Exercise Report-4

For

Course INF 5620/9620 -Numerical Methods for the Partial Differential Equation

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Exercise 8: Compute with a Non-uniform mesh

(a) Derive the linear system for the problem –u''=2 on [0, 1], with u (0) =0 and u (1) =1, using P1 elements and a non-uniform mesh. The vertices have coordinates $x0=0<x_1<...<x_{Nn-1}=1$, and the length of cell number e is $he=x_{e+1}-x_e$

Based on the principle of Galerkin method: (R,v) = 0, ∀ v∈ V

The corresponding Galerkin solution is derived: (u''+2, v) = 0, ∀ v∈ V or (u'',v)=-(2,v) (1)

According to the integration by part:

$$(u'', v) = \int_0^1 u''\, v dx = u'(1)v(1) - u'(0)v(0) - \int_0^1 u'\, v' dx \tag{2}$$

From the equation (2), it can be further derived as

$$(u'', v) = \int_0^1 u''\, v dx = u'(1)v(1) - u'(0)v(0) - (u', v') \tag{3}$$

Thus we can derive the variational form for the initial partial differential equation as following expression

$$(u', v') = (2, v) + u'(1)v(1) - u'(0)v(0) \tag{4}$$

The equation (4) is equivalent to (5) considering V= span {$\psi_{1,}$ $\psi_{2 ...}$ ψ_N}

$$(u', \psi_i') = (2, \psi_i) + u'(1)\psi_i(1) - u'(0)\psi_i(0) \tag{5}$$

It yields:

$$(u', \psi_i') = \begin{cases} (2, \psi_0) - u'(0), & i = 0 \\ (2, \psi_0), & 0 < i < x_{N-1} \\ (2, \psi_0) + u'(1), & i = x_{N-1} \end{cases} \tag{6}$$

Thus based on the equation (6), we can see that the PDE problem –u''=2 can be solved with the variational form (u',v')=(2,v) by using all P1 elements, and the modification for the linear system only need be done on the boundary points.

Thus the linear system for solving this PDE is

$$\sum_j (\psi_i', \psi_j') c_j = (2, \psi_i), \ \ i \in \tau_s \tag{7}$$

Where $A(i,j) = (\psi_i', \psi_j') = \int_0^1 \psi_i'\psi_j' dx$, and $b_i = (2, v) = \int_0^1 2\psi_i dx$

Considering we are using P1 element with non-uniform mesh in our solution, we have the basis function definition

$$\psi_i(x) = \begin{cases} 0, & x < x_{i-1} \\ \frac{x-x_{i-1}}{x_i-x_{i-1}}, & x_{i-1} < x < x_i \\ 1-\frac{x-x_i}{x_{i+1}-x_i}, & x_i < x < x_{i+1} \\ 0, & x > x_{i-1} \end{cases} \tag{8}$$

And corresponding first order derivatives:

$$\psi_i{}'(x) = \begin{cases} 0, & x < x_{i-1} \\ \frac{1}{x_i-x_{i-1}}, & x_{i-1} < x < x_i \\ -\frac{1}{x_{i+1}-x_i}, & x_i < x < x_{i+1} \\ 0, & x > x_{i-1} \end{cases} \tag{9}$$

Thus we can derive the coefficient matrix A (i,j) with following:

For the boundary points (i = 0, and i = N-1):

$$\begin{gathered} A_{0,0} = \int_0^1 \psi_0'\psi_0' dx = \frac{1}{x_1 - x_0} \\ A_{0,1} = \int_0^1 \psi_0'\psi_1' dx = \frac{-1}{x_1-x_0} \\ A_{N-1,N-1} = \int_0^1 \psi_{N-1}'\psi_{N-1}' dx = \frac{1}{x_{N-1} - x_{N-2}}, \\ A_{N-1,N-2} = \int_0^1 \psi_{N-1}'\psi_{N-2}' dx = \frac{-1}{x_{N-1} - x_{N-2}} \end{gathered} \tag{10}$$

Also for all interior points (0<i<N-1), we can derive the A (I,j) with below forms:

$$\begin{gathered} A_{i,i+1} = \int_0^1 \psi_i'\psi_{i+1}' dx = \frac{-1}{x_{i+1} - x_i} \\ A_{i,i} = \int_0^1 \psi_i'\psi_i' dx = \frac{1}{x_i-x_{i-1}} + \frac{1}{x_{i+1}-x_i} \\ A_{i,i-1} = \int_0^1 \psi_i'\psi_{i-1}' dx = \frac{-1}{x_i - x_{i-1}} \end{gathered} \tag{11}$$

Meanwhile we can also derive the coefficient matrix b (i) with following:

For the boundary points (i = 0, and i = N-1):

$$b_0 = 2 * \int_0^1 \psi_0 dx = x_1 - x_0$$

$$b_{N-1} = 2 * \int_0^1 \psi_{N-1} dx = x_{N-1} - x_{N-2} \tag{12}$$

Also for all interior points (0<i<N-1), we can derive the b (I) with below forms:

$$b_i = 2 * \int_0^1 \psi_i dx = x_{i+1} - x_{i-1} \tag{13}$$

So based on the coefficient matrix A(I,j) and b(i), we can derive the linear system:

$$\begin{bmatrix} \frac{1}{x_1-x_0} & \frac{-1}{x_1-x_0} & 0 & 0 & 0 & \dots & 0 & 0 \\ \frac{-1}{x_1-x_0} & \frac{1}{x_1-x_0}+\frac{1}{x_2-x_1} & \frac{-1}{x_2-x_1} & 0 & 0 & \dots & 0 & 0 \\ 0 & \frac{-1}{x_2-x_1} & \frac{1}{x_2-x_1}+\frac{1}{x_3-x_2} & \frac{-1}{x_3-x_2} & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & \dots & \dots & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & \dots & \dots & \dots & \frac{-1}{x_{N-1}-x_{N-2}} & \frac{1}{x_{N-1}-x_{N-2}} \end{bmatrix} * \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ . \\ . \\ . \\ c_{N-2} \\ c_{N-1} \end{bmatrix} = \begin{bmatrix} x_1 - x_0 \\ x_2 - x_0 \\ x_3 - x_1 \\ . \\ . \\ . \\ c_{N-1} - c_{N-3} \\ x_{N-1} - x_{N-2} \end{bmatrix} \tag{14}$$

As we mentioned, to make the linear system applicable for this PDE problem, we need modify the first and last row of the linear system (14) to make it to satisfy with the boundary condition u (0) =0 and u (1) =1. Thus we only need modify the first row to $C_0 = 0$, and last row $C_{N-1} = 1$ and derive the following updated linear system:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \frac{-1}{x_1-x_0} & \frac{1}{x_1-x_0}+\frac{1}{x_2-x_1} & \frac{-1}{x_2-x_1} & 0 & 0 & \dots & 0 & 0 \\ 0 & \frac{-1}{x_2-x_1} & \frac{1}{x_2-x_1}+\frac{1}{x_3-x_2} & \frac{-1}{x_3-x_2} & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & \dots & \dots & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & \dots & \dots & \dots & 0 & 1 \end{bmatrix} * \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ . \\ . \\ . \\ c_{N-2} \\ c_{N-1} \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 - x_0 \\ x_3 - x_1 \\ . \\ . \\ . \\ c_{N-1} - c_{N-3} \\ 1 \end{bmatrix} \tag{15}$$

So by solving the linear system (15), we can derive the coefficient c_i (i = 0,1…N-1)

And the corresponding PDE solution: $u(x) = \sum_{i=0}^{N-1} c_i \psi_i$(x)

(b) It is of interest to compare the discrete equations for the finite element method in a non-uniform mesh with the corresponding discrete equations arising from a finite difference method. Go through the derivation of the finite difference formula u'' (xi) ≈ $[D_xD_xu]_i$ and modify it to find a natural discretization of u''(x_i) on a non-uniform mesh. Compare the finite element and difference discretization:

According to the linear system (15), we can generalize the system as:

$$-\left(\frac{1}{x_i - x_{i-1}}\right)c_{i-1} + \left(\frac{1}{x_i - x_{i-1}} + \frac{1}{x_{i+1} - x_i}\right)c_i - \left(\frac{1}{x_{i+1} - x_i}\right)c_{i+1} = x_{i+1} - x_i \qquad (16)$$

Considering the u(x_i) can be expressed as c_i as the following:

$$u(x_i) = \sum_{i=0}^{N-1} c_i\psi_i\,(x_i) = c_i \qquad (17)$$

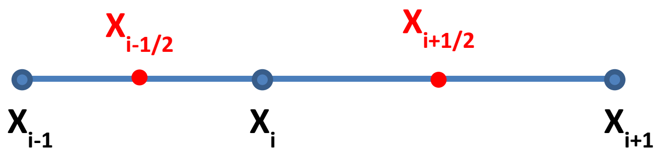
The equation (16) can be rewritten as:

$$-\left(\frac{1}{x_i - x_{i-1}}\right)u(x_{i-1}) + \left(\frac{1}{x_i - x_{i-1}} + \frac{1}{x_{i+1} - x_i}\right)u(x_i) - \left(\frac{1}{x_{i+1} - x_i}\right)u(x_{i+1}) = x_{i+1} - x_i \qquad (18)$$

Then equation (18) can be rearranged as following:

$$\frac{\left(\frac{1}{x_{i+1}-x_i}\right)\left(u(x_{i+1}) - u(x_i)\right) - \left(\frac{1}{x_i - x_{i-1}}\right)\left(u(x_i) - u(x_{i-1})\right)}{x_{i+1} - x_i} = -1 \qquad (19)$$

Meanwhile, we can review the derivation of the finite difference approximation of u''(x_i) based on the non-uniform meshes as below:



$$u''(x_i) = \frac{u'(x_{i+\frac{1}{2}}) - u'(x_{i-\frac{1}{2}})}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}} = \frac{\frac{u(x_{i+1}) - u(x_i)}{x_{i+1} - x_i} - \frac{u(x_i) - u(x_{i-1})}{x_i - x_{i-1}}}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}} \qquad (20)$$

Also considering: $x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} = \frac{1}{2}\,(x_{i+1} - x_{i-1})$ (21)

Thus u''(x_i) can be expressed as:

$$u''(x_i) = \frac{\frac{u(x_{i+1}) - u(x_i)}{x_{i+1} - x_i} - \frac{u(x_i) - u(x_{i-1})}{x_i - x_{i-1}}}{\frac{1}{2}(x_{i+1} - x_{i-1})} \qquad (21)$$

So the PDE –u''=2 can be approximated by the derived non-uniform mesh finite difference method as:

$$u''(x_i) = \frac{\frac{u(x_{i+1})-u(x_i)}{x_{i+1}-x_i} - \frac{u(x_i)-u(x_{i-1})}{x_i-x_{i-1}}}{\frac{1}{2}(\,x_{i+1}-x_{i-1})} = -2 \tag{22}$$

It can be simplified as below:

$$u''(x_i) = \frac{\frac{u(x_{i+1})-u(x_i)}{x_{i+1}-x_i} - \frac{u(x_i)-u(x_{i-1})}{x_i-x_{i-1}}}{(\,x_{i+1}-x_{i-1})} = -1 \tag{23}$$

This is identical to the finite element derived approximation at equation (19), thus the approximation made by finite difference and by the finite element methods are equivalent for this PDE problem.