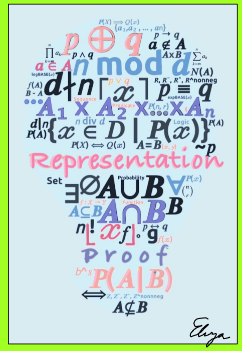
Discrete Structures



Lecture 6: More on predicate calculus

Susan L. Epstein

HUNTER

1

Last time

- **Predicate calculus extends logical representation**
- **Validity ≠ truth**
- Invalid argument forms
- How to express facts in predicate calculus
- How to translate both ways between English and predicate calculus
- How to negate quantified statements
- How to demonstrate uniqueness

Fall 2023 CSCI 150 2/22

2

Today's outline

- Multiple quantifiers in predicate calculus
- Argumentation in predicate calculus
- Formal verbal proofs

Predicate calculus extends logical representation

3

Multiple quantifiers

For x, $D = \{x | x$ is a person$\}$, for y, $E = \{y | y$ is a dog$\}$
$P(x, y)$ means "x loves y"

- $\forall x \in D, \forall y \in E\ P(x, y) \equiv \forall x, y\ P(x, y)$ Everyone loves all dogs
  To prove this, test every possible pair of person and dog
- $\exists x \in D, \exists y \in E \ni P(x, y) \equiv \exists x \in D, y \in E \ni P(x, y)$ Someone loves some dog
  To prove this, test every possible pair of person and dog
- $\forall x \in D, \exists y \in E \ni P(x, y)$ Everyone loves some dog
  To prove this, pick an arbitrary $x \in D$ and find its y
- $\exists x \in D \ni \forall y \in E\ P(x, y)$ Someone loves all dogs
  To prove this, pick an arbitrary $x \in D$ and show it works for every y
- $\forall x \in D, \exists!\, y \in E \ni P(x, y)$ Everyone loves exactly one dog
- $\exists!\, x \in D \ni \forall\, y \in EP(x, y)$ There is only one person who loves all dogs

4

Negation of multiple quantifiers

- Recall that $\sim[\forall x \in D, P(x)] \equiv \exists x \in D \ni\ \sim P(x)$
  $\sim[\exists x \in D \ni D(x)] \equiv \forall x \in D \sim P(x)$

- $\sim(\forall x \in D, \exists y \in E \ni P(x,y))$
  $\equiv \exists x \in D \sim(\exists y \in E \ni P(x,y))$ why?
  $\equiv \exists x \in D \ni \forall y \in E \sim P(x,y)$ why?
  It is false that everyone loves some dog ≡ Someone loves no dogs

  For x, $D = \{x | x$ is a person$\}$,
  for y, $E = \{y | y$ is a dog$\}$
  $P(x,y)$ means "x loves y"

- $\sim(\exists x \in D, \ni \forall y \in E\ P(x,y))$
  $\equiv \forall x \in D \sim(\forall y \in E \ni P(x,y))$ why?
  $\equiv \forall x \in D\ \exists y \in E \ni \sim P(x,y)$ why?
  It is false that someone loves all dogs ≡
  Everyone has some dog they do not love

Fall 2023 CSCI 150 5/22

5

What's the negation of ...?

$\forall x \in D\ \exists y \in E$ such that $x + y = 1$

$\sim(\forall x \in D\ \exists y \in E \ni x + y = 1) \equiv$
$\exists x \in D \ni \sim\exists y \in E \ni x + y = 1 \equiv$
$\exists x \in D \ni \forall y \in E\ x + y \neq 1$

$\exists x \in D$ such that $\forall y \in E\ x + y = -y$

$\sim(\exists x \in D \ni \forall y \in E\ x + y = -y) \equiv$
$\forall x \in D \sim(\forall y \in E\ x + y = -y) \equiv$
$\forall x \in D\ \exists y \in E \ni x + y \neq -y$

Any questions?

Fall 2023 CSCI 150 6/22

6

Today's outline

- ✓ Multiple quantifiers in predicate calculus
- Argumentation in predicate calculus
- Formal verbal proofs

Fall 2023 CSCI 150 7/22

7

Valid argument forms in predicate calculus

- Valid argument form in predicate calculus = for any predicate symbols used in the premises, when the resultant premises are all true the conclusion is true

$A = \{x | x \text{ is alive}\}$ Lola $\in A$

$P(x) = x$ is a dog $Q(x)$ = is a mammal

- Universal modus ponens

$\forall x\, P(x) \rightarrow Q(x)$
$P(a)$ for a particular $a \in A$
$\therefore Q(a)$

All dogs are mammals
Lola is a dog
∴ Lola is a mammal

- Universal modus tollens

$\forall x\, P(x) \rightarrow Q(x)$
$\sim Q(a)$ for a particular $a \in A$
$\therefore \sim P(a)$

Booby $\in A$
All dogs are mammals
Booby is not a mammal
∴ Booby is not a dog

Fall 2023 CSCI 150 8/22

8

Fallacies in predicate calculus

$D = \{x | x \text{ is alive}\}$
$P(x)$ = is a dog
$Q(x)$ = is a mammal
Sally $\in D$

Converse error
$\forall x\ P(x) \rightarrow Q(x)$
$Q(d)$ for a particular $d \in D$
$\therefore P(d)$

All dogs are mammals
Sally is a mammal
∴ Sally is a dog

Inverse error
$\forall x\ P(x) \rightarrow Q(x)$
$\sim P(d)$ for a particular $d \in D$
$\therefore \sim Q(d)$

All dogs are mammals
Sally is not a dog
∴ Sally is not a mammal

Fall 2023 CSCI 150 9/22

9

Predicate calculus and mathematical statements

Not proving these, just getting ready to…

- The sum of 2 positive integers is always positive
  $\forall x, y\ (x > 0 \wedge y > 0) \rightarrow (x + y) > 0$
  $\forall x, y \in \boldsymbol{Z}^{+} (x + y) > 0$
- Every real number except 0 has a multiplicative inverse
  $\forall x \in \boldsymbol{R}\ x \neq 0\ \exists y \in \boldsymbol{R} \ni xy = 1$
- There is no smallest positive real number
  $\forall x \in \boldsymbol{R}^{+}\ \exists y \in \boldsymbol{R}^{+} \ni y < x$

Fall 2023 CSCI 150 10/22

10

There's so much more

- Predicate calculus can be extended
  - Add operators for numbers $=, \neq$
  - Add operators for sets $\cup, \cap$
  - Quantify not just over variables but also over predicates
- Proofs can introduce variables dependent on others for their existence
- Prolog is a programming language that, given the premises of a true theorem in a limited version of predicate calculus, can prove the theorem
- $\exists$ many other valid argument forms and many other ways to construct a proof … we will use them on additional foundational mathematics

Any questions?

11

Today's outline

✓ Multiple quantifiers in predicate calculus

✓ Argumentation in predicate calculus

- Formal verbal proofs

12

Motivation

- A proof is an argument that a statement is true
- Thus far, proofs have been by
  - Truth table
  - Substitution of logical equivalents
  - Substitution into valid argument forms
- Now we look at formal arguments directed at a variety of targets
- Proof formats are intended to help the reader follow the argument

Fall 2023 CSCI 150 13/22

13

What a good (verbal) proof looks like

Theorem: (copy the statement here)

Proof:

Let/Assume/Suppose: Name variables and state what they stand for

be general: any state any assumptions

We must show that…

multiple grammatically correct sentences

Give a reason for every assertion

Clarify your logic: connect statements with reasons Thus Then

Therefore So Hence Consequently It follows that

Display equations and inequalities centered and on separate lines

By definition of By substitution Because Since

QED = *quod erat demonstrandum* = that which we intended to show = □ = ■ (aka tombstone)

Fall 2023 CSCI 150 14/22

14

Proof frame

Theorem: (copy the statement here)

Proof:

Let/Assume/Suppose: Name variables and state what they stand for

be general state any assumptions

We must show that…

multiple grammatically correct sentences

Clarify your logic with a reason for every assertion

Display equations and inequalities clearly

QED

Fall 2023 CSCI 150 15/22

15

Definitions as raw material

- Often a proof relies on formal definitions

  We will use these as an example:

  $n \in \boldsymbol{Z}$ is even iff $\exists k \in \boldsymbol{Z}$ such that $n = 2k$ 30 516

  $n \in \boldsymbol{Z}$ is odd iff $\exists k \in \boldsymbol{Z}$ such that $n = 2k + 1$ 899 -5

- Proofs often rely on properties of sets as well

  We will use these as an example:

  Closure: Set S is closed under operation $\diamond$ iff $\forall x, y \in S,\ x \diamond y \in S$

  $\boldsymbol{Z}$ is closed under addition

  The square of any negative number is positive

Fall 2023 CSCI 150 16/22

16

A correct, formal proof

Theorem: The sum of any 2 odd integers is even.

Proof:

Let x and y be any odd integers.

We must show that $x + y$ is even.

By definition of odd number, $\exists a, b \in \boldsymbol{Z}, x = 2a + 1$ and $y = 2b + 1$.

By substitution, $x + y = 2a + 1 + 2b + 1 = 2a + 2b + 2$
which factors into
$2(a + b + 1)$.

Because $\boldsymbol{Z}$ is closed under addition, $(a + b + 1) \in \boldsymbol{Z}$.

By definition of even number, $2(a + b + 1)$ is even.

And since $x + y = 2(a + b + 1), x + y$ is even.

QED

Fall 2023 CSCI 150 17/22

17

These examples are incorrect!

- Do not argue for a theorem about an infinite set with examples.
  13 is odd and 19 is odd and their sum 32 is even. So…
- Do not reuse the same variable name for 2 different values.
  $\exists k \in \boldsymbol{Z}, x = 2k + 1$ and $y = 2k + 1$.
- Do not skip steps.
  By substitution $x + y = 2(a + b + 1)$
- Do not use circular reasoning.
  Suppose x and y are any odd integers. When any odd integers are added, their sum is even. Hence $x + y$ is even.
- Do not incorporate the conclusion with a variable.
  To show $x + y$ is even we must show $\exists k \in \boldsymbol{Z}, x + y = 2k$
- "any" ≢ "some" Any is like ∀; suggests a particular element
  By definition of odd, $m = 2a + 1$ for any integer a.
- Do not use "if" when you mean "because."
  Suppose p is a prime number. If p is prime, then p cannot be written as a product of two smaller positive integers.

Fall 2023 CSCI 150 18/22

18

Proof by cases skeleton

Theorem: If A_1 or A_2 or… or A_n then C.
Proof:

Let/Assume/Suppose: Name variables and state what they stand for
be general: any state any assumptions

We must show that C is true in each of the following cases:
Case 1: If A_1 then C.
Case 2: If A_2 then C.
…
Case n: If A_n then C

Clarify your logic with a reason for every assertion
Display equations and inequalities clearly

Thus C is true regardless of which is the case.

QED

Fall 2023 CSCI 150 19/22

19

A simple proof by cases

Theorem: If $n \in \boldsymbol{Z}$ then $n^2 \geq n$.

Proof:

Let $n \in \boldsymbol{Z}$.

We must show that $n^2 \geq n$ is true in each of the following cases:

$n = 0$, $n \geq 1$, or $n \leq -1$.

Case 1: $n = 0$

$0^2 = 0 \geq 0$ so true for $n = 0$

Case 2: $n \geq 1$

Multiplying both sides of $n \geq 1$ by n: $n^2 \geq n$ so true for $n \geq 1$

Case 3: $n \leq -1$

Since the square of any negative number is positive, $n^2 > 0$ and $0 > n$, so $n^2 \geq n$ and true for $n \leq -1$

Thus $n^2 \geq n$ is true regardless of which is the case.

QED

Fall 2023 CSCI 150 20/22

20

Proof methods (so far)

Truth table

Sequence of statements with reasons

Valid argument forms (modus ponens, modus tollens,…)

Method of exhaustion

Predicate logic (quantification, existence, uniqueness)

Proof by cases

Fall 2023 CSCI 150 21/22

21

What you should know

★ **Predicate calculus extends logical representation**

- How to translate multiply quantified statements from logic to English and from English to logic
- How to construct and negate multiply quantified predicate statements
- Valid arguments in FOPC
- Basic formal proof structures
- Proof by cases

Any questions?

Next up: *Introduction to number theory*

Time to finish up that Opening sheet!

Problem set 5,6 is due on Thursday, September 21 at 11PM

Fall 2023 CSCI 150 22/22

22