Chapter IV

LINEAR TRANSFORMATIONS

1 Definitions. Properties

Linear Transformations. The Kernel and the Image of a Linear Transformation

Let V and W be two vector spaces over $\mathbb{K}$.

Definition 1.1. *A transformation $f : V \to W$ is* **linear** *if it preserves the operations of vector addition and scalar multiplication:*
1. $f(u + v) = f(u) + f(v)$, for any $u, v \in V$;
2. $f(\alpha u) = \alpha f(u)$, for any $u \in V, \alpha \in \mathbb{K}$.

Let $L(V, W)$ be the set of linear transformations of V in W.

If $f : V \to W$ is a linear transformation, then:
1. $f(0) = 0$;
2. $f(-u) = -f(u)$, for any $u \in V$.

Proposition 1.1. *A transformation $f : V \to W$ is linear iff*

$$f(\alpha u + \beta v) = \alpha f(u) + \beta f(v),$$

for any $u, v \in V, \alpha, \beta \in \mathbb{K}$.

Proposition 1.2. *(***Operations with linear transformations***) Let $f, g \in L(V, W)$, then:*
1. $f + g, \alpha f \in L(V, W)$, for any $\alpha \in \mathbb{K}$.
2. For every $f \in L(V, W)$ and $g \in L(W, Q)$ then the composition $f \circ g \in L(V, Q)$.
3. If $f \in L(V, W)$ is invertible then $f^{-1} \in L(W, V)$.

Definition 1.2. *If $f \in L(V,W)$ is invertible, then f is said to be an* **isomorphism** *of V in W. Two vector spaces V and W are called* **isomorphic** *if there exists at least an isomorphism f of V on W. We denote $V \approx W$.*
If, in addition, $V = W$ then f is called **an automorphism** *of V.*
If $f \in L(V,V)$ then f is called **linear operator**.

Let $Aut(V)$ be the set of all automorphisms of V and $I(V,W)$ be the set of all isomorphism of V in W.

The following results hold:

Proposition 1.3. *The set $L(V,W)$ is a vector space with the usual operations of functions addition and scalar multiplication.*

Proposition 1.4. *The set $I(V,W)$ is a subspace of $L(V,W)$.*

Proposition 1.5. *If f is an isomorphism of V to W then $f^{-1} : W \to V$ is an isomorphism of W to V.*

Proposition 1.6. *The set $Aut(V)$ is a group with the usual functions composition, denoted by $GL(V)$, and called* **the linear group of** *V.*

Proposition 1.7. *Let $S = \{v_1, v_2, ..., v_n\} \subset V$ be a set of vectors and $f : V \to W$ a linear transformation. If*

$$S' = \{f(v_1), f(v_2), ..., f(v_n)\} \subset W$$

is linearly independent then S is linearly independent, too.

Remark 1.1. Let f be a linear transformation of V to W. Then

$$dim f(V) \leq dim V.$$

Remark 1.2. Let f be an isomorphism from V to W and $S = \{v_1, v_2, ..., v_n\}$ a linearly independent set of vectors in V. Then the set

$$S' = \{f(v_1), f(v_2), ..., f(v_n)\} \subset W$$

is linearly independent.

Proposition 1.8. *Two vector spaces, V and W, are isomorphic iff they have the same dimensions.*

$$\mathrm{V} \approx \mathrm{W} \Leftrightarrow \dim \mathrm{V} = \dim \mathrm{W}.$$

Proposition 1.9. *An injective linear transformation $f : V \to W$ such that $dimV = dimW$ is bijective.*

Definition 1.3. **The Kernel of** $f \in L(V,W)$ *is the subset of all vectors from* V *whose images is the zero vector of* W :

$$Kerf = \{x \in \mathrm{V}/f(x) = 0\}.$$

The Image (Range) of $f \in L(V,W)$ *is the set of all vectors from* $w \in W$ *for which there exists vectors in* $v \in V$ *such that* $f(v) = w$:

$$\mathrm{Im}f = \{w \in \mathrm{W}/\ \exists x \in \mathrm{V} : f(v) = w\}.$$

Proposition 1.10. *Properties of* $Kerf$ *and* Imf :
1. $Kerf$ *is a subspace of V.*
2. Imf *is a subspace of W.*
3. If $U \subset V$ *is a subspace of* V *then the set:*

$$f(U) = \{w \in W \mid \exists\, v \in U : f(v) = w\}$$

is a subspace of W.

Proposition 1.11. *A linear transformation is injective iff* $Kerf = 0$. *A linear transformation is onto iff* $Imf = W$.

Proposition 1.12. *If* $f \in L(V,W)$ *then:*

$$dimKerf + dimImf = dimV.$$

Definition 1.4. *The Kernel's dimension is called* **the defect or the nullity of** f *and it is denoted by* $deff = dimKerf$.
The Image's dimension is called **the rank** *of the linear transformation and it is denoted by* $rangf = dimImf$.

Proposition 1.13. *A linear transformation is determined by its action on a basis for V.*

The Matrix of a Linear Transformation. Diagonalization

Consider two bases $B_V = \{v_1, v_2, ..., v_n\}$ for V, $B_W = \{w_1, w_2, ..., w_m\}$ for W, and a linear transformation $f \in L(V,W)$. Then:

Definition 1.5. **The matrix of** $f \in L(V,W)$ **related to the basis** B_V **and** B_W *is the matrix whose columns contains the coordinates of the vectors* $f(v_1), f(v_2), ..., f(v_n)$ *related to the basis* B_W.

We denote this matrix by:

$$\mathrm{A}_{B_\mathrm{V} B_\mathrm{W}} = \begin{pmatrix} \alpha_1^1 & \dots & \alpha_n^1 \\ \alpha_1^2 & \dots & \alpha_n^2 \\ \vdots & & \vdots \\ \alpha_1^m & \dots & \alpha_n^m \end{pmatrix} \in \mathrm{M}_{\mathrm{m}\times\mathrm{n}}(\mathbb{K})$$

The matrix of the linear operator f of V related to the basis B_V is the matrix whose columns are the coordinates of the vectors $f(v_1), f(v_2), ..., f(v_n)$ related to the basis B_V. This matrix is a square one:

$$\mathrm{A}_{B_\mathrm{V}} = \begin{pmatrix} \alpha_1^1 & \dots & \alpha_n^1 \\ \alpha_1^2 & \dots & \alpha_n^2 \\ \vdots & & \vdots \\ \alpha_1^n & \dots & \alpha_n^n \end{pmatrix} \in \mathrm{M}_{\mathrm{n}\times\mathrm{n}}(\mathbb{K}).$$

Consider now the vector $x \in V$. The question is to find the connection between the coordinates of x related to B_V and the coordinates of $f(x)$ related to B_W. First we express

$$f(x) = f(x_1 v_1 + ... + x_n v_n) \qquad = x_1 \underbrace{f(v_1)}_{\in \mathrm{W}} + ... + x_n \underbrace{f(v_n)}_{\in \mathrm{W}}$$

$$= x_1 \left(\alpha_1^1 w_1 + \alpha_1^2 w_2 + ... + \alpha_1^m w_m\right) + ...$$
$$+ x_n \left(\alpha_n^1 w_1 + \alpha_n^2 w_2 + ... + \alpha_n^m w_m\right)$$
$$= \left(x_1 \alpha_1^1 + ... + x_n \alpha_n^1\right) w_1 + \left(x_1 \alpha_1^2 + ... + x_n \alpha_n^2\right) w_2 + ...$$
$$+ (x_1 \alpha_1^m + ... + x_n \alpha_n^m) w_m.$$

then

$$\mathrm{W} \ni f(x) = y_1 w_1 + ... + y_m w_m,$$

and using the uniqueness of the coordinates of a vector related to a basis, we obtain the relations:

$$\begin{cases} y_1 = x_1 \alpha_1^1 + ... + x_n \alpha_n^1 \\ y_2 = x_1 \alpha_1^2 + ... + x_n \alpha_n^2 \\ ... \\ y_m = x_1 \alpha_1^m + ... + x_n \alpha_n^m \end{cases}$$

These are the connections between the coordinates of x related to B_V and the coordinates of $f(x)$ related to B_W.

This formulas admit a matrix representation. Let us denote by:

$$\mathrm{X}_{B_{\mathrm{V}}} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

the column matrix which contains the coordinates of $x \in V$ related to B_V,

$$\mathrm{Y}_{B_{\mathrm{W}}} = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$$

the column matrix which contains the coordinates of $f(x) \in W$ related to B_W and

$$\mathrm{A}_{B_{\mathrm{V}} B_{\mathrm{W}}} = \begin{pmatrix} \alpha_1^1 & \dots & \alpha_n^1 \\ \alpha_1^2 & \dots & \alpha_n^2 \\ \vdots & & \vdots \\ \alpha_1^m & \dots & \alpha_n^m \end{pmatrix} \in \mathrm{M}_{\mathrm{m}\times\mathrm{n}}(\mathbb{K})$$

the matrix for f related to B_V, B_W, then:

$$Y_{B_W} = A_{B_V B_W} \cdot X_{B_V}.$$

Remark 1.3. If $f \in L(V)$ is a linear operator, then $B_V = B_W$, so the above formula becomes:

$$Y_{B_V} = A_{B_V} \cdot X_{B_V}.$$

Now, if we change the basis B_V with B'_V and B_W with B'_W and we denote by $A_{B_V B_W}$ the matrix of f related to B_V, B_W and with $A_{B'_V B'_W}$ the matrix of f related to B'_V, B'_W, we reach the following result:

$$\mathrm{A}_{B'_{\mathrm{V}} B'_{\mathrm{W}}} = \mathrm{T}^{-1}_{B_{\mathrm{W}} B'_{\mathrm{W}}} \cdot \mathrm{A}_{B_{\mathrm{V}} B_{\mathrm{W}}} \cdot \mathrm{T}_{B_{\mathrm{V}} B'_{\mathrm{V}}}.$$

Remark 1.4. If $f \in L(V)$ then, the above relation becomes:

$$\mathrm{A}_{B'_{\mathrm{V}}} = \mathrm{T}^{-1}_{B_{\mathrm{V}} B'_{\mathrm{V}}} \cdot \mathrm{A}_{B_{\mathrm{V}}} \cdot \mathrm{T}_{B_{\mathrm{V}} B'_{\mathrm{V}}}.$$

Proposition 1.14. *If $A_{BB'}$ is the matrix of the linear transformation f and $A_{B'B"}$ is the matrix of g, then the matrix of $g \circ f$ is the product $A_{B'B"} A_{BB'}$.*

2 Solved Problems

1. Prove that $f : \mathbb{R}^3 \to \mathbb{R}^3, f(x, y, z) = (x + y, y + z, z + x)$ is a linear transformation.

   Solution: Let $(x, y, z), (x', y', z') \in \mathbb{R}^3$. We have to verify that the conditions 1 and 2 from Definition 1.1 are satisfied:

   $$f\left[(x, y, z) + \left(x', y', z'\right)\right] = f(x, y, z) + f\left(x', y', z'\right),$$

   for all $(x, y, z), (x', y', z') \in \mathbb{R}^3$, and

   $$f\left[\alpha(x, y, z)\right] = \alpha \cdot f(x, y, z), \forall \alpha \in \mathbb{R}, (x, y, z) \in \mathbb{R}^3.$$

   The first relation becomes:

   $f\left[(x, y, z) + (x', y', z')\right] = f(x + x', y + y', z + z')$
   $= (x + x' + y + y', y + y' + z + z', z + z' + x + x')$
   $= (x + y, y + z, z + x) + (x' + y', y' + z', z' + x')$
   $= f(x, y, z) + f(x', y', z').$

   Similar arguments lead us to the second condition from Definition 1.1, so f is a linear transformation.

2. Find $Kerf$ and Imf of the linear transformation

   $$f : \mathbb{R}^3 \to \mathbb{R}^3, \ \ f(x, y, z) = (x + y, y + z, z + x).$$

   Solution: Using Definition 1.3, $Kerf$ is given by:

   $$\begin{aligned}&Kerf = \{x \in \mathrm{V}/f(x) = 0\}\\ &= \left\{(x, y, z) \in \mathbb{R}^3 \ s.t.\ f(x, y, z) = (0, 0, 0)\right\}\\ &= \left\{(x, y, z) \in \mathbb{R}^3 \ s.t.\ x + y = 0, y + z = 0, x + z = 0\right\}\end{aligned}$$

   which lead us to the following homogeneous system:

   $$\begin{cases} x + y = 0 \\ y + z = 0 \\ x + z = 0. \end{cases}$$

   The system has the unique solution $x = y = z = 0$, so $Kerf = \{0\}$. The image of f is the set given by:

   $$\begin{aligned}\mathrm{Im}f &= \{y \in W/\exists\, x \in V : f(x) = y\}\\ &= \left\{(\alpha, \beta, \gamma) \in \mathbb{R}^3/\exists\,(x, y, z) \in \mathbb{R}^3 \ a.i.\ f(x, y, z) = (\alpha, \beta, \gamma)\right\}\end{aligned}$$

which is equivalently to:

$$\begin{cases} x+y=\alpha \\ y+z=\beta \\ x+z=\gamma. \end{cases}$$

This system is consistent for any $(\alpha,\beta,\gamma)\in\mathbb{R}^3$, so $Imf=\mathbb{R}^3$.

3. Consider $f\in L(\mathbb{R}^2)$ such that $f(1,1)=(2,3)\,,f(1,0)=(1,2)\,.$ Find the analytic form of f.

   Solution: Let $(x_1,x_2)\in\mathbb{R}^2$. The coordinates of (x_1,x_2) related to the basis $\{(1,1),(1,0)\}$ are $\alpha=x_2,\ \beta=x_1-x_2$, so:

$$\begin{aligned} f(x_1,x_2) &= f(x_2\cdot(1,1)+(x_1-x_2)\cdot(1,0)) \\ &= x_2\cdot f(1,1)+(x_1-x_2)\cdot f(1,0) \\ &= x_2\cdot(2,3)+(x_1-x_2)\cdot(1,2) \\ &= (x_1+x_2,2x_1+x_2)\,. \end{aligned}$$

   Thus, the linear transformation is given by:

$$f(x_1,x_2)=(x_1+x_2,2x_1+x_2)\,.$$

4. Find the analytic form of the linear transformation $f\in L(\mathbb{R}^2)$ whose matrix related to the basis $B_{\mathrm{V}}=\{(1,1)\,,(1,0)\}\,,B_{\mathrm{W}}=\{(2,3)\,,(1,2)\}$ is:

$$\mathrm{A}_{B_{\mathrm{V}}B_{\mathrm{W}}}=\begin{pmatrix} 3 & 2 \\ -1 & -1 \end{pmatrix}.$$

   Solution: We have, successively:

$$f(x_1,x_2)=\mathrm{A}_{B_{\mathrm{V}}B_{\mathrm{W}}}\cdot\mathrm{X}_{B_{\mathrm{V}}}$$

$$=\begin{pmatrix} 3 & 2 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x_2 \\ x_1-x_2 \end{pmatrix}$$

$$=\begin{pmatrix} 2x_1+x_2 \\ -x_1 \end{pmatrix}$$

   where (x_2,x_1-x_2) are the coordinates of $(x_1,x_2)\in\mathbb{R}^2$ related to B_V. The entries of this matrix are the coordinates of $f(x_1,x_2)$ related to the basis B_W. We obtain:

$$f(x_1,x_2)=(2x_1+x_2)(2,3)+(-x_1)(1,2)=(3x_1+2x_2,4x_1+3x_2)\,.$$

5. Find the linear transformation $f \in L(\mathbb{R}^2)$ having the matrix:

$$\mathrm{A}_{B_s} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

related to the standard basis.
Solution: Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation. Using the formula from Remark 1.3, we have:

$$\begin{aligned} f(x_1, x_2) &= \mathrm{A}_{B_s} \cdot \mathrm{X}_{B_c} \\ &= \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \begin{pmatrix} x_1 + x_2 \\ 0 \end{pmatrix}, \end{aligned}$$

which means that:

$$f(x_1, x_2) = (x_1 + x_2, 0).$$

3 Exercises

1. Prove or disprove that f is a linear transformation:
a) $f : \mathbb{R}^3 \to \mathbb{R}^3, f(x, y, z) = (2x + y, y + 2z, x - z)$;
b) $f : \mathbb{R}^3 \to \mathbb{R}^3, f(x, y, z) = (xy, y - z, x + 2z)$;
c) $f : \mathbb{R}^2 \to \mathbb{R}^3, f(x, y) = (x - y, y, x)$;
d) $f : \mathbb{R}^2 \to \mathbb{R}^2, f(x, y) = (-x + 3y, y + 2)$.

2. Find the linear transformation $f \in L(\mathbb{R}^3)$ such that $f(v_i) = w_i$, $i = \overline{1, 3}$, where $v_1 = (-2, 3, 1), v_2 = (0, 1, 0), v_3 = (1, -1, 0)$ and $w_1 = (1, 1, 2), w_2 = (1, 1, 1), w_3 = (1, 0, 0)$.

3. Consider $f \in L(\mathbb{R}^3)$ given by:

$$f(x, y, z) = (2x + y, y + 2z, x - z).$$

a) Prove that f is a linear transformation.
b) Find the matrix of f related to the standard basis Bs and the matrix related to the basis:

$$B = \{v_1 = (1, 3, 1), v_2 = (0, 1, 1), v_3 = (1, 1, 0)\}.$$

4. Consider:

$$f : \mathbb{R}_2[X] \to \mathbb{R}^2, f\left(aX^2 + bX + c\right) = (a + b, a - c).$$

a) Prove that f is a linear transformation.
b) Find $Kerf$ and Imf and their dimensions.

5. Consider:

$$f : M_2(\mathbb{R}) \to \mathbb{R}^3, f\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a, b + c, a - d).$$

a) Prove that f is a linear transformation.
b) Find $Kerf$ and Imf and their dimensions.

6. Consider $f : \mathbb{R}^3 \to \mathbb{R}^3$ given by:

$$f(x, y, z) = (-x + 2y - z, x - 2y + 2z, -2x + 4y - 2z).$$

a) Prove that f is a linear transformation.
b) Find $Kerf$ and Imf and a corresponding basis for each subspace.
c) Prove that f is bijective.
d) Find the matrix of f related to the standard basis of $\mathbb{R}^3$.

7. Consider the linear transformation

$$f : \mathbb{R}_2[X] \to \mathbb{R}_2[X],\ f(aX^2 + bX + c) = 2aX^2 + bX.$$

Find the matrix of f related to the basis $B = \left\{1, 1 + X, 1 + X + X^2\right\}$.

8. Consider the linear transformation

$$f : \mathbb{R}^2 \to \mathbb{R}^3, f(x, y) = (2x + y, -2x - y, 4x + 2y).$$

a) Find the kernel of f and its dimension.
b) Find the matrix of f related to the standard basis of $\mathbb{R}^2$ and $\mathbb{R}^3$.

9. Consider the linear transformation

$$f : \mathbb{R}^2 \to M_2(\mathbb{R}), f(x, y) = \begin{pmatrix} x + y & x + y \\ 2x + 2y & 3x + 3y \end{pmatrix}.$$

Find $Kerf$, Imf, their dimensions and the matrix of f related to the standard basis of $\mathbb{R}^2$ and $M_2(\mathbb{R})$. (Exam, 2013)

10. Consider the linear transformation

$$f : \mathbb{R}^2 \to \mathbb{R}_1[X], f(a,b) = 2aX + a - b.$$

a) Find the kernel, the image of f and a corresponding basis for each.
b) Find the matrix of f related to the standard basis of $\mathbb{R}^2$ and $R_1[X]$.
c) Find the matrix of f related to the basis $B_1 = \{e_1 = (1,2), e_2 = (1,2)\}$ and $B_2 = \{f_1 = 2X - 1, f_2 = -X\}$ using two methods.

11. Consider the linear transformation

$$f : \mathbb{R}^3 \to \mathbb{R}^3, f(x,y,z) = (2x + y + z, 2x + 4y + 2z, -2x - y - z).$$

a) Find the kernel of f and the dimension of the image.
b) Find two corresponding basis of the kernel, B_1 and B_2 and the transition matrix from B_1 to B_2.

12. Define $f : \mathbb{R}_2[X] \to \mathbb{R}^3,\ f(p(X)) = (p(0), p(-1), p(1))$.
a) Find the image under f of $p(X) = -1 + 2X$.
b) Show that f is a linear transformation.
c) Find the matrix for f related to the standard basis for $\mathbb{R}_2[X]$ and the standard basis for $\mathbb{R}^3$.

13. Define $f : \mathbb{R}_2[X] \to \mathbb{R}_1[X],\ f(p(X)) = X \int\limits_0^1 tp(t)dt$.
a) Show that f is a linear transformation.
b) Find the kernel and the image of f.
c) Is f bijective? Why? Explain!

14. Let $f : \mathbb{R}_2[X] \to \mathbb{R}_2[X],\ f(p(X)) = Xp'(X)$ be a linear transformation. Find the matrix of f related to the basis

$$B = \left\{1, 1 + X, 1 + X + X^2\right\}.$$

15. Find the linear transformation $f \in L(\mathbb{R}^2)$ such that $f(1,-2) = (2,-1)$, $f(3,5) = (-3,2)$. Find the matrix of f related to the standard basis.

16. Let us consider the linear system:

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ 2x_1 + x_2 - x_3 - x_4 = 0 \\ 6x_1 + 5x_2 + 11x_3 + 3x_4 = 0 \end{cases}$$

a) Find the linear transformation $f \in L(\mathbb{R}^4, \mathbb{R}^3)$ whose kernel is given by the above equations.
b) Find a basis of the kernel, its dimension and enlarge this basis to a basis of $\mathbb{R}^4$.

17. Find the linear transformation $f \in L(\mathbb{R}^2)$ whose matrix related to the basis $B = \{(2,4), (-1,1)\}$ is $A = \begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix}$.