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Глава VI. Неопределенный интеграл

1676. $\displaystyle\int \sqrt{x}\,dx = \frac{2}{3}x\sqrt{x}.$

1677. $\displaystyle\int \sqrt[m]{x^n}\,dx = \int x^{n/m} = \frac{m}{n+m}\int x^{n+m/m}.$

1936. $\displaystyle\int \frac{x\,dx}{\sqrt{1+2x}} = \qquad \left| t=\sqrt{1+2x}, \quad x=\frac{t^2-1}{2}, \quad dx = t\,dt. \right|$

$$= \int \frac{(t^2-1)\cdot t\,dx}{2t} = \frac{t^3}{6} - \frac{t}{2} = \frac{(1+2x)\sqrt{1+2x}}{6} - \frac{\sqrt{1+2x}}{2} = \frac{(x-1)\sqrt{1+2x}}{3}.$$

1941. $\displaystyle\int \frac{dx}{\sqrt{9x^2-6x+2}} = \frac{1}{3}\int \frac{d(3x-1)}{\sqrt{(3x-1)^2+1}} = \frac{1}{3}\ln(3x-1+\sqrt{9x^2-6x+2}).$

1947. $\displaystyle\int \frac{(3x-1)\,dx}{\sqrt{x^2+2x+2}} = \frac{3}{2}\int \frac{d(x^2+2x+2)}{\sqrt{x^2+2x+2}} - 4\int \frac{d(x+1)}{\sqrt{(x+1)^2+1}} =$

$$= 3\sqrt{x^2+2x+2} - 4\ln(x+1+\sqrt{x^2+2x+2}).$$

1954. $\displaystyle\int \frac{\sqrt{x}\,dx}{\sqrt{2x+3}} = \int \sqrt{x}\,d\sqrt{2x+3} = \sqrt{2x^2+3x} - \frac{1}{2}\int \sqrt{2+\frac{3}{x}}\,dx =$

$$= x\sqrt{2+\frac{3}{x}} - \frac{x}{2}\sqrt{2+\frac{3}{x}} + \frac{1}{2}\int x\,d\sqrt{2+\frac{3}{x}} = \frac{x}{2}\sqrt{2+\frac{3}{x}} - \frac{3}{4}\int \frac{dx}{\sqrt{2x^2+3x}} =$$

$$= \frac{\sqrt{2x^2+3x}}{2} - \frac{3}{4\sqrt{2}}\int \frac{d(x+3/4)}{\sqrt{(x+3/4)^2-(3/4)^2}} =$$

$$= \frac{\sqrt{2x^2+3x}}{2} - \frac{3}{4\sqrt{2}}\ln\left(x+\frac{3}{4}+\sqrt{x^2+\frac{3}{2}x}\right).$$

1957. $\displaystyle\int x\sin x\,\cos x\,dx = \int x\sin x\,d\sin x = \frac{1}{2}\int x\,d\sin^2 x =$

$$= \frac{1}{2}\left(x\sin^2 x - \int \sin^2 x\,dx\right) = \frac{x\sin^2 x}{2} - \frac{1}{2}\int \frac{1-\cos 2x}{2}\,dx =$$

$$= \frac{x - x\cos 2x}{4} - \frac{x}{4} + \frac{\sin 2x}{8} = \frac{\sin 2x}{8} - \frac{x\cos 2x}{4}.$$

1966. $\displaystyle\int \frac{dx}{e^x+1} = \qquad \left| t=e^x, \quad dt = e^x dx = t\,dx, \quad dx = \frac{dt}{t} \right|$

$$= \int \frac{dt}{t(t+1)} = \int \frac{dt}{t} - \int \frac{dt}{t+1} = \ln|t| - \ln|t+1| = \ln\frac{e^x}{e^x+1}.$$

1974. $\displaystyle\int \frac{(1+\operatorname{tg} x)\,dx}{\sin 2x} = \quad \left| t=\operatorname{tg} x, \quad x=\operatorname{arctg} t, \quad dx=\frac{dt}{1+t^2}.\right|$

$$= \int \frac{(1+t)(1+t^2)\,dx}{2t(1+t^2)} = \frac{1}{2}\ln t + \frac{1}{2}t = \frac{1}{2}\ln|\operatorname{tg} x| + \frac{1}{2}\operatorname{tg} x.$$

1984. $\displaystyle\int \frac{x^4dx}{\sqrt{(1-x^2)^3}} = \quad |x=\sin u, \quad dx=\cos u.| \quad = \int \frac{\sin^4 u\cdot\cos u}{\cos^3 u}\,du =$

$$= \int \frac{d\operatorname{tg} u}{(1+\operatorname{ctg}^2 u)^2} = \int \frac{\operatorname{tg}^4 u\,d\operatorname{tg} u}{(1+\operatorname{tg}^2 u)^2} = \quad |t=\operatorname{tg} u| \quad = \int \frac{t^4\,dt}{(1+t^2)^2} =$$

$$= \int dt - \int \frac{t^2\,dt}{(1+t^2)^2} - \int \frac{(1+t^2)\,dt}{(1+t^2)^2} = t - \operatorname{arctg} t - \int \frac{t^2\,dt}{(1+t^2)^2} =$$

Отдельно вычислим интеграл

$$-\int \frac{t^2\,dt}{(1+t^2)^2} = \frac{1}{2}\int t\cdot d\left(\frac{1}{1+t^2}\right) = \frac{1}{2}\cdot\frac{t}{1+t^2} - \frac{1}{2}\int\frac{dt}{1+t^2} = \frac{1}{2}\cdot\frac{t}{1+t^2} - \frac{1}{2}\operatorname{arctg} t.$$

Подставим в основной интеграл

$$t - \operatorname{arctg} t + \frac{1}{2}\cdot\frac{t}{1+t^2} - \frac{1}{2}\operatorname{arctg} t = \frac{2t^3+3t}{2(1+t^2)} - \frac{3}{2}\operatorname{arctg} t =$$

Учитывая, что $t=\operatorname{tg} u = \operatorname{tg}\arcsin x = \dfrac{x}{\sqrt{1-x^2}}$, получаем

$$= \left(\frac{2x^3}{(1-x^2)\sqrt{1-x^2}} + \frac{3x}{\sqrt{1-x^2}}\right) \Big/ \left(2+\frac{2x^2}{1-x^2}\right) - \frac{3}{2}\arcsin x =$$

$$= \frac{3x-x^3}{(1-x^2)\sqrt{1-x^2}}\cdot\frac{1-x^2}{2} - \frac{3}{2}\arcsin x = \frac{3x-x^3}{2\sqrt{1-x^2}} - \frac{3}{2}\arcsin x.$$

Еще один вариант решения задачи.

$$\int \frac{x^4dx}{\sqrt{(1-x^2)^3}} = -\frac{1}{2}\int\frac{x^3d(1-x^2)}{\sqrt{(1-x^2)^3}} = \int x^3 d\frac{1}{\sqrt{1-x^2}} = \frac{x^3}{\sqrt{1-x^2}} - 3\int\frac{x^2dx}{\sqrt{1-x^2}} =$$

$$= \frac{x^3}{\sqrt{1-x^2}} + \frac{3}{2}\int\frac{x\,d(1-x^2)}{\sqrt{1-x^2}} = \frac{x^3}{\sqrt{1-x^2}} + 3\int x\,d\sqrt{1-x^2} =$$

$$= \frac{x^3}{\sqrt{1-x^2}} + 3x\sqrt{1-x^2} - 3\int\sqrt{1-x^2}\,dx =$$

Отдельно вычислим интеграл $\displaystyle\int\sqrt{1-x^2}\,dx$. Для этого положим

$$I = \int\sqrt{1-x^2}\,dx = x\sqrt{1-x^2} - \int\frac{-x^2\,dx}{\sqrt{1-x^2}} =$$

$$= x\sqrt{1-x^2} - \int\sqrt{1-x^2}\,dx + \int\frac{dx}{\sqrt{1-x^2}} = x\sqrt{1-x^2} - I + \arcsin x.$$

Отсюда находим: $I = \dfrac{x}{2}\sqrt{1-x^2} + \dfrac{1}{2}\arcsin x.$

Теперь вычисляем сам интеграл задачи:

$$= \frac{x^3}{\sqrt{1-x^2}} + 3x\sqrt{1-x^2} - \frac{3x}{2}\sqrt{1-x^2} - \frac{3}{2}\arcsin x =$$

$$= \frac{2x^3+3x-3x^3}{2\sqrt{1-x^2}} - \frac{3}{2}\arcsin x = \frac{3x-x^3}{2\sqrt{1-x^2}} - \frac{3}{2}\arcsin x.$$

1992. $\displaystyle\int \frac{dx}{(2+x)\sqrt{1+x}} = \qquad \left|t=\sqrt{1+x}, \quad x=t^2-1, \quad dx = 2t\,dt.\right|$
$\displaystyle = \int \frac{2t\,dt}{(t^2+1)t} = 2\,\mathrm{arctg}\, t = 2\,\mathrm{arctg}\sqrt{1+x}.$

2009. $\displaystyle\int \ln(x+\sqrt{1+x^2})\,dx = x\ln(x+\sqrt{1+x^2}) - \int \frac{x\,dx}{\sqrt{1+x^2}} =$
$\displaystyle = x\ln(x+\sqrt{1+x^2}) - \int \frac{d(1+x^2)}{2\sqrt{1+x^2}} = x\ln(x+\sqrt{1+x^2}) - \sqrt{1+x^2}.$

2018. $\displaystyle\int \frac{32x\,dx}{(2x-1)(4x^2-16x+15)} = \int \frac{32x\,dx}{(2x-1)(2x-3)(2x-5)} =$
$\displaystyle = \int \left(\frac{A}{2x-1} + \frac{B}{2x-3} + \frac{C}{2x-5}\right)\,dx = \ldots$
$32x = A(2x-3)(2x-5) + B(2x-1)(2x-5) + C(2x-1)(2x-3).$
$x = 1/2, \quad A = 2. \quad x = 3/2, \quad B = -12, \quad x = 5/2, \quad C = 10.$
$\displaystyle \ldots = \int \left(\frac{2}{2x-1} - \frac{12}{2x-3} + \frac{10}{2x-5}\right)\,dx =$
$\displaystyle = \int \frac{d(2x-1)}{2x-1} - 6\int \frac{d(2x-3)}{2x-3} + 5\int \frac{d(2x-3)}{2x-5}, dx =$
$= \ln|2x-1| - 6\ln|2x-3| + 5\ln|2x-5|.$

2023. $\displaystyle\int \left(\frac{x+2}{x-1}\right)^2 \frac{dx}{x} = \int \left(\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x}\right)\,dx = \ldots$
$(x+2)^2 = Ax(x-1) + Bx + C(x-1)^2.$
$x = 1, \quad B = 9, \quad x = 0, \quad C = 4, \quad x = -1, \quad 1 = 2A - 9 + 16, \quad A = -3.$
$\displaystyle \ldots = \int \left(-\frac{3}{x-1} + \frac{9}{(x-1)^2} + \frac{4}{x}\right)\,dx- == 4\ln|x| - 3\ln|x-1| - \frac{9}{x-1}.$

2034. $\displaystyle\int \frac{x^3-2x^2+4}{x^3(x-2)^2} = \int \left(\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-2} + \frac{E}{(x-2)^2}\right)\,dx = \ldots$
$x^3 - 2x^2 + 4 = Ax^2(x-2)^2 + Bx(x-2)^2 + C(x-2)^2 + Dx^3(x-2) + Ex^3 =$
$= (A+D)x^4 + (-4A+B-2D+E)x^3 + (4A-4B+C)x^2 + (4B-4C)x + 4C.$
$$\begin{cases} A+D=0 \\ -4A+B-2D+E=1 \\ 4A-4B+C=-2 \\ 4B-4C=0 \\ 4C=4 \end{cases} ; \qquad \begin{cases} D=-1/4 \\ E=1/2 \\ A=1/4 \\ B=1 \\ C=1 \end{cases} .$$
$\displaystyle \ldots = \int \left(\frac{1/4}{x} + \frac{1}{x^2} + \frac{1}{x^3} - \frac{1/4}{x-2} + \frac{1/2}{(x-2)^2}\right)\,dx =$
$\displaystyle = \frac{1}{4}\ln|x| - \frac{1}{x} - \frac{1}{2x^2} - \frac{1}{4}\ln|x-2| - \frac{1}{2(x-2)} = \ln\left|\frac{x}{x-1}\right| - \frac{1}{x} - \frac{1}{2x^2} - \frac{1}{2(x-2)}.$

2047. $\displaystyle\int \frac{dx}{1+x^4} = \int \frac{dx}{(x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1)} =$

$$\int \left(\frac{Ax+B}{x^2-\sqrt{2}x+1} + \frac{Cx+D}{x^2+\sqrt{2}x+1} \right) dx = \ldots$$
$1 = (Ax+B)(x^2+\sqrt{2}x+1) + (Cx+D)(x^2-\sqrt{2}x+1).$

$$\begin{cases} A+C=0 \\ \sqrt{2}A+B-\sqrt{2}C+D=0 \\ A+\sqrt{2}B+C-\sqrt{2}D=0 \\ B+D=1 \end{cases}; \qquad \begin{cases} C=-A \\ \sqrt{2}A+B+\sqrt{2}A-B=-1 \\ A+\sqrt{2}B-A+\sqrt{2}B=\sqrt{2} \\ D=1-B \end{cases};$$

$$\begin{cases} C=\frac{1}{2\sqrt{2}} \\ A=-\frac{1}{2\sqrt{2}} \\ B=\frac{1}{2} \\ D=\frac{1}{2} \end{cases}.$$

$$\ldots = \frac{\sqrt{2}}{4}\int \frac{-x+\sqrt{2}}{x^2-\sqrt{2}x+1}\,dx + \frac{\sqrt{2}}{4}\int \frac{x+\sqrt{2}}{x^2+\sqrt{2}x+1}\,dx =$$
$$= -\frac{\sqrt{2}}{8}\int \frac{2x-\sqrt{2}}{x^2-\sqrt{2}x+1}\,dx + \frac{\sqrt{2}}{8}\int \frac{\sqrt{2}}{x^2-\sqrt{2}x+1}\,dx +$$
$$+ \frac{\sqrt{2}}{8}\int \frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1}\,dx + \frac{\sqrt{2}}{8}\int \frac{\sqrt{2}}{x^2+\sqrt{2}x+1}\,dx =$$
$$= -\frac{\sqrt{2}}{8}\ln(x^2-\sqrt{2}x+1) + \frac{1}{4}\int \frac{d(x-\sqrt{2}/2)}{(x-\sqrt{2}/2)^2+1/2} +$$
$$+ \frac{\sqrt{2}}{8}\ln(x^2+\sqrt{2}x+1) + \frac{1}{4}\int \frac{d(x+\sqrt{2}/2)}{(x+\sqrt{2}/2)^2+1/2} =$$
$$= \frac{\sqrt{2}}{8}\ln\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1} + \frac{\sqrt{2}}{4}(\operatorname{arctg}(\sqrt{2}x-1)+\operatorname{arctg}(\sqrt{2}x+1)) =$$
$$= \frac{\sqrt{2}}{8}\ln\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1} + \frac{\sqrt{2}}{4}\operatorname{arctg}\frac{\sqrt{2}x}{1-x^2} = .$$

2053. $$\int \frac{2x\,dx}{(1+x)(1+x^2)^2} = \int \left(\frac{A}{1+x} + \frac{Bx+C}{1+x^2} + \frac{Dx+E}{(1+x^2)^2} \right) dx = \ldots$$
$2x = A(1+x^2)^2 + (Bx+C)(1+x)(1+x^2) + (Dx+E)(1+x).$
$x=-1, \quad -2=4A, \quad A=-1/2.$
$x=i, \quad 2i=(E-D)+(D+E)i, \quad E=1, \quad D=1.$
$2 = \left((Bx+C)(1+x)2x + (Dx+E) + D(1+x)\right)\big|_{x=i};$
$2 = -2(B+C) + 2(C-B)i + i + 1 + 1 + i;$
$B+C=0, \quad B-C=1, \quad B=1/2, \quad C=-1/2.$
$$\ldots = \int \left(-\frac{1}{2(1+x)} + \frac{x-1}{2(1+x^2)} + \frac{x+1}{(1+x^2)^2} \right) dx =$$
$$= -\frac{1}{2}\ln|1+x| + \frac{1}{4}\int \frac{2x\,dx}{1+x^2} - \frac{1}{2}\int \frac{dx}{1+x^2} + \frac{1}{2}\int \frac{2x\,dx}{(1+x^2)^2} + \int \frac{dx}{(1+x^2)^2} =$$
$$= \frac{1}{4}\ln(1+x^2) - \frac{1}{2}\ln|1+x| - \frac{1}{2}\operatorname{arctg} x - \frac{1}{2(1+x^2)} + \int \frac{dx}{(1+x^2)^2}.$$

$$= \frac{1}{4}\ln(1+x^2) - \frac{1}{2}\ln|1+x| - \frac{1}{2}\operatorname{arctg} x - \frac{1}{2(1+x^2)} + \int \frac{dx}{(1+x^2)^2}.$$

Имеем: $\operatorname{arctg} x = \int \frac{dx}{1+x^2} = \frac{x}{1+x^2} + 2\int \frac{x^2\,dx}{(1+x^2)^2} =$

$$= \frac{x}{1+x^2} + 2\int \frac{dx}{1+x^2} - 2\int \frac{dx}{(1+x^2)^2} = \frac{x}{1+x^2} + 2\operatorname{arctg} x - 2\int \frac{dx}{(1+x^2)^2}.$$

Отсюда получаем: $\int \frac{dx}{(1+x^2)^2} = \frac{1}{2}\left(\frac{x}{1+x^2} + \operatorname{arctg} x\right)$.

Ответ: $\frac{1}{4}\ln(1+x^2) - \frac{1}{2}\ln|1+x| - \frac{1-x}{2(1+x^2)}$.

2057. $\int \frac{(4x^2-8x)\,dx}{(x-1)^2(x^2+1)^2} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} + \int\left(\frac{D}{x-1} + \frac{Ex+F}{x^2+1}\right)dx = \ldots$

$$\frac{4x^2-8x}{(x-1)^2(x^2+1)^2} = -\frac{A}{(x-1)^2} + \frac{B(x^2+1)-2x(Bx+C)}{(x^2+1)^2} + \frac{D}{x-1} + \frac{Ex+F}{x^2+1};$$

$4x^2 - 8x = -A(x^4+2x^2+1) + B(x^2-2x+1)(x^2+1) -$
$- 2x(x^2-2x+1)(Bx+C) + D(x-1)(x^4+2x^2+1) +$
$+ (x^2-2x+1)(x^2+1)(Ex+F);$
$x = 1, \quad -4 = -4A, \quad A = 1.$
$x = i, \quad -4-8i = -2i\cdot(-2i)(C+Bi) = -4C - 4Bi,$
$B = 2, \quad C = 1.$
$0x^5 = (D+E)x^5,$
$-8x = (-2B-2C+D-2F+E)x = (-6+D-2F+E)x,$
$0 = -A+B-D+F = 1-D+F.$

$$\begin{cases} D+E=0 \\ D-2F+E=-2 \\ D-F=1 \end{cases}; \qquad \begin{cases} D+E=0 \\ -2F=-2 \\ D-F=1 \end{cases}; \qquad \begin{cases} E=-2 \\ F=1 \\ D=2 \end{cases}.$$

$$\int \frac{(4x^2-8x)\,dx}{(x-1)^2(x^2+1)^2} = \frac{1}{x-1} + \frac{2x+1}{x^2+1} + \int\left(\frac{2}{x-1} + \frac{-2x+1}{x^2+1}\right)dx =$$

$$= \frac{1}{x-1} + \frac{2x+1}{x^2+1} + 2\ln|x-1| - \ln(x^2+1) + \operatorname{arctg} x =$$

$$= \frac{3x^2-x}{(x-1)(x^2+1)} + \ln\frac{(x-1)^2}{x^2+1} - \operatorname{arctg} x.$$

2066. $\int \frac{5-3x+6x^2+5x^3-x^4}{x^5-x^4-2x^3+2x^2+x-1}\,dx = \int \frac{5-3x+6x^2+5x^3-x^4}{(x-1)^3(x+1)^2}\,dx =$

$$= \frac{Ax^2+Bx+C}{(x-1)^2(x+1)} + \int\left(\frac{D}{x-1} + \frac{E}{x+1}\right)dx = \ldots$$

$$\frac{5-3x+6x^2+5x^3-x^4}{(x-1)^3(x+1)^2} =$$

$$= \frac{(x-1)^2(x+1)(2Ax+B) - [2(x^2-1)+(x-1)^2](Ax^2+Bx+C)}{(x-1)^4(x+1)^2} +$$

$$+ \frac{D}{x-1} + \frac{E}{x+1}.$$

$(5-3x+6x^2+5x^3-x^4)(x-1)=(x-1)^2(x+1)(2Ax+B)-$
$-(x-1)(3x+1)(Ax^2+Bx+C)+(x-1)^3(x+1)^2D+(x-1)^4(x+1)E.$
$5-3x+6x^2+5x^3-x^4=(x^2-1)(2Ax+B)-(3x+1)(Ax^2+Bx+C)+$
$+(x-1)^2(x+1)^2D+(x-1)^3(x+1)E.$
$-x^4=(D+E)x^4.\quad 5x^3=(2A-3A-2E)x^3.$
$6x^2=(B-A-3B-2D)x^2.\quad -3x=(-2A-3C-B+2E)x.$
$5=-B-C+D-E.$

$$\begin{cases} D+E=-1 \\ -A-2E=5 \\ -A-2B-2D=6 \\ -2A-B-3C+2E=-3 \\ -B-C+D-E=5 \end{cases};\quad \begin{cases} E=-1-D \\ -A+2D=3 \\ -A-2B-2D=6 \\ -2A-B-3C-2D=-1 \\ -B-C+2D=4 \end{cases};$$

$$\begin{cases} E=-1-D \\ A=2D-3 \\ -2B-4D=3 \\ -B-3C-6D=-7 \\ -B-C+2D=4 \end{cases};\quad \begin{cases} E=-1-D \\ A=2D-3 \\ 2C-8D=-5 \\ -2C-8D=-11 \\ -B=C-2D+4 \end{cases};\quad \begin{cases} E=-2 \\ A=-1 \\ C=3/2 \\ D=1 \\ B=-7/2 \end{cases}.$$

$$\ldots=\frac{3-7x-2x^2}{2(x^3-x^2-x+1)}+\int\left(\frac{1}{x-1}-\frac{2}{x+1}\right)dx=$$
$$=\frac{3-7x-2x^2}{2(x^3-x^2-x+1)}+\ln\frac{|x-1|}{(x+1)^2}.$$

2075. $$\int\frac{dx}{\sqrt[4]{(x-1)^3(x+2)^5}}=\int\frac{\sqrt[4]{x-1}\,dx}{(x-1)(x+2)\sqrt[4]{x+2}}=$$
$$\left|\frac{x-1}{x+2}=t^4,\quad x-1=t^4(x+2),\quad x=\frac{1+2t^4}{1-t^4},\quad x-1=\frac{3t^4}{1-t^4},\right.$$
$$\left.x+2=\frac{3}{1-t^4},\quad dx=\frac{8t^3(1-t^4)+4t^3(1+2t^4)}{(1-t^4)^2}\,dt=\frac{12t^3\,dt}{(1-t^4)^2}.\right|$$
$$=\int\frac{t(1-t^4)^2 12t^3\,dx}{3t^4\cdot 3\cdot(1-t^4)^2}=\frac{4}{3}\int dt=\frac{4}{3}t=\frac{4}{3}\sqrt[4]{\frac{x-1}{x+2}}.$$

2080. $$\int\frac{dx}{\sqrt[3]{1+x^3}}=\quad\left|x^{-3}+1=t^3,\quad x=\frac{1}{\sqrt[3]{t^3-1}},\quad dx=-\frac{t^2\,dt}{\sqrt[3]{(t^3-1)^4}}.\right|$$
$$=-\int\frac{\sqrt[3]{t^3-1}\cdot t^2\,dt}{t\sqrt[3]{(t^3-1)^4}}=-\int\frac{t\,dt}{t^3-1}=-\int\left(\frac{A}{t-1}+\frac{Bt+C}{t^2+t+1}\right)dt=\ldots$$
$t=A(t^2+t+1)+(Bt+C)(t-1).\quad t=1,\quad A=1/3.$
$t=0,\quad 0=C-A,\quad C=A=1/3.$
$t=-1,\quad -1=1/3+(B-1/3)\cdot 2,\quad B=-1/3.$
$$\ldots=-\frac{1}{3}\int\left(\frac{1}{t-1}-\frac{t-1}{t^2+t+1}\right)dt=$$

$$= -\frac{1}{3}\left(\ln|t-1| - \frac{1}{2}\int\frac{(2t+1)\,dt}{t^2+t+1} + \frac{3}{2}\int\frac{dt}{(t+\frac{1}{2})^2+\frac{3}{4}}\right) =$$
$$= -\frac{1}{3}\ln|t-1| + \frac{1}{6}\ln(t^2+t+1) - \frac{1}{2}\cdot\frac{2}{\sqrt{3}}\operatorname{arctg}\frac{2t+1}{\sqrt{3}} =$$
$$= \frac{1}{6}\ln\frac{t^2+t+1}{(t-1)^2} - \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2t+1}{\sqrt{3}}, \quad t = \frac{\sqrt[3]{1+x^3}}{x}.$$

2089. $\int\sqrt[3]{1+\sqrt[4]{x}}\,dx =$
$$\left|1+\sqrt[4]{x} = t^3, \quad x = (t^3-1)^4, \quad dx = 12t^2(t^3-1)^3\,dt\right|$$
$$= 12\int t^3(t^3-1)^3\,dt = 12\left(\frac{t^{13}}{13} - \frac{3t^{10}}{10} + \frac{3t^7}{7} - \frac{t^4}{4}\right), \quad t = \sqrt[3]{1+\sqrt[4]{x}}.$$

2092. $\int\frac{dx}{\cos x\sin^3 x} = \int\frac{d(\sin x)}{(1-\sin^2 x)\sin^3 x} = \int\frac{dt}{(1-t)(1+t)t^3} =$
$$\int\left(\frac{A}{1-t} + \frac{B}{1+t} + \frac{C}{t} + \frac{D}{t^2} + \frac{E}{t^3}+\right)dt = \ldots$$
$1 = A(1+t)t^3 + B(1-t)t^3 + C(1-t^2)t^2 + D(1-t^2)t + E(1-t^2).$
$t = 1, \quad A = 1/2. \quad t = -1, \quad B = -1/2, \quad t = 0, \quad E = 1.$
$0t^3 = (A+B-D)t^3, \quad D = 0. \quad 0t^4 = (A-B-C)t^4, \quad C = 1.$
$$\ldots = \int\left(\frac{1}{2(1-t)} - \frac{1}{2(1+t)} + \frac{1}{t} + \frac{1}{t^3}\right)dt =$$
$$= -\frac{1}{2}\ln|1-t| - \frac{1}{2}\ln|1+t| + \ln|t| - \frac{1}{2}\cdot\frac{1}{t^2} =$$
$$= -\frac{1}{2}\ln|1-t^2| + \ln|t| - \frac{1}{2t^2} = -\frac{1}{2}\ln|1-\sin^2 x| + \ln|\sin x| - \frac{1}{2\sin^2 x} =$$
$$= \ln|\operatorname{tg} x| - \frac{1}{2\sin^2 x}.$$

2099. $\int\operatorname{ctg}^4 x\,dx = \qquad \left|x = \operatorname{arcctg} t, \quad dx = -\frac{dt}{1+t^2} \quad t = \operatorname{ctg} x.\right|$
$$= -\int\frac{t^4}{1+t^2}\,dt = -\int\left(t^2 - 1 + \frac{1}{1+t^2}\right)dt = -\frac{t^3}{3} + t + \operatorname{arcctg} t =$$
$$= \operatorname{ctg} x - \frac{\operatorname{ctg}^3 x}{3} + x.$$

2106. $\int\frac{dx}{a\cos x + b\sin x} = \qquad \left|t = \operatorname{tg}\frac{x}{2}; \quad x = 2\operatorname{arctg} t; \quad dx = \frac{2\,dt}{1+t^2}\right| \qquad =$
$$= \int\frac{2dt}{(1+t^2)\left(a\frac{1-t^2}{1+t^2} + b\frac{2t}{1+t^2}\right)} = \frac{2}{a}\int\frac{dt}{1-\left(t^2 - 2\cdot\frac{b}{a}t\right)} =$$
$$= \frac{2}{a}\int\frac{dt}{1+\frac{b^2}{a^2} - \left(t-\frac{b}{a}\right)^2} = \frac{2}{a}\cdot\frac{a}{2\sqrt{a^2+b^2}}\ln\left|\frac{\frac{\sqrt{a^2+b^2}}{a} + \left(t-\frac{b}{a}\right)}{\frac{\sqrt{a^2+b^2}}{a} - \left(t-\frac{b}{a}\right)}\right| =$$
$$= \frac{1}{\sqrt{a^2+b^2}}\ln\left|\frac{\sqrt{a^2+b^2}+(at-b)}{\sqrt{a^2+b^2}-(at-b)}\right| + C, \text{ где } t = \operatorname{tg}\frac{x}{2}. \text{ И это ответ.}$$

Такую форму ответа можно было бы и оставить, но ответ задачника другой. Это связано с тем, что задачник предполагает другой метод решения. Сначала мы приведем данный ответ к ответу задачника. Для этого мы попытаемся получить под логарифмом тангенс суммы.

$$\frac{1}{\sqrt{a^2+b^2}}\ln\left|\frac{\sqrt{a^2+b^2}+(at-b)}{\sqrt{a^2+b^2}-(at-b)}\right| = \frac{1}{\sqrt{a^2+b^2}}\ln\left|\frac{t+\frac{\sqrt{a^2+b^2}-b}{a}}{\frac{\sqrt{a^2+b^2}+b}{a}-t}\right|.$$

Прибавив константу, мы не изменим ответ, который от константы не зависит.

$$\frac{1}{\sqrt{a^2+b^2}}\ln\left|\frac{t+\frac{\sqrt{a^2+b^2}-b}{a}}{\frac{\sqrt{a^2+b^2}+b}{a}-t}\right| + \frac{1}{\sqrt{a^2+b^2}}\ln\frac{\sqrt{a^2+b^2}+b}{a} =$$

Под логарифм положительная константа $\dfrac{\sqrt{a^2+b^2}+b}{a}$ попадет в виде множителя и вызовет сокращения.

$$= \frac{1}{\sqrt{a^2+b^2}}\ln\left|\frac{t+\frac{\sqrt{a^2+b^2}-b}{a}}{\frac{\sqrt{a^2+b^2}+b}{a}-t}\cdot\frac{\sqrt{a^2+b^2}+b}{a}\right| = \frac{1}{\sqrt{a^2+b^2}}\ln\left|\frac{t+\frac{\sqrt{a^2+b^2}-b}{a}}{1-\frac{a}{\sqrt{a^2+b^2}+b}t}\right| =$$

$$= \frac{1}{\sqrt{a^2+b^2}}\ln\left|\frac{t+\frac{\sqrt{a^2+b^2}-b}{a}}{1-\frac{a(\sqrt{a^2+b^2}-b)}{a^2+b^2-b^2}t}\right| = \frac{1}{\sqrt{a^2+b^2}}\ln\left|\frac{\operatorname{tg}\frac{x}{2}+\frac{\sqrt{a^2+b^2}-b}{a}}{1-\frac{\sqrt{a^2+b^2}-b}{a}\cdot\operatorname{tg}\frac{x}{2}}\right| =$$

$$\frac{1}{\sqrt{a^2+b^2}}\ln\left|\operatorname{tg}\left(\frac{x}{2}+\operatorname{arctg}\frac{\sqrt{a^2+b^2}-b}{a}\right)\right| = \frac{1}{\sqrt{a^2+b^2}}\ln\left|\operatorname{tg}\left(\frac{x}{2}+\frac{\operatorname{arctg}x}{2}\right)\right|.$$

Здесь мы заменили угол, тангенс которого равен $\dfrac{\sqrt{a^2+b^2}-b}{a}$, вдвое большим углом, тангенс которого равен x. Остается вычислить этот тангенс по формуле тангенса двойного угла.

$$x = \frac{2\cdot\frac{\sqrt{a^2+b^2}-b}{a}}{1-\left(\frac{\sqrt{a^2+b^2}-b}{a}\right)^2} = \frac{2(\sqrt{a^2+b^2}-b)a^2}{a(a^2-(\sqrt{a^2+b^2}-b)^2)} =$$

$$= \frac{2(\sqrt{a^2+b^2}-b)a}{a^2-(a^2+b^2-2b\sqrt{a^2+b^2}+b^2)} = \frac{2(\sqrt{a^2+b^2}-b)a}{2b\sqrt{a^2+b^2}-2b^2} = \frac{a}{b}.$$

Окончательный ответ $\dfrac{1}{\sqrt{a^2+b^2}}\ln\left|\operatorname{tg}\dfrac{x+\operatorname{arctg}\frac{a}{b}}{2}\right|+C.$

Он совпадает с приведенным в задачнике.

Интеграл можно взять без использование универсальной тригонометрической подстановки. Ответ задачника подразумевает, что использован именно этот метод.

$$\int\frac{dx}{a\cos x+b\sin x} = \int\frac{dx}{\sqrt{a^2+b^2}\sin\left(x+\operatorname{arctg}\frac{a}{b}\right)} =$$

$$= \frac{1}{\sqrt{a^2+b^2}}\int\frac{d\left(x+\operatorname{arctg}\frac{a}{b}\right)}{\sin\left(x+\operatorname{arctg}\frac{a}{b}\right)} = \frac{1}{\sqrt{a^2+b^2}}\ln\left|\operatorname{tg}\frac{x+\operatorname{arctg}\frac{a}{b}}{2}\right|+C.$$

Здесь использован в качестве табличного следующий интеграл:

$$\int \frac{dx}{\sin x} = \ln\left|\operatorname{tg}\frac{x}{2}\right| + C.$$

В англоязычных руководствах в качестве табличного используется другая форма этого интеграла:

$$\int \frac{dx}{\sin x} = -\ln(\operatorname{ctg} x + \operatorname{cosec} x) + C,$$

что дает нам возможность получить еще одну форму ответа.

2111. $\displaystyle\int \frac{dx}{5+4\sin x} = \quad \left| t = \operatorname{tg}\frac{x}{2}, \quad x = 2\operatorname{arctg} t, \quad dx = \frac{2\,dt}{1+t^2} \right|$

$$= \int \frac{2\,dt}{(5+\frac{8t}{1+t^2})(1+t^2)} = \int \frac{2\,dt}{5t^2+8t+5} = \frac{2}{5}\int \frac{dt}{(t+\frac{4}{5})^2+\frac{9}{25}} =$$

$$= \frac{2}{5}\cdot\frac{5}{3}\operatorname{arctg}\frac{t+\frac{4}{5}}{\frac{3}{5}} = \frac{2}{3}\operatorname{arctg}\frac{5t+4}{3} = \frac{2}{3}\operatorname{arctg}\frac{5\operatorname{tg}\frac{x}{2}+4}{3}.$$

2116. $\displaystyle\int \frac{dx}{5-4\sin x+3\cos x} = \quad \left| t = \operatorname{tg}\frac{x}{2}; \quad x = 2\operatorname{arctg} t; \quad dx = \frac{2\,dt}{1+t^2};\right.$

$$\left.\sin x = \frac{2t}{1+t^2}; \quad \cos x = \frac{1-t^2}{1+t^2}.\right| \quad = \int \frac{\frac{2}{1+t^2}\cdot dt}{5-4\cdot\frac{2t}{1+t^2}+3\cdot\frac{1-t^2}{1+t^2}} =$$

$$= \int \frac{2(1+t^2)\,dt}{(1+t^2)(5+5t^2-8t+3-3t^2)} = \int \frac{2\,dt}{2t^2-8t+8} = \int \frac{dt}{(t-2)^2} = -\frac{1}{t-2} =$$

$$= \frac{1}{2-\operatorname{tg}\frac{x}{2}} + C.$$

2120. $\displaystyle\int \frac{dx}{a^2\sin^2 x+b^2\cos^2 x} = \frac{1}{a^2}\int \frac{d(\operatorname{tg} x)}{\operatorname{tg}^2 x+\left(\frac{b}{a}\right)^2} = \frac{1}{ab}\operatorname{arctg}\left(\frac{a}{b}\operatorname{tg} x\right).$

2123. $\displaystyle\int \sqrt{1+\sin x}\,dx = \int \sqrt{\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}+\cos^2\frac{x}{2}}\,dx =$

$$= \int \left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)\,dx = 2\int \left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)\,d\left(\frac{x}{2}\right) = 2\left(\sin\frac{x}{2}-2\cos\frac{x}{2}\right) + C.$$

2127. $\displaystyle\int \frac{dx}{\sqrt{1-\sin^4 x}} = \quad \left| t = \operatorname{tg} x, \quad dx = \frac{dt}{1+t^2}.\right|$

$$1-\sin^4 x = 1-\left(\frac{1-\cos 2x}{2}\right)^2 = 1-\left(\frac{1-\frac{1-t^2}{1+t^2}}{2}\right)^2 = 1-\left(\frac{t^2}{1+t^2}\right)^2 = \frac{1+2t^2}{(1+t^2)^2}.$$

$$= \int \frac{dx}{\sqrt{1-\sin^4 x}} = \int \frac{(1+t^2)\,dt}{(1+t^2)\sqrt{1+2t^2}} = \frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{t^2+\frac{1}{2}}} =$$

$$= \frac{1}{\sqrt{2}}\ln\left(t+\sqrt{t^2+\frac{1}{2}}\right) = \frac{1}{\sqrt{2}}\ln\left(\sqrt{2}\operatorname{tg} x+\sqrt{2\operatorname{tg}^2 x+1}\right) - \frac{1}{\sqrt{2}}\ln\sqrt{2} =$$

$$= \frac{1}{\sqrt{2}}\ln\left(\sqrt{2}\operatorname{tg} x+\sqrt{2\operatorname{tg}^2 x+1}\right) + C.$$

2139. $\int \operatorname{cth}^2 x\,dx = \int \operatorname{ch} x \cdot \dfrac{d(\operatorname{sh} x)}{\operatorname{sh}^2 x} = -\int \operatorname{ch} x\, d\left(\dfrac{1}{\operatorname{sh} x}\right) =$
$= -\dfrac{\operatorname{ch} x}{\operatorname{sh} x} + \int \dfrac{\operatorname{sh} x}{\operatorname{sh} x}\,dx = x - \operatorname{cth} x.$

2150. $\int \dfrac{e^{2x}dx}{\operatorname{sh}^4 x} = 16\int \dfrac{e^x\,d(e^x)}{(e^x - e^{-x})^4} = 16\int \dfrac{t\,dt}{\left(t-\frac{1}{t}\right)^4} = 16\int \dfrac{\frac{1}{t^3}\,dt}{\left(1-\frac{1}{t^2}\right)^4} =$
$= 8\int \dfrac{d\left(-\frac{1}{t^2}\right)}{\left(1-\frac{1}{t^2}\right)^4} = 8\int \dfrac{d\left(1-\frac{1}{t^2}\right)}{\left(1-\frac{1}{t^2}\right)^4} = -\dfrac{8}{3\left(1-\frac{1}{t^2}\right)^3} = -\dfrac{8t^3}{3\left(t-\frac{1}{t}\right)^3} =$
$= -\dfrac{8(e^x)^3}{3\left(e^x - \frac{1}{e^x}\right)^3} = -\dfrac{e^{3x}}{3\operatorname{sh}^3 x}.$

2154. $\int \dfrac{dx}{x\sqrt{2+x-x^2}}\,dx = \int \dfrac{dx}{x\sqrt{(1+x)(2-x)}}\,dx = \int \sqrt{\dfrac{1+x}{2-x}} \cdot \dfrac{dx}{x(1+x)} =$

$$\left|\frac{1+x}{2-x} = t^2,\quad 1+x = (2-x)t^2,\quad x = \frac{2t^2-1}{t^2+1},\quad x+1 = \frac{3t^2}{t^2+1},\right.$$
$$\left. dx = \frac{(t^2+1)4t - 2t(2t^2-1)}{(t^2+1)^2}\,dt = \frac{6t\,dt}{(t^2+1)^2}\right|$$

$$= \int \frac{t(t^2+1)^2\cdot 6t\,dt}{(2t^2-1)\cdot 3t^2(t^2+1)^2} = \int \frac{dt}{t^2-1/2} = \frac{\sqrt{2}}{2}\ln\left|\frac{t-\sqrt{2}/2}{t+\sqrt{2}/2}\right| =$$
$$= -\frac{\sqrt{2}}{2}\ln\left|\frac{\sqrt{2}t+1}{\sqrt{2}t-1}\right| = -\frac{\sqrt{2}}{2}\ln\left|\frac{2\cdot\frac{1+x}{2-x} + 2\sqrt{2}\cdot\sqrt{\frac{1+x}{2-x}}+1}{2\cdot\frac{1+x}{2-x}-1}\right| =$$
$$= -\frac{\sqrt{2}}{2}\ln\left|\frac{2(1+x)+2\sqrt{2}\cdot\sqrt{2+x-x^2}+2-x}{2(1+x)-(2-x)}\right| =$$
$$= -\frac{\sqrt{2}}{2}\ln\left|\frac{4+x+2\sqrt{2}\cdot\sqrt{2+x-x^2}}{3x}\right| =$$
$$= -\frac{\sqrt{2}}{2}\ln\frac{2\sqrt{2}}{3} - \frac{\sqrt{2}}{2}\ln\left|\frac{\sqrt{2+x-x^2}+\sqrt{2}}{x} + \frac{1}{2\sqrt{2}}\right| =$$
$$= C - \frac{\sqrt{2}}{2}\ln\left|\frac{\sqrt{2+x-x^2}+\sqrt{2}}{x} + \frac{1}{2\sqrt{2}}\right|.$$

2162. $\int \dfrac{dx}{x^2(x+\sqrt{1+x^2})} = -\int \dfrac{(x-\sqrt{1+x^2})}{x^2}\,dx =$
$$= -\int\frac{dx}{x} + \int\frac{\sqrt{1+x^2}}{x^2}\,dx = -\ln|x| + \int\frac{dx}{x^2\sqrt{1+x^2}} + \int\frac{dx}{\sqrt{1+x^2}} =$$
$$= \ln\left|\frac{x+\sqrt{1+x^2}}{x}\right| + \int\frac{dx}{x^2\sqrt{1+x^2}}.$$
Оставшийся интеграл вычисляем подстановкой Абеля.

$$t=\frac{x}{\sqrt{1+x^2}},\quad t^2(1+x^2)=x^2,\quad x^2=\frac{t^2}{1-t^2};\quad t\sqrt{1+x^2}=x,$$
$$dt\sqrt{1+x^2}+\frac{tx\,dx}{\sqrt{1+x^2}}=dx,\quad dt\sqrt{1+x^2}+t^2\,dx=dx,\quad \frac{dt}{1-t^2}=\frac{dx}{\sqrt{1+x^2}}.$$
$$\int\frac{dx}{x^2\sqrt{1+x^2}}=\int\frac{(1-t^2)dt}{t^2(1-t^2)}=\int\frac{dt}{t^2}=-\frac{1}{t}=-\frac{\sqrt{1+x^2}}{x}.$$
Ответ: $\ln\left|\dfrac{x+\sqrt{1+x^2}}{x}\right|-\dfrac{\sqrt{1+x^2}}{x}$.

2170. $$\int\frac{x^4\,dx}{\sqrt{x^2+4x+5}}=(Ax^3+Bx^2+Cx+D)\sqrt{x^2+4x+5}+\lambda\int\frac{dx}{\sqrt{x^2+4x+5}};$$
$$\frac{x^4}{\sqrt{x^2+4x+5}}=(3Ax^2+2Bx+C)\sqrt{x^2+4x+5}+\frac{(Ax^3+Bx^2+Cx+D)(x+2)}{\sqrt{x^2+4x+5}}+$$
$$+\frac{\lambda}{\sqrt{x^2+4x+5}};$$
$$x^4=(3Ax^2+2Bx+C)(x^2+4x+5)+(Ax^3+Bx^2+Cx+D)(x+2)+\lambda;$$
$$1=3A+A,\quad 0=12A+2B+2A+B,\quad 0=15A+8B+C+2B+C,$$
$$0=10B+4C+2C+D;\quad 0=5C+2D+\lambda.$$
$$A=\frac{1}{4},\quad B=-\frac{7}{6},\quad C=-\frac{15}{2}A-5B=-\frac{15}{8}+\frac{35}{6}=\frac{140-45}{24}=\frac{95}{24},$$
$$D=\frac{70}{6}-\frac{6\cdot 95}{24}=\frac{280-570}{24}=-\frac{290}{24}=-\frac{145}{12},$$
$$\lambda=-\frac{5\cdot 95}{24}+\frac{145}{6}=\frac{580-475}{24}=\frac{105}{24}=\frac{35}{8}.$$
Ответ: $\left(\dfrac{1}{4}x^3-\dfrac{7}{6}x^2+\dfrac{95}{24}x-\dfrac{145}{12}\right)\sqrt{x^2+4x+5}+\dfrac{35}{8}\ln\left(x+2+\sqrt{x^2+4x+5}\right)$.

2174. $$\int\frac{(2x+3)\,dx}{(x^2+2x+3)\sqrt{x^2+2x+4}}=$$
$$=\int\frac{(2x+2)\,dx}{(x^2+2x+3)\sqrt{x^2+2x+4}}+\int\frac{dx}{(x^2+2x+3)\sqrt{x^2+2x+4}}.$$
1) $$\int\frac{(2x+2)\,dx}{(x^2+2x+3)\sqrt{x^2+2x+4}}=2\int\frac{d\sqrt{x^2+2x+4}}{x^2+2x+3}=2\int\frac{dt}{t^2-1}=$$
$$=\ln\left|\frac{t-1}{t+1}\right|=\ln\left|\frac{\sqrt{x^2+2x+4}-1}{\sqrt{x^2+2x+4}+1}\right|.$$
2) $$\int\frac{dx}{(x^2+2x+3)\sqrt{x^2+2x+4}}=\dots$$
$$t=\left(\sqrt{x^2+2x+4}\right)'=\frac{x+1}{\sqrt{x^2+2x+4}},\quad x+1=t\sqrt{x^2+2x+4},$$
$$x^2+2x+1=(x^2+2x+4)t^2,\quad x^2+2x+4-3=(x^2+2x+4)t^2,$$
$$x^2+2x+4=\frac{3}{1-t^2},\quad x^2+2x+3=\frac{2+t^2}{1-t^2},$$
$$dx=dt\sqrt{x^2+2x+4}+t^2dx,\quad \frac{dx}{\sqrt{x^2+2x+4}}=\frac{dt}{1-t^2}.$$

$$\ldots = \int \frac{dt}{2+t^2} = -\frac{1}{\sqrt{2}} \operatorname{arcctg} \frac{t}{\sqrt{2}} = -\frac{1}{\sqrt{2}} \operatorname{arcctg} \frac{x+1}{\sqrt{2(x^2+2x+4)}} =$$
$$= -\frac{1}{\sqrt{2}} \operatorname{arctg} \frac{\sqrt{2(x^2+2x+4)}}{x+1} + \frac{1}{\sqrt{2}} \cdot \frac{\pi}{2}.$$
Ответ: $\ln \left| \dfrac{\sqrt{x^2+2x+4}-1}{\sqrt{x^2+2x+4}+1} \right| - \dfrac{1}{\sqrt{2}} \operatorname{arctg} \dfrac{\sqrt{2(x^2+2x+4)}}{x+1} + C.$

2177. $\displaystyle\int x\sqrt[3]{a+x}\, dx = \frac{3}{4}\int x\, d[(a+x)^{4/3}] = \frac{3}{4}x(a+x)^{4/3} - \frac{3}{4}\int (a+x)^{4/3}dx =$
$$= \frac{3}{4}x(a+x)^{4/3} - \frac{9}{28}(a+x)^{7/3}dx = \frac{3(4x-3a)\sqrt[3]{(a+x)^4}}{28}.$$

2185. $\displaystyle\int x^2 \operatorname{sh} x\, dx = \int x^2 d(\operatorname{ch} x) = x^2 \operatorname{ch} x - 2\int x \operatorname{ch} x\, dx =$
$$= x^2 \operatorname{ch} x - 2\int x\, d(\operatorname{sh} x) = x^2 \operatorname{ch} x - 2x \operatorname{sh} x + 2\int \operatorname{sh} x\, dx =$$
$$= x^2 \operatorname{ch} x - 2x \operatorname{sh} x + 2 \operatorname{ch} x.$$

2191. $\displaystyle\int \sin\sqrt{x}\, dx = \quad \left| t=\sqrt{x}, \quad x=t^2 \right| \quad = 2\int t \sin t\, dt =$
$$= -2\int t\, d(\cos t) = -2t\cos t + 2\int \cos t\, dt = -2t\cos t + 2\sin t =$$
$$= 2(\sin\sqrt{x} - \sqrt{x}\cos\sqrt{x}).$$

2197. $\displaystyle\int \frac{dx}{x^3\sqrt{(1+x)^3}} = \int \frac{dx}{x^3(1+x)\sqrt{1+x}} =$
$\left|\sqrt{1+x}=t, \quad x=t^2-1 \quad dx = 2t\, dt.\right|$
$$= \int \frac{2\, dt}{t^2(t^2-1)^3} = \int \left(\frac{A}{t^2} + \frac{B}{t^2-1} + \frac{C}{(t^2-1)^2} + \frac{D}{(t^2-1)^3} \right) dt = \ldots$$
$2 = (t^2-1)^3A + t^2(t^2-1)^2B + t^2(t^2-1)C + t^2D.$
$t^2=0, \quad A=-2. \quad t^2=1, \quad D=2.$
$0t^2 = 3A+B-C+D, \quad B-C=4.$
$0t^4 = -3A-2B+C, \quad 2B-C=6. \quad B=2, \quad C=-2.$
$$\ldots = \int \left(-\frac{2}{t^2} + \frac{2}{t^2-1} - \frac{2}{(t^2-1)^2} + \frac{2}{(t^2-1)^3} \right) dt = \ldots$$
$$I_n = \int \frac{dt}{(t^2-1)^n} = \frac{t}{(t^2-1)^n} + 2n\int \frac{t^2\, dt}{(t^2-1)^{n+1}} =$$
$$= \frac{t}{(t^2-1)^n} + 2n\int \frac{(t^2-1)\, dt}{(t^2-1)^{n+1}} + 2n\int \frac{dt}{(t^2-1)^{n+1}} =$$
$$= \frac{t}{(t^2-1)^n} + 2nI_n + 2nI_{n+1}. \quad I_{n+1} = -\frac{2n-1}{2n}I_n - \frac{t}{2n(t^2-1)^n}.$$
$$I_2 = -\frac{1}{4}\ln\left|\frac{t-1}{t+1}\right| - \frac{t}{2(t^2-1)}. \quad I_3 = \frac{3}{16}\ln\left|\frac{t-1}{t+1}\right| + \frac{3t}{8(t^2-1)} - \frac{t}{4(t^2-1)^2}.$$
$$\ldots = \frac{2}{t} + \ln\left|\frac{t-1}{t+1}\right| + \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + \frac{t}{t^2-1} + \frac{3}{8}\ln\left|\frac{t-1}{t+1}\right| + \frac{3t}{4(t^2-1)} - \frac{t}{2(t^2-1)^2} =$$

$$= \frac{15}{8}\ln\left|\frac{t-1}{t+1}\right| + \frac{8(t^2-1)^2+4t^2(t^2-1)+3t^2(t^2-1)-2t^2}{4t(t^2-1)^2} =$$
$$= \frac{15}{8}\ln\left|\frac{\sqrt{1+x}-1}{\sqrt{1+x}+1}\right| + \frac{8x^2+7(x+1)x-2(x+1)}{4x^2\sqrt{1+x}} =$$
$$= \frac{15x^2+5x-2}{4x^2\sqrt{1+x}} + \frac{15}{8}\ln\left|\frac{\sqrt{1+x}-1}{\sqrt{1+x}+1}\right|.$$

2203. $\displaystyle\int x\ln(1+x^3)\,dx = \frac{1}{2}\int \ln(1+x^3)\,d(x^2) = \frac{x^2}{2}\ln(1+x^3) - \frac{3}{2}\int\frac{x^4dx}{1+x^3}.$
$$\int\frac{x^4dx}{1+x^3} = \int\left(x - \frac{x}{(1+x)(1-x+x^2)}\right)dx = \int\left(x+\frac{A}{1+x}+\frac{Bx+C}{1-x+x^2}\right)dx.$$
$-x = (1-x+x^2)A + (1+x)(Bx+C). \quad x=-1, \quad A = 1/3.$
$x=0, \quad 0 = A+C, \quad C=-1/3. \quad x=1, \quad -1 = A+2B+2C, \quad B=-1/3.$
$$\int\frac{x^4dx}{1+x^3} = \frac{x^2}{2} + \frac{1}{3}\ln|1+x| - \frac{1}{3}\int\frac{x+1}{1-x+x^2}$$

2210. $\displaystyle\int\frac{\sin 2x\,dx}{\cos^4x+\sin^4x} = \int\frac{2\sin x\cos x\,dx}{(1+\operatorname{tg}^4x)\cos^4x} = \int\frac{2\operatorname{tg}x\,d(\operatorname{tg}x)}{1+\operatorname{tg}^4x} =$
$$= \int\frac{d(\operatorname{tg}^2x)}{1+\operatorname{tg}^4x} = \operatorname{arctg}(\operatorname{tg}^2x).$$

2216. $\displaystyle\int\frac{xe^xdx}{\sqrt{1+e^x}} = \quad \left|1+e^x=t^2, \quad x=\ln(t^2-1), \quad dx = \frac{2t\,dt}{t^2-1}.\right|$
$$= \int\frac{\ln(t^2-1)\cdot(t^2-1)\cdot 2t\,dt}{t\cdot(t^2-1)} = 2\int\ln(t^2-1)\,dt =$$
$$= 2t\ln(t^2-1) - 4\int\frac{t^2dt}{t^2-1} = 2t\ln(t^2-1) - 4t - 2\ln\left|\frac{t-1}{t+1}\right| =$$
$$= 2x\sqrt{1+e^x} - 4\sqrt{1+e^x} - 2\ln\left|\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right|.$$

2222. $\displaystyle\int\frac{dx}{\sqrt{1+e^x+e^{2x}}} = \quad \left|e^x = t, \quad x=\ln t, \quad dx = \frac{dt}{t}.\right|$
$$= \int\frac{dt}{t\sqrt{1+t+t^2}} = \quad \left|t=\frac{1}{u}, \quad dt = -\frac{du}{u^2}.\right| \quad = -\int\frac{u\cdot u\,du}{u^2\sqrt{u^2+u+1}} =$$
$$= -\int\frac{d\left(u+\frac{1}{2}\right)}{\sqrt{\left(u+\frac{1}{2}\right)^2+\frac{3}{4}}} = -\ln\left|u+\frac{1}{2}+\sqrt{u^2+u+1}\right| =$$
$$= \ln\left|\frac{1}{u+\frac{1}{2}+\sqrt{u^2+u+1}}\right| = \ln\left|\frac{u+1-\sqrt{u^2+u+1}}{\frac{1}{2}u-\frac{1}{2}+\frac{1}{2}\sqrt{u^2+u+1}}\right| =$$
$$= \ln 2 + \ln\left|\frac{u+1-\sqrt{u^2+u+1}}{u-1+\sqrt{u^2+u+1}}\right| == \ln\left|\frac{1+e^x-\sqrt{1+e^x+e^{2x}}}{1-e^x+\sqrt{1+e^x+e^{2x}}}\right| + C.$$

Далее оцениваем радикал. Имеем: $\sqrt{1+e^x+e^{2x}} = \sqrt{\left(e^x+\frac{1}{2}\right)^2+\frac{3}{4}} < e^x+\frac{1}{2}$;
$\sqrt{1+e^x+e^{2x}} = \sqrt{(e^x+1)^2-e^x} > e^x+1$. Теперь мы обнаруживаем, что числитель

и знаменатель дроби положителен, и мы можем снять знак модуля. Окончательный ответ: $\ln \dfrac{1+e^x-\sqrt{1+e^x+e^{2x}}}{1-e^x+\sqrt{1+e^x+e^{2x}}}+C$.

2225. $\displaystyle\int \frac{(3+x^2)^2x^3dx}{(1+x^2)^3}=\frac{1}{2}\int \frac{(3+x^2)^2x^2d(1+x^2)}{(1+x^2)^3}= \qquad \left|1+x^2=t\right|$

$\displaystyle=\frac{1}{2}\int \frac{(t+2)^2(t-1)\,dt}{t^3}=\frac{1}{2}\int \left(1+\frac{3}{t}-\frac{4}{t^3}\right)dt=$

$\displaystyle=\frac{1}{2}t+\frac{3}{2}\ln|t|+\frac{1}{t^2}=\frac{1}{2}x^2+\frac{3}{2}\ln(x^2+1)+\frac{1}{(x^2+1)^2}+C.$

2227. $\displaystyle\int \frac{dx}{\sin^4 x+\cos^4 x}=\int \frac{dx}{(\operatorname{tg}^4 x+1)\cos^4 x}=\int \frac{(\operatorname{tg}^2 x+1)\,d(\operatorname{tg} x)}{\operatorname{tg}^4 x+1}=$

$\displaystyle=\int \frac{(t^2+1)\,dt}{t^4+1}=\int \frac{\left(1+\frac{1}{t^2}\right)\,dt}{t^2+\frac{1}{t^2}}=\int \frac{d\left(t-\frac{1}{t}\right)}{t^2+\frac{1}{t^2}}=\int \frac{d\left(t-\frac{1}{t}\right)}{\left(t-\frac{1}{t}\right)^2+2}=$

$\displaystyle=\frac{1}{\sqrt{2}}\operatorname{arctg}\frac{t-\frac{1}{t}}{\sqrt{2}}=\frac{\sqrt{2}}{2}\operatorname{arctg}\left[\frac{\sqrt{2}}{2}\left(\frac{\sin x}{\cos x}-\frac{\cos x}{\sin x}\right)\right]=$

$\displaystyle=\frac{\sqrt{2}}{2}\operatorname{arctg}\frac{\sqrt{2}(\sin^2 x-\cos^2 x)}{2\sin x\cos x}=-\frac{\sqrt{2}}{2}\operatorname{arctg}(\sqrt{2}\operatorname{ctg} 2x).$