1 Syntax

- We pick our vocabulary from:

  i. Sets of **constants**, each set specified for an **arity**. E.g. $C_0 = \{a, b, c \ldots\}$, $C_1 = \{W, S, G \ldots\}$, and so on. Again assume a subscripting scheme for constants in order not to run out of symbols no matter how complicated our formulas get.

  ii. A set of **variables**, $V = \{x, y, z, \ldots\}$, again possibly with subscripts.

  iii. A set of **quantifier** symbols $Q = \{\forall, \exists\}$.

  iv. Two sets of **connective symbols** $K_1 = \{\neg\}$ and $K_2 = \{\wedge, \vee, \rightarrow\}$.

  v. We define a set of **terms** $T = C_0 \cup V$.

- We will use upper and lower case Greek letters for variables over predicate symbols and variables, respectively.

- Now we can define the well-formed formulas (wff for short) first-order logic (FOL for short):

 **Definition 1.1** (Well-formed formulas of FOL)

  i. $\Pi \alpha_1, \ldots, \alpha_n$ for $n \geq 1$ is a wff iff $\Pi \in C_n$ and $\alpha_1, \ldots, \alpha_n \in T$.

  ii. $(\phi \wedge \psi)$ is a wff iff ϕ and ψ are wff's;
  likewise for, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$, and $(-\phi)$.

  iii. $\forall \alpha \phi$ and $\exists \alpha \phi$ are wff's iff ϕ is a wff and $\alpha \in V$.

  iv. Nothing else is a wff.

  Example 1.2
  Observe how the formula in (1) is constructed on the basis of definition 1.1.

  $$\forall x((Sx \vee Wx) \rightarrow \exists y(Ky \wedge Txy)) \tag{1}$$

  the tree on the right hand side is simplified by retaining only the structure forming operations as node labels, namely connectives and '.' sign for concatenation. Let us call it the **construction tree** of the formula in (1).
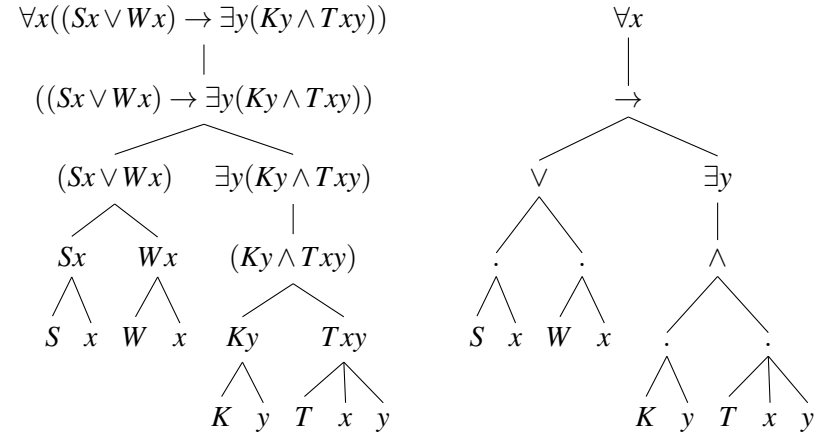
- To keep the number of parentheses manageable, we follow the following convention:

  negation ($-$) and quantifiers ($\forall x, \exists v$, etc.) bind most tightly;

  then comes conjunction ($\wedge$) and alternation ($\vee$)

  finally conditional ($\rightarrow$) binds least tightly.

2 Occurrence, bondage, freedom, and substitution

- We call a **quantifier**, the expression formed by concatenating a quantifier symbol and a variable. E.g. $\forall x$, $\exists z$.

  Occurrence:

- An **occurrence of a variable** in a formula is a leaf (terminal) node in the construction tree of that formula occupied by the variable. Question: how many occurrences does the variable x have in formula (1)?

- Similarly an **occurrence of a quantifier** in a formula is a node in the construction tree occupied by the quantifier.

  Bondage versus freedom:

- An occurrence of a variable α is **bound** in a formula ϕ by an occurrence of a quantifier $\forall \alpha$ (or $\exists \alpha$) if there exists a path from α going up to $\forall \alpha$ (or $\exists \alpha$), and there exists no other occurrence of $\forall \alpha$ (or $\exists \alpha$) along the path.

- An occurrence of a variable α is **free** in a formula ϕ iff α is not bound (by any quantifier) in ϕ.

  Exercise 2.1
  State which occurrences of variables are free and bound in the following formula:

  $$((\exists x Fx \vee \forall x ((Gz \wedge Hx) \to (\exists z Fz \vee Hz))) \to \exists z (Fy \vee Fz))$$

- A formula is **closed** iff it has no free (occurrence of a) variable.

  Substitution:

- Given a formula ϕ,
  $$\phi_{[\beta/\alpha]}$$
  is the formula obtained by substituting the variable β to each and every *free* occurrence of variable α in ϕ.

  Accidental bondage:

- When substituting a variable for another one in a formula, care should be taken NOT to introduce bondages that wouldn't be there if the substitution had not taken place.

  Take for instance the formula

  $$\exists y Lxy$$

  with L designating the binary predicate *loves*, which says there is some entity that x – whatever that is – loves. Substituting y for x in this formula, namely $(\exists y Lxy)_{[y/x]}$, gives $\exists y Lyy$. This says that there exists a self-loving entity. Something different and more specific than our original formula. To avoid such situations we introduce the following definition.

- Variable β is **free for** α in formula ϕ if no free occurrence of α in ϕ stands along a path descending from a quantifier $\forall \beta$ or $\exists \beta$.

  Exercise 2.2
  Give $\phi_{[y/x]}$ for the following ϕ and state whether y is free for x in ϕ:

  1. $\forall z (Px \to Qz)$
  2. $Fx \to \forall x Fx$
  3. $\exists z (\forall x Fx \to Hx)$
  4. $\forall y Fzy \to \exists y Gxyz$

3 Semantics

Definition 3.1 (Model)
A model $\mathscr{M}$ is a structure $\langle D, I, g, \{0,1\} \rangle$ where,

- D is a set of individuals (or entities);
- I is some functional mapping $I : C_0 \mapsto D$ and $I : C_n \mapsto \mathscr{P}(D^n)$, where $\mathscr{P}$ is power set and $D^n = \overbrace{D \times \ldots \times D}^{n \text{ times}}$ (the nth order Cartesian product of D);
- g, the **assignment function**, is some functional mapping $g : V \mapsto D$.

$\square$

Definition 3.2 (Extension of an assignment function)
Given a model $\mathscr{M} = \langle D, I, g, \{0,1\} \rangle$ with an assignment function $g : V \mapsto D$, for an $v \in V$ and $\delta \in D$, $g_{[v \mapsto \gamma]}$ is a function exactly like g, possibly except it maps v to δ. $\square$

Definition 3.3 (Semantics for FOL)
Given a model $\mathscr{M} = \langle D, I, g, \{0,1\} \rangle$,

    i. $[\![\alpha]\!]^g = g(\alpha)$, if $\alpha \in V$;

   ii. $[\![\alpha]\!]^g = I(\alpha)$, if $\alpha \in C_0$;

  iii. $[\![\Pi \alpha_1, \ldots, \alpha_n]\!]^g = 1$, iff $\langle [\![\alpha_1]\!], \ldots, [\![\alpha_n]\!] \rangle \in I(\Pi)$, and 0, otherwise;

iv′. $[\![(\neg\phi)]\!]^g = 1$, iff $[\![\phi]\!] = 0$, and 0, otherwise;

iv″. $[\![(\phi \wedge \psi)]\!]^g = 1$, iff $[\![\phi]\!] = 1$ and $[\![\psi]\!] = 1$, and 0, otherwise;

iv‴. $[\![(\phi \vee \psi)]\!]^g = 0$, iff $[\![\phi]\!] = 0$ and $[\![\psi]\!] = 0$, and 1, otherwise;

iv⁗. $[\![(\phi \to \psi)]\!]^g = 0$, iff $[\![\phi]\!] = 1$ and $[\![\psi]\!] = 0$, and 1, otherwise;

v′. $[\![\forall\alpha\phi]\!]^g = 1$, iff for all $\delta \in D$, $[\![\phi]\!]^{g[\alpha \mapsto \delta]} = 1$, and 0, otherwise;

v″. $[\![\exists\alpha\phi]\!]^g = 1$, iff there exists at least one $\delta \in D$ such that $[\![\phi]\!]^{g[\alpha \mapsto \delta]} = 1$, and 0, otherwise;

<div align="right">□</div>

Exercise 3.4

Express the following sentences in predicate logic:

1. A sample was contaminated.
2. Everything ends.
3. Every semester ends.
4. Every student admires some movie.
5. If an instructor fails, every student passes.
6. No student failed.
7. Some humans love math, but not all who love math are humans.
8. No book is worth reading, except if it is written before 1900.
9. People without friends are unhappy unless they love reading.
10. Only people without friends are happy.
11. All but one student failed.

4   Natural deduction

- All the rules and techniques of natural deduction for propositional logic also apply to predicate logic.

- In addition to them, we introduce introduction and elimination rules for the quantifiers.[1]

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[1]Note that we do not cover terms and identity, therefore you may skip those parts in Huth&Ryan.

The universal quantifier:

Elimination:

$$\frac{\forall x\,\phi}{\phi_{[u/x]}} \; \forall x \, \text{e}$$

provided that u is free for x in ϕ.

Introduction:

$$\frac{\boxed{\begin{array}{c} u \\[4pt] \vdots \\[4pt] \phi_{[u/x]} \end{array}}}{\forall x\,\phi} \; \forall x \, \text{i}$$

The logic of the rule is: if you can prove that a formula holds for an *arbitrary* individual, then it holds for every individual. In order to guarantee that u is arbitrary, it is required that it is "fresh" in the sense that it does not occur anywhere outside of the box.
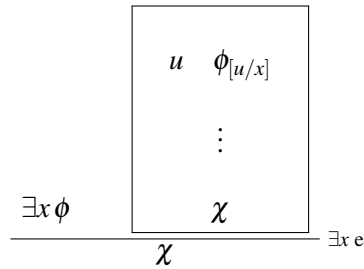
The existential quantifier:

Introduction:

$$\frac{\phi_{[u/x]}}{\exists x\,\phi} \; \exists x \, \text{i}$$

The idea is that if a formula holds for an individual, then you can deduce that there exists something that makes the formula hold.

Elimination:

$$\frac{\exists x\,\phi \quad \boxed{\begin{array}{c} u \quad \phi_{[u/x]} \\[4pt] \vdots \\[4pt] \chi \end{array}}}{\chi} \;\; \exists x\,\text{e}$$

provided that u is free for x in ϕ and u is "fresh" – it does not occur outside of the box.

Here the logic is similar to $\vee$-elimination. You know that ϕ holds for at least one individual, but you do not know which. You assume an arbitrary individual u and that ϕ holds for it. If this assumption leads you to χ, which does not include u, then you can deduce that χ holds.

Let's take a real-world example. Suppose you have 12 friends. You know that at least one of them betrayed you. You sit and think about each, James, Andrew, Matthew, Judas, and others. You find out that *whichever* you pick as the traitor, there is a reason that you are in trouble. As you know that at least one of them *did* betray you, you conclude that you are in trouble.

Exercise 4.1
Prove the following:

1. $\forall x\,(Px \rightarrow Qx), \forall x\,Px \vdash \forall x\,Qx$
2. $\forall x\,\phi \vdash \exists x\,\phi$
3. $\forall x\,(Px \rightarrow Qx), \exists x\,Px \vdash \exists x\,Qx$
4. $\forall x\,(Qx \rightarrow Rx), \exists x\,(Px \wedge Qx) \vdash \exists x\,(Px \wedge Rx)$
5. $\exists x\,Px, \forall x\forall y\,(Px \rightarrow Qy) \vdash \forall y\,Qy$
6. $\vdash \exists x\,(Fx \rightarrow \forall y\,Fy)$