In this note I will explain the theory behind explict finite difference method for diffusion

We can write the diffusion equation as follows

$$\boxed{\frac{\partial F}{\partial t} = D\frac{\partial^2 F}{\partial x^2}}$$

$$F_{(x,t)} = F_{m}^{n}$$

time (n)

Position (m)

$$\frac{\partial F}{\partial t} \simeq \frac{F_m^{n+1} - F_m^{n}}{\Delta t} \quad \text{(Forwart in time)}$$

$$\frac{\partial^2 F}{\partial x^2} \simeq \frac{F_{m-1}^{n} - 2F_m^{n} + F_{m+1}^{n}}{(\Delta x)^2} \quad \text{(Centere in Space)}$$

Because of the above approximations, we call this method FTCS (Forward time center space)

By inserting the difference equations into the differential equations we will have:

$$\frac{F_m^{n+1} - F_m^{n}}{\Delta t} = D\,\frac{F_{m-1}^{n} - 2F_m^{n} + F_{m+1}^{n}}{(\Delta x)^2}$$

$$\Rightarrow F_m^{n+1} = D\,\frac{\Delta t}{\Delta x^2}\left(F_{m-1}^{n} - 2F_m^{n} + F_{m+1}^{n}\right) + F_m^{n}$$

$\Rightarrow$ We can write the above equation in matrix form:

$$F^{n+1} = \frac{D\Delta t}{\Delta x^2}\underbrace{\begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & -2 & 1 & \cdots & \cdots & \cdots & \cdots \\ \cdot & 1 & -2 & 1 & & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \cdots & \cdots & 1 & -2 & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & 1 & -2 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix}}_{\text{tridiagonal matrix}}\begin{pmatrix} F_0^{n} \\ F_1^{n} \\ \vdots \\ F_{m-1}^{n} \\ F_m^{n} \\ F_{m+1}^{n} \\ \vdots \\ F_M^{n} \end{pmatrix} + F^{n}$$

↳ tridiagonal matrix

sparse matrix

Stability Analysis (Von Neumann analysis)

let's start with the update rule:

$$F_j^{n+1} = F_j^n + r\left(F_{j-1}^n - 2F_j^n + F_{j+1}^n\right) \qquad r = \frac{D\Delta t}{\Delta x^2}$$

Now suppose that u_j^n is the values with infinite percision (without round off error due to floating point representation)

and suppose that N_j^n is the solution with round off error ε_j^n.

$$\Rightarrow \quad u_j^n = N_j^n + \varepsilon_j^n$$

Since both u_j^n and N_j^n satisfy the update rule equation, so will do ε_j^n.

$$\rightarrow \quad \varepsilon_j^{n+1} = \varepsilon_j^n + r\left(\varepsilon_{j-1}^n - 2\varepsilon_j^n + \varepsilon_{j+1}^n\right)$$

The Core of Neumen analysis is Fourier transform

Here is the Formula for discrete FT

$$F_{[k]} = \sum_{n=0}^{N-1} x_{[n]} \exp\left(\frac{-i2\pi kn}{N}\right) \quad (FT)$$

$$x_{[n]} = \sum_{n=0}^{N-1} F_{[k]} \exp\left(\frac{i2\pi kn}{N}\right) \quad (IFT)$$

let's start with the derived update rule

$$\varepsilon_j^{n+1} = \varepsilon_j^n + r\left(\varepsilon_{j+1}^n - 2\varepsilon_j^n + \varepsilon_{j-1}^n\right)$$

let's write the inverse Fourier transfor of ε_j^n

$$\boxed{\varepsilon[j\Delta x, n\Delta t] = \varepsilon_j^n = \sum_{k=0}^{N-1} E_{[k,n]} \exp\left(\frac{i2\pi kj}{N}\right)}$$

Now let's plug in the equation above in the update rule

$$\varepsilon(x,t) = \varepsilon(j\Delta x, n\Delta t) = \varepsilon_j^n$$

$$\Rightarrow \begin{cases} \mathcal{E}_j^n = \sum_{h=0}^{N-1} E[h,n] \exp\left(\frac{i 2\pi h j}{N}\right) \\ \mathcal{E}_{j\pm 1}^n = \sum_{h=0}^{N-1} E[h,n] \exp\left(\frac{i 2\pi h (j\pm 1)}{N}\right) \\ \mathcal{E}_j^{n+1} = \sum_{h=0}^{N-1} E[h,n+1] \exp\left(\frac{i 2\pi h j}{N}\right) \end{cases}$$

$$E[h,n] \exp\left(\frac{i 2\pi h j}{N}\right) = \square_j^n \quad ; \quad \exp\left(\frac{2\pi h}{N}\right) = e^{i\eta}$$

plug in the update rule $\Longrightarrow$

$$\sum_h \square_j^{n+1} = \sum_h \square_j^n + r\left(\sum \square_j^n e^{-i\eta} - 2\sum \square_j^n + \sum \square_j^n e^{i\eta}\right)$$

$$= \sum_h \square_j^n + r\left(\sum_h \square_j^n \left(e^{-i\eta} - 2 + e^{i\eta}\right)\right)$$

$$\Rightarrow \square_j^{n+1} = \square_j^n + r\left(\square_j^n \left(e^{-i\eta} - 2 + e^{i\eta}\right)\right)$$

$$\Rightarrow \frac{\square_j^{n+1}}{\square_j^n} = 1 + r\left(e^{-i\eta} - 2 + e^{i\eta}\right) = \frac{E[h,n+1]}{E[h,n]}$$

We call $\frac{E[h,n+1]}{E[h,n]}$ amplification factor (G)

$$\Rightarrow G = 1 + r\left(e^{-iy} - 2 + e^{iy}\right) = 1 - 4r\left(\frac{e^{-iy/2} + e^{iy/2}}{2i}\right)^2$$

$$= 1 - 4r\sin^2\left(\frac{y}{2}\right)$$

$$\rightarrow |G| \leq 1 \Rightarrow \left|1 - 4r\sin^2\left(\frac{y}{2}\right)\right| < 1$$

$$\Rightarrow 4r\sin^2\left(\frac{y}{2}\right) \leq 2$$

$$\Rightarrow \boxed{r \leq \frac{1}{2}} \Rightarrow \boxed{\frac{D\Delta t}{\Delta x^2} \leq \frac{1}{2}}$$