MASTER THESIS

Numerical Study of Modified Newtonian Dynamics

修正ニュートン力学の数値的研究

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And He gave you from all you asked of Him. And if you should count the favor of Allah , you could not enumerate them. Indeed, mankind is [generally] most unjust and ungrateful.
(QS. Ibrahim : (14:34))

Dedicated to :
My family who pray me in their continuous prayers, especially Bapak, Mamak, and Tatae.
My personal teacher (Mr. Norma Sidik Risdianto), someone who always give me best motivation.

above every knower there is a higher Knower.
(QS. Yusuf : 76)

Abstract

In this work we have studied mathematical aspects of particle dynamics model for numerical simulation in astrophysics, especially for the galaxy simulation. In the huge space scale and long time scale of a galaxy, we are required special treatments at numerical simulation. In particular, we focus on Modified Newtonian Dynamics (MOND) model. The MOND model is a galaxy scale modification of the Newtonian Dynamics (ND), and it is a commonly used for the galaxy dynamics. We reformulate the MOND model and derive a solution of rotating N particles with radius R. Using this result, we compare the speed of the rotatiing N particles in MOND with the one in ND. This is important related to the galaxy rotation problem. We also give brief reviews on continuous models which appear in many particle limit, and on numerical schemes for particles dynamics.

Key words: *Newtonian dynamics (ND), Modified Newtonian dynamics (MOND), Euler method, symplectic method*

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Frequently used abbreviations and unity of symbols

ND	Newtonian Dynamics	
MOND	Modified Newtonian Dynamics	
DM	Dark Matter	
AQUAL	A Quadratic Lagrange	
gy	galaxy-year	$\approx 1.4 \times 10^{16}$ [s]
Kpc	Kilo parsec	$\approx 3 \times 10^{20}$ [m]
ly	light-year	$\approx 9.4 \times 10^{15}$ [m]
rad	radian	$\approx 0 \rightarrow 2\pi$
$M_\odot$	Mass of Sun	$\approx 2 \times 10^{30}$ [Kg]
α_0	Critical acceleration	$\approx 1.2 \times 10^{-10}$ [m]

Chapter 1

Introduction

One of interesting issue as an astrophysicist is the information about contents and dynamics of the galaxy. Subject about contents of galaxy has been studied in particle physics, where every single particles have an unique behavior which gives contribution into this universe. For the galaxy using the particle dynamics and its accurate numerical simulation. It is required to understand its mathematical properties. Therefore, in this thesis, we focus on the mathematical framework and basic properties of several particle models which appear in the galaxy simulation.

One of the distinctive features of such galaxy simulation is its huge space and time scales. As we will explain later, even the Newton's law has to be modified in the galaxy scale, and modified Newtonian dynamics (MOND) will be considered in this thesis as one of the main models.

The outline of this thesis is as follows. In the rest of this chapter, we give a general review of dynamics of galaxies, and galaxy rotation problem. In chapter 2 we give an introduction to Newton's dynamics as a fundamental theory of dynamical object that provides an explanation of particle motion. We also show the implementation of rotating system for many particles cases. In the chapter 3, we explain about Modified Newtonian Dynamics (MOND) which emphasize study of this thesis. A short explanation of this model and its implementation in rotating many particle systems also will be given in this chapter. Next, chapter 4 we show the formal derivation as another view to study the particle movement in the system. A brief explanation about the material derivative and the implementation for ND and its modified also will be given in this chapter. The Euler scheme and symplectic scheme are introduced in chapter 5. Then, finally we give a short analysis of two models from the graph speed vs radius in chapter 6.

1.1 Theory of Galaxy Dynamics

Galaxy is a large ensemble of stars and other material that orbits about a shared center and its constituents united by mutual gravitational interactions. Galaxies come in various global forms and internal morphology. In this section we will give an overview of galaxies for mathematical modeling and numerical simulation of galaxy dynamics.

1.1.1 General Review of Galaxy

Galaxies are star-bound systems of gravity, remnants of stars, interstellar gas, dust, and dark matter [16]. The galaxy always moves dynamically. So, by this simulation we can understand past, present and future predictions of galaxies.

We know that each galaxy has different form one and the others. It's easy to identify the type of galaxy we are looking at. We are familiar with galaxies: ellipses, spiral and disks [6]. One of the best known examples is our Milky Way galaxy as an example of spiral galaxy.

Milky Way has many kinds of matter, gas and dust inside it. Observations provide information that our galaxy has weight about 1.5 trillion solar mass around 200 billion stars, including 4 million solar masses from black holes at the center of galaxies.

The other information about our Milky way is the galactic diameter around 10^5 [light-year]. Another scale might sometimes use galactic diameter in [Kpc] scale, where 10^5 [light-year] ≈ 30.7 [Kpc]. In addition, the speed of the star to orbit the center of the galaxy has about ~ 230 [km/s].

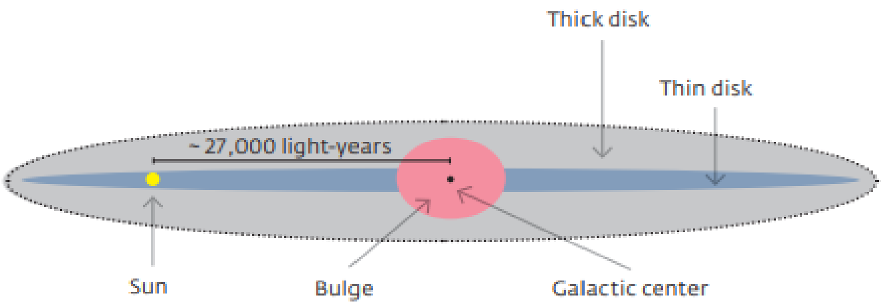

Figure 1.1: Part of Milky way galaxy [8]

Fig.(1.1) shows that our Milky Way galaxy has a complex structure. Each component has a different behavior. As in the center of the galaxy, it remains in the life of the galaxy, which can one day destroy all matter as an active black hole inside that. Then, galactic disk as a medium of many star that make a circular motion. Next for halo galaxies which have the task of guarding globular galaxies. Galaxy content won't come out because of the Halo effect.

On this table we will show an example of the information from our Milky Way galaxy. It can be additional information for us about the actual condition of our galaxy. The detail information can be seen in this table

Table 1.1: Table of Milky Way properties.

Symbols	Meaning	Typical Value
M_{tot}	Total mass of galaxy	1.98×10^{12}[Kg] $\cong 1 \times 10^{12}$ [$M_\odot$] [3]
N	Number of stars	$\sim 2 \times 10^{11}$ [-] [7]
m_i	Average of each star mass	9.94×10^{30}[Kg] $\cong 5$ [$M_\odot$] [8]
R	Radius	$4,63 \times 10^{20}$ [m] $\cong 15$ [Kpc]
$\|v\|$	Speed of star	2.3×10^4 [m/s]$\cong 2.31 \times 10^{-4}$ [Kpc/gy]
ω	Angular velocity	0.11×10^{12} [rad/s] $\cong 0.154 \times 10^{-4}$ [rad/gy]
T	Period of galaxy rotation	1.4×10^{-16} [s] $\cong 1$ [gy]

Actually, some of the value we reshape into another scale to make uniformly scale. It is only an example for our familiar galaxy where in this work we don't actually use that data. We consider our galaxy model as simple as possible in the simulation. Where the main point of the content and character of galaxy we pursuit in suitable case in order to explain dynamics of galaxy.

1.2 Galaxy Rotation Problem

As a great law, Newton's gravity has stuck when tried to explain the rotation of galaxies. The observation which resulted from Fritz Zwicky in 1933 has known that any problem if we refers to Newton's dynamics model [12]. The data observation which obtain look so different if we compare with its prediction. The discrepancy make scientists challenging to fix it and do some analysis as a possible way to answer the problem.

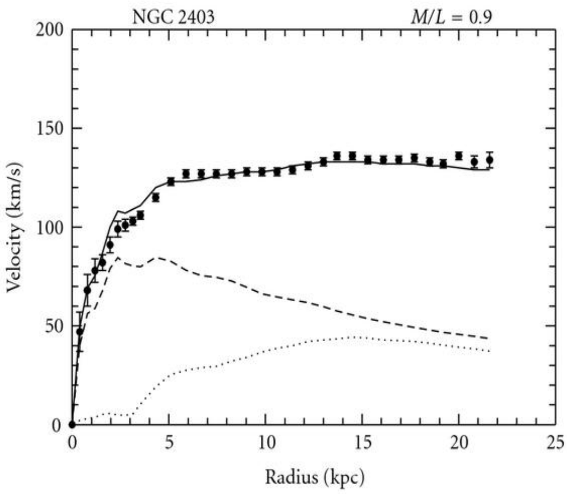

Figure 1.2: Graphic the observation of galaxy rotation velocity [6]

Figure (1.2) is a result of galaxy NGC 2403 observation, where the data observation which show by data point. Meanwhile, the others line are resulted from some prediction of the theory of dynamics. Two of that theory are Newtonian dynamics and its modified. The ND has resulted prediction which represented by dot-line and dash-line plotted. The dot-line indicates for galaxy bulge, and dash-line indicates for galaxy disk. As you can see the ND prediction cannot explain enough about the rotation curve of the content galaxy. This is the big problem of the consistency ND as fundamental law which explain dynamical motion.

This problem commonly discussed as a galaxy rotation problem with the discrepancy of data and its prediction. The two of models which appear trying to solve this problem as by giving additional mass inside the galaxy. The content of our galaxy is not only about visible matter such dust, stars, nebulae and so on, but also any others strange mass that we don't actually know the initial of the object. It is usually called as dark matter (DM)[13]. By giving some additional matters, so it's might be possible as a reason why galaxy so flat or constant of that obtained velocity.

Furthermore, DM is not only the possible way to solve this problem. The other hypothesis has been propose by M. Milgrom to take a modification of ND. The ND maybe do very well in small scale, but in galaxy and larger scale, it is possible to take a little modification in the equation. And actually in Figure (1.2), we can see that the solid line is prediction which resulted by model of the modification of ND. So pretty similar with observation. So, it will be interesting more to know about the theory which usually called as MOdified Newtonian Dynamics (MOND).

1.2.1 The Dark Matter Theory

If we study particle physics, we will find some of the particles which being a fundamental particle to form this universe. All of particle collect in standard model particles. Those have characters and functions which differences of each others. The main idea of those particles that going to solve as a fundamental interaction that work in our universe.

Beside the standard model particles, they usually called as dark matter (DM). DM is stranger particle which we don't have the information about it, just can detect it by the effect that appears. DM is one of interest subject which this decade scientist have been searching information about DM. The reason for this persistence is that dark matter is needed to account for the fact that galaxies don't seem to obey the fundamental laws of physics [13]. However, dark matter searches have remained unsuccessful.

As the previous explanation, DM is one of the possible way to solve the galaxy rotation problem. The stranger mass which gives contribution in large scale, especially in galaxy object, we need to attention more as an possible effect in the content of galaxy. But the actual problem is because we don't really understand what the DM actually is. So, it will be complicated more if we answer by this model [14].

Therefore, based on the main problem that we discuss it in the first, we can skip this way and go to the second model. Furthermore we will neglect this effect of DM in our simulation. So, we can do by natural interaction without DM effect. In spite of the DM hypothesis, it cannot completely enclosed chance to solve by this way. Because maybe someday when the observation of DM has been successfully detect the information about DM, it is so possible to give clarification about this problem.

1.2.2 Modification Newtonian Dynamics Theory

The biggest mystery in cosmology and astrophysics is missing information about dark matter and dark energy. These are an inevitable problem when we study large-scale physics, such as galaxies, the universe and another larger scale physics [15]. Especially about DM, it actually gives a nice reason for some of physics phenomena, in large scale case. But the missing information being a constraint to be a solution for this problem.

For this reason, a scientist named Milgrom, proposed his hypothesis. He said that to solve the DM problem in large-scale, physics need to modify basic Newtonian equations [9]. He said that model as MOdified Newtonian Dynamics (MOND). He proposed this hypothesis caused the real observation of galaxy rotation, which cannot be predicted by Newtonian dynamics equation. As an alternative way to solve this problem, he gives the additional small value in the acceleration of the equation.

In fact, his argument successfully enough to answer that problem if we see the obtained plot in figure(1.2). As we can see MOND can predict velocity of the galaxy content. The solid-line in that figure is indicated MOND predictions. Meanwhile, the dash-line from that figure obtained from Newtonian dynamics as Keplerian prediction.

Basically, this is our main reason to do the simulation of this model. We can do our simulation with considering the real data observation and hopefully also get the good result as a solution of the galaxy rotation problem.

Chapter 2

Newtonian Dynamics

2.1 Introduction

Newtonian dynamics is a mathematical equation that predict the motions of various object which exist in the world around us. The implementation of this model has been familiar in the physical case for small scale interaction until large scale. In this chapter we give a brief explanation about Newtonian dynamics as a fundamental equation that we intent to modify and some of notations in Newton's law.

2.2 Governing Equation of Newtonian Dynamics

2.2.1 Acceleration in Newtonian Dynamics

We consider $X(t) = \{x_i(t)\}_{i=1}^{N} \subset \mathbb{R}^d$ which denotes the positions of N particles in $\mathbb{R}^d (d = 2 \text{ or } 3)$ at time t. We suppose that

$$x_i(t) \neq x_j(t) \qquad (i \neq j) \tag{2.1}$$

The velocity and acceleration of $x_i(t)$ denoted by

$$\begin{cases} v_i(t) := \dot{x}_i(t), \\ a_i(t) := \ddot{x}_i(t), \end{cases}$$

Then, Newton's second law of motions is given as

$$m_i a_i(t) = F_i(t) \qquad (i = 1, 2, \dots, N), \tag{2.2}$$

where m_i is a mass of the particles x_i, and $F_i(t)$ is the total force acting on $x_i(t)$.

Newton's law of universal gravitation takes the form :

$$F_i(t) = G m_i \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{m_j}{r_{ij}(t)^3} x_{ij}(t), \tag{2.3}$$

where G is the gravitational constant and we define $r_{ij}(t) := |x_j(t) - x_i(t)|$, $x_{ij}(t) := x_j(t) - x_i(t)$. If we substitute (2.2) into (2.3), we get the acceleration of particle

$$\begin{aligned} m_i a_i(t) &= G m_i \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{m_j}{r_{ij}(t)^3} x_{ij}(t) \\ a_i(t) &= G \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{m_j}{r_{ij}(t)^3} x_{ij}(t). \end{aligned} \tag{2.4}$$

We define

$$f_i(X) := G \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{m_j}{|x_j - x_i|^3} (x_j - x_i). \qquad \left(X := \{x_j\}_{j=1}^{N} \right), \tag{2.5}$$

then (2.4) becomes

$$\ddot{x}_i(t) = f_i\left(X(t)\right) \qquad (i = 1, 2, \ldots, N). \tag{2.6}$$

To avoid the singularity at $x_i(t) = x_j(t)$, we use a small regularization parameter $\epsilon > 0$ [4]. We replace $r_{ij}(t)$ by $r^{\epsilon}_{ij}(t) = (r_{ij}(t)^2 + \epsilon^2)^{1/2}$ in the numerical simulation. So, becomes

$$a_i(t) = f^{\epsilon}_i(X(t)), \tag{2.7}$$

where $f^{\epsilon}_i(X(t)) := G\sum\limits_{\substack{j=1\\ j\neq i}}^{N} \frac{m_j}{(r^{\epsilon}_{ij})^3} x_{ij}(t)$.

2.3 Conservation of Energy

For a particle system $X(t) = \{x_i(t)\}_{i=1}^{N}$, we denote its kinetic energy by

$$E_k(t) := \sum_{i=1}^{N} \frac{1}{2} m_i |v_i(t)|^2; \tag{2.8}$$

and its potential energy by

$$E_p(t) := \sum_{1\leq i<j\leq N} \frac{m_i m_j G}{|x_j(t) - x_i(t)|}. \tag{2.9}$$

We will prove energy conservation property for the Newtonian dynamics

Theorem 2.3.1 (Conservation of energy)**.** *If $X(t) = \{x_i(t)\}_{i=1}^{N}$ satisfies (2.1) and (2.4). Then $E_k(t) - E_p(t) = Const$ holds.*

Proof.

$$
\begin{aligned}
\frac{d}{dt}E_k(t) &= \frac{d}{dt}\left(\sum_{i=1}^{N}\frac{1}{2}m_i|v_i(t)|^2\right) && (2.10)\\
&= \sum_{i=1}^{N} m_i v_i(t)\cdot \dot{v}_i(t)\\
&= \sum_{i=1}^{N} m_i \dot{x}_i(t)\cdot f_i(X(t)).
\end{aligned}
$$

We define the potential energy as

$$
\mathcal{E}_p(X) := \sum_{1\le i<j\le N}\frac{m_i m_j G}{|x_j - x_i|} \quad \left(X = \{x_i\}_{i=1}^{N}\right). \tag{2.11}
$$

Then, the gradient of $\mathcal{E}_p(X)$ which respect to a variable x_i where $(i = 1, 2, \ldots, N)$ is given by

$$
\begin{aligned}
\nabla_{x_i}\mathcal{E}_p(X) &= \nabla_{x_i}\left(\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{m_i m_j G}{|x_j - x_i|}\right)\\
&= \sum_{\substack{j=1\\ j\neq i}}^{N} m_i m_j G\frac{x_j - x_i}{|x_j - x_i|^3}\\
&= m_i f_i(X).
\end{aligned}
$$

Using this relation, we have

$$
\begin{aligned}
\frac{d}{dt}E_p(t) &= \sum_{i=1}^{N} m_i \dot{x}_i(t)\cdot f_i(X(t))\\
&= \sum_{i=1}^{N} \dot{x}_i(t)\cdot \nabla_{x_i}\mathcal{E}_p(X(t))\\
&= \frac{d}{dt}\mathcal{E}_p(X(t)) = \frac{d}{dt}E_p(t).
\end{aligned}
$$

Hence, we obtain

$$
\frac{d}{dt}\left(E_k(t) - E_p(t)\right) = 0. \tag{2.12}
$$

■

2.4 Rotating Particle System

We consider uniformly located N particles $\{x_i(t)\}_{i=1}^{N} \subset \mathbb{R}^2$ on a circle of radius R with an additional fixed particle at origin $x_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ of mass $m_0 \geq 0$ (commonly as an extra large mass). We set $X(t) = \{x_i(t)\}_{i=0}^{N}$. We suppose that $m_0 > 0$ or $N \geq 2$, and $x_i(t)$ has the following form :

$$\begin{aligned} x_i(t) &= R \begin{pmatrix} \cos\left(\omega t + \frac{2\pi j}{N}\right) \\ \sin\left(\omega t + \frac{2\pi j}{N}\right) \end{pmatrix} \qquad (i = 1, \ldots, N) \\ m_j &= m \qquad (j = 1, \ldots, N), \end{aligned} \tag{2.13}$$

as illustrated in Figure (2.1),

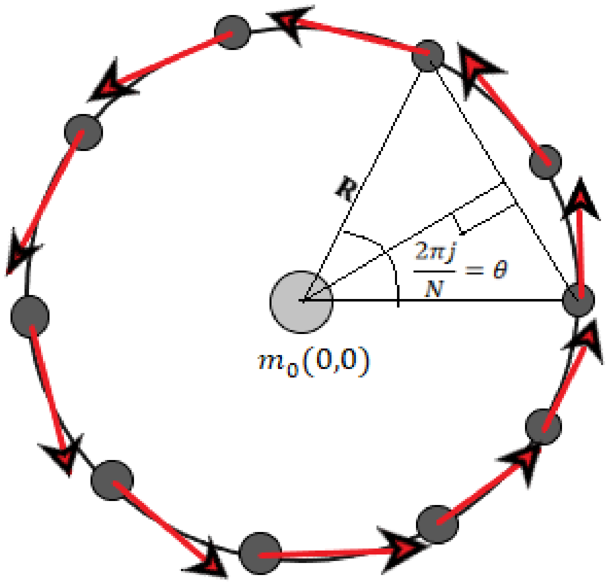

Figure 2.1: Rotating N particles and a fixed particle at origin.

Remark that $r_{ij}(t)$ is given by

$$r_{ij}(t) = 2R\sin\frac{\pi|j-i|}{N} \tag{2.14}$$

From (2.13) we have

$$a_i = -\omega^2 x_i(t). \tag{2.15}$$

On the other hand, it satisfies (2.6) too, where

$$f_i\left(X(t)\right) = mG\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{x_j(t)-x_i(t)}{r_{ij}(t)^3} - m_0 G\frac{x_i(t)}{R^3} \qquad (i=1,\dots,N). \tag{2.16}$$

From the symmetry, it is enough to consider $t=0$, $i=N$:

$$\begin{aligned}
f_N(X(0)) &= mG\sum_{j=1}^{N-1}\frac{x_j(0)-x_N(0)}{r_{jN}(0)^3} - m_0 G\frac{x_N(0)}{R^3}\\
&= \begin{pmatrix} mG\sum_{j=1}^{N-1}\frac{R\left(\cos\frac{2\pi j}{N}-1\right)}{8R^3|\sin\frac{\pi j}{N}|^3} - m_0 G\frac{1}{R^2}\\ 0\end{pmatrix}\\
&= -\frac{G}{R^3}\left(m\sum_{j=1}^{N-1}\frac{\sin^2\frac{\pi j}{N}}{8|\sin\frac{\pi j}{N}|^3} + m_0\right)\begin{pmatrix} R\\ 0\end{pmatrix}\\
&= -\frac{1}{R^3}G\left(m\sum_{j=1}^{N-1}\frac{1}{8|\sin\frac{\pi j}{N}|} + m_0\right)x_N(0)\\
&= -\frac{A}{R^3}x_N(0),
\end{aligned} \tag{2.17}$$

where

$$A := G\left(m\sum_{j=1}^{N-1}\frac{1}{8|\sin\frac{\pi j}{N}|} + m_0\right). \tag{2.18}$$

For general $t\geq 0$, and $i=1,\dots,N$, the equation becomes

$$f_i\left(X(t)\right) = -\frac{A}{R^3}x_i(t). \tag{2.19}$$

From (2.6) and (2.19), we have

$$a_i = -\frac{A}{R^3} x_i(t), \qquad (i = 1, 2, \dots, N)\,. \tag{2.20}$$

Comparing with (2.15) and (2.20), we obtain $\omega^2 R^3 = A$, and

$$\omega = \sqrt{\frac{A}{R^3}}. \tag{2.21}$$

In particular, the speed of $x_i(t)$ is given by

$$|v_i(t)| = \omega R = \sqrt{\frac{A}{R}}. \tag{2.22}$$

This result suggests that the speed of the galaxy becomes slower if the distance from the center becomes larger, in the Newtonian dynamics regime. But this prediction cannot explain the observation as shown in Figure (1.2). We will discuss this problem in chapter (6) again.

Chapter 3

Modified Newtonian Dynamics

3.1 Introduction of MOND

In the 1930's several scientists has discovered the fact that if we only consider the visible matter of content the galaxy, it will be found some problem from Newtons law prediction. Frits Zwicky (1933), then continuing with Horace Babcok in 1939, has been reported that any possible missing mass of the galaxy they observed [9]. Until a Scientist named Vera Rubin, brought his work that the prediction of Newton's law didn't match with his obtained data [17] . The velocity of stars inside of spiral galaxy was different if we did the calculation by Newton's law.

As follow up observation resulted above, at least necessarily two possible answers, the first we need to introduce some additional mass which contribute in the galaxy and the other one that might Newtonian's law cannot be work in large scale, so need to modify it. As a reason that mention it above, in 1983, Mordehai Milgrom was propose that model to modified Newtonian dynamics equation. In his paper, which titled *"A modification of the Newtonian dynamics as a possible alternative to the hidden mass hypothesis"*, he gives the explanation about his modification for the ND [12].

He assume for small acceleration case, ND need to consider small additional value of their acceleration system. It gives an effect in the case which move for small acceleration. The additional value is indicated by a_0 (or in this thesis we write as α_0) with value $\approx 10^{-8}$ [cm/s^2]. He gives that value into ND equation and implement it into the relation of gravity universal force and has resulted the good prediction.

In the first chapter we have been show the plot of galaxy rotation problem and each model to approach that result. Figure (1.2) becomes a valuable reason to work in this model. Therefore in this chapter we will discuss more about MOND and the implementation, how it can be work in the simulation and the other part which describe about this model.

3.2 Governing Equation of MOND

In physics, MOND (Modified Newtonian Dynamics) is actually another way to describe dynamical of added interaction that can't be explain by Newton's law. Milgrom, the name who has propose equation, give adding the small correction in the order of acceleration of the object. He propose the equation caused to answer ND which can't predict well from galaxy observation.

We will show the formulation of MOND equation by

$$F = m\mu\left(\frac{|a|}{\alpha_0}\right)a, \tag{3.1}$$

where $\mu(s)(0 \leq s < \infty)$ is a given function which satisfies

$$\mu(0) = 0, \qquad \lim_{s\to\infty}\mu(s) = 1, \qquad \mu'(s) \geqq 0. \tag{3.2}$$

We introduced $\alpha_0 > 0$ as critical acceleration ($\alpha_0 \sim 10^{-8}$ [cm/s^2] $\approx 1.2 \times 10^{-10}$[km/s^2]).

We remark that when $\mu(s) \equiv 1$ (which does not satisfy $\mu(0) = 0$), (3.1) becomes the ND. A typical choice at $\mu(s)$ in MOND [7] is

$$\mu(s) = \frac{s}{\sqrt{1+s^2}}. \tag{3.3}$$

In this paper, we adopt (3.2) for our $\mu(s)$.

Proposition 3.2.1.

We suppose that $\mu(s)$ is given by (3.3), then The MOND equation (3.1) is equivalent to

$$a = \beta\left(\frac{|f|}{\alpha_0}\right) f, \tag{3.4}$$

where $f := \frac{F}{m}$ and

$$\beta(q) := \sqrt{\frac{1 + \sqrt{1 + \frac{4}{q^2}}}{2}} \qquad (q > 0). \tag{3.5}$$

Proof. From (3.1), we have

$$\mu\left(\frac{|a|}{\alpha_0}\right) a = f \tag{3.6}$$

Setting $s := \frac{|a|}{\alpha_0}$ and $q := \frac{|f|}{\alpha_0}$, we obtain

$$s\mu(s) = q. \tag{3.7}$$

We set

$$\varphi(s) := s\mu(s). \tag{3.8}$$

Then we have $q = \varphi(s)$ and $s = \varphi^{-1}(q)$. From

$$\frac{|a|}{\alpha_0} = \varphi^{-1}\left(\frac{|f|}{\alpha_0}\right), \tag{3.9}$$

and $a = |a|\frac{f}{|f|}$, we obtain

$$a = \alpha_0\varphi^{-1}\left(\frac{|f|}{\alpha_0}\right)\frac{f}{|f|} = \frac{1}{q}\varphi^{-1}(q)f = \beta(q)f, \tag{3.10}$$

where

$$\beta(q) := \frac{1}{q}\varphi^{-1}(q). \tag{3.11}$$

Hence, we have

$$a = \beta\left(\frac{|f|}{\alpha_0}\right) f. \tag{3.12}$$

For the $\mu(s)$ given by (3.2), we can compute

$$\begin{aligned}
\varphi(s) &= \frac{s^2}{\sqrt{1+s^2}}, \\
\varphi^{-1}(q) &= \sqrt{\frac{q^2+\sqrt{q^4+4q^2}}{2}}, \\
\beta(q) &= \frac{1}{q}\varphi^{-1}(q) = \frac{1}{q}\sqrt{\frac{q^2+\sqrt{q^4+4q^2}}{2}} = \sqrt{\frac{1+\sqrt{1+\frac{4}{q^2}}}{2}}, && (3.13)\\
\beta(q) &= \sqrt{\frac{1+\sqrt{1+\frac{4}{q^2}}}{2}}. && (3.14)
\end{aligned}$$

■

3.3 Rotating Particle System for MOND

In section 2.4, we have derived uniformly located rotating particle system for ND case. Then, this section we will consider MOND case. We assume the same situation of (2.13) and (2.19) as in section 2.4. Instead of ND law (2.6), we suppose the following MOND equation:

$$\mu\left(\frac{|a_i(t)|}{\alpha_0}\right) a_i(t) = f_i\left(X(t)\right). \tag{3.15}$$

From (2.19), $|f_i(X(t))| = \frac{A}{R^2} \quad (i = 1, \ldots, N)$ holds. Applying proposition 3.2.1, we have

$$a_i(t) = \beta\left(\frac{|f_i\,(X(t))|}{\alpha_0}\right) f_i\,(X(t)) = -\frac{A}{R^3}\beta\left(\frac{A}{\alpha_0 R^2}\right) x_i(t). \tag{3.16}$$

Similar with ND, we comparing (2.15) and (3.16), we have

$$\omega^2 = \frac{A}{R^3}\beta\left(\frac{A}{\alpha_0 R^2}\right) \tag{3.17}$$

$$\omega = \sqrt{\frac{A}{R^3}\beta\left(\frac{A}{\alpha_0 R^2}\right)}, \tag{3.18}$$

and the speed becomes

$$|v_i(t)| = \omega R = \sqrt{\frac{A}{R}\beta\left(\frac{A}{\alpha_0 R^2}\right)}.$$

Since

$$\begin{aligned} |v_i(t)|^2 &= \frac{A}{R}\beta\left(\frac{A}{\alpha_0 R^2}\right) \\ &= \frac{A}{R}\sqrt{\frac{1+\sqrt{1+\frac{4\alpha_0^2 R^4}{A^2}}}{2}} \\ &= A\sqrt{\frac{1+\sqrt{1+\frac{4\alpha_0^2 R^4}{A^2}}}{2R^2}} \\ &= A\sqrt{\frac{1}{2R^2}+\sqrt{\frac{1}{4R^4}+\frac{\alpha_0^2}{A^2}}}, \end{aligned}$$

we obtain

$$\lim_{R\to\infty} |v_i(t)| = \sqrt{\sqrt{\frac{\alpha_0}{A}}A} = (\alpha_0 A)^{\frac{1}{4}}\,. \tag{3.19}$$

In cotrast to the ND case:

$$\lim_{R\to\infty} |v_i(t)| = 0, \tag{3.20}$$

the above relation (3.19) shows that the speed tends to a positive constant as $R \to \infty$.

3.4 Modified Newtonian Dynamics Via Gravity Potential

If in the previous section we have show the main reason why we use MOND model from the graph. In this section will be given a short type of modification for ND. The other way to modified Newtonian dynamics equation is trough gravity potential. This motivation has solved in some paper [15]. The model mostly called as AQUAL or the other name as QUMOND [15]. We start to modify the ND equation,

$$\begin{aligned} F &= ma \\ m_i G \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{M\left(x_j - x_i\right)}{\left|x_j - x_i\right|^3} &= m_i \ddot{x} \\ m_i f_i &= m_i \ddot{x} \\ \ddot{x} &= f_i \end{aligned} \tag{3.21}$$

Of course $f \equiv G \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{M(x_j - x_i)}{|x_j - x_i|^3}$. Then another way to get this formulation by fundamental gravity potential,

$$E(x) := \frac{1}{4\pi} \frac{1}{|x|} \quad (x \in \mathbb{R}^3, x \neq 0). \tag{3.22}$$

We have a solution by Poisson's equations in the case of acceleration due to an attracting massive object. We write the property as

$$-\Delta E = \delta \ \text{(Dirac's } \delta) \tag{3.23}$$

where the notation $\Delta = \left(\frac{\partial}{\partial X^{(1)}}\right)^2 + \left(\frac{\partial}{\partial X^{(2)}}\right)^2 + \left(\frac{\partial}{\partial X^{(3)}}\right)^2$ or sometime use Laplacian operator ∇^2. Again we define $u(x) := E * f(x) := \int_{\mathbb{R}^3} E(x-y)f(y)dy$. This notation called as convolution of E and f.

$$\begin{cases} -\Delta u = f \\ u(x) = \mathcal{O}\left(\frac{1}{|x|}\right) \text{at} x \to \infty \end{cases} \tag{3.24}$$

So we can write the convolution calculation as follows

$$\begin{aligned} u(x) = E * \delta_{x_0}(x) &= \text{"} \int_{\mathbb{R}^3} E(x-y)\delta_{x_0}(y)dy\text{"} \\ &= E(x-x_0) = \tfrac{1}{4\pi}\tfrac{1}{|x-x_0|}. \end{aligned} \tag{3.25}$$

Next we consider for N-particles case $\{x_i\}_{i=1}^N \subset \mathbb{R}^3$, so the force we have

$$\begin{cases} f &= \sum\limits_{i=1}^N m_i\delta_{x_i}, -\delta U = f \\ u(x) &= E * \left(\sum\limits_{i=1}^N m_i\delta_{x_i}\right) \\ &= \sum\limits_{i=1}^N m_i E * \delta_{x_i} = \sum\limits_{i=1}^N m_i E(x-x_i) = \sum\limits_{i=1}^N \frac{m_i}{4\pi}\frac{1}{|x-x_i|} \end{cases} \tag{3.26}$$

Therefore for gravity force at $x \neq x_i$

$$\begin{aligned} \nabla\left(4\pi G u(x)\right) &= 4\pi G \nabla U(x) \\ &= 4\pi G \sum_{j=1}^N \frac{m_j}{4\pi} \nabla E\left(x-x_j\right) \\ &= 4\pi \sum_{j=1}^N m_j G \nabla E\left(x-x_j\right). \end{aligned} \tag{3.27}$$

where we have gravity potential

$$\begin{aligned}
\nabla E(x) &= -\frac{1}{4\pi}\frac{x}{|x|^3}. \\
&= -G\sum_{i=1}^{N} m_j \frac{x-x_j}{|x-x_j|^3} \\
&= G\sum_{\substack{j=1 \\ j\neq i}}^{N} m_j \frac{x_j-x}{|x_j-x|^3}.
\end{aligned} \tag{3.28}$$

at $x = x_i$; $\sum\limits_{\substack{j=1 \\ j\neq i}}^{N}$ for

$$\text{ND} \approx \begin{cases} \ddot{x}_i & = 4\pi G \nabla u_N(x_i) \\ -\delta u_N & = \sum\limits_{\substack{j=1 \\ j\neq i}}^{N} m_j \delta_{x_j} \\ u(x) & = \mathcal{O}\left(\frac{1}{|x|}\right) \quad \text{as } |x| \to \infty \end{cases} \tag{3.29}$$

Chapter 4

Formal Derivation of Many Particle Limit

4.1 Introduction

Basically in order to explain the particle motions we can consider to look at the case as 'group' of particles that move each times. The 'group' can look as field that indicated the distribution of particles. Start from this condition, we can see that the basic law that work there such a continuity motion case. Therefore we look at a new analogy where the particle motion is essentially as a possible solution in our simulation.

Actually continuity equation governs the conservation of mass/charge/ probability of any closed system. This equation provides us with information about the system. The information is carried from one point to another by a particle (field) wave. In this case the close system is initially given by inside area of radius galaxy. The motivation to work in this way is actually because the dynamical equation of Newton's law, can be obtain by gradient of potential. Therefore another Newtonian modified can also be obtaied by this way. Then, this section we will show another way to modified Newtonian's dynamics such as MOND in another expression called AQUAL.

4.2 Many Particle Limit Cases

In this section we will show the formalism of Newtonian dynamics by using another way. This is the formulation of formal many particles limit for $\mathrm{N}_{particles}\{x_i^N(t)\}_{i=1}^N \subset \mathbb{R}^d$

$$\begin{aligned} m_i^{(N)} &= m_i > 0; \qquad \text{mass of } x_i^N(t) \qquad \text{(depend on } N) \\ \sum_{i=1}^N m_i &= M \qquad \text{(independent of } N) \end{aligned} \tag{4.1}$$

We suppose the following conditions

1. $\underline{\rho(x,t) \geq 0 \text{ limit density as N} \to \infty}$
   for any $\varphi \in C\left(\mathbb{R}^d\right)$,
   $\lim\limits_{N\to\infty}\left(\sum_{i=1}^N m_i\varphi\left(x_i^N(t)\right)\right) = \int_{\mathbb{R}^d}\varphi(x)\rho(x,t)dx.$
2. $\underline{\upsilon(x,t) \in \mathbb{R}^d\text{; limit velocity field.}}$
   $^\exists\upsilon^N(x,t) \in \mathbb{R}^d$ such that $\dot{x}_i^N(t) = \upsilon^N\left(x_i(t),t\right)$,
   and $\lim\limits_{N\to\infty}\upsilon_N(x,t) = \upsilon(x,t)$.
3. $\underline{a(x,t) \in \mathbb{R}^d\text{; limit acceleration field.}}$
   $a(x,t) \in \mathbb{R}^d$ such that $\ddot{x}_i(t) = a^N(x_i(t),t)$
   and $\lim\limits_{N\to\infty}a^N(x,t) = a(x,t)$.

Since we consider the condition

$$C_0^\infty(\mathbb{R}^d) = \{\varphi : \mathbb{R}^d \to \mathbb{R}|_{bounded\ support}^{C^\infty-class}\} \tag{4.2}$$

Then we will give a test function for these term:

- Step 1. For any $\varphi \in C_0^\infty(\mathbb{R}^d)$, we can compute as

$$\begin{aligned} \frac{d}{dt}\left\{\sum_{i=1}^N m_i\varphi(x_i^N(t))\right\} &= \sum_{i=1}^N m_i\nabla\varphi(x_i^N(t))\cdot\dot{x}_i^N(t) \\ &= \sum_{i=1}^N m_i\nabla\varphi(x_i^N(t))\upsilon^N(x_i^N(t),t). \end{aligned} \tag{4.3}$$

We take a limit of (4.3) as $N \to \infty$. Then formally we have

$$\lim\left(l.h.sof(4.3)\right) = \frac{d}{dt}\int_{\mathbb{R}^d} \varphi(x)\rho(x,t)dx \tag{4.4}$$

$$= \int_{\mathbb{R}^d} \varphi(x)\frac{\partial \rho}{\partial t}(x,t)dx. \tag{4.5}$$

Then for right hand side we have

$$\lim\left(r.h.s\ \ of\ \ (4.3)\right) = \int_{\mathbb{R}^d} \nabla\varphi \cdot \upsilon\rho dx \tag{4.6}$$

$$= -\int_{\mathbb{R}^d} \varphi(x)\mathrm{div}\left(\rho(x,t)\upsilon(x,t)\right)dx \tag{4.7}$$

$$= -\int_{\mathbb{R}^d} \varphi(x)\mathrm{div}\left(\rho(x,t)\upsilon(x,t)\right)dx. \tag{4.8}$$

We forward from the right side into left side, then we have

$$\int_{\mathbb{R}^d} \varphi(x)\left\{\frac{\partial \rho}{\partial t}(x,t) + \mathrm{div}\left(\rho(x,t)\upsilon(x,t)\right)\right\}dx = 0 \quad \left(\forall\varphi \in C_0^\infty(\mathbb{R}^d)\right). \tag{4.9}$$

Since $\varphi(x) \neq 0$, so $\{\} = 0$. Then we have relation

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho\upsilon) = 0. \tag{4.10}$$

As we know the last expression is familiar equation that we commonly know as a continuity equation or mass conservation law.

- Step 2 for any $\psi \in C_0^\infty(\mathbb{R}^d, \mathbb{R}^d)$. compute the following

$$
\begin{aligned}
&\frac{d}{dt}\left\{\sum_{i=1}^{N} m_i \psi(x_i^N(t)) \cdot \dot{x}_i^N(t)\right\} \\
&= \sum_{i=1}^{N} m_i \left\{ \left((\dot{x}_i^N(t) \cdot \nabla)\psi\right)(x_i^N(t)) \cdot \ddot{x}_i^N(t) + \right. \\
&\left. \psi\left(x_i^N(t)\right) \cdot \dot{x}_i^N(t) \right\}. \\
&= \sum_{i=1}^{N} m_i \left\{ \left(\left(v_i^N(x_i^N(t), t) \cdot \nabla\right)\psi\right)(x_i^N(t)) \cdot v^N\left(x_i^N(t), t\right) + \right. \\
&\left. \psi\left(x_i^N(t)\right) \cdot a^N\left(x_i^N(t), t\right) \right\}.
\end{aligned} \tag{4.11}
$$

Taking $N \to \infty$, we have

$$
\begin{aligned}
&\frac{d}{dt}\left\{\int_{\mathbb{R}^d} \psi(x) \cdot v(x,t)\rho(x,t)dx\right\} \\
&= \int_{\mathbb{R}^d} \{((v(x,t) \cdot \nabla)\psi(x))\, v(x,t) \qquad (4.12) \\
&+\psi(x) \cdot a(x,t)\}\, \rho(x,t)dx. \qquad (4.13)
\end{aligned}
$$

Then same as the previous calculation, using integration by parts,

$$
\int_{\mathbb{R}^d} \psi \frac{\partial}{\partial t}(\rho v)dx = -\int_{\mathbb{R}^d} \psi \cdot \{\operatorname{div}(\rho v)v + \rho(v \cdot \nabla)v\}\, dx + \int_{\mathbb{R}^d} \psi \cdot (\rho a)dx \quad ({}^\forall \psi \in C_0^\infty)
$$

$$
\frac{\partial}{\partial t}(\rho v) = -\operatorname{div}(\rho v)v - \rho(v \cdot \nabla)^v + \rho a. \tag{4.14}
$$

Applying (4.10), we obtain

$$
\rho \frac{\partial v}{\partial t} = -\rho(v \cdot \nabla)v + \rho a. \tag{4.15}
$$

Summarizing the above, the governing equations in the many particle limit are given as

$$
\begin{cases}
D_t v = a, \\
\rho_t + \operatorname{div}(\rho v) = 0.
\end{cases} \tag{4.16}
$$

Then, the implementation for each theory that we propose in this work, such as Newtonian dynamics, MOND dynamics, and AQUAL, will be shown like this

$$\frac{\partial v}{\partial t} = -(v \cdot \nabla)v + a. \tag{4.17}$$

We denote the material derivative with respect to the velocity filed $v(x,t)$ by

$$D_t := \frac{\partial}{\partial t} + v \cdot \nabla. \tag{4.18}$$

Then we obtain

$$D_t v = a. \tag{4.19}$$

4.3 Material Derivative on Implementation

The most interesting part of each subject that known is the implementation of that equation. As a general equation for movement subject case, we can discuss this part into their application of the equation. In frame to know more idea related to this main topic of master thesis work, we will bring it into some of the case that we use it.

4.3.1 Newtonian Dynamics in Material Motions

As a main topic of our study, first, we need to show the implementation of this concept into ND case. Generally, we can start by writing the dynamical motion of ND as follow

$$\text{ND}: \ \ddot{x}_i(t) = G \sum_{\substack{j=1 \\ j\neq i}}^{N} \boxed{\frac{m_j\,(x_j(t) - x_i(t))}{|x_j(t) - x_i(t)|^3}} =: f_i(t).$$

$$a(x,t) = \lim_{N\to\infty} a^N(x,t). \tag{4.20}$$

$$= \int_{\mathbb{R}^d} G \frac{y-x}{|y-x|^3}\rho(y)dy. \qquad for \ \ y = \begin{pmatrix} y^{(1)} \\ y^{(2)} \\ y^{(3)} \end{pmatrix} \tag{4.21}$$

Then, we simplify as

$$\begin{cases} D_t\upsilon(x,t) = a(x,t) \\ a(x,t) = 4\pi G \ \ E * \rho(x,t) \\ \rho_t(x,t) + \operatorname{div}(\rho(x,t)\upsilon(x,t)) = 0 \\ \rho(x,0) = \rho_0(x) \geqq 0 \\ \upsilon(x,0) = \upsilon_0(x) \in \mathbb{R}^d. \end{cases} \tag{4.22}$$

we denote

$$E(x) = \frac{1}{4\pi}\frac{1}{|x|} \ \text{ in } \ \mathbb{R}^3, \tag{4.23}$$

$$a = 4\pi G\rho \tag{4.24}$$

is also written in the form

$$\begin{cases} -\Delta U = 4\pi G\rho \\ a = \Delta U. \end{cases} \tag{4.25}$$

4.3.2 Modified Newtonian Dynamics in Material Motions

As a traditional modified from Newtonian dynamics equation, MOND has a naturally equation still primarily similar with Newtonian dynamics. In the

Milgrom has first propose of that model actually give a such additional acceleration like in equation (3.1). Therefore in the case of material derivative the modified can be eventually shown as

$$\text{MOND}: \quad \mu\left(\frac{|\ddot{x}_i}{\alpha_0}\right)\ddot{x}_i = f_i \Longleftrightarrow \ddot{x}_i = \beta(f_i) = a^N(x_i^N, t).$$
$$a(x,t) = \beta(4\pi G\nabla(E * \rho)). \tag{4.26}$$

The big motivation such gives the explanation before to solve the constant velocity of galaxy rotating. A while if we conduct from material derivative we can say a simple model which happened at particles motions flow without constraint. But we don't really give the explanation for that case. We only give the another way which have the same motivation as a problem of galaxy rotating motions.

4.3.3 A Quadratic Lagrange (AQUAL) in Material Motions

Same as before, which also as a main topic of this work, where we want to modified Newtonian dynamics equation to solve a prom, AQUAL has been proposed. The basic idea of AQUAL is so related with material derivative way's to solve the particle motion case. The concept which first proposed based on Jacob Backenstein and M. Milgrom work as a represent for this implementation.

AQUAL is one of model from Modified Newtonian Dynamics (MOND) which have same motivation. In AQUAL case, its modified via Lagrangian equation. The theory is abbreviate of A QUAdractic Lagrangian for the basic form of the equation of motion concept. The expression of AQUAL like this

$$\text{AQUAL}: \qquad a = \nabla U$$
$$\text{-div}\left(\mu\left(\frac{|\nabla U|}{\alpha_0}\right)\nabla U\right) = 4\pi G\rho. \tag{4.27}$$

Where U as a potential gravity of dynamical motion. We don't use this equation in the program, but we show as an alternative way to modify ND with the same motivation. Furthermore, it might be better to use in the future work, and we can analyze which one the best concept to solve galaxy problem.

Chapter 5

Numerical Scheme

5.1 Briefly Numerical Analysis and It's Mathematical Notations

Newtonian dynamic and its modified, basically is problem in ordinary differential case. Sometimes we can calculate by manually. But in order to get much calculation, we need work hard very complicated to solve it. So we will solve this problem by numerical calculation.

So many kind of numerical method to solve ordinary differential equation (ODE), but for easiest way we can use by Euler method [11]. This method maybe is not really an efficient method, but many of the ideas involved in the numerical solution of differential equations are introduced most simply with it.

Before we discuss more about the numerical solution in this work, it's important to know the mathematical notation that we are going to use it in simulation. So, this part we will give an mathematical overview for this system. Let's start by giving notation for the particles. We write the particle as

$$X = \{x_i\}_{i=1}^{\mathrm{N}} \quad \subset \mathbb{R}^{\mathrm{d}\times\mathrm{N}} \tag{5.1}$$

where X is particle for i as index of each particles as much as N. Then we set for dynamically system, $f_i(X) := \frac{F_i(X)}{m_i}$, and for indicating the step we show

$$\text{ND}\begin{cases} \dot{x}_i(t) = v_i(t) & (0 \le t \le T), \\ \dot{v}_i(t) = f_i\left(X(t)\right) & (0 \le t \le T), \\ x_i(0) = x_i^0 \;\; ;\text{given} & (i = 1, \ldots \mathrm{N}) \\ v_i(0) = v_i^0 \;\; ;\text{given} & (i = 1, \ldots \mathrm{N}) \end{cases} \tag{5.2}$$

By these notation, we will use it into the program.

5.2 Discretization and The Numerical Scheme

After we fix it the mathematical notation, we also need to attend the time stepping. It is mean that our simulation is depending the function of time, so should fix it. We choose $\Delta t > 0, K := \left[\frac{T}{\Delta t}\right]$, and we consider for our case

$$x_i^k = \begin{pmatrix} x_{\mathrm{i1}}^{\mathrm{k}} \\ x_{\mathrm{i2}}^{\mathrm{k}} \end{pmatrix} \sim x_i(k\Delta t) \tag{5.3}$$

$$v_i = \begin{pmatrix} v_{\mathrm{i1}}^{\mathrm{k}} \\ v_{\mathrm{i2}}^{\mathrm{k}} \end{pmatrix} \sim v_i(k\Delta t) \tag{5.4}$$

where we give notation $(k = 0, 1, \ldots, \mathrm{K}, i = 1, \ldots, \mathrm{N})$.
Time stepping k start from initially, which mean 0, and will be end for K time. It will be implemented into i-th particle. In our code programe we have this conditions

$$\mathrm{X}^{\mathrm{old}} = (x_i^{\mathrm{old}})_{\mathrm{i=1}}^{\mathrm{N}} \,; \mathrm{V}^{\mathrm{old}} = (v_i^{\mathrm{old}})_{\mathrm{i=1}}^{\mathrm{N}} \tag{5.5}$$

$$\mathrm{X}^{\mathrm{new}} = (x_i^{\mathrm{new}})_{\mathrm{i=1}}^{\mathrm{N}} \,; \mathrm{V}^{\mathrm{new}} = (v_i^{\mathrm{new}})_{\mathrm{i=1}}^{\mathrm{N}} \,. \tag{5.6}$$

So this mean that we don't store the previous data. We overwrite the data by new update data of position x and velocity v.

5.2.1 Euler Scheme

As we said before, the numerical method in order to calculate the differential problem is used Euler method. Therefore we should know how actually this method can work. So we will show the numerical scheme for Euler method in this notation

$$\begin{cases} \frac{x_i^{k+1}-x_i^k}{\Delta t} = v_i^k \\ \frac{v_i^{k+1}-v_i^k}{\Delta t} = f_i(X^k) \\ x_i^0 \quad ; v_i^0 \quad ; \text{given.} \end{cases} \tag{5.7}$$

This scheme will be implemented into the program and we discussing more about the result will be comparing by observation data.

5.2.2 Symplectic Scheme

Another scheme that we used in this simulation is symplectic scheme. Generally this method it's better than Euler method. The main reason is about the symplectic always give new velocity update for each step position. So, the result might be closely to the exact.

Then, this is the symplectic scheme which used in the program

$$\begin{cases} \frac{v_i^{k+1}-v_i^k}{\Delta t} = f_i(X^k) \\ \frac{x_i^{k+1}-x_i^k}{\Delta t} = v_i^{k+1} \\ x_i^0 \quad ; v_i^0 \quad ; \text{given.} \end{cases} \tag{5.8}$$

Based two numerical method we will show how more accurate between Euler and symplectic method.

5.3 Galaxy Scaling for Simulation

Galaxy as an object which work in large scale, certainly need big memory to store it. This section we want to show the new scaling in our simulation, that we consider to convert in small scale. Firstly we convert the time scale. In the real observation galaxy time scale in order to indicate one period the galaxy rotations is called galaxy-year [gy] -sometimes cosmic year. This mean one galaxy year is about 1×10^{15}[s]. Basically if use the real scaling of galaxy, it will be crowded enough to calculate it. So we need to convert this scale into small dimension scale as given this calculation

$$\tilde{t} = \frac{t}{C_1} \tag{5.9}$$

for $\tilde{t}$ is indicated by time after re-scaling and we bring out $C_1 = 10^{15}$ as value of galaxy year. So, we can keep C_1 value from calculation and sometimes if we need to show, we can use it in the data that we obtained.

All of unity we need to change into one parameter. So, for the other parameters should be converted also into numerical scale. Therefore we show all of parameters that we have converted as an uniformly into this table

Table 5.1: Table of conversion

Symbols	Meaning	Unity & Scale
G	Gravity constant	6.67×10^{-11} [$\mathrm{m^3Kg^{-1}s^{-2}}$] to 13.37×10^{-11} [$\mathrm{Kpc^3M_\odot{}^{-1}gy^{-2}}$]
M_{total}	Mass of the total particles	$\left(\frac{10^{11}}{N}\right)$ [Kg] to 1 [$M_\odot$]
t	Time scale	1 [$\times 10^{15}$] [s] to 1 [gyear]
R	Radius of Milky way	2.9×10^{20} [m] to 5[Kpc]
v	Velocity	2.3×10^4[m/s] to 2.3×10^{-4} [Kpc/gy]

Finally we get the uniform dimension like on the table(5.1) that we can do for further step with give an initialization of this simulation.

5.4 Initialization of Particles

After we give an uniform scale for our case, the next step is make a condition that we consider as initial position. By this table will be shown some of the initialization

Table 5.2: Table of initialization

Initialization	Physical meaning	Unity	Value
m	Mass of particle	1	$[\mathrm{M}_\odot]$
R	Radius	2.9	[Kpc]
ω	angular velocity	Random value$(0, \pi)$	[rad gy^{-1}]
$x_i^j\,(i,j)$	position of particles	Random uniform	[Kpc]
$v_i^j\,(i,j)$	velocity	$(-\omega R\sin(\omega t), \omega R\cos(\omega t))$	[Kpc gy^{-1}]
t_0	time initialization	0	[gy]
G	Gravity constant	7.14×10^{-8}	$[\mathrm{Kpc}^3 \mathrm{M}_\odot{}^{-1} \mathrm{gy}^{-2}]$

By using the table initialization above (5.2), we can do the simulation for this system.

Chapter 6

Result & Discussion

In this chapter, we show analysis of two models which we considered in the simplest case. The two models are ND and its modification. ND as a fundamental equation has consistent give a nice prediction for dynamical object in small scale. From ND we can understand the dynamical law of each object that movement. We can predict when and where the object will stop, how the trajectory of object's movement and so on. So, it is an important equation to explain the dynamical objects.

On the other hand, we have a fact that ND did not successfully obtain nice prediction for larger scale. We have studied that fact, in section 2 and 3 about the main reason why ND might be needed to modify of the equation. So, we give a short analysis by considering the simple case in order to explain that.

We analyze based on the speed graph of the two models. We also compare the speed value of each model with the radius of the system. Therefore the graph plot of the speed vs radius in this case with consider following parameters:

$$\begin{cases} \alpha_0 = 1 \\ A = 1, \end{cases} \tag{6.1}$$

and obtained the graph as follows

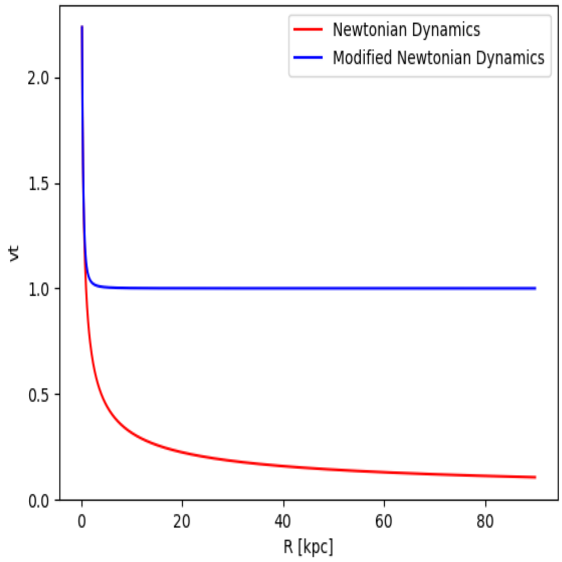

Figure 6.1: Plot v vs R.

Figure (6.1) shows the speed $|v_i(t)|$ vs R, where ND is plotted by redline and MOND is blue-line. We obtained the speed plot for ND is rapidly going to zero. At the moment, MOND obtained speed constant at point 1. The condition which for MOND model has same analysis enough if we refer to Figure (1.2). Where, the speed which obtained has a constant value at $R \to \infty$.

Basically, the Figure (6.1) has been just a simple model to present condition which is gained from two models. The condition which nearly to zero value did not consider in our system. Because for conditioning like in the real system it is a bit complicated, so we just show for simple graph to show the constant value of MOND speed.

Generally, we can say that MOND obtained the constant speed than ND. It can be used to solve galaxy rotation problem which relatively constant even $R \to \infty$. Where ND has the speed which gained limit to zero. Based on this fact, we can say the galaxy rotation problem gives an opportunity to scientist find a valuable solution and meaningful which not contrast with physical law. And MOND has a chance to be an alternative solution to solve that problem.

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