Discrete Structures

Lecture 23: Graph theory

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HUNTER

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Last time

- ⭐ **Proofs may reason combinatorically**
- How to efficiently generate polynomials
- Russell's paradox
- The halting problem

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Today's outline

- Definitions
- Graph models
- Graph properties

The material on graphs and trees in CSCI 150 supersedes your text

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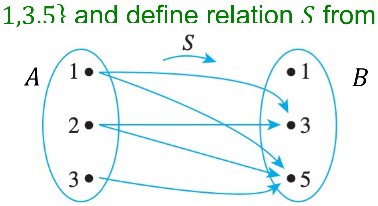
Review: relations

- Binary relation R from set A to set B is a subset of their Cartesian product $A \times B$

  If $A = \{1,23\}, B = \{4,20\}$, $A \times B = \{(1,4),(1,20),(23,4),(23,20)\}$

  Relation $R = \{(23,4),(23,20)\}$ collects $\{(a,b) | a \in A, b \in B, a > b\}$
- Can picture a binary relation with an arrow diagram

  $A = \{1,2,3\}, B = \{1,3.5\}$ and define relation S from A to B to mean $x < y$



This is not a function
Why not?

- Set A is the domain of R and set B is the co-domain of R

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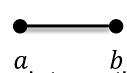
Definitions

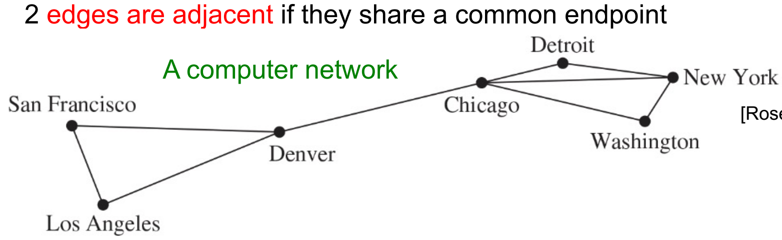
- A graph is a special kind of relation among a set of elements
  - Rooms in a floorplan — Chemical elements
  - Countries — Authors and papers
- **Warning: a graph is NOT a picture** — graphs are often accompanied by pictures because they help us "see" how elements of a set relate to one another, but **graphs are an abstract mathematical concept**
- Graph (V, E) is an ordered pair of sets where
  - V is a finite non-empty set of vertices (aka nodes)
  - E is a set of edges that are pairs of nodes
    $V = \{1,2,3\}$ $\quad$ $E = \{(1,1), (1,2), (1,3)\}$
  - By definition, $E \subseteq V \times V$
    $V = \{1,2,3\}$ $\quad$ $E = \{(3,2)\}$ $\quad$ $E = \emptyset$
  - E can be defined explicitly or implicitly
    $E = \{(1,1), (2,2), (3,3)\}$ $\quad$ $E = \{(a, b) | a = b\}$

Edges

- An edge describes how its endpoints are related to one another
  - V = {rooms in a floorplan} have a connecting door
  - V = {chemical elements} have a chemical bond between them
  - V = {countries} have a common border
  - V = {authors} have co-authored a paper
- The endpoints of $e \in E$ are $\{a,b\}$

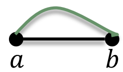


- If $e \in E$, then e connects its endpoints
  - 2 vertices are adjacent if they are endpoints on the same edge
  - a is the neighbor of b and b is the neighbor of a
- An edge is incident on its endpoints
- 2 edges are adjacent if they share a common endpoint



A computer network

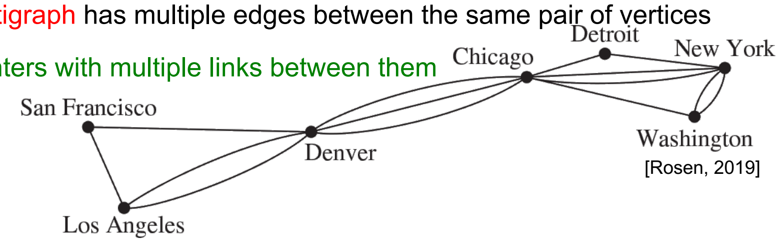
[Rosen, 2019]



Special edges

- A multigraph has multiple edges between the same pair of vertices

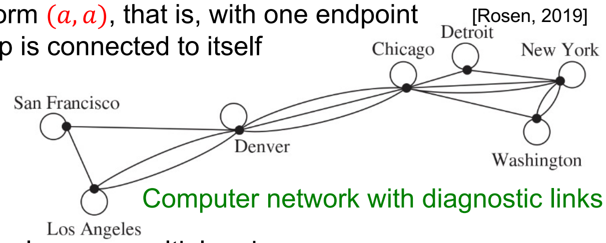
Data centers with multiple links between them



[Rosen, 2019]

- Loop = edge of the form (a, a), that is, with one endpoint
  - Endpoint of a loop is connected to itself

[Rosen, 2019]



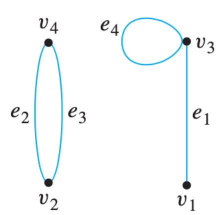
Computer network with diagnostic links

- A simple graph has no loops or multiple edges

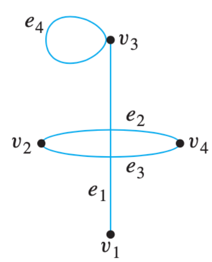
Multiple ways to draw a graph

For $G = (\{v_1, v_2, v_3, v_4\}, \{e_1, e_2, e_3, e_4\})$ where

Edge	Endpoints
e_1	$\{v_1, v_3\}$
e_2	$\{v_2, v_4\}$
e_3	$\{v_2, v_4\}$
e_4	$\{v_3\}$



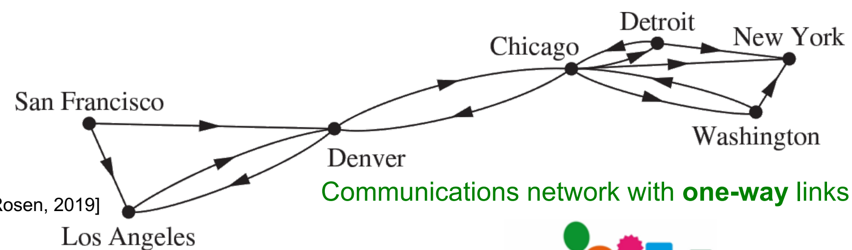
Both of these are drawings of G



Edges may cross in a drawing of a graph without going through a vertex

Directed graphs

- Thus far definitions have been for undirected graphs, where the order of the vertices on an edge does not matter
- Directed graph (aka digraph) (V, E) is an ordered pair of sets where
  - V is a finite non-empty set of vertices (aka nodes)
  - E is a set of edges (aka arcs) that are **ordered** pairs of vertices
  - Edge (a, b) is said to start at a and end at b
- Diagrams of such a network indicate direction with arrows on the edges



[Rosen, 2019]

Communications network with **one-way** links

Any questions?

Today's outline

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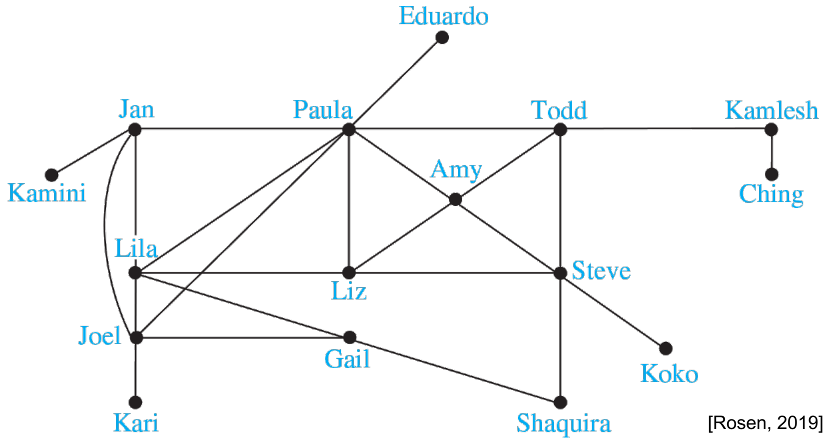
- Graph models
- Graph properties

Graphs model real-world relations

A social network

People who are friends or acquaintances

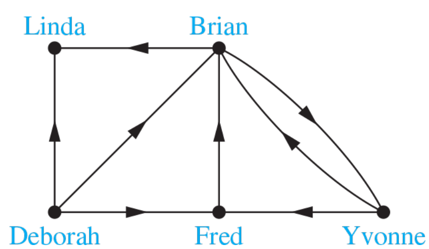


[Rosen, 2019]

For the planet this would be ~8.1 billion nodes and > 1.5 trillion edges

An influence digraph

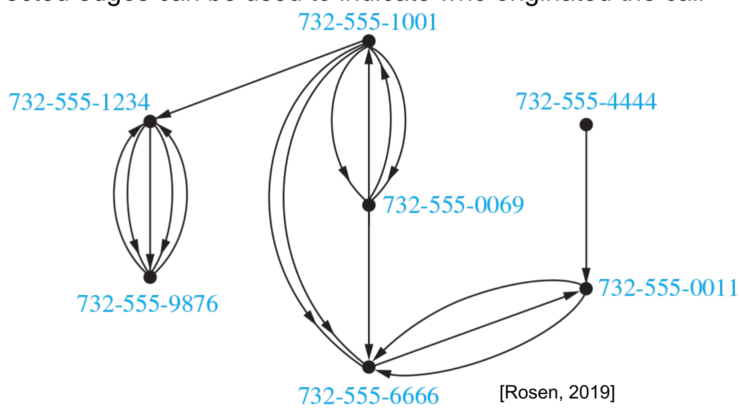
Directed edge from influencer to the one who is influenced



[Rosen, 2019]

Call graphs

- Multiple edges distinguish individual calls
- Directed edges can be used to indicate who originated the call



[Rosen, 2019]

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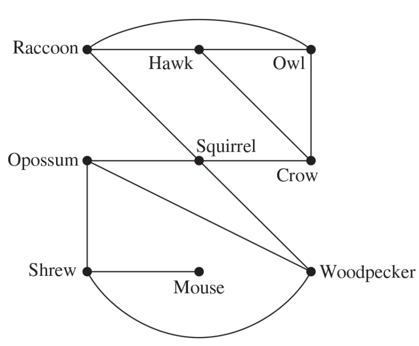
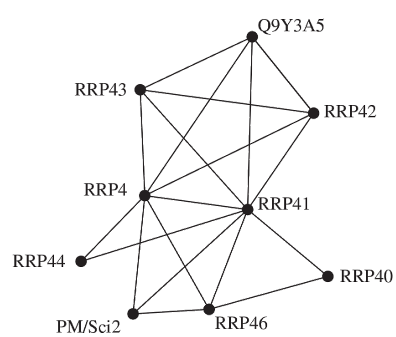
More graph models



Ecological niche overlap



Protein interaction

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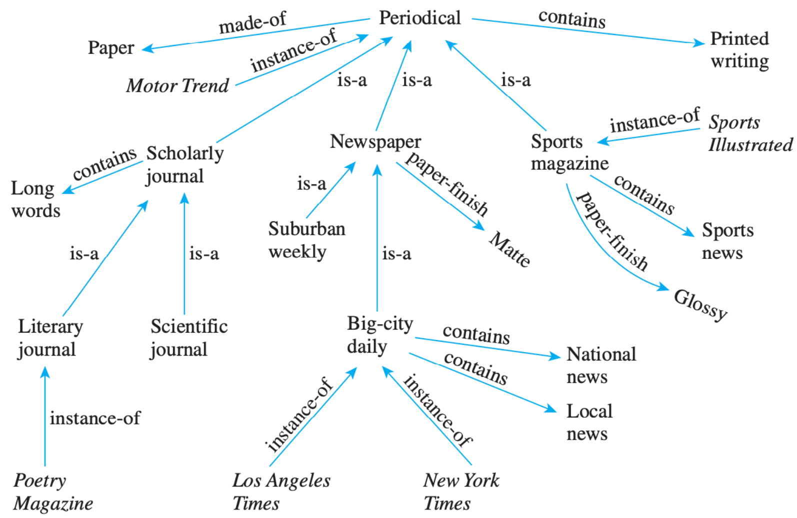
An AI model: An ontological graph



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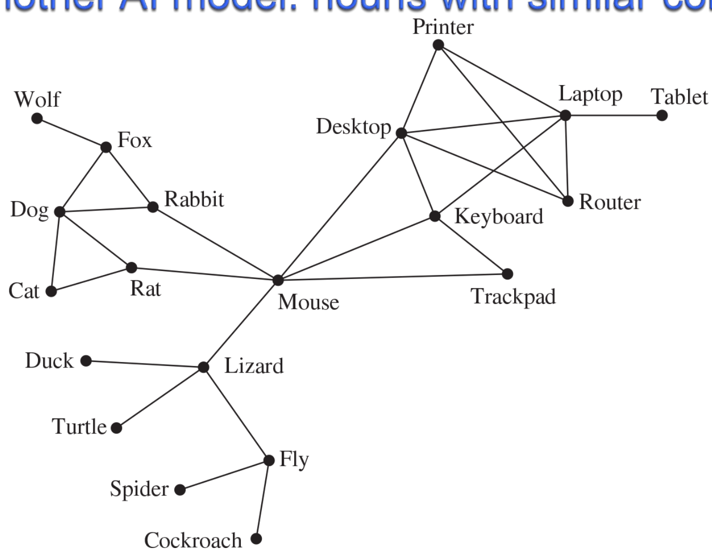
Another AI model: nouns with similar context



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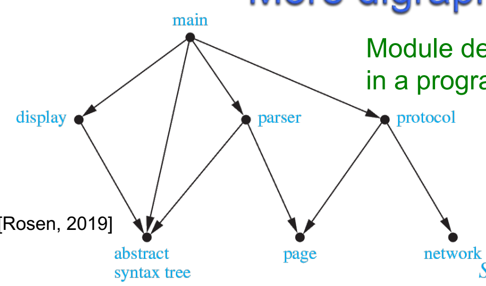
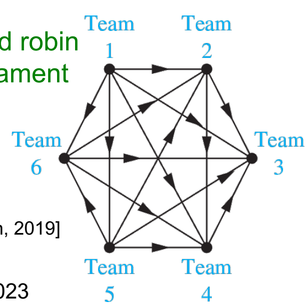
More digraph models

Module dependency in a program



[Rosen, 2019]

Round robin tournament



[Rosen, 2019]

Precedence graphs in concurrent processing

S_1 $a := 0$

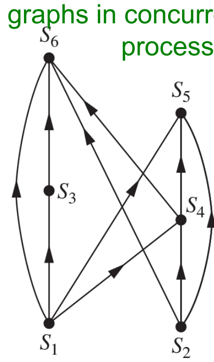
S_2 $b := 1$

S_3 $c := a + 1$

S_4 $d := b + a$

S_5 $e := d + 1$

S_6 $e := c + d$



[Rosen, 2019]

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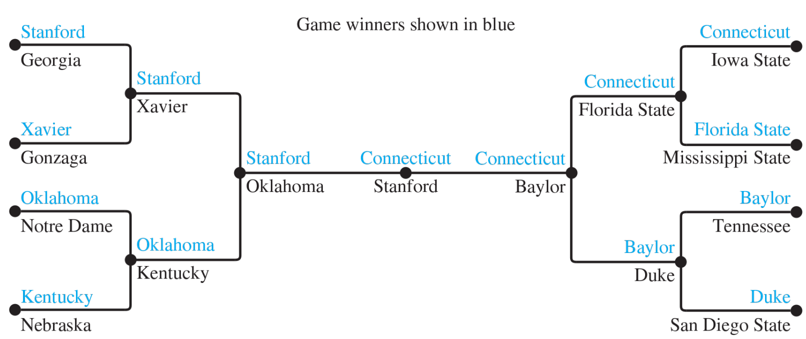
NCAA 2010 women's tournament



[Rosen, 2019]

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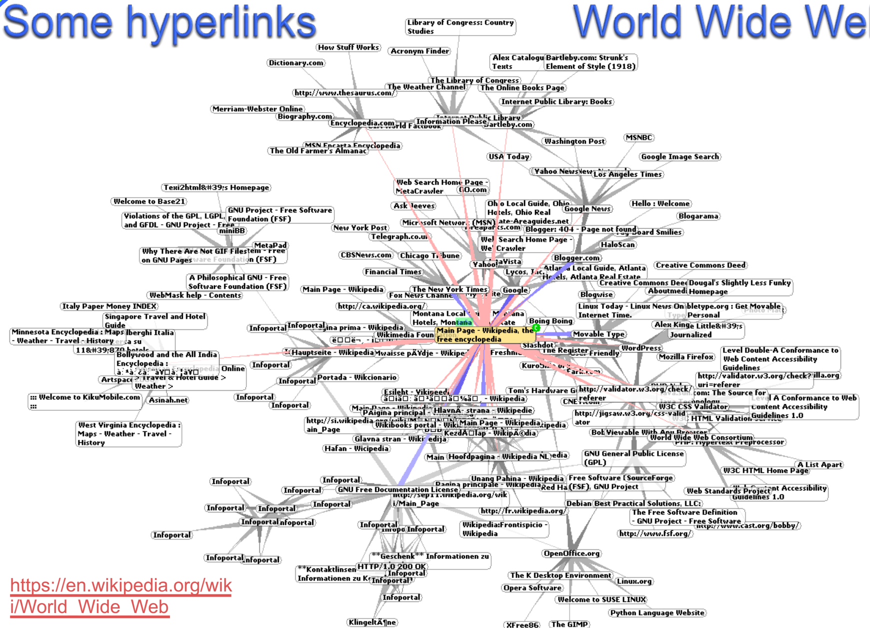
Some hyperlinks World Wide Web



https://en.wikipedia.org/wiki/World_Wide_Web

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Other graph models

- Artistic collaboration
- Authorship collaboration
- Citation graphs
- Airline routes
- Road networks

Any questions?

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Today's outline

- ✓ Definitions
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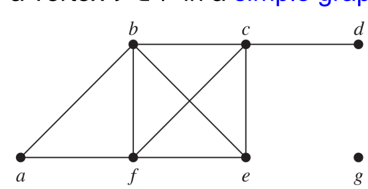
Properties of a vertex

- Recall: if e is an edge in $G = (V, E)$ with endpoints $\{a,b\}$, then a is the neighbor of b and b is the neighbor of a
- Neighborhood $N(v)$ of a vertex $v \in V$ in $G = (V, E)$ is the set of all v's neighbors
- Degree $\deg(v)$ of a vertex $v \in V$ in a simple graph $G = (V, E)$ is $|N(v)|$



[Rosen, 2019]

- An isolated vertex has degree 0
- Total degree of a graph $G = (V, E)$ is the sum $\sum_{v \in V} \deg(v)$ of the degrees of its vertices

Proof skeleton

Theorem: (copy the statement here)

Proof:

Let/Assume/Suppose: Name variables and state what they stand for

be general: any state any assumptions

We must show that…

multiple grammatically correct sentences

Clarify your logic with a reason for every assertion Thus Then

Therefore So Hence Consequently It follows that

By definition of By substitution Because Since

Display equations and inequalities clearly

QED

The handshake theorem

Theorem: If $G = (V, E)$ is any graph, then $\sum_{v \in V} \deg(v) = 2|E|$.
Proof:
Let $G = (V, E)$ be any graph with n vertices and $|\mathrm{E}| = m$ edges.
We must show that m is even.
Let $e \in E$ be any edge. e contributes 1 to the total degree for each of its endpoints and therefore 2 to the total degree.
Since e was chosen arbitrarily, every edge (including any loops) contributes 2 to the total degree.
By definition of total degree, $\sum_{v \in V} \deg(v)$ is the sum of m 2's, that is, $\sum_{v \in V} \deg(v) = 2m$.
By definition of even, $\sum_{v \in V} \deg(v)$ is even.
QED

Corollary

Theorem: The total degree of a graph is even.
Proof:
Let $G = (V, E)$ be any graph with n vertices and m edges.
By the handshake theorem $\sum_{v \in V} \deg(x) = 2m$, and by definition of even number, that is an even number.
QED

Can there be a simple graph with 4 vertices of degree 1,1,3,and 6?

Imagine a social network graph with 9 nodes. Can each person (as represented by a node) have exactly 5 friends?

Proof by contradiction skeleton

Theorem: (copy the statement here)

Proof:

Assume: the negation of the conclusion
be general: any state any assumptions
We will show that this assumption logically leads to a contradiction.

Clarify your logic with a reason for every assertion
Display equations and inequalities clearly

Contradiction. Because the assumption led to a contradiction, negation of the assumption.

QED

A simple graph theory proof

Theorem: Any graph has an even number of vertices of odd degree.
Proof:
Let $G = (V, E)$ be any graph with n vertices and m edges, V_E be its vertices of even degree, and V_O = its vertices of odd degree.
Assume $|V_O|$ is odd.
We will show that this assumption logically leads to a contradiction.
Since $V_E \cup V_O = V$ and V_E and V_O are disjoint they partition V, so

$$|V_E| + |V_O| = n$$

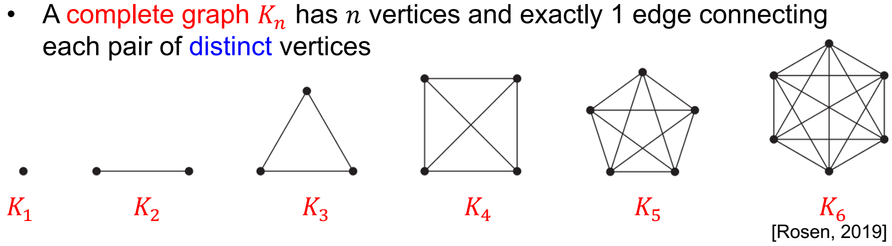
Because the sum of even numbers is even, $\sum_{v \in V_E} \deg(v)$ is even, say, $2k$.
By the handshake theorem, $\sum_{v \in V_E} \deg(v) + \sum_{v \in V_O} \deg(v) = 2m$.
By substitution, $2k + \sum_{v \in V_O} \deg(v) = 2m$ so

$$\sum_{v \in V_O} \deg(v) = 2(m - k)$$

The left side is the sum of an odd number of odd numbers and so is odd, but the right side is even by definition of even. Contradiction.
Because the assumption led to a contradiction, $|V_O|$ is even. **QED**

Complete graphs

- A complete graph K_n has n vertices and exactly 1 edge connecting each pair of distinct vertices



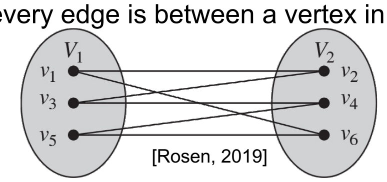
$$K_n = (\{v_1, v_2, \ldots v_n\}, \{(v_i, v_j) | i \neq j\})$$

- How many edges are there in a complete graph on n vertices?

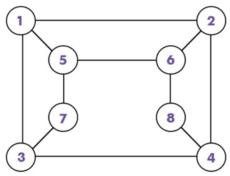
$$C(n, 2)$$

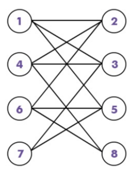
Bipartite graphs

Bipartite graph = simple graph $G = (V, E)$ where V can be partitioned into V_1 and V_2 such that every edge is between a vertex in V_1 and a vertex in V_2

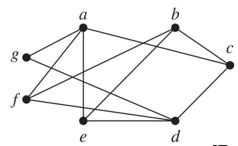


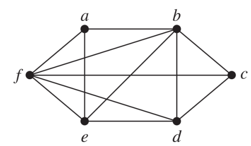
[Rosen, 2019]





Are these graphs bipartite?



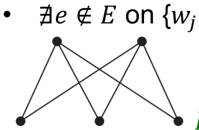


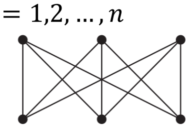
[Rosen, 2019]

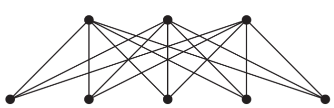
Complete bipartite graphs

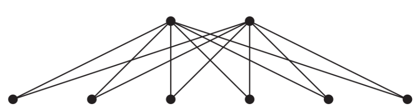
Complete bipartite graph $K_{m,n}$ = simple graph on m distinct vertices v_1, $v_2, \ldots, v_m$ and n distinct vertices $w_1, w_2, \ldots, w_n$ where

- $\exists e \in E$ on $\{v_i, w_j\}$ $\forall i = 1,2, \ldots, m$ and $\forall j = 1,2, \ldots, n$
- $\nexists e \notin E$ on $\{v_i, v_k\}$ for any other $k = 1,2, \ldots, m$
- $\nexists e \notin E$ on $\{w_j, w_l\}$ for any other $l = 1,2, \ldots, n$


$K_{2,3}$


$K_{3,3}$


$K_{3,5}$


$K_{2,6}$

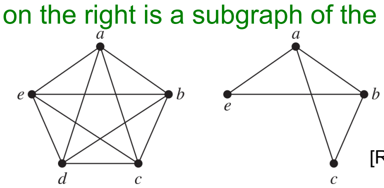
[Rosen, 2019]

- How many vertices in $K_{m,n}$?
- How many edges in $K_{m,n}$?
- What is the degree sum in $K_{m,n}$?

Subgraphs

- $H(W, F)$ is a subgraph of graph $G(V, E)$ iff H is a graph and $W \subseteq V$, $W \neq \emptyset$, and $F \subseteq E$

  The graph on the right is a subgraph of the graph on the left



[Rosen, 2019]

- $H(W, F)$ is a proper subgraph of graph $G(V, E)$ iff $W \subset V$ and $E \subset F$
- How many complete subgraphs are there of a complete graph on n distinct vertices?

$$|2^n| - 1$$

- How many complete proper subgraphs are there of a complete graph on n vertices?

$$|2^n| - 2$$

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Counting subgraphs

How many subgraphs are there of a simple graph on $n \geq 1$ distinct vertices and m distinct edges?

On 1 vertex? $C(1,1)$
On 2 vertices? $C(2,1) + C(2,2)2^{C(2,2)}$
On 3 vertices? $C(3,1)2^{C(1,2)} + C(3,2)2^{C(2,2)} + C(3,3)2^{C(3,2)}$

On 4 vertices? $C(4,1)2^{C(1,2)} + C(4,2)2^{C(2,2)} + C(4,3)2^{C(3,2)} + C(4,4)2^{C(4,2)}$

…

On n vertices: For $C(n,2) \leq m$, $\sum_{\substack{i=1 \\ C(n,2) \leq m}}^{n} C(n,i)2^{C(i,2)}$

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Proof methods (so far)

Truth table
Sequence of statements with reasons
Valid argument forms (modus ponens, modus tollens,…)
Predicate logic (quantification, existence, uniqueness)
Proof by cases
Generalization from the generic particular
Proof by contradiction
Proof by contraposition
Mathematical induction
Strong mathematical induction
Algebraic proof by set theory
Algebraic proof by properties of functions
Algebraic proof by combinatorics
Algebraic proof by graph theory

What you should know

Graphs model real-world relations

- Definition of a graph
- Special kinds of graphs
- Properties of a vertex

Next up: *More on graphs*

Time to finish up that Opening sheet!

Any questions?

Problem set 23,24 is due on Thursday, December 7 at 11PM