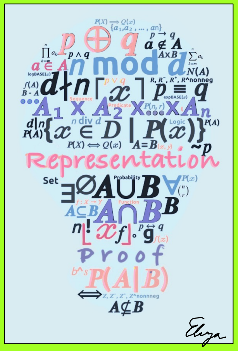
Discrete Structures



Lecture 3: Laws and logical equivalence

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HUNTER

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Last time

★ **Do not invent mathematical notation or language**

- What a proposition is
- Definitions of negation, conjunction, disjunction, and exclusive or
- What conjuncts, disjuncts, tautologies, and contradictions are in formal logic
- How to build and use a truth table

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Today's outline

- Conditional operator
- Laws of propositional equivalence

Logical operator: conditional

- $p \rightarrow q$ denotes the implication (aka conditional proposition) that p implies q

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

A false statement implies anything

- In $p \rightarrow q$, p is the hypothesis and q is the conclusion
- Can read $p \rightarrow q$ as "if p then q"
  p: Mars is made of chocolate q: $2 + 3 = 9$
  $p \rightarrow q$: If Mars is made of chocolate then $2 + 3 = 9$
- **Not** like if-then statement in code, which is executable

Many ways to read a conditional

if p then q
If p, q
q if p
q when p
p implies q
q whenever p

q follows from p
q provided that p
q is necessary for p
a necessary condition for p is q
p is sufficient for q
a sufficient condition for q is p

p: Maria got an A in CSCI 150 q: Maria will find a good job

$p \rightarrow q$: If Maria got an A in CSCI 150 then she will find a good job

For Maria to find a good job it is sufficient that she got an A in CSCI 150

Maria will find a good job if she got an A in CSCI 150

Maria will find a good job unless she did not get an A in CSCI 150

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Only if

- If p then q is written $p \rightarrow q$
- "p only if q" means
  - If q does not happen, then p does not happen
  - If q is not true then p is not true
  - $\sim q \rightarrow \sim p$
- But $\sim q \rightarrow \sim p$ is the contrapositive of $p \rightarrow q$ and so is logically equivalent to it
- Thus "p only if q" is written $p \rightarrow q$ or $\sim q \rightarrow \sim p$

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A really useful logical equivalence

p	$\sim p$	q	$p \to q$	$\sim p \lor q$
T	F	T	T	T
T	F	F	F	F
F	T	T	T	T
F	T	F	T	T

p: Maria got an A in CSCI 150 q: Maria will find a good job
$p \to q$: If Maria got an A in CSCI 150 then she will find a good job

$\sim p \lor q$: Either Maria did not get an A in CSCI 150 or she will find a good job

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Variations on an implication

p: Maria got an A in CSCI 150 q: Maria will find a good job
$p \to q$: If Maria got an A in CSCI 150 then she will find a good job

- Converse of $p \to q$ is $q \to p$ can be read "p only if q" and as "if q then p"

  $q \to p$: If Maria finds a good job then she got an A in CSCI 150

- Inverse of $p \to q$ is $\sim p \to \sim q$

  $\sim p \to \sim q$: If Maria did not get an A in CSCI 150 then she will not find a good job

- Contrapositive of $p \to q$ is $\sim q \to \sim p$

  $\sim q \to \sim p$: If Maria does not find a good job then she did not get an A in CSCI 150


Any questions?

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Equivalent implications

- Must have identical truth table columns
- Which of a conditional proposition and its inverse, converse, and contrapositive are equivalent?

A false statement implies anything

p	$\sim p$	q	$\sim q$	$p \to q$	$\sim p \to \sim q$	$q \to p$	$\sim q \to \sim p$
T	F	T	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	T	F	T	T	T	T	T

- An implication is logically equivalent to its contrapositive

$$(p \to q) \equiv (\sim q \to \sim p)$$

- The converse and inverse of an implication are logically equivalent

$$(q \to p) \equiv (\sim p \to \sim q)$$

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Logical operator: biconditional

- $p \leftrightarrow q$ denotes a biconditional proposition
- Asserts that p and q are logically equivalent

p	q	$p \leftrightarrow q$	$p \to q$	$q \to p$	$(p \to q) \land (q \to p)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

$$p \leftrightarrow q \equiv (p \to q) \land (q \to p)$$

- Can read $p \leftrightarrow q$ as
  - p is necessary and sufficient for q
  - if p then q and conversely
  - p iff q
  - p exactly when q

p: Mars is made of chocolate q: $2 + 3 = 9$

$p \leftrightarrow q$: Mars is made of chocolate if and only if $2 + 3 = 9$

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Why it is called biconditional

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

T
F
F
T

$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$
T	T	
F	T	
T	F	
T	T	

$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

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Comments

- Propositions can get complex
- Order of operations
  - Evaluate ~ first
  - Then evaluate ∧ and ∨
  - Finally evaluate → and ↔
- Use parentheses liberally to keep computation clear
- Evaluate from the inside out
- Logic applies to $\boldsymbol{R}$ but strictly speaking $<, >, \leq, \geq, =, \neq$ **are not symbols in propositional logic**

$$(-x \leq y) \equiv (-x < y) \vee (-x = y)$$
$$(-x \geq y) \equiv (-x > y) \vee (-x = y)$$

Any questions?

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Today's outline

- ✓ Conditional operator
- Laws of propositional equivalence

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De Morgan's Laws apply ~ to ∨ and ∧

$\sim(p \wedge q) \equiv (\sim p \vee \sim q)$ **Proof** by truth table

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim(p \wedge q)$	$\sim p \vee \sim q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

$\sim(p \vee q) \equiv (\sim p \wedge \sim q)$ **Proof** by truth table

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim(p \vee q)$	$(\sim p \wedge \sim q)$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

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DeMorgan's for real numbers

$\sim(p \vee q) \equiv (\sim p \wedge \sim q)$ $\sim(p \wedge q) \equiv (\sim p \vee \sim q)$

What's the negation of $x \leq 3$?

$\equiv (x < 3) \vee (x = 3)$

negation $\equiv \sim((x < 3) \vee (x = 3))$

$\equiv \sim(x < 3) \wedge \sim(x = 3)$

$\equiv (x \geq 3) \wedge (x \neq 3)$

$\equiv (x > 3)$

You should be able to provide a reason for every step here

What is the negation of $1 \leq y \leq 17$ which is a conjunct of disjuncts?

$\sim\Big(\big((1 < y) \vee (1 = y)\big) \wedge \big((y < 17) \vee (y = 17)\big)\Big) \equiv$

$\sim\big((1 < y) \vee (1 = y)\big) \vee \sim\big((y < 17) \vee (y = 17)\big) \equiv$

$\big(\sim(1 < y) \wedge \sim(1 = y)\big) \vee \big(\sim(y < 17) \wedge \sim(y = 17)\big) \equiv$

$\big((1 \geq y) \wedge (1 \neq y)\big) \vee \big((y \geq 17) \wedge (y \neq 17)\big) \equiv$

$(1 > y) \vee (y > 17)$

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Commutative laws for ∨ and ∧

- A binary operator calculates a result for 2 values

  $2 + 5$ $7/9$

- A binary operator is commutative if it produces the same result when its values are interchanged
- Addition and multiplication on $\boldsymbol{R}$ are commutative

  $2 + 5 = 5 + 2$ $2 \times 5 = 5 \times 2$

- Subtraction and division on $\boldsymbol{R}$ are not commutative

  $2 - 5 \neq 5 - 2$ $\frac{7}{9} \neq \frac{9}{7}$

- Logical operators ∨ and ∧ are commutative on propositions

$p \vee q \equiv q \vee p$

$p \wedge q \equiv q \wedge p$

Proof by truth table

p	q	$p \vee q$	$q \vee p$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

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Associative laws for ∨ and ∧

- A binary operator is associative if it produces the same result when its application on 3 arguments are grouped consecutively
- Addition and multiplication on $\boldsymbol{R}$ are associative

  $(2+5)+9=2+(5+9)$ $\qquad$ $(2\times5)\times9=2\times(5\times9)$
- Subtraction and division on $\boldsymbol{R}$ are not associative

  $(2-5)-9\neq 2-(5-9)$ $\qquad$ $(2/5)/9\neq 2/(5\,/9)$
- Logical operators ∨ and ∧ are associative on propositions

$$(p\vee q)\vee r\equiv p\vee(q\vee r)$$
$$(p\wedge q)\wedge r\equiv p\wedge(q\wedge r)$$

Proof of an associative law by truth table

$(p\vee q)\vee r\equiv p\vee(q\vee r)$

p	q	r	$p\vee q$	$(p\vee q)\vee r$	$q\vee r$	$p\vee(q\vee r)$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	T	T	T
T	F	F	T	T	F	T
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	F	T	T	T
F	F	F	F	F	F	F

Distributive laws with ∨ and ∧

- One binary operator is distributive over another if it produces the same result when their application on 3 arguments can be grouped in either order
- Multiplication in $\boldsymbol{R}$ is distributive over addition and over subtraction
  $2{\times}(5+9) = (2{\times}5) + (2{\times}9) \qquad 2{\times}(5-9) = (2{\times}5) - (2{\times}9)$
- Subtraction and division in $\boldsymbol{R}$ are not distributive over addition or multiplication
  $2-(5+9) \neq (2-5)+(2-9) \qquad 2-\left(\frac{5}{9}\right) \neq \frac{2-5}{2-9}$
- Logical operators ∨ and ∧ are distributive over each other

  ∨ is distributive over ∧ $\quad p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

  ∧ is distributive over ∨ $\quad p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

Proof of a distributive law

$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

Identity laws for ∨ and ∧

- If a constant applied with a binary operator does not change the result on any argument, it is an identity for that operator
  0 is the identity for addition and subtraction in $\boldsymbol{R}$
  1 is the identity for multiplication and division in $\boldsymbol{R}$

$\boldsymbol{c}$ is the identity for ∨
Identity law for ∨: $p \vee \boldsymbol{c} \equiv p$

p	$\boldsymbol{c}$	$p \vee \boldsymbol{c}$
T	F	T
T	F	T
F	F	F
F	F	F

Negation of $\boldsymbol{t}$: $\sim\boldsymbol{t} \equiv \boldsymbol{c}$

$\boldsymbol{t}$ is the identity for ∧
Identity law for ∧: $p \wedge \boldsymbol{t} \equiv p$

p	$\boldsymbol{t}$	$p \wedge \boldsymbol{t}$
T	T	T
T	T	T
F	T	F
F	T	F

Negation of $\boldsymbol{c}$: $\sim\boldsymbol{c} \equiv \boldsymbol{t}$

Negation laws for ∨ and ∧

- In propositional logic, the negation law for ∨ is $p \vee \sim p \equiv \boldsymbol{t}$
- In propositional logic, the negation law for ∧ is $p \wedge \sim p \equiv \boldsymbol{c}$

Proofs by truth table

p	$\sim p$	$p \vee \sim p$	$\boldsymbol{t}$	$p \wedge \sim p$	$\boldsymbol{c}$
T	F	T	T	F	F
T	F	T	T	F	F
F	F	T	T	F	F
F	F	T	T	F	F

Idempotent laws for ∨ and ∧

- An idempotent law shows that, for some operator, the identity for a binary operator has no effect on itself

$$0 + 0 = 0 \qquad 1 \times 1 = 1$$

- In propositional logic, the idempotent law for ∨ is $p \vee p \equiv p$

p	$p \vee p$
T	T
F	F

- In propositional logic, the idempotent law for ∧ is $p \wedge p \equiv p$

p	$p \wedge p$
T	T
F	F

Universal bound laws for ∨ and ∧

- A universal bound law shows that for an operator some constant overrides any another value
- In propositional logic, the universal bound law for ∨ is $p \vee \boldsymbol{t} \equiv \boldsymbol{t}$

p	$\boldsymbol{t}$	$p \vee \boldsymbol{t}$
T	T	T
F	T	T

- In propositional logic, the universal bound law for ∧ is $p \wedge \boldsymbol{c} \equiv \boldsymbol{c}$

p	$\boldsymbol{c}$	$p \wedge \boldsymbol{c}$
T	F	F
F	F	F

Remember this?

Evaluate by truth table $((p \wedge q) \vee (p \wedge \sim q)) \wedge \sim p$

p	q	$p \wedge q$	$\sim q$	$p \wedge \sim q$	$(p \wedge q) \vee (p \wedge \sim q)$	$((p \wedge q) \vee (p \wedge \sim q)) \wedge \sim p$
T	T	F	F	F	F	F
T	F	F	T	T	F	F
F	T	F	F	F	F	F
F	F	T	T	F	F	F

Here's a more elegant way that uses these laws

$((p \wedge q) \vee (p \wedge \sim q)) \wedge \sim p$ But every step has to have a reason

$\equiv (p \wedge (q \vee \sim q)) \wedge \sim p$ distributive law $\wedge$ over $\vee$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

$\equiv (p \wedge \boldsymbol{t}) \wedge \sim p$ negation law for $\vee$ $p \vee \sim p \equiv \boldsymbol{t}$

$\equiv p \wedge \sim p$ universal bound law for $\wedge$. $p \vee \boldsymbol{t} \equiv \boldsymbol{t}$

$\equiv \boldsymbol{c}$ negation law for $\wedge$ $p \wedge \sim p \equiv \boldsymbol{c}$

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Absorption laws

- An absorption law allows a simpler expression to generalize over a more restrictive one
- In propositional logic, the absorption law for $\vee$ is $p \vee (p \wedge q) \equiv p$

Proof by truth table

p	q	$p \wedge q$	$p \vee (p \wedge q)$
T	T	T	T
T	F	F	T
F	T	F	F
F	F	F	F

- In propositional logic, the absorption law for $\wedge$ is $p \wedge (p \vee q) \equiv p$

Proof by truth table

p	q	$p \vee q$	$p \wedge (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	F

Any questions?

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Laws are used to simplify propositions

Every equivalence change must have a reason

$(p \wedge q) \vee (\sim p \vee \sim q)$	Given
$((p \wedge q) \vee (\sim p)) \vee \sim q)$	associative law for $\vee$
$((\sim p) \vee (p \wedge q)) \vee \sim q$	commutative law for $\vee$
$((\sim p \vee p) \wedge (\sim p \vee q) \vee \sim q$	distributive law for $\vee$ over $\wedge$
$(\boldsymbol{t} \wedge (\sim p \vee q)) \vee \sim q$	negation law for $\vee$
$(\sim p \vee q) \vee \sim q$	identity law for $\wedge$
$\sim p \vee (q \vee \sim q)$	associative law for $\vee$
$\sim p \vee \boldsymbol{t}$	negation law for $\vee$
$\boldsymbol{t}$	universal bound law for $\boldsymbol{t}$

Each blue expression is addressed by the green law to form the next line

We just proved $(p \wedge q) \vee (\sim p \vee \sim q)$ is a tautology!

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Laws are used to prove equivalences

- Go slowly

Are $\sim(p \vee (\sim p \wedge q))$ and $(\sim p \wedge \sim q)$ logically equivalent?

$\sim(p \vee (\sim p \wedge q))$	given
$\sim p \wedge \sim(\sim p \wedge q)$	De Morgan over $\vee$
$\sim p \wedge (\sim\sim p \vee \sim q)$	De Morgan over $\wedge$
$\sim p \wedge (p \vee \sim q)$	Double negation law
$(\sim p \wedge p) \vee (\sim p \wedge \sim q)$	De Morgan over $\wedge$
$\boldsymbol{c} \vee (\sim p \wedge \sim q)$	negation law for $\wedge$
$(\sim p \wedge \sim q)$	identity law for $\vee$

This is not algebra. Do not treat both sides as if they were equivalent.

Each blue expression is addressed by the green law to form the next line

$$\sim(p \vee (\sim p \wedge q)) \equiv (\sim p \wedge \sim q)$$

Every step in a proof must have a reason

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Proof methods (so far)

Truth table

Sequence of statements with reasons

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What you should know

★ **Every equivalence change must have a reason**

★ **Every step in a proof must have a reason**

- Correctly-spelled names and statements of *all* those laws
- How to prove logical equivalence with a truth table
- How to prove logical equivalence with laws

Next up: *Applications and arguments*

Time to finish up that Opening sheet!

Any questions?

Problem set 3,4 is due on Thursday, September 14 at 11PM

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