Exercise Report-3

For

Course INF 5620/9620 -Numerical Methods for the Partial Differential Equation

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Exercise 2: Compute the deflection of a cable with sine functions

The scaled version of the vertical deflection u is defined as:

$$u'' = 1, \qquad x \in (0,1), u(0) = 0, u'(1) = 0 \qquad (1)$$

(a)Find the exact solution for the deflection u

Since u''(x) = 1, we can derive the u(x) by do the indefinite integration twice as following:

$$u'(x) = \int 1\,dx = x + c0, \qquad u(x) = \int (x + c0)dx = \frac{x^2}{2} + c0 * x + c1 \qquad (2)$$

Then combine with the boundary condition: u(0) =0 and u'(1) = 0, we can derive c0=-1, and c1 = 0,

So the exact solution $u(x) = \frac{x^2}{2} - x$ (3)

The corresponding code is included in the python script.

(b) Based on the basis function: ψj = sin((2i+1)πx/2) ,i=0,1..N, using the Galerkin and least squares method to find coefficients cj in u(x)= ∑ cj ψj,

1) Galerkin method

Based on the principle of Galerkin method: (R,v) = 0, ∀ v∈ V

The corresponding Galerkin solution is derived: (u''-1, v) = 0, ∀ v∈ V or (u'',v)=(1,v) (4)

According to the integration by part:

$$(u'', v) = \int_0^1 u''\, v dx = u'(1)v(1) - u'(0)v(0) - \int_0^1 u'\, v' dx \qquad (5)$$

Also considering $\psi_i(0)=0$, and $u'(1)=0$, the equation (5) can be further derived as

$$(u'', v) = \int_0^1 u''\, v dx = 0 - \int_0^1 u'\, v' dx = -(u', v') \qquad (6)$$

Thus we can derive the variational form for the initial partial differential equation as following expression

$$(u', v') = -(1, v) \tag{7}$$

The equation (7) is equivalent to the linear system

$$\sum_j (\psi_i', \psi_j') c_j = -(1, \psi_i), \;\; i \in \tau_s \tag{8}$$

Where $A(i,j) = (\psi_i', \psi_j') = \int_0^1 \left(sin\frac{(2i+1)\pi x}{2}\right)' \left(sin\frac{(2j+1)\pi x}{2}\right)' dx$, and we can derive the A matrix

$$A(i,j) = \begin{cases} \frac{((2i+1)*\pi)^2}{8}, & i = j \\ 0, & i \neq j \end{cases} \tag{9}$$

and $b_i = -(1, v) = -\int_0^1 sin\frac{(2i+1)\pi x}{2} dx = \frac{-2}{(2i+1)*\pi}$ (10)

Since A(i,j) is diagonal, we can easily derive the $c_j = \frac{b_i}{A(i,j)} = \frac{-16}{((2i+1)*\pi)^3}$ (11)

Thus the approximating solution u(x) based on Galerkin method becomes

$$u(x) = \sum_{i=0}^{N} \frac{-16}{((2i+1)*\pi)^3} * sin\frac{(2i+1)\pi x}{2} \tag{12}$$

2) Least Square Method

Based on the principle of Least Square method: $\left(R, \frac{\partial R}{\partial c_i} = 0\right)$

We can derive the residual R and $\frac{\partial R}{\partial c_i}$

$$R = u'' - 1 = \sum_{j \in \tau_s} c_j \psi_j'' - 1 \tag{13}$$

$$\frac{\partial R}{\partial c_i} = \frac{\partial}{\partial c_i}\left(\sum_{j \in \tau_s} c_j \psi_j'' - 1\right) = \psi_i'' \tag{14}$$

Thus the corresponding solution can be derived as following:

$$\left(\sum_{j \in \tau_s} c_j \psi_j'' - 1, \psi_i''\right) = 0 \tag{15}$$

This yield:

$$\sum_j (\psi_i'', \psi_j'') c_j = (1, \psi_i''), \;\; i \in \tau_s \tag{16}$$

This is equal to a linear system

$$\sum_j A_{ij} c_j = b_i, \;\; i \in \tau_s$$

The corresponding coefficient matrix are given by

$$A_{ij} = (\psi_i'', \psi_j'') = \int_0^1 \left(sin\frac{(2i+1)\pi x}{2}\right)'' \left(sin\frac{(2j+1)\pi x}{2}\right)'' dx \quad (17)$$

Which yield

$$A(i,j) = \begin{cases} \frac{((2i+1)*\pi)^4}{32}, & i = j \\ 0, & i \neq j \end{cases} \quad \begin{matrix}(18)\\(19)\end{matrix}$$

and $b_i = (1, \psi_i'') = \int_0^1 \left(sin\frac{(2i+1)\pi x}{2}\right)'' dx = \frac{-\pi(2i+1)}{2}$

Since A(i,j) is diagonal, we can easily derive the $c_j = \frac{-16}{((2i+1)*\pi)^3}$ (20)

Thus the approximating solution u(x) based on least square method becomes

$$u(x) = \sum_{i=0}^{N} \frac{-16}{((2i+1)*\pi)^3} * sin\frac{(2i+1)\pi x}{2} \quad (21)$$

Thus both the Galerkin and least square method derive the same result, and the decreasing speed of the coefficients' magnitude is:

$$\frac{c_j}{c_{j-1}} = \left(\frac{2i-1}{2i+1}\right)^3 \quad (22)$$

For N=0, the error in the maximum deflection x =1 is:

$$Error(1) = u_e(1) - u(1) = \left(\frac{1}{2} - 1\right) - \left(-\frac{16}{\pi^3}\right) = 0.016$$

(c) Visualize the solution in b for N= 0,1,20

The PDE solution derived from Galerkin and least square methods are displayed below, the corresponding code is also implemented in python.

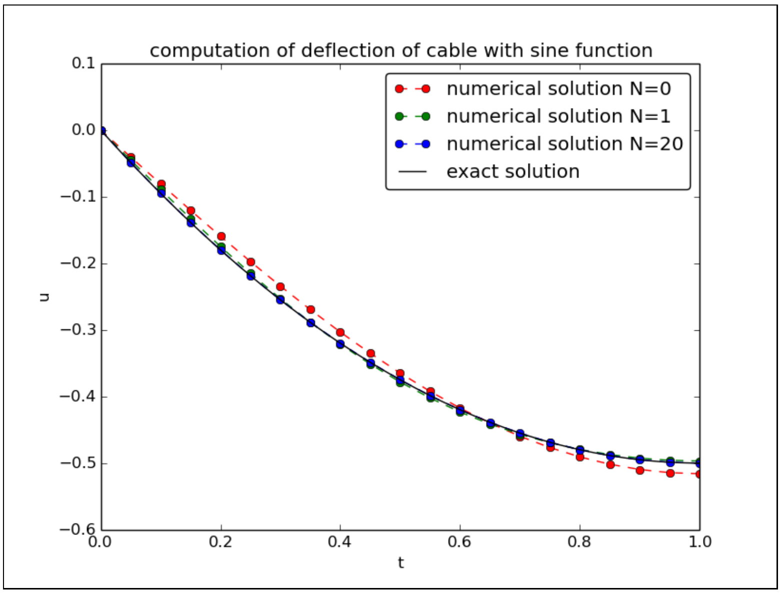

Figure 1 comparison of numerical solution

(D) if chose the basis function : ψj = sin((i+1)πx/2) ,i=0,1..N, Will the method adjust the coefficient such that the additional functions compared with those in b) get vanishing coefficients? Or will the additional basis functions improve the solution? Use Galerkin's method.

Based on the principle of Galerkin method: (R,v) = 0, ∀ v∈ V

The corresponding Galerkin solution is derived: (u''-1, v) = 0, ∀ v∈ V or (u'',v)=(1,v)

According to the integration by part:

$$(u'', v) = \int_0^1 u''\, v dx = u'(1)v(1) - u'(0)v(0) - \int_0^1 u'\, v' dx \tag{22}$$

Also considering $\psi_i(0)=0$, and $u'(1)=0$, the equation (5) can be further derived as

$$(u'', v) = \int_0^1 u''\, v dx = 0 - \int_0^1 u'\, v' dx = -(u', v') \tag{23}$$

Thus we can derive the variational form for the initial partial differential equation as following expression

$$(u', v') = -(1, v) \tag{24}$$

The equation (24) is equivalent to the linear system

$$\sum_j (\psi_i', \psi_j') c_j = -(1, \psi_i), \;\; i \in \tau_s$$

Where $A(i,j) = (\psi_i', \psi_j') = \int_0^1 \left(sin\frac{(\mathrm{i}+1)\,\pi x}{2}\right)' \left(sin\frac{(\mathrm{j}+1)\,\pi x}{2}\right)' dx$, and. Considering the (ψ_i', ψ_j') are not orthogonal in the current case, thus we can derive the A matrix with below form:

$$A(i,j) = \begin{cases} \frac{\left((i+1)*\pi\right)^2}{8}, & i\ = j \quad (25) \\ \int_0^1 \left(sin\frac{(\mathrm{i}+1)\,\pi x}{2}\right)' \left(sin\frac{(\mathrm{j}+1)\,\pi x}{2}\right)' dx\,, & i\ \neq j \quad (26) \end{cases}$$

Also $b_i = -(1, v) = -\int_0^1 \frac{\sin(i+1)\pi \mathrm{x}}{2} dx = \frac{-2}{(i+1)*\pi}$

Thus we can derive the c_i by solving the linear system:

$$\sum_j A_{ij} c_j = b_i, \;\; i \in \tau_s$$

In order to compare the solution with the solutions based on previous basis functions, I choose the N=1 for the numerical comparison of the two approaches based on different basis function:

Based on current basis function, we can derive the c_i with below linear equations

$$\begin{bmatrix} \frac{\pi^2}{8} & \frac{\pi}{3} \\ \frac{\pi}{3} & \frac{\pi^2}{2} \end{bmatrix} * \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} \frac{-2}{\pi} \\ \frac{-1}{\pi} \end{bmatrix}$$

It yields:

$$\begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} \frac{48 - 144\pi}{-16\pi^2 + 9\pi^4} \\ \frac{96 - 18\pi}{-16\pi^2 + 9\pi^4} \end{bmatrix}$$

Thus the u(x) can be derived as:

$$u(x) = \sum_{i=0}^{1} c_i * sin\frac{(\mathrm{i}+1)\,\pi x}{2} = \frac{48 - 144\pi}{-16\pi^2 + 9\pi^4} * \sin(\frac{\pi x}{2}) + \frac{96 - 18\pi}{-16\pi^2 + 9\pi^4} * \sin(\pi x) \qquad (27)$$

Based on previous basis function, we can derive the c_0, the corresponding u(x) and the error at x = 1:

$$c_0 = \frac{-16}{\pi^3};\ c_1 = \frac{-16}{27\pi^3}$$

$$u(x) = \sum_{i=0}^{0} c_i * sin\frac{(2\mathrm{i}+1)\,\pi x}{2} = \frac{-16}{\pi^3} * \sin(\frac{\pi x}{2}) + \frac{-16}{27\pi^3} * \sin(\frac{3\pi x}{2}) \qquad (28)$$

So to compare the accuracy of the two solutions (26) and (27), the visualization is made to compare the solution derived from different basis functions. According to the display, we can see that the solution derived from additional basis function caused more error than the solution derived from the prior basis function.

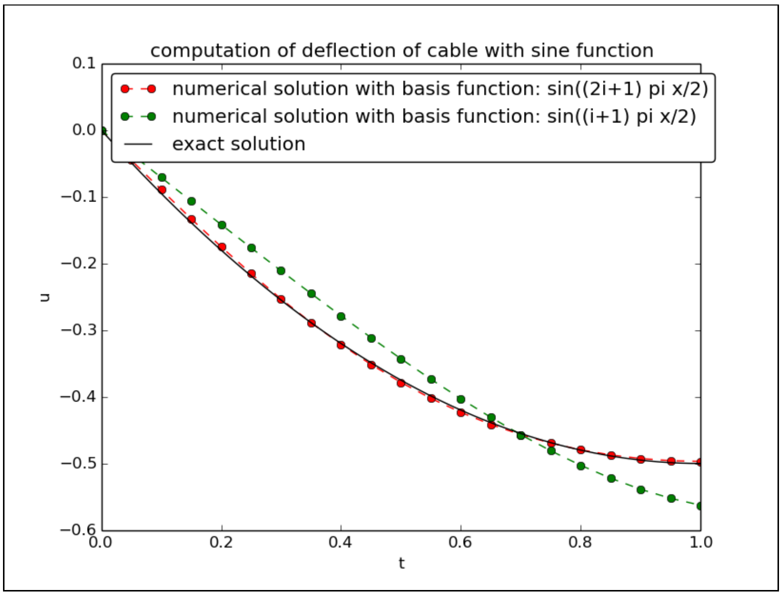

Figure 2 comparison of solution derived from different basis functions

(E) Now we drop the symmetry condition at x=1 and extend the domain to [0, 2] such that it covers the entire (scaled) physical cable. The problem now reads U''=1,x∈(0,2),u(0)=u(2)=0.This time we need basis functions that are zero at x=0 and x=2. The set sin ((i+1)πx/2) from d) is a candidate since they vanish x=2 for any i. Compute the approximation in this case. Why is this approximation without the problem that this set of basis functions introduced in d)?

To solve this problem, I used Galerkin method again in this question.

Based on the principle of Galerkin method: (R,v) = 0, ∀ v∈ V

The corresponding Galerkin solution is derived: (u''-1, v) = 0, ∀ v∈ V or (u'',v)=(1,v)

According to the integration by part:

$$(u'',v) = \int_0^2 u''\, v dx = u'(2)v(2) - u'(0)v(0) - \int_0^2 u'\, v' dx \tag{29}$$

Also considering $\psi_i(0)=0$, and $\psi_i(2)=0$ for the basis function sin ((i+1)πx/2) , the equation (27) can be further derived as

$$(u'',v) = \int_0^2 u''\, v dx = 0 - \int_0^2 u'\, v' dx = -(u',v') \tag{30}$$

Thus we can derive the variational form for the initial partial differential equation as following expression

$$(u',v') = -(1,v) \tag{31}$$

The equation (29) is equivalent to the linear system

$$\sum_j (\psi_i', \psi_j') c_j = -(1, \psi_i), \;\; i \in \tau_s \tag{32}$$

Where $A(i,j) = (\psi_i', \psi_j') = \int_0^1 \left(sin\frac{(i+1)\pi x}{2}\right)' \left(sin\frac{(j+1)\pi x}{2}\right)' dx$, and we can derive the A mattrix

$$A(i,j) = \begin{cases} \frac{((i+1)*\pi)^2}{4}, & i = j \\ 0, & i \neq j \end{cases} \tag{33}$$

and $b_i = -(1,v) = -\int_0^2 sin\frac{(i+1)\pi x}{2} dx = \frac{-2*((-1)^i+1)}{(i+1)*\pi}$ (34)

Since A(i,j) is diagonal in this case, we can easily derive the $c_j = \frac{b_i}{A(i,j)} = \frac{-8*((-1)^i+1)}{((i+1)*\pi)^3}$ (35)

Thus the approximating solution u(x) based on Galerkin method becomes

$$u(x) = \sum_{i=0}^{N} \frac{-8*((-1)^i+1)}{((i+1)*\pi)^3} * \frac{\sin(i+1)\pi x}{2} \quad (36)$$

And the approximating solution u(x) based on Galerkin method (number of basis function is applied: 0,1 and 20) is showing below:

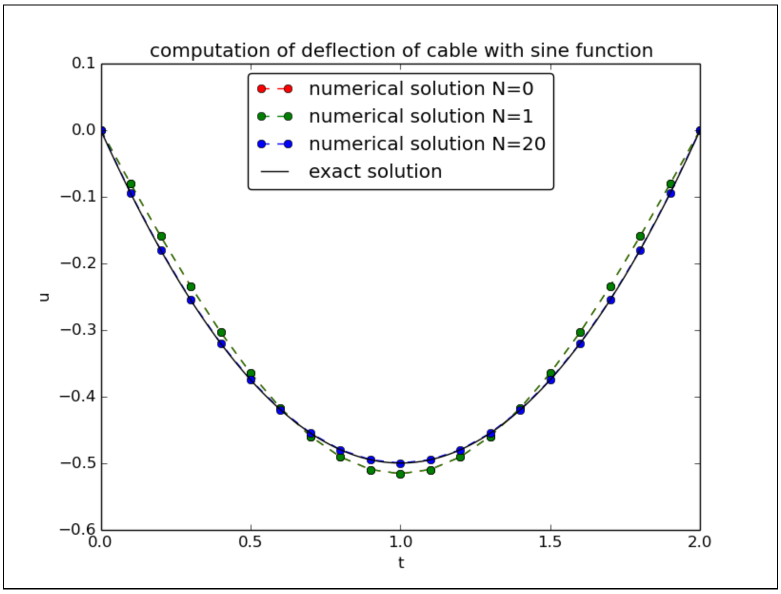

Figure 3 comparison of numerical solution

The python code has included all the display from this exercise, please check the corresponding code for reference.

Exercise5: Compute the deflection of a cable with 2 P1 Elements

Solve the problem for u in Exercise 2: Compute the deflection of a cable with sine functions using two P1 linear elements. Incorporate the condition u(0)=0 by two methods: 1) excluding the unknown at x=0, 2) keeping the unknown at x=0, but modifying the linear system.

$$u'' = 1, x \in (0,1) \;\; u(0) = 0, u'(1) = 0$$

1) **Excluding the unknown at x=0**

Based on the principle of Galerkin method: (R,v) = 0, ∀ v∈ V

The corresponding Galerkin solution is derived: (u''-1, v) = 0, ∀ v∈ V or (u'',v)=(1,v)

According to the integration by part:

$$(u'', v) = \int_0^1 u''\, v dx = u'(1)v(1) - u'(0)v(0) - \int_0^1 u'\, v' dx \tag{37}$$

It equals to:

$$(u'', v) = (u'', \psi_i) = \int_0^1 u''\, \psi_i dx = u'(1)\psi_i(1) - u'(0)\psi_i(0) - \int_0^1 u'\, \psi_i' dx \tag{38}$$

Also considering $u'(1)=0$, $u(0)=0$and in the current case, we will use 2 P1 element for the calculation while excluding the unknown at x =0, so we can define the u(x) as the below form:

$$u(x) = u_0 * \psi_0(x) + \sum_{j=1}^{2} c_j * \psi_j(x) = \sum_{j=1}^{2} c_j * \psi_j(x) \tag{39}$$

Also the equation (38) can be further derived as

$$(u'', v) = (u'', \psi_i) = \int_0^1 u''\, \psi_i dx = -\int_0^1 u'\, \psi_i' dx = -(u', \psi_i') \tag{40}$$

Thus we can derive the variational form for the initial partial differential equation as following expression

$$(u', \psi_i') = -(1, \psi_i')$$

The equation (41) is equivalent to the linear system

$$\sum_j(\psi_i',\psi_j')c_j = -(1,\psi_i),\quad i \in \tau_s \tag{42}$$

Where $A(i,j) = (\psi_i',\psi_j')$, $b(i) = -(1,\psi_i)$

Considering we are using P1 element in our solution, we have the basis function definition

$$\psi_i(x) = \begin{cases} 0, & x < x_{i-1} \\ \frac{(x-x_{i-1})}{h}, & x_{i-1} < x < x_i \\ 1-\frac{(x-x_i)}{h}, & x_i < x < x_{i+1} \\ 0, & x > x_{i-1} \end{cases} \tag{43}$$

And corresponding first order derivative (h = 0.5 in this case)

$$\psi_i{}'(x) = \begin{cases} 0, & x < x_{i-1} \\ \frac{1}{h}, & x_{i-1} < x < x_i \\ -\frac{1}{h}, & x_i < x < x_{i+1} \\ 0, & x > x_{i-1} \end{cases} \tag{44}$$

Thus we can derive the below linear equation to derive the coefficient c:

$$\begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix} * \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{4} \end{bmatrix} \tag{45}$$

By solving this equation, we derived $c_1 = -\frac{3}{8}, c_2 = -\frac{1}{2}$

So the $u(x) = \sum_{i=1}^{2} c_j * \psi_j = -\frac{3}{8}\psi_1 - \frac{1}{2}\psi_2$ (46)

2) keeping the unknown at x=0, but modifying the linear system.

If we keeping the unknown at x =0, then we can define the u(x) as the below form:

$$u(x) = \sum_{J=0}^{2} c_j * \psi_j(x) \tag{47}$$

And we can also derive the variational form for the initial partial differential equation as following expression, but we will modify the final linear system to make the C_0 = 0 to reflect the Dirichlet boundary condition.

$$(u',\psi_i{}') = -(1,\psi_i{}')\text{ , it yields to} A(i,j) = (\psi_i',\psi_j'), b(i) = -(1,\psi_i) \tag{48}$$

Based on the expression of the first order shown from equation (44), we can derive the coefficient matrix A and bi with following form:

$$\begin{bmatrix} 2 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{bmatrix} * \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} \\ -\frac{1}{2} \\ -\frac{1}{4} \end{bmatrix} \tag{49}$$

And in order to reflect the boundary condition at x = 0, we will modify the first row to let $C_0 = 0$, thus the modified linear equation is displayed as equation (50)

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{bmatrix} * \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{2} \\ -\frac{1}{4} \end{bmatrix} \tag{50}$$

By solving these linear equations, we derived $c_0 = 0, c_1 = -\frac{3}{8}, c_2 = -\frac{1}{2}$

Thus the solution $u(x) = \sum_{i=1}^{2} c_j * \psi_j = -\frac{3}{8}\psi_1 - \frac{1}{2}\psi_2$ which is equal to equation (46).