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A Report for
ON FORMAL UNDECIDABLE PROPOSITIONS OF PRINCIPIA MATHEMATICA AND RELATED SYSTEMS
by Kurt Gödel

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1 Introduction

Kurt F. Gödel, born on April 28, 1906 and passed away on January 14, 1978. He was a revolutionary thinker in the fields of logic, mathematics, and philosophy, leading to a significant shift in our comprehension of mathematics and computation. One of his most important work is the article about incompleteness which name On Formally Undecidable Propositions of Principia Mathematica and Related Systems[1]. In this short report, I will introduce the article and explain the purpose, content and the effect of article.

2 ON FORMAL UNDECIDABLE PROPOSITIONS OF PRINCIPIA MATHEMATICA

The article aimed to present and prove groundbreaking results in mathematical logic and the philosophy of mathematics. The main purpose of the article is to demonstrate inherent limitations in formal mathematical systems and to profoundly impact how the foundations of mathematics are understood.

2.1 Incompleteness Theorems

Gödel's incompleteness theorems consist of two fundamental results known as the first and second incompleteness theorems. These two theorems highlight the core of logic, especially handle the inherent limitations of provability within formal axiomatic systems.

2.1.1 First Incompleteness Theorem

Firstly, Gödel's first incompleteness theorem states that in any sufficiently developed formal system, like principia mathematica, there will be always some true mathematical propositions whichs are unprovable in the system itself.

The goal of a formal system is to be complete and consistent.

- **Complete**: If every statement that is true within the system can be proven using its axioms and rules of inference.
- **Consistent**: If no contradictions can be derived from the axioms of the system.

Nevertheless, Gödel's theorem emphasize that a formal system is not able to express basic arithmetic both complete and consistent.

Gödel proved this by encoding mathematical statements into numerical codes, a process known as Gödel numbering. Gödel numbering method makes it possible for mathematics to "talk about itself" by articulating claims about other claims made within the system. His evidence relies heavily on the development of self-referential statements made possible by this encoding. Using this encoding, he constructed a self-referential mathematical statement that essentially states, "This statement is not provable within this system.".

Although not mentioned in the article, this example is a good one to increase clarity: "This statement is false." That's a self-referential statement. If it's true, that would make this statement false. However, the assertion is true if it is untrue. This sentence creates an unsolvable dilemma by referring directly to itself. We can thus assert that "This statement cannot be proven" because of the contradiction if it is neither true nor untrue thanks to Gödel's reference. The fact that this proposition cannot be validated within the system emphasizes its inherent limitation.

The first incompleteness theorem implies that there are always true but unprovable statements in any sufficiently complex mathematical framework. These statements are called undecidable propositions. These are statements that are true, but no proof can be derived from the system's axioms and inference rules. This result shattered the hope of mathematicians, such as David Hilbert, who sought to create a complete and consistent foundation for mathematics.

2.1.2 Second Incompleteness Theorem

Gödel's second incompleteness theorem and the first incompleteness theorem are cumulative. The second one emphasizes that it is not able to prove its own consistency, which system is both consistent and capable of expressing arithmetic. This result leads to those propound:

- Any attempt to prove the consistency of a system, must rely on assumptions which has no intersection with the system itself.
- This introduces an unavoidable reliance on external or higher-order principles, emphasizing the limits of formalism.

In any formal system F with basic arithmetic, a formula $\mathrm{Cons}(F)$ can be created to show its consistency, indicating that no natural number can represent a proof of a contradiction. The Second Incompleteness Theorem by Gödel states that it is impossible to prove the consistency of a formal system F within F, showing the inherent constraints of formal systems.

The second incompleteness theorem emphasizes that the system's own axioms cannot ensure its reliability.

2.2 Impact of the Incompleteness Theorems

Gödel's discoveries reshaped mathematics and philosophy by revealing that:

- The theorems demonstrate that, in spite of its advantages, formalism cannot adequately capture mathematical truth.
- The theorems laid the groundwork for new developments in logic, computer science, and the philosophy of mathematics, including the recognition of undecidability in computational problems.
- The theorems influenced the development of computer science, particularly the concept of Turing machines and the limits of computation.
- The theorems highlighted the inherent limitations of formal systems, influencing the study of computational complexity.

Gödel's work not only revealed the limits of mathematical systems but also inspired deeper inquiries into the nature of truth, logic, and the human pursuit of knowledge.

Gödel's incompleteness theorems made quite an impact on David Hilbert's formalism based program, which sought to provide a universal and logical basis for all mathematics. Its main objective was to prove that every statement of mathematics was logically derivable from some universal axioms that structured the whole of mathematics. It is in regard to this philosophical aspect that it is called formalism. While formalism provides an established way of doing mathematics, the incompleteness theorems provide the contrary view to show that it will fall short in capturing mathematical truth since some true statements will not be capable of being proved in any formal aspect. This restriction shows the possibility that outside of formal systems could exist mathematical facts which are not only restricted to them. The theory is consistent with platonism which is the opposite of formalism, which holds that mathematical facts are objective and exist outside of formal systems and humanity in an abstract, non-physical universe. Platonism's behavior to the mathematical discoveries are that finding fact because they are already exist in the nature, instead of creating them.

Gödel's incompleteness theorems had an important influence on the development of logic and computation theory. The British mathematician Alan Turing introduced the Turing machine as an intellectual device to study the limits of computation in 1936. Turing designed this machine basically to identify undecidable propositions, which are mathematical assertions that cannot be proved true or false within any formal system that can express arithmetic; this concept was already established by Gödel. However, Turing furthered this idea by showing that there cannot exist a general algorithm that is capable of determining if a given proposition is undecidable. This discovery founded computational theory and the study of the limits of computation, whereby it became clear that some problems, such as the halting problem, cannot be solved by any algorithm, no matter how ingenious. The contributions of Turing sound an extension of the Gödel's findings in their indication that some mathematical truths are not calculable by mechanical computation.

3 Conclusion

Gödel's study proved that for any sufficiently powerful formal system, there are true statements that the system itself cannot establish. Besides frustrating the original ambitions of David Hilbert's program to ground mathematics on a complete and consistent basis, this result provided the theoretical basis for computation. His work demarcated what is computable or what is logically deducible and encouraged people such as Alan Turing to define the limits of computation through the concept of Turing machines.

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