École Polytechnique Fédérale de Lausanne

Project: Perpetual Futures

FIN-404 Derivatives

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Chapter 1

Part 1: Documentation

1.1

Perpetual contracts exhibit several distinctions from traditional futures contracts, with the most evident one being the absence of an expiration date, which is a characteristic exclusive to the latter. There are various other dissimilarities as well. Perpetual contracts generally entail lower margin requirements due to the absence of rollover risk associated with finite maturity futures. In contrast, traditional futures necessitate the replacement of contracts with new ones, incurring additional fees, in order to extend a position beyond the initial maturity date. This disparity is absent in perpetual futures since they lack an expiration date, resulting in the position remaining open indefinitely unless it is liquidated.

Another contrast arises in terms of trading fees and liquidity. Perpetual contracts are designed for continuous trading, which often leads to lower trading fees and higher liquidity compared to finite maturity futures. The perpetual contracts' exemption from expiration contributes to this discrepancy.

Furthermore, funding rates contribute to another point of divergence. These rates are utilized to maintain an equilibrium between the price of perpetual contracts and the underlying asset price. They are calculated multiple times per day and depend on the disparity between the perpetual contract price and the spot price of the underlying asset, as well as the interest rates for both assets. If the perpetual contract trades at a premium, traders holding long positions are required to pay funding fees to traders with short positions (and vice versa if the difference favors the other direction). This practice ensures that the perpetual futures align with the spot price of the underlying asset by motivating traders with long positions in the premium-trading asset to sell, thereby avoiding fees and restoring market equilibrium.

Let's delve into the primary perpetual contracts, assuming the BTC/USD pairing for linear and inverse contracts, while using COIN/USD for quanto perpetual futures.

Firstly, we encounter the perpetual *linear* futures. These futures settle in the underlying asset (USD in this case) and possess a linear payout structure. This means that the perpetual contract's price is directly linked to the price of the underlying asset. The calculation for Profit and Loss (PNL) is as follows:

$$PNL = \#\text{Contracts} \times \text{ContractSize} \times (\text{Exit Price} - \text{Entry Price})\ [USD].$$

Next, we have the perpetual *inverse* futures. These futures settle in the cryptocurrency (BTC) and exhibit an inverse payout structure. Consequently, when the price of the underlying asset decreases, the contract price increases (and vice versa). The PNL calculation is expressed as:

$$PNL = \#\text{Contracts} \times \text{ContractSize} \times \left(\frac{1}{\text{Entry Price}} - \frac{1}{\text{Exit Price}}\right) [BTC].$$

Lastly, we encounter the perpetual *quanto* futures. These futures settle in a distinct cryptocurrency (BTC), which differs from the underlying index (COIN/USD). Quanto futures enable investors to adopt long or short positions concerning an exchange rate without necessitating ownership of the underlying assets. As a result, the contract price is influenced by the exchange rate between the underlying and the payout currency. The PNL calculation for quanto futures is given by:

$$PNL = \text{Payout Currency Multiplier} \times \#\text{Contracts} \times (\text{ExitPrice} - \text{Entry Price}) [BTC].$$

The aforementioned formulas draw inspiration from the BitMEX website (*BitMEX - Perpetual Contracts Guide* n.d. and the other guides for perpetuals) as well as an in-class presentation on perpetual futures.

1.2

Perpetual futures exist for several compelling reasons. As mentioned earlier, they offer investors greater flexibility compared to traditional contracts. By eliminating the need for rollovers and the associated costs, perpetual futures appeal to investors seeking long-term asset positions while maintaining flexibility.

Furthermore, perpetual contracts can serve as effective hedging tools. Investors can enter opposite positions in perpetual futures contracts to hedge against undesired price changes, with the added advantage of increased flexibility compared to fixed maturity futures.

These futures contracts also provide investors with access to high levels of leverage, enabling them to amplify potential profits (and losses) significantly. This feature can be particularly enticing for smaller investors with limited capital, as higher leverage allows for the possibility of greater profits. However, it is important to consider the accompanying additional risk and the necessary margin account requirements to prevent position liquidation.

Another advantage of perpetual futures is their continuous trading, which contributes to establishing the fair market value of the underlying asset. Even with funding rates considered, the prices of perpetual contracts provide insight into the underlying asset's overall trend and market valuation.

1.3

For the long position in the perpetual linear futures contract, the associated cash flows are:

- Initial margin paid by the investor to the exchange in order to open a position.

- Daily settlements, based on the change in the Bitcoin price since the previous day. The cash flow paid to (or to be paid by, depending on the sign being positive or negative respectively) the long position holder is:

$$(f_{t+1} - f_t) \times \text{ContractSize} \times \#\text{Contracts}$$

with f_t representing the price of the underlying on day t (the long position holder receives a profit if the price of Bitcoin has gone up, otherwise they incur a loss). These cash flows are only theoretical until a position is closed, or if the balance does not meet the margin call, in which case it might be partially closed.

Initially, as proposed by Robert Shiller (1993) when he imagined the concept of perpetual futures, the daily settlement related to a cash flow producing asset would be:

$$(f_{t+1} - f_t) + (d_{t+1} - r_t \times f_t)$$

with d_{t+1} being the dividend paid by the cash flow on day $t + 1$ and r_t being the return of some low-risk asset between t and $t + 1$.

- Funding, as discussed above, depending on whether the price of the perpetual contract is above (a long position holder has to pay funding to short position holders) or below the underlying asset's price. The funding payment is:

$$\text{FundingPayment} = \text{FundingRate} \times \text{PositionValue}$$

where the funding rate is a function of the premium and the interest rate.

- Finally, when an investor chooses to/has to close the position (or a part of it), they will either receive the remaining amount on their account accounting for the profit or loss, except in the case of total liquidation.

For the long position in the perpetual inverse futures contract, the associated cash flows are:

- Initial margin paid by the investor to the exchange in order to open a position.

- Daily settlements, based on the change in the Bitcoin price since the previous day. The cash flow paid to (or to be paid by, depending on the sign being positive or negative respectively) the long position holder is:

$$(\frac{1}{f_t} - \frac{1}{f_{t+1}}) \times \text{ContractSize} \times \#\text{Contracts}$$

with f_t representing the price of the underlying on day t (the long position holder receives a profit if the Bitcoin price has gone down, otherwise they incur a loss).

- Funding, as already seen above, depending on whether the price of the perpetual contract is below (a long position holder has to pay funding to short position holders) or above the underlying asset's price.

- Closing a position works similarly to linear contracts, except that the formula to calculate profits is different than for perpetual linear futures.

Perpetual linear contracts provide a convenient option for investors seeking a long-term long position on Bitcoin without the need to hold the underlying asset themselves. Calculating profits and losses is relatively straightforward compared to inverse contracts, which can initially pose more complexity for investors to comprehend.

On the other hand, perpetual inverse contracts prove beneficial for investors aiming to establish a long-term long position in USD against Bitcoin and capitalize on a potential decline in the underlying asset's price. They can also serve as hedging instruments, as mentioned earlier, enabling protection against a decrease in Bitcoin's value.

One drawback of both linear and inverse contracts stems from the funding rate, which can be unpredictable and result in additional fees for investors.

1.4

FIGURE 1.1
Bitcoin Futures Open Interest vs Spot.

FIGURE 1.2
Bitcoin Volume vs Spot.

FIGURE 1.3
Ethereum Futures Open Interest vs Spot.

FIGURE 1.4
Ethereum Volume vs Spot.

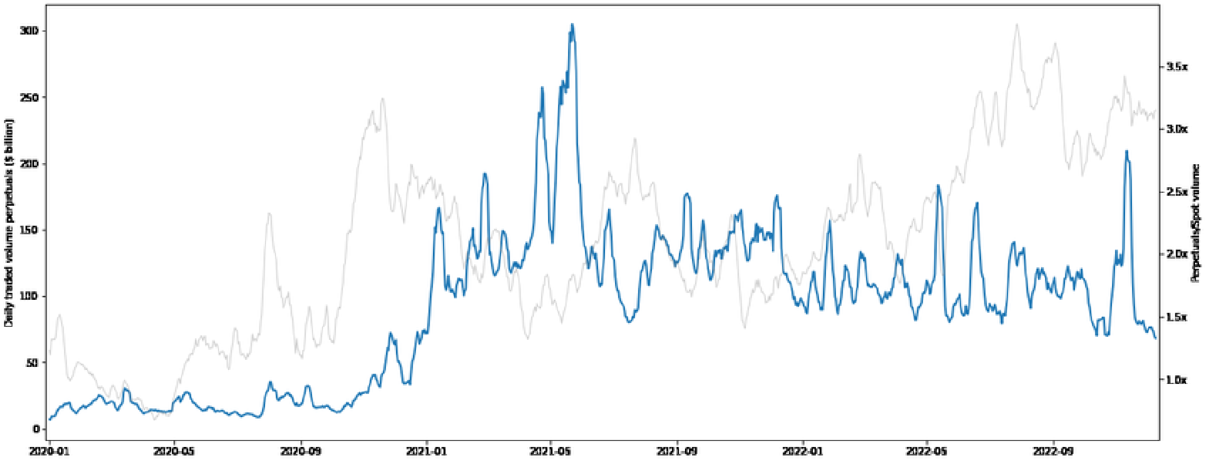

FIGURE 1.5
Perpetuals Volume (blue) and comparison with Spot (grey), He et al. 2022

Over the years, there has been notable growth in crypto futures open interest and volume, reflecting the increasing popularity and maturity of cryptocurrency derivatives markets. These metrics serve as important indicators of market participation and liquidity. To analyze this relationship, we collected data on open interest and volume for both BTC and ETH and plotted them against their spot prices in USD. We also gathered the time series of the perpetuals volume, as well as the multiplier between spot and perpetuals volume. The time series data spans from the start of 2020 to the present (end of 2022 for the perpetuals).

Examining the crypto and perpetual futures markets, we observed that up until May 2021, open interest, spot volume and perpetuals volume all demonstrated an upward trend, indicating a rising interest in crypto derivatives and suggesting market expansion as well as reflecting increased trading activity and indicating enhanced market efficiency and liquidity. However, after this date, the trend reversed for perpetuals (followed by the others in October 2021), with all metrics exhibiting a decline, signaling a slowdown in the cryptocurrency derivatives market.

A significant point of interest lies in the relationship between open interest, volume, and the underlying spot price. In an efficient market, futures prices tend to align with the spot price as the contract approaches its expiration. This alignment, known as the futures' "fair value" or "arbitrage equilibrium," is a foundational principle in derivatives trading. Nonetheless, deviations from this equilibrium can occur due to factors such as market sentiment, supply-demand dynamics, speculation, and external events.

In our last graph, we can see that starting from May 2020, perpetual contracts as a whole have been far more traded (on average two times more) than the currencies themselves, no matter the market conditions. This could be attributed to the reasons stated above (more flexibility and better access to leverage for retail investors, among other factors).

1.5

Perpetual futures can be used for trading commodities, as the underlying assets work somehow like cryptocurrencies: they have (mostly) a very long lifespan, as examples we can take oil and gold. Also, their price is influenced by supply and demand, which makes them easier to track with perpetual futures contracts. The advantages to trade commodities with perpetual futures are similar to the ones with cryptocurrencies: allowing to maintain positions without rollover risk and potentially bringing in more leverage possibilities.

Another application for perpetual futures could be foreign currencies. As the underlying has again a very long lifespan and its value depends on supply and demand, we can reuse the argument from above for this purpose.

One could think that we could apply the concept of perpetual futures to stocks, but in reality, it would not work that well. As stocks represent ownership in a company, they have finite lifespan, unlike the underlying asset in a perpetual futures contract. Another factor is the ease of pricing. As cryptocurrency prices are determined only by market supply and demand, the tracking of the underlying price is easy, but for stocks, it is more complicated, as many other factors, such as financial performance or even macroeconomic trends influence the value of the underlying asset (stock). In addition to this, stock markets are already liquid enough, with lots of different trading instruments (including classic futures).

CHAPTER 2

PART 2: ANALYSIS

2.1 A. EXCHANGE RATE MODELLING

1

The value in a of one share of the b-denominated asset is:
$S^a_{0t} = e^{r_b \Delta t} * x_t$

2

We can now see that from the pov of the investor with a as unit of account, there are the two assets with prices S^a_{0t} and S^b_{0t} (denominated in a and b respectively). However, as seen in part 1. of the analysis, the asset denominated in b originally also holds value $S^a_t = e^{r_b \Delta t} * x_t = e^{r_b \Delta t} * x_0 * \prod_{\tau=1}^{t} U^{\theta_\tau} * D^{1-\theta_\tau}$.
Due to the contribution of θ, which is a Bernoulli process, to the exchange rate, the asset originally denoted in b and converted into a is thus a risky asset S^a_t due to the risky nature of the exchange rate from the pov of the investor.
Also, since the price of S^a_{0t} evolves according to $e^{r_a \Delta t}$, it is riskless from the pov of the investor with rate r_a.

3

As we have seen in the course, no-arbitrage (NA) implies the existence of an equivalent probability measure Q s.t.
$E^Q_t[\frac{S^a_{t+1}}{S^a_{0t+1}}] = \frac{S^a_t}{S^a_{0t}}$
$\frac{S^a_{t+1}(\xi)}{S^a_{0t+1}} = \frac{e^{r_b \Delta t+1} * \xi * x_t}{e^{r_a \Delta t+1}} = e^{(r_b - r_a)\Delta t} * e^{(r_b - r_a)\Delta} * \xi * x_t$
$\frac{S^a_t}{S^a_{0t}} = \frac{e^{r_b \Delta t} * x_t}{e^{r_a \Delta t}} = e^{(r_b - r_a)\Delta t} * x_t$
Now, to fulfill our condition seen above for Q, we require the existence of $q = Q(\xi = U) = Q(\theta_{t+1} = 1)$ s.t.
$q * \frac{S^a_{t+1}(\xi=U)}{S^a_{0t+1}} + (1-q) * \frac{S^a_{t+1}(\xi=D)}{S^a_{0t+1}} = \frac{S^a_t}{S^a_{0t}}$
$\Leftrightarrow q * e^{(r_b - r_a)\Delta t} * e^{(r_b - r_a)\Delta} * U * x_t + (1-q) * e^{(r_b - r_a)\Delta t} * e^{(r_b - r_a)\Delta} * D * x_t = e^{(r_b - r_a)\Delta t} * x_t$
$\Leftrightarrow q * e^{(r_b - r_a)\Delta} * U + (1-q) * e^{(r_b - r_a)\Delta} * D = 1$
$\Leftrightarrow q = \frac{e^{(r_a - r_b)\Delta} - D}{U - D}$
And for this condition to hold (i.e. for q to be a valid probability measure, within $(0, 1)$, we require:
$D < e^{(r_a - r_b)\Delta} < U$

4

Since q units of b are worth $q * x_t$ units of a, q units of a are thus worth $\frac{q}{x_t}$ units of b.
Thus, the value in b of one share of the a-denominated asset is:
$S^b_{0t} * x_t = \frac{e^{r_a \Delta t}}{x_t}$.

5

Again, the market consists of the 2 assets, one per currency.
$S^b_{0t} = e^{r_b \Delta t}$, i.e. it is riskless in terms of currency b with rate r_b.
Now, as seen in part 4, the asset originally denoted in a can be written in terms of b:
$S^b_t = \frac{e^{r_a \Delta t}}{x_t} = e^{r_a \Delta t} * \frac{1}{x_0 * \prod_{\tau=1}^{t} U^{\theta_\tau} * D^{1-\theta_\tau}}$.
Again, θ is a Bernoulli process and contributes to the exchange rate, i.e. to the value of S^b_t, making it a risky asset from the pov of the investor.

6

Looking at part 3, we again require the existence of an equivalent probability measure Q s.t.
$E^Q_t[\frac{S^b_{t+1}}{S^b_{0t+1}}] = \frac{S^b_t}{S^b_{0t}}$
and using the condition found in part 3,
$D < e^{(r_a - r_b)\Delta} < U$
Now,
$\frac{S^b_{t+1}(\xi)}{S^b_{0t+1}} = \frac{e^{r_a \Delta t+1}}{e^{r_b \Delta t+1} * \xi * x_t} = e^{(r_a - r_b)\Delta t} * e^{(r_a - r_b)\Delta} * \frac{1}{\xi * x_t}$
$\frac{S^b_t}{S^b_{0t}} = \frac{e^{r_a \Delta t}}{e^{r_b \Delta t} * x_t} = e^{(r_b - r_a)\Delta t} * \frac{1}{x_t}$
Like in part 3, we thus need some $q = Q(\xi = U) = Q(\theta_{t+1} = 1)$ s.t.
$q * e^{(r_a - r_b)\Delta t} * e^{(r_a - r_b)\Delta} * \frac{1}{U * x_t} + (1-q) * e^{(r_a - r_b)\Delta t} * e^{(r_a - r_b)\Delta} * \frac{1}{D * x_t} = e^{(r_b - r_a)\Delta t} * \frac{1}{x_t}$
$\Leftrightarrow q * e^{(r_a - r_b)\Delta} * \frac{1}{U} + (1-q) * e^{(r_a - r_b)\Delta} * \frac{1}{D} = 1$
$\Leftrightarrow q = \frac{e^{(r_b - r_a)\Delta} - \frac{1}{D}}{\frac{1}{U} - \frac{1}{D}}$
But since by our part 3 assumption, $D < U$, the denominator will be negative, the numerator will have to be negative too i.e.
$e^{(r_b - r_a)\Delta} < \frac{1}{D}$
and to have q within $(0, 1)$,
$\frac{1}{U} < e^{(r_b - r_a)\Delta} < \frac{1}{D}$
$\Leftrightarrow U > e^{(r_a - r_b)\Delta} > D$
which proves that our part 3 condition is enough to guarantee absence of arbitrage and market completeness.

2.2 B. LINEAR FUTURES PRICING

7

We require the (discounted) ex-dividend value of an asset A_t with a-denominated cash flows $\Delta f_T(T)$ (for a FERP) to be 0 for any t, so for the absence of arbitrage, we require:
$\hat{A}_t = 0 = E^{Q_a}_t[\hat{A}^c_{t+1}] = E^{Q_a}_t[A^c_{t+1} * e^{-r_a \Delta}]$
$= e^{-r_a \Delta} * E^{Q_a}_t[\Delta f_{t+1}(T)] = e^{-r_a \Delta} * E^{Q_a}_t[f_{t+1}(T) - f_t(T)]$
and by using the $= 0$ condition,
$f_t(T) = E^{Q_a}_t[f_{t+1}(T)]$

Now, by the law of iterated expectation, as the above condition holds for all t < T:
$E_t^{Q_a}[f_T(T)] = E_t^{Q_a}[E_{t+1}^{Q_a}[f_T(T)]] = E_t^{Q_a}[f_{t+1}(T)] = f_t(T)$
Thus, we have $f_t(T) = E_t^{Q_a}[x_T]$

$f_t(T) = E_t^{Q_a}[x_T] = E_t^{Q_a}[x_t * \prod_{\tau=t+1}^{T} U^{\theta_\tau} * D^{1-\theta_\tau}] = x_t * \prod_{\tau=t+1}^{T} E_t^{Q_a}[(\frac{U}{D})^{\theta_\tau} * D]$
$= x_t * \prod_{\tau=t+1}^{T}(q_a * U + (1-q_a) * D) = x_t * (q_a * (U-D) + D)^{(T-t)}$
and by taking the $q_a = \frac{e^{(r_a-r_b)\Delta}-D}{U-D}$ we found in part A.3,
$f_t(T) = x_t * e^{(r_a-r_b)\Delta(T-t)} = x_t * \psi^{(T-t)}$ and $\psi = e^{(r_a-r_b)\Delta} > 0$

8

We use a similar absence of arbitrage argument as above, but this time taking cash flows $\Delta f_t(T)$ for the perpetual FERP.
$\hat{A}_t = 0 = E_t^{Q_a}[\hat{A}_{t+1}^c] = E_t^{Q_a}[A_{t+1}^c * e^{-r_a\Delta}] = E_t^{Q_a}[e^{-r_a\Delta} * (\Delta f_{t+1}(T) - \lambda(f_{t+1}(T) - X_{t+1}) - \xi X_{t+1})]$
Now, we can prove that M_t is a martingale:
$E_t^{Q_a}[M_{t+1}] = E_t^{Q_a}[M_t + e^{-r_a\Delta} * (\Delta f_{t+1}(T) - \lambda(f_{t+1}(T) - X_{t+1}) - \xi X_{t+1})] = E_t^{Q_a}[M_t] = M_t$
using the equation above, making the additional part equal to 0.

9

Since the same assumptions as above hold, we take the equation from part 8 (removing $e^{-r_a\Delta}$, as it is non-zero):
$E_t^{Q_a}[\Delta f_{t+1}(T) - \lambda(f_{t+1}(T) - X_{t+1}) - \xi X_{t+1}] = E_t^{Q_a}[f_{t+1}(T) - f_t(T) - \lambda(f_{t+1}(T) - X_{t+1}) - \xi X_{t+1}]$
$= -f_t(T) + E_t^{Q_a}[f_{t+1}(T) - \lambda f_{t+1}(T)] + E_t^{Q_a}[\lambda X_{t+1} - \xi X_{t+1}] = 0$, i.e.
$f_t(T) = (1-\lambda) * E_t^{Q_a}[f_{t+1}(T)] + (\lambda x \xi) * E_t^{Q_a}[X_{t+1}]$ as required.
The coefficients to determine are thus $\alpha = 1 - \lambda$ and $\beta = \lambda - \xi$ and are > 0 due to the initial assumption.

10

Taking the equality proven above above,
$f_t(T) = \alpha * E_t^{Q_a}[f_{t+1}(T)] + \beta * E_t^{Q_a}[X_{t+1}]$
we now iterate by replacing each time f_{t+i} by its corresponding value, as the above formula holds for all $t < T$.
We start with $f_{t+1}(T) = \alpha * E_{t+1}^{Q_a}[f_{t+2}(T)] + \beta * E_{t+1}^{Q_a}[X_{t+2}]$ i.e.
$f_t(T) = \alpha * E_t^{Q_a}[\alpha * E_{t+1}^{Q_a}[f_{t+2}(T)] + \beta * E_{t+1}^{Q_a}[X_{t+2}]] + \beta * E_t^{Q_a}[X_{t+1}]$
by the law of iterated expectations,
$f_t(T) = \alpha^2 * E_t^{Q_a}[f_{t+2}] + \alpha * \beta * E_t^{Q_a}[X_{t+2}] + \beta * E_t^{Q_a}[X_{t+1}]$ i.e.
$f_t(T) = \alpha^{t+2-t} * E_t^{Q_a}[f_{t+2}] + \sum_{\tau=t+1}^{t+2} \alpha^{\tau-(t+1)} * \beta * E_t^{Q_a}[X_\tau]$
By recursion, we keep replacing until reaching f_{T-1} to finally get:
$f_t(T) = \alpha^{T-t} * E_t^{Q_a}[f_T] + \sum_{\tau=t+1}^{T} \alpha^{\tau-(t+1)} * \beta * E_t^{Q_a}[X_\tau]$
for all $T >= t = 0, 1, ...$

11

$(y * \alpha^{-t} + f_t(T))_{t=0}^{\infty}$ is a perpetual FERP if for all $T \in \mathbb{N}$, the ex-dividend value of an asset B_t with cash flows $\Delta(y * \alpha^{-t} + f_t(T)) - \lambda * (y * \alpha^{-t} + f_t(T) - X_t) - \xi * X_t$ is 0 at all $t \leq T$.
$\hat{B}_t = e^{r_a\Delta} * E_t^{Q_a}[\Delta(y * \alpha^{-t-1} + f_{t+1}(T)) - \lambda(y * \alpha^{-t-1} + f_{t+1}(T) - X_{t+1}) - \xi X_{t+1}]$
$= e^{r_a\Delta} * E_t^{Q_a}[y * (\alpha^{-t-1} - \alpha{-}t) - \lambda * y * \alpha^{-t-1} + \Delta f_{t+1} - \lambda(f_{t+1}(T) - X_{t+1}) - \xi x_{t+1}]$
$= e^{r_a\Delta} * E_t^{Q_a}[y * (\alpha^{-t-1} - \alpha{-}t - \lambda * \alpha^{-t-1})] + E_t^{Q_a}[\hat{A}_{t+1}^c]$ (second term is 0 due to above)

$= e^{r_a\Delta} * E_t^{Q_a}[y * \alpha^{-t-1} * (1-\alpha-\lambda)] = e^{r_a\Delta} * y * \alpha^{-t-1} * (1-\alpha-\lambda)$
and since $\alpha = 1-\lambda$,
$\hat{B}_t = 0$
i.e. $(y * \alpha^{-t} + f_t(T))_{t=0}^{\infty}$ is a perpetual FERP, as long as $(f_t(T))_{t=0}^{\infty}$ is one too.

The intuition behind that is that the introduction of $y * \alpha^{-t}$ represents a shift/adjustment to the original perpetual FERP price, which can be seen as a scaling or offsetting factor changing over time. As the perpetual FERP price reflects market dynamics, funding rates and other variables, there can be multiple prices reflecting the prevailing market conditions, however, the price process still remains the same FERP. Also, as the horizon is infinite, there is no terminal condition to make a solution unique.

12

From above, we have that: $f_t(T) = \alpha^{T-t} * E_t^{Q_a}[f_T] + \sum_{\tau=t+1}^{T} \alpha^{\tau-(t+1)} * \beta * E_t^{Q_a}[X_\tau]$
and thus,
$f_t^* = \lim_{N\to\infty} \alpha^N * E_t^{Q_a}[f_{t+N}^*] + \sum_{n=1}^{\infty} \alpha^{n-1} * \beta * E_t^{Q_a}[X_{t+n}]$
$= \sum_{n=1}^{\infty} \alpha^{n-1} * \beta * E_t^{Q_a}[X_{t+n}]$ (due to the condition given for f_t^*)
$= \sum_{n=1}^{\infty} \alpha^{n-1} * \beta * E_t^{Q_a}[X_t * \prod_{\tau=t+1}^{t+n}(U^{\theta_\tau} * D^{1-\theta_\tau})]$
$= X_t * \alpha^{-1} * \beta * \sum_{n=1}^{\infty} \alpha^n * (q_a * U + (1-q_a) * D)^n$ (like in Part 7)
Since $\alpha * (q_a * U + (1-q_a) * D) = (1-\lambda) * e^{(r_a - r_b)\Delta}$ and the initial condition states:
$0 < (1-\lambda) * e^{(r_a-r_b)\Delta} < 1$,
we can use the geometric sum formula starting from 1 and we get:
$= X_t * \alpha^{-1} * \beta * \frac{\alpha*(q_a*U+(1-q_a)*D)}{1-\alpha*(q_a*U+(1-q_a)*D)} = X_t * \beta * \frac{(q_a*U+(1-q_a)*D)}{1-\alpha*(q_a*U+(1-q_a)*D)}$ i.e.
$f_t^* = X_t * \beta * \frac{e^{(r_a-r_b)\Delta}}{1-(1-\lambda)*e^{(r_a-r_b)\Delta}}$
Thus, $f_t^* = X_t * \Phi$ and $\Phi = \beta * \frac{e^{(r_a-r_b)\Delta}}{1-\alpha*e^{(r_a-r_b)\Delta}}$

13

In order to replicate f_t^* with a portfolio, we require:
$\pi_{t+1} = \frac{f_{t+1}^*(\xi=U) - f_{t+1}^*(\xi=D)}{S_{0,t+1}^a(\xi=U) - S_{0,t+1}^a(\xi=D)} = \frac{X_t*U*\Phi - X_t*D*\Phi}{e^{r_b\Delta(t+1)}*X_t*U - e^{r_b\Delta(t+1)}*X_t*D} = \Phi * e^{-r_b\Delta(t+1)}$
Now,
$\pi_{0,t+1} = \hat{f}_t^* - \pi_{t+1} * \hat{S}_t^a = e^{-r_a\Delta t} * f_t^* - \Phi * e^{-r_b\Delta(t+1)} * e^{r_a\Delta t} * S_t^a$
$= e^{-r_a\Delta t} * X_t * \Phi - \Phi * e^{-r_b\Delta(t+1)} * e^{r_a\Delta t} * e^{r_b\Delta t} * X_t$
$= X_t * \Phi * e^{-r_a\Delta t} * (1 - e^{-r_b\Delta})$
The actual wealth invested in each of the assets to reproduce the portfolio is:
$\pi_{t+1} * S_{0t}^b = e^{-r_b\Delta} * X_t * \Phi$ in the risky asset and
$\pi_{0,t+1} * S_{0t}^a = (1 - e^{-r_b\Delta}) * X_t * \Phi$ in the risk-free one

2.3 C. INVERSE FUTURES PRICING

14

We have $(i_t(T))$, which is an inverse FERP and like in part 7, we require:
$E_t^{Q_a}[\hat{A}_{t+1}^c] = 0$ i.e.
$E_t^{Q_a}[e^{-r_a\Delta} * \Delta(\frac{1}{i_{t+1}(T)}] = e^{-r_a\Delta} * E^{Q_a}[\frac{1}{i_{t+1}(T)} - \frac{1}{i_t(T)}] = e^{-r_a\Delta} * (E^{Q_a}[\frac{1}{i_{t+1}(T)}] - \frac{1}{i_t(T)}) = 0$ i.e.
$E^{Q_a}[\frac{1}{i_{t+1}(T)}] = \frac{1}{i_t(T)}$
Thus, we can state that i_t^{-1} is a martingale and thus $E^{Q_a}[\frac{1}{i_{t+1}(T)}]^{-1} = i_t(T)$.
We can then proceed in the same way as in part 7 again:

$i_t(T) = E_t^{Q_b}[\frac{1}{x_T}]^{-1} = E_t^{Q_b}[\frac{1}{x_t * \prod_{\tau=t+1}^{T} U^{\theta_\tau} * D^{1-\theta_\tau}}]^{-1} = x_t * (\prod_{\tau=t+1}^{T} E_t^{Q_a}[\frac{1}{(\frac{U}{D})^{\theta_\tau} * D}])^{-1}$
$= x_t * (\prod_{\tau=t+1}^{T} E_t^{Q_b}[(\frac{D}{U})^{\theta_\tau} * \frac{1}{D}])^{-1} = x_t * (\prod_{\tau=t+1}^{T} (\frac{1}{q_b * \frac{1}{U} + (1-q_b) * \frac{1}{D}}))^{-1}$
$= x_t * ((q_b * (\frac{1}{U} - \frac{1}{D}) + \frac{1}{D})^{(T-t)})^{-1}$
and by taking the $q_b = \frac{e^{(r_b - r_a)\Delta} - \frac{1}{D}}{\frac{1}{U} - \frac{1}{D}}$ we found in part A.6,
$i_t(T) = x_t * (e^{(r_b - r_a)\Delta(T-t)})^{-1} = x_t * e^{(r_a - r_b)\Delta(T-t)} = x_t * \psi^{(T-t)}$ and $\psi = e^{(r_a - r_b)\Delta} > 0$
We find the same ψ than in part B.8.

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Now, $(i_t(T))^\infty t = 0$ is a perpetual inverse FERP.
We use a similar absence of arbitrage argument as above, but this time taking cash flows $\Delta\frac{1}{i_t(T)}$ for the perpetual inverse FERP.
$\hat{A}_t = 0 = E_t^{Q_a}[\hat{A}_{t+1}^c] = E_t^{Q_a}[A_{t+1}^c * e^{-r_a\Delta}] = E_t^{Q_a}[e^{-r_a\Delta} * (\Delta(\frac{1}{i_{t+1}(T)}) - \lambda_i(\frac{1}{i_{t+1}(T)} - \frac{1}{X_{t+1}}) - \xi_i \frac{1}{X_{t+1}})]$
Replacing $\frac{1}{i_t(T)}$ by g_t (the (T) is implied) and rearranging, we get:
$E_t^{Q_a}[(g_{t+1} - g_t - \lambda_i(g_{t+1} - \frac{1}{X_{t+1}}) - \xi_i \frac{1}{X_{t+1}})] = 0$ i.e.
$g_t = (1-\lambda_i) * E_t^{Q_a}[g_{t+1}] + (\lambda_i - \xi_i) * E_t^{Q_a}[\frac{1}{X_{t+1}}] = \alpha_i * E_t^{Q_a}[g_{t+1}] + \beta_i * E_t^{Q_a}[\frac{1}{X_{t+1}}]$
with $\alpha_i = (1-\lambda_i)$ and $\beta_i = (\lambda_i - \xi_i)$.
Now, proceeding with the next time steps:
$g_{t+1} = \alpha_i * E_t^{Q_a}[g_{t+2}] + \beta_i * E_t^{Q_a}[\frac{1}{X_{t+2}}]$
Replacing in the expression above, we get:
$g_t = \alpha_i * E_t^{Q_a}[\alpha_i * E_t^{Q_a}[g_{t+2}] + \beta_i * E_t^{Q_a}[\frac{1}{X_{t+2}}]] + \beta_i * E_t^{Q_a}[\frac{1}{X_{t+1}}]$
$= \alpha_i^2 * E_t^{Q_a}[g_{t+2}] + \alpha_i * \beta_i * E_t^{Q_a}[\frac{1}{X_{t+2}}] + \beta_i * E_t^{Q_a}[\frac{1}{X_{t+1}}]$
and by recursion, we get: $g_t = \alpha_i^{T-t} * E_t^{Q_a}[g_T] + \sum_{\tau=t+1}^{T} \alpha_i^{\tau-(t+1)} \beta_i E_t^{Q_a}[\frac{1}{X_\tau}]$
with (as seen above) $\alpha_i = (1-\lambda_i)$ and $\beta_i = (\lambda_i - \xi_i)$.

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We have:
$g_t^* = \lim_{N\to\infty} \alpha_i^N * E_t^{Q_a}[g_{t+N}^*] + \sum_{n=1}^{\infty} \alpha_i^{n-1} * \beta_i * E_t^{Q_a}[\frac{1}{X_{t+n}}]$
$= \sum_{n=1}^{\infty} \alpha_i^{n-1} * \beta_i * E_t^{Q_a}[\frac{1}{X_{t+n}}]$ (due to the condition given for g_t^*)
$= \sum_{n=1}^{\infty} \alpha_i^{n-1} * \beta_i * E_t^{Q_a}[\frac{1}{X_t} * \prod_{\tau=t+1}^{t+n} \frac{1}{U^{\theta_\tau} * D^{1-\theta_\tau}}]$
$= \frac{1}{X_t} * \alpha_i^{-1} * \beta_i * \sum_{n=1}^{\infty} \alpha_i^n * (q_b * (\frac{1}{U} - \frac{1}{D}) + \frac{1}{D})^n$ (like in Part 7)
Since $\alpha_i * (q_b * (\frac{1}{U} - \frac{1}{D}) + \frac{1}{D}) = (1-\lambda_i) * e^{(r_b - r_a)\Delta}$ and the initial condition states:
$0 < (1-\lambda) * e^{(r_b - r_a)\Delta} < 1$,
we can use the geometric sum formula starting from 1 and we get:
$= \frac{1}{X_t} * \alpha_i^{-1} * \beta_i * \frac{\alpha_i * (q_b * (\frac{1}{U} - \frac{1}{D}) + \frac{1}{D})}{1 - \alpha_i * (q_b * (\frac{1}{U} - \frac{1}{D}) + \frac{1}{D})} = \frac{1}{X_t} * \beta_i * \frac{(q_b * (\frac{1}{U} - \frac{1}{D}) + \frac{1}{D})}{1 - \alpha_i * (q_b * (\frac{1}{U} - \frac{1}{D}) + \frac{1}{D})}$ i.e.
$g_t^* = \frac{1}{X_t} * \beta_i * \frac{e^{(r_b - r_a)\Delta}}{1 - (1-\lambda_i) * e^{(r_b - r_a)\Delta}} = \frac{1}{X_t * \Phi_i}$
with $\Phi_i = \frac{1 - \alpha_i * e^{(r_b - r_a)\Delta}}{\beta_i * e^{(r_b - r_a)\Delta}}$

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We have $\Phi = \beta * \frac{e^{(r_a - r_b)\Delta}}{1 - \alpha * e^{(r_a - r_b)\Delta}}$ and $\Phi_i = \frac{1 - \alpha_i * e^{(r_b - r_a)\Delta}}{\beta_i * e^{(r_b - r_a)\Delta}}$.
Thus, $\Phi = 1$ for $\beta = e^{(r_b - r_a)\Delta} - \alpha$ and $\Phi_i = 1$ for $\beta_i = e^{(r_a - r_b)\Delta} - \alpha_i$.
Since $\beta = (\lambda - \xi)$ and $\beta_i = (\lambda_i - \xi_i)$, the conditions become:

$\xi = \lambda - e^{(r_b - r_a)\Delta} + \alpha = 1 - e^{(r_b - r_a)\Delta}$ (since $\alpha = (1 - \lambda)$) and
$\xi_i = \lambda_i - e^{(r_a - r_b)\Delta} + \alpha_i = 1 - e^{(r_a - r_b)\Delta}$ (since $\alpha_i = (1 - \lambda_i)$)
and the choice is independent of the values of λ and λ_i.
Thus, for $\xi = 1 - e^{(r_b - r_a)\Delta}$ and $\xi_i = 1 - e^{(r_a - r_b)\Delta}$, we always have $\Phi = 1$ and $\Phi_i = 1$.

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$\lim_{\Delta \to 0}(\frac{f_t^*}{x_t}) = \lim_{\Delta \to 0} \frac{\Delta * \Lambda * e^{(r_a - r_b)\Delta}}{1 - (1 - \Delta * (\Lambda + \Gamma)) * e^{(r_a - r_b)\Delta}} = \lim_{\Delta \to 0} \frac{(r_a - r_b) * \Delta * \Lambda * e^{(r_a - r_b)\Delta} + \Lambda * e^{(r_a - r_b)\Delta}}{-(r_a - r_b) * (1 - \Delta * (\Lambda + \Gamma)) * e^{(r_a - r_b)\Delta} + (\Lambda + \Gamma) * e^{(r_a - r_b)\Delta}}$
In the last equality, we used L'Hôpital's rule. After applying the limit, we get:
$\lim_{\Delta \to 0}(\frac{f_t^*}{x_t}) = \frac{\Lambda}{\Lambda + \Gamma - (r_a - r_b)} = \frac{\Lambda}{\Lambda + \Gamma - \delta}$
with $\delta = (r_a - r_b)$

As stated previously, $i_t^* = \Theta = \frac{1}{g_t^*}$
$\lim_{\Delta \to 0}(\frac{i_t^*}{x_t}) = \lim_{\Delta \to 0} \frac{1 - (1 - \Delta * (\Lambda_i + \Gamma_i)) * e^{(r_b - r_a)\Delta}}{\Delta * \Lambda_i * e^{(r_b - r_a)\Delta}} = \lim_{\Delta \to 0} \frac{-(r_b - r_a) * (1 - \Delta * (\Lambda_i + \Gamma_i)) * e^{(r_b - r_a)\Delta} + (\Lambda_i + \Gamma_i) * e^{(r_b - r_a)\Delta}}{(r_b - r_a) * \Delta * \Lambda_i * e^{(r_b - r_a)\Delta} + \Lambda_i * e^{(r_b - r_a)\Delta}}$
Again, we used L'Hôpital's rule and applying the limit yields:
$\lim_{\Delta \to 0}(\frac{i_t^*}{x_t}) = \Theta_i = \frac{\Lambda_i + \Gamma_i - (r_b - r_a)}{\Lambda_i} = \frac{\Lambda_i + \Gamma_i + (r_a - r_b)}{\Lambda_i} = \frac{\Lambda_i + \Gamma_i + \delta}{\Lambda_i}$
with $\delta = (r_a - r_b)$
Now, for $\Theta = \Theta_i = 1$, we require $\Gamma = (r_a - r_b)$ and $\Gamma_i = -(r_a - r_b) = -\Gamma$, as in these (unique) cases we get $\Theta = \frac{\Lambda}{\Lambda} = 1$ and $\Theta_i = \frac{\Lambda_i}{\Lambda_i} = 1$.
This means that the implied funding rates are (considering Δ small enough):
For the linear FERP, $\phi(t) = \Delta * (\frac{\Lambda * (\Lambda + \Gamma)}{\Lambda + \Gamma - (r_a) - r_b} - \Lambda)$ or $\phi(t) = \Delta\Gamma = \Delta(r_a - r_b)$ if $\Theta = 1$.
For the inverse FERP, $\phi_i(t) = \Delta * (\frac{\Lambda_i * (\Lambda_i + \Gamma_i) * (\Lambda_i + \Gamma_i + (r_a - r_b))}{\Lambda_i} - \Lambda_i)$ or $\phi_i(t) = \Delta\Gamma_i = -\Delta(r_a - r_b)$ if $\Theta_i = 1$.
On platforms like Binance or BitMEX, the funding rate is calculated with a combination of an interest rate (fixed and potentially adjustable 0.01% per funding interval on Binance and $\frac{r_a - r_b}{funding\ intervals}$, so almost similar to our answer in the case where $\Theta = \Theta_i = 1$, on BitMEX) and the premium (depending on the difference between the perpetual futures and spot prices), which is not reflected in our answer.
The real difference comes thus from the premium, which enforces price convergence between the two futures and spot markets.

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To facilitate the comparison, we created new datasets that capture the differences between the spot rates and the linear/inverse rates for each currency. Plotting these datasets provided an initial insight suggesting a lack of similarity.
We conducted our tests using a 5
Initially, we employed the Mann-Whitney U test, which compares the distributions of two independent groups. If the p-value is below the significance level, we reject the null hypothesis, indicating dissimilarity between the distributions. This non-parametric test eliminates the need for assumptions about normality. Although similar distributions alone are insufficient to establish statistical similarity, they offer valuable insights.
Applying the Mann-Whitney U test to both currencies, we found statistically similar distributions.
Despite the observed departure from normality, we conducted a two-sample t-test to assess the similarity of means, given our dataset size of 365, which is considered sufficient. The variances between the different rates for each currency appeared sufficiently close for the test to be reliable.
Based on these tests, we inferred that the rate distributions are statistically similar.
However, to incorporate potential rate differences into a trading strategy, we needed to further examine the time series of these differences.

In practice, prices will naturally fluctuate over time, but we would expect the differences to follow a Gaussian distribution, or at least have zero mean, allowing us to treat deviations as noise.
Analyzing the means and variances of each difference dataset, we observed that the differences in BTC rates appeared further from 0 compared to ETH rates. However, this discrepancy could be attributed to the significantly higher prices of BTC.
To statistically verify our assumption of a zero-mean Gaussian distribution, we employed the Shapiro-Wilk test.
The test results indicated that the price differences did not follow a Gaussian distribution with a mean of 0, leading us to question the statistical similarity of the rates.
To further investigate, we tested whether the distributions had a mean of 0 using the 1-sample t-test.
Since all null hypotheses were rejected, it became evident that the interest rate component of the funding rate might be misspecified by the exchange, as the spot and futures rates were not equal.

In summary, although the distributions of the rates appeared similar, the differences did not exhibit a zero mean, indicating their dissimilarity. As spot rates, on average, exceeded perpetual futures rates, we devised a strategy involving selling the respective cryptocurrency (ETH or BTC) while simultaneously repurchasing it at the perpetual futures rate. This trade, having zero net initial cost and yielding a positive expected payout (the price difference minus trading fees), presented an arbitrage opportunity.
However, further research is necessary to determine the optimal time window for the trade and obtain accurate information on trading fees, thus enabling practical implementation of the arbitrage strategy.

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