Discrete Structures

Lecture 12: Recursion

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HUNTER

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Last time

- ★ **Strong mathematical induction is a special case of mathematical induction**
- Strong mathematical induction relies on a sequence of consecutive integers for its basis
- Strong mathematical induction can reference any of its bases during proof
- Multiple ways to construct a proof

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Today's outline

- Sequences and recursion
- Solution of recurrence relations by iteration

Factorial

For any $n \in \boldsymbol{Z}^{+}$, $n!$ (read n factorial) is the product of all positive integers $\leq n$

$$n! = \prod_{i=1}^{n} i = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1$$

- By convention, 0! = 1

$$\frac{10!}{9!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 10 = \frac{10(9!)}{9!}$$

$$\frac{n!}{(n-1)!} = \frac{n((n-1)!)}{(n-1)!} = n$$

$$n! = n(n-1)!$$

$$\frac{(n+2)!}{n!} = \frac{(n+2)(n+1)n!}{n!} = (n+2)(n+1)$$

Recurrence relations

- In the sequence $a_0, a_1, a_2, \ldots, a_n$ the predecessors of term a_k are those that come before it $a_0, a_1, a_2, \ldots, a_{k-1}$ where $i \in \boldsymbol{Z}, k - i \geq 0$
- Recurrence relation for a sequence $a_1, a_2, \ldots, a_n$ is a formula that defines each term through its predecessors
  $a_k = 5a_{k-1}$ ∀ integers $k \geq 2$
- Initial conditions for a recurrence relation specify either
  - if $i \in \boldsymbol{N}$ is fixed, values of $a_0, a_1, a_2, \ldots, a_{i-1}$
  - if i depends on k, values of $a_0, a_1, a_2, \ldots, a_m$ for $m \in \boldsymbol{N}$, $m \geq 0$

  ∀ integers $k \geq 2$, define sequence $c_0, c_1, c_2, \ldots$ with $c_k = c_{k-1} + kc_{k-2} + 1$ and $c_0 = 1, c_1 = 2$

  $c_2 = c_{2-1} + 2c_0 + 1 = 2 + 2(1) + 1 = 5$

  $c_3 = c_{3-1} + 3c_{3-2} + 1 = 5 + 3(2) + 1 = 12$

  so sequence begins 1,2,5,12,33,…
- Different initial conditions can generate different sequences even with the same recurrence relation

∀ integers $k \geq 2, a_k = 5a_{k-1}$ with $a_1 = 2$ yields 2,10,50,250, … for $a_1, a_2, \ldots$ while ∀ integers $k \geq 2$, $b_k = 5b_{k-1}$ with $b_1 = 3$ yields 3,15,45,135, … for $b_1, b_2, \ldots$

Recursive definition for addition

- Summation for any $m \in \boldsymbol{Z}$, $\sum_{k=m}^{m} a_k = a_m$
  and for $n > m$ $\sum_{k=m}^{n} a_k = \sum_{k=m}^{n-1} a_k + a_n$

  $\sum_{i=1}^{n+1} \frac{1}{i^3} = \sum_{i=1}^{n} \frac{1}{i^3} + \frac{1}{(n+1)^3}$ $\quad$ $\sum_{k=1}^{n} 3^k + 3^{n+1} = \sum_{k=1}^{n+1} 3^k$
- Note that this definition specifies an order for computation
- Algebra often simplifies computation before you code

Because $\frac{1}{k} - \frac{1}{k+1} = \frac{(k+1)-k}{k(k+1)} = \frac{1}{k(k+1)}$ we can simplify $\sum_{k=1}^{n} \frac{1}{k(k+1)} =$

$$\sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$$

Eliminating an addition loop entirely!

Recursive definition for multiplication

- Multiplication for any $m \in \boldsymbol{Z}$, $\prod_{k=m}^{m} a_k = a_m$ and
  and for $n > m$ $\prod_{k=m}^{n} a_k = \left(\prod_{k=m}^{n-1} a_k\right) \cdot a_k$
  $\prod_{k=2}^{4} k = 2 \cdot 3 \cdot 4 = 24$ $\prod_{i=1}^{1} \frac{i}{i+10} = \frac{1}{11}$
- Note that this definition also specifies an order for computation
- Factorial was defined as
  $$n! = \prod_{i=1}^{n} i = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1$$
  but it also has a recursive definition
  $$n! = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot (n-1)! & \text{if } n \geq 1 \end{cases}$$

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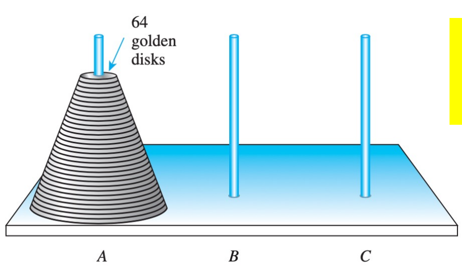
The recursive paradigm

Break down a problem into smaller, easier to solve subproblems and then combine their answers to make a solution to the original problem

The Tower of Hanoi

On the steps of the altar in the temple of Benares, for many, many years Brahmins have been moving a tower of 64 golden disks from one pole to another; one by one, never placing a larger on top of a smaller. When all the disks have been transferred the Tower and the Brahmins will fall, and it will be the end of the world.



How many moves would that take?
Hint: Think recursively!

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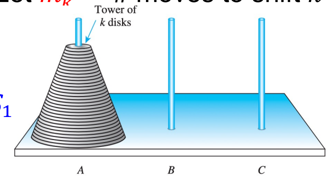
How to shift from pole A to pole C

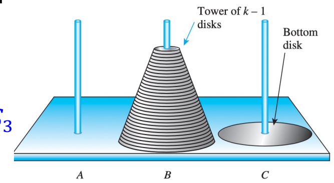
Let m_k = # moves to shift k disks from one pole to another

S_1



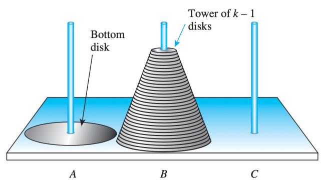
$S_1 \rightarrow S_2$ takes m_{k-1} moves

S_3



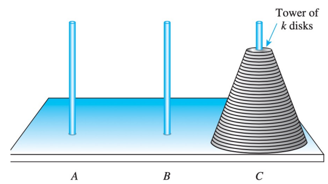
$S_3 \rightarrow S_4$ takes m_{k-1} moves again

S_2



$(S_2) \rightarrow (S_3)$ takes 1 move

S_4



Total most efficient is $m_k = m_{k-1} + 1 + m_{k-1} = 2\, m_{k-1} + 1$ moves

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Tower of Hanoi recursively

$m_k = 2\, m_{k-1} + 1$ where $m_1 = 1$

The terms of the sequences: $1,3,7,15,31,63$ …

```
def TowerOfHanoi(n, source, destination, intermediate):
    if n == 1:
        print("Move disc 1 from pole", source,"to pole", destination)
        return
    TowerOfHanoi(n-1, source, intermediate, destination)
    print("Move disc",n,"from pole", source," intermediate ", destination)
    TowerOfHanoi(n-1, intermediate, destination, source)
```

Worried about the world ending? Call TowerOfHanoi(64, a, b, c):

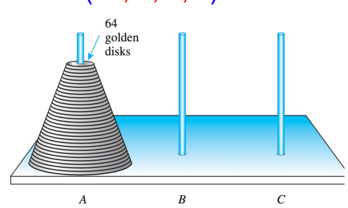
If they move 1 disk / second.

$m_{64} \cong 1.844674 \times 10^{19}$ seconds

$\cong 5.84542 \times 10^{11}$ years

$\cong 584.5$ billion years

The universe is 13.8 ± 0.059 billion years old



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Fibonacci numbers [1202]

A pair of (one male, one female) rabbits is born on 1/1.
Rabbits are not fertile during their first month of life.
Then they give birth to 1 new male/female pair at the end of each month.
Rabbits do not die.
How many rabbits will there be on 12/31?

Recursion! Note that rabbits born in month $k-2$ do not add to the population until month k.
Let F_n = # pairs alive at end of month n
$F_0 = 1$, $F_1 = 1$ and $F_k = F_{k-1} + F_{k-2}$ for all integers $k \geq 2$.
What is F_{12}?
$1,1,2,3,5,8,13,21,34,55,89,144,233, \ldots$
Counting pairs so there will be 466 rabbits.

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Recursion for $$$

The day you were born your fairy godmother invested $100K at 4% for you.

Today you are 21 and if you can tell her the current value you can have it!

The interest is compounded annually.

Let A_n = amount of money at end of year n

A_0 initial amount of money =100K

A_1 amount at end of year $1 = 100K + 0.04(100K) = 100K(1.04)$

A_2 amount at end of year $2 = 100K(1.04)^2$

A_3 amount at end of year $3 = 100K(1.04)^3$

$\ldots$

A_{21} amount at end of year $21 = 100K(1.04)^{21} \cong \$227{,}876.81$

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Interest more generally…

Interest is often paid more often than annually, say k times a year.

An annual rate of $r\%$ paid k times a year accrues at rate $\frac{r}{k}$.

3% compounded quarterly pays $\frac{0.03}{4} = 0.0075$

Let P_k = money on deposit at the end of the kth period $k, k \geq 1$.

Recursively, $P_k = P_{k-1} + P_{k-1}\frac{r}{k} = P_{k-1}\left(1 + \frac{r}{k}\right)$

If you deposit \$10K for a year with 3% interest compounded quarterly, you will have

$$\begin{aligned} P_4 &= 1.0075\, P_3 \\ &= 1.075^2 P_2 \\ &= 1.075^3 P_1 \\ &= 1.075^4 P_0 \cong 10{,}303.39 \end{aligned}$$

an effective interest rate of $.030339 = 3.0339\%$

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Today's outline

✓ Sequences and recursion

- Solution of recurrence relations by iteration

A recursive solution may speed computation

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Motivation

Although a recurrence relation may be relatively simple to understand, its computation can become burdensome

A skydiver's speed increases at about 32.1522 feet/second as they fall.
If there were no air resistance how fast would they be falling in 2 minutes?

$a_k = a_{k-1} + 32.1522$ ft/sec $k \in \boldsymbol{N}, \boldsymbol{k} \geq \mathbf{1}$

You could compute 120 terms of the sequence (0, 32.1522, 64.3044,…)
In 2 minutes they would be falling about 44 miles/hour…

Observe, however, that if you repeatedly substitute into a_4

$a_4 = a_3 + 32.1522$, $a_3 = a_2 + 32.1522$, $a_2 = a_1 + 32.1522$, $a_1 = 32.1522$

$a_0 = 0$

$a_4 = a_2 + 32.1522 + 32.1522 = a_1 + 32.1522 + 32.1522 + 32.1522 =$

$a_0 + 32.1522 + 32.1522 + 32.1522 + 32.1522 =$

$a_0 + 4(32.1522)$

It looks like $a_k = a_0 + k \cdot 32.1522\ldots$

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Solution for an arithmetic sequence

Solution of a recursively defined sequence = an explicit non-recursive form for its terms

- Consider the recurrence relation $a_k = a_{k-1} + b$ for $k \in \boldsymbol{N}, k \geq 0$
- The terms of the arithmetic sequence $a, a + b, a + 2b, \ldots$ can be computed as $a_k = a_0 + (k - 1)b$ for, $k \geq 0$
- This is also the solution to the recurrence relation
- Induction on recurrence relations allows us to prove equations like

$$\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

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Solution for a geometric sequence

Recall a geometric sequence $a, ar, ar^2, \ldots$ has formula $a_k = ar^k$ for $k \geq 0$
The day you were born your fairy godfather invested $100 at 4% for you. The interest is compounded annually. How much is it worth on your 4th birthday?
Again, by repeated substitution into a_4
$a_4 = a_3(1.04),\ a_3 = a_2(1.04),\ a_2 = a_1(1.04),\ a_1 = a_0(1.04),\ a_0 = 100$
$a_4 = a_2(1.04)(1.04) = a_1(1.04)(1.04)(1.04) =$
$a_0(1.04)\,(1.04)(1.04)(1.04) = 100(1.04)^4$
$a_k = a_0(1.04)^4\ k \in \boldsymbol{N}, \boldsymbol{k} \geq \mathbf{1}$

It looks like the terms of the geometric sequence $a, ar, ar^2, \ldots$ can be computed directly as $a_k = a_0 r^k$ for, $k \geq 0$

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Proof skeleton

Theorem: (copy the statement here)
Proof:
Let/Assume/Suppose: Name variables and state what they stand for
be general: any state any assumptions
We must show that…
multiple grammatically correct sentences
Clarify your logic with a reason for every assertion Thus Then
Therefore So Hence Consequently It follows that
By definition of By substitution Because Since
Display equations and inequalities clearly
QED

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Sum of an arithmetic sequence

Theorem: The sum of the first n terms of an arithmetic sequence defined by $(a, a+b, a+2b, \ldots)$ is $an - bn + \frac{nb(n-1)}{2}$.

Proof (algebraic):

We must show that $\sum_{i=1}^{n}(a+(i-1)b) = an - bn + \frac{nb(n-1)}{2}$.

$$\sum_{i=1}^{n}(a+(i-1)b) = \sum_{i=1}^{n}((a-b)+ib) = \sum_{i=1}^{n}(a-b) + \sum_{i=1}^{n-1} ib$$

Since $(a-b)$ and b are constants, the left term is $n(a-b)$.

The sum of the first n integers was proved in Slide set 10 to be $\frac{n(n+1)}{2}$, so the right term is $b\sum_{i=1}^{n-1} i = b\frac{(n-1)n}{2}$.

By substitution $\sum_{i=1}^{n}(a+(i-1)b) = n(a-b) + \frac{nb(n-1)}{2}$

QED

See your text for a proof by induction

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Sum of a geometric sequence

Theorem: The sum of the first n terms of a geometric sequence defined by $(a, ar, ar^2, \ldots)$ is $a\frac{r^n-1}{r-1}$ for any $r \neq 1$ and integer $n \geq 0$.

Proof (algebraic):

We must show that $\sum_{i=0}^{n-1} ar^i = a\frac{r^n-1}{r-1}$.

Since a is a constant, $\sum_{i=0}^{n-1} ar^i = a\sum_{i=0}^{n-1} r^i$

Since $\sum_{i=1}^{n} r^i = \frac{r^{n+1}-1}{r-1}$ was proved in slide set 11,

by substitution $\sum_{i=0}^{n-1} ar^i = a\sum_{i=0}^{n-1} r^i = a\frac{r^n-1}{r-1}$..

QED

What if $r = 1$? Why?

$$\sum_{i=1}^{n} 1^i = ?$$

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Solution for the Tower of Hanoi recurrence

Recall $m_k = 2m_{k-1} + 1$ where $m_1 = 1$

$m_4 = 2m_3 + 1, m_3 = 2m_2 + 1, m_2 = 2m_1 + 1, m_1 = 1$

$m_4 = 2(2m_2 + 1) + 1 = 4m_2 + 2 + 1 = 4(2m_1 + 1) + 2 + 1 =$

$8m_1 + 4 + 2 + 1 = 8 + 4 + 2 + 1$

$m_k = \sum_{i=0}^{k-1} 2^i$ which is the sum of a geometric series so

It looks like $m_k = \frac{2^n - 1}{2-1} = 2^k - 1$

Confirmation: look back at slide 10 where the sequence were calculated as 1,3,7,15,31,63 ...

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Proof by mathematical induction skeleton

Theorem: Let $P(n)$ be (copy $P(n)$ here)
Proof by mathematical induction:
We must show that $P(n)$ is true for all $n \geq$ (state the basis value here)

Basis: Prove some initial case $P(b)$ is true (often but not always, $P(1)$

Inductive step: Assume for some k that $P(k)$is true.

By substitution (state $P(k+1)$ here).
We must show that $P(k+1)$ is true.
(prove that $P(k+1)$ is true)

Since we have proved the basis step and the inductive step, the theorem is true.

QED

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Inductive proof of a recurrence solution

Theorem: Let $P(n)$ be that if $m_1, m_2, m_3, \ldots$ is the sequence defined by $m_k = 2m_{k-1} + 1$ for all integers $k \geq 1$, and $m_1 = 1$, then $m_n = 2^n - 1$ for all integers $n \geq 1$.

Proof by mathematical induction:

We must show that $P(n)$ is true for all $n \geq 1$.

Basis: $P(1)$ We must show that $m_1 = 2^1 - 1$. The left side, m_1, is defined as 1 and the right side is calculated as $2 - 1 = 1$, so $P(1)$ is true.

Inductive step: Assume for some k that $P(k)$ = if $m_1, m_2, m_3, \ldots$ is the sequence defined by $m_k = 2m_{k-1} + 1$ for all integers $k \geq 1$, and $m_1 = 1$, then $m_k = 2^k - 1$ for all integers $n \geq 1$. By substitution $P(k+1)$ is if $m_1, m_2, m_3, \ldots$ is the sequence defined by $m_k = 2m_{k-1} + 1$ for all integers $k \geq 1$, and $m_1 = 1$, then $m_{k+1} = 2^{k+1} - 1$ for all integers $n \geq 1$.

We must show that $P(k+1)$ is true. By definition of the recurrence relation, $m_{k+1} = 2m_k + 1 = 2(2^k - 1) + 1 = 2^{k+1} - 2 + 1 = 2^{k+1} - 1$ which is the right side of $P(k+1)$.

Since we have proved the basis step and the inductive step, the theorem is true. **QED**

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What you should know

Any questions?

A recursive solution may speed computation

- How to interpret and use recursive definitions
- Famous examples of recursion
- How to solve a recurrence relation by iteration

Next up: Introduction to set theory

Time to finish up that Opening sheet!

Problem set 11,12 is due on Monday, October 23 at 11PM

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