Automatic Control
Laboratory practice 9

Exam simulation - notes

July 1, 2024

BIBO stability theory

A generic CT LTI system is BIBO stable if and only if all the poles of its transfer function have negative real part. "If and only if" means that the condition is sufficient and necessary.

The poles of the transfer function are a subset of the eigenvalues of the matrix A. Therefore, if all the eigenvalues of A have negative real part, then the system is BIBO stable.

Internal and BIBO stability of CT systems

Given the LTI dynamical system

$$\dot{x}(t) = \begin{pmatrix} -1 & 3 & 0 & 0 \\ -4 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} x(t) + \begin{pmatrix} -1 \\ 1 \\ 2 \\ 1 \end{pmatrix} u(t)$$

$$y(t) = (0 \;\; 0 \;\; 0 \;\; 1) x(t)$$

the real parts of the eigevalues of A are -1 and 0. The eigenvalue 0 has algebraic multiplicity $\mu = 1$, therefore its minimal multiplicity is $\mu' = 1$. Thus, the system is internally stable (not asymptotically).

The poles of the transfer function are 0 and -1, therefore the system is BIBO unstable.

~~[INF] Internal and BIBO stability of DT systems~~

~~Given the DT LTI dynamical system~~

$$x(k+1) = \begin{pmatrix} -1 & 3 & 0 & 0 \\ -4 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} x(k) + \begin{pmatrix} -1 \\ 1 \\ 2 \\ 1 \end{pmatrix} u(k)$$

$$y(k) = (0 \;\; 0 \;\; 0 \;\; 1)x(k)$$

the magnitude of the eigevalues of A are 3.6056, 1 and 0. Thus, the system is internally unstable.

The poles of the transfer function are 0 and -1, therefore the system is BIBO unstable.

Steady state evaluation

Let us consider the minimal LTI system with transfer function

$$H(s) = \frac{s+2}{s^3 + 6s^2 + 50s + 90}.$$

All the poles of H have negative real part, therefore the system is BIBO stable.

Then, we can compute the steady state response $y_s s(t)$. The problem just asks to compute the maximum magnitude of the output steady state response

$$|y_{ss}(t)|$$

when the input is

$$u(t) = (0.2 + \sin(4t))\varepsilon(t).$$

From the theory, we know that the output steady state response of a step function is a step function. Its amplitude can be computed by using the final value theorem, it is equal to the product of the amplitude $\bar{u}$ of the input and the dc-gain of H. In this case, $0.2 * 0.0222 = 0.0044$.

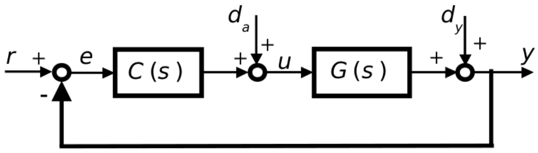
When the input is a sinusoidal function with frequency ω_0 and amplitude $\bar{u}$, the output is of the kind $\bar{u}|H(j\omega_0)|\sin(\dots)$. In this case, $1 * 0.0329 = 0.0329$. In conclusion,

$$|y_{ss}(t)| \leq 0.0044 + 0.0329 = 0.0373.$$

Feedback control stability

Given a plant with trasfer function $G(s) = \frac{2}{(s^2+s-2)}$, first of all we notice that G has poles -2 and 1, therefore it is unstable.

If we consider a feedback control system (see the figure) with controller C with a zero in 1, then we have an unstable zero-pole cancellation and the feedback control system is not stable.



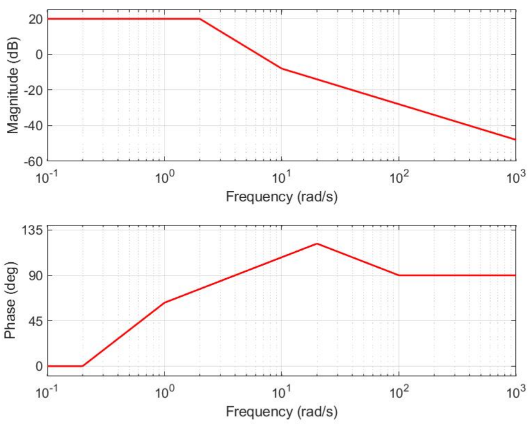
If there is no unstable cancellation, the poles of any relevant transfer function (S, T,...) correspond to the zeros of $1 + L(s)$

If $C(s) = \frac{4}{s}$, $1 + L(s)$ has zeros with positive real part, therefore the system is not stable.

If $C(s) = \frac{5(s+0.5)}{s}$, all the zeros of $1 + L(s)$ have negative real real part, therefore the system is stable.

Analysis of Bode plots

An LTI dynamical system is described by its transfer function $H(s)$ (supposed minimal). The asymptotic Bode diagrams of $H(s)$ are reported below.



It is evident that this system is not stable, because it has a positive pole at frequency 2. Therefore, it is not possible to evaluate the steady state response $y_{ss}(t)$.

~~[INF] Equilibria of non-linear systems~~

~~Given the non linear dynamical system~~

$$\cancel{\begin{cases} \dot{x}_1(t) = x_2(t) + u(t) \\ \dot{x}_2(t) = e^{-(x_1(t)-2)} + u(t) \end{cases}}$$

and the equilibrium input

$$\bar{u} = -0.1,$$

we have to compute the corresponding equilibrium state

$$\bar{x} = (\bar{x}_1 \quad \bar{x}_2)^T .$$

~~We solve the equations~~

$$\cancel{\begin{cases} 0 = \bar{x}_2 - 0.1 \\ 0 = e^{-(\bar{x}_1-2)} - 0.1 \end{cases}}$$

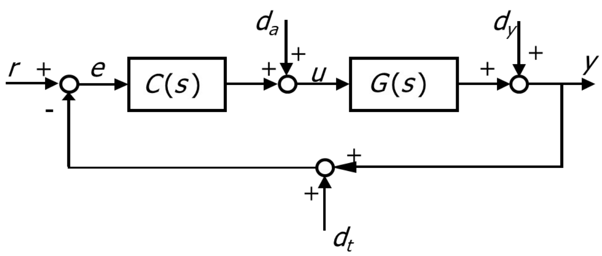
$$\cancel{\Leftrightarrow \begin{cases} \bar{x}_2 = -0.1 \\ x_1 = 2 - \log(0.1) = 4.3026 \end{cases}}$$

~~To evaluate the stability of $(\bar{(x)}, \bar{(u)})$, we compute the matrix A of the linearization around it.~~

~~It results $A = \begin{pmatrix} 0 & 1 \\ -e^{-(\bar{x}_1-2)} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -0.1 & 0 \end{pmatrix} =$ Since its eigevalues have both real part 0, no conclusion can be drawn.~~

ECE Design problem

Consider the following control system



with

$$G(s) = \frac{1.9837}{\left(1 + \frac{s}{142}\right)\left(1 + \frac{s}{14.2}\right)}$$

$d_y(t) = \delta_y \varepsilon(t), \quad |\delta_y| \leq 0.4, \quad d_t(t) = \delta_t \sin(\omega_t t), \quad |\delta_t| \leq 0.1, \quad \omega_t \geq 350\ rad/s.$

Design a controller $C(s)$ that meets the following requirements:

1. $|y^{\infty}_{d_y}| = 0$
2. $|y^{\infty}_{d_t}| \leq 0.0018$
3. $\widehat{S} \leq 15\%$
4. $t_{s,1\%} \leq 0.25$ s

Steady state analysis

The system if type 1.

$C_{ss}(s) = \frac{K_c}{s}$, with $K_c = 1$.

Sinusoidal disturbace analysis

$M_T^{HF} = -35$ dB

$\omega_{c,des} = 35$ rad/s

Transient and requirements analysis

$\zeta = 0.52$, $T_p = 1.06$, $S_s = 3.19$, $\omega_{c,des} = 28$ rad/s.

Design procedure description

From the Nichols plot of $L = C_{ss}G$, we see that $L(j\omega_{c,des})$ has phase around $-164°$ and we have to move it to around $-120°$. Then, we need a phase increase of $44°$ approximately. We can obtain this through a lead network. We select $m_D = 10$ and $\omega_{norm} = 1.2$.

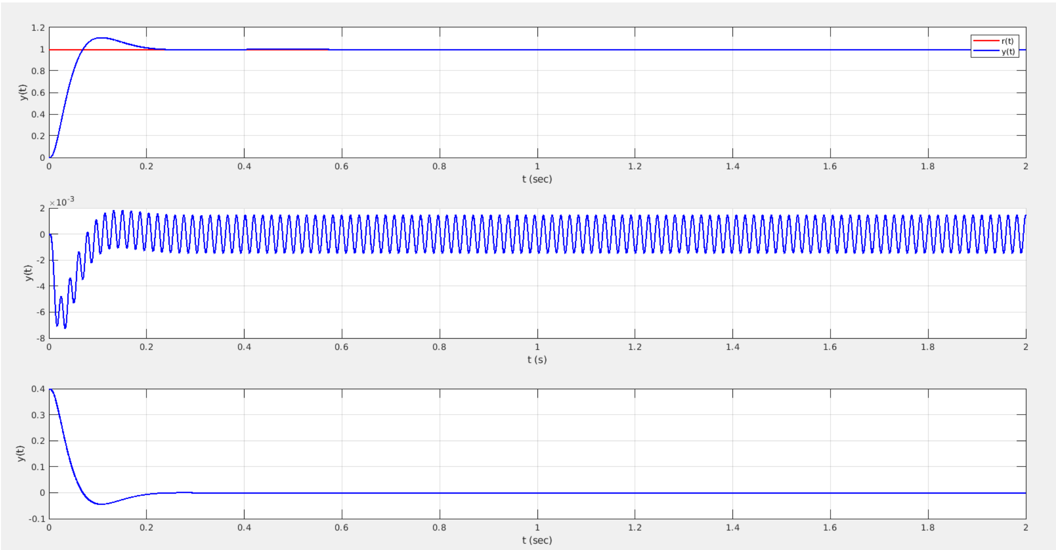
At this point, the phase is ok, while we need a gain increase of 26 dB approximately, corresponding to $K = 20$. This is feasible because there are no constraints on the gain in the steady state analysis.

Design results

$$C(s) = \frac{K_c * K}{s} \frac{1 + s/\omega_D}{1 + s/(m_D \omega_D)}$$

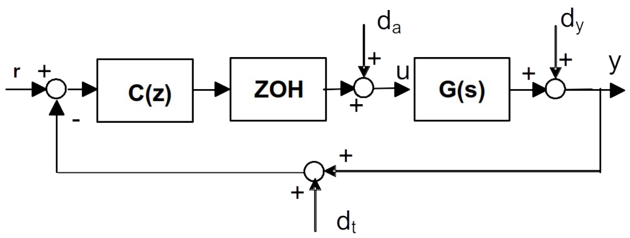
where $K_c = 1$, $K = 20$, $\omega_D = 23.33$, $m_D = 10$

Requirement	1	2	3	4
Value	0	0.0015	10.6	0.07



~~INF Design problem~~

~~Consider the following feedback control system~~



~~with~~

$$G(s) = \frac{120}{s^3 + 15.8s^2 + 12s}, \quad d_a(t) = \delta_a \varepsilon(t), \quad |\delta_a| \leq 0.4$$

$$\cancel{d_t(t) = \delta_t \sin(\omega_t t), \quad |\delta_t| \leq 1, \quad \omega_t \geq 100 \text{ rad/s.}}$$

~~Assuming a sampling time $T_s = 0.015$ s, design a digital controller $C_d(z)$ that meets the following requirements:~~

1. $|y_{d_a}^{\infty}| \leq 0.5$;

2. ~~$|y_{d_t}^{\infty}| \leq 0.8 \cdot 10^{-2}$;~~

3. ~~$\widehat{S} \leq 18\%$;~~

4. ~~$t_r \leq 0.25$ s;~~

~~Steady state analysis~~

~~The system if type 1.~~

~~$C_{ss}(s) = K_c$, with $K_c = 0.8$.~~

~~Sinusoidal disturbace analysis~~

~~$M_T^{HF} = -41.9$ dB~~

~~$\omega_{c,des} = 10$ rad/s~~

~~Transient and requirements analysis~~

~~$\zeta = 0.48$, $T_p = 1.58$, $S_s = 3.52$, $\omega_{c,des} = 7.5$ rad/s.~~

~~Design procedure description~~

~~From the Nichols plot of $L = C_{ss}G$, we see that $L(j\omega_{c,des})$ has phase around $-204°$ and magnitude -20 dB.~~

~~We need a phase lead of about $80°$. Then, we use a real negative zero with $\omega_{norm} = 6$.~~

~~Then, we apply a small gain adjustment ($K = 1.58$).~~

~~Finally, we add a closure pole, far enough from the zero, to make the controller proper.~~

~~Design results~~

$$\cancel{C_0(s) = K_c * K \frac{1 + s/\omega_Z}{1 + s/\omega_P}}$$

~~where $K_c = 0.8$, $K = 1.58$ $\omega_Z = 1.127$, $\omega_P = 100$~~

$$\cancel{C_d(z) = (57.74z - 56.66)/(z - 0.1429)}$$

Requirement	1	2	3	4
Value	0.32	0.0009	16.33	0.2438