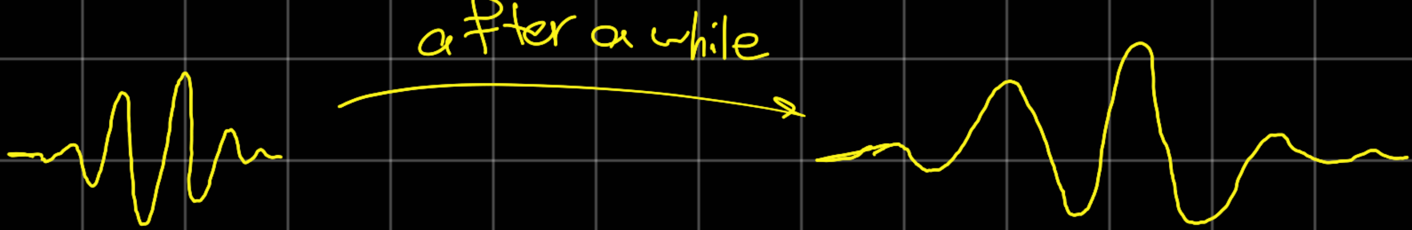
The wave equation is :

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial x^2}$$

every Function with Form $f(kx - \omega t)$ will satisfy this PDE and will propagate in media by velocity v. But The reality is not as simple as that. Suppose the Following Case:



as you Can see the shape of wave has been Changed (because some Frequency components move Faster than the others). So what is v ?! is it the velocity of the general shape ?! or it is the velocity of the Frequency component ?! what will the Function look like after going through media ?!
Here In this note I will talk about all of these topics.

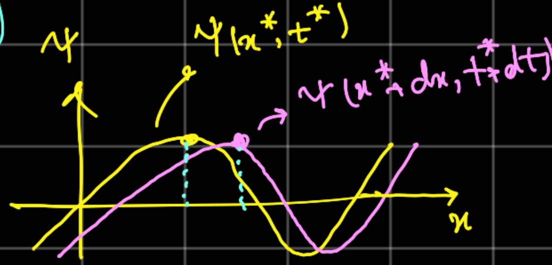
let's assume that the Solution of wave equation has the Form :

$$\psi(x,t) = e^{i(kx-\omega t)}$$

let's First Calculate with what speed this wave will propagate in space ?

Method I : By inserting $\psi(x,t)$ in PDE :

then we will get : $\boxed{v = \frac{\omega}{k}}$



Method II : By using intuition

$$\psi(x^*,t^*) = e^{i(kx^*-\omega t^*)} = a$$

$$\psi(x^*+dx, t^*+dt) = e^{i(k(x^*+dx)-\omega(t^*+dt))} = a$$

$$\Rightarrow kx^* - \omega t^* = k(x^*+dx) - \omega(t^*+dt)$$

$$\Rightarrow \boxed{\frac{dx}{dt} = v = \frac{\omega}{k}}$$

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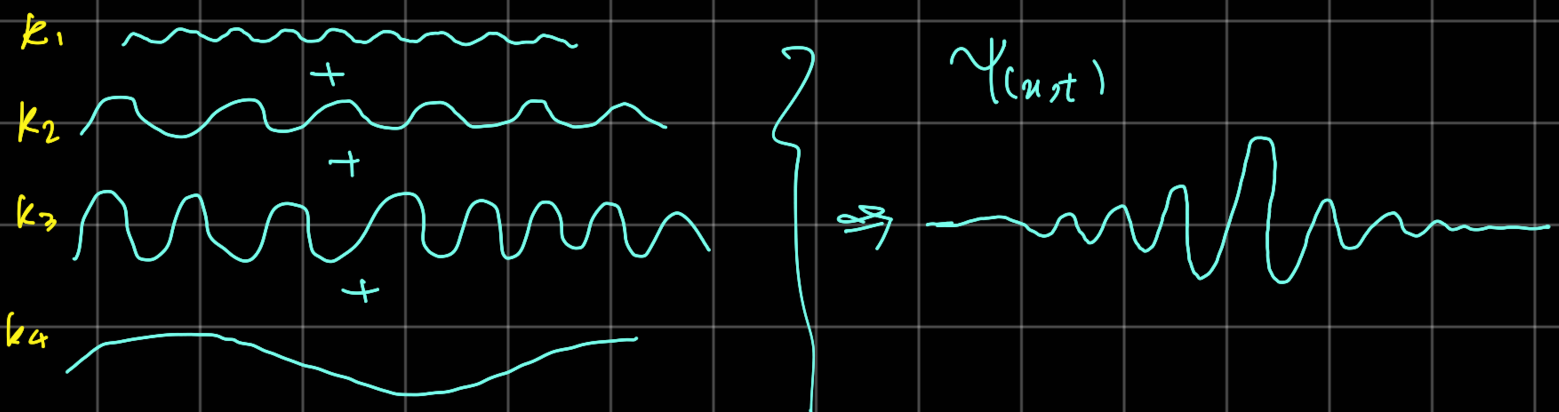
So $\sin(kx-\omega t)$ wave will move with $v = \frac{\omega}{k}$

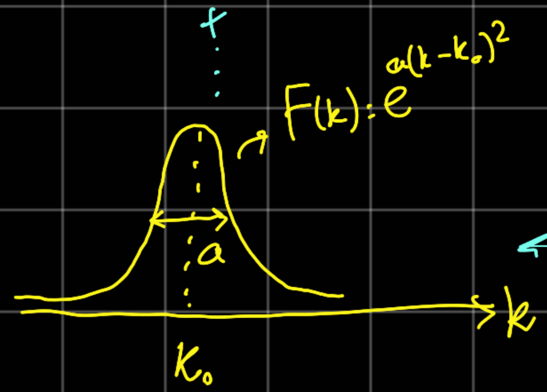
the velocity of a wave is determined by the physics of environment. For example for a wave on a rope the velocity is $\sqrt{\frac{F}{\mu}}$ → string tension, → linear density

So the physics of the media relates ω and k to each other. In the case of wave on a rope we have:

$$\sqrt{\frac{F}{\mu}} = \frac{\omega}{k} \Rightarrow \boxed{\omega = k \times \sqrt{\frac{F}{\mu}}}$$

what if we have a superposition of sine waves on a rope that form a specific shape like:





$$\psi(x,t) = \int F(k)\, e^{i(kx-\omega t)}\, dk$$

As you can see this wave packet is in fact consisting of several spatial frequencies but all of them has temporal frequency ω. So the frequency components (with differen k values) will move with speed $V = \frac{\omega}{k}$

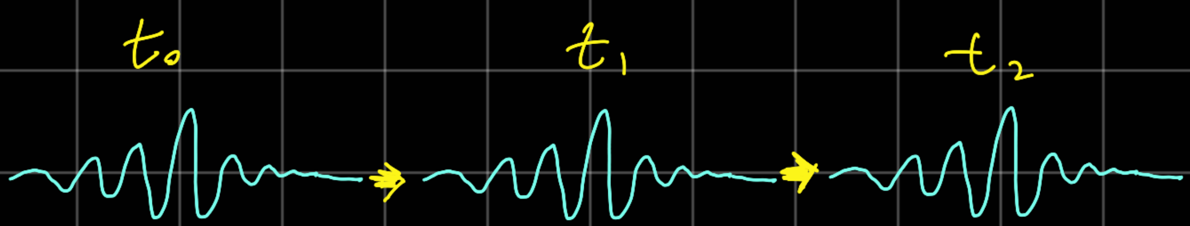
For a wave on a rope we calculated

$$\omega = \sqrt{\frac{F}{\mu}}\, k$$

So the velocity of different phase components will be:

$$\boxed{v = \sqrt{\frac{F}{\mu}}}$$

So for a wave on a rope, all of the frequency components with different k values will move with same velocity, so the over all pattern will be constant.

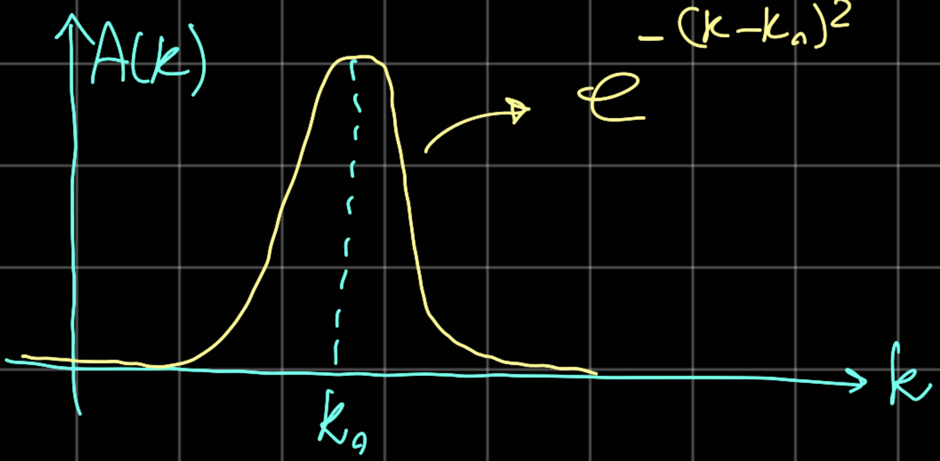


But what if the velocity of frequency comp. with different k, depend on k. let's consider a general case in which ω is an arbitrary function of k ($\omega(k)$).

With this assumption let's evaluate the behaviour of an arbitrary wave form.

$$\psi(x,t) = \int A(k)\, e^{i(kx - \omega t)}\, dk$$

to make things a little bit more simple lets assume $A(k)$ is Centered around k_0 (For exomple a Gussion)



$$e^{-(k-k_0)^2}$$

now since $A(k)$ is Zero except in the vicinity of k_0, So we will have contributions from $e^{i(kx-\omega t)}$ when k is Close to k_0.

So we Can write the taylor series of $\omega(k)$

$$\omega(k) = \overbrace{\omega(k_0)}^{\omega_0} + \overbrace{\frac{\partial \omega}{\partial k}}^{\omega'}(k-k_0)$$

$$\Rightarrow \omega(k) = \omega_0 + \omega'(k-k_0)$$

$$\Rightarrow \psi(x,t) = \int e^{-(k-k_0)^2} \times e^{i(kx - \omega_0 t - \omega'(k-k_0))}$$

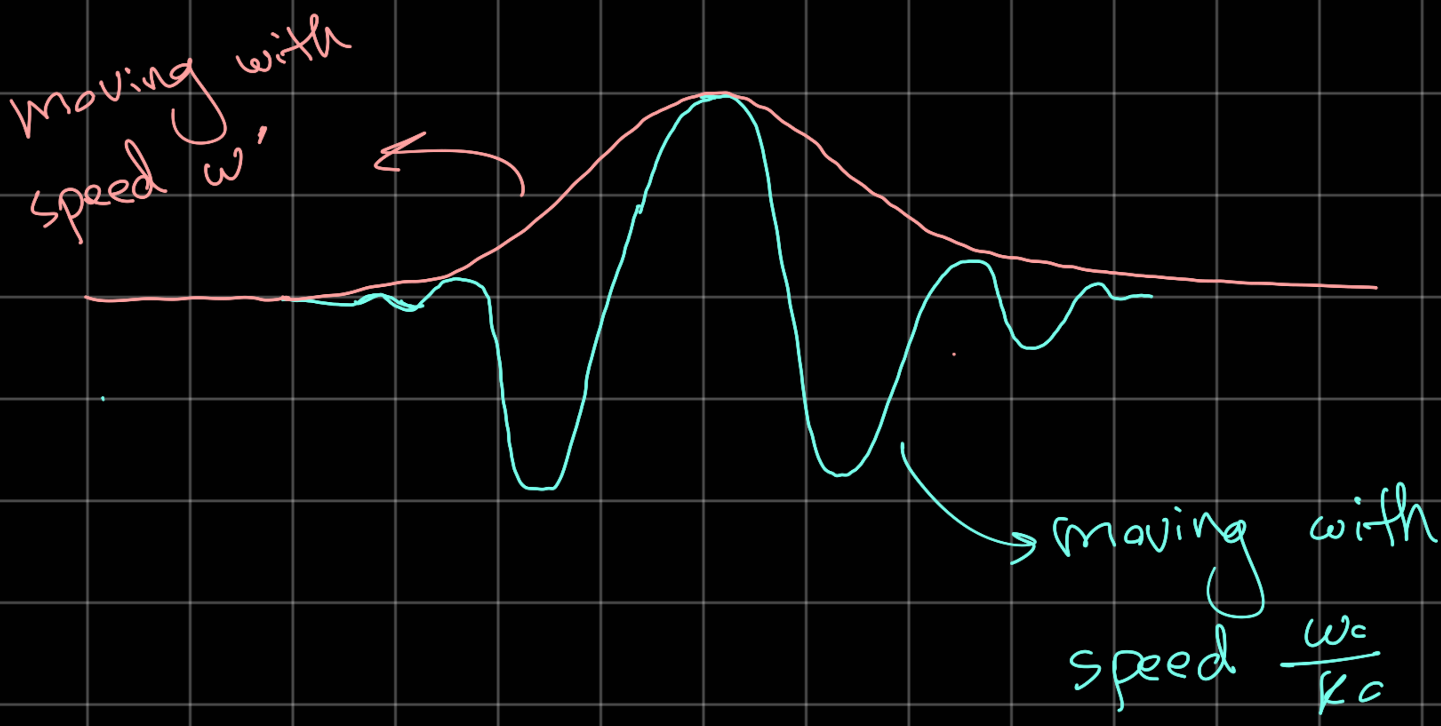
$$\text{exponent} = kx - \omega_0 t - (k-k_0)\omega' t + k_0 x - k_0 x$$
$$= (k-k_0)x - (k-k_0)\omega' t - \omega_0 t + k_0 x$$
$$= (k-k_0)(x-\omega' t) + (k_0 x - \omega_0 t)$$

$$\Rightarrow \Psi_{(x,t)} = e^{i(k_0 x - \omega_0 t)} * \int e^{(k-k_0)^2} * e^{i(k-k_0)(x-\omega' t)}$$

as you can see the original wave is the product of two terms.

$e^{i(k_0 x - \omega_0 t)}$ $\Rightarrow$ Sine wave moving with velocity $\frac{\omega_0}{k_0}$

$\int e^{-(k-k_0)^2} e^{(k-k_0)(x-\omega' t)} dk$ $\Rightarrow$ At first glance this might look like a wave packet that can have different shapes depending of value of k_0. But this is a wave packet with $\underline{\underline{0}}$ as its centeral frequency. no matter what is the value of k_0, the result of integral will look like a simple Guassian
(see theoritical physics / making wave packet. ipynb)
this envelope will be moving with speed ω'



Higher Order terms

As we saw, in writing the taylor expansion of $\omega(k)$ we ignored the higher order terms:

$$\omega(k) = \omega_0 + \omega'(k-k_0) + \frac{\omega''}{2}(k-k_0)^2 + \dots$$

that was because we assumed $A(k)$ is centered around k_0 and is <u>very</u> narrow.

if we inclued the higher degree terms, the we will have group velocity dispersion!

$$\text{envolope} = \int e^{-(k-k_0)^2} \times e^{\hat{i}(k-k_0)\left(x - \left(\omega' + \frac{(k-k_0)\omega''}{2} + \dots\right)t\right)} dk$$

So the velocity of envolope will be:

$$v = \frac{\omega' + \frac{(k-k_0)}{2}\omega'' + \dots}{1} = \omega' + \frac{(k-k_0)}{2}\omega'' + \dots$$

as you can see the velocity of the different wave components (that build the whole envolope) will depend on the wave number of that wave component. Visually, the envlope will get distorted, and practically, the high and low frequency components will get seperated.

Experimental Example.

Example 1.

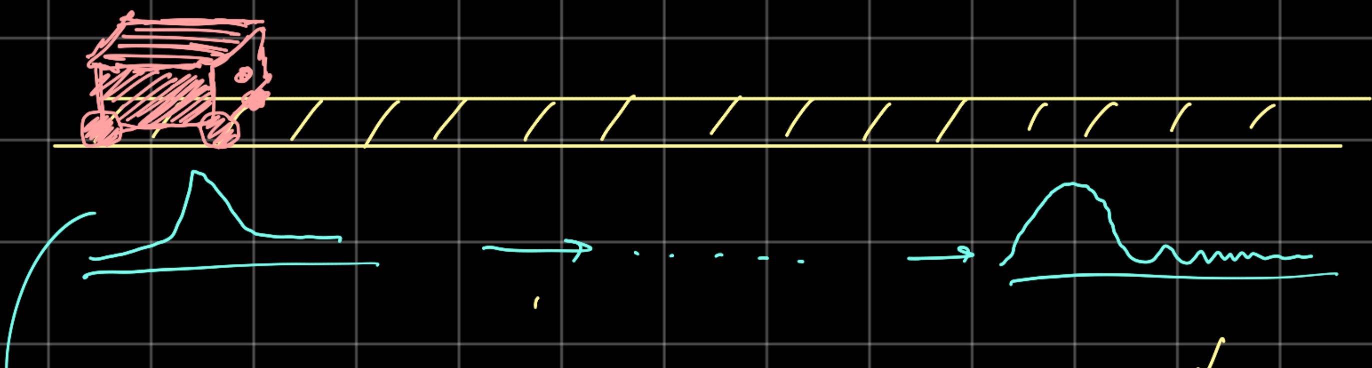
If you have a very tight rope (like the wire above the electric buses) if you disturb it, you will hear some sci_fic sound (similar to when you do a fact freq sweep on a sound generator)

That is because the medium is dispersive and

Causes a speration between high and low freq components of wave as it goes along the wire

Example 2:

The same thing can happen in rail tracks.

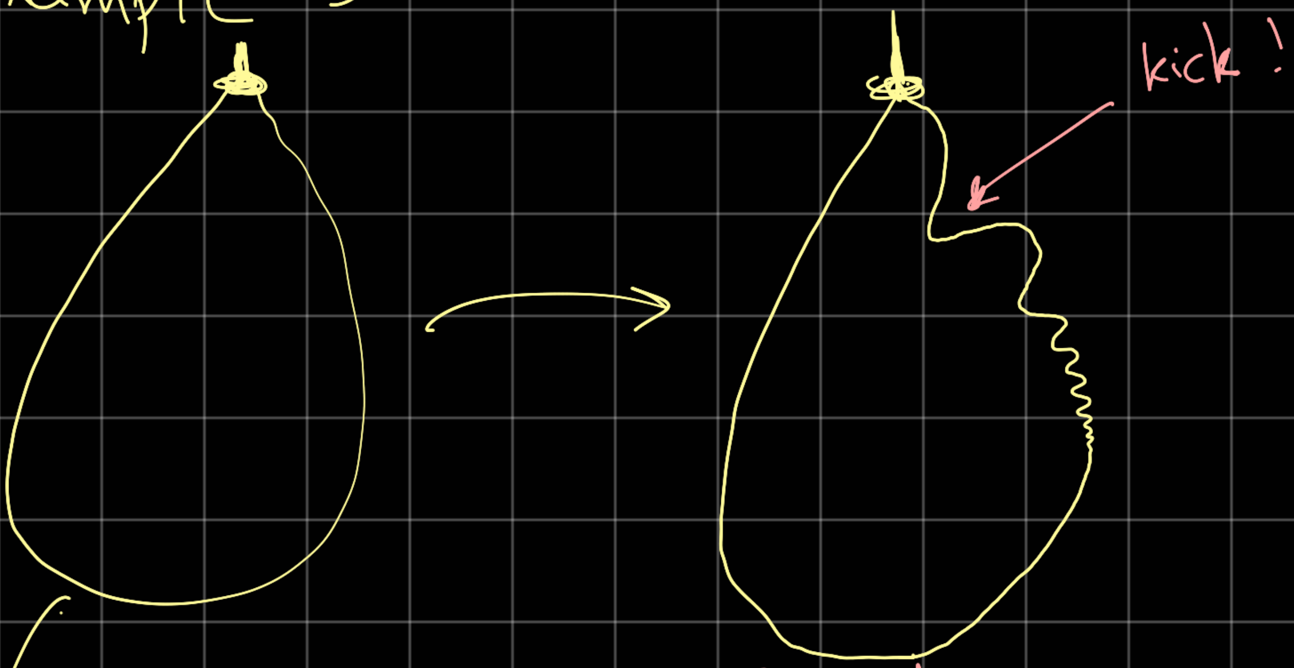


Sound wave in the rail

will be heard as a freq sweep.

(this example was in the last chapter of one of the wave books)

Example 3



balloon full of water

You can find demos on youtube (search slow motion balloon)