Question 1

Properly parenthesize the given formulas:

(a) $\exists x.Px \wedge Qx$

Solution: $(\exists x(Px \wedge Qx))$

(b) $\exists x.Px \wedge \forall y.Qxy \vee Qyx$

Solution: $(\exists x(Px \wedge (\forall y(Qxy \vee Qyx))))$

(c) $\exists x.Px \wedge \neg\forall y.Qxy \vee Qyx$

Solution: $(\exists x(Px \wedge (\neg(\forall y(Qxy \vee Qyx)))))$

Question 2

Simplify the following formulas by eliminating as many parentheses as possible:

(a) $((\neg Qy) \vee (\exists x(Px \to Tx)))$

Solution: $\neg Qy \vee \exists x.Px \to Tx$

(b) $((\neg(Qy \wedge Pz)) \vee ((\exists xPx) \to Tx))$

Solution: $\neg(Qy \wedge Pz) \vee ((\exists xPx) \to Tx)$

(c) $(((\neg(Qy \wedge Pz)) \vee (\exists xPx)) \to Tx)$

Solution: $\neg(Qy \wedge Pz) \vee (\exists xPx) \to Tx$, if the convention is that $\vee$ and $\wedge$ binds more tightly than $\to$; $(\neg(Qy \wedge Pz) \vee (\exists xPx)) \to Tx$, if all binary connectives bind as tightly as every other.

Question 3

Express the following sentences in first-order logic:

(a) A sample was contaminated.

Solution: $\exists x.sample'x \wedge cont'x$

(b) Everything ends.

Solution: $\forall x.ends'x$

(c) Every semester ends.

Solution: $\forall x.sem'x \to ends'x$

(d) Every student admires some movie.

Solution: $\forall x.stu'x \to \exists y.movie'x \wedge admire'xy$

(e) If an instructor fails, every student passes.

Solution: One can understand this sentence in – at least – two ways.
Reading 1: If any instructor fails, every student passes.
$(\exists x.inst'x \wedge fail'x) \rightarrow \forall x.stu'x \rightarrow pass'x$
Reading 2: There is an instructor such that if s/he fails, every student passes.
$\exists x.inst'x \wedge (fail'x \rightarrow \forall y.stu'y \rightarrow pass'y)$

(f) No student failed.

Solution: $\neg\exists x.stu'x \wedge fail'x$

(g) Some humans love math, but not all who love math are humans.

Solution: $(\exists x.hum'x \wedge lovemath'x) \wedge \exists y.lovemath'y \wedge \neg hum'y$

(h) No book is worth reading, except if it is written before 1900.

Solution: $\forall x.book'x \wedge worthread'x \rightarrow wb1900'x$

(i) People without friends are unhappy unless they love reading.

Solution: $\forall x.person'x \wedge \neg(\exists y.friends'xy) \wedge \neg loveread'x \rightarrow \neg happy'x$
this is the uni-directional interpretation of *unless*; some people get a bi-conditional reading with unless. In such a case, in addition to the above, if you are a person without any friends, loving reading guarantees happiness. Here is the logical form:
$\forall x.person'x \wedge \neg(\exists y.friends'xy) \leftrightarrow (\neg loveread'x \rightarrow \neg happy'x)$

(j) Only people without friends are happy.

Solution: This again has more than one reading.

In one reading, let's call this **Reading 1**, the speaker concentrates on people and says that having friends guarantees unhappiness. Therefore if you find some happy person, s/he must be without friends:

$\forall x.person'x \wedge happy'x \rightarrow \neg\exists y.friends'xy$

This formula does not tell anything about unhappy people. What is your intuition? Given the sentence, can there be unhappy people which do not have any friends? If you say yes, than you are getting a bi-conditional reading, which should be rendered as:

$\forall x.person'x \rightarrow ((\exists y.friends'xy) \leftrightarrow \neg happy'x)$

If you get confused, think of this: *Only hard working people succeed.* If this sentence is true, does it follow that whoever works hard should necessarily succeed? If you say yes to this, then you are interpreting it bi-conditionally. But if you think hard working is necessary, but not sufficient for success, then you are having it uni-conditionally – which I believe would be the judgment of most people.
Reading 2 is pragmatically somewhat odd. It says that if anything is happy, it must be a human without friends. This reading excludes other things, say cats, from being happy:
$\forall x.happy'x \rightarrow human'x \wedge \neg(\exists y.friends'xy)$

(k) All but one student failed.

Solution: $\exists x.stu'x \wedge \neg fail'x \wedge \forall y.stu'y \wedge \neg fail'y \rightarrow y = x$
One problem with this formula is that it is true if there is only one student – and that passed. The formula does not require any failing students as long as there is one and only one student who didn't fail. The natural language expression, however, seems to require the existence of failing students, and perhaps at least two of them. It is a matter of debate whether this requirement is pragmatic in nature or should be coded in the conventional semantics. We leave it to pragmatics in this solution.

Question 4
Given the formula,

$$\exists y.Fy \wedge \forall x.Fx \rightarrow Exy \tag{1}$$

where E is a predicate that stands for equality; in the semantics, it yields true if its arguments are interpreted as the same individual, and false otherwise.

(a) Give a model that makes (1) true.

Solution: Any model where there is one and only one F makes the formula true.

(b) Give a model that makes (1) false.

Solution: Any model that has no F or more than one F makes the formula false.

Question 5
A formula P **implies** Q, designated $P \models Q$, if and only if every model that makes P true also makes Q true.

$$\forall x.Fx \rightarrow \exists y.Gxy \models \exists x.Fx \tag{2}$$

Show that the implication in (2) does NOT hold.

Solution: If an implication does not hold, it is easy to show it. Just find a model that makes the implying formula true but the implied formula false. Any model that does not have any F will suffice for the present problem.

Question 6
Given the formula,

$$\exists x.Fx \rightarrow \forall y.Fy \tag{3}$$

Can you find a model with a non-empty domain that makes (3) evaluate to 0?

Solution: No, you can't. The reasoning is as follows. Any model you pick will either have no F or at least one F. For no F, take any non-F, it will make the conditional true, so the formula is true. For models that have at least one F, you can think of two types of models, one is where there is no non-F, and the other has at least one non-F. For no non-F, the consequent of the conditional is true, so the formula is true for any x you pick. For at least one non-F, take that non-F as x, then the conditional is true for that x, therefore the formula is true. These cases exhaust all types of possible models. Therefore it is not possible to come up with a model that makes the formula false.