Seminar 1-2. Sequences and series of real numbers.

Sequences of real numbers

Definition 1

A sequence $(x_n)_{n\in\mathbb{N}} : x_0, x_1, x_2, ..., x_n, x_{n+1}, ...$ is called:

- **increasing** if $x_{n+1} \geq x_n$ for all $n \in \mathbb{N}$;
- **decreasing** if $x_{n+1} \leq x_n$ for all $n \in \mathbb{N}$.

A sequence is **monotonic** if it is either increasing or decreasing.

Remark 1

If the sequence $(x_n)_{n\in\mathbb{N}}$ has strictly positive terms then it is:

- **increasing** if $\frac{x_{n+1}}{x_n} \geq 1$ for all $n \in \mathbb{N}$;
- **decreasing** if $\frac{x_{n+1}}{x_n} \leq 1$ for all $n \in \mathbb{N}$.

Definition 2

The sequence $(x_n)_{n\in\mathbb{N}}$ is **bounded** if there exist two finite numbers m and M such that

$$m \leq x_n \leq M.$$

Remark 2

The sequence $(x_n)_{n\in\mathbb{N}}$ is **bounded** if there exists an $M > 0$ such that $|x_n| \leq M$.

Remark 3

Any **monotonic** and **bounded** sequence is **convergent**. The reverse implication is not true.

Definition 3

The sequence $(x_n)_{n\in\mathbb{N}}$ is **convergent**(has finite limit) if there exists $x \in \mathbb{R}$ such that for all $\epsilon > 0$ exist $N = N(\epsilon) \in \mathbb{N} : |x_n - x| < \epsilon$ for all $n \geq N$. In this case, we denote by $x = \lim\limits_{n\to\infty} x_n$.

Proposition 1 - Squeezing theorem

Let be (x_n) a sequence of real numbers. If there exist two sequences $(a_n)_{n\geq n_0}$ and $(b_n)_{n\geq n_0}$ with

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = x,$$

such that

$$a_n \leq x_n \leq b_n, \quad n \geq n_0,$$

then the sequence (x_n) has limit and

$$\lim_{n\to\infty} x_n = x.$$

Proposition 2 - Ratio test

Let be (x_n) a positive sequence of real numbers such that there exists

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = l \in [0,\infty].$$

1. If $l < 1$ then $\lim_{n\to\infty} x_n = 0$.
2. If $l > 1$ then $\lim_{n\to\infty} x_n = \infty$.

Proposition 3 - Root test

Let be (x_n) a positive sequence of real numbers. If there exists

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = x \in [0,\infty],$$

then

$$\lim_{n\to\infty} \sqrt[n]{x_n} = x.$$

Proposition 4 - Stolz-Cesaro lemma

Let be $x_n = \frac{a_n}{b_n}$ cu $a_n, b_n \in \mathbb{R}$ and $b_n \neq 0$, $n \in \mathbb{N}$. If:

- (b_n) is a strictly monotonically sequence,
- $b_n \to \infty$ or $a_n \to 0$, $b_n \to 0$,
- there exists
$$\lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = x \in \mathbb{R} \cup \{\pm\infty\},$$

then

$$\lim_{n\to\infty} x_n = x.$$

Definition 4

A sequence (x_n) is a **Cauchy sequence** or **fundamental sequence** if and only if for all $\epsilon > 0$ there exist $n_0 = n_0(\epsilon) \in \mathbb{N}$ such that

$$|x_m - x_n| < \epsilon,$$

for all $m, n \geq n_0$.

Proposition 5

If there exist a sequence of nonnegative real numbers $(a_n)_{n \geq n_0}$ with $a_n \to 0$ such that

$$|x_{n+p} - x_n| \leq a_n,$$

for all $n \geq n_0$ and $p \in \mathbb{N}^*$, then (x_n) is a Cauchy sequence.

Theorem 1 - Cauchy's general criterion for convergence of sequences

The sequence (x_n) of real numbers is convergent if and only if (x_n) is a Cauchy sequence.

Series of real numbers

Definition 1

For any sequence $(x_n)_{n \geq n_0}$ we associate a sequence $(S_n)_{n \geq n_0}$, defined by

$$S_n = x_{n_0} + x_{n_0+1} + \cdots + x_n, \quad n \geq n_0.$$

The pair of sequences $((x_n), (S_n))$ is called **series** of real numbers with the **general term** of the series x_n and we denote by $\sum\limits_{n \geq n_0} x_n$. The sequence $(S_n)_{n \geq n_0}$ is called **the sequence of partial sums** associated to the series $\sum\limits_{n \geq n_0} x_n$.

Definition 2

A series $\sum\limits_n x_n$ is called **convergent** if the sequence of partial sums (S_n) is convergent, it means that there exists a real number $S \in \mathbb{R}$ such that

$$\lim_{n \to \infty} S_n = S.$$

If a series is not convergent then it is divergent.
If there exist the limit S of the sequnece $(S_n)_{n \geq n_0}$ it is called the **sum of the series** $\sum\limits_{n \geq n_0} x_n$ and we denote by $\sum\limits_{n=n_0}^{\infty} x_n$.

Example 1 - Geometric series

Let be r a fixed real number. The series
$$\sum_{n\geq 0} r^n$$
is convergent if and only if $r \in (-1, 1)$. If it is convergent, then the sum of the series is
$$\sum_{n\geq 0} r^n = \frac{1}{1-r}.$$
The number r is called the **ratio** of geometric series.

Example 2 - Generalized harmonic series

The series
$$\sum_{n\geq 1} \frac{1}{n^\alpha}, \quad \alpha > 0$$
is divergent if $\alpha \leq 1$ and it is convergent if $\alpha > 1$.

Proposition 1 - Divergence test

If the sequence (x_n) has a nonzero limit or it does not have limit, then the series $\sum\limits_n x_n$ is divergent.

Proposition 2 - Leibniz test

If (α_n) is a sequence of positive real numbers monotonically decreasing and convergent to zero, then the series $\sum\limits_n (-1)^n \alpha_n$ is convergent.

Exercises

1. Study the monotony of the following sequences:

   (a) $x_n = \frac{n-1}{n}, \quad n \in \mathbb{N}^*$;

   (b) $x_n = \frac{5n-1}{5n}, \quad n \in \mathbb{N}^*$;

   (c) $x_n = \frac{n+2}{3^n}, \quad n \in \mathbb{N}$;

   (d) $x_n = \frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n+1)}{2\cdot 5\cdot 8\cdot \ldots \cdot (3n+2)}, \quad n \in \mathbb{N}$.

2. Using definition show that the sequence $x_n = \frac{2n-1}{2n+1}, \quad n \geq 1$ converge at $x = 1$.

3. Compute the limits of the following sequences:

   (a) $\lim\limits_{n\to\infty} \frac{\cos n}{4^n}, \quad n \in \mathbb{N}$;

   (b) $\lim\limits_{n\to\infty} \frac{n^2}{4^n}, \quad n \in \mathbb{N}$;

   (c) $\lim\limits_{n\to\infty} \frac{a^n}{n!}, \quad a \in \mathbb{N}^*$;

   (d) $\lim\limits_{n\to\infty} \frac{\sqrt[n]{n!}}{n}, \quad n \in \mathbb{N}^*$;

   (e) $\lim\limits_{n\to\infty} \frac{1}{n}\left(\frac{1}{\ln 2} + \frac{1}{\ln 3} + \ldots + \frac{1}{\ln n}\right), \quad n \in \mathbb{N}^*$;

   (f) $\lim\limits_{n\to\infty} \left(\frac{n+3}{n+1}\right)^{n-2}, \quad n \in \mathbb{N}$.

4. Use the Cauchy's general criterion for convergence to show the convergence of the sequences:

   (a) $x_n = \frac{\sin x}{2} + \frac{\sin 2x}{2^2} + \ldots + \frac{\sin nx}{2^n}, \quad n \geq 1$;

   (b) $x_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2}$.

5. Using the sequence of partial sums, study the convergence of the series:

   (a) $\sum\limits_{n\geq 1} \frac{1}{n(n+1)}$;

   (b) $\sum\limits_{n\geq 1} \ln(1 + \frac{1}{n})$;

   (c) $\sum\limits_{n\geq 0} \Big[\arctan(n+1) - \arctan n\Big]$.

6. Study the convergence of the following series:

   (a) $\sum\limits_{n\geq 1} \ln \frac{3n+1}{n+1}$;

   (b) $\sum\limits_{n\geq 1} \frac{1}{\sqrt[n]{n}}$;

   (c) $\sum\limits_{n\geq 1} \frac{n}{(n+1)!}$;

   (d) $\sum\limits_{n\geq 2} \ln\left(1 - \frac{1}{n^2}\right)$;

   (e) $\sum\limits_{n\geq 1} (-1)^n \frac{1}{n^2}$;

   (f) $\sum\limits_{n\geq 1} (-1)^n \frac{1}{\ln n}$;

   (g) $\sum\limits_{n\geq 1} (-1)^n \frac{1}{n^2+(-1)^n}$.