin(x+y)dx\} =$$
$$\frac{1}{2}\int_{y=0}^{\pi/2} y\{\frac{1}{2}(\pi-2)sin(y) + cos(y)\}dy = \frac{\pi-2}{2}$$
$$E[X] = \iint_{x,y} xf(x,y)dxdy = \frac{1}{2}\int_{y=0}^{\pi/2} dy\{\int_{x=0}^{\pi/2} xsin(x+y)dx\} =$$
$$\frac{1}{2}\int_{y=0}^{\pi/2}\{\frac{1}{2}(\pi-2)sin(y) + cos(y)\}dy = \frac{\pi}{4}$$
$$E[X^2] = \iint_{x,y} x^2 f(x,y)dxdy = \frac{1}{2}\int_{y=0}^{\pi/2} dy\{\int_{x=0}^{\pi/2} x^2 sin(x+y)dx\} = \frac{1}{8}(\pi^2 + 4\pi - 16)$$
$$\sigma_x^2 = E[X^2] - E^2[X] = \frac{1}{16}(\pi^2 + 8\pi - 32)$$
$$Cov(X,Y) = (\frac{\pi-2}{2}) - \frac{\pi^2}{16}$$
$$\rho(X,Y) = \frac{(\frac{\pi-2}{2}) - \frac{\pi^2}{16}}{\frac{1}{16}(\pi^2 + 8\pi - 32)} = -0.25 \neq -1$$

مسأله ۹

$$p(|X-\mu|>\epsilon)=\frac{VAR(X)}{\epsilon^2}=\frac{VAR(X)}{n\epsilon^2}=\frac{\mu(1-\mu)}{n\epsilon^2}\leq\frac{1}{4}\frac{1}{n\epsilon^2}\leq 1-\eta\Rightarrow n\geq\frac{1}{4(1-\eta)\epsilon^2}=62500$$