Assignment 3

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1. **Efficient portfolios**

   Consider an investor who wants to invest in N risky assets with return $R_i \ \ \forall i = 1, \ldots, N$ with expected return $E[R_i] = \mu_i$ and variance $V[R_i] = \sigma_i^2$, and in a risk-free asset with return R_f. The investor seeks a N-risky asset portfolio weight vector w (and a weight $1 - w^\top \mathbf{1}$ in the risk-free asset), such that her portfolio return $R_p = R_f + w^\top(R - R_f\mathbf{1})$ maximizes her mean-variance objective function $U(w) = E[R_p] - \frac{\gamma}{2}V[R_p]$.

   (a) Show that an optimal portfolio weight vector w is such that for the corresponding mean-variance efficient portfolio return R_p we have

   $$\mu_i - R_f = \gamma\, cov[R_i, R_p] \ \ \forall i = 1, \ldots N$$

   (b) Show that for any such mean-variance efficient portfolio, we have

   $$\mu_i - R_f = \beta_{i,P}(\mu_P - R_f)$$

   where $\beta_{i,P} = \frac{cov(R_i, R_P)}{\sigma_P^2}$ is the linear regression coefficient of return R_i on the mean-variance efficient portfolio return R_P.

   (c) In turn, show that this implies that, if R_p is the return to a mean-variance efficient portfolio, then for any return i we have

   $$R_i = R_f + \beta_i(R_P - R_f) + \epsilon_i$$

   where $cov(R_P, \epsilon_i) = 0$.

   Hint: use the definition of a linear regression

(d) Show that all mean-variance efficient portfolios have the same Sharpe ratio where we define its Sharpe ratio as $SR_p = \frac{\mu_p - R_f}{\sigma_p}$.

2. **Portfolio Math**

   (a) Show that any risky-asset only minimum variance frontier portfolio w can be rewritten as a convex combination of any two arbitrary minimum variance frontier portfolios wa, wb in the sense that $w = \alpha w_a + (1-\alpha) w_b$.

   (b) Let R_{min} denote the return on the global minimum-variance portfolio of risky assets. Let R be the return on any risky asset or portfolio of risky assets, efficient or not. Show that $Cov(R_{min}, R) = Var(R_{min})$. Hint: Consider a portfolio consisting of a fraction w in this risky asset. and a fraction $(1-w)$ in the global minimum-variance portfolio. Compute the variance of the return on this portfolio and realize that the variance has to be minimized for w = 0.

3. **Risk Parity** The optimal mean-variance portfolio is a complex function of estimated means, volatilities, and correlations of asset returns. There are many parameters to estimate. Optimized mean-variance portfolios can give poor results when these inputs are imperfectly estimated. In their paper "leverage aversion and risk- parity" (posted on moodle) Asness, Frazzini, and Pedersen (AFP) suggest that **risk-parity allocation**, which has become widely popular and ignores information in sample-means, dominates the standard mean-variance portfolio because it exploits leverage aversion of investors. Here we will try to replicate some of their findings.

   (a) Following AFP download from CRSP the monthly value weighted CRSP Stock index and the value weighted Bond index from 1960 to 2023. (The respective SQL requests are reported in the appendix below). Compute mean, standard deviation and Sharpe ratio of a portfolio that invests 60% in stocks and 40% in bonds (the 60/40 portfolio). Find the tangency portfolio and give its mean, standard deviation and Sharpe ratio. To compute excess returns assume that the annual risk-free rate is 1%.

   (b) The risk-parity portfolio holds stocks and bonds in proportion to the inverse of their (full-sample) volatility. The levered risk-parity portfolio is levered at the risk-free T-Bill rate such that the portfolio's (full sample) volatility is equal to the volatility of the 60/40 portfolio. Following AFP construct the two portfolios and compute their mean, standard deviation and Sharpe ratio. Plot the RP and RP-unlevered portfolios on the efficient frontier along with the Tangency portfolio and the 60/40 portfolio. What explains the difference between the RP and RP-unlevered portfolio performance?

(c) Why does the RP strategy in Figure 1 of AFP not actually lie on the efficient frontier?

(d) Split the sample in two periods from 1960-1990 and 1990-2023. For both sub-samples, find the optimal portfolio for a mean-variance investor that takes the same risk as a 60/40 portfolio. Also, find the levered RP portfolio with the same standard deviation. Discuss your findings.

SQL Requests

- stocks = db.raw sql("select date,vwretd from crsp.msi where date>=’1960-01-01and date<=’2019-12-31", date cols=[’date’])
- bonds = db.raw sql("select caldt, b2ret from crsp.mcti where caldt>=’1960-01-01’ and caldt<=’2019-12-31’", date cols=[’caldt’])