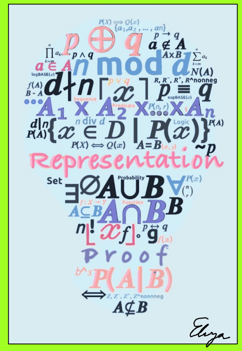
Discrete Structures



Lecture 10: Induction

Susan L. Epstein

HUNTER

1

Last time

★ **Real datasets often have exploitable patterns**

- Why $\sqrt{2}$ is irrational
- Why there are infinitely many primes
- How to represent and manipulate finite and infinite sequences

We can prove some powerful results this way…

2

Today's outline

- Induction for summation
- Induction for other kinds of statements

Induction derives its power from N

Fall 2023 CSCI 150 3/21

3

Proofs and infinite sets

Induction is a proof method for infinite sets that have exploitable regularity

Z an infinite set of logical statements

- Pushing the first domino is guaranteed
- Pieces are close enough to cause a chain reaction

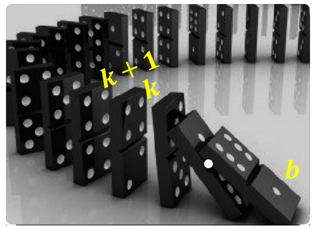
Fall 2023 CSCI 150 4/21

4

Principle of mathematical induction

Let $P(n)$ be a property that is defined for integers n, and let b be a fixed integer. Suppose the following two statements are true:

- $P(b)$ is true b is called the basis
- For all integers $k \geq b$, if $P(k)$ is true then $P(k+1)$ is true



this inductive hypothesis assumes $P(k)$ and works to show that $P(k+1)$ is true

https://depositphotos.com/574666276/stock-video-black-dominoes-falling-row-slow.html

Then the statement "For all integers $n \geq b, P(n)$" Is true.

5

Infinitely many numbered statements

Let $P = \{P(b), P(b+1), \dots, P(k), P(k+1), \dots\}$ be an infinite set of sequentially numbered statements

$P(n) =$ nth statement **to be proved** P_n: $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

$P(b) =$ first statement **to be proved** if $b = 1$ P_1: $1 = \frac{1(1+1)}{2}$

$P(k) =$ kth statement where $n = k$ **assumed** P_k: $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$

$P(k+1) =$ $k+1$st statement where $n = k+1$ **to be proved**

$P(k+1)$: $1 + 2 + \cdots + k + k + 1 = \frac{(k+1)(k+2)}{2}$

To prove: $P(n)$: $\sum_{i=1}^{n} i^2$

Show $b = 1, P(b)$: $\sum_{i=1}^{1} i^2$

Assume $P(k)$: $\sum_{i=1}^{k} i^2$

Show $P(k+1)$: $\sum_{i=1}^{k+1} i^2$

To prove: $P(n)$: $\sum_{i=1}^{n} r^i = \frac{r^{n+1}-1}{r-1}$

Show $b = 1, P(b)$: $\sum_{i=1}^{1} r^i = \frac{r^{1+1}-1}{r-1}$

Assume $P(k)$: $\sum_{i=1}^{k} r^i = \frac{r^{k+1}-1}{r-1}$

Show $P(k+1)$: $\sum_{i=1}^{k+1} r^i = \frac{r^{k+2}-1}{r-1}$

6

Skeleton for a mathematical induction proof

Theorem: Let $P(n)$ be (copy $P(n)$ here)
Proof by mathematical induction:
We must show that $P(n)$ is true for all $n \geq$ (state the basis value here)

Basis: Prove some initial case $P(b)$ is true (often but not always, $P(1)$)

Inductive step: Assume for some k that $P(k)$ is true.
By substitution (state $P(k+1)$ here).
We must show that $P(k+1)$ is true.
(prove that $P(k+1)$ is true)

SECRET: use $P(k)$ to prove $P(k+1)$

Since we have proved the basis step and the inductive step, the theorem is true.

QED

Fall 2023 CSCI 150 7/21

7

Setting up for induction

For each non-negative integer n, let $P(n)$ be the property $5^n - 1$ is divisible by 4.

- What is $P(0)$?
  $4|5^0 - 1$
- What is $P(k)$?
  $4|5^k - 1$
- What is $P(k+1)$?
  $4|5^{k+1} - 1$
- In any proof by mathematical induction what must be shown in the inductive step?
  That if $P(k)$ is true, then $P(k+1)$
  if $4|5^k - 1$ then $4|5^{k+1} - 1$

Fall 2023 CSCI 150 8/21

8

Example 1 (an arithmetic sum)

Theorem: Let $P(n)$ be $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Proof by mathematical induction:
We must show that $P(n)$ is true for all $n \geq 1$.

This is NOT algebra. You may only **work on 1 side at a time**.

Basis: $P(1)$ is $\sum_{i=1}^{1} i \overset{?}{=} \frac{1(1+1)}{2}$

Since the left side is 1 and the right side is $\frac{1(2)}{2} = \frac{2}{2} = 1$, $P(1)$ is true.

Inductive step: Assume for some k that $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$ is true.

By substitution $P(k+1)$: $\sum_{i=1}^{k+1} i = \frac{(k+1)(k+1+1)}{2} = \frac{(k+1)(k+2)}{2}$

We must show $P(k+1)$ is true. By definition of Σ, the left side of P_{k+1} is $\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{k(k+1)+2(k+1)}{2} = \frac{k^2+k+2k+2}{2} = \frac{k^2+3k+2}{2} = \frac{(k+1)(k+2)}{2}$ which is the right side of $P(k+1)$.

Since we have proved the basis step and the inductive step, the theorem is true. **QED**

How can you use $P(k)$ to prove $P(k+1)$?

Fall 2023 CSCI 150 9/21

9

Geometric series

$\forall r \in \boldsymbol{R},\ r^0 = 1$
$0^0 = 1$

- Recall: Geometric sequence $(a, ar, ar^2, \ldots)$ is defined by a first term a and a ratio r

This is different from your text.

- Sum of first n terms in a geometric sequence is $\sum_{i=0}^{n} ar^i = a\frac{r^{n+1}-1}{r-1}$

$$\sum_{i=0}^{4} 5^i = 1 + 5 + 5^2 + 5^3 + 5^4 + 5^5 = \frac{5^5 - 1}{5 - 1}$$

$$\sum_{i=2}^{6} 2^i = 2^2\left(\sum_{i=0}^{4} 2^i\right) = 4\,\frac{2^{4+1}-1}{2-1} = 4(2^5 - 1)$$

$$\sum_{i=2}^{6} 2^i = \left(\sum_{i=0}^{6} 2^i\right) - 1 - 2 = \frac{2^{6+1}-1}{2-1} - 3$$

- Can use the same formula for similar sequences

For $t \geq 2$, $1 + 4 + 4^2 + \ldots + 4^{t+2} = \frac{4^{t+2+1}-1}{4-1} = \frac{4^{t+3}-1}{3}$

For $w \geq 4$, $5^2 + 5^3 + 5^4 + \ldots + 5^w = 5^2(1 + 5^1 + 5^2 + \ldots + 5^{w-2})$

$= 25\left(\frac{5^{w-2+1}-1}{5-1}\right) = 25\left(\frac{5^{w-1}-1}{4}\right)$

Fall 2023 CSCI 150 10/21

10

Proof by mathematical induction skeleton

Theorem: Let $P(n)$ be (copy $P(n)$ here)
Proof by mathematical induction:
We must show that $P(n)$ is true for all $n \geq$ (state the basis value here)

Basis: Prove some initial case $P(b)$ is true (often but not always, $P(1)$)

Inductive step: Assume for some k that $P(k)$is true.
By substitution (state $P(k+1)$ here).
We must show that $P(k+1)$ is true.

How can you use $P(k)$ to prove $P(k+1)$?

(prove that $P(k+1)$ is true)

Since we have proved the basis step and the inductive step, the theorem is true.

QED

Fall 2023 CSCI 150 11/21

11

Example 2 (geometric sum)

This is different from your text.

Theorem: Let $P(n)$ be P_n: $\sum_{i=0}^{n} ar^i = a\frac{r^{n+1}-1}{r-1}$ for any $r \neq 1$ and integer $n \geq 0$.
Proof by mathematical induction:
We must show that $P(n)$ is true for all $n \geq 0$.

This is NOT algebra. You may only **work on 1 side at a time**.

Basis: $P(0)$ is $\sum_{i=0}^{0} ar^i \stackrel{?}{=} a\frac{r^{0+1}-1}{r-1}$

Since the left side is $ar^0 = a$ and the right side is $a\frac{r^1-1}{r-1} = a$, $P(0)$ is true.

Inductive step: Assume for some k that $\sum_{i=0}^{k} ar^i = a\frac{r^{k+1}-1}{r-1}$ is true.

By substitution $P(k+1) = \sum_{i=0}^{k+1} ar^i \stackrel{?}{=} a\frac{r^{k+2}-1}{r-1}$

We must show $P(k+1)$ is true. By definition of Σ, the left side of
$P(k+1)$ is $\sum_{i=0}^{k+1} ar^i = \sum_{i=0}^{k} ar^i + (ar^{k+1}) = a\frac{r^{k+1}-1}{r-1} + ar^{k+1} = a(\frac{r^{k+1}-1}{r-1} + r^{k+1})$
$a\frac{r^{k+1}-1+(r-1)r^{k+1}}{r-1} = a\frac{r^{k+1}-1+r^{k+2}-ar^{k+1}}{r-1} = a\frac{r^{k+2}-1}{r-1}$ = the right side of $P(k+1)$.

Since we have proved the basis step and the inductive step, the theorem is true.

QED

How can you use $P(k)$ to prove $P(k+1)$?

Fall 2023 CSCI 150 12/21

12

Example 3

Theorem: Let $P(n)$ be n cents can be obtained using some number of 3-cent and 5-cent coins for $n \geq 8$.
Proof by mathematical induction:
We must show that $P(n)$ is true for all $n \geq 8$
Basis: $P(8)$ is 8 cents can be obtained using 3-cent and 5-cent coins. Take 1 3-cent and 1 5-cent coin. Since $3 + 5 = 8$, $P(8)$ is true.
Inductive step: Assume for some k that $P(k)$ is true, that is, $\exists\, t, f \in N \ni 3t + 5f = k$. By substitution $P(k+1)$ cents can be obtained using some number of 3-cent and 5-cent coins, that is, $\exists\, t', f' \in N \ni 3t' + 5f' = k + 1$.
We must show $P(k+1)$ is true. There are 2 cases: $f \neq 0$ and $f = 0$.
Case 1: If $f \neq 0$, since $1 = 2 \cdot 3 - 5, k + 1 = 3t + 5f + (2 \cdot 3 - 5) = 3(t + 2) + 5(f - 1)$, so $t' = t + 2$ and $f' = f - 1$. $P(k+1)$ is true for $f \neq 0$.
Case 2: If $f = 0$ then $k = 3u$ for some $u \in N$, $u \geq 3$. But $1 = 2 \cdot 5 - 3 \cdot 3$, so $k + 1 = 3t + 5(0) + (2 \cdot 5 - 3 \cdot 3) = 3(t - 3) + 5(2)$ so $t' = t - 3$ and $f' = 2$. $P(k+1)$ is true for $f = 0$.
Since we have proved the basis step and both cases in the inductive step, the theorem is true.
QED

How can you use $P(k)$ to prove $P(k+1)$?

Fall 2023 CSCI 150 13/21

13

Recognizing the need for induction

- Observe that

$$\frac{1}{1 \cdot 3} = \frac{1}{3}$$
$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} = \frac{2}{5}$$
$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} = \frac{3}{7}$$
$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \frac{1}{7 \cdot 9} = \frac{4}{9}$$

- What's the general formula?

$$\sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}$$

- Can you prove it?

Fall 2023 CSCI 150 14/21

14

Example 4

This is NOT algebra.
You may only
work on 1 side at a time.

Theorem: Let $P(n)$ be $\sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}$

Proof by mathematical induction:

We must show that $P(n)$ is true for all $n \geq 1$.

Basis: $P(1)$ is $\sum_{i=1}^{1} \frac{1}{(2i-1)(2i+1)} = \frac{1}{2\cdot 1+1}$

Since the left side is $\frac{1}{1\cdot 3} = \frac{1}{3}$ and the right side is $\frac{1}{2\cdot 1+1} = \frac{1}{3}$, $P(1)$ is true.

Inductive step: Assume for some k that $\sum_{i=1}^{k} \frac{1}{(2i-1)(2i+1)} = \frac{k}{2k+1}$ is true.

By substitution $P(k+1)$: $\sum_{i=1}^{k+1} \frac{1}{(2i-1)(2i+1)} \stackrel{?}{=} \frac{k+1}{2(k+1)+1} = \frac{k+1}{2k+3}$

We must show $P(k+1)$ is true. By definition of Σ, the left side of $P(k+1)$ is $\sum_{i=1}^{k+1} \frac{1}{(2i-1)(2i+1)} = \sum_{i=1}^{k} \frac{1}{(2i-1)(2i+1)} + \frac{1}{(2(k+1)-1)(2(k+1)+1)} =$

$\frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} = \frac{k(2k+3)+1}{(2k+1)(2k+3)} = \frac{2k^2+3k+1}{(2k+1)(2k+3)} = \frac{(2k+1)(k+1)}{(2k+1)(2k+3)} = \frac{k+1}{2k+3}$

which is the right side of $P(k+1)$.

Since we have proved the basis step and the inductive step, the theorem is true.

How can you use $P(k)$ to prove $P(k+1)$?

QED

15

Today's outline

- ✓ Induction for summation
- Induction for other kinds of statements

16

Example 5 (divisible)/21

Theorem: Let $P(n)$ be $3|2^{2n} - 1\ \forall n \in N, n \geq 0$.
Proof by mathematical induction:
We must show that P_n is true for all $n \geq 0$

Basis: $P(0)$ is $3|2^{2\cdot 0} - 1$. Since $2^{2\cdot 0} - 1 = 2^0 - 1 = 1 - 1 = 0$ and

$3 \cdot 0 = 0, 3|0$, $P(0)$ is true.
Inductive step: Assume for some k that $P(k)$ is true, that is, $3|2^{2k} - 1$.
By substitution $P(k+1)$ is $3|2^{2(k+1)} - 1$.
We must show $P(k+1)$ is true.

Use those definitions!

Since $3|2^{2k} - 1$, $\exists m \in N \ni 2^{2k} - 1 = 3m$ and $2^{2k} = 3m + 1$.
Now $2^{2(k+1)} - 1 = (2^{2k+2}) - 1 = 2^2(2^{2k}) - 1 = 4(3m + 1) - 1 =$
$12m + 4 - 1 = 12m + 3 = 3(4m + 1)$.
Because the integers are closed under multiplication and addition, $4m + 1$ is an integer and $3|2^{2(k+1)} - 1$ so $P(k+1)$ is true.

Since we have proved the basis step and the inductive step, the theorem is true.

How can you use $P(k)$ to prove $P(k+1)$?

QED

Fall 2023 CSCI 150 17/21

17

Example 6 (inequality)

Theorem: Let $P(n)$ be $2n + 1 \leq 2^n\ \forall n \in N, n \geq 3$.
Proof by mathematical induction:
We must show that $P(n)$ is true for all $n \geq 3$.

Basis: $P(3)$ is $2(3) + 1 \overset{?}{\leq} 2^3$. Since $2(3) + 1 = 7,\ 2^3 = 8$, and $7 \leq 8$,
$P(3)$ is true.
Inductive step: Assume for some $k \in N$ that $P(k)$ is true, that is,
$2k + 1 \leq 2^k$.

How can you use $P(k)$ to prove $P(k+1)$?

We must show $P(k+1)$ is true, that is, $2(k+1) + 1 \overset{?}{\leq} 2^{k+1}$.

On the left side, $2(k+1) + 1 = 2k + 3 < 4k + 3$.

On the right side, multiplying the inductive hypothesis $2k + 1 \leq 2^k$, by 2,
$2(2k+1) \leq 2(2^k) = 2^{k+1}$, that is, $4k + 3 \leq 2^{k+1}$. By transitivity of ≤,
$2(k+1) + 1 \leq 2^{k+1}$, so $P(k+1)$ is true.
Since we have proved the basis step and the inductive step, the theorem is true.

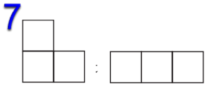
QED

Your text has another version. Take a look!

Fall 2023 CSCI 150 18/21

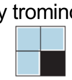
18

Example 7



Tromino = 3 attached same-size squares

Theorem: Let $P(n)$ be for any integer $n \geq 1$, if one square is removed from a $2^n \times 2^n$ checkerboard, it can be completely covered by L-shaped trominoes.
Proof by mathematical induction:
We must show that $P(n)$ is true for all $n \geq 1$.

Basis: $P(1)$ = a $2^1 \times 2^1$checkerboard missing 1 square can be covered completely by trominoes. P_1 is true since all possible such checkerboards are symmetric to 

Inductive step: Assume for some k, $P(k)$ is true, that is, a $2^k \times 2^k$ checkerboard missing 1 square can be covered completely by trominoes.
We must show $P(k+1)$= a $2^{k+1} \times 2^{k+1}$ checkerboard missing 1 square can be covered completely by trominoes is true.

Any ideas?
How can you use P_k to show P_{k+1}?

Any questions?

Proof methods (so far)

Truth table

Sequence of statements with reasons

Valid argument forms (modus ponens, modus tollens,…)

Method of exhaustion

Predicate logic (quantification, existence, uniqueness)

Proof by cases

Generalization from the generic particular

Proof by contradiction

Proof by contraposition

Mathematical induction

What you should know

Induction derives its power from *N*

- Induction is a proof method for infinite sets with cardinality $|N| = \aleph_0$
- Inductive proof has 2 parts: a basis followed by an inductive hypothesis

Next up: *More on induction*

Time to finish up that Opening sheet!

Any questions?

Problem set 9,10 is due on Monday, October 9 at 11PM

From test 1

Provide a reason for each step in the following proof. Reasons must be only the **names** of logical equivalence laws (Theorem 2.1.1)

distributive law

or the **names** of valid argument forms (Table 2.3.1) from your text. Include the relevant line number(s) with each reason.

modus tollens