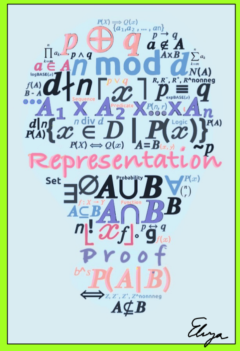
Discrete Structures



Lecture 13: Introduction to set theory

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HUNTER

1

Last time

★ **A recursive solution may speed computation**

- How to interpret and use recursive definitions
- Famous examples of recursion
- How to solve a recurrence relation by iteration

2

Today's outline

- Basic definitions and their role in proofs
- Operations on sets
- Sets of sets
- Sets in proofs

Review: basic ideas

- Set = collection of items called elements
  - Sets are typically named with uppercase letters $A \quad B \quad D$
  - Elements of a set are typically lowercase letters $a \quad b \quad x$
  - $\in$ denotes membership in a set $a \in A$
  - $\notin$ denotes non-membership in a set $a \notin A$
- Two ways to define a set
  - Enumerate its elements in curly brackets { } $\{2,3,5,7,11,\ldots\}$
  - Describe it by a precise rule $C = \{x \mid x \text{ is positive, even and} < 10\}$
- Universal set $\boldsymbol{U}$ = set of all elements
- Empty set $\emptyset$ = set containing no elements
- $\emptyset \neq \{\emptyset\}$

Reminder: Do not invent notation or language.

Sets, subsets and supersets

- Subset $A \subseteq B$ means $\forall x \in A, x \in B$ $\{1,2,3\} \subseteq \{x|x \in \boldsymbol{R}, 0 \leq x < 4\}$
- Proper subset $A \subset B$ means $\forall x \in A, x \in B$ and $\exists x \in B \ni x \notin A$ $\boldsymbol{N} \subset \boldsymbol{Z}$
- Superset $A \supseteq B$ means $\forall x \in B, x \in A$ $\{x|x \in \boldsymbol{R}, 0 \leq x < 4\} \supseteq \{1,2\}$
- Proper superset $A \supset B$ means $\forall x \in b, x \in A$ $\boldsymbol{Q} \supset \boldsymbol{Z}$
- Equality $A = B$ means $A \subseteq B$ and $B \subseteq A$ $\{1,2,3\} = \{x|x \in \boldsymbol{N}, 0 < x < 4\}$
- Negative notation: $\nsubseteq, \not\subset, \nsupseteq, \not\supset, \neq$

Reminder: Do not invent notation or language

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5

Review: cardinality and tuples

- Cardinality $|A|$ = how many elements set contains $|\{1,9,2,3\}| = ?$
- Cartesian product $X \times Y = \{(x, y)|x \in X$ and $y \in Y\}$
  If $A = (4,5\}$ and $B = \{p, q\}$, then $A \times B = \{(4, p), (4, q), (5, p), (5, q)\}$
- $|X \times Y| = |X| \times |Y|$ so if $|X| = m$ and $|Y| = n$, $|X \times Y| = mn$
- Ordered n-tuple = ordered set of n elements formed from n sets $A_1, A_2, \ldots, A_n$ in that order 3-tuple: $(4, 6, -2)$
- Cartesian product $A_1 \times A_2 \times \cdots \times A_n = \{(a, b, \ldots, c)|a \in A_1,\ b \in A_2, \ldots, c \in A_n\}$
  If $A = \{4,5\}$, $B = \{p, q\}$, and $C = \{cat, dog, 9\}$, then $(5, p, cat), \in A \times B \times C$
- $|A_1 \times A_2 \times \cdots \times A_n| = |A_1| \times |A_2| \times \cdots \times |A_n|$ $|A \times B \times C| = ?$

Reminder: Do not invent notation or language

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6

Skeletons for subset arguments

- Since $A \subseteq B$ means $\forall a \in A, b \in B$
  - To prove $A \subseteq B$,
    Let a be any element of A and show? generic particular
    $a \in B$
  - To prove $A \nsubseteq B$,
    Display $a \in A$ and show?
    $a \notin B$ counterexample

Statement: $\boldsymbol{Z} \subseteq \boldsymbol{N}$
Counterexample: $-3 \in \boldsymbol{Z}, -\boldsymbol{3} \notin \boldsymbol{N}$
Thus $\boldsymbol{Z} \nsubseteq \boldsymbol{N}$

- Since $A \subset B$ means $\forall a \in A, a \in B$ and $\exists b \in B \ni b \notin A$
  - To prove $A \subset B$,
    Let a be any element of A and display $b \in B$ and show?
    $a \in B$ and $b \notin A$
  - To prove $A \not\subset B$,
    Display $a \in A$ and show?
    $a \notin B$

Any questions?

7

Today's outline

✓ Basic definitions and their role in proofs
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8

Union and intersection

- Union $A \cup B = \{x | x \in A \text{ or } x \in B\}$
- Intersection $A \cap B = \{x | x \in A \text{ and } |x \in B\}$
- Difference $A - B$ (aka set difference, relative complement) $= \{\boldsymbol{x} | \boldsymbol{x} \in A, x \notin B\}$
- Complement $A^C = \{\boldsymbol{x} | \boldsymbol{x} \notin A\}$
- Properties of $\emptyset$
  - $\emptyset = \{\,\}$
  - $\emptyset \subseteq A$
  - $A \cup \emptyset = A$
  - $A \cap \emptyset = \emptyset$
  - $\emptyset - A = \emptyset$
  - $A - \emptyset = A$

Properties of $\boldsymbol{U}$
- $A \subseteq \boldsymbol{U}$
- $A \cup \boldsymbol{U} = \boldsymbol{U}$
- $A \cap \boldsymbol{U} = A$
- $\boldsymbol{U} - A = A^C$
- $A - \boldsymbol{U} = \emptyset$

$A = \{1,2,3\}$

$A \cup \boldsymbol{Z} = ?$

$\boldsymbol{Q} \cap A = ?$

$A - \boldsymbol{Z} = ?$

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9

A simple set theory proof

Theorem: For any set A, $A \cap \boldsymbol{U} = A$.

Proof:

Let A be any set.

We must show that $A \cap \boldsymbol{U} = A$, that is, that $A \cap \boldsymbol{U} \subseteq A$ and $A \subseteq A \cap \boldsymbol{U}$.

Let x be any element of $A \cap \boldsymbol{U}$. Then by definition of intersection, $x \in A$.

Thus $A \cap \boldsymbol{U} \subseteq A$.

Let x be any element of A. By definition of the universal set $\boldsymbol{U}, A \subseteq \boldsymbol{U}$ and by definition of subset, $x \in \boldsymbol{U}$.

Since $x \in A$ and $x \in \boldsymbol{U}$, by definition of intersection $x \in A \cap \boldsymbol{U}$.

Thus $A \subseteq A \cap \boldsymbol{U}$.

QED

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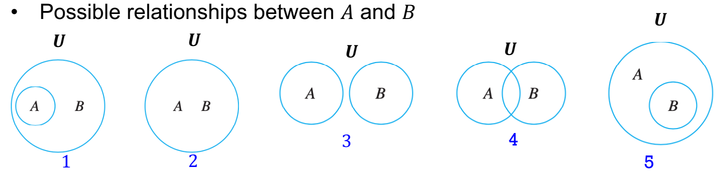
10

Venn diagrams

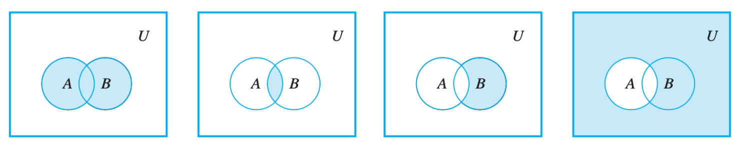
- One circle for each set
- Possible relationships between A and B



proper subset	equal sets	disjoint sets	non-disjoint sets	superset
$A \subset B$	$A = B$	$A \cap B = \emptyset$	$A \cap B \neq \emptyset$	$A \supset B$

11

Venn diagrams for set operations



union	intersection	difference	complement
$A \cup B$	$A \cap B$	$B - A$	A^C

12

Review: intervals are subsets of R

For $a, b \in \boldsymbol{R}$

- Open interval excludes both endpoints
  $(a,b) = \{x \in \boldsymbol{R} | a < x < b\}$ $(-2,7) = \{x \in R | -2 < x < 7\}$
- Closed interval includes both endpoints
  $[a,b] = \{x \in \boldsymbol{R} | a \le x \le b\}$ $[-2,7] = \{x \in R | -2 \le x \le 7\}$
- Half-open interval includes exactly one endpoint
  $(a,b] = \{x \in \boldsymbol{R} \mid a < x \le b\}$ $(-2,7] = \{x \in R | -2 < x \le 7\}$
  $[a,b) = \{x \in \boldsymbol{R} \mid a \le x < b\}$ $[-2,7) = \{x \in R | -2 \le x < 7\}$
- ∞ denotes an interval that is unbounded on the right
  $(a,\infty) = \{x \in \boldsymbol{R} | x > a\}$ $(-2,\infty) = \{x \in R | x > -2\}$
  $[a,\infty) = \{x \in \boldsymbol{R} | x \ge a\}$ $[-2,\infty) = \{x \in R | x \ge -2\}$
- $-\infty$ denotes an interval that is unbounded on the left
  $(-\infty,b) = \{x \in \boldsymbol{R} | x < b\}$ $(-\infty,7) = \{x \in \boldsymbol{R} | x < 7\}$
  $(-\infty,b] = \{x \in \boldsymbol{R} | x \le b\}$ $(-\infty,7] = \{x \in \boldsymbol{R} | x \le 7\}$

WARNING: Delimiters are often reused

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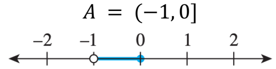
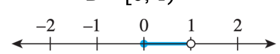
13

Seeing intervals

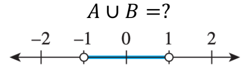
$A = (-1, 0]$



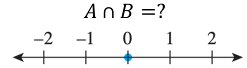
$B = [0, 1)$



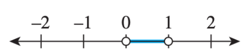
$A \cup B = ?$



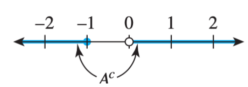
$A \cap B = ?$



$B - A = ?$



$$A^C = \{x | x \notin A\} = \{x \in \boldsymbol{R} | x \notin (-1,0]\} =$$
$$= \{x \in \boldsymbol{R} | \sim((-1 < x) \wedge (x \le 0))\}$$
$$= \{x \in \boldsymbol{R} | \sim(-1 < x) \vee \sim(x \le 0)\}$$
$$= \{x \in \boldsymbol{R} | \ x \le -1) \vee (x > 0)\}$$



Any questions?

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14

Today's outline

- ✓ Basic definitions and their role in proofs
- ✓ Operations on sets
- Sets of sets
- Sets in proofs

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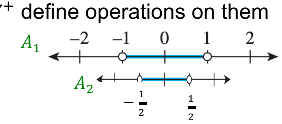
15

Unions of multiple sets

Given sets $A_0, A_1, A_2, \ldots \subseteq \boldsymbol{U}$ where $n \in \boldsymbol{Z}^+$ define operations on them

$A_i = \left\{x \in \boldsymbol{R} \mid x \in \left(-\frac{1}{i}, \frac{1}{i}\right)\right\}$

$A_2 = \left\{x \in \boldsymbol{R} \mid -\frac{1}{2} < x < \frac{1}{2}\right\} = \left(-\frac{1}{2}, \frac{1}{2}\right)$



$$\bigcup_{i=1}^{n} A_i = \{x \mid x \in A_i \text{ for some } i \in \boldsymbol{N}, 0 \le i \le \boldsymbol{n}\}$$

$$\bigcup_{i=1}^{3} A_i = (-1,1) \cup \left(-\frac{1}{2}, \frac{1}{2}\right) \cup \left(-\frac{1}{3}, \frac{1}{3}\right) = (-1,1)$$

$$\bigcup_{i=1}^{\infty} A_i = \{x \mid x \in A_i \text{ for some } i \in \boldsymbol{N}\}$$

$$\bigcup_{i=1}^{\infty} A_i = (-1,1)$$

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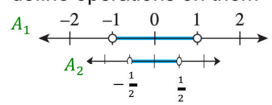
16

Intersections of multiple sets

Given sets $A_0, A_1, A_2, \ldots \subseteq \boldsymbol{U}$ where $n \in \boldsymbol{Z}^+$ define operations on them

$A_i = \left\{x \in \boldsymbol{R} \mid x \in \left(-\frac{1}{i}, \frac{1}{i}\right)\right\}$



$\bigcap_{i=1}^{n} A_i = \{x \mid x \in A_i$ for all $i \in \boldsymbol{N}, 0 \le i \le \boldsymbol{n}\}$

$\bigcap_{i=1}^{3} A_i = (-1,1) \cap \left(-\frac{1}{2}, \frac{1}{2}\right) \cap \left(-\frac{1}{3}, \frac{1}{3}\right) = \left(-\frac{1}{3}, \frac{1}{3}\right)$

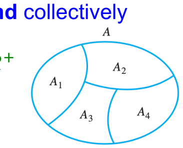
$\bigcap_{i=1}^{\infty} A_i = \{x \mid x \in A_i$ for all $i \in \boldsymbol{N}\}$

$\bigcap_{i=1}^{\infty} A_i = \{0\}$

17

Partitions

- A and B are disjoint iff $A \cap B = \emptyset$ (1,3) and (7,10]
- Sets $A_1, A_2, \ldots, A_n$ are mutually exclusive (aka mutually disjoint, pairwise disjoint, non overlapping) iff $A_i \cap A_j = \emptyset$ for $i \neq j$
  [1,2], [3,4], [5,6],…
- A finite or infinite set $C = \{A_1, A_2, \ldots\}$ of nonempty subsets of a set S is collectively exhaustive iff $\bigcup_i A_i = S$
  [1,2) and [2,3) for [1,3)
- A finite or infinite set $C = \{A_1, A_2, \ldots\}$ of nonempty subsets of a set S partitions S iff the subsets are mutually exclusive **and** collectively exhaustive
  {(0,1),[1,2), [2,3), [3,4),…} partitions R^+



18

Power sets

The power set $\mathcal{P}(A)$ of a set A is the set of all subsets of A

$$\mathcal{P}(A) = \{S | S \subseteq A\}$$

For $A = \{cat, dog\}$, $\mathcal{P}(A) = \{\emptyset, \{\text{cat}\}, \{\text{dog}\}, \{\text{cat.dog}\}\}$

Comments

- Since for any set A, $\emptyset \subseteq A$, $\emptyset \in \mathcal{P}(A)$
- Since for any set A, $A \subseteq A$, $A \in \mathcal{P}(A)$
- $\mathcal{P}(\emptyset) = \{\emptyset\}$
- For any set A, $|\mathcal{P}(A)| > |A|$
- $|\mathcal{P}(A)| = 2^{|A|}$

Any questions?

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19

Today's outline

- ✓ Basic definitions and their role in proofs
- ✓ Operations on sets
- ✓ Sets of sets
- Sets in proofs

Set theory proofs rely on definitions

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20

Relations on subsets

For all sets A, B, C

- $A \cap B \subseteq A$
- $A \cap B \subseteq B$
- $A \subseteq A \cup B$
- $B \subseteq A \cup B$

Theorem: $A \cap B \subseteq A$.

Proof:

Let x be any element of $A \cap B$.

We must show that $x \in A$.

By definition of intersection, if $x \in A \cap B$, then x is an element of both A and B, so $x \in A$.

QED

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21

⊆ Is transitive

Theorem: If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Proof:

Let x be any element of A.

We must show that $x \in C$.

By definition of subset, if $x \in A$ then $x \in B$ and since $B \subseteq C$, again by definition of subset $x \in C$.

QED

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22

Proof methods (so far)

Truth table

Sequence of statements with reasons

Logic (modus ponens, modus tollens,…)

Predicate logic (quantification, vacuous truth)

Generalization from the generic particular

Proof by contradiction

Proof by contraposition

Proof by cases

Mathematical induction

Strong mathematical induction

Proof by set element

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23

What you should know

★ **Set theory proofs rely on definitions**

- How ∅ and $\boldsymbol{U}$ interact with arbitrary sets and with one another
- Intervals on the real number line are sets
- Element arguments facilitate set theory proofs

Any questions?

Next up: *Proofs with set theory*

Time to finish up that Opening sheet!

Problem set 13,14 is due on Monday, October 30 at 11PM

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24