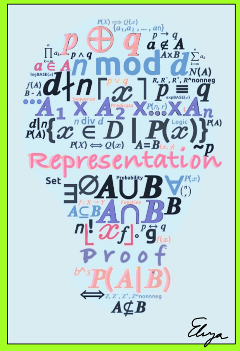
Discrete Structures



Lecture 21: Counting with repetition

Susan L. Epstein

HUNTER

1

Last time

★ **Counting supports function theory**

- How to use the inclusion/exclusion principle
- How to use the pigeonhole principle

Fall 2023 CSCI 150 2/24

2

Review: questions to ask when you count

- How big are the sets that are involved?
- Are the sets involved disjoint?
- Is there inherent order? ≡ is this a permutation or a combination?
- What process would construct an arbitrary element?
- Would the complement be easier to count?
- Does the inclusion / exclusion rule apply?

Review: counting rules (1)

Multiplication rule: If a process consists of k steps that can be performed respectively in $n_1, n_2, \cdots, n_k$ ways, then the entire process can be performed in $n_1 n_2 \cdots n_k$ ways

For any integer $n \geq 1$ and any set S of n elements,

$P(n, r)$: there are $\frac{n!}{(n-r)!}$ permutations of r elements from S

$C(n, r)$: there are $\frac{n!}{(n-r)!r!}$ combinations (ways to select) r elements from S

Addition rule: For any partition $\{A_1, A_2, \dots, A_n\}$ of a finite set A,

$$|A| = |A_1| + |A_2| + \cdots + |A_n|$$

Difference rule: For any finite set A and any subset B of A, $|A - B| = |A| - |B|$

Complement rule: For any finite set $A \subseteq U$, $|A^C| = |U| - |A|$

Review: counting rules (2)

Inclusion/exclusion rules: $|A \cup B| = |A| + |B| - |A \cap B|$ and

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$\left|\bigcup_{i=1}^{n} A_i\right| =$$

$$\sum_{i=1}^{n} |A_i| - \sum_{\text{distinct } i,j=1}^{n} |A_i \cap A_j| +$$

$$\sum_{\text{distinct } i,j,k=1}^{n} |A_i \cap A_j \cap A_{kj}| + \cdots + (-1)^{n-1} \left|\bigcap_{i=1}^{n} A_i\right|$$

Pigeonhole principles:

For any function *f* from a finite set X with n elements to a finite set Y with m elements, if $n > m$, then f is not 1-to-1.

Let f be a function from a finite set X of n elements to a finite set Y of m elements, and $k \in \boldsymbol{Z}^+$. If $k < \frac{n}{m}$, then there is some $y \in Y$ that is the image of at least $k + 1$ elements of X.

Today's outline

- Permutations on multisets
- Combinations from multisets

Repetition with limited quantities

- Multiset $S = \{q_1 \cdot x_1, q_2 \cdot x_2, \dots, q_k \cdot x_k\}$ collection of $\sum_{i=1}^{k} q_i$ elements
  - k kinds of elements $x_1, x_2, \dots, x_k$
  - Each kind x_i has some finite number q_i of identical copies

    letters in BANANA = $\{1 \cdot B, 3 \cdot A, 2 \cdot N\}$

    multiple decks of cards
- Permutations $P(n;\ q_1, q_2, \dots, q_k)$ of a multiset on k kinds of elements where kind i has q_i available copies

  Permutations of BANANA has elements {B,A,N}

  $k = 3, q_1 = 1, q_2 = 3,\ q_3 = 2$ and we want to count $P(6; 1,3,2)$

  6 distinct locations for the letters: _ _ _ _ _ _

  Choose the location for the B: $C(6,1)$

  Choose the location for the A's: $C(5,3)$

  Choose the location for the N's: $C(2,2)$

  By the multiplication rule, that's $C(6,1) \cdot C(5,3) \cdot C(2,2)$

  Does the order in which do this really matter?

7

Permutations and indistinguishable objects

Theorem: The number $P(n;\ q_1, q_2, \dots, q_k)$ of distinguishable permutations of a multiset $S = \{q_1 \cdot x_1, q_2 \cdot x_2, \dots, q_k \cdot x_k\}$ with k kinds of elements $x_1, x_2, \dots, x_k$ and q_i identical copies of kind x_i where $\sum_{i=1}^{k} q_i = n$, is $\frac{n!}{q_1! q_2! \dots q_k!}$

Proof: We must show that $P(n;\ q_1, q_2, \dots, q_k) = \frac{n!}{q_1! q_2! \dots q_k!}$

There are $C(n, q_1)$ ways to position the x_1's, and then $C(n - q_1, q_2)$ ways to position the x_2's,…, $C\left(n - \sum_{i=1}^{j-1} q_i, q_j\right)$ ways to position the x_j's,…,and finally $C\left(n - \sum_{i=1}^{k-1} q_i, q_k\right)$ ways to position the x_k's.

By the multiplication rule this is $C(n, q_1) \cdot C(n - q_1, q_2) \cdots C\left(n - \sum_{i=1}^{k-1} q_i, q_k\right)$

$$= \frac{n!}{(n-q_1)! q_1} \cdot \frac{(n-q_1)!}{(n-q_1-q_2)! q_2!} \cdot \frac{(n-q_1-q_2)!}{(n-q_1-q_2-q_3)! q_3!} \cdots \frac{(n-q_1-q_2-\cdots-q_{k-1})!}{(n-q_1-q_2-\cdots-q_k)! q_k!}$$

$$= \frac{n!}{(n-q_1)! q_1} \cdot \frac{(n-q_1)!}{(n-q_1-q_2)! q_2!} \cdot \frac{(n-q_1-q_2)!}{(n-q_1-q_2-q_3)! q_3!} \cdots \frac{q_k!}{0! q_k!} = \frac{n!}{q_1! q_2! \dots q_k!}$$

QED

Indifferent to the order in which the kinds of elements are placed because multiplication is commutative

8

Examples (1)

How many ways are there to permute the letters in BANANA?

$$P(6;1,3,2)=\frac{6!}{1!3!2!} \ll P(6,6)=6!$$

How many ways are there to permute the letters in MISSISSIPPI?

$$P(11;1,4,4,2)=\frac{11!}{1!\,4!\,4!\,2!}$$

How many ways are there to roll a die 6 times and obtain a sequence of outcomes with one 1, three 5's, and two 6's?

$$P(6;1,3,2)=\frac{6!}{1!\,3!\,2!}$$

How many numbers greater than 3,000,000 can be formed by arrangements of 1, 2, 2, 4, 6, 6, 6 ?

What's the first digit?

$$P(6;1,2,2)+P(6;1,2,3)$$

Examples (2)

How many ways can a deck of cards be broken up into a collection of 4 unordered piles of 13 cards?

Spread them out in a row and then label them with 13 1's, 13 2's, 13 3's, and 13 4's.

$$P(52;13,13,13,13)=\frac{52!}{13!\,13!\,13!\,13!}$$

How many ways can a deck of cards be broken up into 3 unordered piles of 8 cards and 4 piles of 7 cards?

$$P(52;8,8,8,7,7,7,7)=\frac{52!}{8!\,8!\,8!\,7!\,7!\,7!\,7!}$$

How many ways are there to form a sequence of 10 letters from 4 A's, 4 B's, 4 C's, and 4 D's, if each letter must appear at least twice?

That means one of them is used 4 times and the others 2, or 2 of them are used 3 times and the others 2. But which ones?

$$4P(10;4,2,2,2)+C(4,2)\cdot P(10;3,3,2,2)=4\cdot\frac{10!}{4!2!2!2!7!}+\frac{4!}{2!2!}\cdot\frac{10!}{3!3!2!2!}$$

Today's outline

- ✓ Permutations on multisets
- Combinations from multisets

r-combinations with repetition

- $C(n,r) = \frac{n!}{(n-r)!r!}$ counts how to choose r elements from n distinct elements, that is, how many subsets of size r there are
- Let $V(n,r)$ denote the number of r-combinations with unlimited repetition (multisets of size r) that can be built from a set of n distinct repeatable objects

  How many ways can we distribute 7 identical cookies to 4 (distinct) kids? Here $r = 4$ and $n = 7$
- Stars and bars is a counting method for this kind of problem

Counting predicts code complexity

Stars and bars for $V(n, r)$

People and animals are distinct

How many ways can we distribute 7 identical cookies to 4 (distinct) kids?
Label each cookie with a kid's name – how many ways can that be done?
Since the cookies are identical, a labelling like BBBBAAA has the same result as ABABABB and AABBBBA and many others
Alphabetical order would be distinct and count them as 1

cookies are ******* kids are ABCD

put A's cookies here | put B's here | put C 's here | put D's here

Note that only 3 bars are needed to indicate 4 categories

- Represent the identical elements as n stars *****
- Use $r-1$ bars | | | to separate the n identical items into r label groups

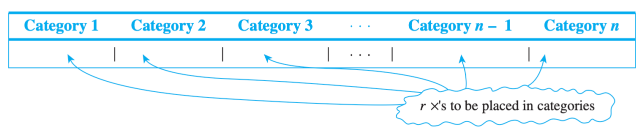
|*|| would give 3 cookies to A, 4 to B and none to the others
||******|* would give 6 cookies to C and 1 to D and none to the others

How many ways are there to arrange $r-1$ bars and n stars?

A formula for $V(n, r)$

- How many ways are there to arrange $r-1$ bars and n stars?
- Can form r distinct categories with only $r-1$ bars

Use $r-1$ bars for r distinct kinds and n stars for identical objects



- That is $n+r-1$ symbols in all _ _ _ _ _ _ _ _ _ _ _ _ _
- Since the stars are identical, just choose spots for them and then put the bars in the empty locations

$$V(n, r) = C(n+r-1, n)$$

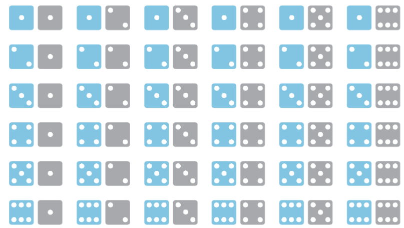
How many ways can we distribute 7 identical cookies to 4 (distinct) kids?

$n = 7, r = 4$ $\quad \dfrac{(7+4-1)!}{(4-1)!\,7!}$

Remember this?

How many outcomes are there when you roll 2 different colored dice?

6^2



If 2 identical dice are rolled, how many different outcomes are there?

$1's \mid 2's \mid 3's \mid 4's \mid 5's \mid 6's$

$V(n,r) = C(n+r-1,n)$

$$n=2 \quad r=6 \quad V(2,6) = C(2+6-1,2) = \frac{7!}{(6-1)!\,2!}$$

Examples (3) $V(n,r) = C(n+r-1,n)$

How many ways are there to fill a box with a dozen donuts chosen from 5 different varieties with the requirement that at least 1 donut of each variety is picked? (The order of the donuts in the box is irrelevant.)

$$n=12 \quad n'=7 \quad r=5 \quad V(7,5) = C(7+5-1,7) = \frac{11!}{4!\,7!}$$

How many ways are there to pick a collection of 10 balls from a pile of red balls, blue balls, and purple balls, if there must be at least 5 red balls?

$$n=10 \quad n'=5 \quad r=3 \quad V(5,3) = C(5+3-1,5) = \frac{7!}{2!\,5!}$$

How many ways are there to pick a collection of 10 coins from very large piles of pennies, nickels, dimes and quarters?

$$n=10 \quad r=4 \quad V(10,4) = C(10+4-1,10) = \frac{13!}{3!\,10!}$$

If 10 identical dice are rolled, how many different outcomes are there?

$$n=10 \quad r=6 \quad V(10,6) = C(10+6-1,10) = \frac{15!}{5!\,10!}$$

How many outcomes are there when you roll 3 different colored dice?

6^3

Examples (4) $V(n,r) = C(n+r-1,n)$

How many ways are there to distribute 20 identical sticks of red licorice and 15 identical sticks of black licorice among 5 children?

$$n_1 = 20 \quad n_2 = 15$$

$$V(20,5) \cdot V(15,5) = C(20+5-1,20) \cdot C(15+5-1,15) = C(24,20) \cdot C(19,15)$$

How many integer solutions are there to $x_1 + x_2 + x_3 + x_4 = 12$ with $x_i \geq 0$?

Distribute 12 1's into the 4 variables

$$n = 12 \quad r = 4 \quad V(12,4) = C(12+4-1,12) = C(15,12)$$

How many solutions with $x_1 \geq 2$, $x_2 \geq 2$, $x_3 \geq 4$, $x_4 \geq 0$?

$$n = 12 \quad n' = 4 \quad r = 4 \quad V(4,4) = C(4+4-1,4) = C(7,4)$$

How many ways are there to pick 10 balls from large piles of identical red, white, and blue balls, plus 1 pink ball, 1 lavender ball, and 1 tan ball?

How many of the singleton colors to use?

$$C(3,3) \cdot V(7,3) + C(3,2) \cdot V(8,3) + C(3,1) \cdot V(9,3) + C(3,0) \cdot V(10,3) =$$

$$1 \cdot C(7+3-1,7) + 3C(8+3-1,8) + 3C(9+3-1,9) + 1 \cdot C(10+3-1,10)$$

$$= C(9,7) + 3C(10,8) + 3C(11,9) + C(12,10)$$

Examples (5) $V(n,r) = C(n+r-1,n)$

How many ways are there to distribute 5 identical apples, 6 identical oranges, 6 identical pears, and 4 identical pineapples among 3 people?

$$V(5,3) \cdot V(6,3) \cdot V(6,3) \cdot V(4,3) = C(7,5) \cdot C(8,6) \cdot C(8,6) \cdot C(6,4)$$

How many ways if each person gets at least one pear?

$$V(5,3) \cdot V(6,3) \cdot V(3,3) \cdot V(4,3) = C(7,5) \cdot C(8,6) \cdot C(5,2) \cdot C(6,4)$$

How many ways are there to distribute 18 chocolate donuts, 12 cinnamon donuts, and 14 powdered sugar donuts among 4 professors if each professor demands at least one donut of each kind?

$$V(14,4) \cdot V(8,4) \cdot V(10,4) = C(17,4) \cdot C(11,4) \cdot C(13,4)$$

How many integer solutions to $x_1 + x_2 + x_3 + x_4 + x_5 = 28$ with $x_i \geq 0$?

$$n = 28 \quad r = 4 \qquad V(28,4) = C(28+4-1,28) = C(31,28)$$

with $x_i > 0$?

$$n = 23 \quad r = 4 \qquad V(23,4) = C(23+4-1,23) = C(26,23)$$

with $x_i > i$?

$$n = 8 \quad r = 4 \qquad V(8,4) = C(8+4-1,4) = C(11,4)$$

Examples (6)

$V(n,r) = C(n+r-1,n)$

How many election outcomes (numbers of votes for each candidate, not who votes for whom) are possible if there are 3 candidates and 30 voters?

$n = 30 \quad r = 3 \qquad V(30,3) = C(30+3-1,30)$

If, in addition, one candidate receives a majority of the votes?

16 is a majority and who wins?

$n = 14 \quad r = 3 \qquad 3V(14,3) = 3C(14+3-1,14)$

How to choose r elements from n

	Order matters	Order does not matter
No repetition	$P(n,r) = \frac{n!}{(n-r)!}$	$C(n,r) = \frac{n!}{(n-r)!\,r!}$
Repetition allowed	n^r	$V(n,r) = C(n-1+r,n)$

Examples (7) $V(n,r) = C(n+r-1,n)$

How many triples (i,j,k) are there where $1 \le i \le j \le k \le n$ and $n \in \boldsymbol{Z}^{+}$?
They all have to be at least 1, so now we are down to $n-3$.
$n' = n-3 \quad r = 3 \qquad V(n-3,3) = C(n-3+3-1, n-3) = C(n-1, n-3)$

How many ways can you assign 300 students to 6 sections of a course?

$$6^{300}$$

How many ways can you assign 300 students to 6 sections of a course if there must be 50 in each class?

$$P(300; 50,50,50,50,50,50)$$

How many ways can you put 300 identical papers in 6 distinct boxes?

$$n = 300 \quad r = 6 \qquad V(300,6) = C(300+6-1, 300)$$

How many ways can you put 300 identical papers in 6 distinct boxes so each box has at least 2 papers?

$$n = 300 \quad n' = 288 \quad r = 6 \qquad V(288,6) = C(288+6-1, 288)$$

Fall 2023 CSCI 150 21/24

21

Examples (8) $V(n,r) = C(n+r-1,n)$

How many ways can you distribute 23 **distinct** books to students Alice, Bob, Charles, Danielle, and Evita?

$$5^{23}$$

How many ways if Alice and Bob get 7 books each and others get 3 each?

$$P(23; 7,7,3,3,3)$$

How many ways if two students get 7 each and the others get 3 each?

$$C(5,2) \cdot P(23; 7,7,3,3,3)$$

How many ways can you distribute 23 **identical** books to students Alice, Bob, Charles, Danielle, and Evita?

$$n = 23 \quad r = 5 \qquad V(23,5) = C(23+5-1, 23)$$

How many ways if Alice and Bob get 7 books each and the others get 3 each?

$$n = 23 \quad n' = 9 \quad n'' = 0 \qquad V(0,5) = 1$$

How many ways if two students get 7 each and the others get 3 each?

$$C(5,2) \cdot 1$$

Fall 2023 CSCI 150 22/24

22

One more example

How many times will the innermost loop be iterated when the algorithm segment below is implemented and run? (Assume n is a positive integer.)

for *k* :=1 **to** *n*
  for *j* := 1 **to** *k*
    for *i* := 1 **to** *j*
      [*the body of the inner loop, with no branching that lead outside it)*
      next *i*
  next *j*
next *k*

$V(n,r) = C(n+r-1,r)$

1 iteration of the innermost loop for every column in table
≡ number of triples of (i,j,k) where $\boldsymbol{i} \le \boldsymbol{j} \le k \le n$

k	1	2 →	→	→	3 →	→	→	→	→	→	⋯	n →	→	→	→	→	→	→
j	1	1	2 →	→	1	2 →	→	3 →	→	→	⋯	1	2 →	→	⋯	n →	→	→
i	1	1	1	2	1	1	2	1	2	3	⋯	1	1	2	⋯	1	⋯	n

Each column above is a trace of variables' values

Category 1	2	3	4	5	Result of the Selection
		× ×		×	(3, 3, 5)
×	×		×		(1, 2, 4)

For $n = 5$, need 4 bars and 3 stars

Thus, $C(n+3-1,3) = C(n+2,3) = \frac{(n+2)!}{(n-1)!3!} = \frac{(n+2)(n+1)n}{6}$

Fall 2023 CSCI 150 23/24

23

What you should know

★ **Counting supports function theory**

- Repetition counting employs identical objects
- Repetition in combinations produces a multiset

Next up: *Polynomial counting*

Time to finish up that Opening sheet!

Any questions?

Problem set 21,22 is due on Thursday, November 30 at 11PM

Fall 2023 CSCI 150 24/24

24