Final exam

Graded

Student

Total Points

67 / 100 pts

Question 1

Card counting

6 / 20 pts

- **+ 20 pts** Correct
- **+ 7 pts** (a) correct. Must explain their formula
- ✔ **+ 3 pts** (a) has valid formula, but no explanation OR partial credit
- **+ 7 pts** (b) correct must explain formula
- ✔ **+ 3 pts** (b) has valid formula but no explanation OR partial credit
- **+ 6 pts** (c) correct with explanation
- **+ 3 pts** (c) valid folmula, no explanation OR partial credit
- **+ 0 pts** Nothing of value/No response

Question 2

Exam questions question

13 / 15 pts

- **+ 15 pts** Correct.
- **+ 5 pts** (a) correct. "Yes" with reason, like "PHP with the questions as pigeons and the topics as pigeonholes"
- ✔ **+ 1 pt** (a) is "yes" but no valid reason
- ✔ **+ 5 pts** (b) correct, "No". Some explaination like "PHP says some topic must have at least two questions, but does not say which"
- **+ 1 pt** (b) is "no" but no valid reason
- ✔ **+ 5 pts** (c) correct. "some questions involve more than one topic"
- **+ 2 pts** (c) is poorly explained, eg "PHP says there must be" which is not quite the same as "there can be" (PHP or not)
- **+ 0 pts** No submission/nothing of value
- **+ 2 pts** reasons in (a) not sufficiently explained.

Question 3

Graphs

15 / 15 pts

✔ **+ 15 pts** Correct

+ 5 pts (a) correct, since any node in the tree could be the root, n ways to do it (or |V| ways)

+ 10 pts (b) correct, with good argument.

+ 4 pts (b) says "two disjoint trees" or similar with no valid explanation.

+ 0 pts Nothing of value/No submission

Question 4

Cardinals

12 / 20 pts

+ 20 pts Correct

✔ **+ 5 pts** (a) correct, mentions both one-one and onto

✔ **+ 5 pts** (b) correct. $A \times B = \{(a, b) | a \in A, b \in B\}$

+ 10 pts (c) correct, must prove there is a one-one and onto map. $f(a, b) = (b, a)$ works.

✔ **+ 2 pts** (c) "yes" with no valid proof

+ 4 pts (c) shows there is a one-one map

+ 4 pts (c) shows there is an onto map

+ 0 pts No submission/Nothing of value

No one said "finite sets".

Question 5

Rudolphic functions

10 / 15 pts

+ 15 pts Correct

✔ **+ 5 pts** (a) correct. "yes" with valid proof

+ 1 pt (a) says "yes" but no valid proof

+ 5 pts (b) correct. "no", with valid counterexample

+ 1 pt (b) says "no) but no valid counterexample

✔ **+ 5 pts** (c) correct, "yes" with proof.

+ 1 pt (c) is "yes" with no proof.

+ 0 pts Nothing of value/No submission

Question 6

Sequence

11 / 15 pts

+ **15 pts** Correct.

+ **3 pts** Correctly identify P(n), "b_n=4^n-3^n" or Proposition, before Basis and Inductive Step.

− **1 pt** Did not identify Basis Step

✔ + **3 pts** Basis Step: Proves P(1) and P(2)

+ **2 pts** Basis Step: Proves only P(1) or P(2)

− **1 pt** Did not identify Inductive Step

✔ + **3 pts** Inductive Step: States P(i) with 1 <= i <= k or assumes just P(k-1) and P(k).

+ **2 pts** Inductive Step: Assumed only P(k).

+ **6 pts** Inductive Step Proof: Correct.

✔ + **5 pts** Inductive Step Proof: Almost, but minor algebraic mistake

+ **4 pts** Inductive Step Proof: Properly used inductive hypothesis.

+ **4 pts** Inductive Step Proof: Proved wrong P(k) correctly.

+ **0 pts** Nothing of value/no submission

1 should be +

2 P(0) is not defined

3 should be >= 2, but close enough

4 missing parentheses

1. (20) A "Remski Deck" is a deck of cards consisting of nine cards in each of 10 denominations, $A, 2, 3, 4, 5, 6, 7, 8, 9$ and 10. In each denomination there are three cards in each of the three suits Harps: (H), Scythes (S) and Roses (R). So there are three different $2H$, three different $10S$, etc. for a total of 90 cards. A "hand" of eight cards is dealt. In a hand, the order the cards are dealt do not matter, and the cards of the same denomination and suit are indistinguishable. A hand is a "pair" if there are two cards of the same denomination and none of the other cards in the hand have the same denomination as the pair or each other. A "straight" consists of 8 cards in numerical order. A "flush" consists of all 8 cards from the same suit. Answer the questions below leaving your answers as combinatorial expressions.

   (a) How many hands contain exactly one pair? (It does not matter if the hand is also a straight or flush).

   (b) How many hands are straights? (It does not matter if the hand is a flush or a pair).

   (c) How many hands are flushes but not straights? (The hand can contain pairs).

a) $\binom{10}{1}\binom{9}{2}\binom{9}{6}\binom{9}{1}^{6}$

b) $\binom{9}{1}^{8}+\binom{9}{1}^{8}+\binom{9}{1}^{8}=3\binom{9}{1}^{8}$

c) $\binom{3}{1}^{2}\binom{10}{8}-3\binom{9}{1}^{8}$

2. (15) An exam has eight questions on five topics (Induction, Strong Induction, Functions, Counting and Graph Theory).

   (a) Must there be at least two questions on some topic? Why or why not?

   (b) Must there be at least two questions on Graph Theory? Why or why not?

   (c) There are 3 questions on Induction, 1 on Strong Induction, 2 on Functions, 3 on Counting and 1 on Graph Theory. Of course $3 + 1 + 2 + 3 + 1 = 10 > 8$. **How is this possible?**

(a) Yes. there are more questions than topics. One there will be one or more questions that share the same topic as all 5 topics must be used.

(b) No, not necessarily. there can be two questions on Induction, two on strong Induction, two on Functions, while only 1 each for counting and Graph theory.

(c) It is possible a question can have multiple topics rather

3. (15) Let $G = \langle V, E \rangle$ be a tree with $|V| = n$.

   (a) How many different rooted trees can be formed from G?

   (b) If one edge is removed from E to form E', and $G' = \langle V, E' \rangle$, is G' a tree? If so, prove it. If not, explain what G' is and prove it.

a) $\underline{n}$ different rooted trees can be formed

b) No because there will be 2 more ~~less~~ vertices than edges (there should be $|E|+1$), causing one to be disconnected and therefore G' is not a tree.

$E' = |E|-1$

4. (20)

   (a) Define (using the properties of functions) what it means for two sets to have the same *cardinality*?

   (b) Consider the sets A and B. What is the set $A \times B$?

   (c) Do $|A \times B|$ and $|B \times A|$ have the same cardinality? If so, prove it (using your definition in (a)), if not give a counterexample.

(a) For two sets to have the same cardinality would mean they have a one-to-one correspondence. Or ~~that they~~ have the same number of elements.

(b) $A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}$

(c) proof

$|A \times B| =$ the number of elements of A $\times$ the number of elements of B

$= |A| \times |B|$

$|B \times A| =$ the number of elements of B $\times$ the number of elements of B

$= |B| \times |A|$

$|A| \times |B| = |B| \times |A|$

$|A \times B| = |B \times A|$

They have the same cardinality

5. (15) A function $f : \mathbb{Z} \to \mathbb{Z}$ is "Rudolphic" (named after Rodolfo, Prince of Guppies) if there are integers i and j, $i \neq 0$ with $f(n) = j + i \cdot n$.

   (a) Are all Rudolphic functions one to one? If so, prove it. If not, give a counterexample.

   (b) Are all Rudolphic functions onto? If so, prove it. If not, give a counterexample.

   (c) Is the set of Rudolphic functions closed with respect to composition? That is, if f and g are Rudolphic, is $f \circ g$? If so, prove it. If not, give a counterexample.

a) $f(n_1) = f(n_2)$

$j + i \cdot n_1 = j + i \cdot n_2$

$i \cdot n_1 = i n_2$

$n_1 = n_2$

All Rudolphic functions are one-to-one

b) $f(n) = j + i \cdot n$, let $y = j + i \cdot n$

$y = j + i \cdot n$ $\quad f(n) = f\left(\frac{y-j}{i}\right) = j + i\left(\frac{y-j}{i}\right)$

$\frac{y-j}{i} = n$ $\quad = j + (y - j)$

$f(n) = y$

All Rudolphic functions are onto

c) $f(n) = j_1 + i_1 n$, $g(n) = j_2 + i_2 n$

$f \circ g = f(g(n)) = f(j_2 + i_2 \cdot n) = j_1 + i_1(j_2 + i_2 n)$

$= j_1 + i_1 j_2 + i_1 i_2 n = (j_1 + i_1 j_2) + i_1 i_2 \cdot n$

$\in \mathbb{Z}$

Follows same form as $j + i \cdot n$, therefore $f \circ g$ is Rudolphic.

therefore the set of Rudolphic functions is closed with respect to composition.

6. (15) Let the sequence $b_1, b_2, b_3, \ldots$ be defined by $b_1 = 1, b_2 = 7$, and

$$b_n = 7 \cdot b_{n-1} - 12 \cdot b_{n-2} \text{ for all } n > 2.$$

Use strong induction to show that $b_k = 4^k - 3^k$ for all $k \in \mathbb{N}^{\geq 1}$. Be sure to identify the proposition you are proving and to indicate what you are doing.

Basis Step:

$P(1) = b_1 = 4^1 - 3^1 = 1$, $P(2): b_2 = 4^2 - 3^2 = 16 - 9 = 7$

Proof by Induction:

Assume $k \in \mathbb{N}^{\geq 1}$ for all $0 \leq m \leq k$

$P(n) = b_n = 4^n - 3^n$

show that $P(k+1) = b_{k+1} = 4^{k+1} - 3^{k+1}$

$$b_{k+1} = 7 \cdot b_k - 12 \cdot b_{k-1}$$
$$= 7 \cdot (4^k - 3^k) - 12 \cdot (4^{k-1} - 3^{k-1})$$
$$= 7 \cdot 4^k - 7 \cdot 3^k - 12 \cdot 4^{k-1} - 3^{k-1}$$
$$- \frac{12 \cdot 4^k}{4} - \frac{12 \cdot 3^k}{3}$$
$$= 7 \cdot 4^k - 7 \cdot 3^k -$$
$$= 7 \cdot 4^k - 7 \cdot 3^k - 3 \cdot 4^k - 4 \cdot 3^k$$
$$= (7-3)4^k - (7-4)3^k$$
$$= 4 \cdot 4^k - 3 \cdot 3^k$$
$$= 4^{k+1} - 3^{k+1}$$

Both the basis and induction step are shown to be true so $b_k = 4^k - 3^k$ for all $k \in \mathbb{N}^{\geq 1}$