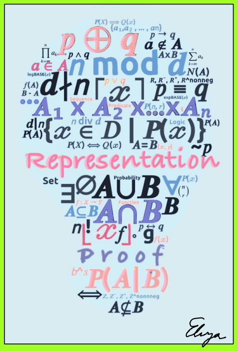
Discrete Structures



Lecture 19: Counting and disjoint sets

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HUNTER

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Last time

★ **There is a lot to think about when you count**

- Process of construction
- Permutation or combination
- Repetition
- Set size
- Disjoint cases

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Review: questions to ask when you count

- How big are the sets that are involved?
- Are the sets involved disjoint?
- Is there inherent order? ≡ is this a permutation or a combination?
- Is repetition okay?
- What process would construct an arbitrary element?
- Are there separate cases?

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Review: counting rules

Multiplication rule: If a process consists of k steps that can be performed respectively in n_1, n_2,⋯, n_k ways, then the entire process can be performed in $n_1 n_2 \cdots n_k$ ways

For any integer $n \geq 1$ and any set S of n elements,

$P(n, r)$: there are $\frac{n!}{(n-r)!}$ permutations of r elements from S

$C(n, r)$: there are $\frac{n!}{(n-r)!r!}$ combinations (ways to select) r elements from S

Addition rule: For any partition $\{A_1, A_2, \dots, A_n\}$ of a finite set A,

$$|A| = |A_1| + |A_2| + \cdots + |A_n|$$

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Today's outline

- More practice
- The difference rule
- Addition and difference

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Examples (1)

A committee is to be chosen from a set of 7 women and 4 men that includes Kamala Harris and Joe Biden. How many ways are there to form it if the committee is:

- of any positive size, but equal numbers of women and men?

$C(7,1)\cdot C(4,1)+C(7,2)\cdot C(4,2)+C(7,3)\cdot C(4,3)+C(7,4)\cdot C(4,4)$

- 4 people and at least 2 are women?

Cases: 2 women, 3 women, or 4 women

$$C(7,2)\cdot C(4,2)+C(7,3)\cdot C(4,1)+C(7,4)$$

- 4 people and one of them must be Kamala Harris?

Choose the others from a pool of 10

$$C(10,3)$$

- 4 people, 2 of each sex, and not both Kamala Harris and Joe Biden?

Count Kamala's committees, Joe's and neither's

$$C(6,1)\cdot C(3,2)+C(6,2)\cdot C(3,1)+C(6,2)\cdot C(3,2)$$

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Examples (2)

How many ways are there to pick a subset of 4 different letters from the alphabet?

$$C(26,4)$$

How many random 6-card hands have exactly 1 pair?

Choose the face value for the pair; then choose the 2 suits for it.

Finish the hand with 4 cards that have another face value and a suit for each of them.

$$C(13,1) \cdot C(4,2) \cdot C(12,4) \cdot [C(4,1)]^4$$

How many random 6-card hands have exactly the same number of hearts and spades?

How many of each?

Choose the hearts and spades and then the other cards

$$C(26,6) + [C(13,1)]^2 \cdot C(26,4) + [C(13,2)]^2 \cdot C(26,2) + [C(13,3)]^2$$

How many 10-bit strings have exactly 4 zeroes?

$$C(10,4) = \frac{10!}{6!\,4!}$$

Examples (3)

How many ways are there to pick 2 cards without replacement from a standard 52-card deck so that the first card is an ace and the second card is not a queen?

Without replacement indicates that this is a sequential process. Pick the first card and then from the depleted deck pick the second.

$$C(4,1) \cdot C(52-1-4,1)$$

So that the first card is a spade and the second card is not a queen?

Pick the first card but now there are 2 cases: that first card was or was not the queen of spades.

$$1 \cdot C(48,1) + C(12,1) \cdot C(47,1)$$

How many ways are there to pair off 10 horses at a stable with some of 20 riders?

$$P(20,10)$$

Examples (4)

How many 5-digit numbers contain exactly one 3?

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Well that depends whether leading zeroes are allowed. If they are

$$5 \cdot 9^4$$

But if they are not, there are 2 cases: whether the 3 is first or not.

$$9^4 + 4 \cdot 8 \cdot 9^3$$

There are 15 different apples and 10 different pears. How many ways are there for Jack to pick an apple or a pear and then for Jill to pick an apple and a pear?

There are 2 cases: Jack picks an apple or he picks a pear.

$$15 \cdot 14 \cdot 10 + 10 \cdot 15 \cdot 9$$

How many non-empty words over our alphabet have no more than 3 letters?

$$26^3 + 26^2 + 26$$

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Python identifiers

In Python, an identifier must begin with _ or a letter, other characters must be alphanumeric or _ and are case sensitive, There are also 33 reserved words. How many Python identifiers have 8 or fewer symbols?

$$53 + 53 \cdot 63 + 53 \cdot 63^2 + 53 \cdot 63^3 + 53 \cdot 63^4 + 53 \cdot 63^5 + 53 \cdot 63^6 + 53 \cdot 63^7 - 33$$

This example pulls in other mathematics we've covered. If you were coding this what should you notice?

The sum of terms in a geometric series: $\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}$

Now the count becomes

$$= \sum_{i=0}^{7} 53 \cdot 63^i - 33 = 53 \sum_{i=0}^{7} 63^i - 33 = 53 \cdot \frac{63^8 - 1}{63 - 1} - 33$$

Better yet you can write a function that tells you how many Python identifiers there are of length x:

$$f(x) = \frac{53}{62}(63^x - 1) - 33$$

(The material in your text on this topic is for IPv4. this is IPv6.)

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Today's outline

- ✓ More practice
- The difference rule
- Addition and difference

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The difference rule

Recall:

- Difference $A – B$ (aka set difference) = $\{\boldsymbol{x} | \boldsymbol{x} \in A \text{ and } x \notin B\}$
- For any partition $\{A_1, A_2, \ldots, A_n\}$ of a finite set A

$$|A| = |A_1| + |A_2| + \cdots + |A_n|$$

Theorem (aka the difference rule): For any finite set A and any subset B of A, $|A – B| = |A| - |B|$

Proof:

Let A be any finite set and B be any subset of A.

We must show that $|A – B| = |A| - |B|$.

By definition of set difference, $\{A - B, B\}$ partitions A.

By the addition rule, $|A - B| + |B| = |A|$

Thus $|A – B| = |A| - |B|$.

QED

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Examples (5)

PINs are made from exactly 5 alphanumeric symbols and may include repeated characters.

How many PINs are possible?

$$36^5$$

How many include at least one repeated symbol?

It is easier to count the complement (no repeated symbols) and subtract.

$$36^5 - 36 \cdot 35 \cdot 34 \cdot 33 \cdot 32$$

What is the probability that a PIN has at least one repeated symbol?

$$\frac{36^5 - 36 \cdot 35 \cdot 34 \cdot 33 \cdot 32}{36^5}$$

What is the probability that a non-empty word over our alphabet has more than 10 letters but less than 20?

Easier to consider probability of complement (between 1 and 10 letters)

$$\frac{26^{11}}{26^{20}} + \frac{26^{12}}{26^{20}} + \frac{26^{13}}{26^{20}} + \frac{26^{14}}{26^{20}} + \frac{26^{15}}{26^{20}} + \frac{26^{16}}{26^{20}} + \frac{26^{17}}{26^{20}} + \frac{26^{18}}{26^{20}} + \frac{26^{19}}{26^{20}}$$

and subtract it from1: $1 - \sum_{i=1}^{10} \frac{26^i}{26^{20}}$

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Different ways to count for the same problem

There are 5 different Spanish books, 6 different French books and 9 different Transylvanian books. How many ways are there to pick 2 books not both in the same language?

Pick the languages (Spanish and French, Spanish and Transylvanian), Spanish, French and Transylvanian) then the books

$$5 \cdot 6 + 5 \cdot 9 + 6 \cdot 9$$

OR pick 2 books and then subtract the ones in the same language.

$$C(20,2) - (C(5,2) + C(6,2) + C(9,2))$$

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Count the complement

- Sometimes it is easier to count the opposite of an event and subtract.
- Recall: complement $A^C = \{x | x \in U, x \notin A\}$
- Set difference rule for finite set A and its subset B: $|A - B| = |A| - |B|$

- Complement of E in S = what could happen instead = $S - E = E^C$
  roll a non-3 pick a heart, club or spade tails

Theorem: For any subset A of finite set U, $|A^C| = |U| - |A|$..
Proof:
We must show that $|A^C| = |U| - |A|$.
By definition of complement, $A^C = U - A$.
By the set difference rule, $|U - A| = |U| - |A|$
Thus $|A^C| = |U| - |A|$.
QED

Examples (6)

Complement of E in S = what could happen instead = $S - E = E^C$

roll a non-3	pick a heart, club or spade	tails
{1,2,4,5,6}	pick a diamond	heads

How many arrangements of the letters a, b, c, d, e are there in which the sequence "ace" does not appear?

$$5! - 3!$$

How many random 6-card hands have exactly 1 pair?
Pick the face value of the pair,
then the 2 suits for it, and then the rest of the hand
which avoids the pair's face value when it chooses 4 others
and then picks the cards for them.

$$C(13,1) \cdot C(4,2) \cdot C(12,4) \cdot [C(4,1)]^4$$

How many random 6-card hands have 1 pair or more?
The complement is no pairs which is much easier to count.
Choose the 6 different face values and then their respective cards.

$$C(52,6) - C(13,6) \cdot [C(4,1)]^6$$

Probability of the complement

- Kolmogorov's axioms:
  Given a probability function $P: \mathcal{P}(S) \to [0,1]$, for all events A and B in S
  - $0 \le P(A) \le 1$
  - $P(\emptyset) = 0$ and $P(S) = 1$
  - If A and B are disjoint $(A \cap B = \emptyset)$ then $P(A \cup B) = P(A) + P(B)$
- Complement $A^C = \{x | x \in U, x \notin A\}$

Theorem: For any event A in S, $P(A^C) = 1 - P(A)$.
Proof:
We must show that $P(A^C) = 1 - P(A)$.
By definition of A^C, $A \cap A^C = \emptyset$ so by axiom 3, $P(A \cup A^C) = P(A) + P(A^C)$
Without loss of generality, take U to be S.
Since $A \cup A^C = U$, by axiom 2, $P(A \cup A^C) = 1$

$$P(A) + P(A^C) = 1$$
$$P(A^C) = 1 - P(A)$$

QED

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Examples (7)

PINs are made from exactly 5 alphanumeric symbols and may include repeated characters.
What is the probability that a PIN has at least one repeated symbol?

$$\frac{36^5 - 36 \cdot 35 \cdot 34 \cdot 33 \cdot 32}{36^5}$$

But also with $P(A^C) = 1 - P(A)$ as $1 - \frac{36 \cdot 35 \cdot 34 \cdot 33 \cdot 32}{36^5}$

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Today's outline

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Examples (8)

How many ways can we choose 5 of 13 people to form a team?

$$C(13,5) = \frac{13!}{8!\,5!}$$

How many ways can we choose 5 of 13 people to form a team if 2 of them must be together, that is, both on the team or off the team?

Either we choose the pair or we don't.

$$C(11,3) + C(11,5) = \frac{11!}{8!\,3!} + \frac{11!}{6!\,5!}$$

How many ways can we choose 5 of 13 people to form a team if two of them must not be together?

Count teams with A but not B, teams with B but not A, and teams with neither.

$$C(11,4) + C(11,4) + \mathrm{C}(11,5)$$

OR use the set difference rule

$$C(13,5) - C(11,3)$$

Example (9)

How many salads with 5 ingredients can be made from a group of 13 that has 7 green vegetables and 6 red vegetables?

$$C(13,5) = \frac{13!}{8!\,5!}$$

How many 5-ingredient salads with can be made if there must be 3 green and 2 red vegetables?

$$C(7,3) \cdot C(6,2) = \frac{7!}{4!3!} \cdot \frac{6!}{4!2!}$$

How many 5-ingredient salads with can be made if there must be at least 1 green vegetable?

$$C(13,5) - C(6,5)$$

OR use the addition rule

$$C(7,1) \cdot C(6,4) + C(7,2) \cdot C(6,3) + C(7,3) \cdot C(6,2) + C(7,4) \cdot C(6,1) + C(6,5) \cdot C(6,0)$$

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What you should know

★ **Set theory supports counting**

- How to use the set difference rule
- How to use complements

Any questions?

Next up: *More counting principles*

Time to finish up that Opening sheet!

Problem set 19,20 is due on Thursday, November 23 at 11PM

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