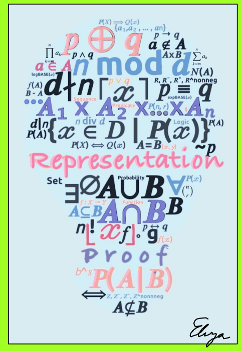
Discrete Structures



Lecture 18: Permutations and combinations

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Review: counting examples

How many 5-place alphanumeric PINs are there?

36^5

How many passwords of length 10 are there, using alphanumeric characters and the special characters @?#$%& ?

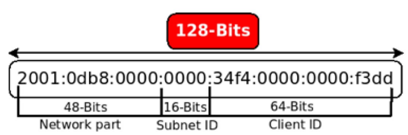
42^{10}

How many elements are there in the Cartesian product $A_1 \times A_2 \times \ldots \times A_n$?

$|A_1| \cdot |A_2| \cdots |A_n|$

IPv6 addresses are 128 bits long.

How many different networks can they support? 2^{48}



How many subnets per network?

2^{16}

How many clients per subnet?

2^{64}

How many addresses?

$2^{48} \cdot 2^{16} \cdot 2^{64}$

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Last time

Sequential decisions multiply possibilities

- What a sample space is and how it affects probabilities
- How to construct items for counting
- The multiplication rule and when to use it

But there are times when the multiplication rule is inapplicable…

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Today's outline

- Permutations
- r-permutations
- Combinations
- Inclusion/exclusion rule

There is a lot to think about when you count

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Review: factorial

For any $n \in \boldsymbol{Z}^+$, $n!$ (read n factorial) is the product of all positive integers $\leq n$

$$n! = \prod_{i=1}^{n} i = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1$$

- By convention, 0! = 1

$$\frac{10!}{9!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 10 = \frac{10(9!)}{9!}$$

$$\frac{n!}{(n-1)!} = \frac{n((n-1)!)}{(n-1)!} = n$$

$$\frac{(n+2)!}{n!} = \frac{(n+2)(n+1)n!}{n!} = (n+2)(n+1)$$

Questions to ask when you count

- How big are the sets that are involved?
- Are the sets involved disjoint?
- Is there inherent order in this problem?
- Is repetition okay?
- What ordered process would construct an arbitrary element?
- Are there separate cases?

Introduction to permutations

- Permutation of a set of objects places them in order
  123 132 213 231 312 321
- A string is a permutation of characters
- Special case: if the elements of the set are letters of an alphabet, any string over them is a word even though it has no meaning
  cat kangaroo sillyputty tca kjakjereij
- How to form a permutation
  - Choose the first element
  - Choose the second element

    …
  - Choose the nth element

Counting permutations of a set

Theorem: Let $P(n)$ be For any integer $n \geq 1$ and any set S of n elements, there are $n!$ permutations of S.

Proof by mathematical induction:
We must show that $P(n)$ is true for all $n \geq 1$.

Basis: $P(1)$ = for any set of 1 element, there are 1! ways to order its elements. Let $S = \{a_1\}$ be any set of 1 element.
Since S can only be ordered in 1 way and $1! = 1, P(1)$ is true.

Inductive step: Assume for some k $P(k)$= there are $k!$ ways to order a set of k elements is true.

By substitution $P(k+1)$ = there are $(k+1)!$ ways to order a set of $k+1$ elements.

(continued)

Counting permutations of a set (continued)

Theorem: Let $P(n)$ be For any integer $n \geq 1$ and any set S of n elements, there are $n!$ permutations of S.

(continued)

By substitution $P(k+1) =$ there are $(k+1)!$ ways to order a set of $k+1$ elements.

We must show $P(k+1)$ is true.

Let $S = \{a_1, a_2, \ldots, a_k, a_{k+1}\}$ be any set of $k+1$ elements.

Then $S' = \{a_1, a_2, \ldots, a_k\}$ is a set of k elements.

Let $b = b_1, b_2, \ldots, b_k$ be an ordering of S'.

There are $k+1$ ways to insert a_{k+1} into b: k ways that put it before some b_i and 1 way that puts it after b_k.

$$\downarrow b_1 \downarrow b_2 \downarrow \ldots \downarrow b_k \downarrow$$

Thus each ordering of S' produces $k+1$ orderings of S.

Since by $P(k)$ there are $k!$ orderings of S', there are $(k+1)\, k! = (k+1)!$ orderings of S.

Since we have proved the basis step and the inductive step, the theorem is true. **QED**

Examples (1)

Consider the letters in the 8-letter word HOLIDAYS

How many ways can they be permuted?

$$8!$$

How many ways can they be permuted if ID must be a substring?

Think about this as putting a rubber band around the I and the D. Now you only have 7 items to permute.

$$7!$$

What is the probability that a random permutation of the letters in HOLIDAYS will contain ID as a substring?

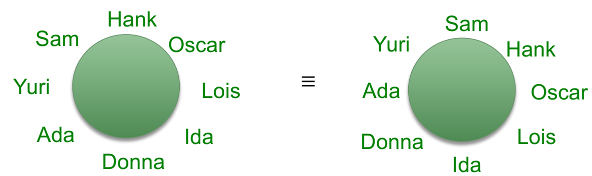
$$\frac{7!}{8!}$$

If you really care about what number results, use a computer.
What CSCI cares about is your thought process

Permutations in a circle

Hank, Oscar, Lois, Ida, Donna, Ada, Yuri, and Sam are going to sit at a circular dinner table. How many different ways can they do that?

Hint: these 2 seating arrangements are identical, that is, rotations of one another. You don't want to count them separately.



To avoid that, fix 1 person in a chair and permute the others.

7!

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Today's outline

- ✓ Permutations
- r-permutations
- Combinations
- Inclusion/exclusion rule

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r-permutations

- An r-permutation of a set S of n elements is a permutation of a subset of size r distinct elements selected from S where $n, r \in \boldsymbol{Z}$ and $0 \leq r \leq n$
- $P(n, r)$ = the number of possible r-permutations of n distinct elements chosen from a set S of n elements
  - Also often written as ${}_nP_r$

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Permutations of a proper subset of a set

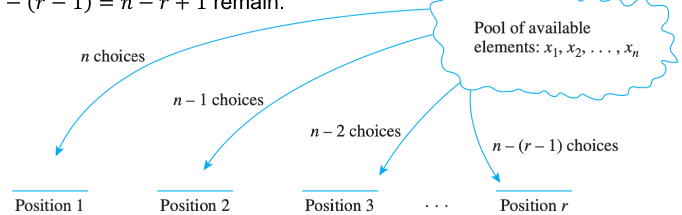
An r-permutation $P(n, r)$ of a set S of n elements is an ordering of r distinct elements selected from S.

Theorem: For any integers $1 \leq r \leq n$ and any set S of n elements, there are $P(n, r) = n(n-1)(n-2)\cdots(n-r+1)$ permutations of S.

This can be proved by induction on n.

Think of this as a process where instead of having n items from which to choose each time, the available item set is reduced by 1 each time.

At the rth selection you have already made $r-1$ choices and so only $n-(r-1) = n-r+1$ remain.



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Examples (2)

$P(n,r) = \frac{n!}{(n-r)!}$ is shorthand for $n(n-1)(n-2)\cdots(n-r+1)$

$$\frac{n!}{(n-r)!} = \frac{n(n-1)(n-2)\cdots(n-r+1)\cancel{(n-r)}\cancel{(n-r-1)}\cdots\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{\cancel{(n-r)}\cancel{(n-r-1)}\cdots\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}$$

$$= n(n-1)(n-2)\cdots(n-r+1).$$

How many 3-letter "words" can be made from the letters of HOLIDAYS?

$$P(8,3) = \frac{8!}{(8-3)!}$$

What is the **probability** that a 4-symbol alphanumeric PIN has only distinct symbols?

$$\frac{P(36,4)}{36^4} = \frac{36!}{36^4(36-4)!}$$

What is the **probability** that such a PIN also ends in 7?

Again, think of this as a process. The PINs are of the form _ _ _7

So you only need to count 3-symbol alphanumeric permutations:

$$\frac{P(35,3)}{35^3} = \frac{35!}{35^3(35-3)!}$$

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Examples (3)

How many 16-digit binary numbers are there?

$$2^{16}$$

How many 4-letter words are there?

$$26^4$$

How many 4-letter words are there with no repeated letters?

$$P(26,4) = \frac{26!}{(26-4)!}$$

How many 4-letter words are there in which the first and last letter are vowels? Think carefully about what this says…

$$5 \cdot 26 \cdot 26 \cdot 5$$

How many 4-letter words are there in which vowels appear only as the first and last letter, if they appear at all?

$$26 \cdot 21 \cdot 21 \cdot 26$$

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Examples (4)

How many different 5-digit numbers are there? (leading zeroes permitted)

$$10^5$$

How many different even 5-digit numbers are there? (leading zeroes permitted)

$$10^4 \cdot 5$$

How many different even 5-digit numbers are there with no leading zeros?

$$9 \cdot 10^3 \cdot 5$$

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Some special cases

- $P(n,0) = \frac{n!}{(n-0)!} = \frac{n!}{n!} = 1$
- $P(n,1) = \frac{n!}{(n-1)!} = n$
- $P(n,n) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$

Any questions?

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Today's outline

- ✓ Permutations
- ✓ r-permutations
- Combinations
- Inclusion/exclusion rule

r-combinations

- An r-combination of a set S of n elements is a subset of size r distinct elements selected from S where $n, r \in \mathbf{Z}$ and $0 \leq r \leq n$,
- $C(n,r)$ = the number of possible r-combinations of n distinct elements
  - Also often written as $\binom{n}{r}$ or ${}_nC_r$ and read "n choose r"
- Every combination of size r can be ordered in $r!$ ways to produce a permutation of r elements
- Thus $P(n,r) = r!\,C(n,r)$ and

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{(n-r)!\,r!} = \binom{n}{r}$$

$$\binom{9}{6} = \frac{9!}{6!3!} = \frac{9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{6\cdot5\cdot4\cdot3\cdot2\cdot1\cdot3\cdot2\cdot1} = \frac{9\cdot8\cdot7}{3\cdot2\cdot1} = 3\cdot4\cdot7 = 84$$

$$\binom{3}{3} = \frac{3!}{3!0!} = 1 = \binom{3}{0}$$

Examples (3)

How many ways are there to choose 7 books from 10 different books?

$$C(10,7) = \frac{10!}{7!\,3!}$$

How many 5-card hands can be formed from a standard 52 card deck?

$$C(52,5) = \frac{52!}{47!\,5!}$$

Card hands are unordered

How many 5-card hands can be formed with 3 but not 4 aces in it?

$$C(4,3) \cdot C(48,2) = \frac{4!}{1!3!} \cdot \frac{48!}{46!2!}$$

A committee is to be chosen from a set of 7 women and 4 men. How many ways are there to form it if the committee has 5 people, 3 women and 2 men?

$$C(7,3) \cdot C(4,2) = \frac{7!}{4!\,3!} \cdot \frac{4!}{2!\,2!}$$

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Some special cases

- $\mathrm{C}(n,0) = \frac{n!}{(n-0)!0!} = \frac{n!}{n!1} = 1$
- $\mathrm{C}(n,1) = \frac{n!}{(n-1)!1!} = \frac{n!}{(n-1)!} = n$
- $\mathrm{C}(n,n) = \frac{n!}{(n-n)!n!} = \frac{n!}{0!n!} = 1$

Any questions?

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Today's outline

✓ Permutations
✓ r-permutations
✓ Combinations
- Inclusion/exclusion rule

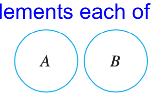
The addition rule

Theorem: For any **partition** $\{A_1, A_2, \dots, A_n\}$ of a finite set A

$$|A| = |A_1| + |A_2| + \cdots + |A_n|$$

Proof is by induction on n.

The basic idea is that the number of distinct elements in disjoint sets is the sum of the number of elements each of them contains.

Think Venn diagrams here. 

How many 4-digit integers are divisible by 5? (no leading zeroes)

Some end in 0 and some end in 5 so

$$(9 \cdot 10 \cdot 10 \cdot 1) + (9 \cdot 10 \cdot 10 \cdot 1)$$

And yes you could have done this the earlier counting way and written

$$(9 \cdot 10 \cdot 10 \cdot 2)$$

Examples (4)

A menu has 3 appetizers, 4 main courses, and 5 desserts. How many ways can you order one thing to eat?
appetizer ≠ main course ≠ dessert

$$3 + 4 + 5$$

How many ways can you order a 3-course meal?
3 choices to make

$$3 \cdot 4 \cdot 5$$

When 2 distinct dice are rolled, how many ways can we get a sum of 6 or 9?

$$5 + 4$$

When 2 distinct dice are rolled, how many ways can we get an odd sum?
Could be a 3 or a 5 or a 7 or a 9 or an 11

$$2 + 4 + 6 + 4 + 2$$

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Examples (5)

A 6-person committee consists of Alice, Ben, Connie, Douglas, Edward, and Felicia. They are to select three officers: a chairperson, a secretary, and a treasurer. No person may hold more than one office.
How many ways can this be done if Alice or Ben must be chairperson?
Count Alice's committees and Ben's separately. Pick the other 2 people and then assign them offices.

$$C(5,2) \cdot 2 + C(5,2) \cdot 2 = \frac{5!}{3!2!} \cdot 2 + \frac{5!}{3!2!} \cdot 2$$

Or choose chair then choose secretary and treasurer.

$$2 \cdot 5 \cdot 4$$

How many ways can this be done if Felicia must be an officer?
Count her committees separately by the office she holds

$$5 \cdot 4 + 5 \cdot 4 + 5 \cdot 4$$

Or choose her job and then choose people for each of the remaining offices

$$C(3,1) \cdot 5 \cdot 4 = \frac{3!}{2!1!} \cdot 5 \cdot 4$$

How many ways can this be done if Douglas and Edward must be officers?
Choose their offices, assign them, and choose 1 person for the remaining office

$$C(3,2) \cdot 2 \cdot C(4,1) = \frac{3!}{2!1!} \cdot 2 \cdot \frac{4!}{3!1!}$$

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What you should know/27

★ **There is a lot to think about when you count**

- Process of construction
- Permutation or combination
- Repetition
- Set size
- Disjoint cases

Any questions?

Next up: *More counting principles*

Time to finish up that Opening sheet!

Problem set 17,18 is due on Thursday, November 16 at 11PM

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The Catalan numbers

- $C_n = \frac{1}{n+1}\binom{2n}{n}$ defines the sequence $1, 2, 5, \ldots$
- $C_k = \frac{1}{k+1}\binom{2k}{k}$ also satisfies the recurrence relation $C_k = \frac{4k-2}{k+1} C_{k-1}$

Proof: $C_k = \frac{1}{k+1}\binom{2k}{k}$ and $C_{k-1} = \frac{1}{k}\binom{2k-2}{k-1}$.

We transform the right side of the recurrence relation:

$$\frac{4k-2}{k+1} C_{k-1} = \frac{4k-2}{k+1} \cdot \frac{1}{k} \cdot \binom{2k-2}{k-1} = \frac{2(2k-1)}{k+1} \cdot \frac{1}{k} \cdot \frac{(2k-2)!}{(k-1)!(2k-2-k+1)!}$$

$$= \frac{2(2k-1)!}{(k+1)k!} \cdot \frac{1}{(k-1)!} = \frac{1}{k+1} \cdot \frac{k}{k} \cdot \frac{2(2k-1)!}{k!(k-1)!} = \frac{1}{k+1} \cdot \frac{2k(2k-1)!}{k(k!)(k-1)!}$$

$$= \frac{1}{k+1} \cdot \frac{(2k)!}{k!k!} = \frac{1}{k+1}\binom{2k}{k} = C_k$$

Catalan numbers are important in many combinatorics proofs

https://en.wikipedia.org/wiki/Catalan_number

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Stirling numbers

- Doubly-ordered sequence is indexed by ordered pairs of integers
- $C(n, r)$ can be thought of as a doubly-ordered sequence for $n, r \in \boldsymbol{N}$, $0 \leq r \leq n$ where index (i, j) indicates that the ith element selected is the jth element in the set
- Let $S_{n,r}$ = number of ways a set of size n can be partitioned into r non-empty subsets
- There are 3 ways to partition $S = \{x_1, x_2, x_3\}$ into 2 non-empty subsets
  $\{x_1, x_2\}\{x_3\}$ $\{x_1, x_3\}\{x_2\}$ $\{x_{21}, x_3\}\{x_1\}$ so $S_{3,2} = 3$
- Stirling numbers are $S_{n,r}$
- $S_{n,1} = 1$
- $S_{n,n} = 1$
- Computation of other values is difficult unless we use recurrence