Assignment 5

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1 Theory: Leverage CAPM

Consider an economy with $n = 1, ..., N$ risky assets with returns R_n. $E[R] = \mu$ is the N-vector of asset expected returns and Σ is the $N \times N$ variance-covariance matrix of returns. Investors $i = 1, ..., I$ with wealth W^i have mean-variance preferences and choose a portfolio of risky assets w^i with return R_P such that $\max E[R_P] - \frac{a^i}{2} V[R_p]$. We define the aggregate wealth as $W = \sum_i W^i$ and the weighted average risk aversion in the economy as $a = \left(\sum_i \frac{W^i}{W} \frac{1}{a^i}\right)^{-1}$.

(a) **Security Market Line without Risk-Free Rate.** Derive the optimal portfolio of a . Then impose the market clearing condition that all investors must hold the market portfolio $\sum_i w^i \frac{W^i}{W} = w^m$. Is the market portfolio mean-variance efficient? Derive the security market line in this economy as a function of the return on the market and the zero-beta portfolio. Under which circumstances is the slope of the SML lower than in the standard CAPM with a risk-free rate?

(b) **Optimal portfolio with Leverage Constraints.** Now assume that a risk-free rate exists with return R_f that is available for investment. However, borrowing at the risk-free rate is constrained to a fraction of l^i of their wealth. Set up the lagrangian and show that the 3-fund separation emerges, i.e. the optimal portfolio is a linear combination of the risk-free rate, the tangency portfolio and the minimum variance portfolio.

(c) **Security Market Line with Leverage Constraints.** Impose the market clearing condition to show that the security market line in the economy with investor-specific leverage constraints l^i is given by

$$\mu_n = R_f + \psi + \beta_n(\mu_M - R_f - \psi) \tag{1}$$

where $\psi = \sum_i \omega^i \frac{\lambda^i}{1+l^i}$ is the weighted tightness of the funding constraints, with $\omega^i = \frac{W^i}{W}\frac{a}{a^i}$ and λ^i the lagrange parameter in the leverage-constrained optimizaton. Also, derive the level of risk aversion a^i for each agent such that in the economy with investor-specific leverage constraints m^i the market portfolio is mean-variance efficient.

2 Empirics: Leverage CAPM & Portfolio Construction

In this question we test the empirical implications of the leverage CAPM (1) and construct an optimal portfolio based on its predictions. To this end, we use the same data as in Problem Set 4.

(a) **Estimating Rolling Betas.** Compute the time-varying $\beta_{t,n}$ for each stock by running monthly rolling 5-year regressions of stock-specific excess returns on the excess market return. Require at least 36 months of observations for each stock. Appendix provides some example code on how to efficiently compute the betas in python.

(b) **Portfolio Sorts.** At every month t, sort all stocks into deciles based on their beta. Then compute monthly returns for 10 decile portfolios that equal weight all stocks in each decile. Plot the average annualized portfolio returns across the 10 deciles in a barplot. Repeat for value-weighted returns. Are the results in line with the prediction of the leverage CAPM (1)?

(c) **Constructing the BAB factor.** Now we construct the betting-against-beta factor as in Frazzini & Pedersen (2014). At every month t, we construct two portfolios, a high-beta portfolio w_H and a low-beta portfolio w_L. (See Appendix for the construction of the portfolios) Construct the BAB factor return as

$$R^{BAB}_{t+1} = \frac{R^H_{t+1} - R_f}{\beta_H} - \frac{R^L_{t+1} - R_f}{\beta_L} \tag{2}$$

where $R^H_{t+1} = w_H^\top R_{t+1}$ is the return on the high-beta portfolio and $R^L_{t+1} = w_L^\top R_{t+1}$ is the return on the low-beta portfolio. $\beta_H = w_H^\top \beta_t$ and $\beta_L = w_L^\top \beta_t$ are the corresponding betas of the portfolios. Report the sharpe ratio and CAPM alpha of the BAB factor.

(d) **Long-Short Portfolios** R_{t+1}^{BAB} is the return to a self-financing (i.e. dollar neutral) long short portfolio. Is R_{t+1}^{BAB} also dollar neutral in terms of risky securities only? Intuitively, describe the difference to the portfolio $R_{t+1}^{H} - R_{t+1}^{L}$.

(e) **Long-Short Portfolios** Given the CAPM alpha and idiosyncratic volatility of the BAB factor R_{t+1}^{BAB}, what is the optimal allocation to the risk free rate, the market porfolio and the BAB factor that maximizes the Sharpe ratio of the portfolio?

(f) **BAB Return and Leverage Constraints** Plot the rolling 10-year average of the BAB factor return R_{t+1}^{BAB}. What is the relationship to the aggregate tightness of the funding constraint ψ?

3 Appendix

- **Construct Rolling Betas**:

  1. The instead of running a *groupby* OLS regression, it will be more efficient to use the built in *cov()* function.
  2. Use *groupby.rolling(60, min_periods=36).cov()* to get the rolling covariance matrix between excess stock returns and excess market returns. Then divide the estimated covariance by the market variance

- **Constructing w_H and w_L**:
  $w_H = k(z - \bar{z})^+$ and $w_L = k(z - \bar{z})^-$ where z is a vector of cross-sectional beta ranks $z_n = rank(\beta_n)$. $\bar{z}$ is the cross-sectional average rank, and $k = \frac{2}{\mathbf{1}^\top |z - \bar{z}|}$ is a normalizing factor. $()^+$ and $()^-$ denote the positive and negative elements of a vector respectively.