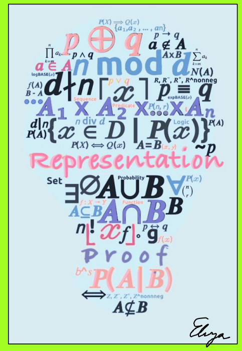
Discrete Structures



Lecture 5: Quantified propositions

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HUNTER

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Last time

- **Validity ≠ truth**
- What an argument form is and how to prove one is valid
- How to use each of many argument forms

modus ponens
$p \rightarrow q$
p
$\therefore q$

modus tollens
$p \rightarrow q$
$\sim q$
$\therefore \sim p$

disjunctive addition
p
$\therefore p \vee q$

propositional proof by cases
$p \vee q$
$p \rightarrow r$
$q \rightarrow r$
$\therefore r$

disjunctive syllogism
$p \vee q$
$\sim p$
$\therefore q$

$p \vee q$
$\sim q$
$\therefore p$

conjunctive simplification
$p \wedge q$
$\therefore p$

$p \wedge q$
$\therefore q$

hypothetical syllogism
$p \rightarrow q$
$q \rightarrow r$
$\therefore p \rightarrow r$

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Today's outline

- Not all arguments are valid
- Basic predicate calculus
- Negation and multiple quantifiers in predicate calculus

BEWARE: argument forms can be incorrect

Fallacy = invalid argument form that represents an error in reasoning

Common fallacies

- Ambiguous premises
- Circular reasoning assumes what is to be proved
- Jumping to a conclusion (inadequate grounds)

An invalid argument form

$p \to (q \vee \sim r)$
$q \to (p \wedge r)$
$\therefore p \to r$

Whenever the premises are true the conclusion must also be true

p	q	r	$\sim r$	$q \vee \sim r$	$p \to (q \vee \sim r)$	$p \wedge r$	$q \to (p \wedge r)$	$p \to r$
T	T	T	F	T	T	T	T	T
T	T	F	T	T	T	F	F	
T	F	T	F	F	F	T	T	
T	F	F	T	T	T	F	T	F
F	T	T	F	T	T	F	F	
F	T	F	T	T	T	F	F	
F	F	T	F	F	T	F	T	T
F	F	F	T	T	T	F	T	T

Converse error

If you do every problem in the text, you will learn discrete math. You learned discrete math. Therefore, you did every problem in the text.

$p =$ you do every problem in the text
$q =$ you learn discrete math

aka affirming the conclusion

$p \to q$
q
$\therefore p$

$p \to q =$ If you do every problem in the text, you will learn discrete math.
$q =$ You learned discrete math.
$p =$ Therefore, you did every problem in the book.
NO!

Inverse error

If you do every problem in the text, you will learn discrete math. You did not do every problem in the text. Therefore, you did not learn discrete math.

p = you do every problem in the text
q = you learn discrete math
$p \rightarrow q$ = If you do every problem in the text, you will learn discrete math.

aka denying the hypothesis

$p \rightarrow q$ $\quad$ $\sim p$ = You did not do every problem in the text.
$\sim p$ $\quad$ $\sim q$ = Therefore, you did not learn discrete math
$\therefore \sim q$ $\quad$ NO!

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GIGO (garbage in, garbage out)

- Distinguish between the validity of an argument and the truth value of its conclusion
- False premises and a valid argument form may produce a false conclusion
- Similarly, some premises with an invalid argument form may produce a true conclusion
- Sound argument = valid and has a true conclusion

Validity ≠ truth


Any questions?

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Today's outline

- ✓ Not all arguments are valid
- Basic predicate calculus
- Negation and multiple quantifiers in predicate calculus

The notation in this section is required for your work.

Please ignore the notation your book uses instead.

Predicate calculus extends logical representation

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Predicates and quantifiers

- Predicate $P(x)$ = sentence with finitely many variables that becomes a proposition when specific values are substituted for those variables
  $P(x)$ = "x is greater than 5" $Q(x, y)$ = "x is queen-of y"
  $P(7) = 7 > 5$ $P(2) = 2 > 5$ $Q(\text{Sonja}, \text{Norway})$ $Q(\text{Sonja}, \text{USA})$
- Use capital letter for predicate with lowercase letters for variables
- Domain of a predicate variable = the set of all values that the variable can assume
  For $P(x)$ = "$x > 5$" the domain of x is some set of numbers
- Truth set of a predicate $P(x) = \{x \in D | P(x)\}$ is all elements in the domain D of predicate variable x that, when substituted for x, make $P(x)$ true (Sonja, Norway) is in the truth set of $Q(x, y)$
  What is the truth set of $P(x)$ = "$\frac{6}{x} \in \boldsymbol{Z}$" when the domain of x is $\boldsymbol{Z}$?
- For a predicate P with domain D, a quantifier describes how much of D is in P's truth set
- Predicate calculus = (aka first-order logic aka First Order Predicate Calculus or FOPC)

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Universal quantification

- Universal quantifier $\forall$ for $P(x)$ with domain D
  - $\forall$ is read "for all"
  - Indicates that all elements of D are in P's truth set
- Universal statement $\forall x \in D, P(x)$ says every element in D makes P true
  For $D = \{a_1, a_2, \ldots, a_n\}, \forall x \in D, P(x) \equiv P(a_1) \wedge P(a_2) \wedge \ldots P(a_n)$
  - $\forall x \in D, P(x)$ is true iff $P(x)$ for every $x \in D$
  - $\forall x \in D, P(x)$ is false iff for at least one $x \in D, P(x)$ is false

  Let H be the set of all Hunter students and $Sleepy(x)$ mean "x is sleepy"
  $\forall x \in H, Sleepy(x)$ means "All Hunter students are sleepy"
- Counterexample = element of D that falsifies $\forall x \in D, P(x)$
- D is important here   Let $A(x)$ mean "$x \geq 0$"
  If $D = \boldsymbol{N}, \forall x \in D, A(x)$ is true, but if $D = \boldsymbol{Z}\ \forall x \in D, A(x)$ is false
- For finite D, can prove $\forall x \in D, P(x)$ by the method of exhaustion = substitute each value in D into P and show that **all** the resultant statement are true

The universal conditional

Let $P(x) = x > 10$ and $Q(x) = x^3 > 1000$

Universal conditional: $\forall x \in D, P(x) \rightarrow Q(x)$ If $x > 10$ then $x^3 > 1000$

Contrapositive: $\forall x \in D, \sim Q(x) \rightarrow \sim P(x)$ If $x^3 \leq 1000$ then $x \leq 10$

Converse: $\forall x \in D, Q(x) \rightarrow P(x)$ If $x^3 > 1000$ then $x > 10$

Inverse: $\forall x \in D, \sim P(x) \rightarrow \sim Q(x)$ If $x \leq 10$ then $x^3 \leq 1000$

- Universal conditional is logically equivalent to its contrapositive:
  $\forall x \in D, P(x) \rightarrow Q(x) \equiv \forall x \in D, \sim Q(x) \rightarrow \sim P(x)$
- But not to its converse or its inverse

Universal necessity and sufficiency

Let $P(x) = x > 10$ and $Q(x) = x^3 > 1000$

$\forall x \in D, P(x) \rightarrow Q(x)$ If $x > 10$ then $x^3 > 1000$

$\forall x \in D, P(x) \rightarrow Q(x)$ means that

- $\forall x \in D, P(x)$ is a sufficient condition for $Q(x)$
  $x > 10$ is sufficient for $x^3 > 1000$
  but so are lots of other conditions $x > 98$ $17 \leq x \leq 28$
- $\forall x \in D, Q(x)$ is a necessary condition for $P(x)$
  $x^3 > 1000$ is necessary for $x > 10$
  but so are lots of other conditions $x^3 > 1001$ $\sqrt{x} > \sqrt{100}$
- $P(x)$ only if $Q(x)$ $x > 10$ only if $x^3 > 1000$

Existential quantification

- Existential quantifier $\exists$ for $P(x)$ with domain D
  - $\exists$ is read "there exists" or "for some"
  - $\exists$ indicates that some element of D is in P's truth set
  - $\nexists$ is read "there does not exist"
- Existential statement $\exists x \in D, P(x)$ states that some element in D makes P true
  - $\exists x \in D, P(x)$ is true iff for some $x \in D$ $P(x)$ is true
  - $\exists x \in D, P(x)$ is false iff no $x \in D$ makes $P(x)$ true

Let H be the set of all Hunter students and $Sleepy(x)$ mean "x is sleepy"
$\exists x \in H, Sleepy(x)$ means "Some Hunter student is sleepy "

- D is important here Consider $Q(x)$ means "$x < 0$"
  If $D = \mathbf{N}, \exists x \in D, Q(x)$ is false, but if $D = \mathbf{Q}, \exists x \in D, Q(x)$ is true
- For any finite D, can prove $\exists x \in D, P(x)$ by the method of exhaustion = substitute each $x \in D$ into P **until** some resultant statement is true
  $\exists x \in \mathbf{Z}, P(x)$ where $P(x)$ means "$x = x^3$ but $x \neq x^2$"

Logic to English

Which of these is a correct translation of $\forall x \in B, x \in T$?
Let $B = \{x | x$ is a basketball player$\}$
Let $T = \{x | x$ is a tall person$\}$

a. Every basketball player is tall.	correct
b. Among all the basketball players, some are tall.	$\exists x \in B, x \in T$
c. Some tall people are basketball players.	$\exists x \in T, x \in B$
d. Anyone who is tall is a basketball player.	$\forall x \in T, x \in B$
e. All people who are basketball players are tall.	correct
f. Anyone who is a basketball player is a tall person.	correct

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English to logic

Let $E = \{x | x$ is an equilateral triangle$\}$
Let $I = \{x | x$ is an isosceles triangle$\}$
$T = \{x | x$ is a triangle$\}$
All equilateral triangles are isosceles.
$\forall x \in T, x \in E \rightarrow x \in I$
$\forall x \in E, x \in I$

Let $H = \{x | x$ is a hatter$\}$
Let $M = \{x | x$ is mad$\}$
Some hatters are mad
$\exists x \in H, x \in M$
$\exists x\ x \in H \wedge x \in M$

If $E'(x)$ means "x is an equilateral triangle"
If $I'(x)$ means "x is an isosceles triangle"
All equilateral triangles are isosceles.
$\forall x, E'(x) \rightarrow I'(x)$

If $H'(x)$ means "x is a hatter"
If $M'(x)$ means "x is mad"
Some hatters are mad
$\exists x\ H'(x) \wedge M'(x)$

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Uniqueness quantification

∋ means "such that"

- Uniqueness quantifier ∃! for $P(x)$ with domain D
  - ∃! is read "there exists exactly one" of
  - Indicates that a single element of D is in P's truth set

  $\exists! x \in D, P(x)$ where $D = \{d | d \text{ is a dog}\}$, $P(x)$ means "I like"
  I like exactly 1 dog
  There is only 1 dog I like
- $\exists! x\ P(x) \equiv \exists x \in D \ni P(x) \wedge \forall y\ P(y) \rightarrow (x = y)$

= is not an operator in predicate calculus

- Uniqueness proof for $\exists!\ P(x)$ **always has 2 parts**
  - $\exists x \in D, P(x)$
  - For $a, b \in D$ when $P(a)$ and $P(b)$, then $a = b$

  $\exists! x \in \boldsymbol{Z} \ni P(x)$ where $P(x)$ means "$x = x^3$ but $x \neq x^2$"
  There is only 1 integer that is equal to its cube not equal to its square

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Vacuous truth

$\forall x \in D, P(x) \rightarrow Q(x)$ is vacuously true (aka true by default)
iff $\forall x \in D, \sim P(x)$

Let $P(n)$ = if $n > 1$ then $n^2 > n$ with $D = \boldsymbol{Z}$
Is $P(0)$ true?
$P(0)$ = if $0 > 1$ then $0^2 > 0$
Because $0 > 1$ is false, $P(0)$ is true why?

$P(n)$ = if $10 \leq n \leq 15$ is a perfect square, then n is a perfect cube with $D = \boldsymbol{Z}$
There is no such n, so $P(n)$ is true

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Translation

- Read carefully, the quantifier may be implicit

  if $x < 0, x^3 < 0$ is really $\forall x \in \boldsymbol{Z}^- \; x^3 < 0$

Highly recommended: Chapter 3's translation examples

3.1.5 — 3.1.9

3.2.1 — 3.2.3

3.3.3 — 3.3.6

Any questions?

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Today's outline

✓ Not all arguments are valid

✓ Basic predicate calculus

- Negation and multiple quantifiers in predicate calculus

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Negation of a universal statement

Logical equivalence $\equiv$ in FOPC means that the statement is true for any predicates with their associated domains

In domain $D = \{x | x \text{ is a person}\}$

$P(x)$ means "x loves ice cream" $Q(x)$ means "x loves cake"

Universal statement: $\forall x \in D, P(x)$ Everyone loves ice cream

Negated universal: $\sim[\forall x \in D, P(x)] \equiv [\exists x \in D \ni \sim P(x)]$

Not everyone loves ice cream $\equiv$ Someone does not love ice cream

Universal conditional: $\forall x \in D, P(x) \rightarrow Q(x)$

Anyone who loves ice cream also loves cake

Negated universal conditional

$\sim[\forall\, x \in D, P(x) \rightarrow Q(x)] \equiv [\exists x \in D \ni \sim(P(x) \rightarrow Q(x))]$ why?

$\equiv [\exists x \in D \ni \sim(\sim P(x) \vee Q(x))]$ why?

$\equiv [\exists x \in D \ni \sim\sim P(x) \wedge \sim Q(x)]$ why?

$\equiv [\exists x \in D, \ni P(x) \wedge \sim Q(x)]$ why?

It is not true that anyone who loves ice cream also loves cake $\equiv$

Someone loves ice cream but does not love cake

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Negation of an existential statement

In domain $D = \{x | x \text{ is a person}\}$

$P(x)$ means "likes cats" $Q(x)$ means "likes dogs"

Existential statement: $\exists x \in D \ni P(x)$ Someone likes cats

Negated existential: $\sim[\exists x \in D \ni P(x)] \equiv [\forall x \in D,\ \sim P(x)]$

There is not someone who likes cats $\equiv$ Everyone dislikes cats

Existential conditional: $\exists x \in D \ni P(x) \rightarrow Q(x)$

There is someone who, if they like cats, necessarily likes dogs

Negated existential conditional

$\sim[\exists x \in D, \ni P(x) \rightarrow Q(x)] \equiv [\forall x \in D, \sim(P(x) \rightarrow Q(x))]$ why?

$\equiv [\forall x \in D, \sim(\sim P(x) \vee Q(x))]$ why?

$\equiv [\forall x \in D, \sim\sim P(x) \wedge \sim Q(x)]$ why?

$\equiv [\forall x \in D, P(x) \wedge \sim Q(x)]$ why?

It is not true that there is someone who, if they like cats also likes dogs

$\equiv$ Everyone likes cats but dislikes dogs.

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Proving quantified statements

- To prove $\exists x \in D \ni P(x)$, find an element in D for which P is true

  There is an integer $n > 5$ such that 2^n-1 is prime
- First, be suspicious

  For all real numbers a and b, if $a < b$ then $a^2 < b^2$

  $-3 < 2$ but $(-3)^2 > 2^2$

Proof methods (so far)

Truth table

Sequence of statements with reasons

Valid argument forms (modus ponens, modus tollens,…)

Method of exhaustion

What you should know

- ⭐ **Predicate calculus extends logical representation**
- ⭐ **Validity ≠ truth**
- Invalid argument forms
- How to express facts in predicate calculus
- How to translate both ways between English and predicate calculus
- How to negate quantified statements
- How to demonstrate uniqueness

Next up: *More on predicate calculus*

Any questions?

Time to finish up that Opening sheet!

Problem set 5,6 is due on Thursday, September 21 at 11PM