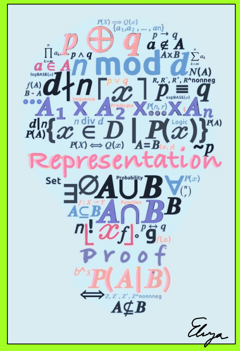
Discrete Structures



Lecture 20: More counting principles

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HUNTER

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Last time

★ **Set theory supports counting**

- How to use the set difference rule
- How to use complements

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Today's outline

- The inclusion /exclusion principle
- The pigeonhole principle

Review: questions to ask when you count

- How big are the sets that are involved?
- Are the sets involved disjoint?
- Is there inherent order? ≡ is this a permutation or a combination?
- What process would construct an arbitrary element?
- Would the complement be easier to count?
- Does the inclusion / exclusion rule apply?

Review: counting rules

Multiplication rule: If a process consists of k steps that can be performed respectively in $n_1, n_2, \cdots, n_k$ ways, then the entire process can be performed in $n_1 n_2 \cdots n_k$ ways

For any integer $n \geq 1$ and any set S of n elements,

$P(n, r)$: there are $\frac{n!}{(n-r)!}$ permutations of r elements from S

$C(n, r)$: there are $\frac{n!}{(n-r)!r!}$ combinations (ways to select) r elements from S

Addition rule: For any partition $\{A_1, A_2, \ldots, A_n\}$ of a finite set A,

$$|A| = |A_1| + |A_2| + \cdots + |A_n|$$

Set difference rule: For any finite set A and any subset B of A, $|A - B| = |A| - |B|$

Complement rule: For any finite set $A \subseteq U$, $|A^C| = |U| - |A|$

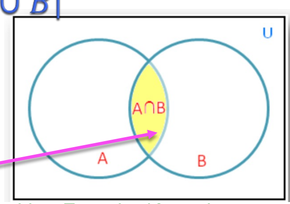
Counting $|A \cup B|$

Consider the union of any 2 sets

- If $(A \cap B) = \emptyset$, then $|A \cup B| = |A| + |B|$
- If $(A \cap B) \neq \emptyset$ then $|A \cup B|$ counts the elements in $A \cap B$ twice

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Double counting



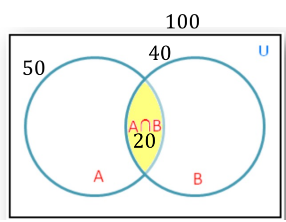
A school with 100 students has 50 students taking French, 40 students taking Chinese, and 20 students taking both languages.

How many students take some language?

$$50 + 40 - 20$$

How many students take no languages?

$$100 - (50 + 40 - 20)$$



Examples (1)

How many integers from 1 through 1000 are multiples of 3 or multiples of 5?

$$\begin{array}{ccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & \dots & 996 & 997 & 998 & 999 \\ & & \updownarrow & & & \updownarrow & & \updownarrow & & & \updownarrow \\ & & 3\cdot 1 & & & 3\cdot 2 & & 3\cdot 332 & & & 3\cdot 333 \end{array}$$

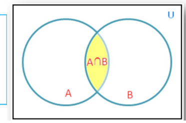
$$\begin{array}{cccccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \dots & 995 & 996 & 997 & 998 & 999 & 1{,}000 \\ & & & & \updownarrow & & & & & \updownarrow & & \updownarrow & & & & & \updownarrow \\ & & & & 5\cdot 1 & & & & & 5\cdot 2 & & 5\cdot 199 & & & & & 5\cdot 200 \end{array}$$

There are $333 - 1 + 1$ multiples of 3 and $200 - 1 + 1$ multiples of 5

What are we double counting?

$66 - 1 + 1$ multiples of 15

$$\begin{array}{cccccccccccccc} 1 & 2 & \dots & 15 & \dots & 30 & \dots & 975 & \dots & 990 & \dots & 999 & 1{,}000 \\ & & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & & \\ & & & 15\cdot 1 & & 15\cdot 2 & & 15\cdot 65 & & 15\cdot 66 & & & \end{array}$$



$|A \cup B| = |A| + |B| - |A \cap B|$ $\quad 333 + 200 - 66$

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Examples (2)

How many 5-letter words begin or end with a vowel?

$$5 \cdot 26^4 + 26^4 \cdot 5 - 5 \cdot 26^3 \cdot 5$$

How many 5-letter words neither begin nor end with a vowel?

~(begin or end) = not begin and not end

$$\underset{\substack{\uparrow \\ \text{not vowel}}}{21 \cdot\times} \quad \underset{\substack{\uparrow \\ \text{any}}}{26^3\times} \quad \underset{\substack{\uparrow \\ \text{Not vowel}}}{21}$$

OR use the complement rule

$$\underset{\substack{\uparrow \\ \text{any.}}}{26^5} - (\underset{\substack{\uparrow \\ \text{vowel 1st}}}{5 \cdot 26^4} + \underset{\substack{\uparrow \\ \text{vowel last}}}{26^4 \cdot 5} - \underset{\substack{\uparrow \\ \text{undo double counting}}}{5 \cdot 26^3 \cdot 5})$$

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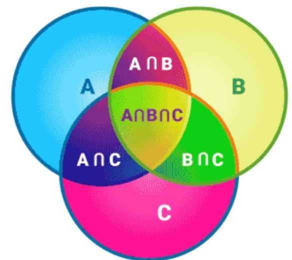
Counting $|A \cup B \cup C|$

Consider the union of any 3 sets
- How often is $(A \cap B)$ counted by $|A \cup B \cup C|$?
- How often is $(A \cap C)$ counted by $|A \cup B \cup C|$?
- How often is $(B \cap C)$ counted by $|A \cup B \cup C|$?
- How often is $(A \cap B \cap C)$ counted by $|A \cup B \cup C|$?

Thus

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$



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Examples (3)

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

Suppose 45% of all newspaper readers like wine, 60% like orange juice, and 55% like tea. Suppose 35% like any given pair of these beverages and 25% like all 3 beverages. What percent of the readers like only wine? $|E|$

$|A| = 25 \quad |B| = 35 - 25 = 10 = |C| = |D|$

$|E| = 45 - |A| - |B| - |C| = 0$

What percent of the readers like exactly 2 of these 3 beverages? $|B| + |C| + |D|$

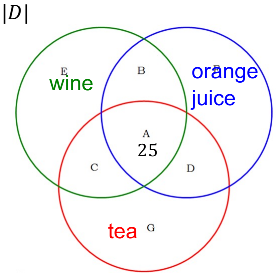
$|B| + |C| + |D| = 10 + 10 + 10 = 30$

What percent of the readers like none of them?

$|F| = 60 - |A| - |B| - |D| = 15$

$|G| = 55 - |A| - |C| - |D| = 10$

$100 - (|A| + |B| + |C| + +|E| + |F| + |G|) = 20$



Make an exception for these problems and work out the values

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Examples (3 another way)

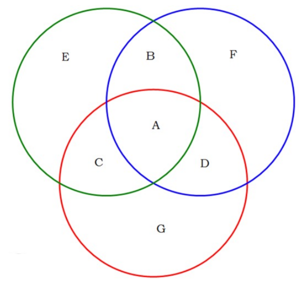
$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

Suppose 45% of all newspaper readers like wine, 60% like orange juice, and 55% like tea. Suppose 35% like any given pair of these beverages and 25% like all 3 beverages.

This describes 7 equations in 7 unknowns

$45 = E + B + A + C$
$60 = B + A + D + F$
$55 = C + A + D + G$
$35 = B + A$
$35 = C + A$
$35 = D + A$
$A = 25$

Since $100 = A + B + C + D + E + F + G$ you can solve this problem algebraically.



Make an exception for these problems and work out the values

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The full inclusion/exclusion rule

$|A \cup B| = |A| + |B| - |A \cap B|$

and

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

are special cases of the full rule

$$\left|\bigcup_{i=1}^{n} A_i\right| =$$
$$\sum_{i=1}^{n}|A_i| - \sum_{\text{distinct } i,j=1}^{n}|A_i \cap A_j| +$$
$$\sum_{\text{distinct } i,j,k=1}^{n} |A_i \cap A_j \cap A_{kj}| + \cdots + (-1)^{n-1}\left|\bigcap_{i=1}^{n} A_i\right|$$

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Examples (4)

How many random 6-card hands have at least one card of each suit?
Look at the complement.
How many hands with 1 suit missing and which suit is that?

$$C(52,6) - C(4,1)C(39,6) + C(4,2)(26,6) - C(4,3)C(13,6)$$

↑ 6-card hands ↑ missing 1 suit ↑ missing 2 suits ↑ missing 3 suits

How many have at least one of each of the 4 values A, K, Q, and J?
Look at the complement.
How many hands with none of some face value and which value is that?

$$C(52,6) - C(4,1)C(48,6) + C(4,2)(44,6) - C(4,3)C(40,6) + C(4,4)C(36,6)$$

↑ 6-card hands ↑ missing 1 ↑ missing 2 ↑ missing 3 ↑ missing all 4

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Today's outline

- ✓ The inclusion /exclusion principle
- The pigeonhole principle

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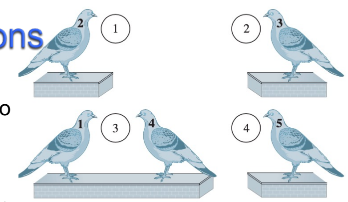
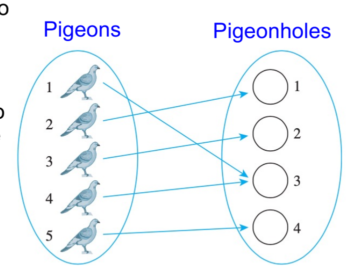
Pigeons



If you have 5 pigeons but only 4 pigeonholes, what happens when they go home to roost at the end of the day? They are going to have to share.

More generally, if you have n pigeons but only $m < n$ pigeonholes, they will have to share too

Pigeonhole principle (aka Dirichlet box principle): A function f from a finite set to a smaller finite set cannot be one-to-one $\equiv$ at least 2 elements in f's domain will have the same image in f's co-domain.



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Proof by contradiction skeleton

Theorem: (copy the statement here)

Proof:

Assume: the negation of the conclusion

be general: any state any assumptions

We will show that this assumption logically leads to a contradiction.

Clarify your logic with a reason for every assertion
Display equations and inequalities clearly

Contradiction. Because the assumption led to a contradiction, negation of the assumption.

QED

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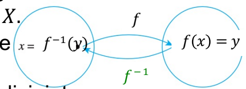
Proof of the pigeonhole principle

Theorem: For any function f from a finite set X with n elements to a finite set Y with m elements, if $n > m$, then f is not one-to-one.

Proof (by contradiction): Let f be any function from a finite set X with n elements to a finite set $Y = \{y_1, y_2, \dots, y_m\}$ with m elements, and $n > m$. For each $y_i \in Y$ the inverse image set $f^{-1}(y_i) = \{x \in X | f(x) = y_i\}$.

Assume that f is one-to-one, and consider all the inverse image sets for all the elements of Y: $f^{-1}(y_1)$, $f^{-1}(y_2)$,…, $f^{-1}(y_m)$. We will show that this assumption logically leads to a contradiction. generic particular function

By definition of function, f gives each $x \in X$ an image in Y, so x is in some inverse image set. $\bigcup_{i=1}^{m} f^{-1}(y_i) = X$.



By definition of function, no $x \in X$ has more than one image in Y. Thus each $x \in X$ is in only one inverse image set, and the inverse image sets are mutually disjoint.

By the addition rule, $|X| = |f^{-1}(y_1)| + |f^{-1}(y_2)| + \cdots + |f^{-1}(y_m)|$.

If f is one-to-one, $|f^{-1}(y_i)| = 1$ and by substitution into those m terms,

$$|X| = 1 + 1 + \cdots 1 = m$$

This contradicts $n > m$. Because the assumption led to a contradiction, f is not one-to-one. **QED**

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Examples (5)

In a group of 6 people, must there be at least 2 who were born in the same month?

no why?

In a group of 13 people, must there be at least 2 who were born in the same month?

yes because by the pigeonhole principle this is $m = 12$ and $n = 13$

Among NYC residents, must there be at least 2 people with the same number of hairs on their heads?

yes by the pigeonhole principle because this is $m < 1{,}000{,}000$ and $n = 8.49$ million

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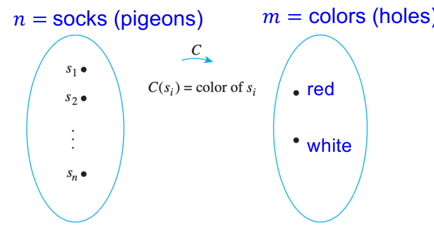
Pick a pair of socks

Your sock drawer contains 10 red socks and 10 white socks. You reach in and pull **some** out without looking at them.
What is the *least* number of socks you must pull out to be sure to get a matched pair?



For the pigeonhole principle this is $m = 20$ and $n = 2$, so any number > 2 will give a matching pair.
The smallest such number is 3

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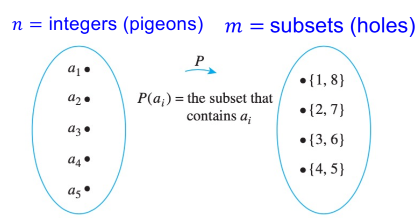
Pick a pair of integers

Consider $A = \{1,2,3,4,5,6,7,8\}$. If 5 integers are selected from A, must at least one pair of them integers sum to 9?
yes the partition $\{1,8\}, (2,7\}, \{3,6\}, \{4,5\}$ shows the ways to sum to 9
If we pick 5 from these 4, by the pigeonhole principle at least 2 must add to 9



If we pick 4 from A instead, will at least 2 add to 9?
no, for example {1,2,3,4}

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Fractions to decimals

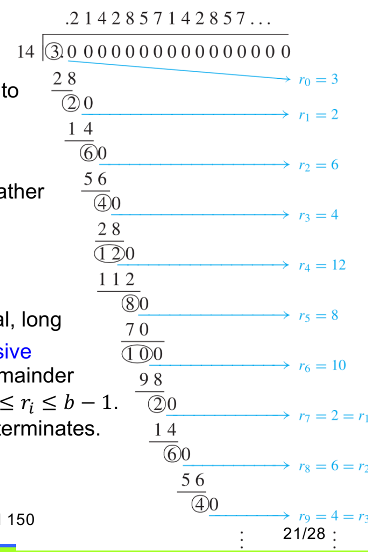
Any rational number can be converted to a decimal by division

$\frac{3}{4} = 0.75 \quad \frac{34}{8} = 4.25 \quad \frac{357}{7} = 51.0$

but some decimal expansions repeat rather than terminate

$\frac{1}{3} = 0.33333333\ldots$

To expand $\frac{a}{b} \in \boldsymbol{Q}, a, b \in \boldsymbol{N}$ into a decimal, long division of a by b produces the successive remainders $r_1, r_2, r_3, \ldots$. The quotient-remainder theorem says these r_i's are integers $0 \leq r_i \leq b-1$. If some $r_i = 0$, the decimal expansion terminates. If not, then what?



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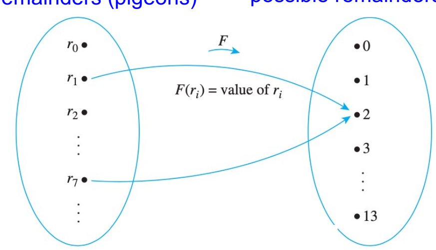
Decimal expansion and pigeonholes

Eventually, the pigeonhole principle promises, if division keeps producing non-zero remainders, since there are only $n = b - 1$ possible remainders, m will eventually be $> b - 1$ and a repeating pattern will arise

generated remainders (pigeons) possible remainders for $\frac{3}{14}$ (holes)



$3/14 = 0.2142857.$

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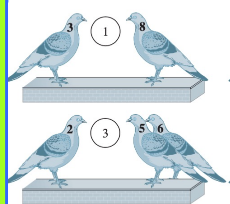
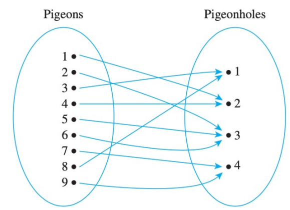
And more generally…

Generalized pigeonhole principle (aka Dirichlet box principle): Let f be a function from a finite set X of n elements to a finite set Y of m elements, and $k \in \mathbf{Z}^{+}$. If $k < \frac{n}{m}$, then there is some $y \in Y$ that is the image of at least $k+1$ elements of X.





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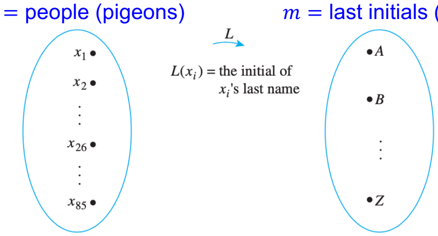
Examples (6)

Show that in a group of 85 people at least 4 must have the same last initial.

By the generalized pigeonhole principle with $n = 85, m = 26, k = 4$, since $3 < 85/26 < 4$ there must be at least 4 people with the same last initial.

n = people (pigeons) m = last initials (holes)



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Contrapositive form

Original: Let f be a function from a finite set X of n elements to a finite set Y of m elements, and $k \in \boldsymbol{Z}^+$. If $k < \frac{n}{m}$, then there is some $y \in Y$ that is the image of at least $k+1$ elements of X.

Contrapositive: Let f be a function from a finite set X of n elements to a finite set Y of m elements, and $k \in \boldsymbol{Z}^+$. If $\nexists y \in Y$ that is the image of at least $k+1$ elements of X, then Y has at most km elements and $n \leq km$.

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One-to-one and onto for finite sets

Theorem: Let X and Y be finite sets with the same number of elements m and f be a function from X to Y. Then f is one-to-one if, and only if, f is onto.

Proof (part 1): We must show that if f is onto then it is one-to-one.

Suppose f is onto. Then by definition of onto, $f^{-1}(y_i) \neq \emptyset$ and $|f^{-1}(y_i)| \geq 1 \; \forall i = 1,2,\dots,m$. As in the proof on slide #17, $X = \bigcup_{i=1}^{m} f^{-1}(y_i)$.

By the addition rule, $|X| = |f^{-1}(y_1)| + |f^{-1}(y_2)| + \cdots + |f^{-1}(y_m)| \geq m$.

If any $|f^{-1}(y_i)| > 1$ then $|X| > m$ which is a contradiction.

Hence each $|f^{-1}(y_i)| = 1$ and f is one-to-one.

Proof (part 2): We must show that if f is one-to-one then it is onto.

Suppose f is one-to-one. Then by definition, $f(x_1), f(x_2), \dots, f(x_m)$ are all distinct. Let S be the set of all elements of Y that are not the image of any element of X.

This theorem is not true for infinite sets!

Then $f(x_1), f(x_2), \dots, f(x_m)$ and S are mutually disjoint and partition Y. By the addition rule $|X| = |f^{-1}(y_1)| + |f^{-1}(y_2)| + \cdots + |f^{-1}(y_m)| + |S| = m + |S|$, so $m = m + |S|$ and $|S| = 0$. That is, there are no elements of Y that are not the image of any element of X, and f is onto. **QED**

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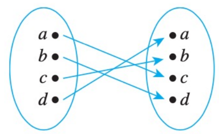
A strong result that required counting

- We just proved that if X and Y are finite sets with the same number of elements and f is a function from X to Y, f is one-to-one if and only if f is onto.
- So if we let X and Y be the same set,

  Any one-to-one function from a set to itself is onto
  Any onto function from a set to itself is one-to-one
- Such functions are permutations of the sets on which they are defined

This arrow definition of a function also defines a permutation of its domain



Counting supports function theory

What you should know

Counting supports function theory

- How to use the inclusion/exclusion principle
- How to use the pigeonhole principle

Any questions?

Next up: *Counting with repeated elements*

Time to finish up that Opening sheet!

Problem set 19,20 is due on Thursday, November 23 at 11PM

Sharing

42 students must share 12 books. Each student uses exactly 1 book, and no book is used by more than 6 students. Show that at least 5 books are used by 3 or more students.

The pigeonholes are ? books $m = 12$
The pigeons are ? students $n = 42$
With a 2-book limit, only 24 students have access to a book, that is.
We need to accommodate the remaining $42 - 24 = 16$ students.
Each pigeonhole (book) can hold at most 6 students.

Let $k =$ number of books shared by 3 or more students
$12 - k =$ number of books shared by at most 2 students.
By the contrapositive form of the generalized pigeonhole principle, at most $2(12 - k) = 24 - 2k$ are served, plus $6k$ if we max out the other books, so $24 + 4k$ students have book access. But if $24 + 4k = 42$, $k \geq 4.5$.