Discrete Structures

Lecture 24: More on graphs

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HUNTER

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Last time

- **Graphs model real-world relations**
- Definition of a graph
- Special kinds of graphs
- Properties of a vertex

Graph theory has important real-world applications

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Today's outline

- Connectivity
- Euler circuits and paths
- Important graph theory concepts

The material on graphs and trees in CSCI 150 supersedes your text

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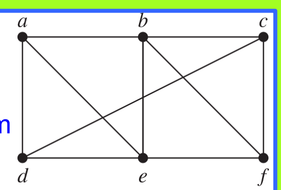
Paths



- Path of length n in an undirected graph $G(V, E)$ from vertex u to vertex v is an alternating sequence $u = v_0 e_1 v_1 e_2 v_2 \dots v_{n-1} e_n v_n = v$ of $n + 1 \in \boldsymbol{Z}$ vertices $v_i \in V, i \in \boldsymbol{N}$ and n edges $e_i \in E$ such that the endpoints of edge e_i are $\{v_{i-1}, v_i\}$

$a(ab)b(bc)c$
length?

abc

- When G is simple, can simply list the vertices
  Is $abedab$ a path?
  Is $deca$ a path?
- Path passes through its vertices and traverses its edges
- Simple path does not include the same edge more than once
  $abedab$ is not simple
- Circuit = path that begins and ends at the same vertex, that is, $u = v$
  $bcfeb$ $febcfebcf$
- Simple circuit = circuit in which the only repeated vertex is $v_0 = v_n$
  $bcfeb$

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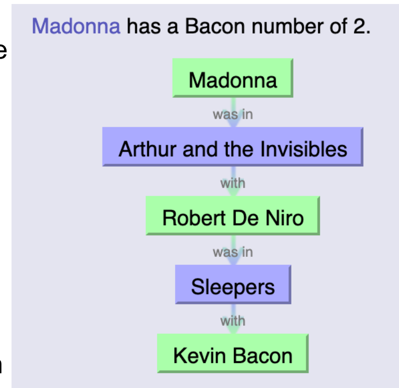
Applications

- 6 degrees of separation = some social scientists estimate the length of the shortest path in an acquaintance graph between you and anyone in the world is about 6
- Erdós number = length of shortest path between mathematician Paul Erdós and yourself in a collaboration graph whose edges denote "collaborated on a long paper"
- Bacon number = length of shortest path between actor Kevin Bacon and another actor in a collaboration graph whose edges denote "acted in the same movie"



https://oracleofbacon.org/

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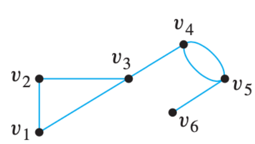
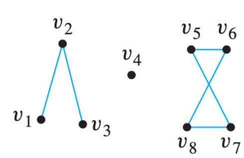
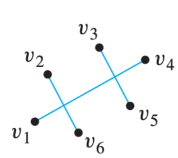
Connectivity in $G(V, E)$

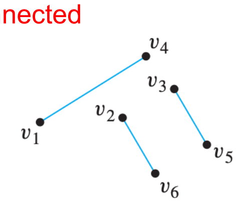
- Vertices $v, w \in V$ are connected iff there is a path from v to w in G





- Recall that edges may cross in a drawing of a graph without going through a vertex
- Graph G is connected iff there is a path between every pair of vertices in V, otherwise it is disconnected





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Proof by cases skeleton

Theorem: If A_1 or A_2 or… or A_n then C.
Proof:

Let/Assume/Suppose: Name variables and state what they stand for
be general: any state any assumptions

We must show that C is true in each of the following cases:
Case 1: If A_1 then C.
Case 2: If A_2 then C.

…

Case n: If A_n then C

Clarify your logic with a reason for every assertion
Display equations and inequalities clearly

Thus C is true regardless of which is the case.

QED

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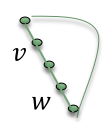
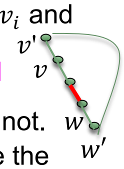
Circuits and connectivity



Lemma: If v, w are part of a circuit in G and one edge is removed from the circuit, there still exists a path from v to w in G.

Proof: Let G be a graph with vertices V and edges E, and let v, w be vertices on some circuit $v_0 e_1 v_1 e_2 v_2 \ldots v_{n-1} e_n v_0$ in G where $v = v_i$ and $w = v_j$. Consider the removal of of some edge (v_k, v_{k+1}).

Case 1



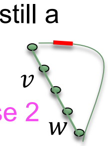
We must show that there is still a path between v and w.

There are 2 cases: the edge was on the path from v to w or it was not.

Case 1: The edge (v_k, v_{k+1}) was on the path from v to w. Let v' be the vertex on the circuit immediately before v, and let w' be the vertex on the circuit immediately after w. Then $ww' \ldots v_{n-1} e_n v_0 e_1 v_1 e_2 v_2 \ldots v'v$ is still a path from w to v, and its reverse is a path from v to w.

Case 2: The edge was not on the path from v to w.
Then the path from v to w was preserved.

Case 2



Thus there is still a path from v to w regardless of which is the case.

QED

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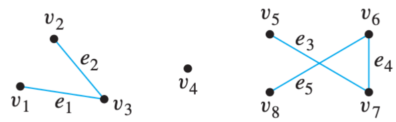
Connected components

Graph $H = (W, F)$ is a connected component of graph $G = (V, E)$ iff H is a subgraph of G, H is connected, and no other connected subgraph $J(X, Y)$ of G has H as a subgraph. (H is a maximal connected component.)



Any questions?

A call graph for 1 day of AT&T calls had 53,767,087 vertices (telephone numbers) and > 170 million edges (calls).
Analysis found >3.7 million connected components, about 75% of which were restricted to calls between 2 numbers that only called each other.
There was also 1 large connected component with 44,989,297 telephone numbers (about 80% of all numbers), where every vertex had a path to any other vertex of length ≤ 20.

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Today's outline

- ✓ Connectivity
- Euler circuits and paths
- Important graph theory concepts

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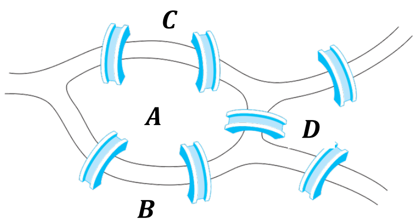
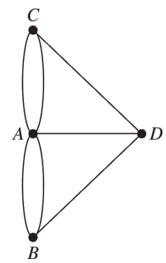
Traverse every edge exactly once

The river Pregel runs through Königsburg (now Kaliningrad, Russia) and 7 bridges cross it at various points. In 1736, the mathematician Leonhard Euler often strolled there after Sunday dinner and began to wonder if it was possible to walk from home and back again while crossing each bridge exactly once.





Is there a circuit in this multigraph that contains every edge?

Things Eulerian

- Eulerian (aka Euler) circuit (aka cycle, tour) for a graph $G(V, E)$ is a circuit that contains at least 1 edge, contains every vertex at least once and contains every edge **exactly** once
- Eulerian graph is a graph that contains an Eulerian circuit
- Note: G may be a directed or an undirected graph
- If a graph $G = (V, E)$ has an Eulerian circuit, then all vertices in V have positive even degree.
- If a graph $G = (V, E)$ is connected and the degree of every vertex of G is a positive even integer, then G has an Eulerian circuit.

Thus

Theorem: A graph G has an Eulerian circuit **iff** G is connected and all its vertices have positive even degree.

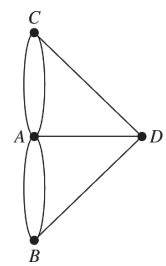
- Eulerian (aka Euler) path for a graph $G(V, E)$ is a path that contains at least 1 edge, contains every vertex at least once and contains every edge **exactly** once

See your text for proofs of this material

Is a graph Eulerian?

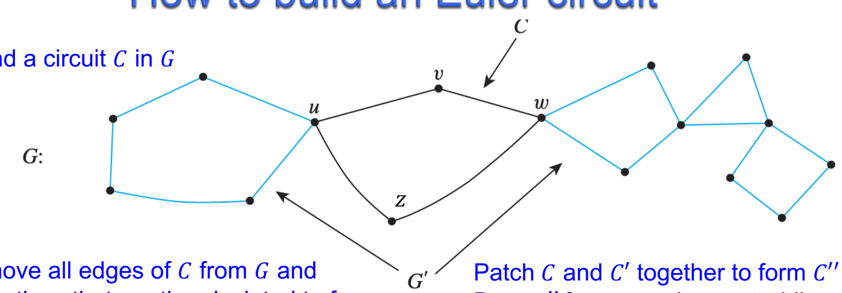
Just proved: If a graph $G = (V, E)$ has an Euler circuit, then all vertices in V have positive even degree.
So the contrapositive must also be true:
Theorem: If any vertex in $G = (V, E)$ has odd degree, then the graph does not have an Eulerian circuit.
Corollary: In a connected graph $G = (V, E)$ there is an Euler path but not an Euler circuit iff there are exactly 2 vertices of odd degree.

Is there a circuit in this multigraph that contains every edge?
Nope. Sorry, Euler.
The Königsberg bridge graph is not Eulerian.
And it doesn't even contain an Eulerian path.



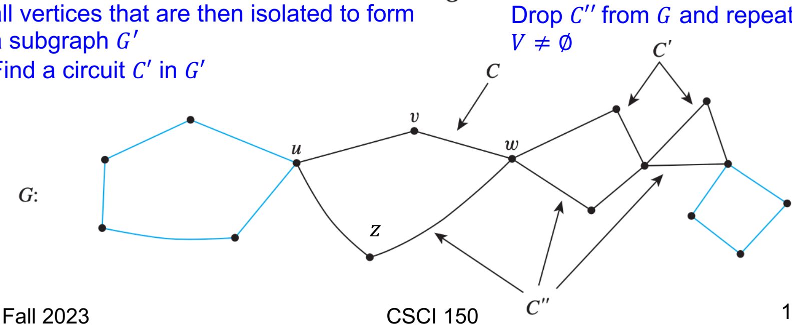
How to build an Euler circuit

Find a circuit C in G



Remove all edges of C from G and all vertices that are then isolated to form a subgraph G'
Find a circuit C' in G'

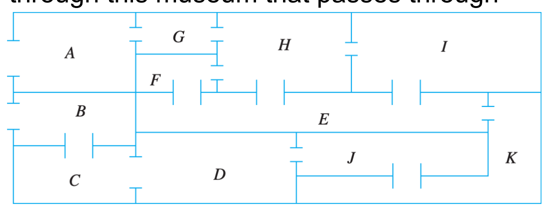
Patch C and C' together to form C''
Drop C'' from G and repeat while $V \neq \emptyset$



Applications of Eulerian circuits

- Travel through a town so that every block is traversed exactly once.
  pick up garbage deliver mail build a power line
  - How to do that?
  - Find an Euler circuit if you have to return where you started
  - Otherwise, find an Euler path
- Circuit layout
- Network multicasting
- DNA sequencing in molecular biology
- Design a tour from A to B through this museum that passes through each door exactly once


Any questions?



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Today's outline

✓ Connectivity

✓ Euler circuits and paths

- Important graph theory concepts

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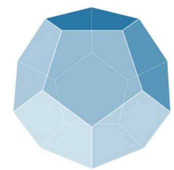
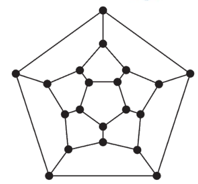
Pass through every vertex exactly once

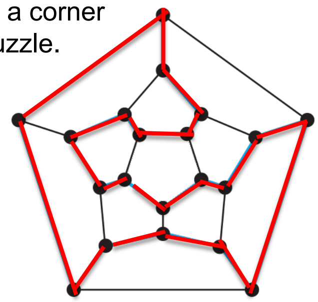
A dodecahedron is a 12-sided solid with pentagonal faces.
In 1859 the Irish mathematician Sir William Hamilton designed a puzzle that labeled each face with a city

Paris London New York

and asked for a trip that would visit each city exactly once and return to the starting point.
Here's a "flattened" version of the dodecahedron where each vertex represents a corner and a solution to the puzzle.







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Things Hamiltonian

- Hamiltonian path in graph $G(V, E)$ = simple path that passes through every vertex exactly once
- Hamiltonian circuit in graph $G(V, E)$= simple circuit that passes through every vertex exactly once
- In general, the existence and detection of Hamiltonian paths and circuits graphs are NP-complete problems (no polynomial-time algorithm is likely to exist for them)
- But there are some sufficient conditions
  - If G is a simple graph with $|V| \geq 3$ vertices all of degree $\geq |V|/2$ then G has a Hamiltonian circuit
  - If G is a simple graph with $|V| \geq 3$ vertices and the degree sum of any two non-adjacent vertices is $\geq |V|$ then G has a Hamiltonian circuit

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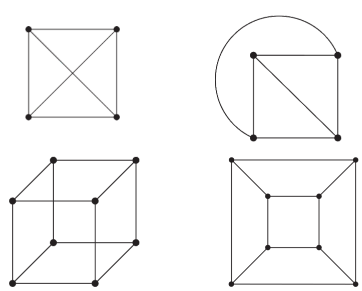
More Hamiltonian results

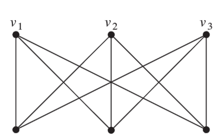
- If there is a vertex of degree 1, there are no Hamiltonian paths or circuits
- If there is a vertex of degree 2, then both edges incident with it must be part of any Hamiltonian path or circuit
- A graph $G(V, E)$ with a Hamiltonian circuit has a connected subgraph $H(W, F)$ such that:
  - $W = V$
  - $|W| = |F|$
  - Every vertex of H has degree 2

Planarity

A graph is planar iff it can be drawn on the plane with no edges that cross one another



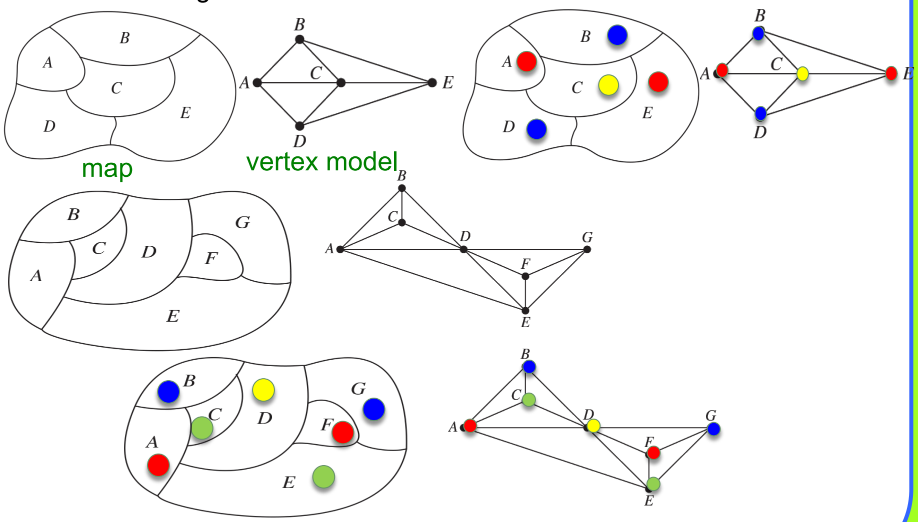
Not planar



Applications to the design of electronic circuits and road networks

Graph coloring

A coloring of a graph assigns a color to each vertex so that no two adjacent vertices are assigned the same color



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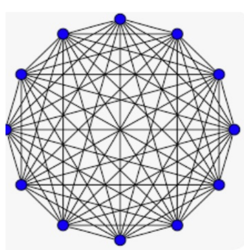
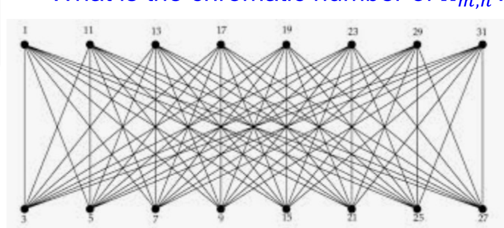
Chromatic number

The chromatic number of a graph = the least number of colors necessary to color it

What is the chromatic number of K_n?



What is the chromatic number of $K_{m,n}$?



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The 4-color theorem

Theorem: The chromatic number of a planar graph is no more than 4. Conjectured by Augustus Guthrie in 1852.

Temporarily "proved" by Alfred Kempe in 1879 until an error was found in the proof in 1890

Proved by Kenneth Appel and Wolfgang Halen in 1976 with a computer that examined 1834 classes of counterexamples and proved that none of them could exist, but errors were found after publication. These were corrected in their 1989 book

Subsequent proofs have reduced the number of counterexample classes to 633 but all of them use a computer

Highly recommended if graph coloring intrigues you:
https://en.wikipedia.org/wiki/Four_color_theorem

Proof methods (so far)

Truth table
Sequence of statements with reasons
Valid argument forms (modus ponens, modus tollens,…)
Predicate logic (quantification, existence, uniqueness)
Proof by cases
Generalization from the generic particular
Proof by contradiction
Proof by contraposition
Mathematical induction
Strong mathematical induction
Algebraic proof by set theory
Algebraic proof by properties of functions
Algebraic proof by combinatorics
Algebraic proof by graph theory
Exhaustive proof by computer

What you should know

★ **Graph theory has important real-world applications**

- Connectivity
- Euler and Hamilton paths and circuits
- Some proofs rely on computers to address their myriad cases

Next up: *Trees*

Time to finish up that Opening sheet!

Any questions?

Problem set 23,24 is due on Thursday, December 7 at 11PM